Theory of Z-transforms

F. BOUVET

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Tutorial objective

Answer to the following questions:

- **What are Z-transforms ?**
- **Why are they so important ?**
- **How are they used for digital regulation ?**

Content

- Introduction
- Z-transforms
  - Definition & properties
  - Examples of Z-transforms
- Relation between Laplace and Z transforms
  - Computation of the Z-transform of a signal from its Laplace transform
  - Mapping from s-plane to z-plane
- Digitally controlled continuous-time systems
  - Modelling of the plant to be controlled
  - Open-loop and closed-loop transfer function
  - Controller algorithm
- Analysis of closed-loop systems
  - Stability of closed-loop systems and robustness
  - Influence of the poles & zeros on the transient behavior
  - Precision of closed-loop systems
- Discrete-time controller synthesis
  - Emulation design
  - Direct discrete-time design

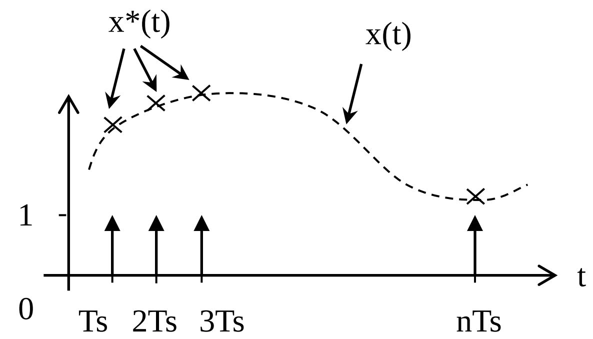
Introduction

- **Reminder about Laplace transform:**
  - Essential mathematical tool for continuous-time system & signal analysis ; stable or unstable
  - The Laplace transform X(s) of a continuous-time causal signal x(t) is given by

$$X(s) = \int_{0-}^{+\infty} x(t)\cdot e^{-st}\cdot dt \qquad s = \sigma + j\cdot w$$

  - Case of discrete-time causal signals:

$$x^*(t) = \sum_{k=0}^{+\infty} x(k)\cdot \delta(t - k\cdot Ts) \qquad x(k) = x(t)\big|_{t=k\cdot Ts}$$



{x(k)}, k∈ℤ: Sequence of sampled values (= 0 ∀ k<0)

Ts: Sampling period (assumed constant)

δ: Dirac delta function

=> The Laplace transform X*(s) of a discrete-time signal x*(t) is given by

$$X^*(s) = \sum_{k=0}^{+\infty} x(k)\cdot e^{-s\cdot k\cdot Ts} \qquad (1)$$

Not a polynomial form…

Z-transform

- **Definition**

With the change of variable $z = e^{s \cdot Ts}$ in eq. (1), we derive the following expression = definition of the Z-transform:

$$\boxed{X(z) = \sum_{k=0}^{+\infty} x(k) \cdot z^{-k}}$$

$\forall$ z $\in \mathbb{C}$ for which X(z) converges

=> Takes the form of a polynomial of the complex variable z

The Z-transform is the discrete-time counter-part of the Laplace transform
$\Rightarrow$ Essential tool for the analysis and design of discrete-time systems

- **Interpretation of the variable z^{-1}**

From Laplace time-shift property, we know that $z = e^{s \cdot Ts}$ is time advance by Ts second
Therefore $z^{-1} = e^{-s \cdot Ts}$ corresponds to unit sample period delay

Z-transform

- **Properties of Z-transforms**

- Linearity $$Z[\lambda \cdot x(k) + \mu \cdot y(k)] = \lambda \cdot X(z) + \mu \cdot Y(z)$$

- Shifting property $$Z[x(k-n)] = z^{-n} \cdot X(z)$$

- Convolution $$Z[x(k) * y(k)] = Z\left[\sum_{n=-\infty}^{n=+\infty} x(n) \cdot y(k-n)\right] = X(z) \cdot Y(z)$$

- Multiply by k property $$Z[k \cdot x(k)] = -z \cdot \frac{d}{dz}\left(X(z)\right)$$

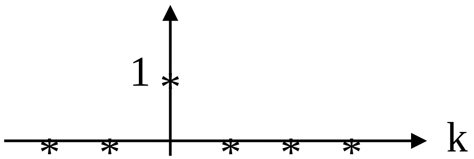
- Final value $$\lim_{k \to \infty} x(k) = \lim_{z \to 1} (z-1) \cdot X(z)$$

Z-transform

- **Examples of Z-transforms**

- Discrete impulse

$$x(k)=\delta(k)=\begin{cases}1 & k=0\\ 0 & k\neq 0\end{cases}$$

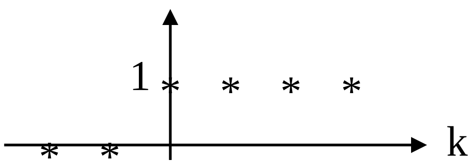


$$X(z)=\sum_{k=0}^{\infty}x(k)\cdot z^{-k}=x(0)+x(1)\cdot z^{-1}+x(2)\cdot z^{-2}+\ldots=x(0)$$

$$\boxed{\Rightarrow X(z)=1}$$

- Discrete step

$$x(k)=\begin{cases}1 & k\geq 0\\ 0 & k<0\end{cases}$$

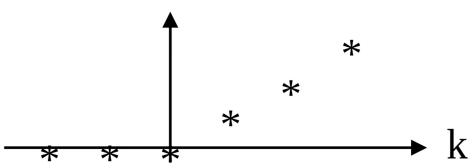


$$X(z)=x(0)+x(1)\cdot z^{-1}+x(2)\cdot z^{-2}+\ldots=1+z^{-1}+z^{-2}+\ldots$$

$$\boxed{\Rightarrow X(z)=\frac{1}{1-z^{-1}}\quad |z>1|}$$

- Discrete ramp

$$x(k)=\begin{cases}k & k\geq 0\\ 0 & k<0\end{cases}$$



$$X(z)=-z\cdot\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)\qquad \boxed{\Rightarrow X(z)=\frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}\quad |z>1|}$$

Z-transform

- **Z-transform Table**

Here:

- $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$
- $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

	Signal, $x[n]$	Z-transform, $X(z)$	ROC
1	$\delta[n]$	1	all z
2	$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
3	$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
4	$e^{-\alpha n}u[n]$	$\frac{1}{1-e^{-\alpha}z^{-1}}$	$\lvert z\rvert > \lvert e^{-\alpha}\rvert$
5	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
6	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
7	$-nu[-n-1]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert < 1$
8	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
9	$-n^2u[-n-1]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert < 1$
10	$n^3u[n]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\lvert z\rvert > 1$
11	$-n^3u[-n-1]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\lvert z\rvert < 1$

	Signal, $x[n]$	Z-transform, $X(z)$	ROC
12	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
13	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
14	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
15	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
16	$n^2a^n u[n]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert > \lvert a\rvert$
17	$-n^2a^n u[-n-1]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert < \lvert a\rvert$
18	$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
19	$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
20	$a^n\cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
21	$a^n\sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Relation between Laplace and Z-transforms

X(s) → X(z) ?

- **Case of signals having only simple poles**

$$X(s) = \sum_{i=1}^{N} \frac{A_i}{s - s_i} \qquad \Rightarrow x(t) = \sum_{i=1}^{N} A_i \cdot e^{s_i \cdot t} \quad t \geq 0$$

By sampling x(t), we obtain the following discrete sequence

$$x(k) = \sum_{i=1}^{N} A_i \cdot e^{s_i \cdot k \cdot Ts} \quad k \geq 0$$

From line 4 of the Z-transform table: $X(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{si \cdot Ts} \cdot z^{-1}} = \sum_{i=1}^{N} \frac{A_i \cdot z}{z - e^{si \cdot Ts}}$

$$\boxed{\Rightarrow X(s) = \sum_{i=1}^{N} \frac{A_i}{s - s_i} \xrightarrow{Z} X(z) = \sum_{i=1}^{N} \frac{A_i \cdot z}{z - e^{si \cdot Ts}}} \qquad (2)$$

