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Roots of quaternion polynomials: Theory and computation

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ABSTRACT

A quaternion polynomial $f(t)$ in the single variable t, is one whose coefficients are in the skew field $\mathbb{H}$ of quaternions. In this manuscript an elementary proof is given of the fact that such an f has a root in $\mathbb{H}$. Moreover, an algorithm is proposed for finding all roots ζ of $f(t)$, along with their multiplicities. The algorithm is based on computing the real part of ζ first, and then using the multiplication rule in $\mathbb{H}$, the imaginary part of ζ is computed via a linear quaternion equation. Several numerical examples are also presented to illustrate the performance of the method.

1. Introduction

In recent years there has been an increasing interest in using quaternions in areas ranging from number theory, robotics, computer aided design, virtual reality and image processing, among others. In that regard, well known problems for real or complex numbers have been considered in the quaternion skew field framework. One such problem is finding the roots of a quaternion polynomial $f(t)$. In [3] Eilenberg and Niven, using a topological degree argument, prove the fundamental theorem of algebra for quaternions, namely that any (general) quaternion polynomial $f(t)$ of positive degree has a root in $\mathbb{H}$.

Computing the roots of quaternion polynomials, however, is not an easy task mainly due to the non commutative structure of $\mathbb{H}$. Perhaps the earliest systematic studies of this problem can be found in the paper of Niven [11]. His method, however, is not very practical since his approach involves solving a system of two non linear equations of degree $2\deg(f)-1$. Currently, several quaternionic root-finding methods are available. De Leo et al. [2] discuss how the companion matrix of a quaternionic polynomial could be used to find the zeros of the polynomial. Serodio et al. [13] present a working method for finding the zeros of a quaternionic polynomial using the companion matrix of the polynomial, a matrix consisting of a column matrix of zeros for the first column, an identity matrix from the second column on and a row vector with its coefficients in the bottom row. Niven's formula in [11] is then used to compute the zeros using the trace and the norm of the eigenvalues of the companion matrix.

Janovska and Opfer [8] propose an algorithm for finding zeros of a quaternionic polynomial along with their types. They use the concept of a companion polynomial following Pogorui and Shapiro [12], whose real and complex roots are transformed into the roots of the quaternionic polynomial using a theorem by the authors. Jia et al. [9] present a method for computing the roots of a monic quaternionic quadratic polynomial using the concept of the real quadratic form. Kalantari [10] develops Newton's and Halley's methods for quaternionic polynomials to find their roots numerically, the methods of

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which are analyzed in detail. Farouki et al. [4] discuss a method for finding the roots of a quadratic quaternionic polynomial and apply this root finding method to the study of quintic curves with rational rotation-minimizing frames.

The purpose of this paper is twofold: First, we present an elementary proof of the fundamental theorem of algebra for quaternions, namely that any (left/right) polynomial $f(t)$ has a root (or a zero) in $\mathbb{H}$. Second, we give an algorithm for computing the roots ζ of $f(t)$. More precisely, we shall first find whether ζ is a real or a spherical zero. Then, we will extract this type of zeros, and thus f will be left only with isolated non real roots. Next, we compute the complex zeros. Finally, we will calculate the roots that belong to $\mathbb{H}-\mathbb{C}$. In addition, for each zero ζ we will compute its multiplicity $\mu(f)(\zeta)$. Several numerical examples are also presented to illustrate the performance and efficiency of the method.

This paper is organized as follows: Section 2 reviews some facts about quaternions and quaternion polynomials. In addition, Section 2 also presents a proof that a quaternion polynomial $f(t)$ has a root over $\mathbb{H}$. Section 3 contains the main algorithm for computing the roots of f, while in Section 4 we illustrate our algorithm by providing a series of examples.

2. Basic results on quaternions and polynomials

In this section we present a brief summary of results pertaining to quaternions and polynomials that are needed for the sequel.

