# 1 Conway-Maxwell Poisson (COM-Poisson, CMP) and Properties

The p.m.f. for Conway-Maxwell Poisson (CMP) is:

$$P(Y = y|\lambda, \nu) = \frac{\lambda^y}{(y!)^\nu} \frac{1}{Z(\lambda, \nu)}$$

$Z(\lambda, \nu) = Z$ is the normalization constant, i.e. $Z(\lambda, \nu) = Z = \sum_{y=0}^{\infty} \frac{\lambda^y}{(y!)^\nu}$, which doesn't have closed form in general. The domain for parameters is $\lambda, \nu > 0$ and $0 < \lambda < 1, \nu = 0$. The parameter $\nu$ controls the dispersion: 1) when $\nu = 1$, the CMP is Poisson distribution, 2) when $\nu < 1$, the distribution is over-dispersed and 3) when $\nu > 1$, the distribution is under-dispersed. When $\nu \to \infty$, the CMP approaches a Bernoulli distribution, while $\nu = 0$, it reduces to a geometric distribution.

To model the mean and dispersion simultaneously, two linear models are used for parameters $\lambda$ and $\nu$, i.e. $\log(\lambda) = \boldsymbol{x'\beta}$ and $\log(\nu) = \boldsymbol{g'\gamma}$, in this manuscript, I further define $\boldsymbol{\theta'} = (\boldsymbol{\beta'}, \boldsymbol{\gamma'})$.

To derive the SSP for CMP regression, the keys are gradient and Hessian for log-likelihood. In the following part of this section, I will define some notations and give some necessary properties for CMP.

Assume there are n independent $Y_i \sim CMP(\lambda_i, \nu_i)$. Denote $\boldsymbol{\eta}_i = (log(\lambda_i), \nu_i)'$, $Z_i = Z(\lambda_i, \nu_i)$ The log-likelihood for $i^{th}$ observation is:

$$l_i(\boldsymbol{\eta}_i) = y_i log(\lambda_i) - log(y_i!)\nu_i - log(Z_i)$$

Since $E(Y_i) = \frac{\partial log(Z_i)}{\partial log(\lambda_i)}$, $Var(Y_i) = \frac{\partial^2 log(Z_i)}{\partial log(\lambda_i)^2}$, $E(log(Y_i!)) = -\frac{\partial log(Z_i)}{\partial \nu_i}$, $Var(log(Y_i!)) = \frac{\partial^2 log(Z_i)}{\partial \nu_i^2}$ and $Cov(Y_i, log(Y_i!)) = -\frac{\partial^2 log(Z_i)}{\partial log(\lambda_i)\partial \nu_i}$, the gradient for $l_i$ (w.r.t $\boldsymbol{\eta}_i$) is:

$$\frac{\partial l_i}{\partial \boldsymbol{\eta}_i} = \begin{pmatrix} y_i - E(Y_i) \\ E(log(Y_i!)) - log(y_i!) \end{pmatrix}$$

And the Hessian for $l_i$ (w.r.t $\boldsymbol{\eta}_i$) is:

$$\frac{\partial^2 l_i}{\partial \boldsymbol{\eta_i} \partial \boldsymbol{\eta_i'}} = \begin{pmatrix} -Var(Y_i) & Cov(Y_i, log(Y_i!)) \\ Cov(Y_i, log(Y_i!)) & -Var(log(Y_i!)) \end{pmatrix}$$

The moments $E(Y_i)$, $Var(Y_i)$, $E(log(Y_i!))$, $Var(log(Y_i!))$ and $Cov(Y_i, log(Y_i!))$ by using the following approximation for normalizing constant $Z_i$:

$$Z_i = \frac{e^{\nu_i \lambda_i^{1/\nu_i}}}{\lambda_i^{\frac{\nu_i - 1}{2\nu_i}}} \left(1 + c_1(\nu_i \lambda_i^{1/\nu_i})^{-1} + c_2(\nu_i \lambda_i^{1/nu_i})^{-2} + \mathcal{O}(\lambda_i^{\frac{-3}{\nu_i}})\right)$$

The approximation works well when $\lambda_i \geq 2$ and $\nu_i \leq 1$, and this can be helpful when updating/ calculating the gradient and hessian matrix.

