Interest Rate Models

10. Mortgage Backed Securities

Andrew Lesniewski

Baruch College
New York

Spring 2017

Outline

Mortgage loans

- *Mortgage backed securities* (MBS) are fixed income instruments collateralized by mortgage loans.
- A *mortgage loan* (or simply a mortgage) is a loan extended to an individual or a corporation with the purpose of financing the purchase of real estate. Two major categories of mortgages are:
  (i) residential,
  (ii) commercial.
- We shall focus on RMBSs, i.e. MBSs backed by residential mortgages. Depending on the size of the loan and credit worthiness of the borrower, residential mortgages fall into two broad categories:
  (i) conforming (or agency),
  (ii) non-agency.

Mortgage loans

- A fixed coupon mortgage is a loan that carries an annual coupon $\widehat{C}$ (say 4.50%), and matures in N months (usually $N = 360$, i.e. 30 years). Denote:

$$c = \frac{\widehat{C}}{12},$$
$$d = \frac{1}{1+c}.$$

- Typically, the principal repayment is amortized over the life of the loan[1]. Specifically, assuming the principal of \$1, the amortization schedule is given by the following set of rules.
- Scheduled monthly payment is

$$m = \frac{c}{1-d^N}.$$

[1] Mortgage loans that repay at maturity are called “balloons”.

Mortgage backed securities

- The principal repayment portion p_j of m for month j is

$$p_j = \frac{cd^{N-j+1}}{1-d^N}\,.$$

- The interest portion i_j of m for month j is

$$i_j = \frac{c\left(1-d^{N-j+1}\right)}{1-d^N}\,.$$

- Balance B_j outstanding at the end of month j is

$$B_j = \frac{1-d^{N-j}}{1-d^N}\,.$$

- The amortization schedule is defined so that the following property holds:

$$i_j = cB_{j-1}\,.$$

Mortgage backed securities

- *Agency pass-throughs* are the basic MBSs: all principal repayments and interest (less servicing and credit spread) of the underlying pool of collateral are paid to the holder.
- *TBA* (to be announced) pass-throughs, or simply TBAs.
- *Mortgage options* = options on TBAs.
- *Structured* MBSs such as IOs, POs, CMOs: cash flows are carved out from the cash flows of the underlying pool of collateral.
- *Constant maturity mortgage* (CMM) products.
- ...

TBAs

- A TBA is a futures contract on a pool of conventional, fixed coupon mortgage loans.
- It carries a coupon C reflective of the coupons on the deliverable loans. The values of C are spaced in 50 bp increments: 3.5%, 4.0%, 4.5%, etc.
- There is a standard delivery date, the *PSA date*, in each month. A vast majority of the trading takes place in either the nearest or the once-deferred month, but the market quotes prices for three or four TBAs settling on the next PSA dates.
- A party long a contract at settlement takes the delivery of a pool of mortgage loans satisfying the good delivery guidelines.

- For example, here is a snapshot of the TBA market on June 2, 2010:

Cpn \ PSA	Jun 2010	Jul 2010	Aug 2010
3.5	95.594	95.211	94.773
4.0	99.125	98.789	98.484
4.5	102.086	101.695	101.359
5.0	104.852	104.430	104.047
5.5	106.906	106.500	106.141
6.0	107.930	107.609	107.328
6.5	108.953	108.656	108.375

- We let $P_C(T)$ denote the price of the TBA with coupon C and settlement date T.

Mortgage options

- Mortgage options are European calls and puts on TBAs.
- They expire one calendar week before the PSA day on the underlying TBA.
- The strikes on the TBA options are standardized: $\mathrm{ATM} \equiv P_C(T)$, $\mathrm{ATM} \pm 1/2$ pt, $\mathrm{ATM} \pm 1$ pt.
- Here is a market snapshot taken on June 2, 2010:

June 2010 expirations

Cpn \ Strike	-1.0	-0.5	ATM	+0.5	+1.0
4.0	0.0326	0.1120	0.2904	0.1016	0.0247
4.5	0.0143	0.0651	0.2279	0.0547	0.0052
5.0	0.0059	0.0299	0.1667	0.0195	N/A
5.5	N/A	0.0104	0.1185	0.0039	N/A

Mortgage options

July 2010 expirations

Cpn \ Strike	-1.0	-0.5	ATM	+0.5	+1.0
4.0	0.4141	0.5755	0.7839	0.5443	0.3555
4.5	0.2852	0.4245	0.6237	0.3828	0.2096
5.0	0.1641	0.2760	0.4570	0.2227	0.0833
5.5	0.0833	0.1628	0.3216	0.1081	0.0195

August 2010 expirations

Cpn \ Strike	-1.0	-0.5	ATM	+0.5	+1.0
4.0	0.6992	0.8737	1.0820	0.8307	0.6133
4.5	0.5078	0.6628	0.8607	0.6029	0.3958
5.0	0.3255	0.4531	0.6341	0.3789	0.1914
5.5	0.1940	0.2930	0.4518	0.2083	0.0677

CMM rates

- A CMM rate is an index representing the yield on a hypothetical TBA pricing at par.
- Spot CMM rate is calculated by interpolation:
    (i) For each coupon C, synthesize a TBA which settles in $T = 30$ days. Its price $P_C(T)$ is defined as the linear interpolation of the two bracketing instruments.
    (i) Synthesize a par TBA with settlement 30 calendar days from today by linearly interpolating the prices $P_C(T)$. Its coupon, expressed in terms of the bracketing coupons C_1 and C_2 (with $C_1 < C_2$),

$$M = w_1 C_1 + w_2 C_2,$$

    is the spot CMM rate.
- Forward CMM rates are quoted in the CMM markets (forward rate agreements and swaps).