2.1. The algebra of quaternions

Let $\mathbb{H}$ denote the skew field of quaternions. The elements of $\mathbb{H}$ are of the form $c=c_0+\mathbf{i}c_1+\mathbf{j}c_2+\mathbf{k}c_3$, where $c_m\in\mathbb{R}$ and $\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1$ and $\mathbf{ij}=-\mathbf{ji}=\mathbf{k}, \mathbf{jk}=-\mathbf{kj}=\mathbf{i}, \mathbf{ki}=-\mathbf{ik}=\mathbf{j}$. If $b=b_0+\mathbf{i}b_1+\mathbf{j}b_2+\mathbf{k}b_3$ is also in $\mathbb{H}$, these formulas give the multiplication rule

$$\begin{aligned}&cb=c_0b_0-c_1b_1-c_2b_2-c_3b_3+\mathbf{i}(c_0b_1+c_1b_0+c_2b_3-c_3b_2)+\\&\mathbf{j}(c_0b_2+c_2b_0+c_3b_1-c_1b_3)+\mathbf{k}(c_0b_3+c_3b_0+c_1b_2-c_2b_1).\end{aligned}$$

The real part of c is $Re(c)=c_0$ while the imaginary part is $Im(c)=\mathbf{i}c_1+\mathbf{j}c_2+\mathbf{k}c_3$. Note that $Re(cb)=Re(bc)$. The norm of c is $|c|=\sqrt{c_0^2+c_1^2+c_2^2+c_3^2}$, its conjugate $c^*=c_0-\mathbf{i}c_1-\mathbf{j}c_2-\mathbf{k}c_3$ while its inverse is $c^{-1}=c^*\cdot|c|^{-2}$, provided that $|c|\neq 0$. c is called an *imaginary unit* if $Re(c)=0$ and $|c|=1$, and $c^2=-1$. Moreover, two quaternions a,b shall be called *similar*, and we denote by $a\sim b$, if $a\eta=\eta b$ for a non zero $\eta\in\mathbb{H}$. Note that $a\sim b$ if and only if $Re(a)=Re(b)$ and $|Im(a)|=|Im(b)|$. Similarity is an equivalence relation in $\mathbb{H}$ and for each $a\in\mathbb{H}$ we denote its equivalence class by $[a]=\{b\in\mathbb{H}\,|\,b\sim a\}$. In that regard, $\mathbb{H}$ is a real normed division (non commutative) algebra.

We may identify $\mathbb{H}$ with $\mathbb{R}^4$ via the map $(x+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)\to(x,y,z,w)$. Let $f:\mathbb{H}\to\mathbb{H}$. In view of this identification, we can also think of f as a map from $\mathbb{R}^4\to\mathbb{R}^4$. Indeed, if $f(x+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)=f_1(x,y,z,w)+\mathbf{i}f_2(x,y,z,w)+\mathbf{j}f_3(x,y,z,w)+\mathbf{k}f_4(x,y,z,w)$ we define $f:\mathbb{R}^4\to\mathbb{R}^4$ by $f(x,y,z,w)=(f_1,f_2,f_3,f_4)$.

2.2. Quaternion polynomials

We will consider polynomial functions $f:\mathbb{H}\to\mathbb{H}$. Due to the non commutative nature of $\mathbb{H}$, polynomials over $\mathbb{H}$ are usually separated into the following three types: *left*, *right* and *general*, [7]. A left polynomial is an expression of the form

$$f(t)=a_nt^n+a_{n-1}t^{n-1}+\cdots+a_0,\ \text{where}\ a_k\in\mathbb{H} \tag{1}$$

while a right polynomial is $g(t)=\sum_{k=0}^m t^k b_k,\ b_k\in\mathbb{H}$. On the other hand, a general polynomial $h(t)$ can be written as $h(t)=a_0ta_1t\cdots ta_n+\phi(t), a_i\in\mathbb{H}, a_i\neq 0$ and $\phi(t)$ is a sum of finite number of similar monomials $b_0tb_1t\cdots tb_k, k<n$.

Here we shall consider, unless otherwise stated, left polynomials only and called them simply *polynomials*. If $g(t)=b_mt^m+b_{m-1}t^{m-1}+\cdots+b_0$ is another polynomial, the product $fg(t)$ is defined in the usual way:

$$fg(t)=\sum_{k=0}^{m+n}c_kt^k,\quad \text{where}\ c_k=\sum_{i=0}^{k}a_ib_{k-i}$$

Note that in this setting the multiplication is performed as if the coefficients were chosen in a commutative field.

