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Authors

Farouki, RT
Dospra, P
Sakkalis, T

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Scalar–vector algorithm for the roots of quadratic quaternion polynomials, and the characterization of quintic rational rotation–minimizing frame curves

Rida T. Farouki*
Department of Mechanical and Aerospace Engineering,
University of California, Davis, CA 95616, USA.

Petroula Dospra and Takis Sakkalis
Mathematics Laboratory, Agricultural University of Athens,
75 Iera Odos, Athens 11855, GREECE.

Abstract

The scalar–vector representation is used to derive a simple algorithm to obtain the roots of a quadratic quaternion polynomial. Apart from the familiar vector dot and cross products, this algorithm requires only the determination of the unique positive real root of a cubic equation, and special cases (e.g., double roots) are easily identified through the satisfaction of algebraic constraints on the scalar/vector parts of the coefficients. The algorithm is illustrated by computed examples, and used to analyze the root structure of quadratic quaternion polynomials that generate quintic curves with rational rotation–minimizing frames (RRMF curves). The degenerate (i.e., linear or planar) quintic RRMF curves correspond to the case of a double root. For polynomials with distinct roots, generating non–planar RRMF curves, the cubic always factors into linear and quadratic terms, and a closed–form expression for the quaternion roots in terms of a real variable, a unit vector, a uniform scale factor, and a real parameter $\tau \in [-1, +1]$ is derived.

*Corresponding author. Phone number: 530–752-1779

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e–mail: farouki@ucdavis.edu, pdospra@aua.gr, stp@aua.gr

1 Introduction

In recent years, the characterization and computation of the quaternion roots of algebraic equations with quaternion coefficients has attracted considerable interest — see [3, 4, 13, 14, 12, 17, 18, 19, 24, 25, 28, 27, 30]. Perhaps the earliest systematic studies of this problem can be found in the papers of Niven [22, 23], Eilenberg and Niven [5], and Gordon & Motzkin [15]. Considerable progress has subsequently been made in elucidating the fundamental nature of quaternion roots, and formulating (numerical) schemes to compute them.

The simplest non–trivial instance of this problem concerns the roots of a monic quadratic equation with quaternion coefficients. Although the non–commutative nature of quaternion products makes this problem much more subtle than its real counterpart, it nevertheless admits an essentially closed–form solution. In the generic case, this solution involves the (unique) positive real root of a cubic equation, which may be obtained by Cardano's method.

This study adopts a different approach to computing the quaternion roots of quadratic equations, based on the scalar–vector quaternion representation. Widespread familiarity with the dot and cross products of vectors leads to an accessible, easily–implemented, and efficient algorithm, accommodating certain special–case instances through simple scalar branch conditions.

The motivation for this study stems from recent investigations of quintic curves with *rational rotation–minimizing frames* [7, 8, 9, 10, 11] — or quintic *RRMF curves*. An RRMF curve $\mathbf{r}(t)$ admits a rational adapted orthonormal frame $(\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t))$ in which $\mathbf{f}_1 = \mathbf{r}'/|\mathbf{r}'|$ is the unit curve tangent, while the normal–plane vectors $\mathbf{f}_2$, $\mathbf{f}_3$ exhibit no instantaneous rotation about $\mathbf{f}_1$ — i.e., the frame angular velocity $\boldsymbol{\omega}$ satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_1 \equiv 0$. Such curves are useful in diverse applications, such as computer animation, robotics, geometric design, and spatial motion control. Quintic RRMF curves are generated by quadratic quaternion polynomials with coefficients that satisfy algebraic constraints [7], and the question arises as to whether the RRMF property can be alternatively characterized in terms of the root structure of such polynomials.

The remainder of this paper is organized as follows. Section 2 introduces the basic problem addressed herein, and briefly reviews earlier work on this problem. The scalar–vector approach to computing the roots of a quadratic quaternion polynomial is then described in Section 3, including identification and treatment of certain special cases. The methodology of Section 3 is then summarized in terms of a simple algorithm in Section 4, and illustrated by representative computed examples. Section 5 applies these results to analyze certain root properties of the quadratic quaternion polynomials that generate quintic RRMF curves. Finally, Section 6 summarizes the key results of this study, and identifies some open problems that deserve further investigation.

2 Quadratic quaternion polynomials

Throughout this paper, we shall work in the *real division quaternion algebra* $\mathbb{H}$ — i.e., with quaternions of the form $\mathcal{Q} = q + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$ where q, q_x, q_y, q_z are real numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{i}\,\mathbf{j} = -\,\mathbf{j}\,\mathbf{i} = \mathbf{k}$, $\mathbf{j}\,\mathbf{k} = -\,\mathbf{k}\,\mathbf{j} = \mathbf{i}$, $\mathbf{k}\,\mathbf{i} = -\,\mathbf{i}\,\mathbf{k} = \mathbf{j}$. Alternative quaternion algebras, based on different conventions for these products, exist [1, 21] — in general, however, they are not division algebras, and non–zero quaternions may not have inverses.

Because of widespread familiarity with the basic vector operations in $\mathbb{R}^3$ (i.e., dot and cross products), the *scalar–vector form* of quaternions provides a highly accessible approach [26] to performing computations on them.[1] We use calligraphic, bold, and italic characters to denote quaternions, vectors in $\mathbb{R}^3$, and scalars (real numbers), respectively. A quaternion $\mathcal{Q}$ is regarded as comprising a *scalar* (or *real*) part $q = \mathrm{scal}(\mathcal{Q})$ and *vector* (or *imaginary*) part $\mathbf{q} = \mathrm{vect}(\mathcal{Q})$. Correspondingly, we write $\mathcal{Q} = (q, \mathbf{q})$ and define the *conjugate*, *modulus*, and *inverse* of $\mathcal{Q}$ by

$$\mathcal{Q}^* \,=\, (q, -\,\mathbf{q})\,, \qquad |\,\mathcal{Q}\,| \,=\, \sqrt{q^2 + |\mathbf{q}|^2}\,, \qquad \mathcal{Q}^{-1} \,=\, \frac{\mathcal{Q}^*}{|\,\mathcal{Q}\,|^2}\,.$$

The sum and product of given quaternions $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ may be compactly expressed [26] as

$$\mathcal{A} + \mathcal{B} \,=\, (a + b\,,\, \mathbf{a} + \mathbf{b})\,, \qquad \mathcal{A}\,\mathcal{B} \,=\, (ab - \mathbf{a}\cdot\mathbf{b}\,,\, a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a}\times\mathbf{b})\,,$$

where $\cdot$ and $\times$ denote the usual vector dot and cross products, the latter being responsible for the non–commutative property of quaternion multiplication. For brevity, we shall henceforth simply write q and $\mathbf{q}$ for pure scalar and pure vector quaternions of the form $(q, \mathbf{0})$ and $(0, \mathbf{q})$.

We now consider, for given quaternion coefficients $\mathcal{B}$ and $\mathcal{C}$, the solutions to the quadratic equation

$$\mathcal{Q}^2 \,+\, \mathcal{B}\,\mathcal{Q} \,+\, \mathcal{C} \,=\, 0 \tag{1}$$

[1]It should be recognized, however, that this approach is anachronistic [2] — the concepts of *scalar* and *vector* originated in the theory of quaternions, rather than vice–versa.

in the quaternion variable $\mathcal{Q}$. Since, in each term, the coefficients appear on the left and powers of $\mathcal{Q}$ on the right, expression (1) defines a *left polynomial* [15] and its solutions are the *right roots* of that polynomial — the algorithm described below can be easily adapted to the case where the coefficients are on the right and powers of $\mathcal{Q}$ on the left, but we shall not pursue this. The existence of at least one root of (1) follows from the "fundamental theorem of algebra" [5, 29] for real quaternions (and also octonions).

There is no loss of generality in assuming the quadratic equation (1) to be monic. The equation $\mathcal{A}\,\mathcal{Q}^2+\mathcal{B}\,\mathcal{Q}+\mathcal{C}=0$ reduces to the form (1) through (left) multiplication with $\mathcal{A}^{-1}$ when $\mathcal{A}\neq 0$, and when $\mathcal{A}=0$ it is linear with the trivial solution $\mathcal{Q}=-\mathcal{B}^{-1}\mathcal{C}$. We also assume that $\mathcal{C}\neq 0$ in (1), since otherwise this equation has the trivial solutions $\mathcal{Q}=0$ and $\mathcal{Q}=-\mathcal{B}$.

