INTERSECTION THEORY OF HYPERQUOT SCHEMES ON CURVES

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Abstract. We study the virtual intersection theory of Hyperquot schemes parameterizing sequences of quotient sheaves of a vector bundle on a smooth projective curve of arbitrary genus. Our results generalize the Vafa–Intriligator formula of Marian–Oprea for Quot schemes and provide a closed formula for virtual counts of maps from the curve to a partial flag variety.

Contents

1. Introduction

Let C be a smooth projective curve and V be a vector bundle of rank n over C. The Quot scheme $\mathsf{Quot}_d(C, r, V)$ is a projective scheme that parametrizes short exact sequences of sheaves on C:

$$0 \to E \hookrightarrow V \twoheadrightarrow F \to 0 \quad \text{such that } \begin{cases} \operatorname{rank} F = n - r, \\ \deg F = d. \end{cases}$$

For a genus g curve C, the Quot scheme was used by [Ber94, BDW96, MO07] to compactify the space of degree d maps from C to the Grassmannian $\mathrm{Gr}(r, n)$ and to prove the Vafa–Intriligator formula for counts of such maps satisfying some incidence conditions.

The Hyperquot scheme is a natural generalization of the Quot scheme. For fixed tuples

$$\mathbf{r} = (r_1, r_2, \ldots, r_k) \quad \text{and} \quad \mathbf{d} = (d_1, d_2, \ldots, d_k)$$

of integers, the Hyperquot scheme $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)$, or $\mathsf{HQ}_{\mathbf{d}}$ for short, parametrizes chains of quotients, equivalently subsheaves,

$$E_1 \hookrightarrow E_2 \hookrightarrow \cdots E_k \hookrightarrow V \twoheadrightarrow F_1 \twoheadrightarrow F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_k \quad \text{such that } \begin{cases} \operatorname{rank} F_j = n - r_j, \\ \deg F_j = d_j, \end{cases}$$

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and E_j is the kernel of $V \twoheadrightarrow F_j$ for each j.

In this article, we establish a formula for virtual intersection numbers on the Hyperquot scheme, generalizing the work of Bertram [Ber94] and Marian–Oprea [MO07]. In the special case when V is trivial and $r_1, r_2, \ldots, r_k$ are distinct, the Hyperquot scheme parametrizes maps from C to the partial flag variety $\mathrm{Fl}(\mathbf{r}, n)$, and our formula provides a virtual count of such maps satisfying certain incidence conditions. Furthermore, when $C = \mathbb{P}^1$, the Hyperquot scheme is a smooth variety and the virtual counts in consideration is an actual count [Kim, CF99a] and played an important role in the study of the quantum cohomology of $\mathrm{Fl}(\mathbf{r}, n)$, see, for example, [Ber97, CF99b, CF99a, Che03, Kim95].

1.1. **Statement of the main formula.** We begin by setting up some notations. Consider the universal sequence of sheaves

$$0 \to \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \to \pi_C^* V \to \mathcal{F}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_k \to 0$$

on $\mathsf{HQ}_{\mathbf{d}} \times C$ where π_C denote the projection map to C. We will compute the top intersection of products of the Chern classes

$$c_i(\mathcal{E}_{j|p}^\vee) \in H^{2i}(\mathsf{HQ}_{\mathbf{d}}, \mathbb{Z}), \tag{1}$$

where $\mathcal{E}_{j|p}$ is the rank r_j locally free sheaf obtained by restricting of $\mathcal{E}_j$ to $\mathsf{HQ}_{\mathbf{d}} \times \{p\}$ for a point $p \in C$.

The Hyperquot schemes $\mathsf{HQ}_{\mathbf{d}}$ are not smooth or irreducible in general. Via the theory of [BF97], one can nevertheless construct a virtual fundamental class in homology, which is a well-behaved replacement for the classical fundamental class. This class was constructed for the Quot scheme in [MO07] and for the Hyperquot scheme in [CFKM14, MR25], yielding

$$[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(\mathsf{HQ}_{\mathbf{d}}, \mathbb{Z}),$$

with virtual dimension equals $\mathrm{vdim} = \sum_{j=1}^{k} \chi(E_j, F_j) - \sum_{j=1}^{k-1} \chi(E_j, F_{j+1})$.

Our formula involves a sum over the solutions of the following system of equations. Consider k tuples of variables $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ where $\mathbf{z}_j := (z_{1,j}, z_{2,j}, \ldots, z_{r_j,j})$, and set $\mathbf{z}_0 = \{\}$, $\mathbf{z}_{k+1} = (0, 0, \ldots, 0) \in \mathbb{C}^n$, as illustrated below.

$$\begin{array}{ccccccc}
\mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \cdots & \cdots & \mathbf{z}_k & \mathbf{z}_{k+1} \\
\hline
 & z_{1,1} & z_{1,2} & \cdots & \cdots & z_{1,k} & 0 \\
 & \vdots & \vdots & & & \vdots & \vdots \\
 & z_{r_1,1} & \vdots & & & \vdots & \vdots \\
 & & z_{r_2,2} & & & \vdots & \vdots \\
 & & & \ddots & & \vdots & \vdots \\
 & & & & \ddots & & \vdots \\
 & & & & & z_{r_k,k} & \vdots \\
 & & & & & & 0
\end{array}$$

For each $1 \leqslant j \leqslant k$, define the polynomial

$$P_j(X) := \prod_{\alpha \in \mathbf{z}_{j+1}} (X - \alpha) + (-1)^{r_j - r_{j-1}} q_j \prod_{\alpha \in \mathbf{z}_{j-1}} (X - \alpha). \tag{2}$$

Definition 1.1. Let $(q_1, q_2, \ldots, q_k)$ be a k-tuple of complex numbers. We define the system of polynomial equations

$$P_j(z_{i,j}) = 0 \quad \text{for each } 1 \leqslant j \leqslant k, \text{ and } 1 \leqslant i \leqslant r_j. \tag{3}$$

We say that a solution $(\mathbf{z}_1, \mathbf{z}_2 \ldots, \mathbf{z}_k) = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \ldots, \boldsymbol{\zeta}_k)$ to the above system is *non-degenerate* if each tuple $\boldsymbol{\zeta}_j = (\zeta_{1,j}, \zeta_{2,j}, \ldots, \zeta_{r_j,j})$ has no repetition.

These equations have appeared in several other contexts; see Section 1.3 for more details.

We are now ready to state our main result, see Theorem 3.1 for an equivariant version.

Theorem 1.1. *Fix the tuple $\mathbf{r} = (r_1, r_2, \ldots, r_k)$, and the vector bundle V on C of rank n with $\deg V = 0$. Then for natural numbers $m_{i,j}$ and a generic $(q_1, \ldots, q_k) \in (\mathbb{C}^*)^k$, we have*

$$\sum_{\mathbf{d}\in\mathbb{N}^k} q_1^{d_1} \cdots q_k^{d_k} \int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} \prod_{j=1}^{k} \prod_{i=1}^{r_j} c_i(\mathcal{E}_{j|p}^{\vee})^{m_{i,j}} = \sum_{\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_k} \prod_{j=1}^{k} \frac{1}{r_j!} \prod_{i=1}^{r_j} e_i(\boldsymbol{\zeta}_j)^{m_{i,j}} \cdot J^{g-1}, \tag{4}$$

where the sum is taken over all non-degenerate solutions $(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k)$ of the equations in Definition 1.1 and $e_i(\boldsymbol{\zeta}_j)$ denote the i-th elementary symmetric polynomial in r_j variables. The factor J is given by

$$J = J(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k) = \prod_{j=1}^{k} \frac{1}{\Delta(\boldsymbol{\zeta}_j)} \cdot \det\left(\frac{\partial P_j(z_{i,j})}{\partial z_{i',j'}}\right)\Bigg|_{(\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_k)=(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\ldots,\boldsymbol{\zeta}_k)},$$

where $\Delta(X_1, \ldots, X_m) := \prod_{a\neq b}(X_a - X_b)$.

Remark 1.1. For fixed $(m_{i,j})$, the expression in (4) is a polynomial in the variables $q_1, q_2, \ldots, q_k$. In practice, this polynomiality allows one to substitute generic complex values for $q_1, q_2, \ldots, q_k$ to solve (3) numerically and then use interpolation to recover the full polynomial.

Remark 1.2. Theorem 1.1 is stated for bundles V of degree zero, but a completely analogous formula holds in the nonzero-degree case. This follows from the fact that every vector bundle V can be reduced to a degree zero bundle by twisting with some line bundle and applying a sequence of elementary modifications. In Proposition 2.3 we show a simple identity relating the virtual class of $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)$ to the one of $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, \widetilde{V})$, where $\widetilde{V}$ is an elementary modification of V, extending the analogous result proven in [Mar25] for Quot schemes.

Remark 1.3. We also prove a compatibility of virtual classes in Proposition 2.2, analogous to [MO07, Theorem 2], keeping V fixed but changing the degree tuple $\mathbf{d}$. This is obtained by twisting ℓ-th subbundle by $\mathcal{O}_C(-\mathrm{pt})$ to obtain an embedding $\iota : \mathsf{HQ}_{\mathbf{d}} \to \mathsf{HQ}_{\mathbf{d}+\delta_\ell}$ where $\delta_\ell = (r_1, \ldots, r_\ell, 0, \ldots, 0)$ and showing

$$\iota_*[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}} = e(\mathcal{E}_\ell^\vee \otimes \mathcal{E}_{\ell+1})_{|p} \cap [\mathsf{HQ}_{\mathbf{d}+\delta_\ell}]^{\mathrm{vir}}.$$

The consequence of this compatibility at the level of the intersection numbers can be directly observed in formula (4).

1.2. **Specializations.** Our work extends results of Bertram [Ber94] and Marian–Oprea [MO07], from whom we also borrow several techniques. We note here that Theorem 1.1 specializes to the Vafa–Intriligator formula for intersection numbers on the Quot scheme $\mathsf{Quot}_d(C, r, \mathcal{O}_C^{\oplus n})$. Indeed, when $k = 1$, the polynomial equation in (2) simplifies to

$$P(X) = X^n + (-1)^r q,$$

and the roots $\mathbf{z} = (z_1, z_2, \ldots, z_r)$ are described using n-th root of unity, and we recover:

Theorem (Vafa–Intriligator formula [Ber94, MO07])**.** *Let $m_1, m_2, \ldots, m_r$ be a tuple of non-negative integers such that $m_1 + 2m_2 + \cdots + rm_r$ equals the expected dimension $nd + r(n-r)(1-g)$ of the Quot scheme $\mathsf{Quot}_d(C, r, \mathcal{O}_C^{\oplus n})$. Then*

$$\int_{[\mathsf{Quot}_d(C,r,\mathcal{O}_C^{\oplus n})]^{\mathrm{vir}}} \prod_{i=1}^{r} c_i(\mathcal{E}_{|p}^{\vee})^{m_i} = \frac{(-1)^{d(r-1)}}{r!} \sum_{\boldsymbol{\zeta}} \prod_{i=1}^{r} e_i(\boldsymbol{\zeta})^{m_i} J^{g-1}(\boldsymbol{\zeta}), \tag{5}$$

where the sum is over all tuples $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_r)$ of distinct nth roots of unity, and

$$J(\boldsymbol{\zeta}) = \prod_{i=1}^{r} n\zeta_i^{n-1} \prod_{1 \leqslant i \neq j \leqslant r} (\zeta_i - \zeta_j)^{-1}.$$

An analogous Vafa–Intriligator formula for fixed-domain Gromov–Witten invariants on the Grassmannian (which are defined using the moduli spaces of stable maps) were computed in [ST97] and the two invariants match.

Remark 1.4. In [MO07], the virtual intersections of classes arising from the Künneth decomposition of $c_i(\mathcal{E})$ on $\mathsf{Quot}_d \times C$ were also considered. One can ask the analogous question for Hyperquot schemes, concerning the intersection Künneth components of $c_i(\mathcal{E}_j)$ on $\mathsf{HQ}_{\mathbf{d}} \times C$. We do not pursue this direction in the present paper; however, our methods may prove useful for such investigations.

The *Hyperquot scheme of points* $\mathsf{HQ}_{\mathbf{d}}(C, V)$ parameterizes successive zero-dimensional quotients of V of degrees $\mathbf{d} = (d_1, d_2, \ldots, d_k)$, i.e., $\mathbf{r} = (n, n, \ldots, n)$. The Hyperquot schemes of points on C are smooth projective varieties. They play an important role in studying the cohomology [MN23] and the derived category [Kru24, MN24] of the Quot schemes of points on C. Their Poincaré polynomials and motives were studied in [MR22].

Corollary 1.1. *Let V be a rank n vector bundle on C, we have*

$$\sum_{\mathbf{d} \in \mathbb{N}^k} \mathbf{q}^{\mathbf{d}} \int_{[\mathsf{HQ}_{\mathbf{d}}(C,V)]} \prod_{j=1}^{k} s_{t_j}(\mathcal{E}_{j|p}) = \prod_{j=1}^{k} \frac{1}{1 - t_j^n \alpha_j},$$

where $s_{t_j}(\mathcal{E}_{j|p})$ are Segre polynomials and $\alpha_j = \sum_{a=j}^{k} \prod_{b=1}^{a} q_b$.

Remark 1.5. See Corollary 7.1 for the Chern class version. The Segre integrals and Euler characteristics of tautological classes on the Quot schemes of points on C were computed in [OP21, OS23]. Our methods may be useful for evaluating analogous invariants for Hyperquot schemes of points.

a non-expert it might seem that we have just computed Segre integrals in Corollary 1.1

In the setting $\mathbf{r} = (1, n-1)$, Theorem 1.1 specializes to a simple expression involving binomial coefficients.

Corollary 1.2. *Fix $\mathbf{r} = (1, n-1)$ and a rank n bundle V on C of degree zero. For a fixed tuple of natural numbers $\ell, m_1, m_2, \ldots, m_{n-1}$,*

$$\sum_{\mathbf{d} \in \mathbb{N}^2} \mathbf{q}^{\mathbf{d}} \int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} c_1(\mathcal{E}_{1|p}^{\vee})^{\ell} \prod_{i=1}^{n-1} c_i(\mathcal{E}_{2|p}^{\vee})^{m_i} = n^g (n-1)^g \sum_{j \in \mathbb{Z}} \binom{d - \overline{g} - m_{n-1}}{jn - \ell - m_{n-1} + \overline{g}} q_1^{\overline{g}+j} q_2^{d - \overline{g} - j},$$

where $\overline{g} := g - 1$ and

$$d := \frac{\ell + \sum_{i=1}^{n-1} i m_i + (2n-3)\overline{g}}{n-1}.$$

Here we set the binomial coefficients $\binom{p}{q} = 0$ *unless* $p, q \in \mathbb{N}$ *satisfying* $q \leqslant p$.

1.3. **Related work and future directions.** When V is trivial and the ranks are distinct, i.e. $0 < r_1 < \cdots < r_k < n$, the Hyperquot scheme $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, \mathcal{O}_C^{\oplus n})$ compactifies $\mathsf{Mor}_{\mathbf{d}}(C, \mathrm{Fl}(\mathbf{r}, n))$ that parameterize morphisms f from C to the partial flag variety $\mathrm{Fl}(\mathbf{r}, n)$ of mulitdegree $\mathbf{d} = (d_1, d_2, \ldots, d_k)$, i.e. the curves class is sent to

$$f_*[C] = \mathbf{d} \in H_2(\mathrm{Fl}(\mathbf{r}, n), \mathbb{Z}) \cong \mathbb{Z}^k.$$

The basis for the homology group above is given, via the universal coefficient isomorphism $H_2(\mathrm{Fl}, \mathbb{Z}) \cong H^2(\mathrm{Fl}, \mathbb{Z})$, by the first Chern classes of the dual bundles to the universal subbundles $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ on $\mathrm{Fl}(\mathbf{r}, n)$.

Problem 1.1. Find conditions under which the virtual intersection numbers on $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, \mathcal{O}_C^{\oplus n})$ obtained in Theorem 1.1 are enumerative, that is,

$$(6) \qquad \int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} \prod_{\ell=1}^{t} c_{i_\ell}(\mathcal{E}^\vee_{j_\ell|p}) = \{f : C \to \mathrm{Fl}(\mathbf{r}, n) \mid f_*[C] = \mathbf{d},\ f(p_\ell) \in Y_{i_\ell, j_\ell} \text{ for all } \ell\},$$

where $p_1, p_2, \ldots, p_t \in C$ are distinct points, and $Y_{i_1,j_1}, Y_{i_2,j_2}, \ldots, Y_{i_t,j_t}$ are the special Schubert subvarieties of $\mathrm{Fl}(\mathbf{r}, n)$ Poincaré duals to the classes $c_{i_1}(\mathcal{A}^\vee_{j_1}), c_{i_2}(\mathcal{A}^\vee_{j_2}), \ldots, c_{i_t}(\mathcal{A}^\vee_{j_t})$, placed in general position so that the right-hand side is finite.

If $C = \mathbb{P}^1$, the Hyperquot scheme in Problem 1.1 is smooth and irreducible, and the enumerativity of the intersection numbers was established in [Kim, CF99a] for all possible values of the multidegree $\mathbf{d}$. The enumerativity of the virtual counts for Grassmannians (i.e. for $\mathsf{Quot}_d(C, r, \mathcal{O}_C^{\oplus n})$) was proved by Bertram [Ber94] for sufficiently large degrees d. More recently, this approach was also used in [MS25] to count maps to Fano hypersurfaces in Grassmannians.

Let us summarize, to the best of our knowledge, the occurrences of our system of equations in the literature. In physics, they arise as the vacuum equations of a two-dimensional gauged linear sigma model; equivalently, they define the critical locus of the twisted effective superpotential; see [GMS+24, 4.6.2] and [KOUY20]. Our equations also appear as relations describing the small quantum cohomology of the Grassmannian in [ST97] and of the partial flag variety in [GK24]. In the Grassmannian case, these are also the Bethe ansatz equations of [GK17, (4.17)].

In particular, we can view formula (4) as summing over the geometric generic fiber of the spectrum of the small quantum cohomology of the partial flag variety $\mathrm{Fl}(\mathbf{r}, n)$ over the affine space of quantum parameters $q_1, \ldots, q_k$. Moreover, the properties of the solutions to the system of equations we derive in Section 4.3 imply that the quantum cohomology of $\mathrm{Fl}(\mathbf{r}, n)$ is generically semisimple and may have applications in other contexts mentioned above.

A different approach to counting the maps in (6) is via fixed-domain Gromov–Witten invariants. When $k = 1$, the approach is known to yield the same intersection numbers as the Quot scheme approach in all degrees, as discussed in Section 1.2, regardless of their enumerativity. Fixed-domain Gromov–Witten invariants are studied for cominuscule homogeneous spaces in [CMP10], and for nonsingular projective varieties in [BP21]. It would be interesting to find a solution to the following:

Problem 1.2. When V is trivial and the ranks are distinct, does our virtual intersection number $\int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} \prod_{\ell=1}^{t} c_{i_\ell}(\mathcal{E}_{j_\ell|p}^{\vee})$ always agree with the degree $\mathbf{d}$ fixed-domain Gromov–Witten invariant on $\mathrm{Fl}(\mathbf{r}, n)$ associated to $c_{i_1}(\mathcal{A}_{j_1}^{\vee}), c_{i_2}(\mathcal{A}_{j_2}^{\vee}), \ldots, c_{i_t}(\mathcal{A}_{j_t}^{\vee})$?

By an unpublished work of Buch and Pandaripande [BP22], fixed-domain Gromov–Witten invariants on smooth projective Fano varieties can be computed in the small quantum cohomology ring and for generalized flag varieties G/P they are always non-negative when the insertions are Schubert classes. This implies that whenever these virtual intersection numbers $\int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} \prod_{\ell=1}^{t} c_{i_\ell}(\mathcal{E}_{j_\ell|p}^{\vee})$ agree with fixed-domain Gromov–Witten invariants, they must be non-negative. This positivity is also reflected in the simplified formulas presented in Corollaries 1.1 (where the ranks are not distinct and V is arbitrary) and 1.2. It is natural to ask whether positivity holds in general. Hence, we pose the following:

Problem 1.3. Prove that each virtual integral in (4) is non-negative and find positive combinatorial formulas for these invariants.

We should write this

We pursue Question 1.1 in our upcoming work...

1.3.1. *Quasimap invariants.* As recalled in Section 1.3, when the ranks are distinct the Hyperquot scheme $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, \mathcal{O}_C^{\oplus n})$ compactifies the space of degree $\mathbf{d}$ maps from C to the partial flag variety $\mathrm{Fl}(\mathbf{r}, n)$. This provides the *quasimap compactification* of $\mathsf{Mor}_{\mathbf{d}}(C, \mathrm{Fl}(\mathbf{r}, n))$ introduced in [CFKM14]. More precisely, $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, \mathcal{O}_C^{\oplus n})$ is known to be the moduli space of 0^+-stable graph quasimaps of degree $\mathbf{d}$ to the partial flag variety $\mathrm{Fl}(\mathbf{r}, n)$ with no marked points, where the target $\mathrm{Fl}(\mathbf{r}, n)$ is given in its standard presentation as a geometric invariant theory (GIT) quotient. In this sense, this paper fits into the literature computing quasimap invariants of Fano quotients of linear spaces, for which formulae of the form (4) are expected to exist [KOUY20].

When the target is a Fano toric variety $V/\!/T$ (presented as a quotient of a linear space V by the action of a torus T), [**?**] proved a Vafa–Intriligator formula for the generating series of genus zero quasimap invariants. In the case of positive quotients of linear spaces by actions of (non-abelian) reductive groups, a similar formula was found by the first author [Ont25] by abelianization, relating the invariants of $V/\!/G$ to those of $V/\!/T$. When the target is $\mathrm{Fl}(\mathbf{r}, n)$ this formula agrees with the genus zero specialization of (4), which we prove by very different methods. As discussed above, higher genus formulae are known when the target is a Grassmannian by [MO07] or a partial Flag variety by Theorem 1.1.