An equivalent representation of f, which will be used frequently in the sequel, is the *quaternion* representation of $f(t)=a(t)+\mathbf{i}b(t)+\mathbf{j}c(t)+\mathbf{k}d(t)=a+\mathbf{i}b+\mathbf{j}(c-\mathbf{i}d)$, where $a,b,c,d\in\mathbb{R}[t]$. If $a_n\neq 0$, n is called the degree of f, while if $c\in\mathbb{H}$, we define $f(c)=a_nc^n+a_{n-1}c^{n-1}+\cdots+a_0$; if $f(c)=0$, c is called a *zero* or a *root* of f. Note that c is a root of f if and only if $f(t)=g(t)(t-c)$, (Theorem 1 of [7]). In [3], Eilenberg and Niven, using a topological degree argument, prove the fundamental theorem of algebra for quaternions, namely that any *general* quaternion polynomial $h(t)$ of positive degree n, has a root in $\mathbb{H}$. Recently, Topuridge [14] and Gentili et al. [6] using a similar argument prove the same result for a regular polynomial over the quaternions and octonions. Here, we will give a new, self contained and constructive *algebraic* proof of the same fact, based on the (usual) fundamental theorem of algebra.

Theorem 2.1 *(FTA for quaternions). Any (left or right) quaternion polynomial of positive degree has a root in* $\mathbb{H}$*.*

Proof. Let $f(t) = a(t) + \mathbf{i}b(t) + \mathbf{j}c(t) + \mathbf{k}d(t)$ be a left polynomial with $\deg(f) > 0$. The conjugate of f^* is defined as $f^* = a(t) - \mathbf{i}b(t) - \mathbf{j}c(t) - \mathbf{k}d(t)$. Note that $ff^* = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}[t]$. We will show that each complex root r of ff^* corresponds to a root ζ_r of f. The idea of the proof is to find a suitable non zero $\gamma = \gamma_1 + \mathbf{i}\gamma_2 + \mathbf{j}\gamma_3 + \mathbf{k}\gamma_4$ so that the polynomial $F(t) = f(t)\,\gamma$ would have r as a root. In that case, $F(t) = g(t)(t - r)$. This equation implies that $f(t) = g(t)\,\gamma^{-1}\,\gamma\,(t - r)\gamma^{-1} = G(t)(t - \zeta_r)$, where $G(t) = g(t)\,\gamma^{-1}$ and $\zeta_r = \gamma\, r \gamma^{-1}$ and thus ζ_r is a root of $f(t)$.

To proceed, let $a(r) = a_1 + \mathbf{i}a_2, b(r) = b_1 + \mathbf{i}b_2, c(r) = c_1 + \mathbf{i}c_2, d(r) = d_1 + \mathbf{i}d_2,\ a_i, b_i, c_i, d_i \in \mathbb{R}$ and let

$$A = a_1^2 - a_2^2 + b_1^2 - b_2^2 + c_1^2 - c_2^2 + d_1^2 - d_2^2, \quad B = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2. \tag{2}$$

Then $A = B = 0$, since r is a root of ff^*. Now we have $F(t) = f(t)\,\gamma = \gamma_1 a - \gamma_2 b - \gamma_3 c - \gamma_4 d + \mathbf{i}(\gamma_2 a + \gamma_1 b + \gamma_4 c - \gamma_3 d) + \mathbf{j}(\gamma_3 a + \gamma_1 c + \gamma_2 d - \gamma_4 b) + \mathbf{k}(\gamma_4 a + \gamma_1 d + \gamma_3 b - \gamma_2 c)$. Then, r is a root of F if and only if $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is a solution of the linear system Σ_r:

$$\begin{bmatrix} a_1 - b_2 & -a_2 - b_1 & d_2 - c_1 & -c_2 - d_1 \\ a_2 + b_1 & a_1 - b_2 & -d_1 - c_2 & c_1 - d_2 \\ c_1 + d_2 & d_1 - c_2 & a_1 + b_2 & -b_1 + a_2 \\ c_2 - d_1 & d_2 + c_1 & a_2 - b_1 & -b_2 - a_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let M_r be the coefficient matrix of the system. A calculation shows that $\det(M_r) = A^2 + 4B^2 = 0$ [from (2)] and thus Σ_r has a non zero solution. Obviously, the same proof works if f is a right polynomial. □

Roots of f are distinguished into two types: (i) *isolated* and (ii) *spherical*. A root c of f is called spherical if and only if its *characteristic* polynomial $q_c(t) = t^2 - 2t\,Re(c) + |c|^2$ divides f; for any such polynomial, call $\alpha_c \pm \mathbf{i}\beta_c$ its complex roots. In that case any quaternion $\gamma \in [c]$ is also a root of f. For example, if $f(t) = t^2 + 1$, any imaginary unit quaternion $c = \alpha_1\,\mathbf{i} + \alpha_2\,\mathbf{j} + \alpha_3\,\mathbf{k}$ is a root of f.

