# 1 Conway-Maxwell Poisson (COM-Poisson, CMP) and Properties

The p.m.f. for Conway-Maxwell Poisson (CMP) is:

$$P(Y=y|\lambda,\nu)=\frac{\lambda^y}{(y!)^\nu}\frac{1}{Z(\lambda,\nu)}$$

$Z(\lambda,\nu)=Z$ is the normalization constant, i.e. $Z(\lambda,\nu)=Z=\sum_{y=0}^{\infty}\frac{\lambda^y}{(y!)^\nu}$, which doesn't have closed form in general. The domain for parameters is $\lambda,\nu>0$ and $0<\lambda<1,\nu=0$. The parameter $\nu$ controls the dispersion: 1) when $\nu=1$, the CMP is Poisson distribution, 2) when $\nu<1$, the distribution is over-dispersed and 3) when $\nu>1$, the distribution is under-dispersed. When $\nu\to\infty$, the CMP approaches a Bernoulli distribution, while $\nu=0$, it reduces to a geometric distribution.

To model the mean and dispersion simultaneously, two linear models are used for parameters $\lambda$ and $\nu$, i.e. $\log(\lambda)=\boldsymbol{x}'\boldsymbol{\beta}$ and $\log(\nu)=\boldsymbol{g}'\boldsymbol{\gamma}$, in this manuscript, I further define $\boldsymbol{\theta}'=(\boldsymbol{\beta}',\boldsymbol{\gamma}')$.

To derive the SSP for CMP regression, the keys are gradient and Hessian for log-likelihood. In the following part of this section, I will define some notations and give some necessary properties for CMP.

Assume there are n independent $Y_i\sim CMP(\lambda_i,\nu_i)$. Denote $\boldsymbol{\eta}_i=(log(\lambda_i),\nu_i)'$, $Z_i=Z(\lambda_i,\nu_i)$ The log-likelihood for $i^{th}$ observation is:

$$l_i(\boldsymbol{\eta}_i)=y_i log(\lambda_i)-log(y_i!)\nu_i-log(Z_i)$$

Since $E(Y_i)=\frac{\partial log(Z_i)}{\partial log(\lambda_i)}$, $Var(Y_i)=\frac{\partial^2 log(Z_i)}{\partial log(\lambda_i)^2}$, $E(log(Y_i!))=-\frac{\partial log(Z_i)}{\partial\nu_i}$, $Var(log(Y_i!))=\frac{\partial^2 log(Z_i)}{\partial\nu_i^2}$ and $Cov(Y_i,log(Y_i!))=-\frac{\partial^2 log(Z_i)}{\partial log(\lambda_i)\partial\nu_i}$, the gradient for $l_i$ (w.r.t $\boldsymbol{\eta}_i$) is:

$$\frac{\partial l_i}{\partial\boldsymbol{\eta}_i}=\begin{pmatrix}y_i-E(Y_i)\\E(log(Y_i!))-log(y_i!)\end{pmatrix}$$

And the Hessian for $l_i$ (w.r.t $\boldsymbol{\eta}_i$) is:

$$\frac{\partial^2 l_i}{\partial\boldsymbol{\eta_i}\partial\boldsymbol{\eta_i}'}=\begin{pmatrix}-Var(Y_i) & Cov(Y_i,log(Y_i!))\\Cov(Y_i,log(Y_i!)) & -Var(log(Y_i!))\end{pmatrix}$$

The moments $E(Y_i)$, $Var(Y_i)$, $E(log(Y_i!))$, $Var(log(Y_i!))$ and $Cov(Y_i,log(Y_i!))$ by using the following approximation for normalizing constant $Z_i$:

$$Z_i=\frac{e^{\nu_i\lambda_i^{1/\nu_i}}}{\lambda_i^{\frac{\nu_i-1}{2\nu_i}}}\left(1+c_1(\nu_i\lambda_i^{1/\nu_i})^{-1}+c_2(\nu_i\lambda_i^{1/nu_i})^{-2}+\mathcal{O}(\lambda_i^{\frac{-3}{\nu_i}})\right)$$

The approximation works well when $\lambda_i \geq 2$ and $\nu_i \leq 1$, and this can be helpful when updating/ calculating the gradient and hessian matrix. Currently, I didn't use this approximation for simplicity.