1.4. **Sketch of proof.** We use deformation invariance of virtual integrals (Proposition 2.1) to reduce to the case where
$$V = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$
and each $M_i \to C$ is a line bundle of degree zero. We then consider the action of $(\mathbb{C}^*)^n$ scaling each M_j and apply Atiyah–Bott localization to reduce the calculation to summing over contributions over the components of the fixed loci, which are products of symmetric products of the curve C. We handle the summation carefully by utilizing the multivariate Lagrange–Bürmann formula, which allows us to prove in genus zero that the generating polynomial of equivariant virtual integrals is a sum over formal power series solutions to the system of equations (see Theorem 4.2). We prove in Proposition 4.4 that this generating polynomial can be evaluated by first specializing $\mathbf{q}$ and the equivariant parameters to generic values, and then summing over degenerate solutions; the proof applies Gröbner degeneration

to the system of equations. In higher genus, we need some intricate intersection-theoretic computations on products of Hilbert schemes of points on C, which we handle in Section 5.

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2. Perfect obstruction theory and virtual fundamental class

In this section, we recall some foundational results about the virtual intersection theory on Hyperquot schemes. Let B be a scheme and consider a vector bundle V on $B \times C$ over B. The relative Hyperquot scheme Use $\mathcal{V}$ in relative setting

$$f : \mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V) \to B \tag{7}$$

can be constructed as a tower of relative Quot schemes as we outline below in the two step case. Let $\mathbf{r} = (r_1, r_2)$ and $\mathbf{d} = (d_1, d_2)$. The first step is to consider the relative Quot scheme $\mathsf{Quot}_{d_2}(C, r_2, V) \to B$. Let $\mathcal{E}_2$ be the universal subsheaf over $\mathsf{Quot}_{d_2}(C, r_2, V) \times C$, and consider the relative Quot scheme $\mathsf{Quot}_{d_1-d_2}(C, r_1, \mathcal{E}_2) \to \mathsf{Quot}_{d_2}(C, r_2, V)$. It is easy to check that the composition

$$\mathsf{Quot}_{d_1-d_2}(C, r_1, \mathcal{E}_2) \to \mathsf{Quot}_{d_2}(C, r_2, V) \to B \tag{8}$$

is the relative Hyperquot scheme $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)$ over B. This construction can be easily iterated for flags of any length.

2.1. Obstruction theory.

On the product $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V) \times C$ we have the universal subsheaves

$$0 =: \mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \hookrightarrow \mathcal{E}_{k+1} := V,$$

where each $\mathcal{E}_i$ is a vector bundle, and the projection $\pi : \mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V) \times C \to \mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)$. As described in [CFKM14] and [MR25, Theorem 3.2] there is a f-relative perfect obstruction theory $\mathbb{E} \to \mathbb{L}_f$, where $\mathbb{L}_f$ denotes the f-relative cotangent complex, for the Hyperquot scheme (7) with

$$\mathbb{E} := \mathrm{Cone}\left[\bigoplus_{i=1}^{k} \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_i) \xrightarrow{\gamma} \bigoplus_{i=1}^{k} \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_{i+1})\right]^\vee \tag{9}$$

where $\mathcal{H}om_\pi$ is the composition of $\mathrm{R}\pi_*$ and $\mathcal{H}om$ in the derived category. The morphism γ is described as follows. By post-composition, the subsheaves define morphisms of sheaves $\alpha_i : \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_i) \to \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_{i+1})$ for every $i \in [k]$. Analogously there are morphisms $\beta_i : \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_i) \to \mathcal{H}om(\mathcal{E}_{i-1}, \mathcal{E}_i)$ given by pre-composition. The arrow is given by

$$\mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_i) \oplus \mathcal{H}om_\pi(\mathcal{E}_{i+1}, \mathcal{E}_{i+1}) \xrightarrow{\alpha_i - \beta_{i+1}} \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_{i+1})$$

for every i.

Remark 2.1. Note that [MR25, Theorem 3.2] is stated in the absolute setting, that is, when $B = \{\mathrm{pt}\}$, but the exactly same argument gives a relative perfect obstruction theory when B is an arbitrary scheme. In the relative setting, the perfect obstruction theory for the Quot scheme was constructed by [Gil11, Kuh24]. One can also construct a perfect obstruction theory for $\mathsf{HQ}_{\mathbf{d}}$ by viewing it as an iterated Quot scheme and applying [Man11, Remark 4.6].

Sketch of construction (see [MR25, Theorem 3.2] for details). Fixed $\mathbf{r} = (r_1, \dots, r_k)$, $\mathbf{d} = (d_1, \dots, d_k)$ and a vector bundle V on $B \times C$ we can consider the moduli Artin stack $\mathfrak{Bun}_i$ of vector bundles of rank r_i and relative degree $\deg V - d_i$ on C. Since these stacks are smooth, the projection

$$\mathrm{pr}_B : B \times \prod_{i=1}^{k} \mathfrak{Bun}_i \to B$$

is smooth and the identity $\mathrm{Id} : \mathbb{L}_{\mathrm{pr}_B} \to \mathbb{L}_{\mathrm{pr}_B}$ of the relative cotangent complex of pr_B defines a relative perfect obstruction theory. From this, thanks to [Man11, Remark 4.6] we just have to describe a relative obstruction theory for the morphism

$$\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V) \to B \times \mathfrak{Bun} \quad : \quad [E_1 \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow V|_{\{b\}\times C}] \mapsto (b, [E_1], \dots, [E_k])$$

where $\mathfrak{Bun} := \prod_{i=1}^{k} \mathfrak{Bun}_i$ and $[E_i] := \mathrm{pt}/\mathrm{Aut}(E_i) \hookrightarrow \mathfrak{Bun}_i$ is the stacky point corresponding to E_i. This is easily achieved by factoring this map as a closed embedding followed by a smooth morphism, as we now recall. Let $\mathfrak{E}_i$ be the universal bundle on $\mathfrak{Bun}_i \times C$ and consider the sheaf $F := \bigoplus_{i=1}^{k} F_i := \bigoplus_{i=1}^{k} \mathcal{H}om(\mathfrak{E}_{i+1}, \mathfrak{E}_i \otimes \omega_\pi)$ on $C \times B \times \mathfrak{Bun}$, where

$$\pi : C \times B \times \mathfrak{Bun} \to B \times \mathfrak{Bun}$$

is the projection along the curve. These can be resolved as

$$0 \to K_i \to M_i \to F_i \to 0 \tag{10}$$

so that K_i and M_i are locally free, satisfy $\pi_* K_i = \pi_* M_i = 0$ and both $\mathrm{R}^1\pi_* K_i$ and $\mathrm{R}^1\pi_* M_i$ are locally free sheaves [Sin24, Lemma 2.8]. Once we set $K := \bigoplus_{i=1}^{k} K_i$ and $M := \bigoplus_{i=1}^{k} M_i$, this implies that the structure morphism

$$p : \operatorname{Spec}\operatorname{Sym}(\mathrm{R}^1\pi_* M) \to B \times \mathfrak{Bun}$$

defines the total space of a vector bundle on $B \times \mathfrak{Bun}$, the inclusion $K \to M$ defines a section κ of the locally free sheaf $(p^*\mathrm{R}^1\pi_* K)^\vee$ and its zero locus

$$Z(\kappa) \xhookrightarrow{i} \operatorname{Spec}\operatorname{Sym}(\mathrm{R}^1\pi_* M) \xrightarrow{p} B \times \mathfrak{Bun} \tag{11}$$

is such that the fiber over $(b, [E_1], \dots, [E_k])$ is $\bigoplus_{i=1}^{k} \mathrm{Hom}(E_i, E_{i+1})$. If, on the other hand, we look at the fiber over the non-stacky point

$$\mathrm{Spec}(\mathbb{C}) \hookrightarrow B \times \mathfrak{Bun}$$

defined by a point $b \in B$ and bundles $E_1, \dots, E_k$ on C, then this fiber is the stack

$$\bigoplus_{i=1}^{k} \frac{\mathrm{Hom}(E_i, E_{i+1})}{\mathrm{Aut}(E_i)}.$$

The Hyperquot scheme sits as the open locus of injective maps inside $Z(\kappa)$, and the factorization (11) induces the perfect obstruction theory

$$\begin{tikzcd}
\mathbb{E} \arrow[d, "\phi"] & = & {[p^*\mathrm{R}^1\pi_* K|_{Z(\kappa)}} \arrow[r, "\mathrm{d}\kappa^\vee"] \arrow[d, "\iota_\kappa"] & {\Omega_{\operatorname{Spec}\operatorname{Sym}(\mathrm{R}^1\pi_* M)}|_{Z(\kappa)}]} \arrow[d, equal] \\
\tau^{[-1,0]}\mathbb{L}_{p\circ i} & = & {[\mathcal{I}/\mathcal{I}^2} \arrow[r, "\mathrm{d}"] & {\Omega_{\operatorname{Spec}\operatorname{Sym}(\mathrm{R}^1\pi_* M)}|_{Z(\kappa)}].}
\end{tikzcd}$$

2.2. **Virtual fundamental class.** Via the construction of [BF97], the perfect obstruction theory recalled above induces a *virtual fundamental class* on the moduli space, namely a Chow class

$$[M]^{\mathrm{vir}} \in \mathrm{A}_{\mathrm{vdim}}(M)$$

of *virtual dimension* equal to the expected dimension from deformation theory

$$\mathrm{vdim} = \chi(\mathbb{E}) = (1-g)\sum_{i=1}^{k} r_i\big(r_{i+1}-r_i\big) + \deg V(r_1-n) + \sum_{i=1}^{k} d_i\big(r_{i+1}-r_{i-1}\big) \tag{12}$$

with $r_0 := 0$ and $r_{k+1} := n$. We remark that $\sum_{i=1}^{k} r_i\big(r_{i+1}-r_i\big) = \dim(\mathrm{Fl}(r_1,\ldots,r_k,n))$.

The presence of the relative perfect obstruction theory implies the deformation invariance of the virtual intersection numbers, via the well known result [Man12, Proposition 3.9].

Lemma 2.1. *Let B be an irreducible projective scheme, let V be a vector bundle on $B \times C$ and pick a point $p \in C$. Let $f : \mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V) \to B$ be the relative Hyperquot scheme endowed with the f-relative perfect obstruction theory (9), and let $\mathcal{E}_1,\ldots,\mathcal{E}_k$ be the corresponding universal subsheaves. For any point $b \in B$ denote with*

$$i_b : \mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V_b) \hookrightarrow \mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)$$

the inclusion of the fiber over b, where $V_b := V|_{\{b\}\times C}$. If α is any cohomology class on $\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)$ obtained from the Chern classes of the bundles $\mathcal{E}_{i|p}^{\vee}$, the function

$$B \longrightarrow \mathbb{Z}, \qquad b \longmapsto \int_{[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V_b)]^{\mathrm{vir}}} i_b^*\alpha$$

is constant on B.

From this we deduce the following useful fact, which shows that we may replace V by a split vector bundle without changing the virtual intersection numbers considered in this paper.

Proposition 2.1. *Let V be a vector bundle on C of rank n, and fix $\mathbf{r} = (r_1,\ldots,r_k)$ and $\mathbf{d} = (d_1,\ldots,d_k)$. Then there exists a split vector bundle $\widetilde{V} \simeq \bigoplus_{i=1}^n M_i$ of the same degree and rank as V such that*

$$\int_{[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)]^{\mathrm{vir}}} \prod_{i,j} c_i(\mathcal{E}_{j|p}^{\vee})^{m_{i,j}} \;=\; \int_{[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},\widetilde{V})]^{\mathrm{vir}}} \prod_{i,j} c_i(\widetilde{\mathcal{E}}_{j|p}^{\vee})^{m_{i,j}}$$

for every $p \in C$ and $m_{i,j} \geqslant 0$, where

$$0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \hookrightarrow V \quad \textit{and} \quad 0 \hookrightarrow \widetilde{\mathcal{E}}_1 \hookrightarrow \cdots \hookrightarrow \widetilde{\mathcal{E}}_k \hookrightarrow \widetilde{V}$$

are the universal subsheaves on $\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)\times C$ and $\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},\widetilde{V})\times C$ respectively. Moreover, we may choose the splitting such that $0 \leqslant \deg(M_i) - \deg(M_j) \leqslant 1$ for all $i \leqslant j$.

Proof. First of all, note that there is a line bundle L and a positive integer N such that V can be realised as a subsheaf of $H := L^{\oplus N}$. Consider the Quot scheme $B := \mathsf{Quot}_{\deg H - \deg V}(C,n,H)$ parameterising subsheaves of H having rank n and degree $\deg V$, so that $b = [V \to H] \in B$. We may choose L of sufficiently large degree, so that the Quot scheme B is irreducible by [PR03] and $n\deg(L) > \deg(V)$. The universal subsheaf $\mathcal{V}$ on $B \times C$ restricts to $\mathcal{V}|_b \cong V$. On the other hand, we show that B contains a point of the form $\widetilde{b} := [\widetilde{V} \hookrightarrow H]$ by starting from the subsheaf $L^{\oplus n} \hookrightarrow H$ and appropriately twisting the summands by $\mathcal{O}_C(-p)$ in a balanced way, until the correct degree $\deg V$. Then the claim follows by Lemma 2.1. □

2.3. **Compatibility of virtual classes.** We now record how the virtual fundamental class behaves under two basic operations on the universal subsheaves:

- twisting $\mathcal{E}_i$ by $\mathcal{O}_C(-p)$ for a point $p \in C$, and
- elementary modification of the ambient bundle V at a point of C.

2.3.1. *Twisting by a point.* Fixed a point $p \in C$, there is an inclusion of Hyperquot schemes

$$\iota : \mathsf{HQ}_{\mathbf{d}} := \mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V) \to \mathsf{HQ}_{\mathbf{d}+\delta_\ell} := \mathsf{HQ}_{\mathbf{d}+\delta_\ell}(C, \mathbf{r}, V)$$

where $\delta_\ell = (r_1, \dots, r_\ell, 0, \dots, 0)$ and the map is defined by the universal property, taking the universal flag of subsheaves $\mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \hookrightarrow V$ on $\mathsf{HQ}_{\mathbf{d}} \times C$ and constructing from it the different flag

$$\mathcal{E}_1(-p) \hookrightarrow \cdots \hookrightarrow \mathcal{E}_\ell(-p) \hookrightarrow \mathcal{E}_{\ell+1} \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \hookrightarrow V.$$

We have the following compatibility result among virtual classes of Hyperquot schemes:

Proposition 2.2. *Let $\mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \hookrightarrow V$ be the universal flag of subsheaves on $\mathsf{HQ}_{\mathbf{d}+\delta_\ell} \times C$. Then*

$$\iota_*[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}} = e(\mathcal{E}_\ell^\vee \otimes \mathcal{E}_{\ell+1})_{|p} \cap [\mathsf{HQ}_{\mathbf{d}+\delta_\ell}]^{\mathrm{vir}}.$$

In particular, for any $Q = \prod_{j=1}^k \prod_{i=1}^{r_j} c_i(\mathcal{E}_{j|p}^\vee)^{m_{i,j}}$, we have the identity

$$\prod_{i=1}^{\ell} q_i^{r_i} \sum_{\mathbf{d} \in \mathbb{Z}^k} q_1^{d_1} \cdots q_k^{d_k} \int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} Q = \sum_{\mathbf{d} \in \mathbb{Z}^k} q_1^{d_1} \cdots q_k^{d_k} \int_{[\mathsf{HQ}_{\mathbf{d}+\delta_l}]^{\mathrm{vir}}} Q \cdot e(\mathcal{E}_{\ell|p}^\vee \otimes \mathcal{E}_{\ell+1|p}).$$

Proof. First of all notice that ι can be seen as the embedding of the zero locus of the section s of the bundle $(\mathcal{E}_l^\vee \otimes \mathcal{E}_{l+1})_{|p}$ on $\mathsf{HQ}_{\mathbf{d}+\delta_\ell}$, obtained by restricting to p the universal map $\mathcal{E}_i \to \mathcal{E}_{i+1}$. Thus we find the fibered diagram

$$\begin{tikzcd}
\mathsf{HQ}_{\mathbf{d}} \arrow[r, "\iota"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\square"] & \mathsf{HQ}_{\mathbf{d}+\delta_\ell} \arrow[d, "s"] \\
\mathsf{HQ}_{\mathbf{d}+\delta_\ell} \arrow[r, "0"] & \mathrm{Tot}((\mathcal{E}_l^\vee \otimes \mathcal{E}_{l+1})_{|p}).
\end{tikzcd}$$

The resolutions (10) used to construct the relative perfect obstruction theory on $g_\delta : \mathsf{HQ}_{\mathbf{d}+\delta_\ell} \to \mathfrak{Bun}$ can be obtained from the resolutions (10) used for $g : \mathsf{HQ}_{\mathbf{d}} \to \mathfrak{Bun}$ by twisting the summand corresponding to $i = \ell$ by $\mathcal{O}_C(-p)$. After some computations this implies the commutativity of the diagram of perfect obstruction theories

$$\begin{tikzcd}
\mathrm{L}\iota^* \mathbb{E}_{\mathbf{d}+\delta_\ell} \arrow[r, "\alpha"] \arrow[d, "\iota^* \phi_{\mathbf{d}+\delta_\ell}"] & \mathbb{E}_{\mathbf{d}} \arrow[d, "\phi_{\mathbf{d}}"] \\
\mathrm{L}\iota^* \mathbb{L}_{g_\delta} \arrow[r] & \mathbb{L}_g,
\end{tikzcd}$$

where the top horizontal arrow α is induced by the morphism among the triangles (9) defined by the inclusions $\mathcal{E}_i(-\delta^i_\ell p) \hookrightarrow \mathcal{E}_i$ Here, $\delta^i_\ell = 1$ if $i = \ell$, otherwise zero, different from the definition of the tuple δ_ℓ. you're right! I didn't notice this notation clash

$$\begin{tikzcd}
\bigoplus_i \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_i) \arrow[r, "\iota^*\gamma_\delta"] & \bigoplus_i \mathcal{H}om_\pi(\mathcal{E}_i(-\delta^i_\ell p), \mathcal{E}_{i+1}) \arrow[r] & \mathrm{L}\iota^*\mathbb{E}^\vee_{\mathbf{d}+\delta_\ell} \arrow[r, "+1"] & {} \\
\bigoplus_i \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_i) \arrow[u, "\sim"] \arrow[r, "\gamma"] & \bigoplus_i \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_{i+1}) \arrow[u] \arrow[r] & \mathbb{E}^\vee_{\mathbf{d}} \arrow[u, "\alpha^\vee"] \arrow[r, "+1"] & {}
\end{tikzcd}$$

where $\mathcal{E}_i$ is the i-th universal subsheaf on $\mathsf{HQ}_{\mathbf{d}}$. By a straightforward computation, the cones of the diagram above form an exact triangle

$$0 \to \mathcal{H}om(\mathcal{E}_{\ell|p}, \mathcal{E}_{\ell+1|p}) \to \mathrm{Cone}(\alpha^\vee) \xrightarrow{+1}$$

Implying $\mathrm{Cone}(\alpha^\vee) \simeq \mathcal{H}om(\mathcal{E}_{\ell|p}, \mathcal{E}_{\ell+1|p})$. Dualizing we find the exact triangle

$$\mathrm{L}\iota^*\mathbb{E}_{\mathbf{d}+\delta_\ell} \to \mathbb{E}_{\mathbf{d}} \to \mathcal{H}om(\mathcal{E}_{\ell|p}, \mathcal{E}_{\ell+1|p})^\vee[1] \to +1$$

which, after using [Aut, Tag 08SJ] to show $\mathcal{H}om(\mathcal{E}_{\ell|p}, \mathcal{E}_{\ell+1|p})^\vee[1] \simeq \iota^*\mathbb{L}_0$, implies $0^![\mathsf{HQ}_{\mathbf{d}+\delta_\ell}]^{\mathrm{vir}} = [\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}$ via [BF97, Proposition 5.10], where $0^!$ is the Gysin pullback along the zero section of $(\mathcal{E}_l^\vee \otimes \mathcal{E}_{l+1})_{|p}$. The claim then follows from [Ful84, Theorem 6.2]:

$$\iota_*[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}} = \iota_* 0^![\mathsf{HQ}_{\mathbf{d}+\delta_\ell}]^{\mathrm{vir}} = 0^* s_*[\mathsf{HQ}_{\mathbf{d}+\delta_\ell}]^{\mathrm{vir}} = e(\mathcal{E}^\vee_{\ell|p} \otimes \mathcal{E}_{\ell+1|p}) \cap [\mathsf{HQ}_{\mathbf{d}+\delta_\ell}]^{\mathrm{vir}},$$

since $0^* s_*$ corresponds to capping with the Euler class of the bundle. □

2.3.2. *Elementary Modifications.* Let V be a vector bundle on C and fix a point $p \in C$, consider a surjection of linear spaces $V_{|p} \twoheadrightarrow \mathbb{C}$, inducing a short exact sequence

$$0 \to \widetilde{V} \xrightarrow{\alpha} V \to \mathcal{O}_p \to 0, \tag{13}$$

where $\mathcal{O}_p$ denotes the skyscraper sheaf at p. The mapping

$$[S_1 \xrightarrow{f_1} \dots \xhookrightarrow{f_{k-1}} S_k \xhookrightarrow{f_k} \widetilde{V}] \mapsto [S_1 \xhookrightarrow{f_1} \dots \xhookrightarrow{f_{k-1}} S_k \xhookrightarrow{\alpha\circ f_k} V]$$

induces a morphism of Hyperquot schemes

$$j : \mathsf{HQ}_{\mathbf{d-1}}(C, \mathbf{r}, \widetilde{V}) \to \mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)$$

where $\mathbf{1}$ denotes the vector $(1, \dots, 1) \in \mathbb{Z}^k$. This is a closed embedding, which can be realized as the zero locus of the section of $\mathcal{E}^\vee_{k|p}$ given by the tautological morphism

$$\mathcal{E}_k \to \pi_C^* V \to \ \pi_C^*\mathcal{O}_p$$

restricted to $\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V) \times \{p\}$, where π_C denotes the projection to the curve C.