Remark 2.1. The polynomial $f(t)$ has a spherical root if and only if $f(t)$ has roots $\alpha + \mathbf{i}\beta, \alpha - \mathbf{i}\beta,\ \alpha, \beta \in \mathbb{R}, \beta \neq 0$.

Theorem 2.1, as well as the representation of f as in (1), allows us to factor f into a product of linear factors $(t - c_i)$, $c_i \in \mathbb{H}$. Indeed, if c_1 is a root of f, then $f(t) = f_1(t)(t - c_1)$ and since $f_1(t)$ has a root (from Theorem 2.1), simple induction shows that

$$f(t) = a_n(t - c_n)(t - c_{n-1}) \cdots (t - c_1),\ c_j \in \mathbb{H} \tag{3}$$

A word of caution: In this factorization, while c_1 is necessarily a root of f, c_j, $j = 2, \cdots n$, might not be roots of f. For example, the polynomial $f(t) = (t + \mathbf{k})(t + \mathbf{j})(t + \mathbf{i}) = t^3 + (\mathbf{i} + \mathbf{j} + \mathbf{k})t^2 + (-\mathbf{i} + \mathbf{j} - \mathbf{k})t + 1$, has only one root, namely $t = -\mathbf{i}$. Theorem 2.1 of [5] provides a more detailed version of the above factorization.

If we write f in the form $f(t) = a(t) + \mathbf{i}b(t) + \mathbf{j}c(t) + \mathbf{k}d(t)$ we see that f has no spherical roots if $\gcd(a, b, c, d) = 1$. Such an f will be called *primitive*. It is known (Corollary 3.3, [15]) that a primitive $f(t)$ of degree n, has at most n distinct roots in $\mathbb{H}$.

Let $f(t)$ be as in (1) and let (3) be one of its factorizations. Then the conjugate of f takes the form

$$f^*(t) = (t - c_1^*)(t - c_2^*) \cdots (t - c_n^*)a_n^* \tag{4}$$

We now have:

Proposition 2.1. *Let f be a* primitive *polynomial and let $\zeta_1, \zeta_2, \cdots, \zeta_k$ be its distinct roots. Let also $r_1, \cdots, r_m$ be the distinct and pairwise non conjugate complex roots of ff^*. Then, $k = m$ and*

$$[\zeta_1] \cup [\zeta_2] \cup \cdots \cup [\zeta_k] = [r_1] \cup [r_2] \cup \cdots \cup [r_m] \tag{5}$$

Proof. Let ζ be a root of f. Then, from (4) we get that $(t - \zeta)(t - \zeta^*)$ is a factor of ff^*. But the roots of $(t - \zeta)(t - \zeta^*)$ are $Re(\zeta) \pm \mathbf{i}\,|Im(\zeta)|$ which in turn are similar to ζ. Conversely, let r be a root of ff^*. Then, from the proof of Theorem 2.1 and Theorem 4, page 221 of [7] there exists a *unique* root $\zeta_r \in \mathbb{H}$ of f similar to r. This shows that $k = m$ and the proof is now complete. □

Definition 2.1. Let $\phi(t) \in \mathbb{C}[t]$ and $r \in \mathbb{C}$ be a root of ϕ. We denote by $\mu(\phi)(r)$ the multiplicity of r. Now let $c \in \mathbb{H}$ be a root of f and let $m = \mu(ff^*)(\alpha_c + \mathbf{i}\beta_c)$. Then, (1) if c is isolated, we define its multiplicity $\mu(f)(c)$, as a root of f, to be m; (2) if c is spherical, its multiplicity is set to be $2m$.

Note that the notion of multiplicity in Definition 2.1 agrees with the one given in Definition 2.6 of [5], page 23. From Definition 2.1 we have:

Criterion 1. *Let c be a root of the* primitive *polynomial f. Then c has multiplicity k if and only if* $\gcd(q_c^k, ff^*) = q_c^k$. *In particular, $f(t)$ has simple roots–i.e. roots of multiplicity one–if* $\gcd(a^2+b^2+c^2+d^2, aa'+bb'+cc'+dd') = 1$.

