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Abstract

The polynomials with quaternion coefficients have two kind of roots: isolated and spherical. A spherical root generates a class of roots which contains only one complex number z and its conjugate $\bar{z}$, and this class can be determined by z. In this paper, we deal with the complex roots of quaternion polynomials. More precisely, using Bézout matrices, we give necessary and sufficient conditions, for a quaternion polynomial to have a complex root, a spherical root, and a complex isolated root. Moreover, we compute a bound for the size of the roots of a quaternion polynomial.

Keywords: Quaternion polynomial; Bézout Matrices; Spherical Root; Isolated Root.
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1 Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, respectively. We denote by $\mathbb{H}$ the skew field of real quaternions. Its elements are of the form $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, where $x_1, x_2, x_3, x_4 \in \mathbb{R}$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The *conjugate* of q is defined as $\bar{q} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. The *real* and the *imaginary part* of q are $\mathrm{Re}q = x_0$ and $\mathrm{Im}q = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, respectively. The *norm* $|q|$ of q is defined to be the quantity

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Two quaternions q and q' are said to be *congruent* or *equivalent,* written $q \sim q'$, if for some quaternion $w \neq 0$ we have $q' = wqw^{-1}$. By [27], we have $q \sim q'$ if and only if $\mathrm{Re}q = \mathrm{Re}q'$ and $|q| = |q'|$. The *congruence class* of q is the set

$$[q] = \{q' \in \mathbb{H}/ \ q' \sim q\} = \{q' \in \mathbb{H}/ \ \mathrm{Re}q = \mathrm{Re}q', \ |q| = |q'|\}.$$

Note that every class $[q]$ contains exactly one complex number z and its conjugate $\bar{z}$, which are $x_0 \pm \mathbf{i}\sqrt{x_1^2 + x_2^2 + x_3^2}$.

Let $\mathbb{H}[t]$ be the polynomial ring in the variable t over $\mathbb{H}$. Every polynomial $f(t) \in \mathbb{H}[t]$ is written as $a_0 t^n + a_1 t^{n-l} + \cdots + a_n$ where n is an integer ≥ 0 and $a_0, \ldots, a_n \in \mathbb{H}$ with $a_0 \neq 0$. The addition and the multiplication of polynomials are defined in the same way as the commutative case, where the variable t is assumed to commute with quaternion coefficients [17, Chapter 5, Section 16]. For every $q \in \mathbb{H}$ we define the evaluation of $f(t)$ at q to be the element

$$f(q) = a_0 q^n + a_1 q^{n-l} + \cdots + a_n.$$

Note that the evaluation at q is not in general a ring homomorphism from $\mathbb{H}[t]$ to $\mathbb{H}$.

We say that a quaternion q is a zero or a root of $f(t)$ if $f(q) = 0$. In 1941, Niven proved that the "Fundamental Theorem of Algebra" holds for quaternion polynomials [19], and in 1944 Eilenberg and I. Niven proved this result for general quaternion polynomials [4]. Recent proofs can be found in [9] and [16, Theorem 5.1]. The roots of a quaternion polynomial $f(t)$ and its expression as a product of linear factors have been investigated in several papers [7, 8, 10, 13, 16, 21, 22, 23, 24, 26].

A root q of $f(t)$ is called *spherical* if $q \notin \mathbb{R}$ and for every $r \in [q]$ we have $f(r) = 0$. Otherwise, it is called *isolated.* If two elements of a class are zeros of $f(t)$, then [10, Theorem 4] implies that all elements of this class are zeros of $f(t)$ and so spherical roots of $f(t)$. Since every class contains exactly one complex number z and its conjugate $\bar{z}$, the pairs of complex numbers $\{z, \bar{z}\}$ which are roots of $f(t)$ determine all the spherical roots of $f(t)$.

In this paper we study the complex roots of quaternion polynomials using Bezout matrices. First, we determine the degree of the highest degree complex right factor of a quaternion polynomial, and we give a necessary and sufficient condition for a quaternion polynomial to have a complex root. Next, we give necessary and sufficient conditions for a quaternion polynomial to have a spherical root, and also to have a complex isolated root. Finally, we give bounds for the size of a root of a quaternion polynomial which are sharper than the bound give in [20, Theorem 4.2] in case where the roots of polynomial are quite large.