=> A pole s_i in X(s) yields a pole $z_i = e^{s_i \cdot Ts}$ in X(z)

$$\boxed{s_i \xrightarrow{Z} z_i = e^{s_i \cdot Ts}}$$

Relation between Laplace and Z-transforms

- **General case**

$$\boxed{X(z) = \sum_{s_i = poles\ of\ X(s)} \text{Residues}\left\{ \mathrm{X(s)} \cdot \frac{1}{1\text{-}\mathrm{z}^{-1} \cdot e^{s \cdot Ts}} \right\}_{s=s_i}} \qquad (3)$$

Calculation of the residue at the pole s_j of multiplicity *m:*

$$\text{Residue}\left\{ \mathrm{X(s)} \cdot \frac{1}{1\text{-}\mathrm{z}^{-1} \cdot e^{s \cdot Ts}} \right\}_{s=s_j} = \frac{1}{(m-1)!} \cdot \lim_{s \to s_j} \frac{d^{m-1}}{ds^{m-1}} \left[(s-s_j)^m \cdot \mathrm{X(s)} \cdot \frac{1}{1 - \mathrm{z}^{-1} \cdot e^{s \cdot Ts}} \right]$$

For a simple pole (m = 1):

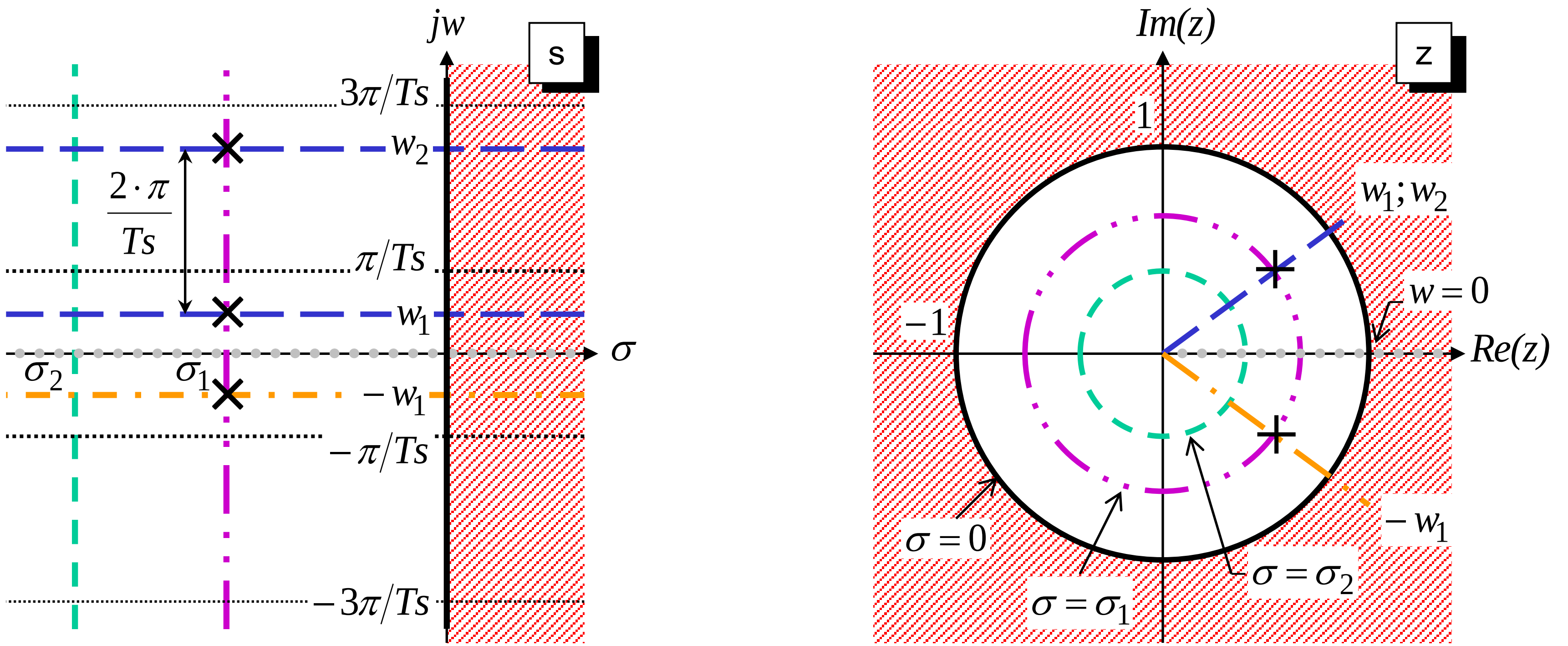
$$\text{Residue}\left\{ \mathrm{X(s)} \cdot \frac{1}{1\text{-}\mathrm{z}^{-1} \cdot e^{s \cdot Ts}} \right\}_{s=s_j} = \lim_{s \to s_j} (s-s_j) \cdot \mathrm{X(s)} \cdot \frac{1}{1 - \mathrm{z}^{-1} \cdot e^{s \cdot Ts}}$$

An example of calculation will be given in the next chapter

Relation between Laplace and Z-transforms

- **Mapping from s-plane to z-plane**

Since $z_i = e^{s_i \cdot Ts} = e^{\sigma_i \cdot Ts} \cdot e^{j \cdot w_i \cdot Ts}$ we can map the s-plane to the z-plane as below:



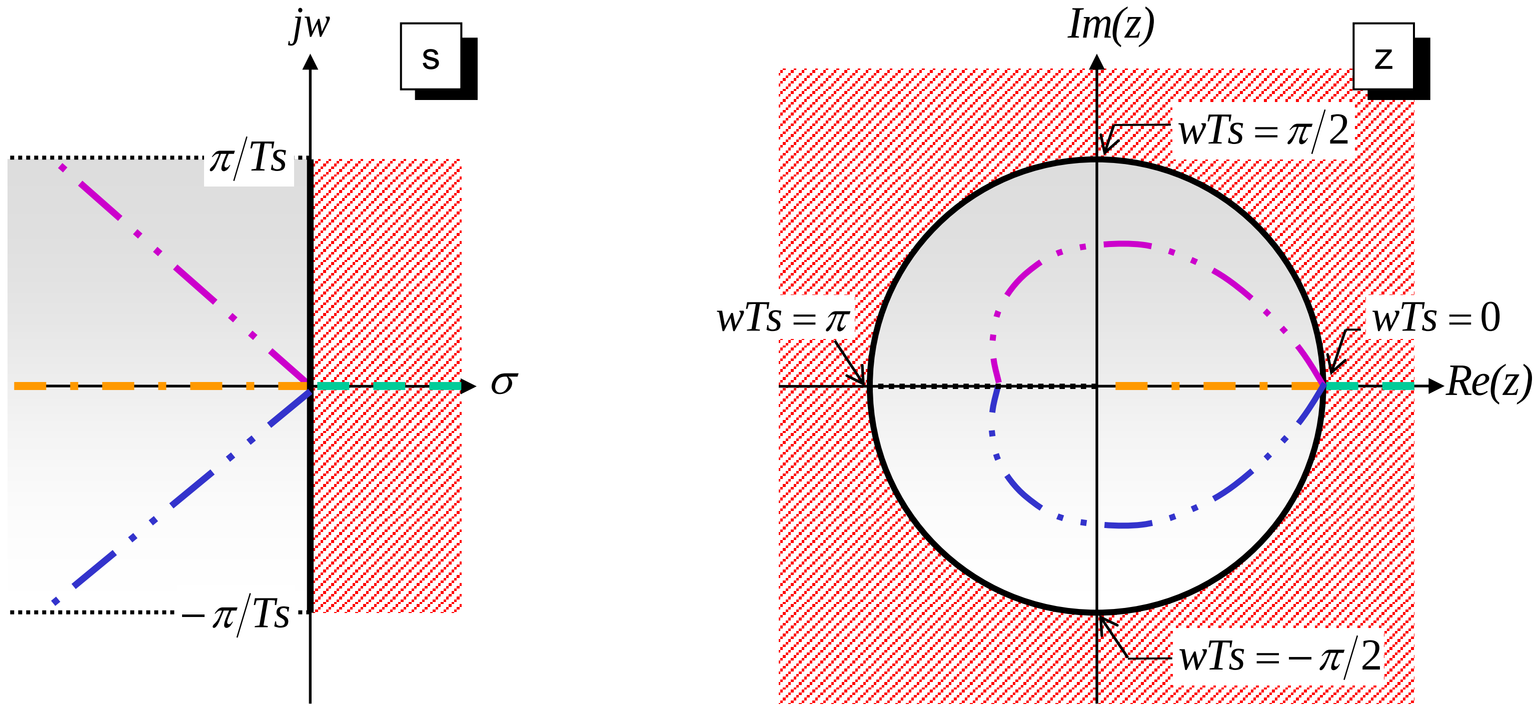
NB: 2 poles in the s-plane which imaginary part differ by $2\pi/Ts$ map to the same pole in the z-plane