3. The main algorithm

In this section we present the main algorithm for the computation of zeros of $f(t)$. This algorithm is based primarily on Proposition 2.1 and the fact that if c is an *imaginary unit*, $c^2 = -1$. To begin let $f(t)$ be written as $f(t) = a(t) + \mathbf{i}b(t) + \mathbf{j}(c(t) - \mathbf{i}d(t))$ and let $g(t) = \gcd(a, b, c, d)$. Then, the *real* and *spherical* zeros of f are precisely the real and complex roots of g. Thus, dividing f by g we can assume from now on that f is primitive, and thus f has a finite number of isolated zeros in $\mathbb{H} - \mathbb{R}$.

Next, we extract the complex roots of f. Let then $\zeta = \alpha + \mathbf{i}\beta$ be a zero of f. Then, $f = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d = (a_1 + \mathbf{i}b_1 + \mathbf{j}c_1 + \mathbf{k}d_1)(t - \alpha - \mathbf{i}\beta)$. Thus, $a = (t-\alpha)a_1 + \beta b_1, b = (t-\alpha)b_1 - \beta a_1, c = (t-\alpha)c_1 - \beta d_1$ and $d = (t-\alpha)d_1 + \beta c_1$. These equations imply $(a + \mathbf{i}b)(\zeta) = (c - \mathbf{i}d)(\zeta) = 0$, so we get that the complex roots of f are precisely the complex roots of $\phi(t) = \gcd(a + \mathbf{i}b, c - \mathbf{i}d)$.

Now, we move on computing the roots of f that are pure quaternions by considering the real polynomial ff^*. Using a complex root finding algorithm, we find the distinct roots $r_1, \cdots, r_k$ of ff^*. Let $r = \alpha \pm \mathbf{i}\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta > 0$ be one of these roots. Proposition 2.1 shows that there exists a unique root ζ_r of f that is similar to r. Obviously $\zeta_r = \zeta_{r^*}$. So $Re(\zeta_r) = \alpha$ and $|Im(\zeta_r)| = \beta$. Write $Im(\zeta_r) = \beta(\mathbf{i}x_1 + \mathbf{j}y_1 + \mathbf{k}z_1) = \beta c$. Obviously, c is an imaginary unit, so $c^2 = -1$. Introduce a new variable $s = (\alpha + \beta t)$ and consider the polynomial $h(t) = f(\alpha + \beta t)$. Then,

$$f(t)|_{t=\zeta_r} = 0 \quad \text{if and only if} \quad h(c) = 0 \tag{6}$$

Let $h(t) = b_n t^n + b_{n-1}t^{n-1} + \cdots + b_0$. Then

$$h(c) = b_0 + b_1 c - b_2 - b_3 c + \cdots + b_n \gamma \tag{7}$$

where $\gamma = (-1)^k, (-1)^k c$ if $n = 2k, 2k+1$. If we set $\mathcal{A} = b_1 - b_3 + \cdots$, $\mathcal{B} = -b_0 + b_2 + \cdots$, then upon setting $h(c) = 0$, Equation (7) gives

$$\mathcal{A}\, c = \mathcal{B} \tag{8}$$

Thus, $\zeta_r = \alpha + \beta \mathcal{A}^{-1}\mathcal{B}$ is the root of $f(t)$ that comes from the root r of ff^*.

Thus we see that in the case where r is computed exactly, this procedure gives the exact root ζ_r of f. On the other hand, if $r_1 = \alpha_1 + \mathbf{i}\beta_1$ is an approximation of the actual root r of ff^*, this method would produce $\zeta_{r_1} = \alpha_1 + \beta_1 c_1 = \alpha_1 + \mathbf{i}x_0 + \mathbf{j}y_0 + \mathbf{k}z_0$, where $c_1 = {\mathcal{A}_1}^{-1}\mathcal{B}_1$, with $\mathcal{A}_1, \mathcal{B}_1$ defined in an analogous way as $\mathcal{A}$ and $\mathcal{B}$. To find a better approximation of ζ_r, we use Newton's method [1] on the system $f_1 = f_2 = f_3 = f_4 = 0$ with initial point (value) $(\alpha_1, x_0, y_0, z_0)$.