Proposition 2.3. *The virtual fundamental classes of the two Hyperquot schemes satisfy the following compatibility condition:*

$$j_*[\mathsf{HQ}_{\mathbf{d-1}}(C, \mathbf{r}, \widetilde{V})]^{\mathrm{vir}} = e(\mathcal{E}^\vee_{k|p}) \cap [\mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)]^{\mathrm{vir}}.$$

In particular, for any $Q = \prod_{j=1}^k \prod_{i=1}^{r_j} c_i(\mathcal{E}^\vee_{j|p})^{m_{i,j}}$, *we have the identity*

$$\sum_{\mathbf{d}\in\mathbb{Z}^k} q_1^{d_1+1} \cdots q_k^{d_k+1} \int_{[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},\widetilde{V})]^{\mathrm{vir}}} Q = \sum_{\mathbf{d}\in\mathbb{Z}^k} q_1^{d_1} \cdots q_k^{d_k} \int_{[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)]^{\mathrm{vir}}} Q \cdot c_{r_k}(\mathcal{E}^\vee_{k|p}).$$

Proof. Let $\mathbb{E}$ be the obstruction complex in the perfect obstruction theory of $\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)$ and denote by $\widetilde{\mathbb{E}}$ the one for $\mathsf{HQ}_{\mathbf{d-1}}(C,\mathbf{r},\widetilde{V})$. By the same argument used in the proof of Proposition 2.2 we see that the cone

$$\mathfrak{C} := \mathrm{Cone}(\widetilde{\mathbb{E}}^\vee \to \mathrm{L}j^*\mathbb{E}^\vee)$$

fits into the exact triangle

$$\mathcal{H}om_\pi(\mathcal{E}_k,\widetilde{V}) \to \mathcal{H}om_\pi(\mathcal{E}_k,V) \to \mathfrak{C} \to +1,$$

hence by (13)

$$\mathfrak{C} \simeq \mathcal{H}om_\pi(\mathcal{E}_k,\pi_C^*\mathcal{O}_p) \simeq \mathrm{R}\pi_*(\mathcal{E}_k^\vee \otimes \pi_C^*\mathcal{O}_p) \simeq \mathrm{R}\pi_*\mathrm{R}j_*(j^*\mathcal{E}_{k|p}^\vee) \simeq j^*\mathcal{E}_{k|p}^\vee.$$

After dualization, we find the exact triangle

$$\mathrm{L}j^*\mathbb{E} \to \widetilde{\mathbb{E}} \to j^*\mathcal{E}_{k|p}[1] \to +1,$$

which allows us to apply [BF97, Proposition 5.10] to obtain

$$0^![\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)]^{\mathrm{vir}} = [\mathsf{HQ}_{\mathbf{d-1}}(C,\mathbf{r},\widetilde{V})]^{\mathrm{vir}},$$

where 0 is the zero section of the vector bundle $(\mathcal{E}_{k|p}^\vee)$ on $\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)$. The rest of the proof follows as in Proposition 2.2. □

2.4. **A vanishing criterion.** Recall that the Hyperquot scheme $\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)$ is constructed iteratively using relative Quot schemes

$$\mathsf{Quot}_{d_1-d_2}(C,r_1,\mathcal{E}_2) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{k-2}} \mathsf{Quot}_{d_{k-1}-d_k}(C,r_{k-1},\mathcal{E}_k) \xrightarrow{\phi_{k-1}} \mathsf{Quot}_{d_k}(C,r_k,V) \xrightarrow{\phi_k} \{\mathrm{pt}\},$$

with relative virtual dimensions

$$\begin{aligned}\mathrm{vdim}\phi_j &= \chi(E_j,E_{j+1}) - \chi(E_j,E_j)\\ &= \big((1-g)\,r_j - e\big)(r_{j+1}-r_j) + r_{j+1}d_j - r_jd_{j+1}.\end{aligned}$$

We note the following vanishing result, not immediate from the formula in Theorem 1.1, which follows from the properties of virtual pullbacks.

Remark 2.2. When the virtual dimension of an intermediate Quot scheme $\mathsf{Quot}_{d_j-d_{j+1}}(C,r_j,\mathcal{E}_{j+1})$ is negative, i.e. when $\sum_{i=j}^k \mathrm{vdim}\phi_i < 0$, the virtual class $[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)]^{\mathrm{vir}}$ is zero and all virtual integrals on the Hyperquot scheme vanish:

$$\int_{[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)]^{\mathrm{vir}}} \prod_{j=1}^k\prod_{i=1}^{r_j} c_i(\mathcal{E}_{j|p}^\vee)^{m_{i,j}} \;=\; 0.$$

Remark 2.3. Notice that similar vanishings need not hold when the virtual dimension of a single morphism ϕ_j is negative. For instance, let C be a curve of genus 13, let V be a rank-3 bundle of degree 0, and take $\mathbf{r}=(1,2)$. Then Corollary 1.2 implies

$$\sum_{d_1,d_2\geqslant 0} q_1^{d_1}q_2^{d_2}\int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} 1 \;=\; 6^{13}\big(q_1^{10}q_2^8 + 20\,q_1^9q_2^9 + q_1^8q_2^{10}\big)\,.$$

When $d_1=d_2=9$, the relative virtual dimensions are $\mathrm{vdim}\phi_2=3$ and $\mathrm{vdim}\phi_1=-3$, yet the virtual integral of 1 equals $20\cdot 6^{13}\neq 0$.

2.5. **Counting maximal subsheaves.** I am considering to move this to the Explicit Formulas section, as indeed we are talking about specific Hyperquot schemes (those whose base of the tower is virtually zero dimensional) and since we need the Vafa–Intriligator formula (more precisely, the fact that the integrals only depend on the numerical invariants) to prove the Corollary Recall the coefficient 6^{13} of $q_1^{10}q_2^8$ in the polynomial computed in Remark 2.3. We now give a geometric interpretation of this number.

Recall that, if V is a general stable bundle V of degree e and $\operatorname{vdim}\mathsf{Quot}_d(C,r,V)=0$, the Quot scheme $\mathsf{Quot}_d(C,r,V)$ is a smooth zero-dimensional scheme consisting of all the rank-r subbundles of V having maximal slope [Hol04]. Let $m(n,e,r,g)=\deg[\mathsf{Quot}_d(C,r,V)]$ be the number of such subbundles. Then we can prove the following

Corollary 2.1. *Let V be a vector bundle on C of rank n and degree e, and assume that $g\geqslant 2$. Fix $\mathbf{r}=(r_1,\dots,r_k)$ and $\mathbf{d}=(d_1,\dots,d_k)$ so that $\operatorname{vdim}\phi_k=0$. For any rank-$r_k$ vector bundle E_k of degree $e-d_k$, we have*

$$\int_{[\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)]^{\mathrm{vir}}}\prod_{j=1}^{k-1}\prod_{i=1}^{r_j}c_i(\mathcal{E}_{j|p}^\vee)^{m_{i,j}} \;=\; m(n,e,r_k,g)\cdot\int_{[\mathsf{HQ}_{\mathbf{d}'}(C,\mathbf{r}',E_k)]^{\mathrm{vir}}}\prod_{j=1}^{k-1}\prod_{i=1}^{r_j}c_i(\mathcal{E}_{j|p}^\vee)^{m_{i,j}},$$

where $\mathbf{d}'=(d_1-d_k,\dots,d_{k-1}-d_k)$ and $\mathbf{r}'=(r_1,\dots,r_{k-1})$.

Proof. The Vafa–Intriligator formula of Theorem 1.1 shows that, fixed the monomial in the Chern classes, the virtual integral on the Hyperquot scheme only depends on $g,n,e,\mathbf{r}$ and $\mathbf{d}$. If $g\geqslant 2$ there exists a stable vector bundle of degree e and rank n, hence we can assume that V is a general stable bundle and that $\mathsf{Quot}_{d_k}(C,r_k,V)=\{V_1,\dots,V_m\}$ with $m:=m(n,e,r_k,g)$. In this case we can compute the pushforward to $\mathsf{Quot}_{d_k}(C,r_k,V)$ as

$$(\phi_{k-2}\circ\cdots\circ\phi_1)_*\prod_{j=1}^{k-1}\prod_{i=1}^{r_j}c_i(\mathcal{E}_{j|p}^\vee)^{m_{i,j}}=\sum_{j=1}^{m}[V_j]\int_{[\mathsf{HQ}_{\mathbf{d}'}(C,\mathbf{r}',V_j)]^{\mathrm{vir}}}\prod_{j=1}^{k-1}\prod_{i=1}^{r_j}c_i(\mathcal{E}_{j|p}^\vee)^{m_{i,j}}. \tag{14}$$

Since the integrals on the right only depend on $g,\mathbf{d}$ and $\mathbf{r}$, we can assume that $V_j=E_k$ for all j, and the formula follows by pushing forward (14) to a point, which algebraically consists in setting $[V_j]$ to 1 for all j. □

Remark 2.4. We can now easily compute, for a curve of genus 13 and a rank-3 bundle V of degree zero, the virtual integral

$$\int_{[\mathsf{HQ}_{(10,8)}(C,(1,2),V)]^{\mathrm{vir}}}1=m(3,0,2,13)\cdot m(2,-8,1,13)=3^{13}\cdot 2^{13}=6^{13},$$

where the numbers of maximal subbundles are computed by [Hol04, Corollary 4.5]. Indeed one can easily check that $\operatorname{vdim}\phi_1=\operatorname{vdim}\phi_2=0$ in this case.

Remark 2.5. Note that the hypothesis $g\geqslant 2$ is needed to ensure the existence of a stable bundle of rank n and degree e. Strictly speaking, such stable bundles also exist for $g=1$ in the coprime case $\gcd(n,e)=1$ by work of Atiyah (see [**?**, Thm 1 and below]), but in this case the Quot scheme has virtual dimension $\operatorname{vdim}\mathsf{Quot}_d(C,r,V)=nd+(r-n)e$, which is never zero since n and e are coprime.

3. Torus action and fixed loci

By deformation invariance property of virtual integrals proved in Proposition 2.1, the calculation reduces to the case of split vector bundles, so we can assume that

$$V = M_1 \oplus M_2 \oplus \cdots \oplus M_n \tag{15}$$

where each M_i is a line bundle of degree zero over C.

3.1. **Equivariant formulation.** In this section we state the equivariant version of our main Theorem 1.1. Let the torus $T = (\mathbb{C}^*)^n$ act on V by scaling the fiber of each line bundle summand M_j by t_j^{-1}. This induces a T action on the Hyperquot scheme $\mathsf{HQ}_{\mathbf{d}} := \mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)$, by sending a point $[E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow V]$ to the composition

$$E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow V \xrightarrow{t} V$$

for every element $t \in T$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the weights of the action of T on $V^\vee = M_1^\vee \oplus M_2^\vee \oplus \cdots \oplus M_n^\vee$ or, more precisely, set

$$\varepsilon_i := c_1^T(M^\vee_{i|\mathrm{pt}}) \in H_T^*(\mathrm{pt}),$$

for any point $\mathrm{pt} \in C$.

We now define an equivariant analogue of the system of equation (3), which is a key ingredient of the formula that we want to prove. Consider the tuples of variables $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ as before with $\mathbf{z}_0 = \{\}$ and $\mathbf{z}_{k+1} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, as illustrated below.

$$\begin{array}{ccccccc}
\mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \cdots & \cdots & \mathbf{z}_k & \mathbf{z}_{k+1} \\
\hline
 & z_{1,1} & z_{1,2} & \cdots & \cdots & z_{1,k} & \varepsilon_1 \\
 & \vdots & \vdots & & & \vdots & \vdots \\
 & z_{r_1,1} & \vdots & & & \vdots & \vdots \\
 & & z_{r_2,2} & & & \vdots & \vdots \\
 & & & \ddots & & \vdots & \vdots \\
 & & & & \ddots & \vdots & \vdots \\
 & & & & & z_{r_k,k} & \vdots \\
 & & & & & & \varepsilon_n
\end{array}$$

Let $q_1, q_2 \ldots, q_k$ be formal variables, and define the following polynomials for each $1 \leqslant j \leqslant k$;

$$P_j^T(X) := \prod_{\alpha \in \mathbf{z}_{j+1}} (X - \alpha) + (-1)^{r_j - r_{j-1}} q_j \prod_{\alpha \in \mathbf{z}_{j-1}} (X - \alpha). \tag{16}$$

Definition 3.1. For fixed values of $(\varepsilon_1, \ldots, \varepsilon_n; q_1, \ldots, q_k) \in \mathbb{C}^n \times (\mathbb{C}^*)^k$, we define the system of polynomial equations in the variables $z_{s,j}$

$$P_j^T(z_{s,j}) = 0 \quad \text{for each } 1 \leqslant j \leqslant k, \text{ and } 1 \leqslant s \leqslant r_j. \tag{17}$$

We say that a solution $(\mathbf{z}_1, \mathbf{z}_2 \ldots, \mathbf{z}_k) = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \ldots, \boldsymbol{\zeta}_k)$ to the above system is *non-degenerate* if each tuple $\boldsymbol{\zeta}_j = (\zeta_{1,j}, \zeta_{2,j}, \ldots, \zeta_{r_j,j})$ has no repetition.

The T-equivariant version of our main theorem is

Theorem 3.1. *Let V be as in (15), then for natural numbers $m_{s,j}$ and values of $(\varepsilon_1, \ldots, \varepsilon_n; q_1, \ldots, q_k)$ in a Zariski open subset of $\mathbb{C}^n \times (\mathbb{C}^*)^k$ intersecting $0 \times (\mathbb{C}^*)^k$, we have*

$$\sum_{\mathbf{d} \in \mathbb{N}^k} q_1^{d_1} q_2^{d_2} \cdots q_k^{d_k} \int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} \prod_{j=1}^{k} \prod_{s=1}^{r_j} c_s^T(\mathcal{E}_{j|p}^\vee)^{m_{s,j}} = \sum_{\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k} \prod_{j=1}^{k} \frac{1}{r_j!} \prod_{s=1}^{r_j} e_s(\boldsymbol{\zeta}_j)^{m_{i,j}} \cdot J^{g-1}$$

where the sum is taken over all non-degenerate solutions $(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k)$ of the equations in (17) and $e_i(\boldsymbol{\zeta}_j)$ denote the i-th elementary symmetric polynomial. The factor J is given by

$$J = \prod_{j=1}^{k} \frac{1}{\Delta(\boldsymbol{\zeta}_j)} \cdot \det\left(\frac{\partial P_j^T(z_{s,j})}{\partial z_{u,\ell}}\right)\Bigg|_{(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k) = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k)},$$

where $\Delta(X_1, \ldots, X_m) := \prod_{a \neq b}(X_a - X_b)$ and the second factor is the Jacobian of (16).

We will first prove a version of this theorem where the solutions are viewed as formal power series in the q-variables (see Theorem 4.2). The version as stated above will be proved in Proposition 4.4 when $C = \mathbb{P}^1$ and in Section 6.3 for arbitrary genus g.

3.2. **Fixed loci.** We now describe the fixed loci for the torus action introduced in the previous section. A point $[E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow V] \in \mathsf{HQ}_{\mathbf{d}}$ is fixed under the torus action if and only if each of the subsheaves split as a direct sum of line bundles

$$E_j = K_{1,j} \oplus K_{2,j} \oplus \cdots \oplus K_{r_j,j} \quad \text{for } 1 \leqslant j \leqslant k;$$

and the inclusion $E_j \hookrightarrow E_{j+1}$ respects the splitting. More explicitly, this means that there exists a permutation $\sigma \in S_n$ such that for all $1 \leqslant s \leqslant r_j$,

$$K_{s,j} \hookrightarrow K_{s,j+1} \quad \forall\ 1 \leqslant j \leqslant k-1$$

and each summand $K_{s,k}$ of E_k maps into $M_{\sigma(s)} \subset V$.

To each torus-fixed point $[E_{\mathbf{r}} \hookrightarrow V] \in \mathsf{HQ}_{\mathbf{d}}$, we associate two pieces of data:

- The *splitting degree*, a tuple recording the degrees of the line bundles $K_{s,j}^\vee$:

$$D = (d_{s,j}) \quad \text{for } 1 \leqslant j \leqslant k,\ 1 \leqslant s \leqslant r_j.$$

Note that $d_j = \sum_{s=1}^{r_j} d_{s,j}$ for all $1 \leqslant j \leqslant k$. Should we also impose $d_{s,j} \geqslant d_{s,j+1}$, otherwise the fixed locus is empty

- A permutation

(18) $$\sigma \in \mathfrak{S}_{\mathbf{r}} := S_n / \left(S_{r_1} \times S_{r_2 - r_1} \times \cdots \times S_{r_k - r_{k-1}} \times S_{n - r_k}\right)$$

that records the choice of summands in V used to split the vector bundle E_k. The quotient by the subgroup $S_{r_{j+1} - r_j}$ accounts for the fact that reordering the line bundle summands in E_{j+1}/E_j yields the same splitting data up to permutation of degrees (here, we set $r_0 = 0$ and $r_{k+1} = n$).

For every class $\sigma \in \mathfrak{S}_{\mathbf{r}}$, fix once and for all a representative in S_n, which by an abuse of notation we will still denote with

(19) $$\sigma \in S_n.$$

This allows to write $\sigma(s)$ for $1 \leqslant s \leqslant k$. Chosen a pair (σ, D), the corresponding fixed locus $\mathrm{X}_{\sigma,D}$ is the product of Hilbert schemes of points of the curve

$$\mathrm{X}_{\sigma,D} = \prod_{j=1}^{k} \prod_{s=1}^{r_j} C^{[d_{s,j} - d_{s,j+1}]} \subset \mathsf{HQ}_{\mathbf{d}}(C, \mathbf{r}, V)$$

which can be described graphically by the diagram:

$$
\begin{array}{ccccccccc}
C^{[d_{1,1}-d_{1,2}]} & \times & C^{[d_{1,2}-d_{1,3}]} & \times & \cdots & \times & C^{[d_{1,k-1}-d_{1,k}]} & \times & C^{[d_{1,k}]} \\
\times & & \times & & \cdots & & \times & & \times \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\times & & \times & & \cdots & & \times & & \times \\
C^{[d_{r_1,1}-d_{r_1,2}]} & \times & C^{[d_{r_1,2}-d_{r_1,3}]} & \times & \cdots & \times & C^{[d_{r_1,k-1}-d_{1,k}]} & \times & C^{[d_{r_1,k}]} \\
 & & \times & & \cdots & & \times & & \times \\
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \times & & \cdots & & \times & & \times \\
 & & C^{[d_{r_2,2}-d_{r_2,3}]} & \times & \cdots & \times & C^{[d_{r_2,k-1}-d_{r_2,k}]} & \times & C^{[d_{r_2,k}]} \\
 & & & & & & \times & & \times \\
 & & & & \ddots & & \vdots & & \vdots \\
 & & & & & & \times & & \times \\
 & & & & & & C^{[d_{r_{k-1},k-1}-d_{r_{k-1},k}]} & \times & C^{[d_{r_{k-1},k}]} \\
 & & & & & & & & \times \\
 & & & & & & & & \vdots \\
 & & & & & & & & \times \\
 & & & & & & & & C^{[d_{r_k,k}]}
\end{array}
$$

For the s-th row, let j_s be the smallest j such that E_j has rank r_j at least s. The scheme

$$C^{[d_{s,j_s}-d_{s,j_s+1}]} \times \cdots \times C^{[d_{s,k-1}-d_{s,k}]} \times C^{[d_{s,k}]}$$

parametrizes sequences of injections of invertible sheaves

$$K_{s,j_s} \hookrightarrow K_{s,j_s+1} \hookrightarrow \cdots \hookrightarrow K_{s,k-1} \hookrightarrow K_{s,k} \hookrightarrow K_{s,k+1} := M_{\sigma(s)}.$$

Note that the permutation σ doesn't play a role in describing geometry of the fixed $X_{\sigma,D}$, but it determines how $X_{\sigma,D}$ embeds in the $\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)$. It is crucial too keep track of σ, for example, in the calculation of virtual normal bundles in the next section.

3.3. **Virtual normal bundle.** Recall the universal subsheaves

$$\mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_j \hookrightarrow \cdots \hookrightarrow \mathcal{E}_{k-1} \hookrightarrow \mathcal{E}_k \hookrightarrow \mathcal{E}_{k+1} := \pi_C^* V \tag{20}$$

on $\mathsf{HQ}_{\mathbf{d}} \times C$, and let π and π_C denote the projections to $\mathsf{HQ}_{\mathbf{d}}$ and C respectively. Consider the virtual tangent bundle of $\mathsf{HQ}_{\mathbf{d}}$, namely the K-theory class of the dual of the obstruction complex in (9):

$$T^{\mathrm{vir}} = \left[\bigoplus_{j=1}^{k} \mathcal{H}om_\pi(\mathcal{E}_j, \mathcal{E}_{j+1}) - \bigoplus_{j=1}^{k} \mathcal{H}om_\pi(\mathcal{E}_j, \mathcal{E}_j) \right], \tag{21}$$

where $\mathcal{H}om_\pi$ is the composition of the functors $R\pi_*$ and $\mathcal{H}om$. The restriction of the universal subsheaves (20) to the fixed loci $\mathrm{X}_{\sigma,D} \times C$ is best seen in the following diagram:

$$\begin{array}{ccccccccccc}
\mathcal{K}_{1,1} & \longrightarrow & \mathcal{K}_{1,2} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_{1,k-1} & \longrightarrow & \mathcal{K}_{1,k} & \longrightarrow & \mathcal{K}_{1,k+1} \\
\oplus & & \oplus & & \cdots & & \oplus & & \oplus & & \oplus \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\oplus & & \oplus & & \cdots & & \oplus & & \oplus & & \oplus \\
\mathcal{K}_{r_1,1} & \longrightarrow & \mathcal{K}_{r_1,2} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_{r_1,k-1} & \longrightarrow & \mathcal{K}_{r_1,k} & \longrightarrow & \mathcal{K}_{r_1,k+1} \\
 & & \oplus & & \cdots & & \oplus & & \oplus & & \oplus \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \oplus & & \cdots & & \oplus & & \oplus & & \oplus \\
 & & \mathcal{K}_{r_2,2} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_{r_2,k-1} & \longrightarrow & \mathcal{K}_{r_2,k} & \longrightarrow & \mathcal{K}_{r_2,k+1} \\
 & & & & & & \oplus & & \oplus & & \oplus \\
 & & & & \ddots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \oplus & & \oplus & & \oplus \\
 & & & & & & \mathcal{K}_{r_{k-1},k-1} & \longrightarrow & \mathcal{K}_{r_{k-1},k} & \longrightarrow & \mathcal{K}_{r_{k-1},k+1} \\
 & & & & & & & & \oplus & & \oplus \\
 & & & & & & & & \vdots & & \vdots \\
 & & & & & & & & \oplus & & \oplus \\
 & & & & & & & & \mathcal{K}_{r_k,k} & \longrightarrow & \mathcal{K}_{r_k,k+1}
\end{array}$$

where $\mathcal{E}_j$ restricts to j-th column for all $1 \leqslant j \leqslant k$, i.e.,

$$\mathcal{E}_j\big|_{\mathrm{X}_{\sigma,D}\times C} = \mathcal{K}_{1,j} \oplus \cdots \oplus \mathcal{K}_{r_j,j}, \quad \text{and} \quad \mathcal{K}_{s,k+1} := \pi_C^* M_{\sigma(s)} \ \forall s \in \{1,2,\ldots,n\}.$$

The virtual tangent bundle of $\mathsf{HQ}_{\mathbf{d}}$ restricts to the following K-theory class on $\mathrm{X}_{\sigma,D}$:

$$T^{\mathrm{vir}}\big|_{\mathrm{X}_{\sigma,D}} = \pi_* \sum_{j=1}^{k} \left(\sum_{s\leqslant r_j, u\leqslant r_{j+1}} \mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j+1} - \sum_{s,u\leqslant r_j} \mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j} \right),$$

where π_* denote the push forward in K-theory for the projection $\pi : \mathrm{X}_{\sigma,D} \times C \to \mathrm{X}_{\sigma,D}$.