Bijective mapping between both planes => $Im(X(s)) \in [-\pi/Ts;+\pi/Ts]$

$$\Rightarrow \frac{\pi}{Ts} > \max_i \left\{ \left| Im(s_i) \right| \right\}$$

Relation between Laplace and Z-transforms

- **Mapping from s-plane to z-plane**

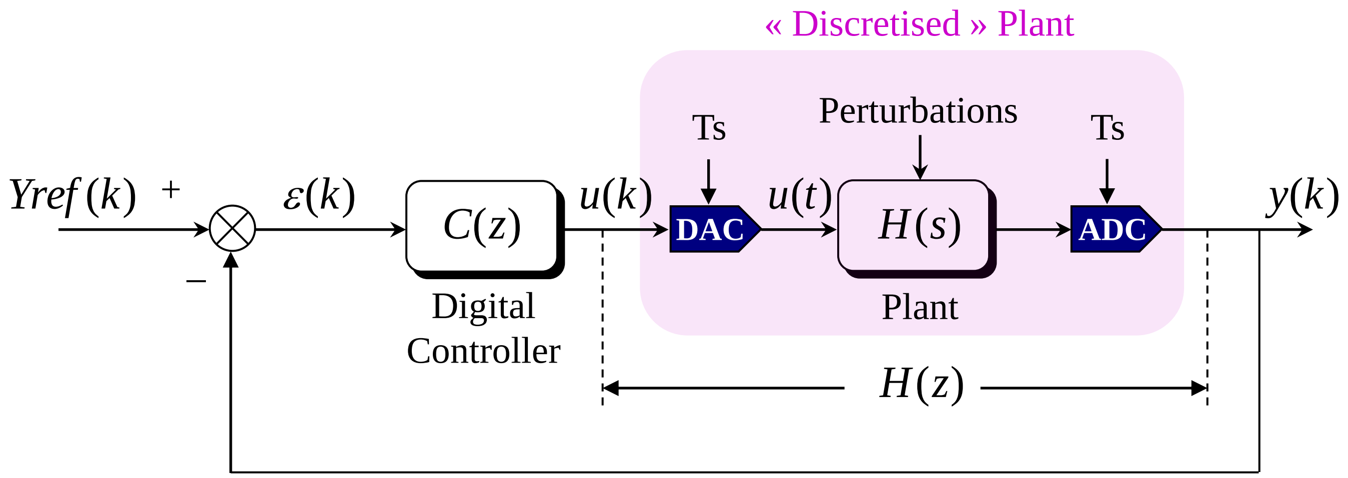


Legend:

solid line	$s = jw$	$\lvert z \rvert = 1$
green dashed line	$s = \sigma \geq 0$	$z = r, \quad r \geq 1$
orange dash-dot line	$s = \sigma \leq 0$	$z = r, \quad 0 \leq r \leq 1$
magenta and blue dash-dot-dot lines	$\begin{cases} s = -\zeta w_n \pm j w_n \sqrt{1-\zeta^2} \\ \zeta = \text{Constant}, \quad w_n \text{ varies} \end{cases}$	Logarithmic spiral
dotted line	$s = \sigma \pm j\pi/Ts$	$z = -r$

Digitally controlled continuous-time systems

- **Modelling of the plant to be controlled**



Model of the DAC = Zero-order hold (ZOH)

→ Converts u(k) to u(t) by holding each sample value for one sample interval

$$u(t) = u(k), \quad k \cdot Ts \le t \le (k+1) \cdot Ts$$

NB: Delay introduced by the ZOH = Ts/2

The Laplace transform transfer function of the ZOH is

$$H_{ZOH}(s) = \frac{1 - e^{-s \cdot Ts}}{s}$$

$$\boxed{\Rightarrow H(z) = \left(1 - z^{-1}\right) Z\left[\frac{H(s)}{s}\right]} \qquad (4)$$

If H(s) has poles $s = s_i$, then H(z) has poles $z = e^{s_i \cdot Ts}$. But the zeros are unrelated.

Digitally controlled continuous-time systems

$\neq$ ways to calculate $Z\left[\frac{H(s)}{s}\right]$:

- Partial fraction decomposition + use z-transform table

- If $\frac{H(s)}{s}$ has only simple poles, use Eq. (2):

$$\frac{H(s)}{s} = \frac{A_1}{s} + \sum_{i=2}^{N} \frac{A_i}{s - s_i} \xrightarrow{Z} X(z) = \frac{A_1}{1 - z^{-1}} + \sum_{i=2}^{N} \frac{A_i}{1 - e^{si \cdot Ts} \cdot z^{-1}}$$

- Use Eq. (3): $H(z) = \sum_{s_i = poles\ of\ H(s)/s} \text{Residues} \left\{ \frac{H(s)}{s} \cdot \frac{1}{1\text{-}z^{-1} \cdot e^{s \cdot Ts}} \right\}_{s = s_i}$

- $H(z)$ → Ask Matlab: Function ‘c2d’

Syntax

sysd = c2d(sys,Ts)

Description

sysd = c2d(sys,Ts) discretizes the continuous-time LTI model sys using zero-order hold on the inputs and a sample time of Ts seconds.

Digitally controlled continuous-time systems

If $Y(z)=Z[y(k)]$; $Yref(z)=Z[Yref(k)]$; $E(z)=Z[\varepsilon(k)]$; $U(z)=Z[u(k)]$

- **Open-loop transfer function:** $\boxed{\frac{Y(z)}{E(z)}=C(z)\cdot H(z)}$

- **Closed-loop transfer function:** $\boxed{\frac{Y(z)}{Yref(z)}=\frac{C(z)\cdot H(z)}{1+C(z)\cdot H(z)}}$

- **Controller algorithm:**

Transfer function of the digital controller

$$C(z)=\frac{U(z)}{E(z)}=\frac{b_0+b_1\cdot z^{-1}+\ldots+b_p\cdot z^{-p}}{1+a_1\cdot z^{-1}+\ldots+a_n\cdot z^{-n}}$$

$$\Rightarrow\left(1+a_1\cdot z^{-1}+\ldots+a_n\cdot z^{-n}\right)\cdot U(z)=\left(b_0+b_1\cdot z^{-1}+\ldots+b_p\cdot z^{-p}\right)\cdot E(z)$$

Hence $\quad u(k)+a_1\cdot u(k-1)+\ldots+a_n\cdot u(k-n)=b_0\cdot\varepsilon(k)+b_1\cdot\varepsilon(k-1)+\ldots+b_p\cdot\varepsilon(k-p)$

