OSM Lab 2017: Math Pset 4

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Problems from the Book

6.1

Let us define the following functions.

$$\begin{aligned} f(\mathbf{w}) &= e^{-\mathbf{w}^T\mathbf{x}} \\ G(\mathbf{w}) &= \mathbf{w}^T\mathbf{x} - \mathbf{w}^T A\mathbf{w} + \mathbf{w}^T A\mathbf{y} \\ H(\mathbf{w}) &= \mathbf{y}^T\mathbf{w} - \mathbf{w}^T\mathbf{x} \end{aligned}$$

Now we can write our optimization problem in the usual form:

$$\begin{aligned} \min \quad & -f(\mathbf{w}) \\ s.t. \quad & -G(\mathbf{w}) \leq a \\ & H(\mathbf{w}) = b \end{aligned}$$

6.5

We know that profits can be defined as $0.07x + 0.05y$, where x is the number of milk bottles, and y is the number of plastic knobs. We also have the following constraints on plastic ($4x + 3y \leq 240$) and on labor ($2x + y \leq 100$). We can write these constraints into a matrix:

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

We can then write our problem into the standard form of an optimization problem:

$$\begin{aligned} \min \quad & -0.07x - 0.05y \\ s.t. \quad & A\begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 240 \\ 100 \end{bmatrix} \end{aligned}$$

5.5

Consider the function

$$f(x, y) = 3x^2y + 4xy^2 + xy$$

We can take the partial derivatives and construct the Hessian

$$\begin{aligned} D[f(x,y)] &= \begin{bmatrix} 6xy + 4y^2 + y & 3x^2 + 8xy + x \end{bmatrix} \\ D^2[f(x,y)] &= \begin{bmatrix} 6y & 8y + 6x + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix} \end{aligned}$$

Solving the equation for $D[f(x,y)] = 0$ gives the following pairs of points, $(0,0), (0, -\frac{1}{4}), (-\frac{1}{3}, 0), (-\frac{1}{9}, -\frac{1}{12})$. We can then evaluate the Hessian to decide if these points are local maxima, minima or saddle points.

$$
\begin{aligned}
D^2[f(0,0)] &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
D^2[f(0, \frac{-1}{4})] &= \begin{bmatrix} -3/2 & -1 \\ -1 & 0 \end{bmatrix} \\
D^2[f(\frac{-1}{3}, 0)] &= \begin{bmatrix} 0 & -1 \\ -1 & -8/3 \end{bmatrix} \\
D^2[f(\frac{-1}{9}, \frac{-1}{12})] &= \begin{bmatrix} -1/2 & -1/3 \\ -1/3 & -2/3 \end{bmatrix}
\end{aligned}
$$

Looking at the Hessian, we see that (0, -1/4) and (-1/9, -1/12) are both local maxima, since their Hessian is negative definite. In addition, we note that (0,0) and (-1/3, 0) are saddle points.

6.11

Consider $f(x) = ax^2 + bx + c$, $a > 0, b, c \in mathbbR$. Note that the unique minimizer of f is given by $x = -\frac{b}{2a}$. We can obtain this by solving the first order conditions and noting that $f''(x) > 0$.

Now consider the Newton iteration starting at any x_0.

$$
\begin{aligned}
x_1 &= x_0 - \frac{f'(x_0)}{f''(x_0)} \\
&= x_0 - \frac{2ax_0 + b}{2a} \\
&= \frac{-b}{2a}
\end{aligned}
$$