

# Gibbs Sampling Problem 1

Ganchao Wei

5/26/2021

## 1. Gibbs Sampling for a Bivariate Normal Distribution

### 1.1 Notations and Full Conditional Distributions

Let  $\mathbf{X} = (X_1, X_2)' \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , where  $\boldsymbol{\mu} = (\mu_1, \mu_2)'$  and  $\boldsymbol{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ . Therefore, the full conditional distributions of  $X_1$  and  $X_2$  are:

$$X_1|X_2 = x_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

$$X_2|X_1 = x_1 \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

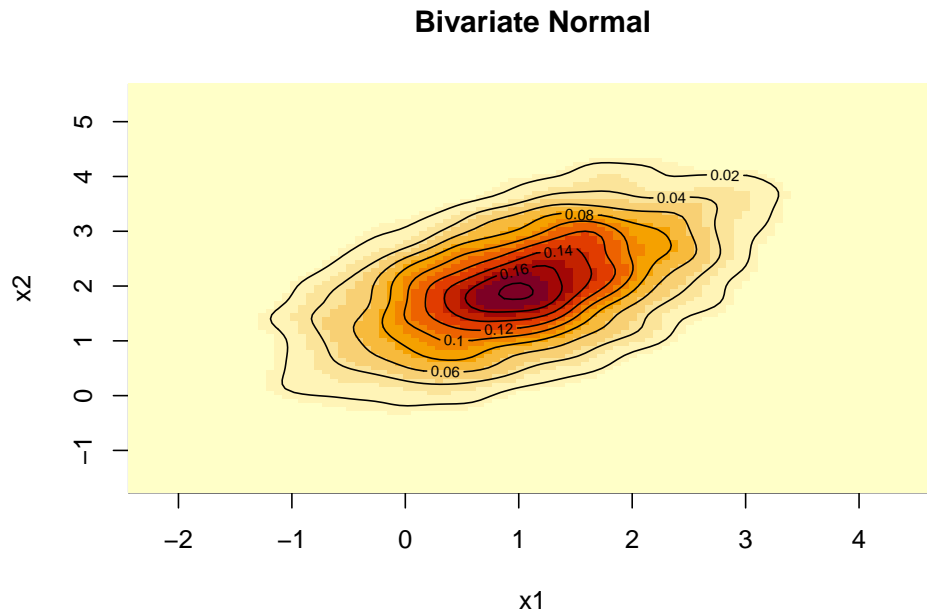
### 1.2 Simulation

Here, I set  $\boldsymbol{\mu} = (1, 2)'$  and  $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ . To visualize the distribution, let's draw 5000 samples directly from the distribution, with contours shown (from kernel density estimation).

```
set.seed(1)

mu <- c(1, 2)
Sig <- matrix(c(1, .5,
                .5, 1), nrow = 2, byrow = T)
BVNs <- mvrnorm(5000, mu, Sig)

## plot
BVNs.kde <- kde2d(BVNs[, 1], BVNs[, 2], n = 100)
image(BVNs.kde, main = 'Bivariate Normal',
      xlab = 'x1', ylab = 'x2')
contour(BVNs.kde, add = T)
```



