

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

$$\begin{aligned} \hat{r} &= (\hat{r} \cdot \hat{x}) \hat{x} + (\hat{r} \cdot \hat{y}) \hat{y} + (\hat{r} \cdot \hat{z}) \hat{z} \\ &= \frac{\partial x}{\partial r} \hat{x} + \frac{\partial y}{\partial r} \hat{y} + \frac{\partial z}{\partial r} \hat{z} \\ &= \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= (\hat{\theta} \cdot \hat{x}) \hat{x} + (\hat{\theta} \cdot \hat{y}) \hat{y} + (\hat{\theta} \cdot \hat{z}) \hat{z} \\ &= \frac{\partial x}{r \partial \theta} \hat{x} + \frac{\partial y}{r \partial \theta} \hat{y} + \frac{\partial z}{r \partial \theta} \hat{z} \\ &= \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} + (-\sin \theta) \hat{z} \end{aligned}$$

$$\begin{aligned} \hat{\phi} &= (\hat{\phi} \cdot \hat{x}) \hat{x} + (\hat{\phi} \cdot \hat{y}) \hat{y} + (\hat{\phi} \cdot \hat{z}) \hat{z} \\ &= \frac{\partial x}{r \sin \theta \partial \phi} \hat{x} + \frac{\partial y}{r \sin \theta \partial \phi} \hat{y} + \frac{\partial z}{r \sin \theta \partial \phi} \hat{z} \\ &= -\sin \phi \hat{x} + \cos \phi \hat{y} \end{aligned}$$

為什麼可以這樣算？

以 $\hat{r} \cdot \hat{x}$ 來說

\hat{r} 和 \hat{x} 都是 unit vector

$\hat{r} \cdot \hat{x}$ 可以視為

\hat{x} 在 \hat{r} 上的投影

也就是說

$\hat{x} \cdot \hat{r}$ 這個數字可以看作是
當 r 增加 1 個單位時

x 對應的增量

也就是 $\frac{\partial x}{\partial r}$

以 $\hat{\theta} \cdot \hat{x}$ 來說

就是當 θ 方向上的距離增加

1 個單位時， x 對應的增量， $\frac{\partial x}{r \partial \theta}$

$\hat{\phi} \cdot \hat{x}$

ϕ 方向上的距離增加 1 個單位，

x 對應的增量 $\frac{\partial x}{r \sin \theta \partial \phi}$

其亦同理。

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$$\begin{aligned}\hat{x} &= (\hat{x} \cdot \hat{r}) \hat{r} + (\hat{x} \cdot \hat{\theta}) \hat{\theta} + (\hat{x} \cdot \hat{\phi}) \hat{\phi} \\ &= \frac{\partial x}{\partial r} \hat{r} + \frac{\partial x}{r \partial \theta} \hat{\theta} + \frac{\partial x}{r \sin \theta \partial \phi} \hat{\phi} \\ &= \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} + (-\sin \phi) \hat{\phi}\end{aligned}$$

$$\begin{aligned}\hat{y} &= (\hat{y} \cdot \hat{r}) \hat{r} + (\hat{y} \cdot \hat{\theta}) \hat{\theta} + (\hat{y} \cdot \hat{\phi}) \hat{\phi} \\ &= \frac{\partial y}{\partial r} \hat{r} + \frac{\partial y}{r \partial \theta} \hat{\theta} + \frac{\partial y}{r \sin \theta \partial \phi} \hat{\phi} \\ &= \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}\end{aligned}$$

$$\begin{aligned}\hat{z} &= (\hat{z} \cdot \hat{r}) \hat{r} + (\hat{z} \cdot \hat{\theta}) \hat{\theta} + (\hat{z} \cdot \hat{\phi}) \hat{\phi} \\ &= \cos \theta \hat{r} + (-\sin \theta) \hat{\theta}\end{aligned}$$

$$\begin{aligned}x &= s \cos \phi \\ y &= s \sin \phi \\ z &= z\end{aligned}$$

$$\begin{aligned}\hat{s} &= (\hat{s} \cdot \hat{x}) \hat{x} + (\hat{s} \cdot \hat{y}) \hat{y} + (\hat{s} \cdot \hat{z}) \hat{z} \\ &= \frac{\partial x}{\partial s} \hat{x} + \frac{\partial y}{s \partial \phi} \hat{y} + \frac{\partial z}{\partial s} \hat{z} \\ &= \hat{x} \cos \phi + \hat{y} \sin \phi\end{aligned}$$

$$\begin{aligned}\hat{\phi} &= \hat{x} (\hat{\phi} \cdot \hat{x}) + \hat{y} (\hat{\phi} \cdot \hat{y}) + \hat{z} (\hat{\phi} \cdot \hat{z}) \\ &= \hat{x} \frac{\partial x}{s \partial \phi} + \hat{y} \frac{\partial y}{s \partial \phi} + \hat{z} \frac{\partial z}{s \partial \phi} \\ &= \hat{x} (-\sin \phi) + \hat{y} \cos \phi\end{aligned}$$

$$\hat{z} = \hat{z}$$

$$\begin{aligned}\hat{x} &= (\hat{x} \cdot \hat{s}) \hat{s} + (\hat{x} \cdot \hat{\phi}) \hat{\phi} + (\hat{x} \cdot \hat{z}) \hat{z} \\ &= \frac{\partial x}{\partial s} \hat{s} + \frac{\partial x}{s \partial \phi} \hat{\phi} + \frac{\partial x}{\partial z} \hat{z} \\ &= \cos \phi \hat{s} + (-\sin \phi) \hat{\phi}\end{aligned}$$

$$\begin{aligned}\hat{y} &= (\hat{y} \cdot \hat{s}) \hat{s} + (\hat{y} \cdot \hat{\phi}) \hat{\phi} + (\hat{y} \cdot \hat{z}) \hat{z} \\ &= \frac{\partial y}{\partial s} \hat{s} + \frac{\partial y}{s \partial \phi} \hat{\phi} + \frac{\partial y}{\partial z} \hat{z} \\ &= \sin \phi \hat{s} + \cos \phi \hat{\phi}\end{aligned}$$

$$\hat{z} = \hat{z}$$

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