**Dynamic modeling of spike count data with Conway-Maxwell Poisson variability**

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# Abstract

# Introduction

Drift…

(Rokni et al. 2007)

(Chestek et al. 2007)

(Stevenson et al. 2011)

(Tomko and Crapper 1974)

Stability…

(Dickey et al. 2009)

(Steinmetz et al. 2021) [shows that correcting for electrode drift reduces some instability]

[some amount of drift is functional] adaptation/plasticity…

(Lesica et al. 2007)

Variability…

(Ghanbari et al. 2019)

(Stevenson 2016)

(Fenton and Muller 1998)

(Barbieri et al. 2001)

(Maimon and Assad 2009)

(Churchland et al. 2010)

(Churchland et al. 2011)

(Eden et al. 2004)

(Brown et al. 2001)

(DeWeese and Zador 1998)

# Methods

Here we consider a dynamic GLM with Conway-Maxwell Poisson (CMP) observations to describe time-varying spike counts. We first introduce the model. Although the CMP distribution allows us to flexibly model non-Poisson variability, one major challenge with using this model is that there are not closed-form solutions for the CMP likelihood. Here, we infer fit the model using a global Gaussian approximation, and we discuss several additional technical challenges that arise when using the CMP distribution with a dynamic GLM.

## Dynamic Conway-Maxwell Poisson Model

A count observation , such as the spike count for a neuron, is assumed to follow the CMP distribution, with parameters and . The probability mass function (pmf) of CMP is

, where is the normalizing constant. The parameter controls different dispersion patterns, i.e. equi- (), over- () or under-dispersion (). Three common distributions occur as special cases: 1) the Poisson (), 2) geometric (), and 3) Bernoulli ().

For multiple observations up to steps, such as simultaneous spike counts from neurons, denote the counts at time bin as , for The corresponding CMP parameters at are and . Previous work has examined the CMP-GLM (Chatla and Shmueli 2018; Sellers and Shmueli 2010), and here we focus on the dynamic version of this GLM. The CMP parameters at are modeled by two log-linear models, and , with and , and and denote known predictors. Under the CMP-GLM, the parameters are static. Here, we assume that they progress linearly with a Gaussian noise.

The observations follow conditionally independent CMP distributions, given the state vector .

While the state vector evolves linearly with Gaussian noise:

Given the initial state mean and covariance … , linear dynamics , and process covariance .

## Inference by Gaussian approximation

To fit the model to data we need to estimate the time-varying state vector In this section, we first assume **F** and **Q** are known. Since the observations are CMP distributed, we cannot estimate in closed form. Instead, here we approximate it by a multivariate Gaussian distribution, , with . The parameters of this Gaussian are found by a global Laplace approximation, i.e., and . The log-posterior is given by:

, where is the log-likelihood. The log-posterior is concave (Gupta et al. 2014), and the Markovian structure of the state vector dynamics makes it possible to optimize by Newton-Raphson (NR) in time (Paninski et al. 2010). After the Newton update, we can further quantify the uncertainty for the CMP parameters and the underlying rates (see Appendix).

There are several technical challenges involved with performing the Newton update with CMP observations. Firstly, in order to find the gradient and Hessian we need to calculate moments of and , which have no closed forms (Shmueli et al. 2005). We can calculate these moments by truncated summation. However, when and , truncated summation is computationally costly since we need many steps for accurate approximation. In this case, we approximate the moments using previous (Chatla and Shmueli 2018; Gaunt et al. 2019) asymptotic results (see Appendix). A second challenge is that the Hessian is not robust to outliers. Outliers often result in the Hessian being close to singular or even positive-definite (see details in Appendix). To ensure robustness, we use Fisher scoring where the observed information is replaced by the expected information. Finally, a third challenge is that the Newton updates take a long time to converge if the initial state estimate is far from the maximum of the posterior, especially when is large. To resolve this issue, we use a smoothing estimate with local Gaussian approximation as a “warm start”. Forward filtering for a dynamic Poisson model has been previously described (Eden et al. 2004), and, here we implement CMP filtering following the same rationale. Let and be the mean and variance for the one-step prediction density and and be mean and variance for the posterior density, then the filtering update for step is given by

Here, to again ensure robustness, we use Fisher scoring when updating the state covariance. We then find smoothed estimates using a backward pass (Rauch-Tung-Striebel smoother). Although doing smoothing is fast, the estimates can be inaccurate, especially when there are large changes in the state vector. In the forward filtering stage, the Gaussian approximation at each step is conducted locally at the recursive prior . This will be statistically inefficient when the recursive prior is too far away from the posterior mode, or when there is a large change in the state vector. Moreover, Fisher scoring reduces the efficiency of the smoother even further. The smoother provides reasonable initial estimates, but estimation accuracy is substantially improved by using Newton’s method to find the global Laplace approximation for the posterior.