We note that, because of the non–commutative nature of the quaternion product, the familiar "completing the square" process cannot be employed to compute the roots of (1). In particular,

$$\mathcal{Q}^2 + \mathcal{B}\,\mathcal{Q} + \mathcal{C} \neq (\mathcal{Q}+\tfrac{1}{2}\mathcal{B})^2 - \tfrac{1}{4}\mathcal{B}^2 + \mathcal{C}\,,$$

since, in general, we have

$$(\mathcal{Q}+\tfrac{1}{2}\mathcal{B})^2 - \tfrac{1}{4}\mathcal{B}^2 = \mathcal{Q}^2 + \tfrac{1}{2}(\mathcal{B}\,\mathcal{Q}+\mathcal{Q}\,\mathcal{B}) \neq \mathcal{Q}^2 + \mathcal{B}\,\mathcal{Q}\,.$$

Moreover, although equation (1) has unique roots $\mathcal{Q}_1$, $\mathcal{Q}_2$ it does *not* admit a factorization of the form

$$(\mathcal{Q}-\mathcal{Q}_1)(\mathcal{Q}-\mathcal{Q}_2) = 0 \qquad \text{or} \qquad (\mathcal{Q}-\mathcal{Q}_2)(\mathcal{Q}-\mathcal{Q}_1) = 0\,.$$

Because of the non–commutative product, greater care must be exercised in analyzing and interpreting the roots of quaternion polynomials [20].

It is still possible to uniquely reconstruct the polynomial from its roots if $\mathcal{Q}_1\neq\mathcal{Q}_2$. From $\mathcal{Q}_1^2+\mathcal{B}\,\mathcal{Q}_1+\mathcal{C}=\mathcal{Q}_2^2+\mathcal{B}\,\mathcal{Q}_2+\mathcal{C}=0$ we obtain

$$\mathcal{B} = (\mathcal{Q}_1^2-\mathcal{Q}_2^2)(\mathcal{Q}_2-\mathcal{Q}_1)^{-1} = \frac{(\mathcal{Q}_1^2-\mathcal{Q}_2^2)(\mathcal{Q}_2^*-\mathcal{Q}_1^*)}{|\mathcal{Q}_2-\mathcal{Q}_1|^2}\,,$$

and from this we have $\mathcal{C}=-(\mathcal{Q}_1+\mathcal{B})\mathcal{Q}_1=-(\mathcal{Q}_2+\mathcal{B})\mathcal{Q}_2$. However, this is not true in the case of double roots. The distinct polynomials $\mathcal{Q}^2-(\mathbf{i}+\mathbf{j})\,\mathcal{Q}-\mathbf{k}$ and $\mathcal{Q}^2-(\mathbf{k}+\mathbf{i})\,\mathcal{Q}+\mathbf{j}$, for example, both have $\mathcal{Q}=\mathbf{i}$ as a double root.

A few recent studies specifically address computation of the quaternion roots of (1). Huang and So [17] present detailed case–by–case formulae for the roots, dependent on the nature of the coefficients. Jia et al. [19] approach the problem through analysis of an equivalent real quadratic form. The solution presented herein differs from prior studies by systematically employing the scalar–vector quaternion representation, which yields simple formulae and special–case conditions expressed in terms of the familiar vector dot and cross products. The solution derived in this manner is amenable to analyzing the root structure of those quadratic quaternion polynomials that generate quintic space curves with rational rotation–minimizing frames (see Section 5).

3 Scalar–vector solution for roots

Setting $\mathcal{B} = (b, \mathbf{b})$, $\mathcal{C} = (c, \mathbf{c})$, and $\mathcal{Q} = (q, \mathbf{q})$, equation (1) may be reduced to the scalar and vector components

$$q^2 - |\mathbf{q}|^2 + b\,q - \mathbf{b}\cdot\mathbf{q} + c = 0\,, \tag{2}$$

$$(2q + b)\,\mathbf{q} + q\,\mathbf{b} + \mathbf{b}\times\mathbf{q} + \mathbf{c} = \mathbf{0}\,, \tag{3}$$

which can be regarded as a system of four quadratic equations in the scalar part q and (the three components of) the vector part $\mathbf{q}$ of $\mathcal{Q}$.

Remark 1. As observed by Niven [22] one may also assume, without loss of generality, that $\mathcal{B}$ in (1) is a pure vector (imaginary) quaternion, by making the substitution $(q, \mathbf{q}) \to (q - \frac{1}{2}\,b, \mathbf{q})$, so that $(b, \mathbf{b}) \to (0, \mathbf{b})$ and $(c, \mathbf{c}) \to (c - \frac{1}{4}\,b^2, \mathbf{c} - \frac{1}{2}\,b\,\mathbf{b})$. However, we omit this assumption here, since it does not significantly simplify the solution of equations (2)–(3).

Before analyzing the general solution of (1), we first treat the degenerate special case in which $\mathcal{B}$ and $\mathcal{C}$ are pure scalars (i.e., real numbers).

Lemma 1. *When the coefficients $\mathcal{B}$ and $\mathcal{C}$ are both real, i.e., $\mathbf{b} = \mathbf{c} = \mathbf{0}$, the solutions of the quadratic equation* (1) *are*

- *the double real root $\mathcal{Q} = \frac{1}{2}(-b, \mathbf{0})$ when $b^2 - 4c = 0$;*
- *the two real roots $\mathcal{Q} = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c}, \mathbf{0})$ when $b^2 - 4c > 0$;*
- *the "spherical root" $\mathcal{Q} = \frac{1}{2}(-b, \sqrt{4c - b^2}\,(\lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}))$, where λ, μ, ν are real numbers satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$, when $b^2 - 4c < 0$.*

Proof. When $\mathbf{b} = \mathbf{c} = \mathbf{0}$, equations (2)–(3) reduce to

$$q^2 \,-\, |\mathbf{q}|^2 \,+\, b\,q \,+\, c \,=\, 0 \qquad \text{and} \qquad (2q+b)\,\mathbf{q} \,=\, \mathbf{0}\,.$$

The second equation implies that $\mathbf{q} = \mathbf{0}$ or $q = -\,\frac{1}{2}\,b$. In the former case, the first equation reduces to $q^2 + b\,q + c = 0$, with no real roots if $b^2 - 4c < 0$; a double root $q = -\,\frac{1}{2}\,b$ if $b^2 - 4c = 0$; and distinct roots $q = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$ if $b^2 - 4c > 0$. In the latter case, the first equation gives $|\mathbf{q}|^2 = c - \frac{1}{4}\,b^2$, which is satisfied by any vector of the form

$$\mathbf{q} \,=\, \tfrac{1}{2}\,\sqrt{4c - b^2}\,(\lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k})$$

with $\lambda^2 + \mu^2 + \nu^2 = 1$ when $b^2 - 4c < 0$; by $\mathbf{q} = \mathbf{0}$ when $b^2 - 4c = 0$; and by no real vector when $b^2 - 4c > 0$. ■

Henceforth, we focus on equation (1) with $\mathcal{B}, \mathcal{C}$ not both real. We consider two categories of solutions $(q, \mathbf{q})$ to the system (2)–(3), namely, roots with (i) $q \neq -\,\frac{1}{2}\,b$; and (ii) $q = -\,\frac{1}{2}\,b$. We call roots in categories (i) and (ii) the *generic* and *singular* quaternion roots of (1). We shall see that the singular case corresponds to roots with identical scalar parts, and it encompasses the case of *double* roots of (1) as a proper subset.