If we use models $\log(\lambda_i) = \boldsymbol{x_i}'\boldsymbol{\beta}$ and $\log(\nu_i) = \boldsymbol{g_i}'\boldsymbol{\gamma}$, by using chain rule, the gradient for $l_i$ (w.r.t $\boldsymbol{\theta}$) is:

$$\frac{\partial l_i}{\partial \boldsymbol{\theta}} == \begin{pmatrix} [y_i - E(Y_i)]\boldsymbol{x_i} \\ \nu_i[E(log(Y_i!)) - log(y_i!)]\boldsymbol{g_i} \end{pmatrix}$$

And the Hessian for $l_i$ (w.r.t $\boldsymbol{\theta}$) is:

$$\frac{\partial^2 l_i}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta'}} = \begin{pmatrix} -Var(Y_i)\boldsymbol{x_i}\boldsymbol{x_i}' & \nu_i Cov(Y_i, log(Y_i!))\boldsymbol{x_i}\boldsymbol{g_i}' \\ \nu_i Cov(Y_i, log(Y_i!))\boldsymbol{g_i}\boldsymbol{x_i}' & -\nu_i[\nu_i Var(log(Y_i!) - E(log(Y_i!)) + log(y_i!))]\boldsymbol{g_i}\boldsymbol{g_i}' \end{pmatrix}$$

## 2 SSP for CMP

By denoting $a_i = [y_i - E(Y_i|\hat{\boldsymbol{\theta}}_{MLE})]$ and $b_i = \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_i!)]$, we define

$$\boldsymbol{V}_c = \frac{1}{rn^2}\sum_{i=1}^{n}\frac{1}{\pi_i}\begin{pmatrix} a_i^2\boldsymbol{x_i}\boldsymbol{x_i}' & a_i b_i \boldsymbol{x_i}\boldsymbol{g_i}' \\ a_i b_i \boldsymbol{g_i}\boldsymbol{x_i}' & b_i^2\boldsymbol{g_i}\boldsymbol{g_i}' \end{pmatrix}$$

where $r$ is the subsample size, with subsampling probabilities $\pi_i$ for all data points.

Further we denote the observed information matrix as

$$\boldsymbol{M}_X = \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix} A_i & B_i \\ B_i^{'} & C_i \end{pmatrix}$$

where

$$A_i = Var(Y_i|\hat{\boldsymbol{\theta}}_{MLE})\boldsymbol{x_i}\boldsymbol{x_i}'$$
$$B_i = -\nu_i(\hat{\boldsymbol{\theta}}_{MLE})Cov(Y_i, log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE})\boldsymbol{x_i}\boldsymbol{g_i}'$$
$$C_i = \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[\nu_i(\hat{\boldsymbol{\theta}}_{MLE})Var(log(Y_i!|\hat{\boldsymbol{\theta}}_{MLE}) - E(log(Y_i!|\hat{\boldsymbol{\theta}}_{MLE})) + log(y_i!))]\boldsymbol{g_i}\boldsymbol{g_i}'$$

Then follow the same steps as in OSMAC, we can show that as $n \to \infty$ and $r \to \infty$, conditional on full data matrix $\mathcal{F}_n = \boldsymbol{X}, \boldsymbol{y}$ in probability,

$$\boldsymbol{V}^{-1/2}(\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}_{MLE}) \to N(0, \boldsymbol{I})$$

where $\boldsymbol{V} = \boldsymbol{M}_X^{-1}\boldsymbol{V}_c\boldsymbol{M}_X^{-1}$ and $\tilde{\boldsymbol{\theta}}$ is the sub-sampling estimates for $\boldsymbol{\theta}$
Then under A-optimality criterion, the sub-sampling probabilities (SSPs) $\pi_i^{mMSE}$ are proportional to $||\boldsymbol{M}_X^{-1}\begin{pmatrix} a_i\boldsymbol{x}_i \\ b_i\boldsymbol{g}_i \end{pmatrix}||$, while under L-optimality criterion, $\pi_i^{mVc} \propto$