## Estimating process noise

For the applications to neural data examined here, we assume that . However, we still need to estimate the process noise . When is small, especially when , different values will have a substantial influence on estimation. One possible way to estimate is to use an Expectation Maximization (EM) algorithm (Macke et al. 2011). However, using the Laplace approximation for during the E-step breaks the usual guarantee of non-decreasing likelihoods in EM, and, hence, may lead to divergence. To avoid that, we could sample the posterior directly by MCMC. However, the lack of closed-form moments for the CMP distribution makes sampling computationally intensive. Here, to estimate robustly and quickly, we instead assume is diagonal and estimate it by maximizing the prediction likelihood in the filtering stage, as in (Wei and Stevenson 2021).

## Neural Data

[need to add section here with description and citations for V1 and HC data, similar to Neural Comp paper or can also look at the J Comp Neuro CMP paper. This should also include details on and ]

I prefer to move this section to application part (reason: this is the data we use, but the not the method we propose. If we collect the data by ourself, surely we should write it here. But we are just using it…)

# Results

## Figure 1

Code: [figure1\_singleNu\_dirShift.m](https://github.com/weigcdsb/COM_POISSON/blob/main/demo/figure1/figure1_singleNu_dirShift.m)

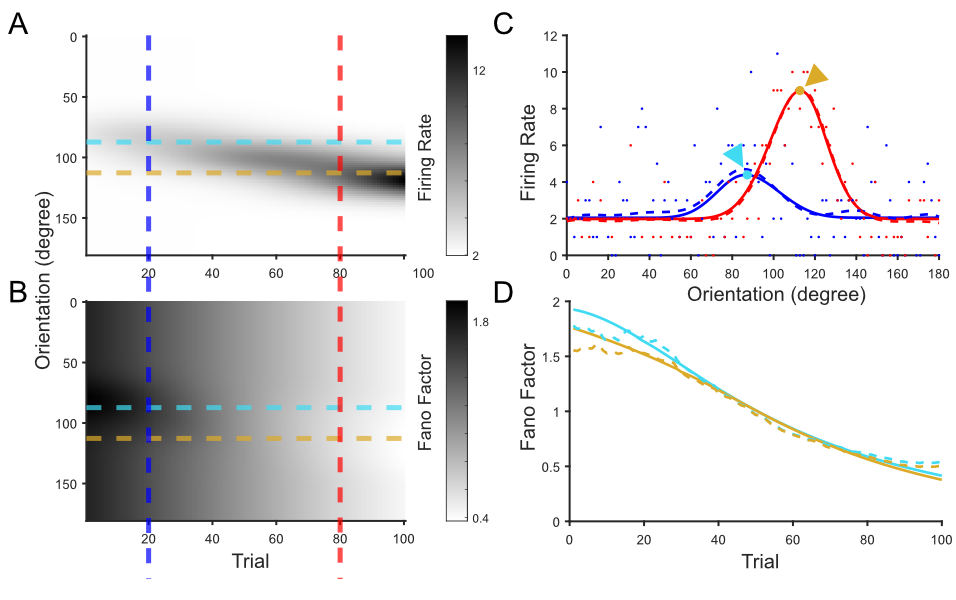


Figure 1. A simulated neuron with shifting firing pattern. (A) The place field of the neuron shifted linearly, with the response amplitude increasing. (B) The neuron changes from over-dispersion to under-dispersion, , with linear changing (0.3679 ~ 2.7183) in log-scale. This leads to a decreasing Fano factor from 1.94 to 0.38. (C) The shifting pattern of the neural activity can be clearly viewed by slicing the 20th (blue) and 80th trial (red). (D) To show the decreasing pattern in Fano factor, we investigate the peak response orientation of 20th (cyan, 87.27 degree)and 80th trial (yellow, 112.73 degree). The dashed lines are fitted values.