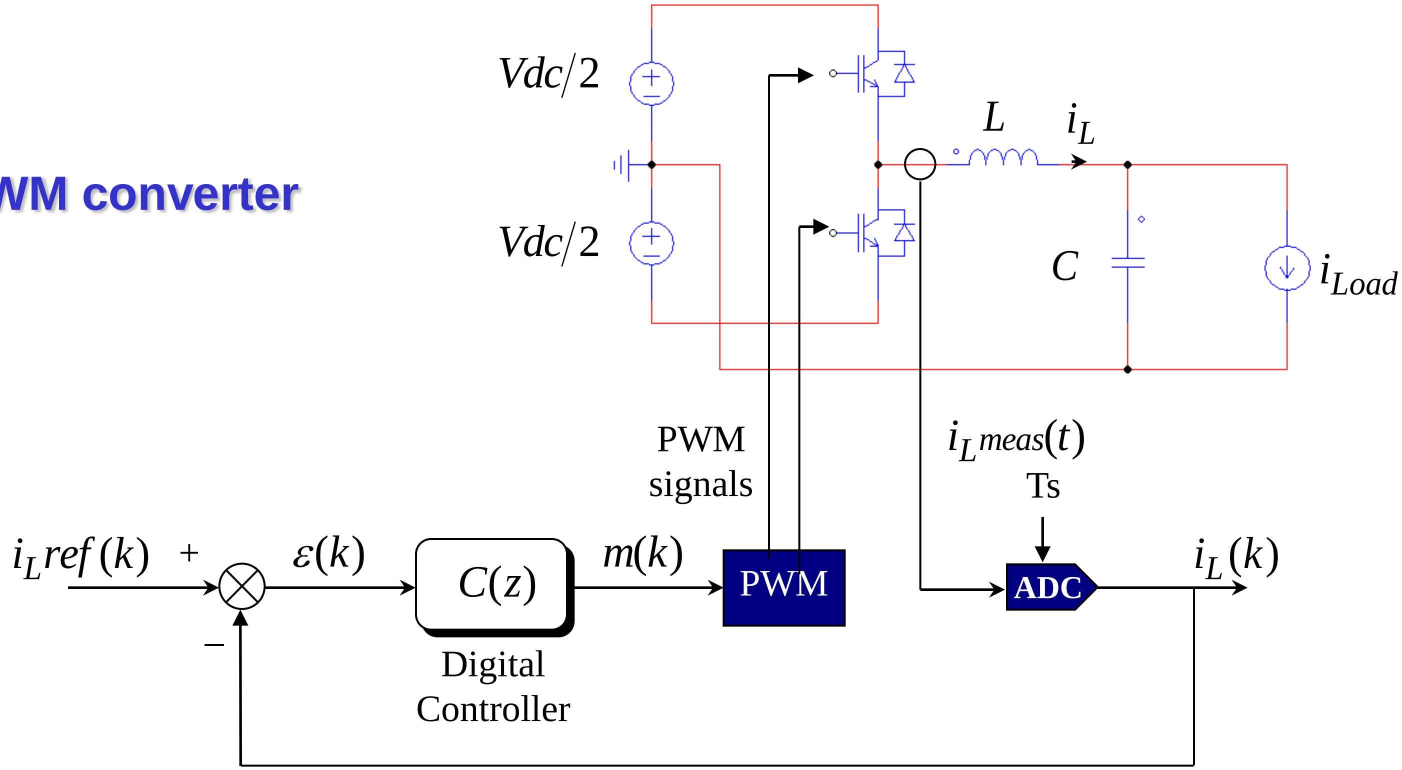
$\left(Z[x(k-n)]=z^{-n}\cdot X(z)\right)$

$$\boxed{\Rightarrow u(k)=b_0\cdot\varepsilon(k)+b_1\cdot\varepsilon(k-1)+\ldots+b_p\cdot\varepsilon(k-p)-a_1\cdot u(k-1)-\ldots-a_n\cdot u(k-n)}$$

= Difference equation, where current output is dependent on current input and previous inputs and outputs

Digitally controlled continuous-time systems

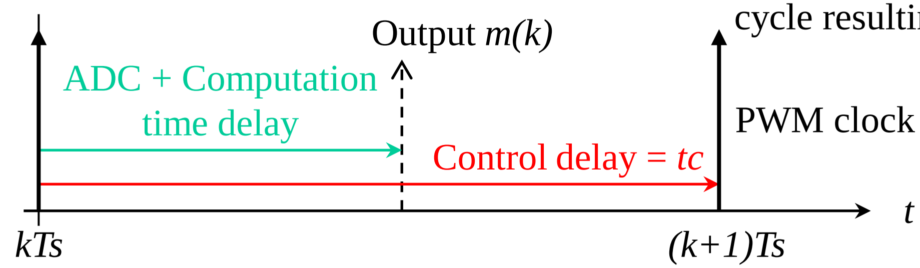
- **Example:**
  Half bridge PWM converter



Controller output =
modulation index (-1 < *m* < 1)

Sampling

Update of PWM signal duty
cycle resulting from *m(k)*

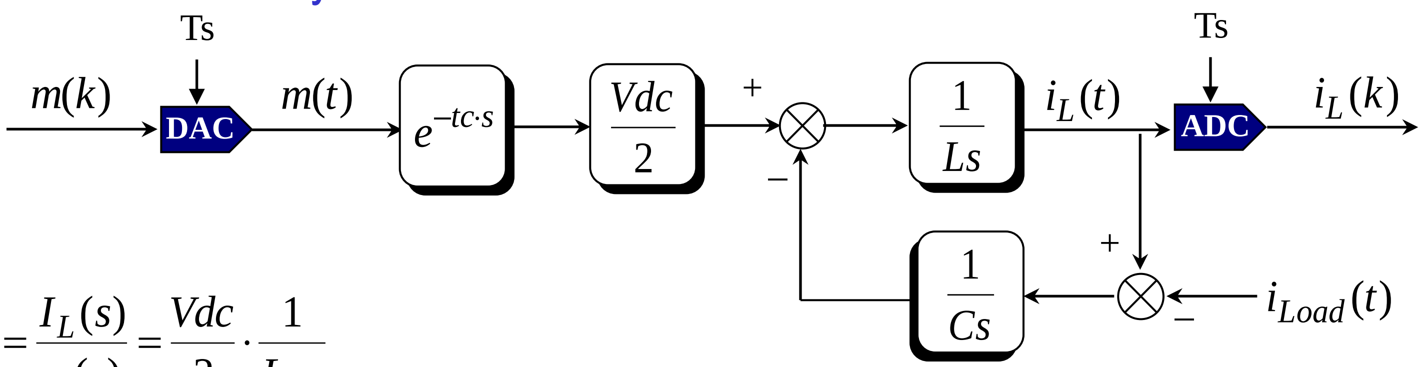


H(z) ?

Digitally controlled continuous-time systems

Modelling of the PWM converter

The PWM can be modelled by a DAC



$$H(s)=\frac{I_L(s)}{m(s)}=\frac{Vdc}{2}\cdot\frac{1}{L\cdot s}$$

$$\Rightarrow H(z)=\left(1-z^{-1}\right)\cdot Z\left[\frac{H(s)}{s}\right]=\left(1-z^{-1}\right)\cdot\frac{Vdc}{2\cdot L}\cdot Z\left[\frac{e^{-tc\cdot s}}{s^2}\right]=z^{-n}\cdot\left(1-z^{-1}\right)\cdot\frac{Vdc}{2\cdot L}\cdot Z\left[\frac{e^{\theta\cdot s}}{s^2}\right]$$

with $tc=n\cdot Ts-\theta$ $(n\in IN)$

Using Eq. (3), with a double pole:

$$Z\left[\frac{e^{\theta\cdot s}}{s^2}\right]=\frac{1}{(2-1)!}\cdot\lim_{s\to 0}\frac{d^{2-1}}{ds^{2-1}}\left[(s-0)^2\cdot\frac{e^{\theta\cdot s}}{s^2}\cdot\frac{1}{1-z^{-1}\cdot e^{s\cdot Ts}}\right]$$

$$=\lim_{s\to 0}\frac{d}{ds}\left[e^{\theta\cdot s}\cdot\frac{1}{1-z^{-1}\cdot e^{s\cdot Ts}}\right]=\frac{\theta\cdot\left(1-z^{-1}\right)+Ts\cdot z^{-1}}{\left(1-z^{-1}\right)^2}$$

In this example $n=1$ and $\theta=0$

$$\boxed{\Rightarrow H(z)=\frac{Vdc\cdot Ts}{2\cdot L}\cdot\frac{z^{-2}}{1-z^{-1}}}$$

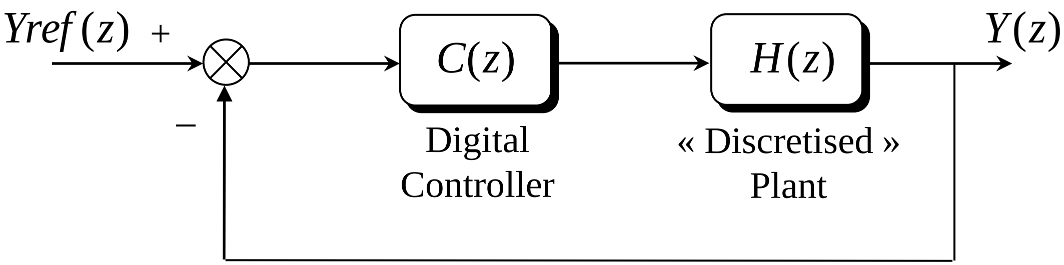
Analysis of closed-loop systems

Closed control loop performance criteria:

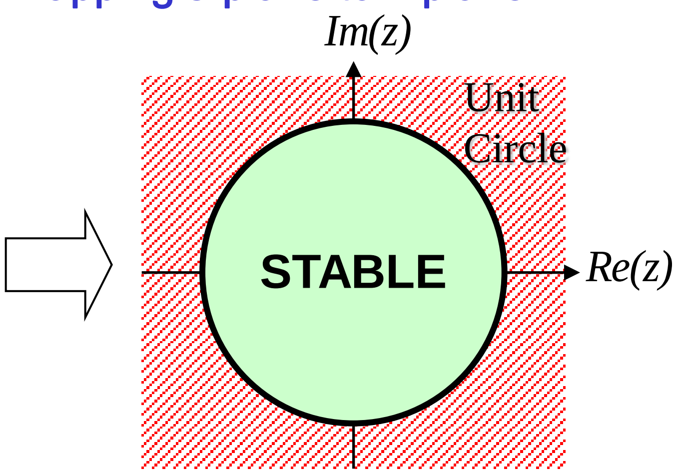
- Stability & robustness
- Static & dynamic behavior

The system behavior & stability is determined by the roots of the closed-loop TF

- Stability of closed-loop systems



Mapping s-plane to z-plane



z-plane stable pole location

Closed-loop transfer function:

$$CL(z) = \frac{Y(z)}{Yref(z)} = \frac{C(z)\cdot H(z)}{1 + C(z)\cdot H(z)}$$

Poles of the closed-loop TF = Roots of the characteristic equation $1 + C(z)\cdot H(z) = 0$

Analysis of closed-loop systems

- **Robustness**

Open-loop transfer function:

$$OL(z) = C(z) \cdot H(z)$$

Phase margin Φ_M :

$$\boxed{\Phi_M = 180° + \arg\left[OL\left(e^{j \cdot w_{cr} \cdot Ts}\right)\right]}$$

Where w_{cr} **is such that** $\left|OL\left(e^{j \cdot w_{cr} \cdot Ts}\right)\right| = 1$

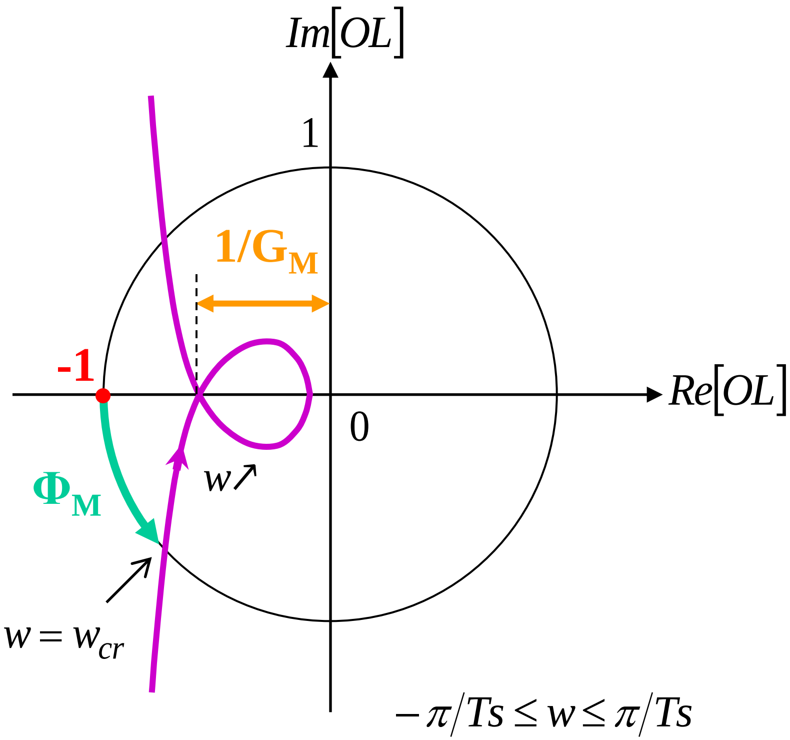
Gain margin G_M :

$$\boxed{G_M = \frac{1}{\left|OL\left(e^{j \cdot w_{\pi} \cdot Ts}\right)\right|}}$$

Where w_{π} **is such that**

$$\arg\left[OL\left(e^{j \cdot w_{\pi} \cdot Ts}\right)\right] = -\pi \quad rad$$

Typically: $\Phi_M = \pm 40°$, $G_M = \pm 6\text{dB}$



Matlab plots and margins → Use functions ‘nyquist’ , ‘bode’ , ‘margin’

Analysis of closed-loop systems

- **Influence of the poles on the transient behavior**

Will be illustrated with the step response analysis of a system CL(z) having only simple poles:

Step response @ sampling instants:

$$y(k) = CL(1) + \sum_{\substack{i=1 \\ real \\ poles}}^{n} c_i \cdot z_i^{\,k} + \sum_{\substack{j=1 \\ complex \\ conjugate \\ poles}}^{m} |c_j| \cdot |z_j|^k \cdot \cos(k \cdot \theta_j + \varphi_j)$$

Dependent on pole locations

- Steady-state:

$$\lim_{k \to \infty} y(k) = CL(1)$$

- Contribution of real poles z_i :

Sum of exponential terms
$$\begin{cases} \xrightarrow[k\to\infty]{} 0 & if\ |z_i| < 1 \\ \xrightarrow[k\to\infty]{} \pm\infty & if\ |z_i| > 1 \end{cases}$$

- Contribution of complex conjugate poles z_j :

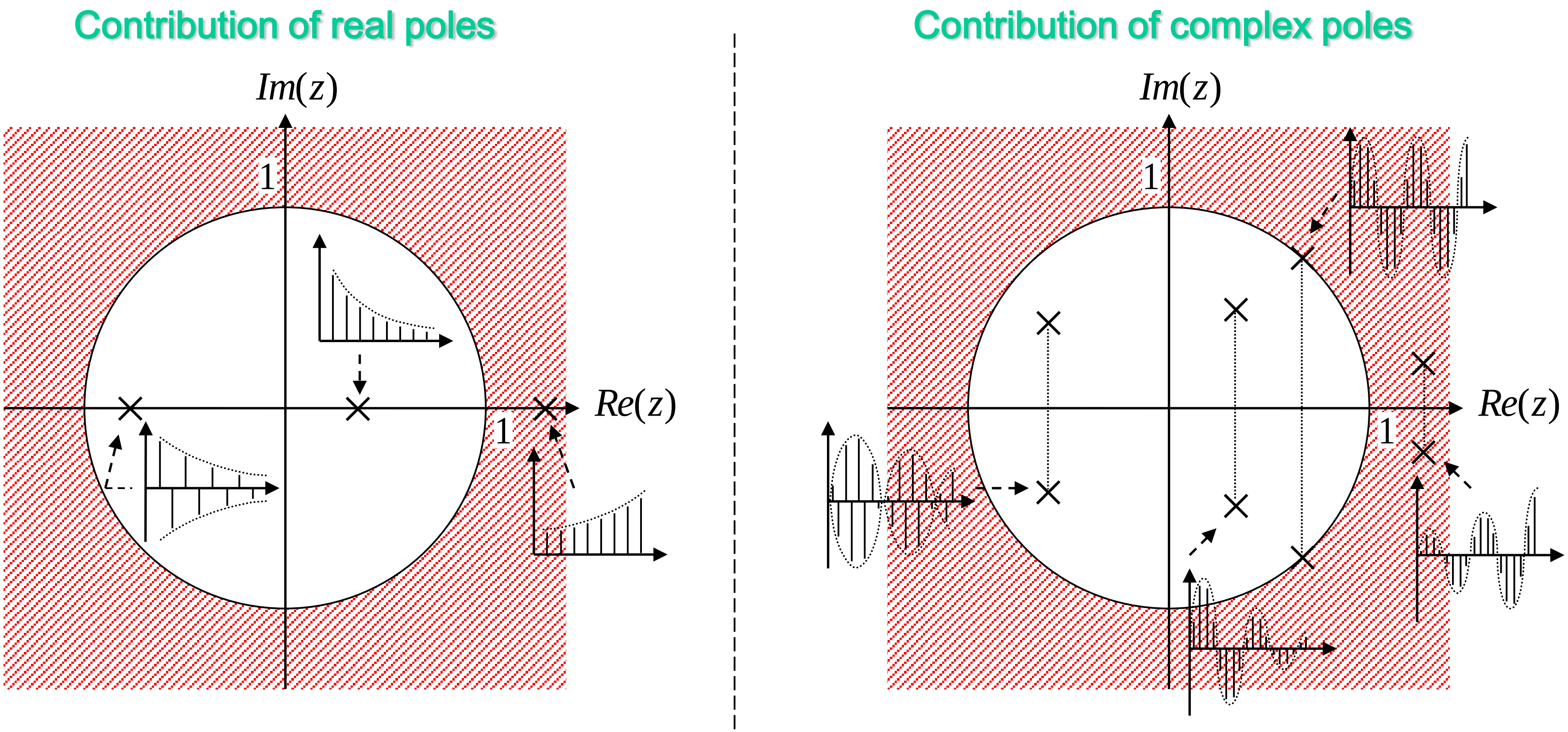
Oscillating regime
$$\begin{cases} \xrightarrow[k\to\infty]{} 0 & if\ |z_j| < 1 \\ \xrightarrow[k\to\infty]{} \pm\infty & if\ |z_j| > 1 \end{cases}$$