If we use models $\log(\lambda_i) = \boldsymbol{x_i}'\boldsymbol{\beta}$ and $\log(\nu_i) = \boldsymbol{g_i}'\boldsymbol{\gamma}$, by using chain rule, the gradient for $l_i$ (w.r.t $\boldsymbol{\theta}$) is:

$$\frac{\partial l_i}{\partial \boldsymbol{\theta}} == \begin{pmatrix} [y_i - E(Y_i)]\boldsymbol{x_i} \\ \nu_i[E(log(Y_i!)) - log(y_i!)]\boldsymbol{g_i} \end{pmatrix}$$

And the Hessian for $l_i$ (w.r.t $\boldsymbol{\theta}$) is:

$$\frac{\partial^2 l_i}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} = \begin{pmatrix} -Var(Y_i)\boldsymbol{x_i}\boldsymbol{x_i}' & \nu_i Cov(Y_i, log(Y_i!))\boldsymbol{x_i}\boldsymbol{g_i}' \\ \nu_i Cov(Y_i, log(Y_i!))\boldsymbol{g_i}\boldsymbol{x_i}' & -\nu_i[\nu_i Var(log(Y_i!) - E(log(Y_i!)) + log(y_i!))]\boldsymbol{g_i}\boldsymbol{g_i}' \end{pmatrix}$$

# 2 SSP for CMP

By denoting $a_i = [y_i - E(Y_i|\hat{\boldsymbol{\theta}}_{MLE})]$ and $b_i = \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_i!)]$, we define

$$\boldsymbol{V}_c = \frac{1}{rn^2} \sum_{i=1}^{n} \frac{1}{\pi_i} \begin{pmatrix} a_i^2 \boldsymbol{x_i}\boldsymbol{x_i}' & a_i b_i \boldsymbol{x_i}\boldsymbol{g_i}' \\ a_i b_i \boldsymbol{g_i}\boldsymbol{x_i}' & b_i^2 \boldsymbol{g_i}\boldsymbol{g_i}' \end{pmatrix}$$

where $r$ is the subsample size, with subsampling probabilities $\pi_i$ for all data points.

Further we denote the observed information matrix as

$$\boldsymbol{M}_X = \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} A_i & B_i \\ B_i^{'} & C_i \end{pmatrix}$$

where

$$A_i = Var(Y_i|\hat{\boldsymbol{\theta}}_{MLE})\boldsymbol{x_i}\boldsymbol{x_i}'$$
$$B_i = -\nu_i(\hat{\boldsymbol{\theta}}_{MLE})Cov(Y_i, log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE})\boldsymbol{x_i}\boldsymbol{g_i}'$$
$$C_i = \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[\nu_i(\hat{\boldsymbol{\theta}}_{MLE})Var(log(Y_i!|\hat{\boldsymbol{\theta}}_{MLE}) - E(log(Y_i!|\hat{\boldsymbol{\theta}}_{MLE})) + log(y_i!))]\boldsymbol{g_i}\boldsymbol{g_i}'$$

Then follow the same steps as in OSMAC, we can show that as $n \to \infty$ and $r \to \infty$, conditional on full data matrix $\mathcal{F}_n = \boldsymbol{X}, \boldsymbol{y}$ in probability,

$$\boldsymbol{V}^{-1/2}(\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{MLE}) \to N(0, \boldsymbol{I})$$

where $\boldsymbol{V} = \boldsymbol{M}_X^{-1}\boldsymbol{V}_c\boldsymbol{M}_X^{-1}$ and $\tilde{\boldsymbol{\theta}}$ is the sub-sampling estimates for $\boldsymbol{\theta}$
Then under A-optimality criterion, the sub-sampling probabilities (SSPs) $\pi_i^{mMSE}$

are proportional to $||\boldsymbol{M}_X^{-1}\begin{pmatrix} a_i\boldsymbol{x}_i \\ b_i\boldsymbol{g}_i \end{pmatrix}||$, while under L-optimality criterion, $\pi_i^{mVc} \propto ||\begin{pmatrix} a_i\boldsymbol{x}_i \\ b_i\boldsymbol{g}_i \end{pmatrix}||$. In other words:

$$\pi_i^{mMSE} = \frac{||\boldsymbol{M}_X^{-1}\begin{pmatrix} [y_i - E(Y_i|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_i \\ \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_i!)]\boldsymbol{g}_i \end{pmatrix}||}{\sum_{j=1}^n ||\boldsymbol{M}_X^{-1}\begin{pmatrix} [y_j - E(Y_j|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_j \\ \nu_j(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_j!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_j!)]\boldsymbol{g}_j \end{pmatrix}||}$$

$$\pi_i^{mVc} = \frac{||\begin{pmatrix} [y_i - E(Y_i|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_i \\ \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_i!)]\boldsymbol{g}_i \end{pmatrix}||}{\sum_{j=1}^n ||\begin{pmatrix} [y_j - E(Y_j|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_j \\ \nu_j(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_j!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_j!)]\boldsymbol{g}_j \end{pmatrix}||}$$

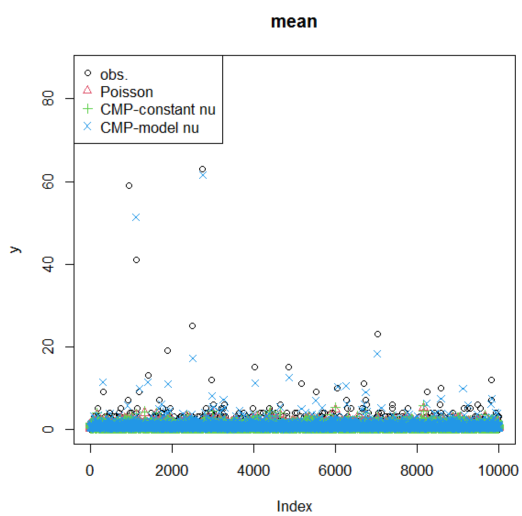
# 3 Necessity for fitting CMP model

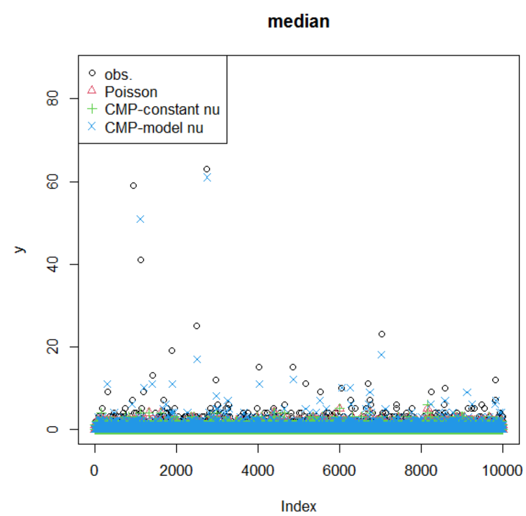
Here I first generate the data by CMP distribution.

There are $n = 10000$ independent observations, with $Y_i \sim CMP(\lambda_i, \nu_i)$ for $i = 1, 2, ..., n$. The $\lambda_i$ and $\nu_i$ are modeled as:

$$\lambda_i = \exp(x_{i1})$$
$$\nu_i = \exp(1 + g_{i1})$$

where $x_{i1} \overset{i.i.d.}{\sim} N(0, 1)$ and $g_{i1} \overset{i.i.d.}{\sim} N(1, 1)$ The following two plots show the fitted mean and median for 1) Poisson regression, 2) CMP regression, with constant $\nu_i = \nu$ (constant CMP) and 3) CMP with $\nu_i$ modeled by $g_{i1}$ (full CMP).



The MSEs ($\frac{(Y_i-\bar{Y})^2}{n}$) for mean are 1) Poisson: 2.269, 2) constant CMP: 2.273 and 3) full CMP: 0.391. The Poisson model and constant dispersion CMP model are not good for handling extreme observations. Therefore, it's necessary to take dispersion into account and CMP is a good choice.

The running time for these three: 1)Poisson: 0.03s, 2) constant CMP: 2.40s and 3) full CMP: 3.59s. This shows that CMP regression is much more computational expensive than regular Poisson regression, which suggests the necessity of sub-sampling.

# 4 Sub-sampling in CMP

To fit the CMP regression model, although we can use regular Newton-Raphson method to maximize the likelihood, the information matrix may not be stable. Therefore (as far as I know), there are two methods to deal with that: 1) use direct optimization strategy, such as L-BFGS-B and 2) optimize $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in an alternative way, i.e. hold one part fixed when fitting another. The alternating method reduces the problem into a two-step Newton-Raphson/ IRWLS problem.