Then the torus weights for each term in the above expression are given by $\varepsilon_{\sigma(u)} - \varepsilon_{\sigma(s)}$. The virtual normal bundle for $\mathrm{X}_{\sigma,D} \subset \mathsf{HQ}_{\mathbf{d}}$, denoted $N_{\sigma,D}^{\mathrm{vir}}$, is the K-theory class on $\mathrm{X}_{\sigma,D}$ given by moving part of the virtual tangent bundle, i.e., by imposing the condition $s \neq u$:

$$N_{\sigma,D}^{\mathrm{vir}} = \pi_* \sum_{j=1}^{k} \left(\sum_{\substack{s\leqslant r_j, u\leqslant r_{j+1} \\ s\neq u}} \mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j+1} - \sum_{\substack{s,u\leqslant r_j \\ s\neq u}} \mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j} \right). \tag{22}$$

Note that the stationary part of the virtual tangent bundle, that is, when $s = u$, yields the classical tangent bundle on $\mathrm{X}_{\sigma,D}$.

4. Integrals over Hyperquot scheme on $\mathbb{P}^1$

In this section we will illustrate the proof of Theorem 1.1 when $C = \mathbb{P}^1$ and V is a trivial bundle of rank n. The Hyperquot scheme $\mathsf{HQ}_{\mathbf{d}} = \mathsf{HQ}_{\mathbf{d}}(\mathbb{P}^1, \mathbf{r}, \mathcal{O}_{\mathbb{P}^1}^{\oplus n})$ is a smooth projective variety of the expected dimension, and the virtual class coincides with the usual fundamental class. We will return to the general case in Section 6, after introducing additional tools that are needed.

In this section we apply the Atiyah–Bott localization formula to reduce the calculation to summing over contributions over the fixed loci, that are products of projective spaces. This leaves us with a complicated residue calculation, which we tackle using the multivariate Lagrange-Bürmann formula and prove Theorem 3.1 in genus zero setting. Specializing to non-equivariant setting requires a bit more care, we argue it carefully in the Section 4.3.

4.1. **Equivariant Euler class in genus zero.** In K-theory, the class of the classical tangent bundle over HQ_d equals the virtual tangent bundle given in (21). Recall that the fixed loci $\mathrm{X}_{\sigma,D}$ defined in Section 3.2 are products of Hilbert schemes of points on $\mathbb{P}^1$, and therefore isomorphic to products of projective spaces

$$\mathrm{X}_{\sigma,D} \cong \prod_{j=1}^{k} \prod_{s=1}^{r_j} \mathbb{P}^{d_{s,j}-d_{s,j+1}},$$

with the convention that $d_{s,k+1} = 0$ for all s. Note that over $\mathrm{X}_{\sigma,D} \times \mathbb{P}^1$, we have

$$\mathcal{K}_{s,j} = \left(\boxtimes_{\ell \geqslant j} \mathcal{O}_{\mathbb{P}^{d_{s,\ell}-d_{s,\ell+1}}}(-1) \right) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-d_{s,j}). \tag{23}$$

Let

$$x_{s,j} \in H^2(\mathbb{P}^{d_{s,j}-d_{s,j+1}}, \mathbb{Z}) \quad \text{for all } 1 \leqslant j \leqslant k,\ 1 \leqslant s \leqslant r_j$$

be the hyperplane classes, and use the same notation for their pullbacks in $H^2(\mathrm{X}_{\sigma,D}, \mathbb{Z})$ under the projection map. The equivariant first Chern class of the dual of $\mathcal{K}_{s,j}$ is given by

$$c_1^T(\mathcal{K}_{s,j}^\vee) = z_{s,j} \otimes 1 + 1 \otimes (d_{s,j} \cdot [\mathrm{pt}]) \in H_T^2(\mathrm{X}_{\sigma,D} \times \mathbb{P}^1)$$

under the Künneth isomorphism, where $[\mathrm{pt}]$ is the class of a point $p \in \mathbb{P}^1$ and

$$z_{s,j} := c_1^T(\mathcal{K}_{s,j|p}^\vee) = x_{s,j} + x_{s,j+1} + \cdots + x_{s,k} + \varepsilon_{\sigma(s)}, \tag{24}$$

where $\mathcal{K}_{s,j|p}$ denote the restriction of $\mathcal{K}_{s,j}$ to $\mathrm{X}_{\sigma,D} \times \{p\}$.

For convenience, we define the degree r_j polynomials

$$R_j(z) := (z - z_{1,j})(z - z_{2,j}) \cdots (z - z_{r_j,j}) \quad \text{for } 1 \leqslant j \leqslant k, \tag{25}$$

$$R_0(z) = 1, \quad \text{and} \quad R_{k+1}(z) := (z - \varepsilon_1) \cdots (z - \varepsilon_n).$$

In what follows, we will work with the ring obtained by inverting the equivariant parameters in $H_T^*(\mathrm{pt})$, which we denote by

$$\Gamma := \mathbb{C}[\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_1^{-1}, \ldots, \varepsilon_n^{-1}], \tag{26}$$

I'm worried that we ght have to work over $\varepsilon_1, \ldots, \varepsilon_k$), see below.

where $\varepsilon_j = c_1^T(M_{j|p}^\vee)$. We shall also use bold symbols as a shorthand notation for tuples, for instance, we write $\mathbf{x} := (x_{s,j})_{j \leqslant k,\ s \leqslant r_j}$.

Proposition 4.1. *The equivariant Euler class of the normal bundle is given by*

$$\frac{1}{e_T(N_{\sigma,D}^{\mathrm{vir}})} = (-1)^{\beta} \prod_{j=1}^{k} \prod_{s \leqslant r_j} x_{s,j}^{d_{s,j}-d_{s,j+1}+1} \frac{R_j'(z_{s,j}) R_{j-1}(z_{s,j})^{d_{s,j}}}{R_{j+1}(z_{s,j})^{d_{s,j}+1}} \quad \in \Gamma[\mathbf{x}], \tag{27}$$

where the sign is given by

$$\beta = \sum_{j \leqslant k} d_j(r_j - r_{j-1} - 1).$$

One should be a bit careful here. This is an expression in a field of rational functions, NOT in the honest equivarian cohomology of the fixed locus. In other words, here the x's are formal variables: if they were cohomology classes we would have $x_{s,j}^{d_{s,j}-d_{s,j+1}+1} = 0$ on $\mathbb{P}^{d_{s,j}-d_{s,j+1}}$

Proof. Using the description in (23) and the projection formula, we note that the equivariant Euler classes of the pushforwards appearing in (22) are given by

$$e_T\left(\pi_*\left(\mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j}\right)\right) = (z_{s,j} - z_{u,j})^{d_{s,j}-d_{u,j}+1};$$
$$e_T\left(\pi_*\left(\mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j+1}\right)\right) = (z_{s,j} - z_{u,j+1})^{d_{s,j}-d_{u,j+1}+1}.$$

We now carefully substitute the equivariant Euler classes computed above into the formula for the normal bundle $N_{\sigma,D}^{\mathrm{vir}}$ in (22), noting that equivariant Euler class is multiplicative.

We first consider the terms in the same column (i.e. keeping the subscript j fixed):

$$\begin{aligned}\prod_{\substack{s,u\leqslant r_j\\ s\neq u}} e_T\left(\pi_*\left(\mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j}\right)\right) &= (-1)^{d_j(r_j-1)} \prod_{\substack{s,u\leqslant r_j\\ s\neq u}} (z_{s,j} - z_{u,j})\\ &= (-1)^{d_j(r_j-1)} \prod_{s\leqslant r_j} R_j'(z_{s,j}).\end{aligned}$$

In the first equality, we used that $(z_{s,j} - z_{u,j})^{d_{s,j}-d_{u,j}}$ cancel each other by swapping the role of s and u, leaving behind the sign $(-1)^{d_j(r_j-1)}$.

We then consider the terms involving two consecutive columns:

$$\begin{aligned}\prod_{\substack{s\leqslant r_j\\ u\leqslant r_{j+1}\\ s\neq u}} e_T\left(\pi_*\left(\mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{u,j+1}\right)\right) &= \prod_{\substack{s\leqslant r_j\\ u\leqslant r_{j+1}\\ s\neq u}} (z_{s,j} - z_{u,j+1})^{d_{s,j}-d_{u,j+1}+1}\\ &= \prod_{s\leqslant r_j}\left(\frac{R_{j+1}(z_{s,j})}{x_{s,j}}\right)^{d_{s,j}+1} \cdot \prod_{u\leqslant r_{j+1}}\left(\frac{(-1)^{r_j}x_{u,j}}{R_j(z_{u,j+1})}\right)^{d_{u,j+1}},\end{aligned}$$

where we set $x_{u,j} := 1$ for $u > r_j$.

here's where the trick happens. In honest cohomology we're not allowed to divide by $x_{s,}$ (and by $R_j(z_{u,j+1})$ too

In the second term of the last expression, we substitute the index $j \to j-1$. Collecting the above two formulas with the correct sign governed by (22) for $1 \leqslant j \leqslant k$, we get the required identity. □

4.2. **Atiyah–Bott localization.** Recall that $\mathsf{HQ}_d = \mathsf{HQ}_{\mathbf{d}}(\mathbb{P}^1, \mathbf{r}, V)$ for a split vector bundle $V = M_1 \oplus \cdots \oplus M_n$, where $M_i^\vee \cong \mathcal{O}_{\mathbb{P}^1}$ endowed with equivariant weight ε_i for each $1 \leqslant i \leqslant n$.

We now use the Atiyah-Bott localization formula, see [AB84]: Let Q be a cohomology class in top degree $H^{2e}(\mathsf{HQ}_{\mathbf{d}})$ that is T-equivariant. Then

Since we insisted to wor in cohomology, the reference [EG98] is not really relevant. I think [AB84] is more appropriate

$$\int_{\mathsf{HQ}_{\mathbf{d}}} Q = \sum_{\sigma,D} \int_{\mathrm{X}_{\sigma,D}} \frac{Q|_{\mathrm{X}_{\sigma,D}}}{e_T(N_{\sigma,D}^{\mathrm{vir}})}.$$

I'm not sure all such classes are polynomials z-classes. For example can add ϵ classes freely

Note that the equivariant Euler class in the denominator is invertible in the equivariant cohomology (once the ε_i are inverted).

Is this true? Is it enoug to invert just the ϵ_i? Fo example I'm not sure if $f(\mathbf{x}) + \epsilon_1 - \epsilon_2$ would be invertible there. Are there linear factors in th denominator that are of this form, or are they al of the form $f(\mathbf{x}) + \varepsilon_i$, fo which inverting ε_i woul suffice?

To evaluate the integral, we integrate the top-power of the hyperplane class on each projective space $\mathbb{P}^{d_{s,j}-d_{s,j+1}}$, which amounts to extracting the coefficient of $x_{s,j}^{d_{s,j}-d_{s,j+1}}$ for all $1 \leqslant j \leqslant k$ and $1 \leqslant s \leqslant r_j$. Therefore, using (27), we express the integral as a residue:

$R'(z_{s,k})$ should be $R'(z_{s,j})$?

$$\int_{[\mathsf{HQ}_{\mathbf{d}}]} Q = (-1)^{\beta} \sum_{\sigma,D} \operatorname*{Res}_{\mathbf{x}=\mathbf{0}} \widetilde{Q}(\mathbf{z}) \prod_{\substack{j\leqslant k\\ s\leqslant r_j}} \frac{R'_j(z_{s,k})R_{j-1}(z_{s,j})^{d_{s,j}}}{R_{j+1}(z_{s,j})^{d_{s,j}+1}}, \tag{28}$$

where the residue is taken over each $x_{s,j}=0$ for $1\leqslant j\leqslant k$ and $1\leqslant a\leqslant r_j$; and $\widetilde{Q}(\mathbf{z})$ is the polynomial in $(z_{s,j})_{j\leqslant k,\ s\leqslant r_j}$ describing the class Q in the Chern roots of $\mathcal{E}_j$'s. Observe that if we enlarge the set D to include all $d_{s,j}$'s such that $\sum_{a=1}^{r_j} d_{s,j}=d_j$, allowing $d_{s,j}<d_{s,j+1}$ for $j\leqslant k-1$, then the new contributions are all zero. This can be seen by showing that when $d_{s,j}<d_{s,j+1}$, the expression above does not have Laurent part in $x_{s,j}$: observe that a factor $x_{s,j}$ without any equivariant parameter, appears in the denominator $R_{j+1}(z_{s,j})$ with multiplicity $1+d_{s,j}$, which gets canceled by the factor $x_{s,j}$ appearing in the numerator $R_j(z_{s,j+1})$ with multiplicity $d_{s,j+1}$.

We now apply the following multivariate Lagrange–Bürmann formula, see [Ges87, Theorem 2] for instance, to simplify the above residue:

Theorem 4.1 (Lagrange–Bürmann formula)**.** *Consider multivariate formal power series $\Phi_1,\dots,\Phi_N$ in variables $h_1,\dots,h_N$, each with non-zero constant term. Then the following change of variables*

$$q_i=\frac{h_i}{\Phi_i}\quad \textit{for } 1\leqslant i\leqslant N,$$

has unique inverses $h_i(q_1,q_2,\dots,q_N)$, which are multivariate power series for each $1\leqslant i\leqslant N$. Moreover, for any Laurent series Ψ in $h_1,\dots,h_N$,

$$\sum_{m_1,\dots,m_N\geqslant 0} q_1^{m_1}\cdots q_N^{m_N}\operatorname{Res}_{\mathbf{h}=0}\left(\Psi\cdot\prod_{i=1}^{N}\left(\frac{\Phi_i}{h_i}\right)^{m_i}\right)=\Psi\cdot\det\left(q_i^{-1}\frac{\partial q_i}{\partial h_j}\right)^{-1}_{1\leqslant i,j\leqslant N}, \tag{29}$$

where Ψ on the right-hand side is understood to be expressed in terms of the variables $q_1,\dots,q_N$.

Proposition 4.2. *For any fixed $\sigma\in\mathfrak{S}_{\mathbf{r}}$ and splitting degree D*

$$\operatorname*{Res}_{\mathbf{x}=\mathbf{0}} \widetilde{Q}(\mathbf{z}) \prod_{\substack{j\leqslant k\\ s\leqslant r_j}} \frac{R'_j(z_{s,k})R_{j-1}(z_{s,j})^{d_{s,j}}}{R_{j+1}(z_{s,j})^{d_{s,j}+1}} = [\mathbf{q}^D]\,\widetilde{Q}(\mathbf{z}) \prod_{\substack{j\leqslant k\\ s\leqslant r_j}} \frac{R'_j(z_{s,j})}{R_{j-1}(z_{s,j})}\det\left(\frac{dq_{s,j}}{dx_{u,i}}\right)^{-1}, \tag{30}$$

where we use the following change of variables to express the right hand side in terms of the new variables

$$q_{s,j}=\frac{R_{j+1}(z_{s,j})}{R_{j-1}(z_{s,j})}, \tag{31}$$

and $\mathbf{q}^D := \prod_{j\leqslant k,\ s\leqslant r_j} q_{s,j}^{d_{s,j}}$.

Proof. The calculation involves two steps. First, let

$$\Psi=\widetilde{Q}(\mathbf{z})\prod_{s\leqslant r_j, j\leqslant k}\frac{R'_j(z_{s,j})}{R_{j+1}(z_{s,j})}\quad\text{and}\quad \Psi_{s,j}=\frac{x_{s,j}R_{j-1}(z_{s,j})}{x_{s,j-1}R_{j+1}(z_{s,j})}.$$

Here, $x_{s,j}:=1$ for $s>r_j$. For all $j\leqslant k$ and $s\leqslant r_j$, the expressions $\Psi_{s,j}$ are multivariate formal power series in variables $\mathbf{x}$ with non-zero constant terms in the equivariant parameters. Expanded how?

For all $j \leqslant k$ and $s \leqslant r_j$, we define

$$\Phi_{s,j} := \Psi_{s,j}\Psi_{s,j-1}\cdots\Psi_{s,1},$$

where $\Psi_{s,j} := 1$ when $s > r_j$, which are again multivariate power series with non-zero constant terms. Thus the left-hand side of (30) equals

$$\begin{aligned}
&\operatorname{Res}_{\mathbf{x}=0}\left(\Psi\cdot\prod_{\substack{j\leqslant k\\ s\leqslant r_j}}\left(\frac{R_{j-1}(z_{s,j})}{R_{j+1}(z_{s,j})}\right)^{d_{s,j}}\right) &&= \operatorname{Res}_{\mathbf{x}=0}\left(\Psi\cdot\prod_{\substack{j\leqslant k\\ s\leqslant r_j}}\left(\Psi_{s,j}\frac{x_{s,j-1}}{x_{s,j}}\right)^{d_{s,j}}\right)\\
&= \operatorname{Res}_{\mathbf{x}=0}\left(\Psi\cdot\prod_{\substack{j\leqslant k\\ s\leqslant r_j}}\left(\frac{\Phi_{s,j}/x_{s,j}}{\Phi_{s,j-1}/x_{s,j-1}}\right)^{d_{s,j}}\right) &&= \operatorname{Res}_{\mathbf{x}=0}\left(\Psi\cdot\prod_{\substack{j\leqslant k\\ s\leqslant r_j}}\left(\frac{\Phi_{s,j}}{x_{s,j}}\right)^{d_{s,j}-d_{s,j+1}}\right)\\
&= \left[\prod_{\substack{j\leqslant k\\ s\leqslant r_j}} t_{s,j}^{d_{s,j}-d_{s,j+1}}\right]\Psi\det\left(t_{s,j}^{-1}\frac{\partial t_{s,j}}{\partial x_{u,i}}\right)^{-1},
\end{aligned}$$

where in the last equality we use Theorem 4.1 for the change of variables

$$t_{s,j} = \frac{x_{s,j}}{\Phi_{s,j}} \quad \text{for all } j \leqslant k,\ s \leqslant r_j. \tag{32}$$

Next, we make another change of variables

$$q_{s,j} = t_{s,j}/t_{s,j-1} \quad \text{for all } j \leqslant k,\ s \leqslant r_j,$$

where we set $t_{s,j} = 1$ if $s > r_j$. Observe that

$$t_{s,j} = q_{s,j}q_{s,j-1}\cdots q_{s,1} \quad \text{for all } j \leqslant k,\ s \leqslant r_j, \tag{33}$$

where we set $q_{s,m} = 1$ if $s > r_m$. Hence the residue equals

$$\left[\prod_{\substack{j\leqslant k\\ s\leqslant r_j}} q_{s,j}^{d_{s,j}}\right]\Psi\det\left(t_{i,j}^{-1}\frac{\partial t_{i,j}}{\partial x_{\ell,m}}\right)^{-1}, \tag{34}$$

and the right hand side in the expression above is still a multivariate Laurent series in $q_{s,j}$ ($j \leqslant k$, $s \leqslant r_j$). Furthermore, observe that when expressed in the variables $\mathbf{x}$,

$$q_{s,j} = \frac{x_{s,j}\Phi_{s,j-1}}{x_{s,j-1}\Phi_{s,j}} = \frac{R_{j+1}(z_{s,j})}{R_{j-1}(z_{s,j})},$$

which is not a multivariate power series but a Laurent series in the variables $\mathbf{x}$. Note that

$$\det\left(t_{s,j}^{-1}\frac{\partial t_{s,j}}{\partial q_{b,m}}\right)^{-1} = \prod_{j\leqslant k,\ s\leqslant r_j} q_{s,j} = \prod_{j\leqslant k,\ s\leqslant r_j}\frac{R_{j+1}(z_{s,j})}{R_{j-1}(z_{s,j})},$$

and hence (34) equals

$$\left[\mathbf{q}^D\right]\Psi\prod_{j\leqslant k, s\leqslant r_j}\frac{R_{j+1}(z_{s,j})}{R_{j-1}(z_{s,j})}\det\left(\frac{\partial q_{s,j}}{\partial x_{b,m}}\right)^{-1}.$$

□

Remark 4.1. In the above proof, the Lagrange–Bürmann formula does not apply directly because the final change of variables in (31) does not satisfy the conditions of Theorem 4.1. This subtlety is also present in the proof of Theorem 4.2 below and is crucial for justifying that the system of polynomial equations in (39) is solvable (as a system of power series) and has the correct number of solutions.