Then 5000 Gibbs samples are drawn, with the initial value  $\mathbf{x}^{(1)} = \boldsymbol{\mu}$ . The code for Gibbs sampling are as follows. The contours of kernel density is overlaid by the samples from the Gibbs sampler.

```
GS_BVN <- function(mu, Sig, nGS, x0){
  X <- matrix(NA, nrow = nGS, ncol = 2)
  X[1, ] <- x0
  for(i in 2:nGS){

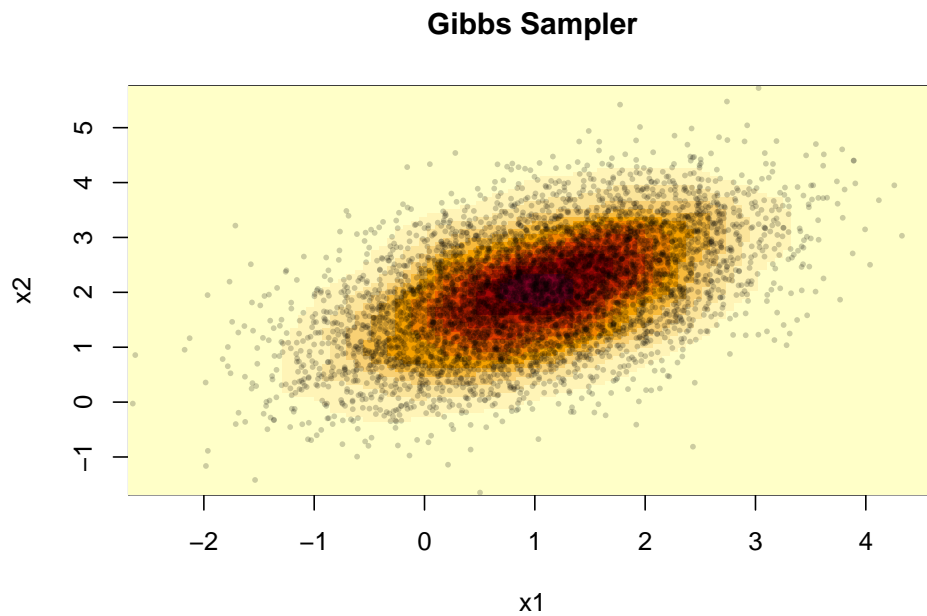
    ## update x1
    mu_tmp1 <- mu[1] + Sig[1, 2] %*% solve(Sig[2, 2]) %*% (X[i-1, 2] - mu[2])
    sig_tmp1 <- Sig[1, 1] - Sig[1, 2] %*% solve(Sig[2, 2]) %*% Sig[2, 1]
    X[i, 1] <- rnorm(1, mu_tmp1, sqrt(sig_tmp1))

    ## update x2
    mu_tmp2 <- mu[2] + Sig[2, 1] %*% solve(Sig[1, 1]) %*% (X[i, 1] - mu[1])
    sig_tmp2 <- Sig[2, 2] - Sig[2, 1] %*% solve(Sig[1, 1]) %*% Sig[1, 2]
    X[i, 2] <- rnorm(1, mu_tmp2, sqrt(sig_tmp2))
  }
  return(X)
}

set.seed(2)
x0 <- mu
nGS <- 5000
X <- GS_BVN(mu, Sig, nGS, x0)

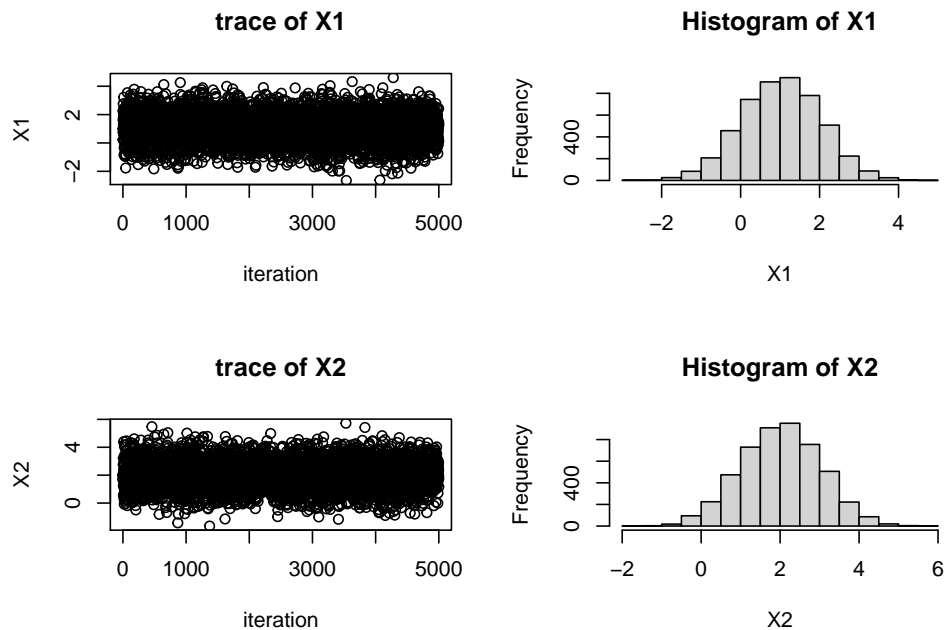
## plot
X.kde <- kde2d(X[, 1], X[, 2], n = 100)
image(X.kde, main = 'Gibbs Sampler',
      xlab = 'x1', ylab = 'x2')
lines(X[, 1], X[, 2], type = 'p',
```

```
col = alpha(1, 0.2), pch = 16, cex = 0.5)
```