3.1 Generic roots

For category (i) roots with $2q + b \neq 0$, analysis through the `MAPLE` computer algebra system reveals that equation (3) may be solved to express $\mathbf{q}$ in terms of q, b, c and $\mathbf{b}, \mathbf{c}$ as

$$\mathbf{q} \,=\, \frac{1}{2q+b}\left[\,\frac{(2q+b)\,\mathbf{b}\times\mathbf{c} \,-\, (2q+b)^2\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{b}}{(2q+b)^2 + |\mathbf{b}|^2} \,-\, q\,\mathbf{b}\,\right]. \tag{4}$$

By writing $x = (2q+b)^2$ and substituting (4) into (2), further analysis using `MAPLE` indicates that the latter equation can be factorized to obtain

$$(x + |\mathbf{b}|^2)(x^3 + a_2x^2 + a_1x + a_0) \,=\, 0\,, \tag{5}$$

where

$$\begin{aligned} a_2 &\,=\, 2\,|\mathbf{b}|^2 - b^2 + 4c\,, \\ a_1 &\,=\, (|\mathbf{b}|^2 - b^2 + 4c)\,|\mathbf{b}|^2 \,-\, |b\,\mathbf{b} - 2\,\mathbf{c}|^2\,, \\ a_0 &\,=\, -\,(b\,|\mathbf{b}|^2 - 2\,\mathbf{b}\cdot\mathbf{c})^2\,. \end{aligned} \tag{6}$$

For a category (i) solution of equation (1), we must have $x = (2q+b)^2 > 0$. So the first factor in (5) cannot vanish, and therefore the cubic

$$x^3 + a_2 x^2 + a_1 x + a_0 \tag{7}$$

must possess a positive real root. *Cardano's method* [31] offers a closed–form solution for the roots of this cubic. However, useful insight into the number of its positive roots can be deduced, without actually computing them, by inspection of the coefficients using *Descartes Law of Signs* [31].

Lemma 2. *If $a_0 \neq 0$, the cubic defined by* (6)–(7) *has one positive real root.*

Proof : Descartes Law of Signs states that the number of positive real roots of a polynomial is less than the number of its coefficient sign changes by an even amount. Now the cubic (7) is monic, and from (6) we have $a_0 < 0$ when $a_0 \neq 0$ (the case $a_0 = 0$ is treated in Section 3.2 below). Hence, the number of possible coefficient sign changes may be categorized[2] as follows:

(a) there is one sign change if (a_2, a_1) have signature $(+,+)$ or $(+,-)$;

(b) there are two sign changes if (a_2, a_1) have signature $(-,+)$.

We show that case (b) is impossible. From (6), the conditions $a_2 < 0$ and $a_1 > 0$ are equivalent to

$$c < \frac{b^2 - 2\,|\mathbf{b}|^2}{4} \qquad \text{and} \qquad c > \frac{b^2|\mathbf{b}|^2 - |\mathbf{b}|^4 + |\,b\,\mathbf{b} - 2\,\mathbf{c}\,|^2}{4\,|\mathbf{b}|^2}\,.$$

In order for these inequalities to be consistent, we must have

$$b^2|\mathbf{b}|^2 - 2\,|\mathbf{b}|^4 \,>\, b^2|\mathbf{b}|^2 - |\mathbf{b}|^4 + |\,b\,\mathbf{b} - 2\,\mathbf{c}\,|^2\,,$$

or, equivalently,

$$|\mathbf{b}|^4 \,+\, |\,b\,\mathbf{b} - 2\,\mathbf{c}\,|^2 \,<\, 0\,.$$

Since this is clearly impossible, the cubic defined by (6)–(7) has one coefficient sign change, and thus one positive real root, when $a_0 \neq 0$. ■

Let ρ be the positive root of (7) when $a_0 \neq 0$. Since $\rho = (2q+b)^2$, this yields two distinct values

$$q = \frac{-\,b \pm \sqrt{\rho}}{2} \tag{8}$$

[2] We do not explicitly address the cases $a_1 = 0$ or $a_2 = 0$, since in these instances the number of sign changes cannot exceed the indicated amounts.

for the scalar parts q of the roots $\mathcal{Q}$ of (1), with corresponding vector parts $\mathbf{q}$ specified by (4). Now using (8) we can re–write (4) as

$$\mathbf{q} \,=\, \frac{\mathbf{b}\times\mathbf{c}}{\rho+|\mathbf{b}|^2} - \frac{\mathbf{b}}{2} \pm \frac{1}{\sqrt{\rho}}\left[\frac{b\,\mathbf{b}}{2} - \frac{\rho\,\mathbf{c}+(\mathbf{b}\cdot\mathbf{c})\,\mathbf{b}}{\rho+|\mathbf{b}|^2}\right], \tag{9}$$

and thus from (8)–(9) the two quaternion roots of (1) can be expressed as

$$\mathcal{Q} \,=\, \left(-\frac{b}{2}\,,\,\frac{\mathbf{b}\times\mathbf{c}}{\rho+|\mathbf{b}|^2} - \frac{\mathbf{b}}{2}\right) \pm \frac{1}{\sqrt{\rho}}\left(\frac{\rho}{2}\,,\,\frac{b\,\mathbf{b}}{2} - \frac{\rho\,\mathbf{c}+(\mathbf{b}\cdot\mathbf{c})\,\mathbf{b}}{\rho+|\mathbf{b}|^2}\right), \tag{10}$$

where ρ is the unique positive root of (7) with $a_0 \neq 0$.

Remark 2. If $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent, the roots (10) reduce to

$$\mathcal{Q} \,=\, -\frac{1}{2}\,(b,\mathbf{b}) \pm \frac{1}{2\sqrt{\rho}}\,(\rho, b\,\mathbf{b}-2\,\mathbf{c})\,,$$

so the vector parts of both roots are also linearly dependent on $\mathbf{b}$ and $\mathbf{c}$.

3.2 Singular roots

For category (ii) with $2q+b=0$, we have $q=-\frac{1}{2}\,b$ and equation (3) becomes

$$\mathbf{b}\times\mathbf{q} \,=\, \tfrac{1}{2}\,b\,\mathbf{b} \,-\, \mathbf{c}\,.$$

Now the quantities b, $\mathbf{b}$, $\mathbf{c}$ cannot be freely specified if this equation is to be satisfied. Specifically, since $\mathbf{b}\cdot(\mathbf{b}\times\mathbf{q})=0$, we must have

$$\tfrac{1}{2}\,b\,|\mathbf{b}|^2 \,-\, \mathbf{b}\cdot\mathbf{c} \,=\, 0\,, \tag{11}$$

which is equivalent to $a_0=0$ in (6), i.e., $x=0$ is a root of (7). Note that (11) is automatically satisfied when $\mathbf{b}=\mathbf{0}$. However, equation (3) with $2q+b=0$ and $\mathbf{b}=\mathbf{0}$ can only be satisfied when we also have $\mathbf{c}=\mathbf{0}$, which corresponds to the case of real coefficients treated in Lemma 1. When condition (11) is satisfied with $\mathbf{b}\neq\mathbf{0}$, we have

$$\mathbf{q} \,=\, \frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}|^2} \,+\, \gamma\,\mathbf{b}\,, \tag{12}$$

where γ is a free parameter. Substituting (12) and $q=-\frac{1}{2}\,b$ into (2), and noting that $|\mathbf{b}|\neq 0$, then gives the quadratic equation

$$|\mathbf{b}|^6\gamma^2 \,+\, |\mathbf{b}|^6\gamma \,+\, |\mathbf{b}\times\mathbf{c}|^2 + \tfrac{1}{4}\,b^2|\mathbf{b}|^4 - c\,|\mathbf{b}|^4 \,=\, 0 \tag{13}$$

in γ. In order for (13) to have real roots, we require that

$$|\mathbf{b}|^6 - 4\,|\mathbf{b}\times\mathbf{c}|^2 - b^2|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^4 \,\geq\, 0\,, \tag{14}$$

Now from (11) we have $b^2|\mathbf{b}|^4 = 4\,(\mathbf{b}\cdot\mathbf{c})^2$, and using the identity

$$|\mathbf{b}\times\mathbf{c}|^2 \,+\, (\mathbf{b}\cdot\mathbf{c})^2 \,=\, |\mathbf{b}|^2|\mathbf{c}|^2\,,$$

the condition (14) can be reduced to

$$|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^2 - 4\,|\mathbf{c}|^2 \,\geq\, 0\,. \tag{15}$$

In summary, category (ii) roots exist only when conditions (11) and (15) are *both* satisfied, i.e., $a_0 = 0$ and (13) has a real root γ. The quaternion roots of (1) can then be expressed as

$$\mathcal{Q} \,=\, \left(-\,\frac{b}{2}\,,\,\frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}|^2} - \frac{\mathbf{b}}{2}\right) \,\pm\, \left(0\,,\,\frac{\sqrt{|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^2 - 4\,|\mathbf{c}|^2}}{2\,|\mathbf{b}|^2}\,\mathbf{b}\right)\,. \tag{16}$$

Comparing (10) and (16), we see that singular roots differ from generic roots in having identical scalar parts. If $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent, both roots have vector parts linearly dependent on $\mathbf{b}$ (see Remark 2).