The following is a formal version of Theorem 3.1. A formal solution $(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k) = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k)$ to the system of equations in (16) is a solution such that each $\zeta_{s,j} \in \Gamma[\![q_1, q_2, \ldots, q_k]\!]$ is a formal power series with coefficients in Γ. We say that $(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k)$ is non-degenerate if each $\boldsymbol{\zeta}_j = (\zeta_{1,j}, \zeta_{2,j}, \ldots, \zeta_{r_j,j})$ has no repeated constant term in Γ.

Theorem 4.2. *Let V be as in (15), then for any tuple $m_{s,j}$, for $1 \leqslant j \leqslant k$ and $1 \leqslant s \leqslant r_j$,*

$$\sum_{\mathbf{d}\in\mathbb{N}^k} q_1^{d_1} q_2^{d_2} \cdots q_k^{d_k} \int_{[\mathsf{HQ}_\mathbf{d}]^{\mathrm{vir}}} \prod_{j=1}^{k} \prod_{s=1}^{r_j} c_s^T(\mathcal{E}_{j|p}^\vee)^{m_{s,j}} = \sum_{\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_k} \prod_{j=1}^{k} \frac{1}{r_j!} \prod_{i=1}^{r_j} e_s(\boldsymbol{\zeta}_j)^{m_{s,j}} \cdot J^{g-1} \tag{35}$$

where the sum is taken over all formal non-degenerate solutions $(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k)$ of the equations in (16) and $e_s(\boldsymbol{\zeta}_j)$ denote the s-th elementary symmetric polynomial. The factor J equals

$$J = \prod_{j=1}^{k} \frac{1}{\Delta(\boldsymbol{\zeta}_j)} \cdot \det\left(\frac{\partial P_j^T(z_{s,j})}{\partial z_{u,\ell}}\right)\Bigg|_{(\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_k)=(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\ldots,\boldsymbol{\zeta}_k)},$$

where $\Delta(X_1, \ldots, X_m) \coloneqq \prod_{a\neq b}(X_a - X_b)$ and the second factor is the Jacobian of (16).

Proof of Theorem 4.2 in genus zero setting. Identity (28) and Proposition 4.2 imply that

$$\int_{[\mathsf{HQ}_\mathbf{d}]} Q = (-1)^\beta \sum_{\sigma,D} [\mathbf{q}^D] \widetilde{Q}(\mathbf{z}) \prod_{\substack{j\leqslant k\\ s\leqslant r_j}} \frac{R_j'(z_{s,j})}{R_{j-1}(z_{s,j})} \det\left(\frac{dq_{s,j}}{dx_{u,i}}\right)^{-1}, \tag{36}$$

where σ runs over all the elements of $\mathfrak{S}_\mathbf{r}$ and $D = (d_{s,j})_{j\leqslant k,\ s\leqslant r_j}$ over the set of all splitting degrees that satisfy $\sum_{a=1}^{r_j} d_{s,j} = d_j$. Recall how the expression on the right-hand side is written in terms of the variables $\mathbf{q} \coloneqq (q_{s,j})_{j\leqslant k, s\leqslant r_j}$: the Lagrange–Bürmann formula implies that using the first change of coordinates in (32),

$$x_{s,j} = t_{s,j} F_{a,j}(\mathbf{t}, \varepsilon_\sigma) \quad \text{for all } j \leqslant k,\ s \leqslant r_j,$$

where $F_{a,j}$ is a multivariate power series in variables $\mathbf{t} \coloneqq (t_{s,j})_{j\leqslant k,\ s\leqslant r_j}$ with coefficients in Γ and a non-zero constant term. Using the second change of variables in (33), we observe that

$$x_{s,j} = q_{s,j} q_{s,j-1} \cdots q_{s,1} \widetilde{F}_{a,j}(\mathbf{q}, \varepsilon_\sigma) \quad \text{for all } j \leqslant k,\ s \leqslant r_j, \tag{37}$$

where $\widetilde{F}_{a,j}$ is a multivariate power series with non-zero constant term and $q_{s,m} = 1$ if $s > r_m$. Therefore, the expression on the right-hand side of (36) is a Laurent series in the variables $\mathbf{q}$ with coefficients in Γ.

Since we want to sum over all the D's satisfying $\sum_{i=1}^{r_j} d_{i,j} = d_j$, and since the Laurent series on the right in (36) does not depend on D, we may make the substitution we can specialise the variables as

$$q_j \coloneqq q_{1,j} = \cdots = q_{r_j,j}$$

to obtain

$$\int_{[\mathsf{HQ}_{\mathbf{d}}]} Q = (-1)^{\beta}[q_1^{d_1} q_2^{d_2} \cdots q_k^{d_k}] \sum_{\sigma} \widetilde{Q}(\mathbf{z}) \prod_{\substack{j\leqslant k \\ s\leqslant r_j}} \frac{R_j'(z_{s,j})}{R_{j-1}(z_{s,j})} \det \left(\frac{dq_{s,j}}{dx_{u,i}} \right)^{-1}. \tag{38}$$

Recall the change of variables in (31), and thus define the polynomials

$$\widetilde{P}_j(z) := R_{j+1}(z) - q_j R_{j-1}(z),$$

for $1 \leqslant j \leqslant k$ (recall (25)). Consider the system of equations

$$\widetilde{P}_j(z_{i,j}) = 0 \quad \text{for all } 1 \leqslant i \leqslant r_j,\ 1 \leqslant j \leqslant k. \tag{39}$$

For any permutation $\sigma \in S_n$, the multivariate power series obtained from (37):

$$\zeta_{s,j} = \sum_{\ell \geqslant j} q_\ell q_{\ell-1} \cdots q_{m_s} \widetilde{F}_{s,\ell}(\mathbf{q}, \varepsilon_\sigma) + \varepsilon_{\sigma(s)} \in \Gamma[\![q_1, q_2, \ldots, q_k]\!] \quad \text{for all } j \leqslant k,\ s \leqslant r_j, \tag{40}$$

where m_s is the smallest index such that $r_{m_s} \geqslant s$, gives us a formal non-degenerate solution $(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k) = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k)$ for (39).

To see that all non-degenerate solutions are of this form, we apply Hensel's Lemma [Bou72, III, §4.5, Cororllary 2] to the ring $A := \Gamma[\![q_1, q_2, \ldots, q_k]\!]$ with ideal $\mathfrak{m} := \langle q_1, \ldots, q_k \rangle$. It implies that for any solution $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k) \in A^{\sum_{j=1}^k r_j}$ to the system (39) mod $\mathfrak{m}^{\sum_{j=1}^k r_j}$ such that $\det(\partial P_j(z_{i,j})/\partial z_{k,l}) \mid_{\mathbf{z}=\mathbf{a}}$ is invertible in A, there exists a unique solution $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k)$ to the system (39) such that $\boldsymbol{\zeta} \equiv \mathbf{a} \bmod \mathfrak{m}^{\sum_{j=1}^k r_j}$. Solving (39) mod $\mathfrak{m}^{\sum_{j=1}^k r_j}$ is equivalent to solving the system

$$R_{j+1}(z_{i,j}) = 0 \quad \text{for all } 1 \leqslant i \leqslant r_j,\ 1 \leqslant j \leqslant k \tag{41}$$

mod $\mathfrak{m}^{\sum_{j=1}^k r_j}$, and we easily see that the only non-degenerate solutions $\mathbf{a}$ are of the form $a_{s,j} \equiv \varepsilon_{\sigma(s)} \bmod \mathfrak{m}$ for all $j \leqslant k,\ s \leqslant r_j$, where for some permutation $\sigma \in S_n$ (up to further permuting each $\mathbf{a}_j$ by S_{r_j}). For such $\mathbf{a}$, the constant term in $\det(\partial P_j(z_{i,j})/\partial z_{k,l}) \mid_{\mathbf{z}=\mathbf{a}}$ is $\det(\partial R_{j+1}(z_{i,j})/\partial z_{k,l}) \mid_{z_{i,j}=\varepsilon_{\sigma(i)}}$. Note that $(\partial R_{j+1}(z_{i,j})/\partial z_{k,l}) \mid_{z_{i,j}=\varepsilon_{\sigma(i)}}$ is triangular with diagonal entries being nonzero elements of $\mathbb{C}(\varepsilon_1, \ldots, \varepsilon_n)$. It follows that $\det(\partial P_j(z_{i,j})/\partial z_{k,l}) \mid_{\mathbf{z}=\mathbf{a}}$ is invertible in A. Therefore, we have found all non-degenerate solutions to the system (39).

Note that $\prod_{j=1}^k S_{r_j} \times \mathfrak{S}_n$ acts transitively and freely on the set of non-degenerate solutions of (39). The action of $\prod_{j=1}^k S_{r_j}$ does not affect the summands in (38). Since we consider the identity element of $\prod_{j=1}^k S_{r_j}$ while constructing the solution in (40), this justifies the factor $\prod_{j=1}^k 1/r_j!$ in the formula (35) when summing over all non-degenerate solutions.

We conclude the proof by a simple change of variables to write Jacobian in the desired form, and substituting $q_j \to (-1)^{r_j - r_{j+1}+1} q_j$, which cancels $(-1)^{\beta}$ in (38).

□

Note that the final expression must be polynomial in both ε and q variables. In the next section, we prove that this polynomial can be evaluated by first specializing the ε and q variables in (39) to values in an open subset of $\mathbb{C}^n \times (\mathbb{C}^*)^k$ intersecting $0 \times (\mathbb{C}^*)^k$, and then summing over its non-degenerate solutions, as stated in Theorem 3.1. In the process, we prove that we still obtain the correct number of non-degenerate solutions when we specialize the ε and q variables in this way.

4.3. **Equations.** Consider a chain of integers $0 = r_0 < r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k \leqslant r_{k+1}$ and set $R := \sum_{j=1}^{k} r_j$. We introduce a set of R variables $z_{i,j}$ for $j \in [k]$ and $i \in [r_j]$ and, fixed some complex numbers $\mathbf{q} := (q_1, \ldots, q_k)$, we consider the equations

$$\prod_{b=1}^{r_{j+1}} (z_{b,j+1} - z_{i,j}) - q_j \prod_{s=1}^{r_{j-1}} (z_{i,j} - z_{s,j-1}) = 0 \tag{$\mathrm{P}_{j,i}(\mathbf{q})$}$$

for every $j \in [k]$ and $i \in [r_j]$, where we use the notation $z_{i,k+1} = 0$ for all i. We denote with $X(\mathbf{q})$ the closed subscheme of the affine space $\mathbb{A}^R$ cut by the equations ($\mathrm{P}_{j,i}(\mathbf{q})$) and with $N(\mathbf{q})$ the cardinality of the set of closed points in $X(\mathbf{q})$, counted without multiplicity. We are interested in proving the following result:

Proposition 4.3. *For a generic choice of $\mathbf{q} := (q_1, \ldots, q_k) \in (\mathbb{C}^*)^k$, the system ($\mathrm{P}_{j,i}(\mathbf{q})$) admits the expected number of distinct solutions computed by Bézout's theorem, namely*

$$N(\mathbf{q}) = \prod_{j=1}^{k} r_{j+1}^{r_j}.$$

Moreover, the number of solutions satisfying $z_{i,j} \neq z_{\ell,j}$ for all $j \in [k]$ and all $i \neq \ell$ is $\prod_{j=0}^{k} \frac{r_{j+1}!}{(r_{j+1}-r_j)!}$.

4.3.1. *Idea of the proof.* Before attacking the problem by *Gröbner degeneration*, let's explain the idea. We will construct a family of closed subschemes of $\mathbb{P}^R$

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & \mathbb{P}^R \times (\mathbb{C}^*)^k \times \mathbb{A}^1 \arrow[d, "\pi_{q,t}"] \\ & (\mathbb{C}^*)^k \times \mathbb{A}^1 \end{tikzcd}$$

where the generic fiber $\mathcal{X}_{|(\mathbf{q},t)}$, for $t \neq 0$ is isomorphic to $X(\tilde{\mathbf{q}})$ for some $\tilde{\mathbf{q}}$ depending on $\mathbf{q}$ and t, while the special fiber above $(\mathbf{q}, 0)$ is a nice smooth zero dimensional scheme having the expected number of points. We will show that this family is smooth at every point of the form $(\mathbf{q}, 0)$ on the base, and since smoothness of projective morphisms is an open property on the base we can move all the results to nearby fibers of the form $\mathcal{X}_{|(q,t)}$ with $t \neq 0$, thus proving the claimed result.

4.3.2. *Construction of the degeneration.* For every $j \in [k]$ define

$$\rho_j := r_{j+1} - r_{j-1} \in \mathbb{N}.$$

Consider the homogeneous version of ($\mathrm{P}_{j,i}(\mathbf{q})$), which is

$$\prod_{b=1}^{r_{j+1}} (z_{b,j+1} - z_{i,j}) - q_j Z_0^{\rho_j} \prod_{s=1}^{r_{j-1}} (z_{i,j} - z_{s,j-1}) = 0 \tag{$\mathrm{HP}_{i,j}(\mathbf{q})$}$$

once we introduce the extra variable Z_0. Consider the $\mathbb{C}^*$-action on $\mathbb{P}^R \times (\mathbb{C}^*)^k$ given by

$$t \cdot z_{i,j} := t^j z_{i,j}, \quad t \cdot Z_0 := Z_0 \quad \text{and} \quad t \cdot q_j := t^{j\rho_j + r_{j-1}} q_j.$$

We construct the family over $\mathbb{A}^1\backslash 0 = \mathrm{Spec}(\mathbb{C}[t]_t)$ given by the zero locus of the perturbed equations

$$\prod_{b=1}^{r_{j+1}} t^j(tz_{b,j+1} - z_{i,j}) - t^{j\rho_j + r_{j-1}} q_j Z_0^{\rho_j} \prod_{s=1}^{r_{j-1}} t^{j-1}(tz_{i,j} - z_{s,j-1}) = 0$$

inside $\mathbb{P}^R \times (\mathbb{C}^*)^k \times (\mathbb{A}^1\backslash 0)$.

Remark 4.2. This is the family over $\mathbb{C}^* = \mathbb{A}^1\backslash 0$ obtained by moving the set of solutions to ($\mathrm{HP}_{i,j}(\mathbf{q})$) via the $\mathbb{C}^*$-action we just defined.

There is a classical way to extend this family (which is not flat a priori) to the whole $\mathbb{A}^1$ via [Eis95, Theorem 15.17], and the resulting family is the so-called *Gröbner degeneration* of ($\mathrm{HP}_{i,j}(\mathbf{q})$). This is obtained by factoring out all the extra powers of t, so that each equation admits a monomial of degree zero and no term of negative degree. Our equations become

$$(\mathrm{HP}_{i,j}(\mathbf{q},t)) \qquad \prod_{b=1}^{r_{j+1}} (tz_{b,j+1} - z_{i,j}) - q_j Z_0^{\rho_j} \prod_{s=1}^{r_{j-1}} (tz_{i,j} - z_{s,j-1}) = 0$$

and their zero locus $\mathcal{X} \hookrightarrow \mathbb{P}^R \times (\mathbb{C}^*)^k \times \mathbb{A}^1$ defines a scheme over $(\mathbb{C}^*)^k \times \mathbb{A}^1$ whose generic fiber is isomorphic to the zero locus of ($\mathrm{HP}_{i,j}(\mathbf{q})$) in $\mathbb{P}^R$, while the special fiber above $(\mathbf{q}, 0)$ is the zero locus of

$$(\mathrm{HP}_{i,j}(\mathbf{q},0)) \qquad (-z_{i,j})^{r_{j+1}} - (-1)^{r_{j-1}} q_j Z_0^{\rho_j} \prod_{s=1}^{r_{j-1}} z_{s,j-1} = 0.$$

4.3.3. *Properties of the fibers.*

Lemma 4.1. *For every $\mathbf{q} \in (\mathbb{C}^*)^k$, there is a Zariski open neighborhood U of $0 \in \mathbb{A}^1$ so that the fiber $\mathcal{X}_{|(\mathbf{q},t)}$ doesn't meet the hyperplane at infinity $Z_0 = 0$ for every $t \in U$.*

Proof. The statement follows from $\mathcal{X}_{(\mathbf{q},0)} \cap \{Z_0 = 0\} = \varnothing$, so we'll prove this. Clearly $\rho_1 > 0$ by definition, so the equation ($\mathrm{HP}_{i,1}(\mathbf{q},0)$) specialised to $Z_0 = 0$ implies $z_{i,1} = 0$ for all i. Now fix $j > 1$ and assume $z_{\ell,j-1} = 0$ for every ℓ. The equation ($\mathrm{HP}_{i,j}(\mathbf{q},0)$) implies $z_{i,j} = 0$ for every i, hence by induction we obtain that if a point of $\mathbb{P}^R$ satisfies both $Z_0 = 0$ and the equations ($\mathrm{HP}_{i,j}(\mathbf{q},0)$), then $z_{i,j} = 0$ for all i and j, which is a contradiction. □

Lemma 4.2. *Consider the projection $\mathcal{X} \to (\mathbb{C}^*)^k \times \mathbb{A}^1$. Given any $\mathbf{q}$ and a generic $t \neq 0$, the fiber $\mathcal{X}_{|(\mathbf{q},t)}$ is isomorphic to $X(t \cdot \mathbf{q})$.*

Proof. For a generic t, by Lemma 4.1 there are no solutions at $Z_0 = 0$, so we can work in the affine space $\mathbb{A}^R = \{X_0 \neq 0\}$. The global automorphism of $\mathbb{A}^R$ given by

$$z_{i,j} \mapsto t^j z_{i,j}$$

maps the equations defining $X(t \cdot \mathbf{q})$ into the equations defining $\mathcal{X}_{|(\mathbf{q},t)}$. □

Lemma 4.3. *For every $\mathbf{q} \in (\mathbb{C}^*)^k$ the fiber $\mathcal{X}_{|(\mathbf{q},0)}$ is zero dimensional, smooth, it consists of $\prod_{i=1}^k r_{i+1}^{r_i}$ distinct points and it doesn't intersect any coordinate hyperplane in $\mathbb{P}^R$. Moreover, the number of points in $\mathcal{X}_{|(\mathbf{q},0)}$ that satisfy $z_{i,j} \neq z_{\ell,j}$ for all $i \neq \ell$ and $j \in [k]$ is precisely $\prod_{j=0}^k \frac{r_{j+1}!}{(r_{j+1}-r_j)!}$.*

Proof. By Lemma 4.1 we can work in affine coordinates setting $Z_0 = 1$. The system ($\mathrm{HP}_{i,j}(\mathbf{q}, 0)$) can be solved explicitly. For $j = 1$ we find

$$-z_{i,1} \text{ is a } r_2\text{-th root of } q_1.$$

Then from $z_{l,1}$ we can recover $z_{i,2}$ via ($\mathrm{P}_{i,2}(\mathbf{q}, 0)$) and so on:

$$-z_{i,j} \text{ is a } r_{j+1}\text{-th root of } (-1)^{r_{j-1}} q_j \prod_{s=1}^{r_{j-1}} z_{s,j-1}.$$

Since $q_j \neq 0$ for every j, the number of such choices is precisely $\prod_{j=0}^{k} r_{j+1}^{r_j}$, and in particular $z_{i,j} \neq 0$ for every point in $\mathcal{X}^*_{|(\mathbf{q},0)}$. Finally, counting solutions with $z_{i,j} \neq z_{\ell,j}$ yields the second part of the claim. □

4.3.4. *Smoothness of the family and proof of Proposition 4.3.*

Lemma 4.4. *The projection*

$$(\pi_q, \pi_t) : \mathcal{X} \to (\mathbb{C}^*)^k \times \mathbb{A}^1.$$

is smooth at every point of $(\mathbb{C}^*)^k \times 0$.

Proof. Consider the Jacobian matrix $J(\mathbf{z}, \mathbf{q}, t)$ for the equations ($\mathrm{HP}_{i,j}(\mathbf{q}, 0)$). The $R \times R$ submatrix $J_{\partial_z}(\mathbf{z}, \mathbf{q}, 0)$ of $J(\mathbf{z}, \mathbf{q}, 0)$ corresponding to the z-derivatives $\partial/\partial_{z_{i,j}}$ is lower triangular, hence its determinant is the product of the diagonal terms:

$$\det J_{\partial_z}(\mathbf{x}, \mathbf{q}, 0) = \prod_{j=1}^{k} \prod_{i=1}^{r_j} (-1)^{r_{j+1}} r_{j+1} z_{i,j}^{r_{j+1}-1}$$

which, by Lemma 4.3, is non-zero at every point of $\mathcal{X}_{|(\mathbf{q},0)}$. By the Jacobian criterion, this implies the claim. □

We know that the locus of points $(\mathbf{q}, t)$ in $(\mathbb{C}^*)^k \times \mathbb{A}^1$ so that (π_q, π_t) is smooth at $(\mathbf{q}, t)$ is open since the map (π_q, π_t) is projective[1]. Hence for every $\mathbf{q} \in (\mathbb{C}^*)^k$ and a generic $t \in \mathbb{C}^*$ the point $(\mathbf{q}, t)$ is a regular value of this map.