A pseudo code for the algorithm is given as follows:

Input : $f(t) = 0$, quaternionic polynomial equation
Output: all zeros of $f(t) = 0$
Set $f(t) = a(t) + \mathbf{i}b(t) + \mathbf{j}(c(t) - \mathbf{i}d(t))$;
Compute $g(t) = GCD(a(t), b(t), c(t), d(t))$;
Compute the real and spherical zeros of $f(t)$ *by finding the real and complex roots of* $g(t)$;
Compute $f = f/g$;
Find the distinct roots $r_1, r_2, \cdots, r_k$ *of* $ff^* = 0$;
for $i \leftarrow 1$ **to** k **do**
  Set $\alpha = Re(r_i)$ *and* $\beta = |Im(r_i)|$;
  Set $h(c) = f(\alpha + \beta c)$;
  Find c that satisfies $h(c) = 0$ *using Eq. (8) to produce* $\zeta_{r_i} = \alpha + \beta c$;
  Set $f(\alpha + \beta c) = f_1(x, y, z) + \mathbf{i}f_2(x, y, z) + \mathbf{j}f_3(x, y, z) + \mathbf{k}f_4(x, y, z)$ *with* $c = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$;
  Use Newton's method on the system $f_1 = f_2 = f_3 = f_4 = 0$ *to find a better approximation of the zero with the initial value* z_{r_i};
end

Algorithm 1: Computation of the zeros of a quaternion polynomial.

4. Examples

Example 1. Consider the quaternion polynomial in [13].

$$f(t) = t^6 + (\mathbf{i} + 3\mathbf{k})t^5 + (3 + \mathbf{j})t^4 + (5\mathbf{i} + 15\mathbf{k})t^3 + (-4 + 5\mathbf{j})t^2 + (6\mathbf{i} + 18\mathbf{k})t - 12 + 6\mathbf{j}.$$

Table 1
Roots of Example 2.

roots of $ff^*(t)$	roots of $f(t)$
$-1.26112 \pm 4.56864\mathbf{i}$	$-1.26112 - 1.92547\mathbf{i} + 4.10532\mathbf{j} - 0.557994\mathbf{k}$
$0.930191 \pm 0.73780\mathbf{i}$	$0.930191 + 0.278693\mathbf{i} - 511865\mathbf{j} - 0.452428\mathbf{k}$
$-1.07301 \pm 0.49536\mathbf{i}$	$-1.07301 + 0.464099\mathbf{i} - 0.092359\mathbf{j} - 0.146527\mathbf{k}$
$1.14205 \pm 0.12193\mathbf{i}$	$1.14205 + 0.0805848\mathbf{i} + 0.0778339\mathbf{j} - 0.0480975\mathbf{k}$
$-0.65287 \pm 0.92714\mathbf{i}$	$-0.65287 - 0.01858\mathbf{i} + 0.883947\mathbf{j} - 0.279083\mathbf{k}$
$0.21713 \pm 1.06495\mathbf{i}$	$0.21713 - 0.245867\mathbf{i} + 1.02274\mathbf{j} + 0.166336\mathbf{k}$
$-0.38874 \pm 0.96394\mathbf{i}$	$-0.38874 + 0.0886184\mathbf{i} - 0.465936\mathbf{j} - 0.839191\mathbf{k}$
$0.28474 \pm 0.81268\mathbf{i}$	$0.28474 - 0.455772\mathbf{i} - 0.33153\mathbf{j} + 0.585472\mathbf{k}$
$0.60157 \pm 0.57023\mathbf{i}$	$0.60157 + 0.212445\mathbf{i} + 0.442665\mathbf{j} - 0.289963\mathbf{k}$
$-0.79994 \pm 0.18175\mathbf{i}$	$-0.79994 - 0.03740\mathbf{i} + 0.16540\mathbf{j} - 0.06495\mathbf{k}$

First, we compute the polynomials $a(t)$, $b(t)$, $c(t)$ and $d(t)$ of $f(t) = a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}$. Then $\gcd(a, b, c, d) = g(t) = t^4 + 5t^2 + 6$. By dividing f with g, $f(t)$ becomes $(t^2 - 2) + \mathbf{i}t + \mathbf{j} + 3\mathbf{k}t$. The roots of $ff^* = t^4 + 6t^2 + 5$ are $\pm\mathbf{i}$, $\pm\mathbf{i}\sqrt{5}$.