Here I use the package implementing L-BFGS-B, and modify the objective likelihood function a bit to allow for optimization of weighted log-likelihood. To be more specific, I change:

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sum(y*log(lambda) - nu*lgamma(y+1) - logz)
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to

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sum(weights*(y*log(lambda) - nu*lgamma(y+1) - logz))
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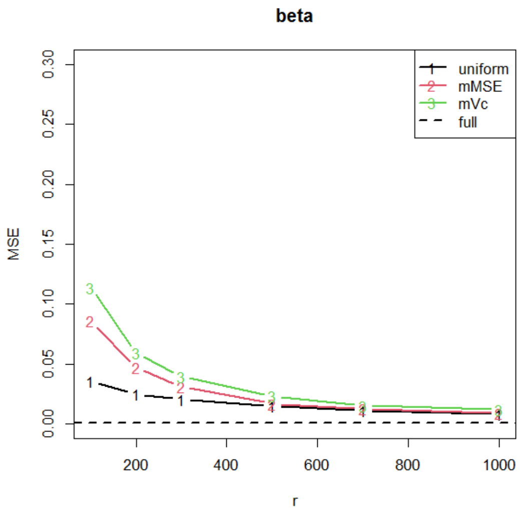
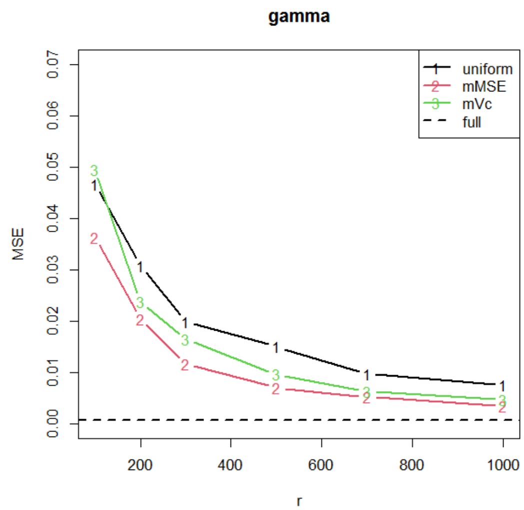
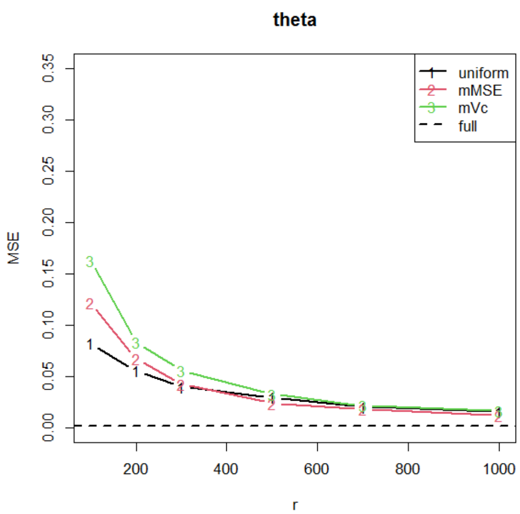
The weights are normalized such that the summation of weights equals to sample size. Therefore, I can use the modified functions to maximizes the weighted likelihood.

The gradient and hessian are calculated by truncated summation (500 steps), according to estimated p.m.f. by package ($\hat{f}(y|\lambda_i, \nu_i)$). For example, $E(Y_i) \approx \sum_{y=0}^{500} y \cdot \hat{f}(y|\lambda_i, \nu_i)$

Then I use the bootstrap ($B = 500$) to calculate the MSE for $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and the overall $\boldsymbol{\theta}$. In some cases (rare), bad subsamples will make the algorithm crash down (e.g. infinite function for L-BFGS-B). Here, I simply discarded those bad bootstrap samples (This will cause problems?) Well...




It seems that the gradient and hessian (numerically evaluated) is not stable, and the variations for evaluating gradient and hessian are larger than the variation for uniform subsampling, especially for $\boldsymbol{\beta}$. ($\boldsymbol{\gamma}$ is always better than $\boldsymbol{\beta}$, in the limited simulations)