Proof of Proposition 4.3. Given such a regular value $(\mathbf{q}, t)$ with $t \neq 0$ we have that $\mathcal{X}_{|(\mathbf{q},t)}$ is smooth of dimension zero, so by Bézout's theorem it is the union of $\prod_{j=1}^{k} r_{j+1}^{r_j}$ distinct closed points, and the proof of the first part of the Proposition follows from Lemma 4.2. Consider the symmetric group $S_{\mathbf{r}} \coloneqq \prod_{j=1}^{k} S_{r_j}$ acting on $\mathbb{A}^R$ permuting the same level coordinates. The ideal generated by the equations ($\mathrm{HP}_{i,j}(\mathbf{q}, t)$) is $S_{\mathbf{r}}$-invariant, hence $\mathcal{X}$ inherits an induced action. Fixed $(\mathbf{q}, 0) \in (\mathbb{C}^*)^k \times \mathbb{A}^1$, from Lemma 4.3 we know that the number of points in $\mathcal{X}_{|(\mathbf{q},0)}$ satisfying $z_{i,j} \neq z_{\ell,j}$ is $\prod_{j=0}^{k} \frac{r_{j+1}!}{(r_{j+1}-r_j)!}$. This implies that the fiber $\mathcal{X}_{|(\mathbf{q},0)}$ contains precisely $\prod_{j=0}^{k} \binom{r_{j+1}}{r_j}$ free $S_{\mathbf{r}}$-orbits. By smoothness, in a complex analytic neighborhood U of $(\mathbf{q}, 0)$ the family $\mathcal{X}_{|U}$ is the disjoint union of copies of U indexed by the points of $\mathcal{X}_{|(\mathbf{q},0)}$. The group $S_{\mathbf{r}}$ acts by shuffling these copies, and the action is completely determined by the action on the fiber above $\mathcal{X}_{|(\mathbf{q},0)}$, therefore for every $(\mathbf{q}, t) \in U$ with $t \neq 0$ we have that $X(t \cdot \mathbf{q})$ has precisely $\prod_{j=0}^{k} \binom{r_{j+1}}{r_j}$ free $S_{\mathbf{r}}$-orbits, concluding the proof. □

[1] A projective morphism is closed, hence the image of the critical locus of our map is closed and misses the origin of $\mathbb{A}^1$, so we take its complement.

4.3.5. *Equivariant version.* Let $\mathbb{A}^n$ be the space of some additional variables $\varepsilon_1, \ldots, \varepsilon_n$. Consider the family over $(\mathbb{C}^*)^k \times \mathbb{A}^n$ of closed subschemes of $\mathbb{A}^R$ given by the same equations ($\mathrm{P}_{j,i}(\mathbf{q})$), but this time interpreted as if $z_{i,k+1} := \varepsilon_i$ for all $i = 1, \ldots, n$. If we define

$$\delta := \bigcup_{j=1}^{k} \bigcup_{1 \leqslant i \neq \ell \leqslant r_j} \{z_{i,j} = z_{\ell,j}\} \subset \mathbb{A}^R$$

then we can consider the induced family of closed subschemes of $\mathbb{A}^R \backslash \delta$

$$\begin{tikzcd} \mathcal{X}' \arrow[r, hook, "i"] \arrow[dr, "p"'] & (\mathbb{A}^R \backslash \delta) \times (\mathbb{C}^*)^k \times \mathbb{A}^n \arrow[d, "p_{q,\varepsilon}"] \\ & (\mathbb{C}^*)^k \times \mathbb{A}^n \end{tikzcd}$$

Lemma 4.5. *There is an open subscheme* $\mathrm{Reg}(p) \subset \mathbb{A}^n \times (\mathbb{C}^*)^k$*, which is nonempty and intersects the subspace* $\varepsilon_1 = \cdots = \varepsilon_n = 0$*, so that* p *is étale on* $\mathrm{Reg}(p)$*. Moreover, if* f *is a regular function on* $p^{-1}(\mathrm{Reg}(p)) \subset \mathcal{X}'$*, then the function*

$$\mathrm{Tr}_p(f) : \mathrm{Reg}(p) \to \mathbb{C} \quad : \quad (\mathbf{q}, \boldsymbol{\varepsilon}) \mapsto \sum_{(\mathbf{x},\mathbf{q},\boldsymbol{\varepsilon}) \in p^{-1}(\mathbf{q},\boldsymbol{\varepsilon})} f(\mathbf{x}, \mathbf{q}, \boldsymbol{\varepsilon})$$

is a regular function on $\mathrm{Reg}(p)$*.*

Proof. The first part of the claim follows by considering first the homogenised equations cutting a family of subschemes of $\mathbb{P}^R$, showing that there are no solutions on the hyperplane at infinity, and that this family is smooth at a generic point of the form $(\mathbf{q}, 0)$ by the non-equivariant Lemma 4.4. Then projectivity ensures that smoothness is generic on the base. Moreover, we know that for a generic point of the base, the number of solutions in $\mathbb{A}^R \backslash \delta$ stays equal to $\prod_{j=0}^{k} \frac{r_{j+1}!}{(r_{j+1}-r_j)!}$, hence $\mathcal{X}'$ is finite étale over those points. The second part follows by applying the trace map $\mathrm{Tr}_p : p_* \mathcal{O}_{\mathcal{X}'} \to \mathcal{O}_{\mathbb{A}^n \times (\mathbb{C}^*)^k}$ to f [Aut, Tag 0BVH]. ☐

4.3.6. *Formal and algebraic equations.* Fix $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{C}^*$ with $\varepsilon_i \neq \varepsilon_j$ for all $i \neq j$ and consider the corresponding equivariant equations. A simple computation proves the following

Lemma 4.6. *Fixed* $q_1 = \cdots = q_k = 0$*, the equations can be solved in* $\mathbb{A}^R \backslash \delta$*. In this case, the number of solutions is* $\prod_{j=0}^{k} \frac{r_{j+1}!}{(r_{j+1}-r_j)!}$ *and the* x*-Jacobian is invertible over the solutions.*

This implies, by the implicit function theorem, that the algebraic solutions in $\mathbb{A}^R \backslash \delta$ can be upgraded to formal solutions $\boldsymbol{\zeta}_{\boldsymbol{\varepsilon}}(\mathbf{q}) \in \mathbb{C}[\![q_1, \ldots, q_k]\!]^R$. Not only that; there is an analytic neighborhood $U \subseteq (\mathbb{C}^*)^k$ of the origin so that for every $\mathbf{q} \in U$ the formal solutions above converge, and $\boldsymbol{\zeta}_{\boldsymbol{\varepsilon}}(\mathbf{q})$ are precisely all the solutions in $\mathbb{A}^R \backslash \delta$ of the system corresponding to $(\mathbf{q}, \boldsymbol{\varepsilon})$. Then we find the following

Lemma 4.7. *Fix* $\boldsymbol{\varepsilon} \in (\mathbb{C}^*)^n$*. There is a bijection between the solutions* $\boldsymbol{\zeta}_{\boldsymbol{\varepsilon}}$ *of the equivariant equations in* $\mathbb{C}[\![q_1, \ldots, q_k]\!]^R$ *such that* $\boldsymbol{\zeta}_{\boldsymbol{\varepsilon}}(0) \in \delta$*, and the solutions in* $\mathbb{A}^R \backslash \delta$ *for* $\mathbf{q}$ *small enough.*

Proof. By the proof of Theorem 4.2 in genus zero, the cardinality of the two sets is the same, equal to $\prod_{j=0}^{k} \frac{r_{j+1}!}{(r_{j+1}-r_j)!}$. Then the discussion above shows that lifting through the

Add ref to end of localisation section whe this is proven for formal solutions.

implicit function theorem gives an injective map from the set of algebraic solutions to the set of formal ones, and the inverse is given by letting the series converge. □

The equivariant Vafa–Intriligator formula of Theorem 3.1 holds true when the right-hand side is evaluated, fixed $\boldsymbol{\varepsilon} \in (\mathbb{C}^*)^n$, at the formal solutions of the equivariant system of equations. By Lemma 4.7 we can prove the following

Proposition 4.4. *Let $(\mathbf{q}, \boldsymbol{\varepsilon}) \in \mathrm{Reg}(p)$. Then the generating polynomial of equivariant virtual integrals $\mathbf{B}^{\mathbf{m}}_{g,\mathbf{r},V}$ can be evaluated at $(\mathbf{q}, \boldsymbol{\varepsilon})$ by computing the right-hand side of the equivariant Vafa–Intriligator formula by summing over the solutions in $\mathbb{A}^R \backslash \delta$ of the equivariant system of equations with fixed $(\mathbf{q}, \boldsymbol{\varepsilon})$.*

Proof. First of all notice that the equality that we want to prove is

$$\mathbf{B}^{\mathbf{m}}_{g,\mathbf{r},V}(\mathbf{q}, \boldsymbol{\varepsilon}) = \left(\prod_{j=1}^{k} \frac{1}{r_j!} \right) \mathrm{Tr}_p \left(J(\mathbf{z})^{g-1} \prod_{i,j} e_i(\mathbf{z}_j)^{m_{ij}} \right)$$

for a generic choice of $(\mathbf{q}, \boldsymbol{\varepsilon})$. Note that the left-hand side is a polynomial, while the right-hand is a rational function by Lemma 4.5. The discussion above shows that there is a small analytic nonempty open subset of $\mathbb{A}^n \times (\mathbb{C}^*)^k$ where the claim holds, but then the claim holds in general. □

5. Integrals over products of Hilbert schemes on curves

We collect notation and lemmas for intersection–theoretic computations on products of Hilbert schemes of points on curves. Although these results are essential for the proof of Theorem 1.1 for Hyperquot schemes on genus-g curves, experts may wish to skip this section on a first reading and return to it as needed.

Here C is a smooth projective curve of genus g and consider, given a non-negative integer m and a degree vector $\mathbf{d} \in \mathbb{N}^m$, the product of Hilbert schemes

$$X_{\mathbf{d}} := \prod_{i=1}^{m} C^{[d_i]}.$$

This is a smooth projective variety of dimension $\sum_{i=1}^{m} d_i$ and, on $X_{\mathbf{d}} \times C$, we can consider the tautological ideal sheaves $\mathcal{L}_i$ pulled back from $C^{[d_i]} \times C$. We will be interested in some cohomology classes on $X_{\mathbf{d}}$. In this section, we will denote with H^* the singular cohomology with real coefficients.

5.1. x-classes and y-classes.

For every $i \in \{1, \ldots, m\}$ we can consider the Künneth decomposition of the cohomology of $C^{[d_i]} \times C$:

$$H^*(C^{[d_i]} \times C) \simeq H^*(C^{[d_i]}) \otimes_{\mathbb{R}} H^*(C).$$

We define the following x and y classes through the Künneth decomposition of the first Chern class of a tautological bundle. Having fixed a symplectic basis $1, \delta_1, \ldots, \delta_{2g}, \eta$ for $H^*(C)$ we define $x_i \in H^2(C^{[d_i]})$ and $y_i^j \in H^1(C^{[d_i]})$ to be the classes satisfying

$$c_1(\mathcal{L}_i^\vee) = x_i \otimes 1 + \sum_{j=1}^{2g} y_i^j \otimes \delta_j + d_i \otimes \eta. \tag{42}$$

Let's recall a useful fact about intersections of x and y-classes, see [Tha92] or [Sin24, Section 6] for instance:

Lemma 5.1. *On a Hilbert scheme $C^{[d]}$, fix a positive integer $l \leqslant d$ and consider an integral of the form*

$$\int_{C^{[d]}} x^{d-l} \prod_{i=1}^{2l} y^{f(i)}$$

for a function $f : [2l] \to [2g]$. It is non-zero if and only if there is an injective function $h : [l] \to [g]$ so that

$$\prod_{i=1}^{2l} y^{f(i)} = \pm \prod_{j=1}^{l} y^{h(j)} \wedge y^{g+h(j)}.$$

Moreover, in that case

$$\int_{C^{[d]}} x^{d-l} \prod_{j=1}^{l} y^{h(j)} \wedge y^{g+h(j)} = \int_{C^{[d]}} x^{d} = 1.$$

By a slight abuse of notation we will still denote with x_i and y_i^j the pullbacks of these classes from $C^{[d_i]}$ to $X_{\mathbf{d}}$.

5.2. **Theta classes.** Consider a vector $\mathbf{w} = (w^1, \ldots, w^m) \in \mathbb{Z}^m$. The tensor product

$$\mathcal{L}_{\mathbf{w}} := \bigotimes_{i=1}^{m} \mathcal{L}_i^{\otimes w^i}$$

defines, after dualisation, a line bundle $\mathcal{L}_{\mathbf{w}}^{\vee}$ on $X_{\mathbf{d}} \times C$ of relative degree $\langle \mathbf{w}, \mathbf{d} \rangle := \sum_i w^i d_i$, hence it induces a morphism

$$\pi_{\mathbf{w}} : X_{\mathbf{d}} \to \operatorname{Pic}^{\langle \mathbf{w}, \mathbf{d} \rangle}(C)$$

satisfying

$$(\pi_{\mathbf{w}} \times \mathrm{Id}_C)^* \mathcal{P} = \mathcal{L}_{\mathbf{w}}^{\vee} \tag{43}$$

for some particular choice of the Poincaré line bundle $\mathcal{P}$ on $\operatorname{Pic}^{\langle \mathbf{w}, \mathbf{d} \rangle}(C) \times C$, by the universal property of the Jacobian of C. We also consider the Künneth decomposition of the first Chern class of the dual of this Poincaré bundle[2]:

$$c_1(\mathcal{P}^{\vee}) = x \otimes 1 + \sum_{j=1}^{2g} y^j \otimes \delta_j + \langle \mathbf{w}, \mathbf{d} \rangle \otimes \eta. \tag{44}$$

Definition 5.1. The *theta class* on $X_{\mathbf{d}}$ corresponding to w is the pullback of the theta class on the target Jacobian variety:

$$\theta_{\mathbf{w}} := \pi_{\mathbf{w}}^* \theta.$$

Remark 5.1. From this definition it's clear that $\theta_{\mathbf{w}}^j = 0$ for every $j > g$.

[2]It's immediate to see that the y-classes are invariant under tensoring $\mathcal{P}$ with a line bundle coming from the Jacobian, so they are independent of the choice of the Poincaré bundle. On the other hand, the class x changes by the first Chern class of the line bundle.

We have a straightforward characterization of this theta class in terms of y-classes:

Lemma 5.2. *Given $\mathbf{w} \in \mathbb{Z}^m$, the following equality holds true:*

$$\theta_{\mathbf{w}} = \sum_{j=1}^{g} \left(\sum_{i=1}^{m} w^i y_i^j \right) \wedge \left(\sum_{i=1}^{m} w^i y_i^{j+g} \right).$$

In particular $\theta_{\mathbf{w}} = \theta_{-\mathbf{w}}$.

Proof. It is well known [ACGH13, page 335] that

$$\theta = \sum_{j=1}^{g} y^j \wedge y^{j+g}.$$

Then the thesis follows from (42), (43), and (44), which indeed imply

$$\pi_{\mathbf{w}}^* y^j = -\sum_{i=1}^{m} w^i y_i^j$$

for every $j \in \{1, \ldots, 2g\}$. □

5.3. **Tautological pushforwards.** We consider the projection

$$\pi : X_{\mathbf{d}} \times C \to X_{\mathbf{d}}$$

and, given a vector $\mathbf{w} \in \mathbb{Z}^m$, we look at the class in $K^0(X_{\mathbf{d}})$ given by the pushforward $\pi_*[\mathcal{L}_{\mathbf{w}}]$. The following result follows from a standard application of the Grothendieck–Riemann–Roch theorem:

Lemma 5.3. *The Chern character of $\pi_*[\mathcal{L}_{\mathbf{w}}]$ is*

$$\mathrm{ch}(\pi_*[\mathcal{L}_{\mathbf{w}}^\vee]) = e^{\langle \mathbf{w}, \mathbf{x} \rangle}(1 - g + \langle \mathbf{w}, \mathbf{d} \rangle - \theta_{\mathbf{w}})$$

where we set $\langle \mathbf{w}, \mathbf{x} \rangle := \sum_i w_i x_i$.

Proof. Grothendieck–Riemann–Roch tells us that the Chern character of $\pi_*[\mathcal{L}_{\mathbf{w}}^\vee]$ is equal to

$$\pi_* \left(\mathrm{ch}(\mathcal{L}_{\mathbf{w}}^\vee) \cap \mathrm{Td}(T_\pi) \right), \tag{45}$$

where T_π is the (pullback of the) tangent bundle of the curve. Notice that, since π is a projection, π_* in cohomology is just integration over C. This means that, in order to compute the pushforward (45) we can just compute the Künneth decomposition of the argument in terms of the symplectic basis of $H^*(C)$ and extract the coefficient of η. We know that

$$\mathrm{Td}(T_C) = 1 + (1 - g)\eta,$$

so we are left to computing $\mathrm{ch}(\mathcal{L}_{\mathbf{w}}^\vee)$ in terms of the x and y classes defined in (42). Let $\mathcal{P}'$ be a Poincaré bundle on $\mathrm{Pic}^{\langle \mathbf{w}, \mathbf{d} \rangle}(C)$ such that there is a point $p \in C$ satisfying

$$\mathcal{P}'_{|_{\mathrm{Pic}^{\langle \mathbf{w}, \mathbf{d} \rangle}(C) \times p}} \simeq \mathcal{O}_{\mathrm{Pic}^{\langle \mathbf{w}, \mathbf{d} \rangle}(C)}.$$

This means that $c_1(\mathcal{P}')$ has trivial component coming from $H^2(\mathrm{Pic}^{\langle \mathbf{w}, \mathbf{d} \rangle}(C))$, and by (43)

$$c_1(\mathcal{L}_{\mathbf{w}}^\vee) = \langle \mathbf{w}, \mathbf{x} \rangle + (\pi_{\mathbf{w}} \times \mathrm{Id}_C)^* c_1(\mathcal{P}'),$$

which at the level of Chern characters reads

$$\mathrm{ch}(\mathcal{L}_{\mathbf{w}}^{\vee}) = e^{\langle \mathbf{w},\mathbf{x}\rangle}(\pi_{\mathbf{w}} \times \mathrm{Id}_C)^*\mathrm{ch}(\mathcal{P}').$$

The Chern character of this Poincaré line bundle is

$$\mathrm{ch}(\mathcal{P}') = 1 + \langle \mathbf{w}, \mathbf{d}\rangle\eta + \gamma - \theta \otimes \eta,$$

as computed in [ACGH13, pag 336], where γ is a class in $H^1(\mathrm{Pic}^{\langle \mathbf{w},\mathbf{d}\rangle}(C)) \otimes H^1(C)$. Therefore we find

$$\mathrm{ch}(\mathcal{L}_{\mathbf{w}}^{\vee}) = e^{\langle \mathbf{w},\mathbf{x}\rangle}(1 + \langle \mathbf{w}, \mathbf{d}\rangle\eta + (\pi_{\mathbf{w}} \times \mathrm{Id}_C)^*\gamma - \theta_{\mathbf{w}} \otimes \eta).$$

Putting all together, the Chern character of $\pi_*[\mathcal{L}_{\mathbf{w}}^{\vee}]$ is the coefficient of η in

$$e^{\langle \mathbf{w},\mathbf{x}\rangle}(1 + \langle \mathbf{w}, \mathbf{d}\rangle\eta + (\pi_{\mathbf{w}} \times \mathrm{Id}_C)^*\gamma - \theta_{\mathbf{w}} \otimes \eta)(1 + (1-g)\eta),$$

which gives the claimed expression after noticing that γ doesn't contribute since its component coming from C is of odd degree. □

From this we can obtain an expression for the total Chern class of this pushforward:

Proposition 5.1. *The total Chern class of $\pi_*[\mathcal{L}_{\mathbf{w}}]$ is*

$$c(\pi_*[\mathcal{L}_{\mathbf{w}}^{\vee}]) = (1 + \langle \mathbf{w}, \mathbf{x}\rangle)^{1-g+\langle \mathbf{w},\mathbf{d}\rangle} \exp\left(-\frac{\theta_{\mathbf{w}}}{1 + \langle \mathbf{w}, \mathbf{x}\rangle}\right).$$

Proof. This follows from Lemma 5.3 and the following standard fact: for every vector bundle V, if $\mathrm{ch}(V) = (k + \alpha)\mathrm{ch}(L)$ for $k \in \mathbb{Z}$, $\alpha \in H^2(X)$ and a line bundle L, then $c(V) = (1 + c_1(L))^k \exp(\frac{\alpha}{1+c_1(L)})$. To see that this holds true, apply the splitting principle and let $\omega_1, \ldots, \omega_k$ be the Chern roots of $V \otimes L^{-1}$, so that

$$\log(c(V \otimes L^{-1})) = \sum_{i=1}^{k} \log(1 + \omega_i) \quad \text{and} \quad \mathrm{ch}(V \otimes L^{-1}) = \sum_{i=1}^{k} \exp(\omega_i) = k + \alpha.$$

From the expression for the Chern character we find that $\sum_{i=1}^{k} \omega_i^l$ is non-zero if and only if $l = 0$ or $l = 1$, in which case we find $\sum_{i=1}^{k} \omega_i = \alpha$. Then expanding the logarithms we find that $\log(c(V \otimes L^{-1})) = \alpha$ and therefore

$$\prod_{i=1}^{k}(1 + \omega_i) = c(V \otimes L^{-1}) = \exp(\alpha).$$

This readily implies that

$$\begin{aligned} c(V) &= \prod_{i=1}^{k}(1 + \omega_i + c_1(L)) = (1 + c_1(L))^k \prod_{i=1}^{k}\left(1 + \frac{\omega_i}{1 + c_1(L)}\right) \\ &= (1 + c_1(L))^k \exp\left(\frac{\alpha}{1 + c_1(L)}\right), \end{aligned}$$

concluding the proof. □

5.4. **Substitution rules.** Here we study some integrals of the form

$$\int_{X_{\mathbf{d}}} P(x_1, \dots, x_m) Q(\theta_{\mathbf{w}_1}, \dots \theta_{\mathbf{w}_n})$$

for specific choices of a polynomial Q in n variables. In this expression, the weights $\mathbf{w}_1, \dots, \mathbf{w}_n \in \mathbb{Z}^m$ are possibly not distinct and P is a polynomial in m variables. Our aim in this section is to find a class $R_Q(\mathbf{w}_1, \dots, \mathbf{w}_n, x_1, \dots, x_m)$, involving the weights $\mathbf{w}_i$ and the x-classes, so that the integral above is equal to

$$\int_{X_{\mathbf{d}}} P(x_1, \dots, x_m) R_Q(\mathbf{w}_1, \dots, \mathbf{w}_n, x_1, \dots, x_m).$$

Notice that we don't aim at an equality at the level of cohomology classes; we just look for classes that integrate to the same number.