For the root $\pm\mathbf{i}$, $h(t) = f(t)$. Let $c = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ be a unit quaternion and set $h(c) = 0$. Then, from (8) we get $(\mathbf{i} + 3\mathbf{k})c = 3 - \mathbf{j}$ and thus $c = -0.6\mathbf{i} - 0.8\mathbf{k}$. Similarly, for $\pm\mathbf{i}\sqrt{5}$, we get the root $-\mathbf{i} - 2\mathbf{k}$.

Finally, since the roots of $g(t) = t^4 + 5t^2 + 6$ are $\pm\sqrt{2}\mathbf{i}$ and $\pm\sqrt{3}\mathbf{i}$, the roots of $f(t)$ are

$$\zeta_1 = -0.6\mathbf{i} - 0.8\mathbf{k},\ \zeta_2 = -\mathbf{i} - 2\mathbf{k},\ \zeta_3 = [\pm\sqrt{2}\mathbf{i}],\ \zeta_4 = [\pm\sqrt{3}\mathbf{i}],$$

which are the same as those in [13]. Note that ζ_1, ζ_2 are isolated while ζ_3, ζ_4 are spherical roots, and $\mu(f)(\zeta_1) = \mu(f)(\zeta_2) = 1$, $\mu(f)(\zeta_3) = \mu(f)(\zeta_4) = 2$.

Example 2. Consider the polynomial

$$\begin{aligned} f(t) &= t^{10} + (1 + 2\mathbf{i} - 4\mathbf{j})t^9 - (3.1\mathbf{i} + \mathbf{k})t^8 + (2.5\mathbf{j} + 2.1\mathbf{k})t^7 \\ &\quad + (3 - \mathbf{i})t^6 - 1.7t^5 - (\mathbf{i} + \mathbf{j})t^4 - 7.2t^3 - \mathbf{j}t + 2.9(\mathbf{j} - \mathbf{k}) - 4. \end{aligned}$$

When $f(t)$ is rewritten in the form $f(t) = a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}$, the GCD of $a(t)$, $b(t)$, $c(t)$ and $d(t)$ is 1. Therefore, $f(t)$ is a primitive polynomial. The roots of ff^* are computed using Mathematica and are shown in the first column of Table 1.

First, take the root $0.930101 \pm 0.73780\mathbf{i}$. We define $h(t) = f(0.930191 + 0.73780t)$.

Next, compute $h(c) = 0$, where $c = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. From (8) we get $(-15.4225 + 8.70632\mathbf{i} - 5.32619\mathbf{j} - 3.84412\mathbf{k})c + (9.34124 + 5.22197\mathbf{i} - 14.5781\mathbf{j} - 5.42478\mathbf{k}) = 0$ and thus $c = 0.377706\mathbf{i} - 0.693516\mathbf{j} - 0.613238\mathbf{k}$. Then, we have $\alpha + \beta c$ = $0.930191 + 0.73780c$. This value is given as the initial value for solving $f_1 = f_2 = f_3 = f_4 = 0$ using Newton's method, and a root $0.930191 + 0.278693\mathbf{i} - 0.511865\mathbf{j} - 0.452428\mathbf{k}$ is calculated, which is almost the same as that in [13]. Using the same process, we get the remaining 9 roots of $f(t)$ listed in Table 1. In addition, the multiplicity of each root of f is equal to 1.

Example 3. Consider the following polynomial of degree 12.

$$\begin{aligned} f(t) &= t^{12} + (\mathbf{i} + \mathbf{j})t^{11} + (10.4 + \mathbf{i} + 3.5\mathbf{j} + \mathbf{k})t^{10} + t^9 \\ &\quad + (5 + 3\mathbf{i} + \mathbf{k})t^8 + \mathbf{i}t^7 + \mathbf{j}t^6 + t^4 + (3 + \mathbf{j} + \mathbf{k})t^3 \\ &\quad + t^2 + (\mathbf{i} + \mathbf{j})t + 0.4 + 5\mathbf{i} + 4\mathbf{j} + 0.8\mathbf{k}. \end{aligned}$$