=> Damped oscillations (if $|z_j| < 1$)

=> Undamped oscillations (if $|z_j| > 1$)

Analysis of closed-loop systems

- Influence of the poles on the transient behavior



NB: Poles closer to origin → Faster transient regime

Analysis of closed-loop systems

- **Particular case: 2nd order systems**

A popular technique for controller design is pole placement.

A common strategy consists to derive the controller parameters from a pole placement such that the closed-loop behaves like a 2nd order system.

Continuous-time theory:

The design specifications imply constrainsts on the cut-off frequency w_n and the damping ratio ζ

$$CLref(s) = \frac{w_n^{\,2}}{s^2 + 2\cdot\zeta\cdot w_n\cdot s + w_n^{\,2}}$$

Rise time (10% → 90%): $t_r \approx 1.8/w_n$

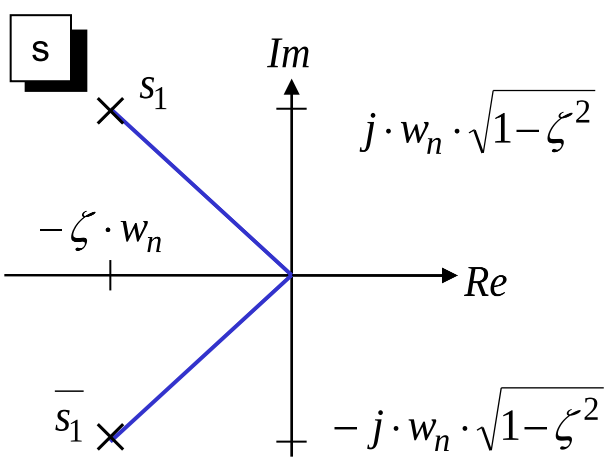
Peak overshoot: $M_p \approx e^{-\pi\cdot\zeta/\sqrt{1-\zeta^2}}$

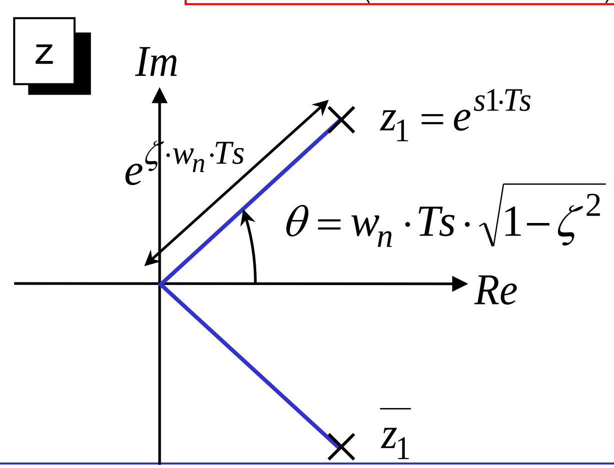
Settling time (to 1%): $t_s = 4.6/\zeta\cdot w_n$

Discrete closed-loops:

Pole mapping from s-plane to z-plane:

$$\boxed{s_1 = -w_n\cdot\left(\zeta + j\cdot\sqrt{1-\zeta^2}\right) \;\Rightarrow\; z_1 = e^{-w_n\cdot Ts\cdot\left(\zeta + j\cdot\sqrt{1-\zeta^2}\right)}}$$





$$\boxed{t_r\ ,\ M_p \Rightarrow \zeta, w_n \Rightarrow z_1, \overline{z_1}}$$

$$\boxed{\begin{aligned} & t_s = 4.6/\zeta\cdot w_n \\ \Rightarrow\ & |z_1| < e^{-4.6\cdot Ts/t_s} \approx 0.01^{Ts/t_s}\end{aligned}}$$

=> condition for Ts

Analysis of closed-loop systems

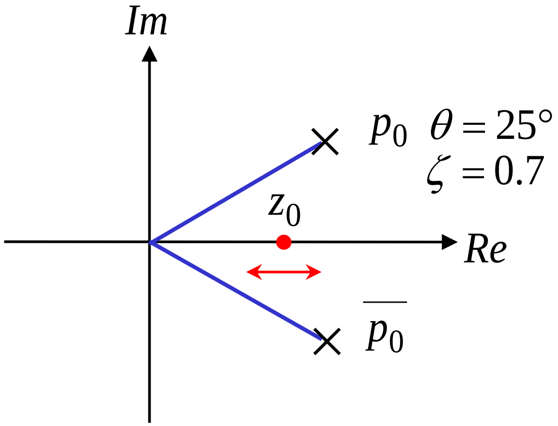
- **Particular case: 2nd order systems**

Influence of a zero

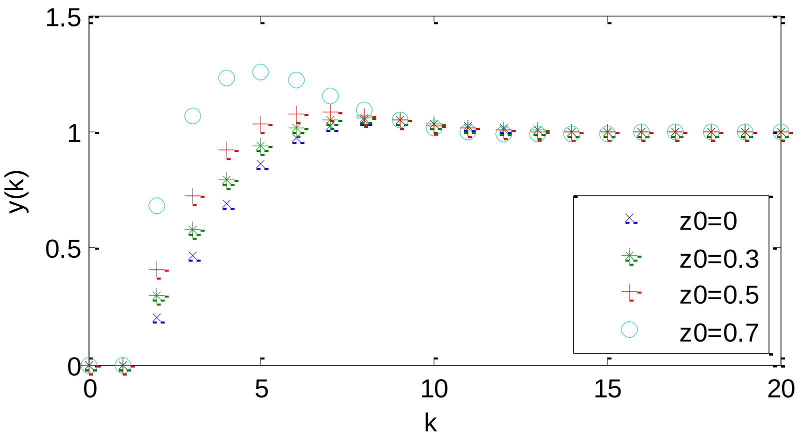
$$CL(z) = K \cdot \frac{z - z_0}{(z - p_0)\cdot(z - \overline{p_0})}$$

where $z_0 \in \mathbb{R}$ and $K = (1 - p_0)\cdot(1 - \overline{p_0})/1 - z_0$

($\rightarrow$ unitary static gain)



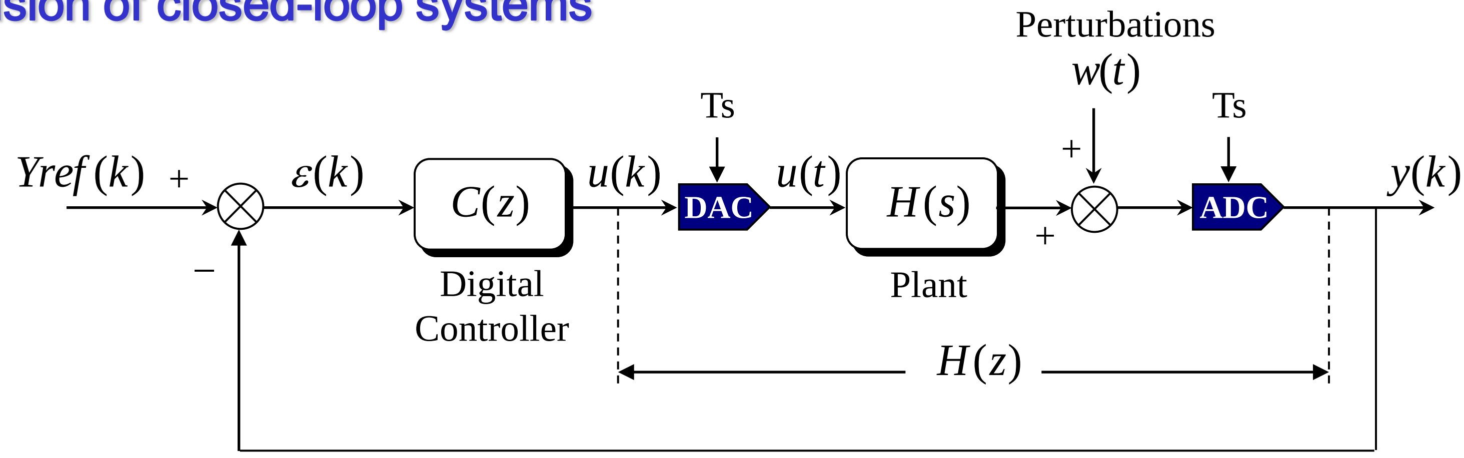
Discrete step response



- Increasing overshoot when the zero is moving towards +1 → Take care…
- The reference tracking performance can be improved by designing appropriate zeros in the closed-loop transfer function.