We can see the sampling distribution from the Gibbs sampler matches the true distribution well. Further, the 1-D sample traces and histograms are shown below:

```
par(mfrow = c(2, 2))
plot(X[, 1], main = 'trace of X1',
     xlab = 'iteration', ylab = 'X1')
hist(X[, 1], xlab = 'X1', main = 'Histogram of X1')
plot(X[, 2], main = 'trace of X2',
     xlab = 'iteration', ylab = 'X2')
hist(X[, 2], xlab = 'X2', main = 'Histogram of X2')
```



```
par(mfrow = c(1, 1))
```

The trace plots show that there are no heavy auto-correlation issues, and the 1D histograms show that the marginal distributions are normal.

## 2. Body Temperature

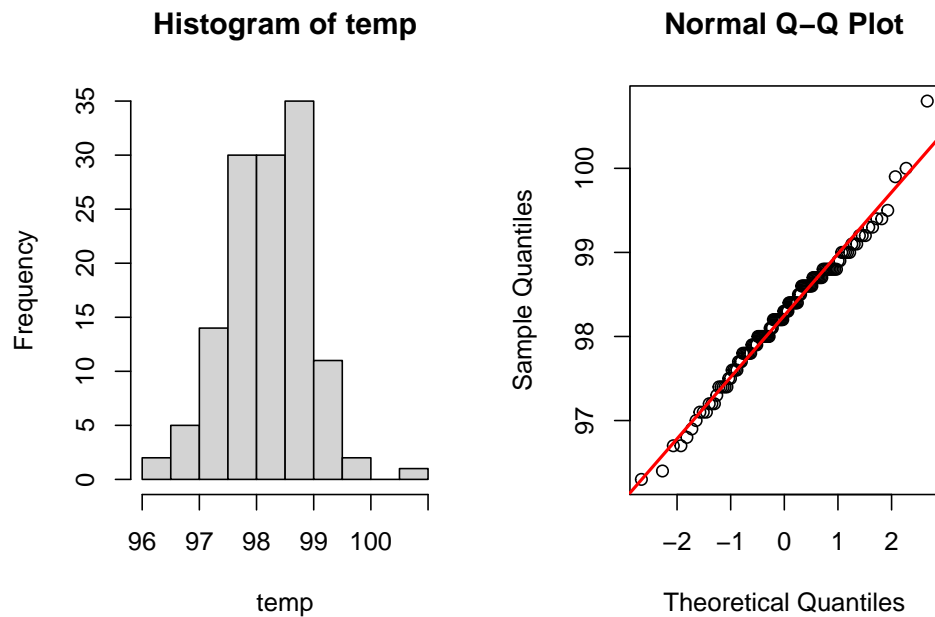
Currently, we focus on temperature only.

```
d2 <- read.table('bodytemp.txt')
temp <- d2$temperature
```

### 2.1 Exploratory Analysis

The histogram and normal Q-Q plot are show below.

```
par(mfrow = c(1, 2))
hist(temp)
qqnorm(temp)
abline(a = mean(temp), b= sd(temp),
       col = 'red', lwd = 2)
```



```
par(mfrow = c(1, 1))
```

We can see that the distribution of body temperature is somewhat normal. The p-value for Shapiro-Wilk test is 0.23, which is larger than 0.05. Therefore, we may assume that the body temperature follows a normal distribution.

```
shapiro.test(temp)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  temp
## W = 0.98658, p-value = 0.2332
```

The mean and variance:

```
temp.mean <- mean(temp)
temp.var <- var(temp)
temp.n <- length(temp)

cat('sample mean:', temp.mean, '\n')
```

```
## sample mean: 98.24923
```

```
cat('sample variance:', temp.var)
```

```
## sample variance: 0.5375575
```