2 Bézout Matrices

In 1971, Barnett computed the degree (resp. coefficients) of the greatest common divisor of several univariate polynomials with coefficients in an integral domain by means of the rank (resp. linear dependencies of the columns) of several matrices involving theirs coefficients [1, 2]. In this section we recall a formulation of Barnett's results using Bézout matrices [3] which we shall use for the presentation of our results. We could equally use another formulation of Barnett's results given in [3] or to use another approach [6, 15, 25], but we have chosen the formulation with Bézout matrices as more simple and quite efficient in computations.

Let F be a field of characteristic zero and $P(x)$, $Q(x)$ polynomials in $F[x]$ with $d = \max\{\deg P, \deg Q\}$. Consider the polynomial

$$\frac{P(x)Q(y) - P(y)Q(x)}{x - y} = \sum_{i,j=0}^{d-1} c_{i,j} x^i y^j.$$

The *Bézout matrix* associated to $P(x)$ and $Q(x)$ is:

$$\operatorname{Bez}(P, Q) = \begin{pmatrix} c_{0,0} & \cdots & c_{0,d-1} \\ \vdots & \vdots & \vdots \\ c_{d-1,0} & \cdots & c_{d-1,d-1} \end{pmatrix}.$$

The *Bezoutian* associated to $P(x)$ and $Q(x)$ is defined as the determinant of the matrix $\operatorname{Bez}(P, Q)$ and it will be denoted by $\operatorname{bez}(P, Q)$. Let $n = \deg P$, $m = \deg Q$ and p_0 the leading coefficient of $P(x)$. If $n \geq m$, then

$$\operatorname{bez}(P, Q) = (-1)^{n(n-1)/2} p_0^{n-m} R(P, Q),$$

where $R(P, Q)$ is the well known Sylvester resultant of $P(x)$ and $Q(x)$ [2, 11]. Furthermore, we have $\operatorname{bez}(P, Q) = 0$ if and only if $\deg(\gcd(P, Q)) \geq 1$.

Now, let $P(x)$, $Q_1(x), \ldots, Q_k(x)$ be a family of polynomials in $F[x]$ with $n = \deg P$ and $\deg Q_j \leq n - 1$ for every $j \in \{1, \ldots, k\}$. Set

$$\mathcal{B}_P(Q_1, \ldots, Q_k) = \begin{pmatrix} \operatorname{Bez}(P, Q_1) \\ \vdots \\ \operatorname{Bez}(P, Q_k) \end{pmatrix}.$$

We have the following formulation of Barnett's theorem.

Lemma 1 *The degree of the greatest common divisor of polynomials $P(x)$, $Q_1(x), \ldots, Q_k(x)$ verifies the following formula:*

$$\deg(\gcd(P, Q_1, \ldots, Q_k)) = n - \operatorname{rank} \mathcal{B}_P(Q_1, \ldots, Q_k).$$

Proof. See [3, Theorem 3.2].

Moreover, the matrix $\mathcal{B}_P(Q_1, \ldots, Q_t)$ can provide the greatest common divisor of $P(x)$, $Q_1(x), \ldots, Q_k(x)$ [3, Theorem 3.4].

3 Complex Roots

In this section we give a necessary and sufficient condition for a quaternion polynomial to have a complex root and in this case we determine the solutions of a quadratic equation.

Let $Q(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ be a monic polynomial with $\deg Q = n \geq 1$. Then there are $f(t), g(t) \in \mathbb{C}[t]$ with $f(t)g(t) \neq 0$ such that

$$Q(t) = f(t) + \mathbf{k}g(t).$$

We write

$$f(t) = f_1(t) + f_2(t)\mathbf{i} \quad \text{and} \quad g(t) = g_1(t) + g_2(t)\mathbf{i},$$

where $f_1(t), f_2(t), g_1(t), g_2(t) \in \mathbb{R}[t]$. Since $Q(t)$ is monic of degree n, we have $\deg f_1 = \deg f = n$ and $\deg f_2$, $\deg g_1$, $\deg g_2$ are smaller than n.