CMM rates

- A *CMM FRA* is structured as follows:
  - (i) The counterparties agree on the contract rate K.
  - (ii) The spot CMM rate M is fixed two business days before the start date T. The net rate $M - K$ is applied to the notional over the accrual period $[T, T_{\mathrm{pay}}]$.
  - (iii) The payment of the net amount is made on T_{pay}.
- A *CMM swap* is a multi-period version of a CMM FRA.
- The break even value of K is called the *CMM rate* $M_0(T)$.
- We let $\mathcal{M}_0$ denote the curve $[0, \infty) \ni T \to M_0(T)$, the *CMM curve* (or mortgage rate curve).
- Generally, there is a good deal of liquidity in the CMM markets for T out to about a year, and it is possible to get quotes for settlements further out.

Modeling framework

- The primary source of risk of MBSs is the interest rates risk modeled by a term structure model.
- The underlying interest rates dynamics is modeled by a term structure model whose state variables are denoted by $X_1(t), \ldots, X_k(t)$. Underlying this dynamics is a probability space $\big(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathrm{P}\big)$ with a (multi-dimensional) Wiener process driving the interest rates dynamics.
- The model describes the evolution of the forward curve $\mathcal{C}(t)$. Its detailed specification will play no role in the following, it can be any model such as the Hull-White model, LMM, SABR/LMM, ... It is of practical importance that the model is properly specified and capable of accurate calibration to the market data.

Modeling framework

- We assume the following about the model.
  (i) The initial value $X_1(0), \ldots, X_k(0)$ of the dynamics is given by the current LIBOR forward curve $\mathcal{C}_0$. Here, $\mathcal{C}_0$ is defined in terms of a smooth (say, twice continuously differentiable) function $f_0 : \mathbb{R}_+ \to \mathbb{R}$, representing the instantaneous forward rate. The rates dynamics depends smoothly on $\mathcal{C}_0$.
  (ii) The current interest rates volatility $\mathcal{V}_0$, represented as a smooth surface $\sigma_0 : \mathbb{R}_+^2 \to \mathbb{R}$, is mapped smoothly on the dynamics of the interest rates process. Ideally, in order to accurately take into account the volatility smile, one should represent $\mathcal{V}_0$ as a three dimensional object $\sigma_0 : \mathbb{R}_+^3 \to \mathbb{R}$. This mapping is such that the model prices correctly relevant benchmark interest rates options.

Modeling framework

- A key feature of MBSs is a variety of event risks embedded in the collateral. In particular, a borrower has the right to *prepay* the loan.
- This prepayment risk is modeled by a random time $\mathcal{T}$, the *time to event*. $\mathcal{T}$ is not necessarily a stopping time with respect to $(\mathscr{F}_t)_{t\geq 0}$. A borrower's decision to prepay is driven by factors which are partly exogenous to the interest rates.
- The probability of prepayment depends only on the information up to time t,

$$\mathsf{P}\left(\mathcal{T} > t \,|\, \mathscr{F}_\infty\right) = \mathsf{P}\left(\mathcal{T} > t \,|\, \mathscr{F}_t\right).$$

There exists a doubly stochastic Cox process $\lambda\left(t\right)$ such that

$$\mathsf{P}\left(\mathcal{T} > t \,|\, \mathscr{F}_t\right) = \exp\left(-\int_0^t \lambda\left(s\right) ds\right).$$

Modeling framework

- The intensity process $\lambda(t)$ represents the conditional prepayment probability density. The quantity $S(0,t) = \exp\left(-\int_0^t \lambda(s)\, ds\right)$ is the prepayment survival probability. Note that $S(0,t)$ itself is random.
- Key among the factors affecting $\lambda(t)$ is the mortgage rate curve $\mathcal{M}(t)$.
- Mortgage rate process is modeled as a diffusion defined on a suitable probability space $\left(\Omega_1, (\mathscr{G}_t)_{t\geq 0}, \mathrm{P}_1\right)$. The prepayment intensity $\lambda(t)$ depends exogenously on the mortgage rate curve at time t:

$$\lambda(t) = \lambda(t, \mathcal{M}(t)).$$

Modeling framework

- Prepayment behavior shows the following characteristics:
  - (i) The intensity of prepayments λ increases, as the mortgage rate M goes down, and decreases, as it goes up.
  - (ii) If C is the coupon on the existing loan, λ approaches finite saturation levels λ_0 and λ_1, as $C - M \to \infty$ and $C - M \to -\infty$, respectively.
- This pattern of prepayments is fairly well captured by the logistic function (the "S-curve") model of the prepayment intensity:

$$\lambda(t) = \lambda_0 + (\lambda_1 - \lambda_0)\ \frac{1}{1 + \gamma e^{\kappa(M(t) - C)}}\ .$$

  Here, $0 < \lambda_0 < \lambda_1$, and $\gamma, \kappa > 0$.
- Later in this lecture we will discuss more sophisticated prepayment models.