3.3 Double roots

In the generic case ($a_0 \neq 0$) the roots (10) are necessarily distinct, since $\rho > 0$ and hence the scalar parts differ. In the singular case ($a_0 = 0$) the roots (16) have coincident scalar parts, but their vector parts are usually different since equation (13) generically yields two distinct γ values in expression (12).

Clearly, equation (1) admits a double root only in the singular case when (11) is satisfied, with the further requirement that (13) has a double root γ, so that the vector parts (12) of the roots coincide, as well as the scalar parts. Now equation (13) has a double root when its discriminant vanishes, which means that (15) holds with equality. Hence, the two conditions

$$\tfrac{1}{2}\,b\,|\mathbf{b}|^2 \,=\, \mathbf{b}\cdot\mathbf{c} \qquad \text{and} \qquad |\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^2 = 4\,|\mathbf{c}|^2 \tag{17}$$

together specify when equation (1) has a double root. If these conditions are satisfied, the double root is defined by the first term on the right in (16), and using the first condition in (17) it can be expressed as

$$\mathcal{Q} \,=\, \left(-\,\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|^2}\,,\,\frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}|^2} - \frac{\mathbf{b}}{2}\right)\,. \tag{18}$$

Note that, although b and c do not appear explicitly in (18), the double root depends on them implicitly through the satisfaction of conditions (17).

4 Algorithm & computed examples

The preceding analysis of the roots of the quadratic quaternion equation (1) is summarized in the following algorithm.

Algorithm

input: quaternion coefficients $\mathcal{B} = (b, \mathbf{b})$ and $\mathcal{C} = (c, \mathbf{c})$

1. if conditions (11) and (15) are both satisfied, go to step 4;
2. compute the unique positive real root ρ of the cubic (7);
3. compute two quaternion roots from (10) and go to **output**;
4. if condition (15) is satisfied with equality go to step 6;
5. compute two quaternion roots from (16) and go to **output**;
6. compute the double quaternion root from (18);

output: two quaternion roots $\mathcal{Q} = (q, \mathbf{q})$.

The following simple examples serve to illustrate the above algorithm.

Example 1. Consider the quadratic equation (1) with $\mathcal{B} = (0, \mathbf{j})$, $\mathcal{C} = (0, \mathbf{k})$. Since $b = c = 0$ and $\mathbf{b} = \mathbf{j}$, $\mathbf{c} = \mathbf{k}$ the cubic (7) becomes

$$x^3 + 2x^2 - 3x \;=\; 0\,,$$

with roots $x = -3, 0, 1$. Thus, from the positive root we obtain $(2q+b)^2 = 1$, and hence $q = \pm\frac{1}{2}$. Expression (4) then gives the corresponding vector parts as $\mathbf{q} = \frac{1}{2}(\mathbf{i} - \mathbf{j} \mp \mathbf{k})$. Hence, in this case, we have the generic right roots

$$\mathcal{Q}_1 \;=\; \tfrac{1}{2}\,(1, \mathbf{i} - \mathbf{j} - \mathbf{k}) \qquad \text{and} \qquad \mathcal{Q}_2 \;=\; \tfrac{1}{2}\,(-1, \mathbf{i} - \mathbf{j} + \mathbf{k})\,, \tag{19}$$

and one can easily verify that both satisfy $\mathcal{Q}^2 + (0, \mathbf{j})\,\mathcal{Q} + (0, \mathbf{k}) = 0$.

Example 2. For equation (1) with the coefficients

$$\mathcal{B} \;=\; \left(-\,2, \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}\right) \qquad \text{and} \qquad \mathcal{C} \;=\; \left(2, \frac{\mathbf{j}-\mathbf{k}}{\sqrt{2}}\right)\,,$$

the cubic (7) becomes

$$x^3 + 6x^2 - 3x - 4 \;=\; 0\,,$$

with the positive root $x = 1$ and negative roots $\frac{1}{2}\,(-\,7 \pm \sqrt{33})$. The positive root gives $(2q+b)^2 = 1$, and since $b = -\,2$ the roots have scalar parts $q = \frac{1}{2}$ or $\frac{3}{2}$. From (4), the corresponding vector parts are then $\mathbf{q} = \frac{1}{2}\,(-\,\mathbf{i} + \sqrt{2}\,\mathbf{j})$ and $-\,\frac{1}{2}\,(\mathbf{i} + 2\sqrt{2}\,\mathbf{j} + \sqrt{2}\,\mathbf{k})$. Hence, we have the generic right roots

$$\mathcal{Q}_1 \;=\; \tfrac{1}{2}\,(1, -\,\mathbf{i} + \sqrt{2}\,\mathbf{j}) \qquad \text{and} \qquad \mathcal{Q}_2 \;=\; \tfrac{1}{2}\,(3, -\,\mathbf{i} - 2\sqrt{2}\,\mathbf{j} - \sqrt{2}\,\mathbf{k})$$

and one can verify that, for the given coefficients, they both satisfy (1).

Example 3. Consider now equation (1) with $\mathcal{B} = (2, \mathbf{j})$ and $\mathcal{C} = (1, \mathbf{j})$. Since $b = 2$, $c = 1$ and $\mathbf{b} = \mathbf{c} = \mathbf{j}$, the cubic (7) becomes

$$x^3 + 2x^2 + x \;=\; 0\,,$$

with roots $x = -1, -1, 0$. Since none of these roots is positive, there are no generic quaternion roots. For the root $x = 0$, we investigate the existence of singular roots. Since $b = 2$ and $\mathbf{b} = \mathbf{c} = \mathbf{j}$, condition (11) is satisfied. Equation (13) then becomes $\gamma^2 + \gamma = 0$, with real solutions $\gamma = -1$ and 0, for which (12) gives vector parts $\mathbf{q} = -\,\mathbf{j}$ and $\mathbf{q} = \mathbf{0}$ associated with the scalar part $q = -\,\frac{1}{2}\,b = -\,1$. Hence, we have the two singular right roots

$$\mathcal{Q}_1 \;=\; (-1, -\mathbf{j}) \qquad \text{and} \qquad \mathcal{Q}_2 \;=\; (-1, \mathbf{0})\,,$$

which both satisfy $\mathcal{Q}^2 + (2, \mathbf{j})\,\mathcal{Q} + (1, \mathbf{j}) = 0$.

Example 4. For equation (1) with $\mathcal{B} = (2, \mathbf{j})$ and $\mathcal{C} = (\frac{7}{4}, \mathbf{j} + \mathbf{k})$ the cubic (7) becomes

$$x^3 + 5x^2 \;=\; 0\,,$$

with roots $x = -\,5, 0, 0$. Since this has no positive roots, equation (1) has no generic quaternion roots in this case. For the singular root corresponding to $x = 0$, the scalar part is $q = -\,\frac{1}{2}\,b = -1$, and one can verify that both of the conditions (17) are satisfied, so this must define a double quaternion root. Equation (13) reduces to

$$\gamma^2 + \gamma + \tfrac{1}{4} \,=\, 0\,,$$

and has, as expected, the double root $\gamma = -\frac{1}{2}$. The corresponding vector part is then determined as $\mathbf{q} = \mathbf{i} - \frac{1}{2}\,\mathbf{j}$ from expression (12). Hence,

$$\mathcal{Q}_1 \;=\; (-\,1, \mathbf{i} - \tfrac{1}{2}\,\mathbf{j})$$

is the only quaternion root in this case, and it defines a *double* (right) root.