$|| \begin{pmatrix} a_i \boldsymbol{x}_i \\ b_i \boldsymbol{g}_i \end{pmatrix} ||$. In other words:

$$\pi_i^{mMSE} = \frac{||\boldsymbol{M}_X^{-1} \begin{pmatrix} [y_i - E(Y_i|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_i \\ \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_i!)]\boldsymbol{g}_i \end{pmatrix} ||}{\sum_{j=1}^n ||\boldsymbol{M}_X^{-1} \begin{pmatrix} [y_j - E(Y_j|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_j \\ \nu_j(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_j!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_j!)]\boldsymbol{g}_j \end{pmatrix} ||}$$

$$\pi_i^{mVc} = \frac{|| \begin{pmatrix} [y_i - E(Y_i|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_i \\ \nu_i(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_i!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_i!)]\boldsymbol{g}_i \end{pmatrix} ||}{\sum_{j=1}^n || \begin{pmatrix} [y_j - E(Y_j|\hat{\boldsymbol{\theta}}_{MLE})]\boldsymbol{x}_j \\ \nu_j(\hat{\boldsymbol{\theta}}_{MLE})[E(log(Y_j!)|\hat{\boldsymbol{\theta}}_{MLE}) - log(y_j!)]\boldsymbol{g}_j \end{pmatrix} ||}$$

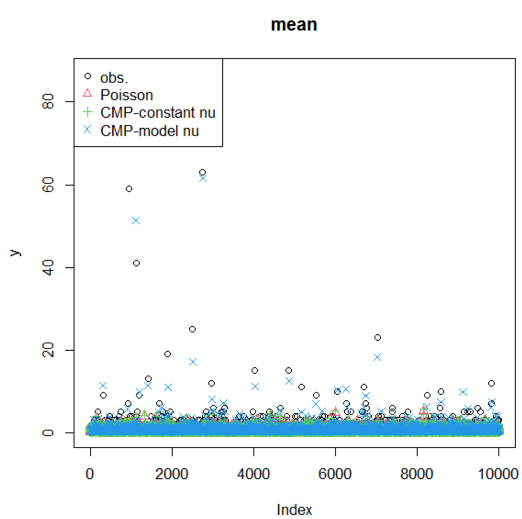
# 3 Necessity for fitting CMP model

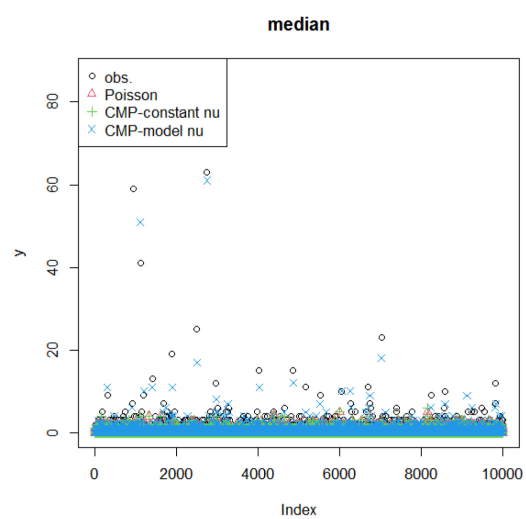
Here I first generate the data by CMP distribution.

There are $n = 10000$ independent observations, with $Y_i \sim CMP(\lambda_i, \nu_i)$ for $i = 1, 2, ..., n$. The $\lambda_i$ and $\nu_i$ are modeled as:

$$\lambda_i = \exp(x_{i1})$$
$$\nu_i = \exp(1 + g_{i1})$$

where $x_{i1} \overset{i.i.d.}{\sim} N(0, 1)$ and $g_{i1} \overset{i.i.d.}{\sim} N(1, 1)$ The following two plots show the fitted mean and median for 1) Poisson regression, 2) CMP regression, with constant $\nu_i = \nu$ (constant CMP) and 3) CMP with $\nu_i$ modeled by $g_{i1}$ (full CMP).




The MSEs ($\frac{(Y_i - \bar{Y})^2}{n}$) for mean are 1) Poisson: 2.269, 2) constant CMP: 2.273 and 3) full CMP: 0.391. The Poisson model and constant dispersion CMP model are not good for handling extreme observations. Therefore, it's necessary to take dispersion into account and CMP is a good choice.