Valuation of a TBA

- The holder of a TBA is short an American call: the cash flows on a mortgage are uncertain because of the borrower's right to prepay. For calculation of the TBA's PV, the cash flows should be discounted by the survival probability S.
- Let T denote the TBA's settlement date, and let $T_1, \ldots, T_N$ denote the payment dates. The scheduled cash flow on the date T_j is denoted by C_j, and B_j denotes the outstanding balance. The price of a TBA is then given by[2]

$$\begin{aligned} P(T) = \mathrm{E}\Big[\sum\nolimits_j Z(T, T_j) \\ \times \Big(S(T, T_{j-1})\, C_j + (S(T, T_{j-1}) - S(T, T_j))\, B_j \Big) \Big]. \end{aligned}$$

- Here E is the expected value under the risk neutral measure, and $Z(T, T_j)$ denotes the discount factor.

[2] For simplicity, we assume here that the TBA is brand new. For seasoned pools, it is necessary to take into account the *factor*, which accounts for the already repaid fraction of the principal.

Valuation of a TBA

- Alternatively, we can write this as

$$P(T) = \mathrm{E}\Big[\sum\nolimits_j Z^{\pi}(T, T_j)\left(C_j + \overline{\lambda}(T_{j-1}, T_j)\, B_j\right)\Big].$$

- Here, $Z^{\pi}(T, T_j) = Z(T, T_j)\, S(T, T_{j-1})$ is the *prepaying discount factor*, and

$$\begin{aligned}\overline{\lambda}(T_{j-1}, T_j) &= \frac{S(T, T_{j-1}) - S(T, T_j)}{S(T, T_{j-1})}\\ &\simeq \int_{T_{j-1}}^{T_j} \lambda(t)\, dt\end{aligned}$$

  is the conditional probability of prepayment in month j, also known as the single month mortality (SMM).

Valuation of a TBA

- The scheduled cash flow on a TBA is $C_j = p_j + i_j^{\mathrm{net}}$, where p_j is the scheduled principal repayment, and i_j^{net} is the interest less the servicing and credit spread F,
$$i_j^{\mathrm{net}} = \frac{\widehat{C} - F}{12}\, B_{j-1}.$$
- The rate $C = \widehat{C} - F$ is the *net coupon*. We can thus recast the valuation formula as:
$$P(T) = \mathrm{PO}(T) + \mathrm{IO}(T).$$

Valuation of a TBA

- The first term on the RHS is the principal only (PO),

$$\mathrm{PO}\,(T) = \mathsf{E}\Big[\sum\nolimits_j Z^{\pi}\,(T, T_j)\,(p_j + \overline{\lambda}\,(T_{j-1}, T_j)\,B_j)\Big].$$

  Note that the cash flows consist of the scheduled and prepaid principal repayments.

- The second term is the interest only (IO),

$$\begin{aligned}\mathrm{IO}\,(T) &= \mathsf{E}\Big[\sum\nolimits_j Z^{\pi}\,(T_0, T_j)\,i_j^{\mathrm{net}}\Big]\\ &= \frac{C}{12}\,\mathsf{E}\Big[\sum\nolimits_j Z^{\pi}\,(T, T_j)\,B_{j-1}\Big].\end{aligned}$$

CMM valuation

- In order to define the forward CMM rate we will make the following completeness assumption about the TBA market: For each coupon $C > 0$ and each settlement date $T > 0$, there exists a traded TBA of that coupon and maturity.
- Since this assumption is in practice violated, TBAs for arbitrary settlements and coupons have to be created synthetically:
  (i) For settlements not exceeding the longest traded PSA date, interpolate the coupons and settlement dates.
  (ii) For settlement dates past the longest traded PSA date, we model the TBA prices based on the currently calibrated term structure and prepayment models.

CMM valuation

- From the TBA valuation formula,

$$C = \frac{P_C(T) - \mathrm{PO}_C(t)}{L^{\mathrm{IO}}(T)},$$

where

$$L_C^{\mathrm{IO}}(T) = \frac{1}{12} \sum\nolimits_j Z^{\pi}(T, T_j)\, B_{j-1}$$

is the IO annuity associated with the specified TBA.

