$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{aligned}\hat{r} &= (\hat{r}\cdot\hat{x})\hat{x} + (\hat{r}\cdot\hat{y})\hat{y} + (\hat{r}\cdot\hat{z})\hat{z} \\ &= \frac{\partial x}{\partial r}\hat{x} + \frac{\partial y}{\partial r}\hat{y} + \frac{\partial z}{\partial r}\hat{z} \\ &= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\end{aligned}$$

$$\begin{aligned}\hat{\theta} &= (\hat{\theta}\cdot\hat{x})\hat{x} + (\hat{\theta}\cdot\hat{y})\hat{y} + (\hat{\theta}\cdot\hat{z})\hat{z} \\ &= \frac{\partial x}{r\partial\theta}\hat{x} + \frac{\partial y}{r\partial\theta}\hat{y} + \frac{\partial z}{r\partial\theta}\hat{z} \\ &= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} + (-\sin\theta)\hat{z}\end{aligned}$$

$$\begin{aligned}\hat{\phi} &= (\hat{\phi}\cdot\hat{x})\hat{x} + (\hat{\phi}\cdot\hat{y})\hat{y} + (\hat{\phi}\cdot\hat{z})\hat{z} \\ &= \frac{\partial x}{r\sin\theta\partial\phi}\hat{x} + \frac{\partial y}{r\sin\theta\partial\phi}\hat{y} + \frac{\partial z}{r\sin\theta\partial\phi}\hat{z} \\ &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\end{aligned}$$

為什麼可以這樣算？

以 $\hat{r}\cdot\hat{x}$ 來說

$\hat{r}$和 $\hat{x}$ 都是 unit vector

$\hat{r}\cdot\hat{x}$ 可以視為

$\hat{x}$ 在 $\hat{r}$ 上的投影

也就是說

$\hat{x}\cdot\hat{r}$ 這個數字可以看作是

當 r 增加 1個單位時

x 對應的增量

也就是 $\frac{\partial x}{\partial r}$

以 $\hat{\theta}\cdot\hat{x}$ 來說

就是當 $\hat{\theta}$方向上的距離增加

1個單位時，x 對應的增量，$\frac{\partial x}{r\partial\theta}$

$\hat{\phi}\cdot\hat{x}$

$\hat{\phi}$方向上的距離增加1個單位，

x 對應的增量 $\frac{\partial x}{r\sin\theta\partial\phi}$

其餘同理。

球座標 ⟷ 卡氏座標

$$\hat{x} = (\hat{x}\cdot\hat{r})\,\hat{r} + (\hat{x}\cdot\hat{\theta})\,\hat{\theta} + (\hat{x}\cdot\hat{\phi})\,\hat{\phi}$$

$$= \frac{\partial x}{\partial r}\,\hat{r} + \frac{\partial x}{r\partial\theta}\,\hat{\theta} + \frac{\partial x}{r\sin\theta\,\partial\phi}\,\hat{\phi}$$

$$= \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} + (-\sin\phi)\,\hat{\phi}$$

$$\hat{y} = (\hat{y}\cdot\hat{r})\,\hat{r} + (\hat{y}\cdot\hat{\theta})\,\hat{\theta} + (\hat{y}\cdot\hat{\phi})\,\hat{\phi}$$

$$= \frac{\partial y}{\partial r}\,\hat{r} + \frac{\partial y}{r\partial\theta}\,\hat{\theta} + \frac{\partial y}{r\sin\theta\,\partial\phi}\,\hat{\phi}$$

$$= \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

$$\hat{z} = (\hat{z}\cdot\hat{r})\,\hat{r} + (\hat{z}\cdot\hat{\theta})\,\hat{\theta} + (\hat{z}\cdot\hat{\phi})\,\hat{\phi}$$

$$= \cos\theta\,\hat{r} + (-\sin\theta)\,\hat{\theta}$$

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\hat{s} = (\hat{s}\cdot\hat{x})\,\hat{x} + (\hat{s}\cdot\hat{y})\,\hat{y} + (\hat{s}\cdot\hat{z})\,\hat{z}$$

$$= \frac{\partial x}{\partial s}\,\hat{x} + \frac{\partial y}{\partial s}\,\hat{y} + \frac{\partial z}{\partial s}\,\hat{z}$$

$$= \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi} = \hat{x}(\hat{\phi}\cdot\hat{x}) + \hat{y}(\hat{\phi}\cdot\hat{y}) + \hat{z}(\hat{\phi}\cdot\hat{z})$$

$$= \hat{x}\,\frac{\partial x}{s\partial\phi} + \hat{y}\,\frac{\partial y}{s\partial\phi} + \hat{z}\,\frac{\partial z}{s\partial\phi}$$

$$= \hat{x}(-\sin\phi) + \hat{y}\cos\phi$$

$$\hat{z} = \hat{z}$$

$$\hat{x} = (\hat{x}\cdot\hat{s})\,\hat{s} + (\hat{x}\cdot\hat{\phi})\,\hat{\phi} + (\hat{x}\cdot\hat{z})\,\hat{z}$$

$$= \frac{\partial x}{\partial s}\,\hat{s} + \frac{\partial x}{s\partial\phi}\,\hat{\phi} + \frac{\partial x}{\partial z}\,\hat{z}$$

$$= \cos\phi\,\hat{s} + (-\sin\phi)\,\hat{\phi}$$

$$\hat{y} = (\hat{y}\cdot\hat{s})\,\hat{s} + (\hat{y}\cdot\hat{\phi})\,\hat{\phi} + (\hat{y}\cdot\hat{z})\,\hat{z}$$

$$= \frac{\partial y}{\partial s}\,\hat{s} + \frac{\partial y}{s\partial\phi}\,\hat{\phi} + \frac{\partial y}{\partial z}\,\hat{z}$$

$$= \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}$$

$$\hat{z} = \hat{z}$$

柱座標 $\longleftrightarrow$ 卡氏座標