The running time for these three: 1)Poisson: 0.03s, 2) constant CMP: 2.40s and 3) full CMP: 3.59s. This shows that CMP regression is much more computational expensive than regular Poisson regression, which suggests the necessity of sub-sampling.

# 4 Sub-sampling in CMP

To fit the CMP regression model, although we can use regular Newton-Raphson method to maximize the likelihood, the information matrix may not be stable. Therefore (as far as I know), there are two methods to deal with that: 1) use direct optimization strategy, such as L-BFGS-B and 2) optimize $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in an alternative way, i.e. hold one part fixed when fitting another. The alternating method reduces the problem into a two-step Newton-Raphson/ IRWLS problem.

Here I use the package implementing L-BFGS-B, and modify the objective likelihood function a bit to allow for optimization of weighted log-likelihood. To be more specific, I change:

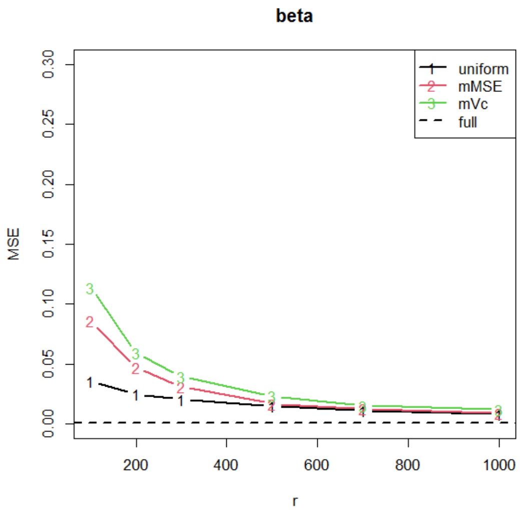
```
sum(y*log(lambda) - nu*lgamma(y+1) - logz)
```

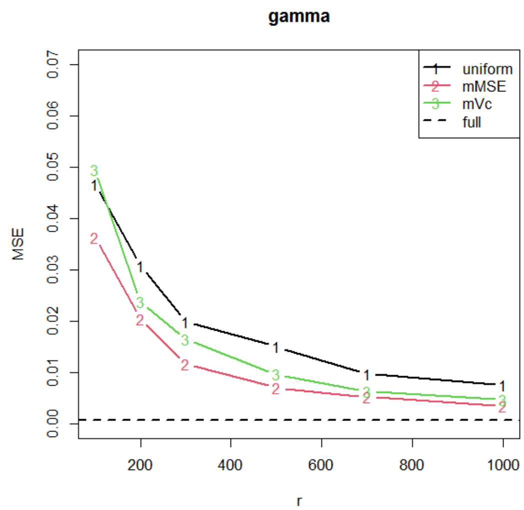
to

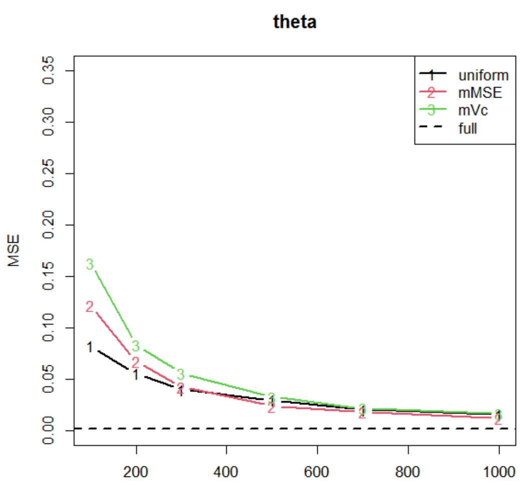
```
sum(weights*(y*log(lambda) - nu*lgamma(y+1) - logz))
```

The weights are normalized such that the summation of weights equals to sample size. Therefore, I can use the modified functions to maximizes the weighted likelihood.

Then I use the bootstrap ($B = 500$) to calculate the MSE for $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and the overall $\boldsymbol{\theta}$. Well...




It seems that the gradient and hessian (numerically evaluated) is not stable, and the variations for evaluating gradient and hessian are larger than the variation for uniform subsampling, especially for $\boldsymbol{\beta}$. ($\boldsymbol{\gamma}$ is always better than $\boldsymbol{\beta}$, in the limited simulations)