- Let $Y(T)$ denote the coupon on the par TBA. Then

$$Y(T) = \frac{1 - \mathrm{PO}(T)}{L^{\mathrm{IO}}(T)},$$

where $L^{\mathrm{IO}}(T) = L_{Y(T)}^{\mathrm{IO}}(T)$ is the IO annuity associated with the par TBA.

CMM valuation

- As a consequence, $Y(T)$ is a tradable asset, if the IO annuity is used as a numeraire, and its dynamics is given by a martingale. The corresponding martingale measure Q^{IO} is called the *IO measure*.
- As far as we know, no well developed market for CMM swaptions currently exists. Should such a market ever come to existence, the IO measure introduced above would be the natural martingale measure for CMM swaption valuation, very much like the swap measure is the natural martingale measure for interest rate swaption valuation. For example, the price of a CMM receiver swaption struck at K would be
$$\mathrm{RecSwpt}_T = L^{\mathrm{IO}}(T)\,\mathrm{E}^{\mathrm{Q}^{\mathrm{IO}}}\left[\max\left(S(T) - K, 0\right)\right].$$

CMM valuation

- The CMM rate is the contract rate K in the definition of a CMM FRA. This leads to the following definition:

$$M(T \mid T_{\text{pay}}) = \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}}\left[Y(T_{\text{fix}} + \vartheta)\right],$$

  where ϑ is the 30 day settlement delay on the underlying TBA.
- We will be assuming that the payment on the FRA is made on the start date T, and, for simplicity, we will neglect the small convexity correction coming from the fact that the rate fixes on T_{fix} rather than on T.
- We thus define the forward CMM rate for the date T as

$$M(T) = \mathsf{E}^{\mathsf{Q}_T}\left[Y(T + \vartheta)\right].$$

Empirical Model of a TBA

- We now describe a simple empirical model of a TBA [5]. To motivate, consider a mortgage which pays a continuous stream of cash.
- By $\widehat{C}$ we denote the annual coupon on the mortgage, and assume that its term is T_m years.
  (i) The (constant) payment $dm(t)$ in $[t,\ t+dt]$ is
  $$dm(t) = \frac{\widehat{C}\, dt}{1 - e^{-\widehat{C} T_m}}\ .$$
  (ii) The outstanding balance at time t is
  $$B(t) = \frac{1 - e^{-\widehat{C}(T_m - t)}}{1 - e^{-\widehat{C} T_m}}\ .$$
  (iii) The interest portion $di(t)$ of the payment $dm(t)$ is
  $$di(t) = \widehat{C} B(t)\, dt\ .$$

Empirical Model of a TBA

- Consider now a TBA collateralized by a pool of such mortgages, and assume that :
  (i) All cash flows are discounted on a constant interest rate r.
  (ii) The prepayment intensity is time independent, $\lambda = \lambda(r, \widehat{C})$.
- The price of the TBA is given by the integral

$$P_C(T) = \int_T^{T_m} e^{-(r+\lambda)t} \big[dm(t) + (\lambda - F) B(t)\, dt \big],$$

  where F denotes the servicing and credit spread.
- This integral is closed (albeit lengthy) form. Assuming that $T = 0$, and $T_m \to \infty$,

$$P = \frac{C + \lambda}{r + \lambda}\ .$$

Empirical Model of a TBA

- Note that $P_C = 1$, if and only if $r = \widehat{C} - F \equiv C$. We expand $P = P(r)$ around $r = C$,

$$P(r) \simeq 1 - D(r - C),$$

where

$$D = -\frac{d}{dr} \log P(r)\,\Big|_{r=C}$$

is the *duration*.

- This yields:

$$D \simeq \frac{1}{r + \lambda}\ .$$

- Explicitly, we can write the duration in the form:

$$D = D_0 + (D_1 - D_0)\ \frac{1}{1 + \Gamma e^{-\kappa(r-C)}}\ .$$

Empirical Model of a TBA

- The graph below shows the duration function in the empirical model.

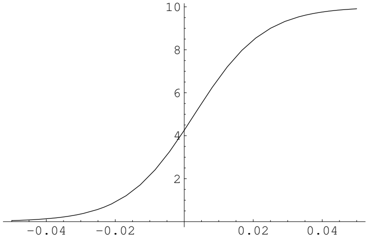

Figure: 1. The model duration function

Empirical Model of a TBA

- This leads us to the following model of a TBA.
- Let $M = M(T)$ denote the CMM rate on the TBA settlement date T. The duration of a TBA is assumed to be a function of

$$X(T) = M(T) - C.$$

- We parameterize the duration of a TBA by means of a logistic function of X:

$$\mathbf{D}(X) = L + (U - L)\ \frac{1}{1 + e^{-\kappa(X-\Delta)}}\ .$$

- Integrating $\mathbf{D}(X)$ and exponentiating the result yields the following shape of the TBA model price function:

$$\mathbf{P}(X) = \mu\ \exp\left(-\frac{(L+U)\,X}{2}\right)\ \left\{\frac{\cosh\frac{\kappa\Delta}{2}}{\cosh\frac{\kappa(X-\Delta)}{2}}\right\}^{\frac{U-L}{\kappa}}\ .$$

Empirical Model of a TBA

- The graph below shows the price function in the empirical model.