Set

$$D(t) = \gcd(f_1(t), f_2(t), g_1(t), g_2(t)) \quad \text{and} \quad E(t) = \gcd(f(t), g(t)).$$

The polynomial $D(t)$ divides $f_1(t)$ and $f_2(t)$, whence we get that $D(t)$ divides $f(t)$. Similarly, we deduce that $D(t)$ divides $g(t)$. It follows that $D(t)$ divides $E(t)$.

The polynomial $B(t) \in \mathbb{H}[t]$ is called a *right factor* of $Q(t)$ if there exists $C(t) \in \mathbb{H}[t]$ such that $Q(t) = C(t)B(t)$. Note that q is a root of $Q(t)$ if and only if $t - q$ is a right factor of $Q(t)$, i.e. there exists $g(t) \in \mathbb{H}[t]$ such that $Q(t) = g(t)(t-q)$ [17, Proposition 16.2]. We shall determine the monic right factors of $Q(t)$ in $\mathbb{C}[t]$ having the highest degree.

Theorem 1 *The only monic right factor of $Q(t)$ in $\mathbb{C}[t]$ having the highest degree is $E(t)$ and its degree is $n - \mathrm{rankBez}(f, g)$. Furthermore, if $z \in \mathbb{R}$, then z is a root of $Q(t)$ if and only if it is a root of $D(t)$.*

Proof. Let $G(t)$ be a right factor of $Q(t)$ in $\mathbb{C}[t]$ having the highest degree. Then there are $a(t), b(t) \in \mathbb{C}[t]$ such that $Q(t) = (a(t) + \mathbf{k}b(t))G(t)$. Then $f(t) = a(t)G(t)$ and $g(t) = b(t)G(t)$. It follows that $G(t)$ divides $E(t)$. On the other hand, there are $f_1(t), g_1(t) \in \mathbb{C}[t]$ such that $f(t) = f_1(t)E(t)$ and $g(t) = g_1(t)E(t)$. Then $Q(t) = (f_1(t) + \mathbf{k}g_1(t))E(t)$. Since $G(t)$ divides $E(t)$, $G(t)$ and $E(t)$ are monic and $G(t)$ is a highest degree right factor of $Q(t)$ in $\mathbb{C}[t]$, we deduce that $G(t) = E(t)$. By Lemma 1, we have $\deg E = n - \mathrm{rankBez}(f, g)$.

Suppose that $z \in \mathbb{R}$. Then $Q(z) = 0$ if and only if $f(z) = g(z) = 0$. Since $f_1(z), f_2(z), g_1(z), g_2(z) \in \mathbb{R}$, we obtain that $f(z) = g(z) = 0$ is equivalent to $f_1(z) = f_2(z) = g_1(z) = g_2(z) = 0$, and so to $D(z) = 0$.

Corollary 1 *The polynomial $\mathcal{Q}(t)$ has a complex root if and only if we have $R(f, g) = 0$ or equivalently $\mathrm{bez}(f, g) = 0$.*

Proof. By Theorem 1, $Q(t)$ has a complex root if and only $\deg E > 0$. Further, we have that $\deg E > 0$ if and only $R(f,g) = 0$ which is equivalent to $\mathrm{bez}(f,g) = 0$.

Corollary 2 *The polynomial $Q(t)$ has at most $n - \mathrm{rankBez}(f,g)$ complex roots.*

The case of a quadratic quaternion equation has been studied in [12, 14, 19, 5]. The next corollary provides theirs solutions in the special case where one of them is complex.