5 Analysis of quintic RRMF curves

A *Pythagorean–hodograph* (PH) *curve* $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a polynomial curve with the distinctive property [6] that the components of its derivative $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ satisfy the Pythagorean condition

$$x'^2(t) + y'^2(t) + z'^2(t) \;=\; \sigma^2(t) \tag{20}$$

for some polynomial $\sigma(t)$. A PH curve may be generated from a quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ through the product[3]

$$\begin{aligned}\mathbf{r}'(t) \;&=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \;=\; [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i} \\ &\qquad + 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} \,+\, 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k}\,.\end{aligned} \tag{21}$$

Remark 3. For any quaternion $\mathcal{Q} = (q, \mathbf{q})$ and vector $\mathbf{v}$, the product $\mathcal{Q}\,\mathbf{v}\,\mathcal{Q}^*$ yields a pure vector $\tilde{\mathbf{v}}$, namely

$$\tilde{\mathbf{v}} \;=\; (q^2 - |\mathbf{q}|^2)\,\mathbf{v} + 2\,(\mathbf{q}\cdot\mathbf{v})\,\mathbf{q} + 2q\,(\mathbf{q}\times\mathbf{v})\,.$$

One can always express $\mathcal{Q}$ in the form $|\mathcal{Q}|\,(\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$, where $\mathbf{n}$ is a unit vector, and $\tilde{\mathbf{v}}$ is obtained geometrically by rotating $\mathbf{v}$ through angle θ about an axis defined by the direction $\mathbf{n}$, and scaling it by $|\mathcal{Q}|^2$.

Remark 3 allows one to interpret the hodograph (21) as being generated through a continuum of rotations/scalings of the fixed unit vector $\mathbf{i}$. It should be kept in mind when interpreting the conditions (24) and (27) below.

[3]The use of the unit vector $\mathbf{i}$ in the product (21) is merely conventional — replacing it with any other unit vector corresponds to only a change of coordinates.

There has been considerable recent interest [7, 9, 10, 11] in a special subset of the PH curves, known as the *rational rotation–minimizing frame* (RRMF) *curves*. These curves possess rational orthonormal adapted frames $(\mathbf{t},\mathbf{u},\mathbf{v})$, where $\mathbf{t}=\mathbf{r}'/|\mathbf{r}'|$ is the curve tangent and $\mathbf{u}$, $\mathbf{v}$ span the curve normal plane at each point, with a frame angular velocity $\boldsymbol{\omega}$ that satisfies $\boldsymbol{\omega}\cdot\mathbf{t}\equiv 0$. The angular velocity governs the frame variation through the relations

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \boldsymbol{\omega}\times\mathbf{t}\,, \qquad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = \boldsymbol{\omega}\times\mathbf{u}\,, \qquad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s} = \boldsymbol{\omega}\times\mathbf{v}\,,$$

where s is the curve arc length, and the condition $\boldsymbol{\omega}\cdot\mathbf{t}$ implies that $\mathbf{u}$ and $\mathbf{v}$ exhibit no instantaneous rotation about $\mathbf{t}$ — or, equivalently, the derivatives of $\mathbf{u}$ and $\mathbf{v}$ are always parallel to $\mathbf{t}$. It was shown in [16] that the existence of polynomials $a(t)$, $b(t)$ such that

$$\frac{uv'-u'v-pq'+p'q}{u^2+v^2+p^2+q^2} \;=\; \frac{ab'-a'b}{a^2+b^2}\,. \tag{22}$$

is a sufficient and necessary condition for (21) to define an RRMF curve.

Planar PH curves are trivially RRMF curves, and we are concerned here only with true space curves. The simplest non–planar RRMF curves are [8] of degree 5. Quintic PH curves are conventionally defined by substituting a quadratic Bernstein–form polynomial

$$\mathcal{A}(t) \;=\; \mathcal{A}_0(1-t)^2+\mathcal{A}_1 2(1-t)t+\mathcal{A}_2 t^2 \tag{23}$$

into (21), and integrating. Then satisfaction of the constraint

$$\mathrm{vect}(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*) \;=\; \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^* \tag{24}$$

by the coefficients of (23) was shown in [7] to be sufficient and necessary for the PH quintic to be an RRMF curve satisfying (22) with $\deg(a(t),b(t))=2$.

Remark 4. The existence of non–planar RRMF quintics satisfying (22) with $\deg(a,b)=1$ was discovered in [11] — they are called Class II RRMF quintics, as distinct from Class I RRMF quintics satisfying (22) with $\deg(a,b)=2$. We focus on the latter here, since a characterization of Class II curves, analogous to the coefficient constraint (24) for Class I curves, is not yet known.

Now the condition (24) on the coefficients of the polynomial (23) evidently constrains its (quaternion) roots, and the question arises as to whether the

RRMF quintics can be alternatively characterized by means of a special root structure of the quaternion polynomials generating them. The methodology used in [7] to derive (24) does not easily extend to degree 7 or higher–order PH curves, and a root–structure characterization of the RRMF curves may offer an alternative approach to the study of higher–order curves.

In the formulation (21) of PH curves, the parameter t is interpreted as a *real* variable. However, when we speak of the roots of $\mathcal{A}(t)$, this restriction is relaxed to allow t to assume any *quaternion* value. To apply the methods of Section 3 to determine the roots of (23), we convert it to the power form

$$\mathcal{A}(t) \,=\, \mathcal{A}\,t^2 + \mathcal{B}\,t + \mathcal{C}\,, \tag{25}$$

where

$$\mathcal{A}_0 \,=\, \mathcal{C}\,, \qquad \mathcal{A}_1 \,=\, \tfrac{1}{2}\,\mathcal{B} + \mathcal{C}\,, \qquad \mathcal{A}_2 \,=\, \mathcal{A} + \mathcal{B} + \mathcal{C}\,. \tag{26}$$

Lemma 3. *If the polynomial $\mathcal{A}(t)$ is represented in power form, the condition*

$$\mathrm{vect}(\mathcal{A}\,\mathbf{i}\,\mathcal{C}^*) \,=\, \tfrac{1}{4}\,\mathcal{B}\,\mathbf{i}\,\mathcal{B}^* \tag{27}$$

on its coefficients is sufficient and necessary for the PH quintic specified by (21) *and* (25) *to be an RRMF curve.*

Proof : Substitute (26) into (24) and simplify. ∎

Now the derivation of the constraint (24) in [7] was facilitated by assuming that $\mathcal{A}_0 = (1, \mathbf{0})$ in (23). This condition, achieved by (left) multiplication of (23) with $\mathcal{A}_0^{-1}$, amounts to imposing a scaling/rotation on $\mathbf{r}(t)$ that maps any (non–zero) initial derivative $\mathbf{r}'(0)$ to the unit vector $\mathbf{i}$ — the curve $\mathbf{r}(t)$ is then said to be in *canonical form*. Since the canonical–form assumption amounts to choosing a particular coordinate system, it incurs no loss of generality and does not alter whether or not a given PH quintic is an RRMF curve.

In the present context, it is convenient to employ a different normalization for the polynomial $\mathcal{A}(t)$. In particular, to analyze its quaternion roots by the methodology of Section 3, we convert (25) into a *monic* polynomial by (left) multiplication with $\mathcal{A}^{-1}$. Again, this amounts to imposing a scaling/rotation on $\mathbf{r}(t)$. Although its geometrical meaning is less evident than the canonical–form transformation of (23), the assumption that (25) is monic likewise incurs no loss of generality: it does not affect the RRMF nature of a given PH curve, and does not alter the roots of (25). A PH curve generated by (21) is said to be in *normal form* [11] when $\mathcal{A}(t)$ is a monic quaternion polynomial.

Henceforth, we confine our attention to curves specified by (21) and (25) with leading coefficient $\mathcal{A} = (1, \mathbf{0})$ and we write the scalar and vector parts of $\mathcal{B}$ as b and $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, and of $\mathcal{C}$ as c and $\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$. With this convention, the quintic RRMF curves may be characterized as follows.