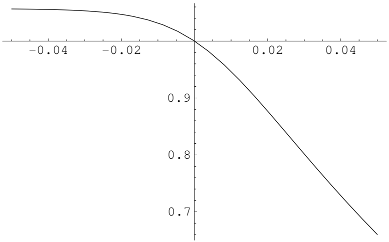

Figure: 2. The model price function

Empirical Model of a TBA

- The convexity of a TBA is given by the following function:

$$\mathbf{K}(X) = \mathbf{D}(X)^2 - \frac{\kappa(U-L)}{4}\,\frac{1}{\cosh^2\frac{\kappa(X-\Delta)}{2}}\,.$$

- The price of a TBA in the empirical model is given by

$$P_C(T) = \mathrm{P}(M(T) - C)\,.$$

  The model parameters L, U, κ and Δ and $\mu = \mathbf{P}(0)$ are determined to fit to the market.

- Note that the key assumption behind this model is that the price of a TBA depends on the forward mortgage rate for the same settlement as that of the TBA (rather than on the entire CMM curve).
- This is somewhat unsatisfying, as one might believe that prepayment behavior depends on the entire rates outlook (i.e. forward rates) rate than the spot rate only.

Empirical Model of a TBA

- The graph below shows the price function in the empirical model.

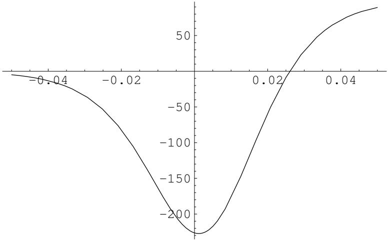

Figure: 3. The model convexity function

Valuation of Mortgage Options

- Calls and puts struck at K and expiring at T are valued according to

$$\begin{aligned}\mathrm{Call}\left(P_C, K\right) &= Z\left(T\right) \mathrm{E}^{\mathrm{Q}_T}\left[\max\left(P_C\left(T+\delta\right)-K,\ 0\right)\right],\\ \mathrm{Put}\left(P_C, K\right) &= Z\left(T\right) \mathrm{E}^{\mathrm{Q}_T}\left[\max\left(K-P_C\left(T+\delta\right),\ 0\right)\right],\end{aligned}$$

  where δ is the time lag between the option expiration and the settlement date of the underlying TBA.

- Note that:
  (i) These valuation formulas do not lead to closed form expressions.
  (ii) The expectations are evaluated by means of Monte Carlo simulations based on a term structure model, and they require substantial computational resources.

Valuation of Mortgage Options

- Since mortgage options have short expirations, it is a common practice to value them within a single rate framework.
- We assume that the CMM rate follows a normal SABR process,

$$\begin{aligned} dM(t) &= \sigma(t)\, dW(t), \\ d\sigma(t) &= \alpha\sigma(t)\, dB(t), \end{aligned}$$

  where $\mathsf{E}[dW(t)\, dB(t)] = \rho dt$, and $M(0) = M_0, \sigma(0) = \sigma_0$.
- The system has the following strong solution

$$M(t) = M_0 + \sigma_0 \int_0^t \exp\left(-\frac{\alpha^2 s}{2} + \alpha B(s)\right) dW(s),$$

  which can easily and efficiently be implemented by a Monte Carlo simulation.

Valuation of Mortgage Options

- Consider a coupon C TBA whose price process P_C is given by

$$P_C(t) = \mathbf{P}(M(t) - C),$$

where $\mathbf{P}(x)$ is the model price function.

- Let T denote the option expiration. Shifting from the IO measure to the forward measure Q_T yields

$$\frac{dP_C(t)}{P_C(t)} = \sigma(t) D_C(t) dW(t).$$

- Under Q_T,

$$\mathsf{E}^{\mathsf{Q}_T}[P_C(t)] = P_{C,0},$$

where $P_{C,0}$ denotes here the current market price.

Valuation of Mortgage Options

- Since
  (i) $P_C(t)$ is given by an explicit expression,
  (ii) the Monte Carlo paths are generated from a simple CMM dynamics rather than a term structure model,

  the computation is rapid and accurate.
- The model can easily be calibrated:
  (i) Choose the parameter μ so that the mean of the price distribution is the market price of the underlying TBA.
  (ii) Choose the parameters σ_0, α and ρ of the SABR model to match the option premia. These parameters may show a dependence on the coupon C.

Valuation of Mortgage Options

- The graph below shows the volatility smile on the FNMA 4.5 options of various expirations.

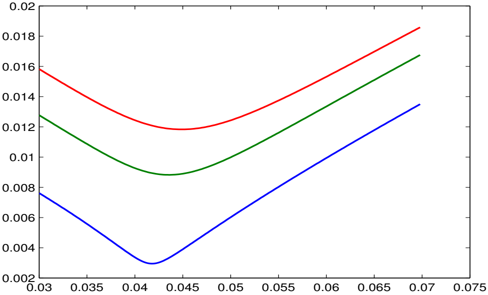

Figure: 4. CMM smile on FNMA 4.5

Valuation of Mortgage Options

- The graph below shows the volatility smile on the FNMA 5 options of various expirations.