Analysis of closed-loop systems

- **Precision of closed-loop systems**



Same conclusions as for continuous-time closed-loops

- Precision versus the input

To achieve zero steady-state error, we require

→ at least 1 integrator (pole @ z =1) in the open-loop TF $C(z)\cdot H(z)$ for a step input

→ at least 2 integrators in the open-loop TF for a ramp input

→ …

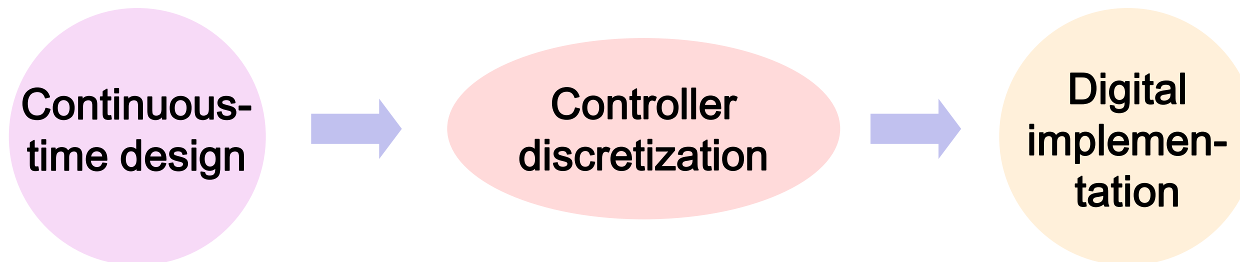
- Perturbation rejection

To reject disturbances of class N → at least N+1 integrators in $C(z)\cdot H(z)$

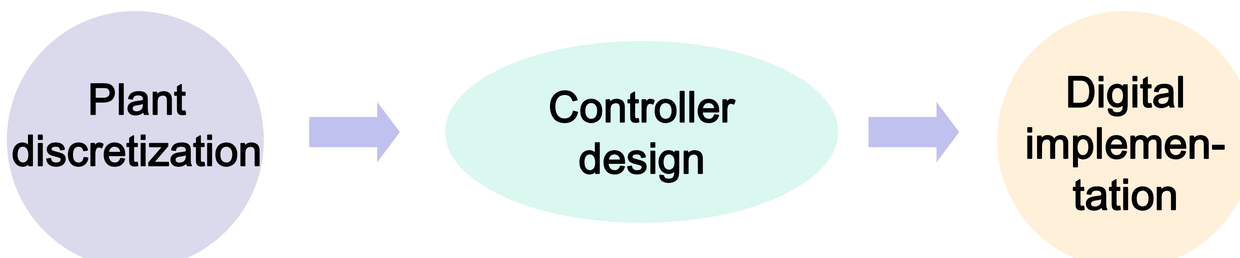
Discrete-time controller synthesis

2 main ways to synthesize discrete-time controllers:

1. Emulation design



2. Direct discrete-time design



Discrete-time controller synthesis

- **Emulation design**

1rst step: Continuous-time controller design. At this stage the sampling is ignored. But the control + ZOH delays should be taken into account (→ preserve phase margin)

2nd step: Discretization of the continuous-time controller (Followed by simulations to check ≠ Methods: performances)

- Approximate s , i.e $C(s) \to C(z)$
- Pole-zero matching

3rd step: Derivation of the controller algorithm (difference equation)

Approximation methods:

- Euler $\boxed{s \to \frac{1}{Ts} \cdot \left(1 - z^{-1}\right)}$

- Tustin's or bilinear approximation $\boxed{s \to \frac{2}{Ts} \cdot \frac{\left(1 - z^{-1}\right)}{\left(1 + z^{-1}\right)}}$

Example: Discretization of a PI controller using Tustin's approximation

$$C(s) = Kp \cdot \left(1 + \frac{1}{Ti \cdot s}\right) \quad \Rightarrow \quad C(z) = Kc \cdot C(s)\Big|_{s=\frac{2}{Ts}\cdot\frac{(1-z^{-1})}{1+z^{-1}}} = Kc \cdot Kp \cdot \left(\frac{1 + Ts/(2 \cdot Ti) + (-1 + Ts/(2 \cdot Ti)) \cdot z^{-1}}{1 - z^{-1}}\right)$$

$$\lim_{z \to 1} (z-1) \cdot C(z) = \lim_{s \to 0} s \cdot C(s) \quad \Rightarrow \quad Kc = 1/Ts$$

Matlab → sysd = c2d(sys,Ts,'tustin')

Discrete-time controller synthesis

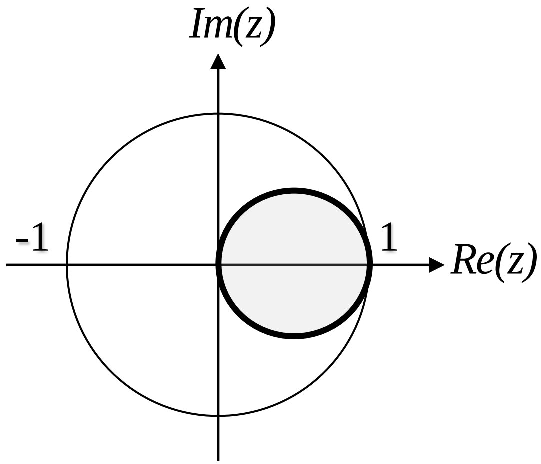
- **Comparison between Euler and Tustin's approx.**

- Stability

Half-plane

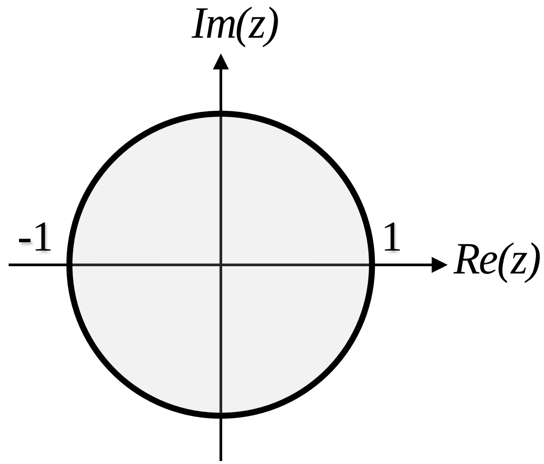
$Re[s] < 0$

Euler



An unstable continuous-time system can be mapped to a stable discrete system

Tustin



Perfect correspondence

- Mapping of the poles

Euler: $s - s_0 \rightarrow (z \cdot (1 - s_0 \cdot Ts) - 1)/Ts \cdot z$ **=>** $z_0 = 1/1 - s_0 \cdot Ts = 1 + (s_0 \cdot Ts) + (s_0 \cdot Ts)^2 + (s_0 \cdot Ts)^3 + \ldots$

Tustin: $s - s_0 \rightarrow (z \cdot (2 - s_0 \cdot Ts) - (2 + s_0 \cdot Ts))/Ts \cdot (z+1)$

=> $z_0 = (1 + s_0 \cdot Ts/2)/(1 - s_0 \cdot Ts/2) = 1 + (s_0 \cdot Ts) + \frac{1}{2} \cdot (s_0 \cdot Ts)^2 + \frac{1}{4} \cdot (s_0 \cdot Ts)^3 + \ldots$