Corollary 3 *Let $Q(t) = t^2 + q_1 t + q_0$ be a quadratic polynomial of $\mathbb{H}[t] \setminus \mathbb{C}[t]$ with no real factor. Set $q_1 = b_1 + \mathbf{k}c_1$ and $q_0 = b_0 + \mathbf{k}c_0$, where $b_0, b_1, c_0, c_1 \in \mathbb{C}$. Then $Q(t)$ has one complex root if and only if*

$$c_0^2 - c_0 b_1 c_1 + b_0 c_1^2 = 0.$$

In this case $c_0 c_1 \neq 0$, and the roots of $Q(t)$ are

$$q = -\frac{c_0}{c_1}, \qquad \sigma = (q - \bar{p})^{-1} p (q - \bar{p}),$$

where $p = -(b_0 c_1 / c_0 + \mathbf{k}c_1)$.

Proof. Let $f(t) = t^2 + b_1 t + b_0$ and $g(t) = c_1 t + c_0$. We have

$$R(f,g) = c_0^2 - c_0 b_1 c_1 + b_0 c_1^2$$

and by Corollary 1, $Q(t)$ has a complex root if and only if the above quantity is zero.

Suppose now that $Q(t)$ has a complex root q. If $c_1 = 0$, then the equality $R(f,g) = 0$ implies $c_0 = 0$ and hence $Q(t) \in \mathbb{C}[t]$ which is a contradiction. Thus $c_1 \neq 0$. If $c_0 = 0$, then we deduce $b_0 = 0$, and so t is a factor of $Q(t)$ which is a contradiction. Therefore $c_0 \neq 0$.

By Theorem 1, we have $g(q) = 0$ and $f(q) = 0$. It follows that $q = -c_0/c_1$ and $f(t) = (t - b_0/q)(t - q)$. Thus, we have the factorization

$$Q(t) = (t - p)(t - q),$$

where $p = -(b_0 c_1 / c_0 + \mathbf{k}c_1)$. If $p = \bar{q}$, then we have $b_0 c_1 / c_0 + \mathbf{k}c_1 = \bar{c}_0 / \bar{c}_1$. It follows that $c_1 = 0$ which is a contradiction. Thus, [23, Lemma 1] yields

$$Q(t) = (t - (p - \bar{q}) q (p - \bar{q})^{-1})(t - (q - \bar{p})^{-1} p (q - \bar{p})).$$

Hence, the other root of $Q(t)$ is $\sigma = (q - \bar{p})^{-1} p (q - \bar{p})$.

4 Spherical and Complex Isolated Roots

In this section, we give necessary and sufficient conditions for a quaternion polynomial to have a spherical root and to have a complex isolated root. First, we consider the spherical roots.

Theorem 2 *Let $z \in \mathbb{C} \setminus \mathbb{R}$. The following are equivalent:*
(a) The number z is a spherical root of $Q(t)$.
(b) The number z and its conjugate $\bar{z}$ are common roots of $f(t)$ and $g(t)$.
(c) The number z is a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$.

Proof. If z is a spherical root of $Q(t)$, then its conjugate $\bar{z}$ is also a root of $Q(t)$. Thus Theorem 1 implies that z and $\bar{z}$ are common roots of $f(t)$ and $g(t)$. If this holds, then the polynomial $(t-z)(t-\bar{z})$ is a factor of $f(t)$ and $g(t)$. It follows that $(t-z)(t-\bar{z})$ is a factor of $f_1(t), f_2(t), g_1(t), g_2(t)$. Hence z is a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$. Finally, if z is a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$, then $\bar{z}$ is also a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$. Hence z and $\bar{z}$ are roots of $Q(t)$ and so, they define the same spherical root.

Corollary 4 *If $\mathcal{Q}(t)$ has no real factor, then it has only isolated roots.*

Proof. Suppose that $Q(t)$ has a spherical root ρ. Then there is a complex number $z \in [\rho]$. It follows that z is a spherical root of $Q(t)$ and so, Theorem 2(b) implies that z and its conjugate $\bar{z}$ are common roots of $f(t)$ and $g(t)$. Thus, the real polynomial $(t-z)(t-\bar{z})$ is a common factor of $f(t)$ and $g(t)$. Therefore, $Q(t)$ has the real factor $(t-z)(t-\bar{z})$ which is a contradiction. Hence $Q(t)$ has only isolated roots.