Proposition 1. *A PH quintic defined by* (21) *and* (25) *with* $\mathcal{A} = (1, \mathbf{0})$ *is an RRMF curve if and only if* $\mathcal{C} = (c, \mathbf{c})$ *can be expressed in terms of* $\mathcal{B} = (b, \mathbf{b})$ *and a free scalar parameter* ξ *as*

$$\mathcal{C} \,=\, \left(\, \tfrac{1}{4}(b^2 - |\mathbf{b}|^2 + 2\, b_x^2) \,,\, \tfrac{1}{2}(\xi\, \mathbf{i} + b\, \mathbf{b} + b_x\, \mathbf{i} \times \mathbf{b}) \,\right) \,. \tag{28}$$

Proof : For a PH quintic defined by (21) and (25) with $\mathcal{A} = (1, \mathbf{0})$ the RRMF condition (27) reduces to

$$c\, \mathbf{i} + \mathbf{c} \times \mathbf{i} \,=\, \tfrac{1}{4}\,(\,(b^2 - |\mathbf{b}|^2)\, \mathbf{i} + 2\, b_x\, \mathbf{b} + 2\, b\, \mathbf{b} \times \mathbf{i}\,)\,. \tag{29}$$

Taking the dot product of both sides with $\mathbf{i}$ gives

$$c \,=\, \tfrac{1}{4}\,(b^2 - |\mathbf{b}|^2 + 2\, b_x^2)\,, \tag{30}$$

and substituting this into (29) and making some re–arrangements, we obtain

$$(\mathbf{c} - \tfrac{1}{2}\, b\, \mathbf{b} - \tfrac{1}{2}\, b_x\, \mathbf{i} \times \mathbf{b}) \times \mathbf{i} \,=\, \mathbf{0}\,.$$

Hence, we can express $\mathbf{c}$ in terms of b, $\mathbf{b}$, and a real parameter ξ as

$$\mathbf{c} \,=\, \tfrac{1}{2}\,(\xi\, \mathbf{i} + b\, \mathbf{b} + b_x\, \mathbf{i} \times \mathbf{b})\,, \tag{31}$$

and the quaternion coefficient $\mathcal{C} = (c, \mathbf{c})$ has the stated form (28). ∎

Now any linear or planar locus is trivially an RRMF curve, and since we are only interested in space curves, we first identify instances of $\mathbf{b}$ and ξ that define straight lines or plane curves. These cases are discounted in analyzing the roots of the polynomials (25) that generate spatial RRMF curves.

Proposition 2. *With* $\mathcal{A} = (1, \mathbf{0})$ *and* $\mathcal{C}$ *given by* (28)*, substituting* (25) *into* (21) *generates straight lines when* $(b_y, b_z) = (0, 0)$*, and planar curves other than straight lines when* $b_x = \xi = 0$ *and* $(b_y, b_z) \neq (0, 0)$*.*

Proof : With $\mathcal{A} = (1, \mathbf{0})$ and $\mathcal{C}$ specified by (28), the components of $\mathcal{A}(t) = u(t) + v(t)\, \mathbf{i} + p(t)\, \mathbf{j} + q(t)\, \mathbf{k}$ are given by

$$\begin{aligned}
u(t) \,&=\, t^2 \,+\, b\, t \,+\, \tfrac{1}{4}\,(b^2 + b_x^2 - b_y^2 - b_z^2)\,, \\
v(t) \,&=\, b_x t \,+\, \tfrac{1}{2}\,(\xi + b\, b_x)\,, \\
p(t) \,&=\, b_y t \,+\, \tfrac{1}{2}\,(b\, b_y - b_x b_z)\,, \\
q(t) \,&=\, b_z t \,+\, \tfrac{1}{2}\,(b\, b_z + b_x b_y)\,.
\end{aligned}$$

Since $\deg(u) = 2 > \deg(v, p, q)$ we say [11] that $\mathcal{A}(t)$ is in *normal form*, and Proposition 1 of [11] then shows that the curve defined by (21) degenerates to a straight line if and only if $(p(t), q(t)) \equiv (0, 0)$ and to a planar curve other than a straight line if and only if

$$(p^2 + q^2)\,(uv' - u'v) \;+\; (u^2 + v^2)\,(pq' - p'q) \;\equiv\; 0 \tag{32}$$

with $(p(t), q(t)) \not\equiv (0, 0)$. Clearly, $(b_y, b_z) = (0, 0)$ is a sufficient and necessary condition for $(p(t), q(t) \equiv (0, 0)$. Conversely, $(p(t), q(t)) \not\equiv (0, 0)$ if $(b_y, b_z) \neq (0, 0)$, and using `MAPLE` we find that (32) becomes

$$-\,\frac{b_y^2 + b_z^2}{32}\,(c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0) \;\equiv\; 0\,, \tag{33}$$

where

$$\begin{aligned}
c_4 &= 48\,b_x\,, \quad c_3 \;=\; 32\,(\xi + 3\,b\,b_x)\,, \\
c_2 &= 24\,(2\,b\,\xi + 3\,b^2b_x + b_x^3)\,, \quad c_1 \;=\; 24\,(b^2 + b_x^2)(\xi + b\,b_x)\,, \\
c_0 &= 4\,b_x\xi^2 \;+\; 4\,b\,(b^2 + 3\,b_x^2)\,\xi \;+\; b_x\,[\,3\,b^4 + 6\,b^2b_x^2 - b_x^4 + (b_y^2 + b_z^2)^2\,]\,.
\end{aligned}$$

From c_4 we must have $b_x = 0$ if (33) is satisfied. Substituting $b_x = 0$, we find using `MAPLE` that this condition reduces to

$$-\,\frac{b_y^2 + b_z^2}{8}\,\xi\,(2t + b)^3 \;\equiv\; 0\,.$$

Since $b_y^2 + b_z^2 \neq 0$, the condition for a planar curve (other than a straight line) corresponds to $b_x = \xi = 0$. ∎

We make the following remarks concerning the degeneration of an RRMF quintic to a straight line, or a plane curve other than a straight line.

Remark 5. The condition $b_y = b_z = 0$ for degeneration to a straight line is automatically satisfied if $\mathbf{b} = \mathbf{0}$.

Remark 6. When $\mathcal{A} = (1, \mathbf{0})$ and $\mathcal{C}$ is given by (28) with $b_y = b_z = 0$, the polynomial (25) reduces to

$$\mathcal{A}(t) \;=\; \big(\,t^2 + b\,t + \tfrac{1}{4}(b^2 + b_x^2)\,,\,[\,b_xt + \tfrac{1}{2}(\xi + b\,b_x)\,]\,\mathbf{i}\,\big)$$

— i.e., the vector part of $\mathcal{A}(t)$ has no $\mathbf{j}$ or $\mathbf{k}$ component. Thus, degeneration to a straight line occurs when $\mathcal{A}(t)$ is equivalent to a *complex* polynomial. If $b_x = \xi = 0$, on the other hand, we have

$$\mathcal{A}(t) \;=\; \big(\,t^2 + b\,t + \tfrac{1}{4}(b^2 - b_y^2 - b_z^2)\,,\,(t + \tfrac{1}{2}\,b)(b_y\mathbf{j} + b_z\mathbf{k})\,\big)\;,$$

so degeneration to a plane curve other than a straight line occurs when the vector part of $\mathcal{A}(t)$ has no $\mathbf{i}$ component.

From Proposition 1, it is evident that the roots of a quadratic quaternion polynomial (25) with $\mathcal{A} = (1, \mathbf{0})$ that defines an RRMF quintic curve depend on only the quaternion coefficient $\mathcal{B} = (b, \mathbf{b})$ and the scalar parameter ξ. The coefficients (6) of the cubic (7) can be expressed in terms of b, $\mathbf{b}$, and ξ as

$$a_2 \,=\, |\mathbf{b}|^2 + 2\, b_x^2 \,, \qquad a_1 \,=\, b_x^2(|\mathbf{b}|^2 + b_x^2) - \xi^2 \,, \qquad a_0 \,=\, -\, b_x^2\, \xi^2 \,, \tag{34}$$

and we observe that these coefficients do *not* depend on b. In fact, the cubic (7) has a very special structure when the quaternion polynomial (25) satisfies the condition (27), and thus generates a quintic RRMF curve through (21).

Lemma 4. *For a quadratic quaternion polynomial* (25) *with* $\mathcal{A} = (1, \mathbf{0})$ *and* $\mathcal{C}$ *given by* (28)*, the cubic equation* (7) *specified by the coefficients* (34) *admits the factorization*

$$(x^2 + (|\mathbf{b}|^2 + b_x^2)x - \xi^2)\,(x + b_x^2) \,=\, 0\,. \tag{35}$$

Proof : Expanding (35) yields the cubic (7) with the coefficients (34). ∎

Thus, computing the roots of the quadratic quaternion polynomials that generate quintic RRMF curves does not require solution of a cubic equation, indicating a special structure to these quaternion roots. If $b_x \neq 0$ and $\xi \neq 0$, the only positive root of (35) is

$$\rho \,=\, \sqrt{\xi^2 + \tfrac{1}{4}(|\mathbf{b}|^2 + b_x^2)^2} \,-\, \tfrac{1}{2}(|\mathbf{b}|^2 + b_x^2)\,, \tag{36}$$

and this determines two generic quaternion roots of (25), specified by (10). If $b_x = 0$ or $\xi = 0$, however, then $a_0 = 0$ and (25) may possess singular roots. We consider first the case where $x = 0$ is a root of (35). The following result characterizes the instance $b_x = \xi = 0$.