To be compared to $z_0 = e^{s_0 \cdot Ts} = 1 + (s_0 \cdot Ts) + \frac{1}{2} \cdot (s_0 \cdot Ts)^2 + \frac{1}{6} \cdot (s_0 \cdot Ts)^3 + \ldots$ **≠ poles**

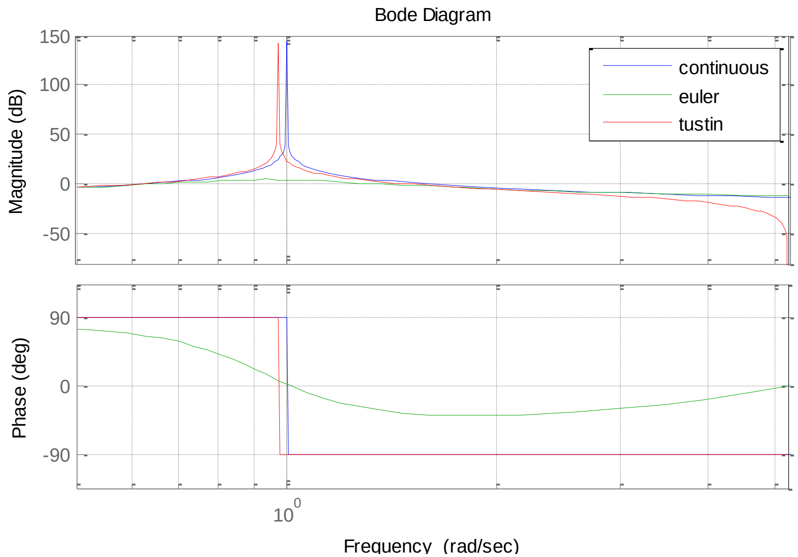
Discrete-time controller synthesis

• Comparison between Euler and Tustin's approx.

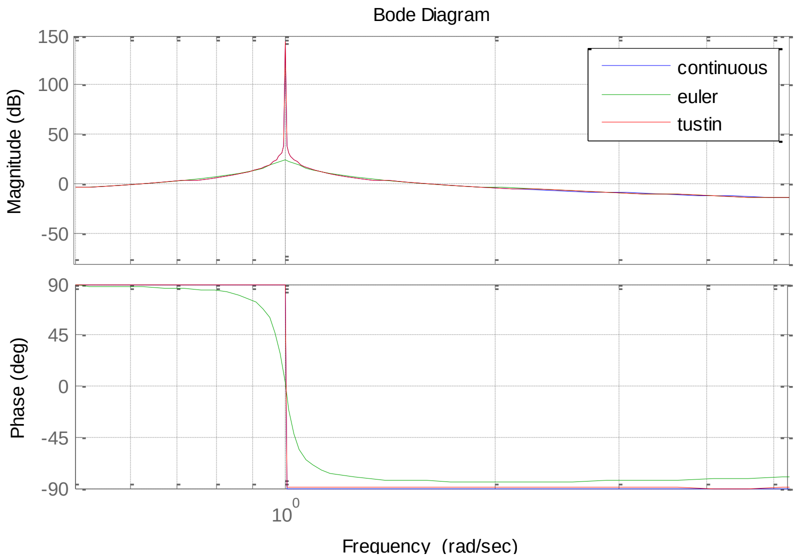
- Pole and zero locations not preserved → Frequency response is changed
- Increasing the sampling frequency → Smaller approximation errors

Example 1: $$C(s) = \frac{s}{1+s^2}$$

Ts = 0.6



Ts = 0.06

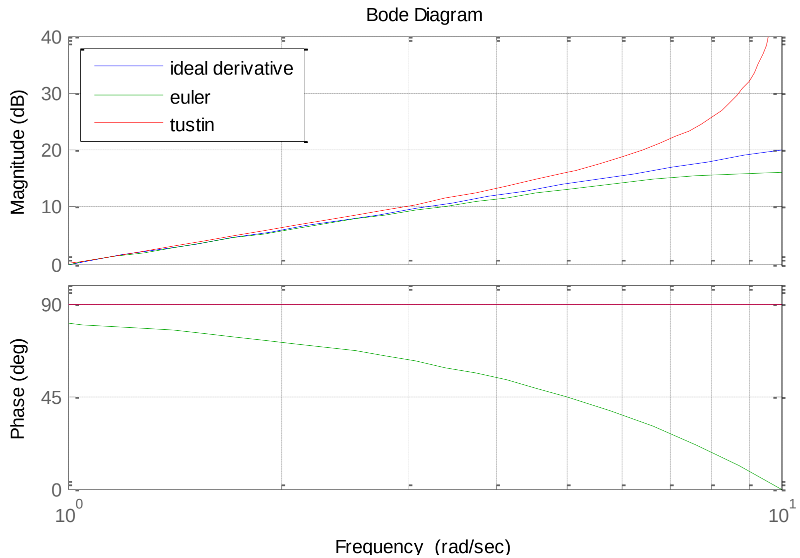


=> Better result with Tustin

Discrete-time controller synthesis

- **Comparison between Euler and Tustin's approx.**

Example 2: Ideal derivative



Euler: Filtering effect @ high frequencies

Tustin:
Magnitude → ∞ when w → π/Ts
Noise amplification @ high frequencies

=> Euler more appropriate for discretization of high-pass filters

=> Tustin more appropriate for discretization of low-pass filters

- **Other discretization method = Matched transform**

Matlab →
sysd = c2d(sys,Ts,'matched')

$$C(s) = K \cdot \frac{\prod_j (s - r_j)}{\prod_j (s - \sigma_j)} \xrightarrow{Z} C(z) = Kc \cdot \frac{\prod_j \left(z - e^{r_j \cdot Ts}\right)}{\prod_j \left(z - e^{\sigma_j \cdot Ts}\right)}$$

Kc is set so to obtain the same static gain

No frequency distorsion => Well-adapted for the discretization of transfer functions including resonances (ex: notch filter, …)

Discrete-time controller synthesis

- **Direct discrete-time design**

- A system controlled using an emulation controller always suffer the degradation of performance compared with its continuous-time counter-part

- To reduce the degree of degradation, very fast sampling can be needed, as

{ADC – Digital controller – DAC} should behave the same as the analogue controller (e.g. PID type)

$\Rightarrow$ **Bad use of the potentialities of the digital controller**

In this case, direct discrete-time design offers an alternative solution, since in this design the sampling is considered from the beginning of the design process

1rst step: Discretization of the continuous-time plant (Cf. chapter 3)

2nd step: Choice of controller type (PID, RST, …) and computation of the controller parameters using for example a pole placement method (Cf. chapter 4)

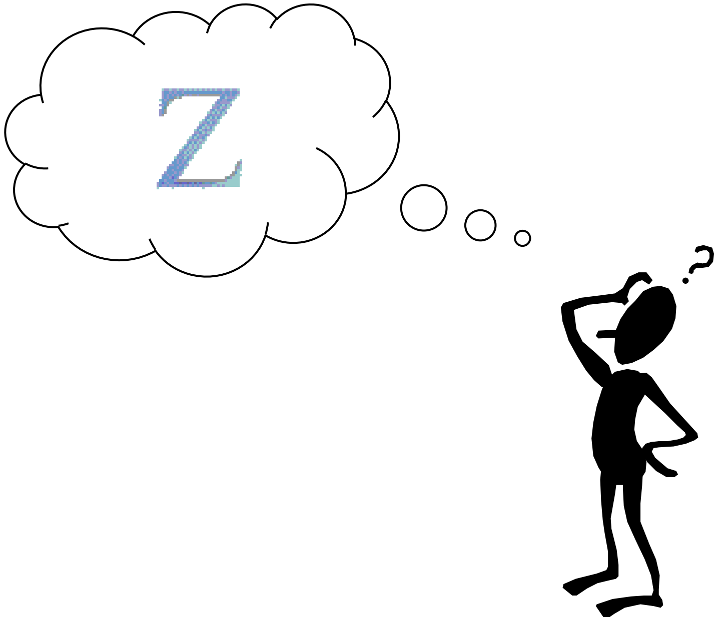
Ts : Choice based on the bandwidth of the closed-loop system $F^{CL}{}_B$

$$\boxed{\frac{1}{Ts} = (6 \ \ to \ \ 25) \cdot F^{CL}{}_B}$$

3rd step: Derivation of the controller algorithm (difference equation)

More in the next tutorials…

Questions



Thank you for your attention.
Questions ?