## 2.2 Gibbs Sampler

Denote  $X_i \stackrel{i.i.d.}{\sim} N(\theta, \sigma^2)$ , and  $\mathbf{X} = (x_1, \dots, x_n)'$ . Here, we use independent priors for  $\theta$  and  $\sigma^2$  as follows:

$$\begin{aligned}\theta &\sim N(\mu_0, \tau_0^2) \\ \tilde{\sigma}^2 = 1/\sigma^2 &\sim \text{Gamma}(\alpha_0, \beta_0)\end{aligned}$$

Here,  $\tilde{\sigma}^2 = 1/\sigma^2$  is the “precision”,  $\alpha_0$  is the shape parameter, and  $\beta_0$  is the rate parameter. Therefore, we can get the full conditional distributions for  $\theta$  and  $\tilde{\sigma}^2$ :

$$\begin{aligned}p(\theta|\tilde{\sigma}^2, \mathbf{x}) &\propto p(\mathbf{x}|\theta, \tilde{\sigma}^2)p(\theta|\tilde{\sigma}^2)p(\tilde{\sigma}^2) \\ &\propto p(\mathbf{x}|\theta, \tilde{\sigma}^2)p(\theta) \\ &\propto N\left(\frac{\mu_0/\tau_0^2 + n\mathbf{x}\tilde{\sigma}^2}{1/\tau_0^2 + n\tilde{\sigma}^2}, (1/\tau_0^2 + n\tilde{\sigma}^2)^{-1}\right) \\ p(\tilde{\sigma}^2|\theta, \mathbf{x}) &\propto p(\mathbf{x}|\theta, \tilde{\sigma}^2)p(\tilde{\sigma}^2|\theta)p(\theta) \\ &\propto p(\mathbf{x}|\theta, \tilde{\sigma}^2)p(\tilde{\sigma}^2) \\ &\propto \text{Gamma}\left(\alpha_0 + \frac{n}{2}, \beta_0 + \frac{\sum_{i=1}^n (x_i - \theta)^2}{2}\right)\end{aligned}$$

To help with interpretation, rewrite  $\alpha_0 = \frac{\nu_0}{2}$  and  $\beta_0 = \frac{\nu_0\sigma_0^2}{2}$ . Here,  $\nu_0$  can be viewed as the prior sample size, and  $\sigma_0^2$  can be viewed as the prior sample variance. Further, since  $\sum_{i=1}^n (x_i - \theta)^2 = \sum_{i=1}^n (x_i - \bar{\mathbf{x}})^2 + \sum_{i=1}^n (\bar{\mathbf{x}} - \theta)^2 = (n-1)s^2 + n(\bar{\mathbf{x}} - \theta)^2$ , where  $s^2$  and  $\bar{\mathbf{x}}$  are sample variance and mean. So, the full conditional distribution for  $\tilde{\sigma}^2$  can be written as:

$$p(\tilde{\sigma}^2|\theta, \mathbf{x}) \propto \text{Gamma}\left(\frac{\nu_0 + n}{2}, \frac{\nu_0\sigma_0^2 + (n-1)s^2 + n(\bar{\mathbf{x}} - \theta)^2}{2}\right)$$

OK, let's begin to do the Gibbs sampling!