Lemma 5. *If the polynomial* (25) *with* $\mathcal{A} = (1, \mathbf{0})$ *and* $\mathcal{C}$ *given by* (28) *has a double root, the quintic RRMF curve specified by* (21) *degenerates to a planar curve or a straight line.*

Proof : With $\mathcal{C}$ given by (28), the conditions (17) for a double root become

$$b_x\xi \,=\, 0 \qquad \text{and} \qquad \xi^2 \,+\, 2\, b b_x \xi \,-\, b_x^2(2b_x^2 + b_y^2 + b_z^2) \,=\, 0\,.$$

The first condition implies that $b_x = 0$ or $\xi = 0$, whereas the second condition cannot be satisfied if $b_x = 0 \neq \xi$ or $b_x \neq 0 = \xi$, so we must have $b_x = \xi = 0$. By Proposition 2, the curve defined by (21) is a straight line when $(b_y, b_z) = (0,0)$ and a planar curve other than a straight line when $(b_y, b_z) \neq (0,0)$ — the double root of (25) in the former case is $\mathcal{Q} = (-\frac{1}{2}\,b, \mathbf{0})$, and in the latter it is $\mathcal{Q} = (-\frac{1}{2}\,b, -\frac{1}{2}\,(b_y\,\mathbf{j} + b_z\,\mathbf{k}))$. ■

Consider now the cases in which just one of b_x and ξ is zero. If $b_x = 0 \neq \xi$, we obtain $c = \frac{1}{4}\,(b^2 - |\mathbf{b}|^2)$ and $\mathbf{c} = \frac{1}{2}\,(\xi\,\mathbf{i} + b\,\mathbf{b})$ from (30) and (31), and thus $|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^2 - 4\,|\mathbf{c}|^2 = -\,\xi^2$. Since condition (15) is obviously not satisfied, there are no singular roots. On the other hand, the quadratic factor in (35) has a single positive real root

$$\rho \;=\; \sqrt{\xi^2 + \tfrac{1}{4}\,|\mathbf{b}|^4} - \tfrac{1}{2}\,|\mathbf{b}|^2\,,$$

corresponding to the specialization $b_x = 0$ of (36). There are then two generic quaternion roots, defined with this ρ value in (10). Finally, when $\xi = 0 \neq b_x$, equation (13) reduces to

$$|\mathbf{b}|^2\left[\,|\mathbf{b}|^4(\gamma + \tfrac{1}{2})^2 \;-\; \tfrac{1}{4}\,b_x^2(|\mathbf{b}|^2 + b_x^2)\,\right] \;=\; 0\,,$$

and assuming that $\mathbf{b} \neq \mathbf{0}$ (See Remark 5) it has the two solutions

$$\gamma \;=\; \frac{-\,|\mathbf{b}|^2 \pm b_x\sqrt{|\mathbf{b}|^2 + b_x^2}}{2\,|\mathbf{b}|^2}\,.$$

Expression (16) for the two singular roots thus reduces to

$$\mathcal{Q} \;=\; \left(-\frac{b}{2}\,,\,\frac{b_x|\mathbf{b}|^2\mathbf{i} - (|\mathbf{b}|^2 + b_x^2)\,\mathbf{b}}{2\,|\mathbf{b}|^2}\right) \pm \left(0\,,\,\frac{b_x\sqrt{|\mathbf{b}|^2 + b_x^2}\,\mathbf{b}}{2\,|\mathbf{b}|^2}\right). \qquad (37)$$

For the generic roots, with the positive root of (35) defined by (36) when $(b_x, \xi) \neq (0,0)$, we have

$$\mathbf{b}\cdot\mathbf{c} \;=\; \tfrac{1}{2}\,(b_x\xi + b\,|\mathbf{b}|^2)\,,$$
$$\mathbf{b}\times\mathbf{c} \;=\; \tfrac{1}{2}\,(\xi\,\mathbf{b}\times\mathbf{i} + b_x\mathbf{b}\times(\mathbf{i}\times\mathbf{b})) \;=\; \tfrac{1}{2}\,(b_x|\mathbf{b}|^2\mathbf{i} - b_x^2\mathbf{b} - \xi\,\mathbf{i}\times\mathbf{b})\,.$$

Substituting into (10) and simplifying then gives the roots as

$$\begin{aligned}\mathcal{Q} \;=\;& \left(-\frac{b}{2}\,,\,\frac{b_x|\mathbf{b}|^2\,\mathbf{i} - b_x^2\,\mathbf{b} - \xi\,\mathbf{i}\times\mathbf{b}}{2(\rho + |\mathbf{b}|^2)} - \frac{\mathbf{b}}{2}\right)\\ &\pm\;\frac{1}{\sqrt{\rho}}\left(\frac{\rho}{2}\,,\,-\,\frac{\rho\,\xi\,\mathbf{i} + b_x\xi\,\mathbf{b} + \rho\,b_x\,\mathbf{i}\times\mathbf{b}}{2(\rho + |\mathbf{b}|^2)}\right).\end{aligned} \qquad (38)$$

Lemma 6. *The singular roots* (37) *are the formal limit of the generic roots* (38)*, as* $\xi \to 0$*.*

Proof : First, note from (36) that $\rho \to 0$ as $\xi \to 0$, and otherwise ρ increases monotonically with $|\xi|$. Setting $\xi = \rho = 0$ in the first term of (38), it clearly reduces to the first term of (37). Likewise, the scalar part of the second term of (38) is zero when $\rho = 0$, and thus agrees with the scalar part of the second term of (37). The vector part of the second term in (38) requires more careful analysis. First, it is clear that the $\mathbf{i}$ and $\mathbf{i} \times \mathbf{b}$ terms in this vector part vanish as $\rho \to 0$. For the $\mathbf{b}$ term, we use (36) to write ξ in terms of ρ as

$$\xi \,=\, \pm\sqrt{\rho^2 + (|\mathbf{b}|^2 + b_x^2)\rho}\,,$$

and we then have

$$\pm \lim_{\rho\to 0} \frac{b_x \xi\, \mathbf{b}}{2\sqrt{\rho}\,(\rho + |\mathbf{b}|^2)} \;=\; \pm \lim_{\rho\to 0} \frac{b_x \sqrt{\rho + |\mathbf{b}|^2 + b_x^2}\; \mathbf{b}}{2(\rho + |\mathbf{b}|^2)} \;=\; \pm \frac{b_x \sqrt{|\mathbf{b}|^2 + b_x^2}\; \mathbf{b}}{2\,|\mathbf{b}|^2}\,.$$

Hence, the generic roots (38) converge to the singular roots (37) as $\rho \to 0$ (and hence $\xi \to 0$). ∎

Lemma 7. *For each* ξ *value, the roots* (38) *scale linearly with the quaternion coefficient* $\mathcal{B} = (b, \mathbf{b})$*.*

Proof : We invoke the parameter transformation $\xi \to \psi$ defined by

$$\xi \;=\; \tfrac{1}{2}\,(|\mathbf{b}|^2 + b_x^2)\,\tan\psi\,, \tag{39}$$

specifying a one–to–one map between $\xi \in (-\infty, +\infty)$ and $\psi \in (-\frac{1}{2}\pi, +\frac{1}{2}\pi)$. Then from (36) we have

$$\rho \;=\; \tfrac{1}{2}\,(|\mathbf{b}|^2 + b_x^2)\,(\sec\psi - 1)\,. \tag{40}$$

Hence $\xi \to \lambda^2\,\xi$ and $\rho \to \lambda^2\rho$ for each ψ when $\mathcal{B} = (b, \mathbf{b}) \to \lambda\,\mathcal{B} = (\lambda b, \lambda\mathbf{b})$ and we see that the roots (38) then scale as $\mathcal{Q} \to \lambda\,\mathcal{Q}$. ∎

By Lemma 7, a particular scaling can be imposed on $\mathcal{B} = (b, \mathbf{b})$ without altering the roots of equation (1) in an essential manner. For simplicity, we assume henceforth that[4] $|\mathbf{b}| = 1$, i.e.,

$$b_x^2 + b_y^2 + b_z^2 \;=\; 1\,. \tag{41}$$

[4]Recall from Remark 5 that we require $\mathbf{b} \neq \mathbf{0}$ for a true space curve.