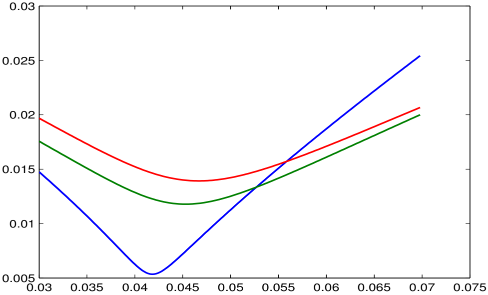

Figure: 5. CMM smile on FNMA 5

Valuation of Mortgage Options

- The graph below shows the volatility smile on the FNMA 5.5 options of various expirations.

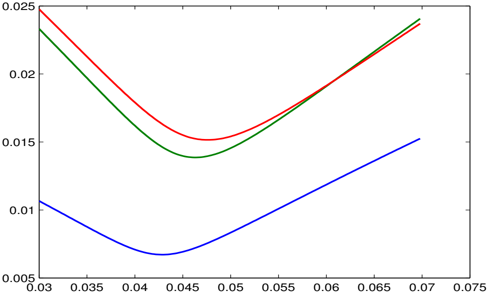

Figure: 6. CMM smile on FNMA 5.5

Event risk embedded in MBSs

- There is no useful concept of an "implied prepayments" measure in the MBS markets, and modeling prepayments relies on the analysis of historical data and projecting these data forward.
- The mathematical framework used in prepayment modeling is *survival analysis*.
- The main challenge in modeling event risk affecting an MBS is to capture the heterogeneity of the collateral loans. The heterogeneity may be:
  - (i) Observable and possible to capture in terms of a small number of factors. This is conveniently modeled by the Cox model.
  - (ii) Unobservable and difficult to capture in terms of measurable factors. We model it by implying a distribution of unobservable factors. The relevant model is the *frailty / threshold* model.

Modeling observed heterogeneity

- Observed heterogeneity is described by means of the Cox (or "proportional hazards") model:

$$\lambda(t\,|\,V) = \lambda_0\,(t)\exp\Big(\sum\nolimits_j \beta_j V_j\,(t)\Big).$$

- Here,
  (i) $\lambda_0\,(t)$ is the baseline hazard.
  (ii) $V_1, \ldots, V_m$ are the *factors* affecting the event. They may be deterministic (constant or time dependent) or stochastic.
  (iii) $\beta_1, \ldots, \beta_m$ are the *factor loadings*.

Modeling unobserved heterogeneity

- The unobserved heterogeneity is described by the probability distribution $\mathrm{F}(x)$ on a set X. The elements of X are the unobserved factors affecting the probability of an event to occur.
- Each borrower has intensity $\lambda(t, x)$. Their survival probability is

$$S(t,x) = \exp\Big(-\int_0^t \lambda(s,x)ds\Big).$$

- The population survival probability is

$$\begin{aligned} S(t) &= \exp\Big(-\int_0^t \lambda(s)\, ds\Big) \\ &= \int_X S(t,x)d\mathrm{F}(x)\,. \end{aligned}$$

Modeling unobserved heterogeneity

- In the frailty / threshold model, the heterogeneity space X is two-dimensional, $x = (f, c)$. Here,
  (i) f = *frailty*, c = *threshold*.
  (ii) Borrower's intensity process
  $$\lambda(t, x) = f \, 1_{C(t) \geq c} \, \lambda_0(t) \, .$$
  (iii) $C(t)$ is the *incentive process*.

Modeling prepayment risk

- Decomposition of $S(t)$ into independent *competing risks*:
  (i) Turnover (Cox model)
  (ii) Refinancing (frailty / threshold model)
  (iii) ...
- Then $S(t) = S_1(t)\, S_2(t) \ldots$, where $S_j(t)$ are the survival probabilities corresponding to the *latent* risks. Factors affecting $S_j(t)$:
  (i) Interest rates
  (ii) Housing prices (HPI)
  (iii) Loan characteristics (size, age, FICO score)
  (iv) Unobserved heterogeneity characteristics of borrowers
  (v) Macroeconomic factors

Modeling mortgage default risk

- The loss process of an MBS is a multi-state process which is modeled as a continuous time Markov chain. The underlying state space is assumed to be of the form:

$$S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix} = \begin{bmatrix} \text{Current} \\ \text{60+ Days Delinquent} \\ \text{In Foreclosure} \\ \text{REO} \\ \text{Liquidated} \end{bmatrix}.$$

Modeling mortgage default risk

- The generator $D(t)$ of the Markov chain is of given by:

$$D(t) = \begin{bmatrix} -\lambda_1(t) & \lambda_1(t) & 0 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2(t) & -\lambda_2(t) & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -\lambda_{n-1}(t) & \lambda_{n-1}(t) \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

  where $\lambda_j(t)$ are the hazard rates for the transitions $S_j \to S_{j+1}$.