The neural responses are nonstationary over time, not only in mean but also in variance. There is growing evidence that the neural variability also changes over time, depending on an animal’s alertness or motivation, as well as, the specific stimuli or behavior.

Here, we simulated a neuron, with place field, the response region of orientation, shifting over 100 trials. The response amplitude also increases along the time at the same time (panel A and C in Figure 1). Meanwhile, the dispersion pattern also changes from over-dispersion to under-dispersion, controlled by linear changing (0.3679 ~ 2.7183) in log-scale. This leads to a decreasing Fano factor from 1.94 to 0.38 (panel B in Figure 1). We then fitted the simulated data, with the same generated predictors: be 10-knots cubic spline basis expansion of the orientation and , at trial . The panel C and D in figure 1 show the fitting results for the selected slices.

## Simulation

Ideas…

* Q determines the timescales in the state estimates… we can’t track arbitrarily fast changes. Changes in dispersion are somewhat hard to track.
* (Fig 2) Practical issue – most neuroscientists only look at the mean, but you could imagine the dispersion changing while the mean is constant… (Scott … Pillow Neurips – NB-LDS constant dispersion param?). Also illustrate how CMP-DGLM can track over- and under-dispersion.

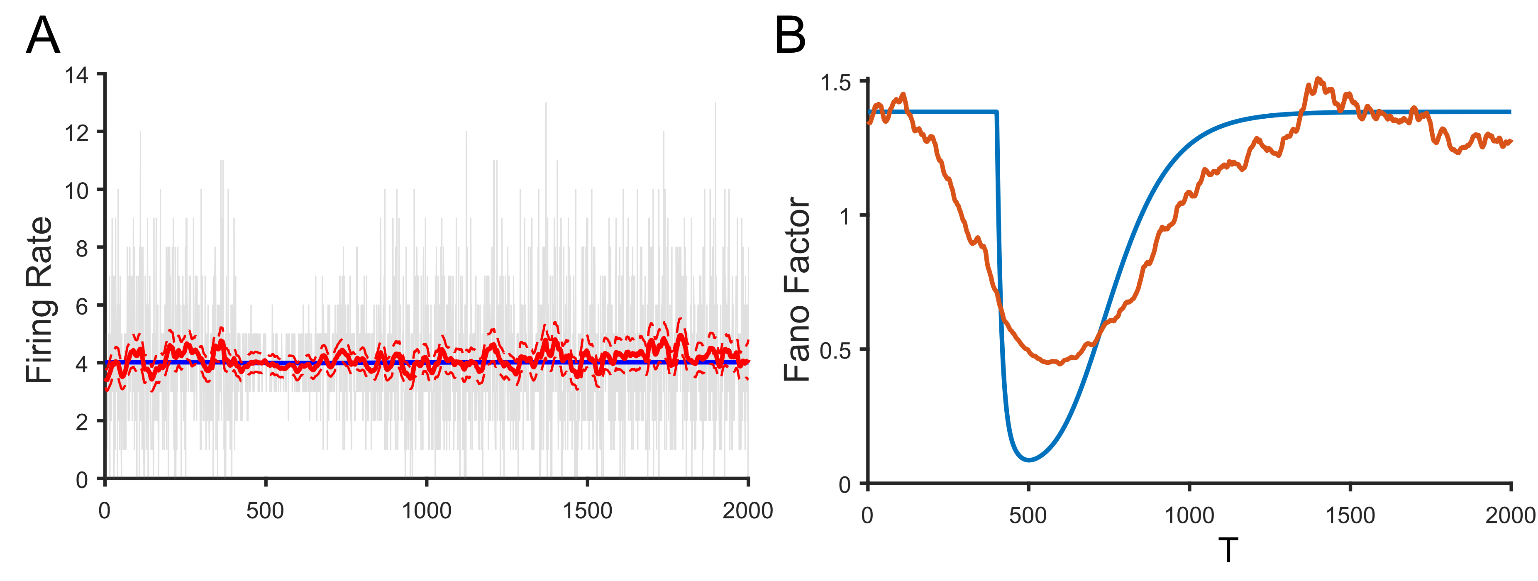


Figure 2. Constant firing rate with non-stationary variability. The mean firing rate is held constant at 4, but there is a sharp decrease in variability at around 500. The dispersion recovers back exponentially afterwards. (A) The background grey line shows the simulated spikes, with the ground truth firing rate in blue. The solid red line shows the fitted value, and the dashed line shows one standard deviation from the MAP. The standard deviations are calculated by truncated summation, but the approximations give essentially the same results (details in appendix- quantifying uncertainties). (B) The fitted Fano factor is shown in red, overlaid by the ground truth in blue.