Remark 1 Since a spherical root of $Q(t)$ has in its class a number $z \in \mathbb{C}\setminus\mathbb{R}$, Theorem 2 yields that we can find the spherical roots of $Q(t)$ by computing all the common complex roots of $f_1(t), f_2(t), g_1(t), g_2(t)$.

Theorem 3 *Suppose that the quaternion polynomial $Q(t)$ has no real root. The following are equivalent:*
(a) The polynomial $Q(t)$ has a spherical root.
(b) $\deg D(t) > 0$.
(c) $n > \operatorname{rank} \mathcal{B}_{f_1}(f_2, g_1, g_2)$.

Proof. Suppose that $Q(t)$ has a spherical root q. Let z and $\bar{z}$ be the only complex numbers of the class of q. Then we have $Q(z) = Q(\bar{z})$, whence we get

$$f(z) = f(\bar{z}) = 0 \quad \text{and} \quad g(z) = g(\bar{z}) = 0.$$

It follows that the real polynomial $(t-z)(t-\bar{z})$ divides $f(z)$ and $g(z)$ and hence $D(t)$. Therefore $\deg D(t) > 0$.

Conversely, suppose that $\deg D(t) > 0$. Then $D(t)$ has a root $z \in \mathbb{C}$. If $z \in \mathbb{R}$, then z is a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$ and hence z is a root of $Q(t)$. Since $Q(t)$ has no real root we have a contradiction. Thus $z \notin \mathbb{R}$ and so, its conjugate $\bar{z}$ is also a root of $D(t)$. It follows that z and $\bar{z}$ are roots of $Q(t)$. By Theorem 2, the class of z is a spherical root of $\mathcal{Q}(t)$.

Finally, by Lemma 1 we have

$$\deg D = n - \operatorname{rank} \mathcal{B}_{f_1}(f_2, g_1, g_2)$$

and so, the equivalence of (b) and (c) follows.

Remark 2 In the above theorem, the hypothesis that $Q(t)$ has no real root, implies that $D(t)$ has not a real root and so, if $\deg D > 0$, then we have that $\deg D$ is even.

Theorem 4 *Suppose that the quaternion polynomial $Q(t)$ has no real root. The following are equivalent:*
(a) The polynomial $Q(t)$ has an isolated complex root.
(b) $\deg E > \deg D$.
(c) $\operatorname{rank} \operatorname{Bez}(f, g) < \operatorname{rank} \mathcal{B}_{f_1}(f_2, g_1, g_2)$.

Proof. Let z be an isolated complex root of $Q(t)$. By Theorem 1, z is a common root of $f(t)$ and $g(t)$. Since the root z is isolated, $\bar{z}$ is not a common root of these two polynomials. Thus, the real polynomial $(t-z)(t-\bar{z})$ is not a common factor of $f_1(t), f_2(t), g_1(t), g_2(t)$. Hence z is not a root of $D(t)$. Since $D(t)$ divides $E(t)$, we deduce that $\deg E > \deg D$.

Conversely, suppose that $\deg E > \deg D$. Then $E(t)$ has a complex root z which is not a root of $D(t)$. If z is a spherical root, then $\bar{z}$ is also a common root of $f(t)$ and $g(t)$. It follows that $(t-z)(t-\bar{z})$ is a common factor of $f_1(t), f_2(t), g_1(t), g_2(t)$ and hence divides $D(t)$. Therefore z is a root of $D(t)$ which is a contradiction. Thus, z is an isolated root of $Q(t)$.

By Lemma 1, we have

$$\deg D = n - \operatorname{rank} \mathcal{B}_{f_1}(f_2, g_1, g_2)$$

and

$$\deg E = n - \operatorname{rankBez}(f, g).$$

Thus, we have $\deg E > \deg D$ if and only if

$$\operatorname{rankBez}(f, g) < \operatorname{rank} \mathcal{B}_{f_1}(f_2, g_1, g_2).$$

5 Bounds for the Size of the Roots

In [20, Section 4] some bounds for the roots of quaternion polynomials are given. In this section we compute new bounds which are sharper than the bound give in [20, Theorem 4.2], in case where the roots of polynomial are quite large.