- Here, for $i = 1, \ldots, 4$:
  (i) $1 - \lambda_i(t)\, dt$ is the conditional probability of staying in state S_i.
  (ii) $\lambda_i(t)\, dt$ is the conditional probability of transition $S_i \to S_{i+1}$.
  (iii) For tractability, we ban all transitions of the form $S_i \to S_{i+2}$, $S_i \to S_{i-1}$, etc. This is an approximation that is borne out by the data.

Modeling mortgage default risk

- Break up the problem into a sequence of survival processes:

$$D = D_{S_1 \to S_2} + D_{S_2 \to S_3} + D_{S_3 \to S_4} + D_{S_4 \to S_5} \ .$$

- Here, each summand is the generator of a survival Markov chain

$$\begin{bmatrix} -\lambda_j(t) & \lambda_j(t) \\ 0 & 0 \end{bmatrix}$$

embedded in a 5×5 matrix.

Modeling mortgage default risk

- The cumulative time to event is (approximately)

$$T = T_{S_1 \to S_2} + T_{S_2 \to S_3} + T_{S_3 \to S_4} + T_{S_4 \to S_5}.$$

- The composite CDF can be expressed as the convolution of the constituent CDFs,

$$Q(t) = Q_{S_1 \to S_2} * Q_{S_2 \to S_3} * Q_{S_3 \to S_4} * Q_{S_4 \to S_5}(t).$$

  Rapid computation of multiple convolutions can be done by means of the FFT algorithm.

Modeling mortgage default risk

- Factors affecting the T_i's:
  (i) Combined loan to value (CLTV)
  (ii) FICO score
  (iii) Property state (may be grouped in categories)
  (iv) Log of loan size
  (v) Mortgage servicer
  (vi) ...

Modeling mortgage default risk

- The *severity*, or *loss given a default* (LGD), is given by:

$$L(T) = B(1 - R(T)) - A(T),$$

  where B is the balance on the loan, $A(T)$ is the cash advanced, and $R(T)$ is the recovery rate.

- We model the recovery rate by

$$R(T) = \sum_i \alpha_i X_i,$$

  where X_i are the factors affecting LGD such as:
  (i) Time in foreclosure and REO
  (ii) CLTV
  (iii) ...

HPI modelling

- Housing prices enter both the prepayment and credit models, and they impact they way the market perceives the values of mortgage backed securities.
- The 2007 - 2008 financial crisis was largely caused by a bubble in housing pricing and the market's belief in continuing appreciation of house prices.
- The lesson learned from the housing prices bust is that they should be regarded as stochastic, and can depreciate as well as appreciate.
- Modeling HPI is an inexact science:
  - (i) There is no liquid forward looking market on housing prices.
  - (ii) Historical prices are noisy and affected by a variety of exogenous factors (such as e.g. geography).

HPI modelling

- There exists a number of HPI indices; the most popular among them are:
  (i) *FHFA indices* are based on the repeat-sale method.
  (ii) *Case-Shiller indices* are also based on the repeat-sale method. CME trades futures on one of the CS indices.
  (iii) Radar Logic indices (RPX) computes the price per square feet. There is a futures market on RPX.
- Projecting future HPI levels can be done by means of one the following approaches:
  (i) A stochastic process with a mean everting component.
  (ii) Kalman filter.

Calibration of the Time to Event Model

- Calibration is done by means of maximum likelihood estimation (MLE) on recent historical data. This can be framed as minimization of the distance between the observed event frequencies and model probability distribution.
- Calculations are distributed on the computational cluster when dealing with large data sets.
- The "goodness of fit" measure of the calibration is the *Kullback-Leibler divergence* between the observed probability distribution $\{p_i\}$ and the model probability distribution $\{p_i(\theta)\}$:

$$D(p\|p(\theta)) = \sum_i p_i \log\left(\frac{p_i}{p_i(\theta)}\right).$$

Then

(i) $D(p\|p(\theta)) \geq 0$.
(ii) "Best" value of $\theta \Rightarrow$ minimum KL divergence.

OAS

- We see from the discussion above that modeling of MBSs is a hybrid of two approaches:
  (i) Risk neutral valuation (Q -measure)
  (ii) Historical valuation (P - measure).
- As a result, model prices of securities typically do not match market prices.
- Furthermore, model prices produced by various models disagree among each other.

