OSM Lab 2017: Math Pset 5

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Problems from the Book

7.1

We will prove that if S is a non-empty subset of V, then $conv(S)$ is convex.

Consider $z, y \in conv(S)$. Then we know by definition that $z = \sum_{i=1}^n \lambda_{i,z} x_i$ and $y = \sum_{i=1}^n \lambda_{i,y} x_i$. Now consider:

$$\begin{aligned}\lambda z + (1-\lambda)y &= \lambda \sum_{i=1}^n \lambda_{i,z} + (1-\lambda)\sum_{i=1}^n \lambda_i x_{i,y} \\ &= \sum_{i=1}^n (\lambda\lambda_{i,z} + (1-\lambda)\lambda_{i,y})x_i\end{aligned}$$

Now since $\sum_{i=1}^n \lambda\lambda_{i,z} + (1-\lambda)\lambda_{i,y} = 1$, it follows that $\lambda z + (1-\lambda)y$ is in $conv(S)$. As this holds for every z, y and any $\lambda \in [0,1]$, it follows that $conv(S)$ is convex.

7.2

(i) Consider $x, y \in P$, where P is a hyperplane defined by a, b. Now:

$$\langle z = \lambda x + (1-\lambda)y, a\rangle = \lambda\langle x, a\rangle + (1-\lambda)\langle x, b\rangle = b$$

So $z \in P$, and therefore P is convex.

(ii) Consider $x, y, \in H$, where H is again the half space defined by a, b. Now:

$$\langle z = \lambda x + (1-\lambda)y, a\rangle = \lambda\langle x, a\rangle + (1-\lambda)\langle y, a\rangle \leq b$$

So $z \in H$, and therefore H is convex.

7.4

We will use parts (i) - (iv) to prove hte following theorem. Let $C \subset \mathbb{R}^n$ be nonempty, closed and convex. A point $\mathbf{p} \in C$ is the projection of $\mathbf{x}$ onto C iff

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y}\rangle \geq 0 \quad \forall \mathbf{y} \in C$$

(i)

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{p}-\mathbf{y}\rangle + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \quad \text{From the bilinearity and symmetry of inner products} \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle\end{aligned}$$

(ii) From (i), we know that if $2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \geq 0$, then

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &\geq \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 \\ &> \|\mathbf{x}-\mathbf{p}\|^2 \quad \text{Since } \|\mathbf{p}-\mathbf{y}\|^2 > 0 \text{ when } \mathbf{y} \neq \mathbf{p}\end{aligned}$$

This is ($\Rightarrow$)

(iii) If $\mathbf{z} = \lambda\mathbf{y} + (1-\lambda)\mathbf{p}$, $\lambda \in [0,1]$, then

$$\begin{aligned}
\|\mathbf{x}-\mathbf{z}\|^2 &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{z}\|^2 + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{z}\rangle \\
&\text{Substituting } \mathbf{z} = \lambda\mathbf{y} + (1-\lambda)\mathbf{p} \\
&= \|\mathbf{x}-\mathbf{p}\|^2 + \lambda^2\|\mathbf{y}-\mathbf{p}\|^2 + 2\lambda\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle
\end{aligned}$$

(iv) Now let $\mathbf{p}$ be a projection of $\mathbf{x}$ onto the convex set C. Since C is convex, for any $\mathbf{y} \in C$, we can define $\mathbf{z} = \lambda\mathbf{y} + (1-\lambda)\mathbf{p} \in C$.

Now since $\mathbf{p}$ is a projection, we know that

$$\begin{aligned}
\|\mathbf{x}-\mathbf{p}\| &\leq \|\mathbf{x}-\mathbf{z}\| \\
\|0 &\leq \mathbf{x}-\mathbf{z}\|^2 - \|\mathbf{x}-\mathbf{p}\|^2 \\
\text{From (iii)} \\
&= \lambda^2\|\mathbf{y}-\mathbf{p}\|^2 + 2\lambda\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \\
&= \lambda\|\mathbf{y}-\mathbf{p}\|^2 + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle
\end{aligned}$$

Now since $\lambda\|\mathbf{y}-\mathbf{p}\|^2 > 0$, it follows that $\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle > 0$, and so we have ($\Leftarrow$).

7.6

We will prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then the set $S = \{\mathbf{x}\mathbb{R}^n | f(\mathbf{x}) \leq c\} \subset \mathbb{R}^n$ is a convex set.