Let

$$Q(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n.$$

be a quaternion polynomial. We define the *height* of $Q(t)$ to be the quantity

$$H(Q) = \max\{1, |a_1/a_0|, \ldots, |a_n/a_0|\}.$$

We write $Q(t) = f(t) + \mathbf{k}g(t)$, where $f(t), g(t) \in \mathbb{C}[t]$, and

$$f(t) = f_1(t) + f_2(t)\mathbf{i}, \quad g(t) = g_1(t) + g_2(t)\mathbf{i},$$

where $f_1(t), f_2(t), g_1(t), g_2(t) \in \mathbb{R}[t]$. We set

$$\mathcal{H}_1 = \min\{H(f), H(g)\} \quad \text{and} \quad \mathcal{H}_2 = \min\{H(f_1), H(f_2), H(g_1), H(g_2)\}.$$

Theorem 5 *Suppose that the polynomial $Q(t)$ is monic and ρ is a root of $Q(t)$. If ρ is a spherical root, then*

$$|\rho| < 1 + \mathcal{H}_2^{1/2}.$$

If ρ is an isolated complex root, then

$$|\rho| < 1 + \mathcal{H}_1.$$

In the general case, we have

$$|\rho| < 1 + H(Q).$$

Proof. Suppose that ρ is a spherical root of $Q(t)$. Then there is $z \in \mathbb{C} \setminus \mathbb{R}$ in the class of ρ which is also a root of $Q(t)$. By Theorem 2, z is a common complex root of $f_1(t), f_2(t), g_1(t), g_2(t)$. Thus, [18, Corollary 3] implies that $|z| < 1 + \mathcal{H}_2^{1/2}$. Since $|\rho| = |z|$, we obtain $|\rho| < 1 + \mathcal{H}_2^{1/2}$.

Suppose that ρ is an isolated root. If $\rho \in \mathbb{C}$, then Theorem 1 implies that ρ is a common root of $f(t)$ and $g(t)$. Hence [18, Corollary 2] yields $|\rho| < 1 + \mathcal{H}_1$.

Suppose next that ρ is an isolated non-complex root. If $|\rho| \leq 1$, then the result is true. Suppose that $|\rho| > 1$. Since ρ is a root of $Q(t)$, there is $G(t) \in \mathbb{H}[t]$ such that $Q(t) = G(t)(t - \rho)$. Write

$$G(t) = t^{n-1} + b_1 t^{n-2} + \cdots + b_{n-1}.$$

Then

$$Q(t) = G(t)(t - \rho) = t^n + (b_1 - \rho)t^{n-1} + (b_2 - b_1\rho)t^{n-2} + \cdots + b_{n-1}\rho.$$

It follows that

$$a_1 = b_1 - \rho, \quad a_2 = b_2 - b_1\rho, \quad a_3 = b_3 - b_2\rho, \ldots, \quad a_n = b_{n-1}\rho.$$

Let i be the smallest index such that $H(G) = |b_i|$. Then we have

$$H(Q) \geq |b_i - b_{i-1}\rho| \geq ||b_i| - |b_{i-1}\rho|| \geq |H(G) - |b_{i-1}\rho|| > |H(G)(1 - |\rho|)|,$$

whence we deduce the result.

Remark 3 In case where $a_0 = 1$, [20, Theorem 4.2] yields that the roots ρ of $Q(t)$ satisfy

$$|\rho| \leq \max\{1, \sum_{i=1}^{n} |a_i|\}.$$

If $\sum_{i=1}^{n} |a_i| > 1 + H(Q)$, then Theorem 5 gives a sharper bound.

Corollary 5 *Let $Q(t) \in \mathbb{H}[t] \setminus \mathbb{H}$ be a monic polynomial. Then $Q(t)$ has at most a finite number of roots $\mathbf{x}$ of the form $\mathbf{x} = x_1 + x_2 i + x_3 j + x_4 k$, where x_1, x_2, x_3, x_4 are integers.*

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