According to <https://www.webmd.com/first-aid/normal-body-temperature>, the normal body temperature of an adult ranges from 97F to 99F, I set the prior parameters as  $\mu_0 = 98, \tau_0^2 = 1, \nu_0 = 4, \sigma_0^2 = 0.5$ . Draw 5000 samples, with initial values be  $\theta^{(1)} = \bar{\mathbf{x}}$  and  $\tilde{\sigma}^{2(1)} = 1/s^2$ . The code are as follows. In the code, I further denote  $\phi = (\theta, \tilde{\sigma}^2)'$

```
nGS <- 5000
PHI <- matrix(NA, nrow = nGS, ncol = 2)

## prior
mu0 <- 98
tau20 <- 1
nu0 <- 4
sig20 <- 0.5

## initialization
PHI[1, ] <- c(temp.mean, 1/temp.var)

## GS
set.seed(3)
for(i in 2:nGS){

  ## update theta
  mun <- (mu0/tau20 + temp.n*temp.mean*PHI[i-1, 2])/
    (1/tau20 + temp.n*PHI[i-1, 2])
  tau2n <- 1/(1/tau20 + temp.n*PHI[i-1, 2])
```

```

PHI[i, 1] <- rnorm(1, mun, sqrt(tau2n))

## update sig2
alpn <- (nu0 + temp.n)/2
betan <- (nu0*sig20 + (temp.n-1)*temp.var + temp.n*(temp.mean - PHI[i, 1])^2)/2
PHI[i, 2] <- rgamma(1, alpn, betan)
}

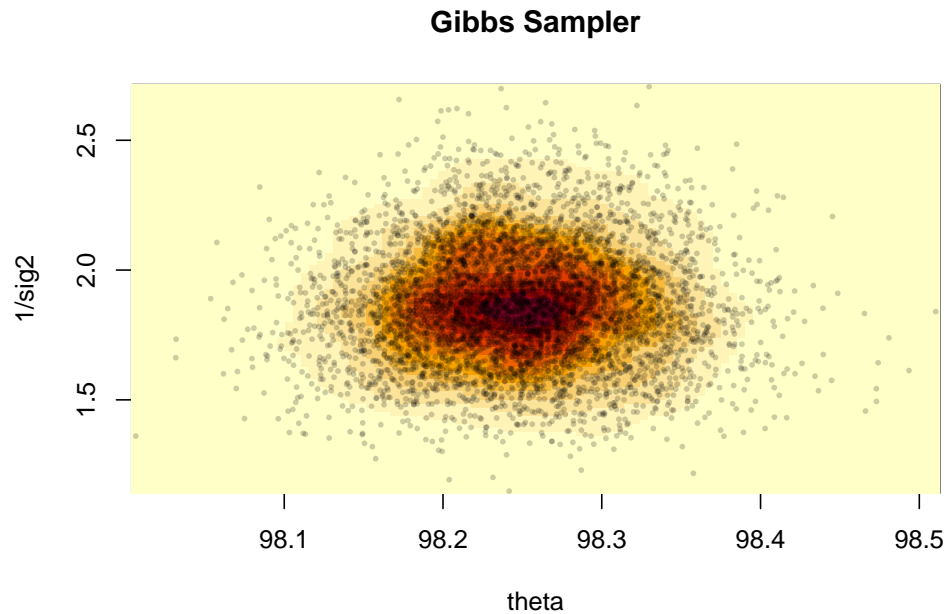
```

Again, let's first see the samples from the posterior distribution by the Gibbs sampler. The contours of kernel density is overlaid by the samples from the Gibbs sampler.

```

PHI.kde <- kde2d(PHI[, 1], PHI[, 2], n = 100)
image(PHI.kde, main = 'Gibbs Sampler',
      xlab = 'theta', ylab = '1/sig2')
lines(PHI[, 1], PHI[, 2], type = 'p',
      col = alpha(1, 0.2), pch = 16, cex = 0.5)

```



The traces and histograms are shown below. Gibbs samples mean, 95% HPD 0.25 quantile and 0.975 quantile are overlaid in the histograms.

```

par(mfrow = c(2, 2))
plot(PHI[, 1], main = 'trace of theta',
     xlab = 'iteration', ylab = 'theta')
hist(PHI[, 1], xlab = 'theta',
     main = 'Histogram of theta')
theta.hdi <- hdi(PHI[, 1], credMass = 0.95)
theta.quantile <- quantile(PHI[, 1], c(0.025, 0.975))

abline(v = mean(PHI[, 1]), lwd = 2, col = 'red')