Proof. Consider any $\mathbf{x}_1, \mathbf{x}_2$ in S, and any $\lambda \in [0,1]$. Now:

$$f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda)f(\mathbf{x}_2) \leq c$$

So clearly $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ is in S, and so S is convex. □

7.7

We will prove that for any convex set C, and convex functions $f_1, ..., f_k$ taking C to $\mathbb{R}$, and for any $\lambda_1, ..., \lambda_k \geq 0$, the function

$$f(\mathbf{x}) = \sum_{i=1}^{k} \lambda_i f_i(\mathbf{x})$$

is convex.

Proof. Consider any $\mathbf{x}_1, \mathbf{x}_2$ in C, $\lambda \in [0,1]$.

$$\begin{aligned}
\lambda f(\mathbf{x}_1) + (1-\lambda)f(\mathbf{x}_2) &= \lambda\sum_{i=1}^{k} \lambda_i f_i(\mathbf{x}_1) + (1-\lambda)\sum_{i=1}^{k} \lambda_i f_i(\mathbf{x}_2) \\
&= \sum_{i=1}^{k} \lambda_i(\lambda f_i(\mathbf{x}_1) + (1-\lambda)f_i(\mathbf{x}_2)) \\
&\geq \sum_{i=1}^{k} \lambda_i(f_i(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2)) \\
&= f(\lambda\mathbf{x}_1 + (1-\lambda)(\mathbf{x}_2))
\end{aligned}$$

And so f is a convex function. □

7.13

Lets assume $f(x) < M$ and f is convex. Now consider $x, y \in \mathbb{R}^n$. We will prove this by contradiction. If f were not constant, then there exists some x, y such that $f(x) > f(y)$.

Then, $f(x) \leq \lambda(f(\lambda x + (1-\lambda)y)) + (1-\lambda(f(y)))$. But since $f(x) > f(y)$,we have $f(x) - f(y) + \lambda f(y) \leq \lambda(f(\lambda x + (1-\lambda)y))$. Now note that as $\lambda \to \infty$, this implies that we have $\lim_{\lambda\to\infty} f(\lambda x + (1-\lambda y)) \geq \lim_{\lambda\to\infty} \frac{f(x)-f(y)}{\lambda} + f(y) = \infty$.

However, this implies that f is unbounded, a contradiction.

7.20

We will prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $-f$ is also convex, then f is affine.

Consider $x, y \in \mathbb{R}^n$.

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda)f(y) \quad \text{Since } f \text{ is convex} \\ f(\lambda x + (1-\lambda)y) &\geq \lambda f(x) + (1-\lambda)f(y) \quad \text{Since } -f \text{ is convex} \\ \implies f(\lambda x + (1-\lambda)y) &= \lambda f(x) + (1-\lambda)f(y) \end{aligned}$$

Now clearly f is a linear transformation, and so our function f is an affine function with $L = f$ and $c = 0$.

7.21

We will show that if $D \subset \mathbb{R}$ with $f : \mathbb{R}^n \to D$, and if $\phi : D \to \mathbb{R}$ is a strictly increasing function, then $\mathbf{x}^*$ is a local minimizer for $\phi \circ f(\mathbf{x})$ subject to constraints $GandH$ if and only if $\mathbf{x}^*$ is a local minimizer for the $f(\mathbf{x})$ subject to constraints $GandH$

$\Leftarrow$

Now if $\mathbf{x}^*$ is a local minimizer of $\phi \circ f$, since ϕ is strictly increasing, this implies that for all $\mathbf{x}$ fulfilling our constraints, $f(\mathbf{x}) \geq f(\mathbf{x}^*)$, and so $\mathbf{x}^*$ is a local minimizer of f subject to the same constraints.

$\Rightarrow$

Now if $\mathbf{x}^*$ is a local minimizer of f, then it follows that for all $\mathbf{x}$ subject to our constraints, $f(\mathbf{x}) \geq f(\mathbf{x}^*)$. Now since ϕ is strictly increasing, it follows that $\phi \circ f(\mathbf{x}) \geq \phi \circ f(\mathbf{x}^*)$ for all $\mathbf{x}$ subject to our constraints, and so it follows that $\mathbf{x}^*$ is a local minimizer of $\phi \circ f$ subject to our constraints.