OAS

- Since the 1980's the MBS market adopted the convention according to which the discrepancy between the market and model price of a security is eliminated by means of an additional spread applied to the discounting rate. For historical reasons, this spread is called the *option adjusted spread* (OAS).
- Specifically, consider an MBS whose (contingent) cashflow on a coupon date T_j is $C_j(T_j)$. Assuming that we use the spot measure Q_0 for valuation, its model price is given by
$$P = \mathsf{E}^{\mathsf{Q}_0}\Big[\sum_j e^{-\int_0^{T_j} r(t)\,dt} C_j(T_j)\Big]. \tag{1}$$
- Adding the OAS spread to r we replace this expression with
$$P(OAS) = \mathsf{E}^{\mathsf{Q}_0}\Big[\sum_j e^{-\int_0^{T_j} (r(t)+OAS)\,dt} C_j(T_j)\Big]. \tag{2}$$

OAS

- We calculate the OAS of a security by requiring that its market price $\overline{P}$ matches the OAS-adjusted model price,

$$P(OAS) = \overline{P}, \tag{3}$$

  and solving this equation for *OAS* (using, for example, the secant method).
- Securities with $OAS > 0$ are referred to as *cheap*, while those with $OAS < 0$ are referred to as *rich*. This terminology reflects the fact that, relative to the model, their market prices are too low and too high, respectively.
- For example, Trust IOs are typically cheap, while Trust POs are typically rich.
- The richness / cheapness measure is the the basis for many MBS managers' investment strategies.
- The concept of OAS can also be extended to account for credit events, and it is referred to as *credit OAS*.

CMM rate in terms of mortgage options

- The CMM rate can be replicated in terms of mortgage options. This is reminiscent of replicating a CMS rate in terms of interest rate swaptions discussed in a previous lecture.
- Since

$$\begin{aligned} P_C(T) &< 1, \quad \text{if } Y(T) > C, \\ P_C(T) &= 1, \quad \text{if } Y(T) = C, \\ P_C(T) &> 1, \quad \text{if } Y(T) < C, \end{aligned}$$

we can write

$$\begin{aligned} Y(T) &= \int_0^{Y(T)} dC \\ &= \int_0^{\infty} \Theta(1 - P_C(T))\, dC, \end{aligned}$$

where $\Theta(x)$ denotes the step function.

CMM rate in terms of mortgage options

- From the definition of the CMM rate,

$$M_0(T) = \mathrm{E}^{\mathrm{Q}_T}\left[Y(T+\vartheta)\right],$$

we infer that

$$M_0(T) = \int_0^\infty \mathrm{E}^{\mathrm{Q}_T}\left[\Theta\left(1 - P_C(T+\vartheta)\right)\right] dC.$$

- Finally, we note that the integrand in the integral above equals, up to the discount factor, to the price of a digital put on a TBA struck at par.

CMM rate in terms of mortgage options

- This proves the following *CMM replication formula*. Let $\mathrm{DigiPut}_T\left(P_C\left(t\right), K\right)$ denote the price of a digital put option on a TBA which expires on T and is struck at K. Then

$$M_0\left(T\right) = \frac{1}{Z\left(T\right)} \int_0^\infty \mathrm{DigiPut}_T\left(P_C\left(T+\vartheta\right), 1\right) dC.$$

- In practice, the prices of digital puts are approximated as put spreads:

$$\begin{aligned}&\mathrm{DigiPut}_T\left(P_C\left(T+\vartheta\right), 1\right)\\ &\quad\approx \frac{\mathrm{Put}_T\left(P_C\left(T+\vartheta\right), 1+\epsilon\right) - \mathrm{Put}_T\left(P_C\left(T+\vartheta\right), 1-\epsilon\right)}{2\epsilon},\end{aligned}$$

  where ϵ is a strike spread.

CMM rate in terms of mortgage options

- Let now $\Delta_{\text{DigiPut}_T}(P_C(t), K)$ denote the price delta of the digital put above, and let

$$w_C(T) = -\frac{1}{Z(T)}\, \Delta_{\text{DigiPut}_T}(P_C(T+\vartheta), 1)\, P_C(T+\vartheta)\, D_C(T+\vartheta).$$

- Since the delta of a digital put is non-negative, $w_C(T) \geq 1$. Furthermore, as a consequence of the replication formula,

$$\int_0^\infty w_C(T)\, dC = 1.$$

- The number $w_C(T)$ represents thus the dynamic weight of the coupon C TBA in the replication of the CMM rate.

References

- Birman, J., and MacDonald, I.: Introduction to CMM products, Goldman Sachs (2005).
- Blanchet-Scalliet, C., El Karoui, N., and Martellini, L.: Dynamic asset pricing theory with uncertain time-horizon, J. Economic Dynamics & Control, **29**, 1737 1764 (2005).
- Davidson, A., and Levin, A.: *Mortgage Valuation Models: Embedded Options, Risk, and Uncertainty*, Oxford University Press (2014).
- Hayre, L.: *Salomon Smith Barney Guide to Mortgage-Backed and Asset-Backed Securities*, Wiley (2001).
- Prendergast, J. R.: The complexities of mortgage options, J. Fixed Income, **12**, 7 - 24 (2003).
- Velayudham, K., and Nashikkar, A.: Constant maturity mortgages (CMM): applications and modeling, Barclays Capital (2010).
- Zhou, F., Subramanian, P., and Chen, L.: Constant maturity mortgages, Lehman Brothers (2005).