When $f(t)$ is rewritten in the form $f(t) = a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}$, the GCD of $a(t)$, $b(t)$, $c(t)$ and $d(t)$ is 1. Therefore, $f(t)$ is primitive. Next, we compute the roots of ff^*. Then we have 24 complex roots as follows:

$$-0.881354 \pm 0.16309\mathbf{i},\ -0.79032 \pm 0.377873\mathbf{i},\ -0.64289 \pm 0.755039\mathbf{i},$$

$$-0.486075 \pm 0.85199\mathbf{i},\ -0.47028 \pm 2.57154\mathbf{i},\ -0.123811 \pm 1.00366\mathbf{i},$$

$$0.143886 \pm 0.971923\mathbf{i},\ 0.366947 \pm 0.935865\mathbf{i},\ 0.549355 \pm 3.96952\mathbf{i},$$

$$0.618292 \pm 0.685211\mathbf{i},\ 0.857708 \pm 0.427965\mathbf{i},\ 0.865443 \pm 0.250769\mathbf{i}.$$

For the first root $-0.881354 + 0.16309\mathbf{i}$, we compute $h(t) = f(-0.881354 + 0.16309t)$. Substituting t with a unit quaternion $c = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ in $h(t)$ we get the equation $h(c) = 0$, from which we find $(-4.87285 - 0.686484\mathbf{i} - 0.783928\mathbf{j} - 0.36198\mathbf{k})c + (-1.01034 + 4.16181\mathbf{i} + 2.56889\mathbf{j} + 0.144493\mathbf{k}) = 0$.

Thus $c = 0.87294\mathbf{i} + 0.476837\mathbf{j} + 0.102914\mathbf{k}$. The corresponding quaternion root is $-0.881354 + 0.142369\mathbf{i} + 0.0777673\mathbf{j} + 0.0167842\mathbf{k}$. This root is taken as the initial value for solving $f_1 = f_2 = f_3 = f_4 = 0$ using Newton's method, which computes $-0.881354 + 0.142369\mathbf{i} + 0.0777673\mathbf{j} + 0.0167842\mathbf{k}$. For verification, we evaluate $f(t)$ with the computed root, and obtain $f(-0.881354 + 0.142369\mathbf{i} + 0.0777673\mathbf{j} + 0.0167842\mathbf{k}) = 0$.

Using the same process, we get the remaining 11 roots of $f(t)$ as follows:

$$-0.79032 - 0.325395\mathbf{i} - 0.157088\mathbf{j} - 0.110585\mathbf{k}, -0.64289 + 0.673752\mathbf{i} + 0.317314\mathbf{j} + 0.124318\mathbf{k},$$

$$-0.486075 - 1.43923\mathbf{i} - 0.741123\mathbf{j} - 0.394858\mathbf{k}, -0.47028 + 1.42705\mathbf{i} + 1.69981\mathbf{j} - 1.29884\mathbf{k},$$

$$-0.123811 + 0.719797\mathbf{i} + 0.173398\mathbf{j} + 0.677607\mathbf{k}, 0.143886 - 0.280941\mathbf{i} - 0.182025\mathbf{j} - 0.912455\mathbf{k},$$

$$0.366947 + 0.481672\mathbf{i} + 0.790495\mathbf{j} + 0.105746\mathbf{k}, 0.549335 - 1.89111\mathbf{i} - 3.44312\mathbf{j} + 0.570716\mathbf{k},$$

$$0.618292 - 0.627908\mathbf{i} - 0.220284\mathbf{j} - 0.163464\mathbf{k}, 0.857708 + 0.285561\mathbf{i} + 0.317722\mathbf{j} - 0.0257371\mathbf{k},$$

$$0.865443 - 0.198484\mathbf{i} - 0.147069\mathbf{j} - 0.0431311\mathbf{k}.$$

The number of iterations required for Newton's method to converge ranges from 3 to 25. Here, the termination tolerance is 10^{-7}. Specifically, the root $0.549355 - 1.89111\mathbf{i} - 3.44312\mathbf{j} - 0.570716\mathbf{k}$ is found after 25 iterations. Finally, the multiplicity of each root of f is equal to 1.

5. Conclusion

In this paper roots of a (left) quaternion polynomial $f(t)$ are investigated. Using the power as well as the quaternion representation of $f(t)$, a new, constructive and elementary proof of the FTA for quaternions is derived. Then, an algorithm is given to compute the roots of f along with their multiplicities. Note that the algorithm is especially suitable for symbolic root finding methods. Several numerical examples are also presented to illustrate the performance of the method. Finally, we believe that the same approach can be considered for polynomials over the octonion ring $\mathbb{O}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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