```

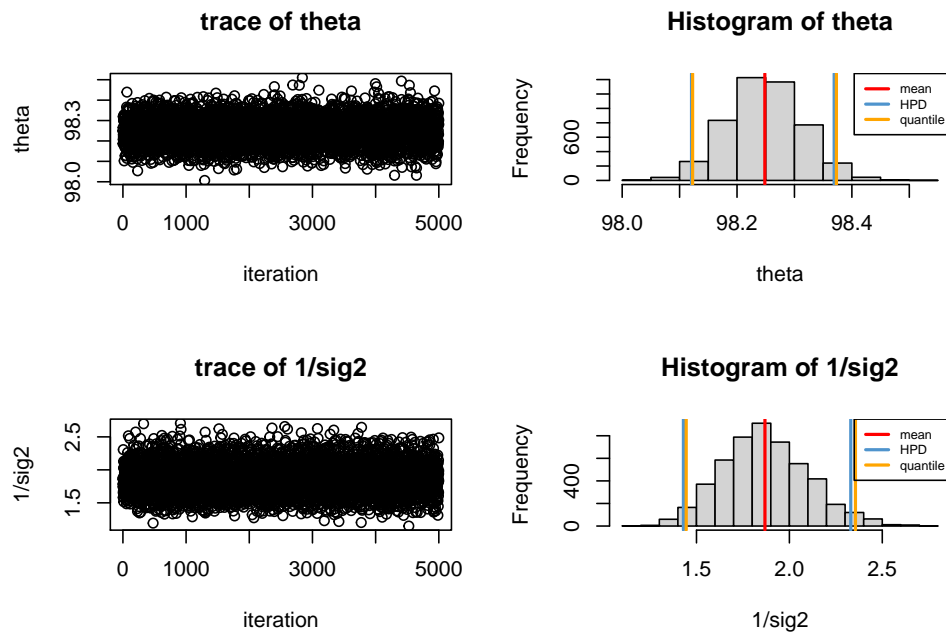
```

abline(v = theta.hdi[1], lwd = 2, col = 'steelblue3')
abline(v = theta.hdi[2], lwd = 2, col = 'steelblue3')
abline(v = theta.quantile[1], lwd = 2, col = 'orange')
abline(v = theta.quantile[2], lwd = 2, col = 'orange')
legend('topright', legend = c('mean', 'HPD', 'quantile'),
      lwd = 2, col = c('red', 'steelblue3', 'orange'), cex = 0.6)

plot(PHI[, 2], main = 'trace of 1/sig2',
     xlab = 'iteration', ylab = '1/sig2')
hist(PHI[, 2], xlab = '1/sig2',
     main = 'Histogram of 1/sig2')
invSig2.hdi <- hdi(PHI[, 2], credMass = 0.95)
invSig2.quantile <- quantile(PHI[, 2], c(0.025, 0.975))

abline(v = mean(PHI[, 2]), lwd = 2, col = 'red')
abline(v = invSig2.hdi[1], lwd = 2, col = 'steelblue3')
abline(v = invSig2.hdi[2], lwd = 2, col = 'steelblue3')
abline(v = invSig2.quantile[1], lwd = 2, col = 'orange')
abline(v = invSig2.quantile[2], lwd = 2, col = 'orange')
legend('topright', legend = c('mean', 'HPD', 'quantile'),
      lwd = 2, col = c('red', 'steelblue3', 'orange'), cex = 0.6)

```



```

par(mfrow = c(1, 1))

```

The trace plots show that there are no heavy auto-correlation issues. The histograms show that the posterior distributions are somewhat symmetric for both parameters.

The posterior mean, 95% HPD and 95% symmetric credible interval for  $\theta$ :

```
## posterior mean: 98.24841
```



```
## 95% HPD: [ 98.12076 , 98.36916 ]
```

```
## 95% symmetric credible interval: [ 98.12267 , 98.37308 ]
```

While the posterior mean, 95% HPD and 95% symmetric credible interval for  $\tilde{\sigma}^2$ :

```
## posterior mean: 1.868408
```

```
## 95% HPD: [ 1.429549 , 2.330347 ]
```

```
## 95% symmetric credible interval: [ 1.444683 , 2.355617 ]
```

Further, the values for variance  $\sigma^2$ :

```
## posterior mean: 0.5434473
```

```
## 95% HPD: [ 0.4151132 , 0.6779238 ]
```

```
## 95% symmetric credible interval: [ 0.4245172 , 0.6921935 ]
```

The HPD and symmetric credible intervals are close in both parameters.

Also, notice that the observation mean & precision are:

```
## sample mean: 98.24923
```

```
## sample variance: 0.5375575
```

```
## sample precision: 1.860266
```

## 2.3 Conclusion

From the above analysis, the 95% HPD for the posterior mean is [98.121, 98.369] and 95% HPD for the posterior variance is [0.415, 0.678]. Since 98.6 is not in 95% HPD of the mean, the normal body temperature is not 98.6.