Definition 5.2. Let α_1, α_2 be two cohomology classes on $X_{\mathbf{d}}$ such that

$$\int_{X_{\mathbf{d}}} \alpha_1 \wedge P(x_1, \dots, x_m) = \int_{X_{\mathbf{d}}} \alpha_2 \wedge P(x_1, \dots, x_m)$$

for every polynomial P in the x-classes. We denote this equivalence relation with

$$\alpha_1 \rightsquigarrow \alpha_2$$

and call it a *substitution rule*.

We state here the first substitution rule, which is just the result of a rather lengthy linear algebra computation:

Lemma 5.4. *Let n be a positive integer and consider n weight vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in \mathbb{Z}^m$, from which we construct the cohomology-valued $n \times n$ matrix M whose entries are*

$$M_a^b := \sum_{c=1}^{m} w_a^c w_b^c x_c.$$

Then

$$\prod_{i=1}^{n} \theta_{\mathbf{w}_i} \rightsquigarrow \sum_{A_1 \sqcup \dots \sqcup A_g = [n]} \prod_{j=1}^{g} \det(M_{A_j, A_j})$$

is a substitution rule, where the sum is over all ordered partitions of $[n]$ in g (possibly empty) subsets, and M_{A_i, A_i} denotes the submatrix of M obtained by only considering the rows and the columns indexed by elements of A_i.

Proof. By Lemma 5.2 we can express the product of theta classes in terms of the y-classes. By considering sums over n-tuples as sums over functions from $[n]$ we can write

$$\prod_{i=1}^{n} \theta_{\mathbf{w}_i} = \sum_{i:[n]\to[g]} \sum_{j:[n]\to[m]} \sum_{k:[n]\to[m]} \Sigma_{i,j,k},$$

where the summand $\Sigma_{i,j,k}$ is

$$\Sigma_{i,j,k} := \prod_{h=1}^{n} \left(w_h^{j(h)} w_h^{k(h)} \right) y_{j(h)}^{i(h)} \wedge y_{k(h)}^{i(h)+g}.$$

By Lemma 5.1 we know which of these summands will have a chance[3] at contributing to the integral. More precisely we know that the integral of $P(x_1, \dots, x_m)\Sigma_{i,j,k}$ vanishes unless there is a permutation $\sigma \in S_n$ such that

(1) $i(h) = i(\sigma(h))$ for every $h \in [n]$, and
(2) $k(h) = j(\sigma(h))$ for every $h \in [n]$.

If such a σ exists, then the corresponding summand is

$$\Sigma_{i,j,k} := \Sigma_{i,j,j\circ\sigma} = \prod_{h=1}^{n} \left(w_h^{j(h)} w_h^{j(\sigma(h))}\right) y_{j(h)}^{i(h)} \wedge y_{j(\sigma(h))}^{i(\sigma(h))+g}.$$

By a simple reordering of the variables we see that

$$\prod_{h=1}^{n} y_{j(h)}^{i(h)} \wedge y_{j(\sigma(h))}^{i(\sigma(h))+g} = (-1)^{\sigma} \prod_{h=1}^{n} y_{j(h)}^{i(h)} \wedge y_{j(h)}^{i(h)+g},$$

therefore we can rewrite the summand $\Sigma_{i,j,\sigma}$ as

$$\Sigma_{i,j,\sigma} = \operatorname{sgn}(\sigma) \prod_{h=1}^{n} \left(w_h^{j(h)} w_h^{j(\sigma(h))}\right) y_{j(h)}^{i(h)} \wedge y_{j(h)}^{i(h)+g}.$$

By Lemma 5.1 we find the substitution rule for the summand $\Sigma_{i,j,\sigma}$

$$\Sigma_{i,j,\sigma} \rightsquigarrow \operatorname{sgn}(\sigma) \prod_{h=1}^{n} w_h^{j(h)} w_h^{j(\sigma(h))} x_{j(h)}.$$

Putting all together we obtain that

$$\prod_{i=1}^{n} \theta_{w_i} \rightsquigarrow \sum_{i:[n]\to[g]} \sum_{j:[n]\to[m]} \sum_{\sigma\in(S_n)_i} \operatorname{sgn}(\sigma) \prod_{h=1}^{n} w_h^{j(h)} w_h^{j(\sigma(h))} x_{j(h)}, \tag{46}$$

where $(S_n)_i$ is the stabiliser of i for the action of S_n on $\operatorname{Hom}([n],[m])$, or in other words is the set of σ satisfying condition 1 above. Let's simplify this expression. Keeping i and σ fixed, consider the sum

$$\begin{aligned} \sum_{j:[n]\to[m]} \prod_{h=1}^{n} w_h^{j(h)} w_h^{j(\sigma(h))} x_{j(h)} &= \sum_{j:[n]\to[m]} \prod_{h=1}^{n} w_h^{j(h)} w_{\sigma^{-1}(h)}^{j(h)} x_{j(h)} \\ = \sum_{j_1,\dots,j_n=1}^{m} \prod_{h=1}^{n} w_h^{j_h} w_{\sigma^{-1}(h)}^{j_h} x_{j_h} &= \prod_{h=1}^{n} \sum_{j=1}^{m} w_h^j w_{\sigma^{-1}(h)}^j x_j = \prod_{h=1}^{n} M_h^{\sigma^{-1}(h)}. \end{aligned}$$

This means that the substitution rule (46) has right-hand side equal to

$$\sum_{i:[n]\to[g]} \sum_{\sigma\in(S_n)_i} \operatorname{sgn}(\sigma) \prod_{h=1}^{n} M_h^{\sigma^{-1}(h)}. \tag{47}$$

Now notice that a function $i : [n] \to [g]$ corresponds to an ordered partition $A_1 \sqcup \cdots \sqcup A_g = [n]$ via $A_l := i^{-1}(l)$, and the group $(S_n)_i$ corresponds to the product $\prod_{l=1}^{g} S_{A_l}$ of

[3] meaning that it can give a non-zero contribution depending on the shape of P.

automorphisms groups of the subsets forming the partition. Thus we find that (47) is equal to

$$\sum_{A_1\sqcup\cdots\sqcup A_g=[n]}\ \sum_{\sigma\in\prod_{l=1}^g S_{A_l}} \operatorname{sgn}(\sigma)\prod_{h=1}^n M_h^{\sigma^{-1}(h)} = \sum_{A_1\sqcup\cdots\sqcup A_g=[n]}\ \prod_{l=1}^g \sum_{\sigma_l\in S_{A_l}} (-1)^{\sigma_l}\prod_{h\in A_l} M_h^{\sigma_l^{-1}(h)}$$

$$= \sum_{A_1\sqcup\cdots\sqcup A_g=[n]}\ \prod_{l=1}^g \det(M_{A_j,A_j}),$$

where the last equality is the usual expansion of the determinant. □

5.4.1. *The monomial substitution rule.* In Lemma 5.4 we found a substitution rule once the monomial in the theta classes is expressed in the form $\prod_{i=1}^n \theta_{\mathbf{w}_i}$, where each theta class appears with power one but the vectors $\mathbf{w}_i$ are allowed to appear repeated multiple times. We now find a more explicit formula for the presentation of monomials of the form $\prod_{i=1}^n \theta_{\mathbf{w}_i}^{u_i}$, where powers are allowed:

Proposition 5.2. *Let n be a positive integer, consider n vectors $\mathbf{w}_1,\dots,\mathbf{w}_n\in\mathbb{Z}^m$ and a n-tuple of non-negative integers $u=(u_1,\dots,u_n)$. Then the substitution rule*

$$\prod_{i=1}^n \theta_{\mathbf{w}_i}^{u_i} \rightsquigarrow \prod_{i=1}^n u_i! \sum_{(A_1,\dots,A_g)\in\mathcal{P}(u)}\ \prod_{j=1}^g \det(M_{A_j,A_j})$$

holds true, where M is the same matrix of Lemma 5.4 and $\mathcal{P}(u)$ is the set of g-tuples $(A_1,\dots,A_g)$ of subsets of $[n]$ satisfying the property

$$i \text{ belongs to exactly } u_i \text{ sets among } A_1,\dots,A_g \text{ for every } i\in[n]. \tag{48}$$

Proof. Consider the multiset U whose underlying set is $[n]$ and where i is repeated u_i times. In other words, we consider the set

$$U := \{1_1,\dots,1_{u_1},2_1,\dots,2_{u_2},\dots,n_1,\dots,n_{u_n}\},$$

which is endowed with the map $f: U\to[n]$ given by $f(i_j):=i$. Since the monomial we are trying to substitute is

$$\prod_{i=1}^n \theta_{\mathbf{w}_i}^{u_i} = \prod_{i\in U}\theta_{\mathbf{w}_{f(i)}},$$

by Lemma 5.4 we find the substitution rule

$$\prod_{i=1}^n \theta_{\mathbf{w}_i}^{u_i} \rightsquigarrow \sum_{\widetilde{A}_1\sqcup\cdots\sqcup\widetilde{A}_g=U}\ \prod_{j=1}^g \det(\widetilde{M}_{\widetilde{A}_j,\widetilde{A}_j}) \tag{49}$$

where $\widetilde{M}$ is the matrix whose rows and columns are indexed by elements of U and whose entries are $\widetilde{M}_i^j = M_{f(i)}^{f(j)}$. This means that the right-hand side of (49) is

$$\sum_{\widetilde{A}_1\sqcup\cdots\sqcup\widetilde{A}_g=U}\ \prod_{j=1}^g \det(M_{f(\widetilde{A}_j),f(\widetilde{A}_j)}). \tag{50}$$

This immediately shows that the function f must be injective on every $\widetilde{A}_j$ in order for $\det(\widetilde{M}_{\widetilde{A}_j,\widetilde{A}_j})$ to be non-zero, hence we can restrict the sum over partitions of U so that each $\widetilde{A}_i$ contains at most one copy of each element of $[n]$. Clearly the function

$$\left\{\text{partitions } \widetilde{A}_1 \sqcup \cdots \sqcup \widetilde{A}_g = U \text{ s.t. } f \text{ is injective on each } \widetilde{A}_i\right\} \xrightarrow{f} \mathcal{P}(u)$$

has fibers of cardinality $\prod_{i=1}^n u_i!$, since given a partition of U one can shuffle all the copies of each element of $[n]$ to produce another partition of U mapping to the same subsets of $[n]$. Therefore (50) coincides with the claimed substitution rule once we rename $A_j := f(\widetilde{A}_j)$. $\square$

5.4.2. *The exponential substitution rule.* In our applications, we will substitute polynomials in the theta classes that appear as truncated exponentials. For this kind of polynomials, the substitution rule simplifies:

Proposition 5.3. *Let $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_m]$ be polynomials in the x-classes, from which we form the diagonal $n \times n$ matrix*

$$C := \mathit{diag}\,(f_1, \ldots, f_n)\,.$$

Then we have the substitution rule

$$\prod_{i=1}^n \exp(f_i \theta_{\mathbf{w}_i}) \rightsquigarrow \det(\mathbb{I}_n + CM)^g.$$

Proof. By the definition of the exponential (note that $\theta_{\mathbf{w}_i}^{g+1} = 0$ as discussed in Remark 5.1) and Proposition 5.2 we find that

$$\begin{aligned}\prod_{i=1}^n \exp(f_i \theta_{\mathbf{w}_i}) &= \sum_{u_1,\ldots,u_n=0}^{g} \left(\prod_{i=1}^n \frac{f_i^{u_i}}{u_i!}\right) \prod_{j=1}^n \theta_{\mathbf{w}_j}^{u_j} \\ &\rightsquigarrow \sum_{u_1,\ldots,u_n=0}^{g} \left(\prod_{i=1}^n f_i^{u_i}\right) \sum_{(A_1,\ldots,A_g)\in\mathcal{P}(u)} \prod_{j=1}^g \det(M_{A_j,A_j}) \\ &= \sum_{u_1,\ldots,u_n=0}^{g} \sum_{(A_1,\ldots,A_g)\in\mathcal{P}(u)} \prod_{j=1}^g \det((CM)_{A_j,A_j}).\end{aligned}$$

Note that

$$\bigcup_{u_1,\ldots,u_n=0}^{g} \mathcal{P}(u) = \mathcal{P}([n])^g,$$

where the right-hand side is the Cartesian product of g copies of the power set of $[n]$. Thus we have the substitution rule

$$\begin{aligned}\prod_{i=1}^n \exp(f_i \theta_{\mathbf{w}_i}) &\rightsquigarrow \sum_{A_1,\ldots,A_g\in\mathcal{P}([n])} \prod_{j=1}^g \det((CM)_{A_j,A_j}) \\ &= \left(\sum_{A\in\mathcal{P}([n])} \det((CM)_{A,A})\right)^g = \det(\mathbb{I}_n + CM)^g,\end{aligned}$$

where the last equality is the well know expansion of the determinant of $\mathbb{I} + Z$ as the sum of the principal minors of Z. □

Finally, we can reshape this result into the most useful form for our applications:

Corollary 5.1 (Exponential substitution rule)**.** *Let $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_m]$ be polynomials in the x-classes, and consider the diagonal $m \times m$ matrix*

$$X := diag\,(x_1, \ldots, x_m)\,.$$

The substitution rule

$$\prod_{i=1}^{n} \exp(f_i \theta_{\mathbf{w}_i}) \rightsquigarrow \det(\mathbb{I}_m + XG(c))^g$$

holds true, where $G(c)$ is the $m \times m$ matrix

$$G(c) := \sum_{i=1}^{n} f_i \mathbf{w}_i^T \cdot \mathbf{w}_i,$$

and the weight vectors $\mathbf{w}_i \in \mathbb{Z}^m$ are considered as row vectors.

Proof. Consider the $n \times m$ matrix W with entries $W_a^b := w_a^b$, or, in other words, the one obtained by treating the weights $\mathbf{w}_i$ as rows of a matrix. Then we immediately find that the matrix M defined in Lemma 5.4 is

$$M = WXW^T,$$

hence by Proposition 5.3 we find

$$\prod_{i=1}^{n} \exp(f_i \theta_{\mathbf{w}_i}) \rightsquigarrow \det(\mathbb{I}_n + CWXW^T)^g.$$

By Sylvester's determinant theorem, stating that

$$\det(\mathbb{I}_n + AB) = \det(\mathbb{I}_m + BA) \tag{51}$$

for any $n \times m$ matrix A and $m \times n$ matrix B, we obtain that

$$\prod_{i=1}^{n} \exp(f_i \theta_{\mathbf{w}_i}) \rightsquigarrow \det(\mathbb{I}_m + XW^T CW)^g$$

concluding the proof. □

5.4.3. *The Jacobian substitution rule.* Let $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{C}$. If, defined $\langle \mathbf{x}, \mathbf{w} \rangle := \sum_{j=1}^{m} w^j x_j$, we fix the constants of Corollary 5.1 as

$$f_i = (\langle \mathbf{x}, \mathbf{w}_i \rangle + \varepsilon_i)^{-1},$$

we can give a nice description of the substitution rule:

Corollary 5.2. *We have the substitution rule*

$$\prod_{i=1}^{n} \exp\left(\frac{\theta_{\mathbf{w}_i}}{\langle \mathbf{x}, \mathbf{w}_i \rangle + \varepsilon_i}\right) \rightsquigarrow \prod_{j=1}^{m} x_j^g \det\left(\frac{1}{t_j} \frac{\partial t_j}{\partial x_k}\right)^g,$$

where $t : \mathbb{C}^m \dashrightarrow \mathbb{C}^m$ *is defined as*

$$t_j(\mathbf{x}) := x_j \prod_{i=1}^{n} (\langle \mathbf{x}, \mathbf{w}_i \rangle + \varepsilon_i)^{w_i^j} \qquad \forall j \in [m].$$

Proof. It's immediate to compute that

$$\frac{\partial t_j}{\partial x_k} = t_j(\mathbf{x}) \left(\frac{\delta_{j,k}}{x_j} + \sum_{i=1}^{n} \frac{w_i^j w_i^k}{\langle \mathbf{x}, \mathbf{w}_i \rangle + \varepsilon_i} \right) = \frac{t_j(\mathbf{x})}{x_j} \left(\delta_{j,k} + x_j G(f)_j^k \right).$$

where we used the notation of Corollary 5.1. □

6. Virtual integrals in higher genus

We now turn our attention to the higher genus case. In this section, we apply virtual localization [GP99] to reduce the integrals in Theorem 3.1 to a sum of integrals over products of symmetric products of curves. The extra difficulty in higher genus calculation, compared to genus zero case, is in addressing the combinatorics of intersecting several theta classes and x-classes in the localization process. The combinatorics of these integrals is addressed in Section 5.

6.1. **Cohomology classes on $\mathrm{X}_{\sigma,D}$.** Let us first recall the description of the fixed locus $X_{\sigma,D}$ as a product of symmetric products of curves:

$$\mathrm{X}_{\sigma,D} \cong \prod_{j=1}^{k} \prod_{s=1}^{r_j} C^{[d_{s,j} - d_{s,j+1}]}$$

We adopt the notation in Section 5.

6.1.1. *Equivariant Chern classes.* Fix a point $p \in C$. For a vector bundle $\mathcal{L}$ on a product scheme $X \times C$, we will denote its restriction to $X \times \{p\}$ by $\mathcal{L}_{|p}$.

Let $\mathcal{L}_{s,j}$ be the universal ideal subsheaf over the product $C^{[d_{s,j} - d_{s,j+1}]} \times C$ and its pullback to $X_{\sigma,D} \times C$, in a slight abuse of notation. Note that

$$\mathcal{L}_{s,j}^\vee = \mathcal{K}_{s,j}^\vee \otimes \mathcal{K}_{s,j+1} \tag{52}$$

and let

$$x_{s,j} := c_1\left(\mathcal{L}_{s,j|p}^\vee\right) \in H^2(X_{\sigma,D}) \quad \text{for all } 1 \leqslant j \leqslant k \text{ and } \ 1 \leqslant s \leqslant r_j.$$

The torus acts trivially on the fixed locus $\mathrm{X}_{\sigma,D}$, and the equivariant cohomology is

$$H_T^*(\mathrm{X}_{\sigma,D}) \cong H^*(\mathrm{X}_{\sigma,D}) \otimes H_T^*(\mathrm{pt})$$

where $H_T^*(\mathrm{pt}) = \mathbb{C}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]$, and recall the definition of Γ in (26) obtained by inverting the equivariant parameters ε's in $H_T^*(\mathrm{pt})$.

Using (52), we obtain

$$z_{s,j} := c_1^T\left(\mathcal{K}_{s,j|p}^\vee\right) = x_{s,j} + x_{s,j+1} + \cdots + x_{s,k} + \varepsilon_{\sigma(s)}$$

where $c_1^T\left(\mathcal{K}_{s,k+1|p}^\vee\right) = c_1^T\left(\pi_C^* M_{s|p}^\vee\right) = \varepsilon_{\sigma(s)}$. To be consistent, we define $z_{s,k+1} = \varepsilon_i$.

6.1.2. *Theta classes.* Let $\mathbf{w}$ be a vector that assigns an integer to each Hilbert scheme in $X_{\sigma,D}$, that is,

$$\mathbf{w} \in \mathbb{Z}^{r_1} \oplus \cdots \oplus \mathbb{Z}^{r_k}.$$

We define the line bundle

$$\mathcal{L}_{\mathbf{w}} = \bigotimes_{s,j} \mathcal{L}_{s,j}^{\otimes w^{s,j}}.$$

Recall that $\mathcal{L}_{\mathbf{w}}^{\vee}$ induces a natural morphism

$$\pi_{\mathbf{w}} : X_{\sigma,D} \to \mathrm{Pic}^{\langle \mathbf{w}, \mathbf{e} \rangle}(C)$$

where $\mathbf{e} \in \mathbb{Z}^{r_1} \oplus \cdots \oplus \mathbb{Z}^{r_k}$ is the dimension vector given by $e^{s,j} = d_{s,j} - d_{s,j+1}$. For any vector $\mathbf{w}$, we have

$$\theta_{\mathbf{w}} := \pi_{\mathbf{w}}^{*}\theta \in H^2(\mathrm{X}_{\sigma,D}, \mathbb{Z})$$

where $\theta \in \mathrm{Pic}^{\langle \mathbf{w}, \mathbf{e} \rangle}(C)$ is the theta class. Note that $\theta_{\mathbf{w}} = \theta_{-\mathbf{w}}$ and $\theta_{\mathbf{w}}^{g+1} = 0$.

Remark 6.1. We have $\mathcal{K}_{s,j} \otimes \mathcal{K}_{s,k+1}^{\vee} = \mathcal{L}_{s,j} \otimes \cdots \otimes \mathcal{L}_{s,k} = \mathcal{L}_{w_{s,j}}$ for the associated vector

$$w_{s,j} = \left(0^{(1)}, \ldots, 0^{(j-1)}, \delta_s^{(j)}, \delta_s^{(j+1)}, \ldots, \delta_s^{(k)}\right) \in \mathbb{Z}^{r_1} \oplus \cdots \oplus \mathbb{Z}^{r_k},$$

where $0^{(j)}$ is the zero vector in $\mathbb{Z}^{r_j}$ and $\delta_s^{(j)} = (0, \ldots, 0, 1, 0 \ldots, 0) \in \mathbb{Z}^{r_j}$ with 1 placed at s^{th} position.

6.2. **Euler class of virtual normal bundle.** All the cohomology classes that will appear in the localization calculation are expressions of the $\mathbf{x}$ and θ classes. In particular, Proposition 5.1 implies

Proposition 6.1. *Let $1 \leqslant j \leqslant k$, $s \leqslant r_j$, $u \leqslant r_{j+1}$ and $s \neq u$, then the equivariant Euler class is given by*

$$e_T\left(\pi_*\left(\mathcal{K}_{s,j}^{\vee} \otimes \mathcal{K}_{u,j}\right)\right) = (z_{s,j} - z_{u,j})^{d_{s,j} - d_{u,j} + 1 - g} \exp\left(-\frac{\theta_{w_{s,j} - w_{u,j}}}{z_{s,j} - z_{u,j}}\right); \tag{53}$$

$$e_T\left(\pi_*\left(\mathcal{K}_{s,j}^{\vee} \otimes \mathcal{K}_{u,j+1}\right)\right) = (z_{s,j} - z_{u,j+1})^{d_{s,j} - d_{u,j+1} + 1 - g} \exp\left(-\frac{\theta_{w_{s,j} - w_{u,j+1}}}{z_{s,j} - z_{u,j+1}}\right). \tag{54}$$

Recall the definitions in (25), of the polynomials

$$R_j(z) = (z - z_{1,j})(z - z_{2,j}) \cdots (z - z_{r_j,j}) \quad \text{for all } 0 \leqslant j \leqslant k+1.$$

We will carefully use the equivariant Chern classes computed in Proposition 6.1 and substitute them in the formula for the virtual normal bundle $N_{\sigma,D}^{\mathrm{vir}}$ in (22), noting that equivariant Euler class is multiplicative.