We then control the mean firing rate to be constant, but make the variance decrease at first and recover back exponentially. Intuitively, the more variability in neural activities, the more variance in the firing rate.

Maybe we can compare CMP vs. Poisson here. Basically, there’s no “free lunch”… CMP dynamic model allow us to track FF, but it sacrifice the estimation accuracy for mean a bit.

Use the same Q for (9.3998e-04).

Left: The mean firing rate. Blue: CMP fit; red: Poisson fit.

- similar, but the blue one (CMP) varies a bit.

Right: comparison of variance of mean firing rate, Orange: CMP; blue: Poisson (lognormal variance)



When we turn on the Q estimation for Poisson, the variance should be even smaller (because the ground truth is constant ). Optimized Q = 6.2826e-07.





## Application

We next applied our method to two datasets: 1) Utah array extracellular recordings of visually evoked activity from anesthetized macaque primary visual cortex (referred as the V1 dataset), and 2) recordings from different rat hippocampal regions while the rat was performing linear maze task (referred as the HC data).

V1 data

In the V1 dataset (Kohn and Smith 2016), the spiking activity was recorded while presenting a movie of sinusoidal gratings with ~100 different drift directions (300ms each, 30s movie in total). The trial was replicated 120 times. For further details on how the data were obtained, see (Kelly et al. 2010; Smith and Kohn 2008). Considering the sequential measurement effect, we analyze the data with single neuron input. The predictors and are circular cubic spline basis expansion of the grating directions. The basis number for is 5, while it varies between 1 and 3 for model comparisons.

Code: [v1\_comparison\_nan.m](https://github.com/weigcdsb/COM_POISSON/blob/main/demo/v1/v1_comparison_nan.m); [compare\_all\_na.m](https://github.com/weigcdsb/COM_POISSON/blob/main/demo/v1/compare_all_na.m); [demo\_v1\_pd.m](https://github.com/weigcdsb/COM_POISSON/blob/main/demo/v1/demo_v1_pd.m)



Hippocampus data

In the HC dataset, the rat was running back and forth along the 250cm linear track. The recording holds up to ~66 min. The input predictors are 10-knots cubic spline basis expansion of positions, and the for all .

Code: [hc\_comparison\_v2.m](https://github.com/weigcdsb/COM_POISSON/blob/main/demo/hc/hc_comparison_v2.m);



Should we separate the direction (-200 ~ 200) or ignore the direction (0 ~ 200)? I prefer to use -200 ~ 200 (left plot) rather than the right one. Since the direction is different, the (empirical) mean and FF should be separated.

# Discussion

Omitted variables can increase the apparent variability of observations via the law of total variance. For example, in the hippocampus, place cell firing is highly variable on different passes through the field (cite Fenton). This may be partially due to joint selectivity to position, speed, and head direction, as well as the influence of theta phase. Here, rather than model these distinct covariates assuming Poisson observations, we allow the variability to be non-Poisson.

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# Appendix

## Approximation of moments for Conway-Maxwell Poisson distribution

In order to efficiently estimate the state-vector for the dynamical CMP model, we make use of a previous for the pmf . Assume and , let , and

Then the moments are (The highlights can be found in the reference, the covariance is calculated by myself):

Here we use this approximation when … and use a truncated summation otherwise.

## Gradient and Hessian of the log-posterior

We estimate the state vector by maximizing the log-posterior with Newton-Raphson updates. Denote , the -th update of NR algorithm is .

The gradient:

, where , and .

The Hessian:

, where

When , the Hessian may be ill-conditioned or even positive-definite. Because of the factorial, the “outliers” are common. To ensure the robustness, do Fisher scoring, i.e. replace the observed information by the expected information , so that .

## Quantifying Uncertainties

After convergence, we have an approximation of the log-posterior , and we can use this approximation to quantify the state uncertainty as well as uncertainty about the mean rate at each time.

To find the uncertainty in CMP parameters, let , then . Denote the variance of CMP parameters as , and .

The conditional mean firing rate is , whose variance can be calculated by the Delta method:

We can calculate the moments as in the first section of the appendix, or we can use a simpler approximation when or . Then and .