Now setting $\tau = \tan \frac{1}{2}\psi \in [-1, +1]$ we have

$$\tan\psi \;=\; \frac{2\,\tau}{1-\tau^2} \qquad \text{and} \qquad \sec\psi \;=\; \frac{1+\tau^2}{1-\tau^2}\,, \tag{42}$$

and we note that $\rho = \xi\,\tau$. Using (39)–(42), the scalar part of (38) can then be written as

$$q \;=\; \frac{1}{2}\left(-\,b \pm |\tau|\,\sqrt{\frac{1+b_x^2}{1-\tau^2}}\,\right), \tag{43}$$

while the vector part reduces to

$$\begin{aligned} \mathbf{q} \;=\; & \frac{-\,b_x(b_x^2+\tau^2)\,\mathbf{i} - (1+b_x^2)\,[\,(b_y-b_z\tau)\,\mathbf{j} + (b_z+b_y\tau)\,\mathbf{k}\,]}{2\,(1+b_x^2\tau^2)} \\ & \mp\;\mathrm{sign}(\tau)\sqrt{\frac{1+b_x^2}{1-\tau^2}}\;\frac{(b_x^2+\tau^2)\,\mathbf{i} + b_x(1-\tau^2)\,[\,(b_y-b_z\tau)\,\mathbf{j} + (b_z+b_y\tau)\,\mathbf{k}\,]}{2\,(1+b_x^2\tau^2)}\,. \end{aligned} \tag{44}$$

Note that $\mathbf{q}$ does not depend on b. When $\tau = 0$, expressions (43)–(44) agree with the singular roots (37) under the assumption $|\mathbf{b}| = 1$. As $\tau \to \pm 1$, on the other hand, $q \to \pm\infty$ and $\mathbf{q}$ increases without bound in the direction $\pm\mathbf{i}$. The preceding results may be summarized as follows.

Proposition 3. *The quadratic quaternion polynomials that generate quintic RRMF curves are characterized by roots of the form $\mathcal{Q} = \lambda(q, \mathbf{q})$ where $\lambda > 0$ is a scale factor, while q and $\mathbf{q}$ depend on a real value b, a unit vector $\mathbf{b}$, and a real parameter $\tau \in [-1, +1]$ through expressions* (43)–(44)*.*

The above arguments are illustrated by means of the following example.

Example 5. With the choices

$$b \;=\; -1\,, \qquad \mathbf{b} \;=\; \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}\,, \qquad \xi \;=\; 1\,,$$

Proposition 1 gives, for a quintic RRMF curve,

$$c \;=\; 0 \qquad \text{and} \qquad \mathbf{c} \;=\; \frac{\sqrt{2}\,\mathbf{i}-\mathbf{j}-\mathbf{k}}{2\sqrt{2}}\,.$$

With $\mathcal{A} = (1, \mathbf{0})$, $\mathcal{B} = (b, \mathbf{b})$, $\mathcal{C} = (c, \mathbf{c})$, the polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ defined by (25) has the components

$$u(t) \;=\; t^2 - t\,, \quad v(t) \;=\; \frac{1}{2}\,, \quad p(t) \;=\; \frac{2\,t-1}{2\sqrt{2}}\,, \quad q(t) \;=\; \frac{2\,t-1}{2\sqrt{2}}\,,$$

and generates the Pythagorean hodograph

$$x'(t) = t^4 - 2\,t^3 + t\,, \quad y'(t) = \frac{4\,t^3 - 6\,t^2 + 4\,t - 1}{2\sqrt{2}}\,, \quad z'(t) = \frac{-\,4\,t^3 + 6\,t^2 - 1}{2\sqrt{2}}$$

which satisfies (20) with $\sigma(t) = t^4 - 2\,t^3 + 2\,t^2 - t + \frac{1}{2}$, and (22) with $a(t) = t^2 - t + \frac{1}{2}$, $b(t) = \frac{1}{2}$. Since $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' \not\equiv 0$, the resulting RRMF quintic is a true space curve. From (39) and (42) we obtain

$$\tan\psi \,=\, 2 \qquad \text{and} \qquad \tau \,=\, \frac{\sqrt{5}-1}{2}\,,$$

so the scalar and vector parts of the roots $\mathcal{Q} = (q, \mathbf{q})$ become

$$q \,=\, \frac{\sqrt{2} \pm \sqrt{\sqrt{5}-1}}{2\sqrt{2}}\,, \quad \mathbf{q} \,=\, \frac{\mp\,(\sqrt{5}-1)^{3/2}\,\mathbf{i} - (3-\sqrt{5})\,\mathbf{j} - (1+\sqrt{5})\,\mathbf{k}}{4\sqrt{2}}\,,$$

and one can verify that these roots satisfy (1) with

$$\mathcal{B} \,=\, \left(-1, \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}\right) \qquad \text{and} \qquad \mathcal{C} \,=\, \left(0, \frac{\sqrt{2}\,\mathbf{i}-\mathbf{j}-\mathbf{k}}{2\sqrt{2}}\right).$$

Figure 1 illustrates the quintic RRMF curve $\mathbf{r}(t)$ constructed in this manner, together with the normal–plane vectors for the Frenet frame and the rational RMF (the curve tangent is common to both frames).

6 Closure

A simple algorithm to determine the (right) roots of a quadratic quaternion polynomial has been developed, based on the scalar–vector representation of quaternions. Widespread familiarity with the vector dot and cross products makes the algorithm easy to understand and implement. Special cases (such as double roots, or distinct roots with coincident scalar parts) are identified and appropriately handled through simple branch conditions. The algorithm is robust and computationally efficient, and the scalar–vector description of the roots offers better geometrical insight into their structure.

As an alternative to existing coefficient constraints [7], the algorithm was employed to characterize the quadratic quaternion polynomials that generate

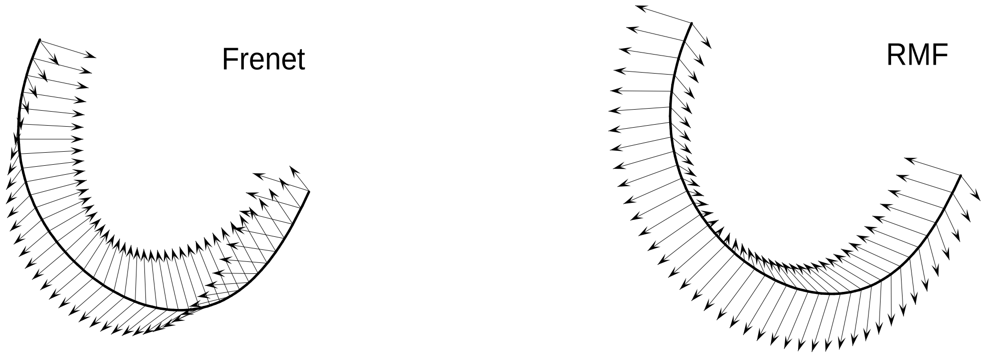

Figure 1: The quintic RRMF curve $\mathbf{r}(t)$ constructed in Example 1, comparing the variation of the normal–plane vectors for the Frenet frame (left) and the rational rotation–minimizing frame (right) over the parameter domain $[\,0,1\,]$.

quintic curves with rational rotation–minimizing frames (RRMF curves), in terms of their root structure. Trivial (linear or planar) quintic RRMF curves correspond to polynomials with a double root. For polynomials with distinct roots, a closed–form description of the roots in terms of uniform scale factor, a quaternion with unit vector part, and a parameter $\tau \in [\,-1,+1\,]$ was derived. The five degrees of freedom embodied in the roots may prove useful in terms of developing new methods to construct quintic RRMF curves that satisfy geometrical constraints, such as rigid–body motion interpolants [9].

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