OSM Lab 2017: Math Pset 5

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Problems from the Book

7.1

We will prove that if S is a non-empty subset of V, then $conv(S)$ is convex.

Consider $z, y \in conv(S)$. Then we know by definition that $z = \sum_{i=1}^n \lambda_{i,z} x_i$ and $y = \sum_{i=1}^n \lambda_{i,y} x_i$. Now consider:

$$\begin{aligned}\lambda z + (1-\lambda)y &= \lambda \sum_{i=1}^n \lambda_{i,z} + (1-\lambda)\sum_{i=1}^n \lambda_i x_{i,y} \\ &= \sum_{i=1}^n (\lambda\lambda_{i,z} + (1-\lambda)\lambda_{i,y})x_i\end{aligned}$$

Now since $\sum_{i=1}^n \lambda\lambda_{i,z} + (1-\lambda)\lambda_{i,y} = 1$, it follows that $\lambda z + (1-\lambda)y$ is in $conv(S)$. As this holds for every z, y and any $\lambda \in [0,1]$, it follows that $conv(S)$ is convex.

7.2

(i) Consider $x, y \in P$, where P is a hyperplane defined by a, b. Now:

$$\langle z = \lambda x + (1-\lambda)y, a\rangle = \lambda\langle x, a\rangle + (1-\lambda)\langle x, b\rangle = b$$

So $z \in P$, and therefore P is convex.

(ii) Consider $x, y, \in H$, where H is again the half space defined by a, b. Now:

$$\langle z = \lambda x + (1-\lambda)y, a\rangle = \lambda\langle x, a\rangle + (1-\lambda)\langle y, a\rangle \leq b$$

So $z \in H$, and therefore H is convex.

7.4

We will use parts (i) - (iv) to prove hte following theorem. Let $C \subset \mathbb{R}^n$ be nonempty, closed and convex. A point $\mathbf{p} \in C$ is the projection of $\mathbf{x}$ onto C iff

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y}\rangle \geq 0 \quad \forall \mathbf{y} \in C$$

(i)

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{p}-\mathbf{y}\rangle + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \quad \text{From the bilinearity and symmetry of inner products} \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle\end{aligned}$$

(ii) From (i), we know that if $2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \geq 0$, then

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &\geq \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 \\ &> \|\mathbf{x}-\mathbf{p}\|^2 \quad \text{Since } \|\mathbf{p}-\mathbf{y}\|^2 > 0 \text{ when } \mathbf{y} \neq \mathbf{p}\end{aligned}$$

This is $(\Rightarrow)$

(iii) If $\mathbf{z} = \lambda\mathbf{y} + (1-\lambda)\mathbf{p}$, $\lambda \in [0,1]$, then

$$\begin{aligned}
\|\mathbf{x}-\mathbf{z}\|^2 &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{z}\|^2 + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{z}\rangle \\
&\text{Substituting } \mathbf{z} = \lambda\mathbf{y} + (1-\lambda)\mathbf{p} \\
&= \|\mathbf{x}-\mathbf{p}\|^2 + \lambda^2\|\mathbf{y}-\mathbf{p}\|^2 + 2\lambda\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle
\end{aligned}$$

(iv) Now let $\mathbf{p}$ be a projection of $\mathbf{x}$ onto the convex set C. Since C is convex, for any $\mathbf{y} \in C$, we can define $\mathbf{z} = \lambda\mathbf{y} + (1-\lambda)\mathbf{p} \in C$.

Now since $\mathbf{p}$ is a projection, we know that

$$\begin{aligned}
\|\mathbf{x}-\mathbf{p}\| &\leq \|\mathbf{x}-\mathbf{z}\| \\
\|0 &\leq \mathbf{x}-\mathbf{z}\|^2 - \|\mathbf{x}-\mathbf{p}\|^2 \\
\text{From (iii)} \\
&= \lambda^2\|\mathbf{y}-\mathbf{p}\|^2 + 2\lambda\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \\
&= \lambda\|\mathbf{y}-\mathbf{p}\|^2 + 2\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle
\end{aligned}$$

Now since $\lambda\|\mathbf{y}-\mathbf{p}\|^2 > 0$, it follows that $\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle > 0$, and so we have $(\Leftarrow)$.