We first consider the terms with the same column (second) index:

$$\begin{aligned}\prod_{\substack{s,u \leqslant r_j \\ s \neq u}} e_T\left(\pi_*\left(\mathcal{K}_{s,j}^{\vee} \otimes \mathcal{K}_{u,j}\right)\right) &= (-1)^{d_j(r_j - 1)} \prod_{\substack{s,u \leqslant r_j \\ s \neq u}} (z_{s,j} - z_{u,j})^{1-g} \\ &= (-1)^{d_j(r_j-1)} \prod_{s \leqslant r_j} R_j'(z_{s,j})^{1-g}.\end{aligned}$$

In the first equality, we used that the exponential terms in (53) and $(z_{s,j} - z_{u,j})^{d_{s,j} - d_{u,j}}$ cancel each other by swapping the role of s and u, leaving behind the sign $(-1)^{d_j(r_j-1)}$.

We then consider the terms involving two consecutive columns:

$$
\begin{aligned}
&\prod_{\substack{s\leqslant r_j\\ u\leqslant r_{j+1}\\ s\neq u}} e_T\left(\pi_*\left(\mathcal{K}_{s,j}^\vee\otimes\mathcal{K}_{u,j+1}\right)\right)\\
&=\prod_{\substack{s\leqslant r_j\\ u\leqslant r_{j+1}\\ s\neq u}}(z_{s,j}-z_{u,j+1})^{d_{s,j}-d_{u,j+1}+1-g}\exp\left(-\frac{\theta_{w_{s,j}-w_{u,j+1}}}{z_{s,j}-z_{u,j+1}}\right)\\
&=\prod_{s\leqslant r_j}\left(\frac{R_{j+1}(z_{s,j})}{x_{s,j}}\right)^{d_{s,j}+1-g}\cdot\prod_{u\leqslant r_{j+1}}\left(\frac{(-1)^{r_j}x_{u,j}}{R_j(z_{u,j+1})}\right)^{d_{u,j+1}}\cdot\prod_{\substack{s\leqslant r_j\\ u\leqslant r_{j+1}\\ s\neq u}}\exp\left(-\frac{\theta_{w_{s,j}-w_{u,j+1}}}{z_{s,j}-z_{u,j+1}}\right),
\end{aligned}
$$

where we set $x_{u,j}:=1$ for $u>r_j$.

In the second term of the last expression, we substitute the index $j\to j-1$. Collecting the above two formulas with the correct sign governed by (22) for $1\leqslant j\leqslant k$, we prove:

Proposition 6.2. *The equivariant Euler class of the virtual normal bundle is given by*

$$
\frac{1}{e_T(N_{\sigma,D}^{\mathrm{vir}})}=(-1)^\beta J(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\varepsilon})\cdot\prod_{j=1}^k\prod_{s\leqslant r_j}x_{s,j}^{d_{s,j}-d_{s,j+1}+1}\frac{R_j'(z_{s,j})R_{j-1}(z_{s,j})^{d_{s,j}}}{R_{j+1}(z_{s,j})^{d_{s,j}+1}}, \tag{55}
$$

where ths sign is given by

$$
\beta=\sum_{j\leqslant k}d_j(r_j-r_{j-1}-1)
$$

and the following class consisting of z-classes with exponent g and the theta classes

$$
J(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\varepsilon})=\prod_{j=1}^k\prod_{s\leqslant r_j}\left(\frac{R_{j+1}(z_{s,j})}{x_{s,j}R_j'(z_{s,j})}\right)^g\prod_{\substack{u\leqslant r_{j+1}\\ u\neq s}}\exp\left(\frac{\theta_{w_{s,j}-w_{u,j+1}}}{z_{s,j}-z_{u,j+1}}\right).
$$

6.3. **Proof of Theorem 3.1.** The deformation invariance in Proposition 2.1 allows us to assume V in Theorem 1.1 is a split vector bundle $V=M_1\oplus M_2\oplus\cdots\oplus M_n$, where each M_i has degree zero. In this section, we will prove the equivariant formula, Theorem 3.1 for all genera, that concludes the proof of Theorem 1.1 by specializing the equivariant parameter (described in Section 4.3).

We start by using the substitution rule we prove in Section 5.4 to take care of $J(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\varepsilon})$.

Lemma 6.1. *Fix σ and D. Fix σ and D. For any polynomial $H(\mathbf{x},\boldsymbol{\varepsilon})\in\boldsymbol{\Gamma}[\mathbf{x}]$, we have*

$$
\int_{\mathrm{X}_{\sigma,D}}H(\mathbf{x},\boldsymbol{\varepsilon})J(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\varepsilon})=\int_{\mathrm{X}_{\sigma,D}}H(\mathbf{x},\boldsymbol{\varepsilon})J(\mathbf{x},\boldsymbol{\varepsilon})^g,
$$

where

$$
J(\mathbf{x},\boldsymbol{\varepsilon})=\prod_{\substack{j\leqslant k\\ s\leqslant r_j}}\frac{R_{j-1}(z_{s,j})}{R_j'(z_{s,j})}\cdot\det\left(\frac{\partial q_{s,j}}{\partial x_{u,m}}\right)\quad\textit{for}\quad q_{s,j}:=(-1)^{r_j-r_{j-1}-1}\frac{R_{j+1}(z_{s,j})}{R_{j-1}(z_{s,j})}.
$$

Proof. We first recall from Proposition 6.2,

$$J(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\varepsilon}) = \prod_{j=1}^{k}\prod_{s\leqslant r_j}\left(\frac{R_{j+1}(z_{s,j})}{x_{s,j}R_j'(z_{s,j})}\right)^g \prod_{\substack{u\leqslant r_{j+1}\\ u\neq s}} \exp\left(\frac{\theta_{w_{s,j}-w_{u,j+1}}}{z_{s,j}-z_{u,j+1}}\right).$$

Let us define $\varepsilon_{su} = \varepsilon_u - \varepsilon_s$ and the vectors

$$w_{suj} \coloneqq w_{s,j} - w_{u,j+1} \in \mathbb{Z}^{r_1}\oplus\cdots\oplus\mathbb{Z}^{r_k}.$$

Then

$$\frac{\theta_{w_{s,j}-w_{u,j+1}}}{z_{s,j}-z_{u,j+1}} = \frac{\theta_{w_{suj}}}{\langle x, w_{suj}\rangle + \varepsilon_{su}}.$$

We are now in the setting to apply Corollary 5.2. We set

$$t_{c,i}(\mathbf{x},\boldsymbol{\varepsilon}) \coloneqq x_{c,i}\prod_{\substack{s\leqslant r_j,\ j\leqslant k\\ u\leqslant r_{j+1},\ s\neq u}} (\langle x, w_{suj}\rangle + \varepsilon_{su})^{w^{c,i}_{suj}} \quad \text{for } c\leqslant r_i,\ i\leqslant k.$$

Applying Corollary 5.2, we obtain that in the integral,

$$\prod_{\substack{s\leqslant r_j,\ j\leqslant k\\ u\leqslant r_{j+1},\ s\neq u}} \exp\left(\frac{\theta_{w_{s,j}-w_{u,j+1}}}{z_{s,j}-z_{u,j+1}}\right)$$

can be replaced by

$$\prod_{c\leqslant r_i,\ i\leqslant k} x_{c,i}^g \det\left(\frac{1}{t_{c,i}(\mathbf{x},\boldsymbol{\varepsilon})}\frac{\partial t_{c,i}(\mathbf{x},\boldsymbol{\varepsilon})}{\partial x_{c',i'}}\right)^g.$$

Note that

$$t_{c,i}(\mathbf{x},\boldsymbol{\varepsilon}) = x_{c,i}\frac{\prod_{\substack{j\leqslant i,\ r_j\geqslant c\\ u\leqslant r_{j+1},\ u\neq c}}(z_{c,j}-z_{u,j+1})}{\prod_{\substack{j+1\leqslant i,\ s\leqslant r_j\\ r-j+1\geqslant c,\ s\neq c}}(z_{s,j}-z_{c,j+1})} = (-1)^{\alpha(c,i)}\prod_{j\leqslant i,\ r_j\geqslant c} q_{c,j}$$

for some sign $\alpha(c,i)$. The Lemma follows from the chain rule and the following identity

$$\det\left(\frac{1}{t_{c,i}(\mathbf{x},\boldsymbol{\varepsilon})}\frac{\partial t_{c,i}(\mathbf{x},\boldsymbol{\varepsilon})}{\partial q_{c',i'}}\right) = \prod_{c\leqslant r_i,\ i\leqslant k}\frac{1}{q_{c,i}}.$$

□

The key advantage of Lemma 6.1 is that it reduces our task to integrals in the x–classes on each fixed locus $\mathrm{X}_{\sigma,D}$; by Lemma 5.1, these integrals are computed via residues. The combinatorics of summing the contributions over all fixed loci mirrors the genus–zero analysis in Section 4.2. We briefly recall that setup and then complete the proof of Theorem 4.2, which is the formal version. Theorem 3.1 and 1.1 follows by the discussion in Section 4.3.

Proof of Theorem 4.2. Let $Q = \prod_{j=1}^{k}\prod_{i=1}^{r_j} c_i(\mathcal{E}^\vee_{j|p})^{m_{i,j}}$. The virtual localization formula of [GP99] implies

$$\int_{[\mathsf{HQ}_{\mathbf{d}}(C,Fl(\mathbf{r},n))]^{\mathrm{vir}}} Q = \sum_{\sigma,D}\int_{\mathrm{X}_{\sigma,D}}\frac{Q|_{\mathrm{X}_{\sigma,D}}}{e_T(N^{\mathrm{vir}}_{\sigma,D})},$$

where the equivariant Euler class $e_T(N^{\mathrm{vir}}_{\sigma,D})$ of the virtual normal bundle was computed in Proposition 6.2. Explicitly, this gives

$$(-1)^{\beta}\sum_{\sigma,D}\int_{\mathrm{X}_{\sigma,D}} Q|_{X_{\sigma,D}}\cdot J(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\varepsilon})\cdot\prod_{j=1}^{k}\prod_{s\leqslant r_j} x_{s,j}^{d_{s,j}-d_{s,j+1}+1}\frac{R'_j(z_{s,j})R_{j-1}(z_{s,j})^{d_{s,j}}}{R_{j+1}(z_{s,j})^{d_{s,j}+1}}.$$

Using Lemma 6.1, the factor $J(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\varepsilon})$ can be replaced by $J(\mathbf{x},\boldsymbol{\varepsilon})^g$, which depends only on the x-variables and the genus g. Since the rule for integrating x-classes is identical to the $C=\mathbb{P}^1$ case, the above integral reduces to a residue:

$$\int_{[\mathsf{HQ}_{\mathbf{d}}(C,Fl(\mathbf{r},n))]} Q = (-1)^u\sum_{\sigma,D}\operatorname*{Res}_{\mathbf{x}=\mathbf{0}} Q|_{X_{\sigma,D}}\prod_{\substack{j\leqslant k\\ i\leqslant r_j}}\frac{R'_j(z_{i,j})\,R_{j-1}(z_{i,j})^{d_{i,j}}}{R_{j+1}(z_{i,j})^{d_{i,j}+1}}\cdot J(\mathbf{x},\boldsymbol{\varepsilon})^g.$$

Note that the new factor $J(\mathbf{x},\varepsilon)^g$ is independent of the splitting data D, and consequently the Lagrange–Bürmann argument from the genus-zero case in Section 4.2 applies unchanged. The proof is thus completed by following the proof of Theorem 3.1 for $C=\mathbb{P}^1$ verbatim, with the additional observation that $J(\mathbf{x},\varepsilon)=J$ when expressed in terms of the z-variables. □

7. Explicit formulas

In this section, we specialize Theorem 1.1 to several cases and obtain explicit combinatorial formulas. In two basic instances, the Hyperquot scheme of points on C and the two-step type $\mathbf{r}=(1,n-1)$, we derive simple combinatorial expressions that manifest the positivity of the virtual intersection numbers.

Definition 7.1. Fix a genus g curve C, ranks $\mathbf{r}=(r_1,r_2,\ldots,r_k)$, and a vector bundle V of degree e and rank n. For any tuple of natural numbers

$$\mathbf{m}=\{m_{i,j}\in\mathbb{N} : 1\leqslant j\leqslant k,\ 1\leqslant i\leqslant r_j\},$$

we collect the corresponding integrals of powers Chern classes in (1) on the Hyperquot schemes $\mathsf{HQ}_{\mathbf{d}}$ into a multivariate Laurent polynomial

$$\mathbf{B}^{\mathbf{m}}_{g,\mathbf{r},n,e}(q_1,q_2,\ldots,q_k)=\sum_{\mathbf{d}\in\mathbb{Z}^k} q_1^{d_1}q_2^{d_2}\cdots q_k^{d_k}\int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}}\prod_{j=1}^{k}\prod_{i=1}^{r_j} c_i(\mathcal{E}^{\vee}_{j|p})^{m_{i,j}}, \tag{56}$$

as we vary the multidegree $\mathbf{d}$. Note that when V is the trivial bundle, (56) is a polynomial.

7.1. Explicit formulas for Hyperquot scheme of points on C.

Fix a positive integer k and vector bundle V of rank n on a genus g curve C. For a tuple of non-negative integers $\mathbf{d}=(d_1,d_2,\ldots,d_k)$, let $\mathsf{HQ}_{\mathbf{d}}:=\mathsf{HQ}_{\mathbf{d}}(C,\mathbf{r},V)$ denote the Hyperquot scheme of points, i.e. $\mathbf{r}=(n)^k:=(n,n,\ldots,n)$, parametrizing successive zero-dimensional quotients

$$V\to F_1\to F_2\to\cdots F_k,$$

where F_j has support of length d_j for all $1\leqslant j\leqslant k$. Note that for $\mathsf{HQ}_{\mathbf{d}}$ to be non-empty, the lengths of the support for subsequent quotients must decrease or stay the same at each step, that is, $d_1\geqslant d_2\geqslant\cdots\geqslant d_k$. In this case, $\mathsf{HQ}_{\mathbf{d}}$ is a smooth irreducible scheme of dimension $d_1 n$, see [MR22, Theorem 1.4, Proposition 2.1].

Corollary 7.1. *For any tuple* $\mathbf{m} = (m_{i,j} : 1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant n)$,

$$\mathbf{B}^{\mathbf{m}}_{g,(n)^k,n,e}(q_1, q_2, \ldots, q_k) = \begin{cases} \prod_{j=1}^k \alpha_j^{m_{n,j}} & \text{if } m_{i,j} = 0 \quad \forall i < n, 1 \leqslant j \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\alpha_j := q_1 q_2 \cdots q_j (1 + q_{j+1} + q_{j+1}q_{j+2} + \cdots + q_{j+1}q_{j+2}\cdots q_k) \quad \text{for } 1 \leqslant j \leqslant k. \tag{57}$$

Note that the above formula does not depend on the genus g of C and degree e of V.

Proof. Let us first fix $e = 0$, and apply Theorem 1.1. It suffices to assume $(q_1, q_2, \ldots, q_k)$ is a tuple of general complex numbers. Consider the system of variables $\mathbf{z}_j = (z_{1,j}, \ldots, z_{n,j})$ for $1 \leqslant j \leqslant k$. We are required to solve the systerm of equations $P_j(z_{i,j}) = 0$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$ defined in (2):

$$P_1(X) = \prod_{i=1}^n (X - z_{i,2}) + (-1)^n q_1; \qquad P_k(X) = X^n + q_k \prod_{i=1}^n (X - z_{i,k-1});$$
$$P_j(X) = \prod_{i=1}^n (X - z_{i,j+1}) + q_j \prod_{i=1}^n (X - z_{i,j-1}) \quad \text{for} \quad 2 \leqslant j \leqslant k-1.$$

By Proposition 4.3 there is a unique solution $(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k) = (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k)$ to this system of equations up to permuting entries in each $\boldsymbol{\zeta}_j$.

Consider the tuple $\mathbf{m} = \mathbf{0}$, then the polynomial

$$\mathbf{B}^{\mathbf{0}}_{g,(n)^k,n,0}(q_1, q_2, \ldots, q_k) = 1$$

since the Hyperquot scheme of points on C is (non-empty) of zero dimension exactly when $d_1 = d_2 = \cdots = d_k = 0$. On the other hand, Theorem 1.1 implies

$$\mathbf{B}^{\mathbf{0}}_{g,(n)^k,n,0}(q_1, q_2, \ldots, q_k) = J(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k)^{g-1}$$

evaluated at the unique solution. Hence, we find $J(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k) = 1$, that is, the integral does not depend on the genus g.

Fix the unique non-degenerate solution $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_k$ and let

$$R_j(X) := (X - \zeta_{1,j})(X - \zeta_{2,j}) \cdots (X - \zeta_{n,j})$$

for each $1 \leqslant j \leqslant k$. Since $\boldsymbol{\zeta}_j$ is the complete list of roots of $P_j(X)$ for each $1 \leqslant j \leqslant k$, we have the following relations:

$$\begin{aligned} R_1(X) &= R_2(X) + (-1)^n q_1 \\ (1 + q_2)R_2(X) &= R_3(X) + q_2 R_1(X) \\ &\vdots \\ (1 + q_{k-1})R_{k-1}(X) &= R_k(X) + q_{k-1}R_{k-2}(X) \\ (1 + q_k)R_k(X) &= X^n + q_k R_{k-1}(X) \end{aligned}$$

Solving the above linear relations for the monic polynomials $R_j(X)$ for $1 \leqslant j \leqslant k$, we obtain

$$R_j(X) = X^n + (-1)^n \alpha_j,$$

where α_j is defined in (57), and hence the elementary symmetric polynomials $e_n(\boldsymbol{\zeta}_j) = \alpha_j$ and $e_i(\boldsymbol{\zeta}_j) = 0$ whenever $i < n$.

Now, for arbitrary degree e vector bundle V, we consider the elementary modification $\widetilde{V}$, which has degree $e-1$, see Section 2.3.2. Apply Proposition 2.3, we obtain the identity

$$\mathbf{B}^{\mathbf{m}+\delta_{k,n}}_{g,(n)^k,n,e}(q_1,q_2,\ldots,q_k) = (q_1q_2\cdots q_k)\cdot \mathbf{B}^{\mathbf{m}}_{g,(n)^k,n,e-1}(q_1,q_2,\ldots,q_k)$$

where $\delta_{k,n}=(0,\ldots,0,1)$, i.e. 1 is added to $m_{k,n}$ in the tuple $\mathbf{m}$. The assertion now follow by noting that $\alpha_k=(q_1q_2\cdots q_k)$ and the result for $e=0$. $\square$

Remark 7.1. Note that for any vector bundle E on a projective scheme X, the Segre series $s_t(E)=1/(1+c_1(E)t+\cdots+c_n(E)t^n)$. The result about the Segre integrals in Corollary 1.1 is a direct consequence of Corollary 7.1, by expressing the Segree series in terms of the inverse of the Chern polynomial.

7.2. **Explicit formulas for $\mathbf{r}=(1,n-1)$.** Fix a smooth curve C of genus g, V be a vector bundle of rank n and degree zero. Let $\mathbf{r}=(1,n-1)$. For any multi-degree $\mathbf{d}=(d_1,d_2)$, the expected dimension of the quot scheme $\mathsf{Quot}_{\mathbf{d}}=\mathsf{Quot}_{\mathbf{d}}(C,\mathbf{r},V)$ is given by

$$\mathrm{vdim}=(n-1)(d_1+d_2)-(2n-3)(g-1).$$

For any non-negative integers ℓ and $m_1,m_2,\ldots,m_{n-1}$, we are interested in evaluating the virtual invariants

$$\mathbf{B}^{\ell,\mathbf{m}}_{g,\mathbf{r},n,0}(q_1,q_2):=\sum_{d_1,d_2\in\mathbb{N}} q_1^{d_1}q_2^{d_2}\int_{[\mathsf{HQ}_{\mathbf{d}}]^{\mathrm{vir}}} c_1(\mathcal{E}^\vee_{1|p})^{\ell}\prod_{i=1}^{n-1} c_i(\mathcal{E}^\vee_{2|p})^{m_i},$$

where we denote $(\ell,\mathbf{m})=(\ell,m_1,\ldots,m_{n-1})$, instead of calling it $\mathbf{m}$.

To apply Theorem 1.1, we must first solve a system of equations; we return to the proof of the theorem at the end of this section. Let $x=z_{11}$ and $y_i=z_{i,n-1}$ for $1\leqslant i\leqslant n-1$. The system of equations (see (3)) is

$$P_1(x)=P_2(y_1)=\cdots=P_2(y_{n-1})=0,$$

where

$$P_1(X)=\prod_{i=1}^{n-1}(X-y_i)-q_1,\qquad P_2(Y)=Y^n+(-1)^nq_2(Y-x). \tag{58}$$

Lemma 7.1. *Fix a tuple (q_1,q_2) of general complex numbers. Every non-degenerate solution of (58),*

$$(x=\zeta,\ y_1=\eta_1,\ldots,y_{n-1}=\eta_{n-1}),$$

is of the form: let w be an n-th root of q_1/q_2,

(i) ζ is an $(n-1)$-th root of $q_1(1+w^{-1})$,

(ii) $\eta_1,\eta_2,\ldots,\eta_{n-1}$ are distinct roots of

$$Y^{n-1}+\left(-\frac{\zeta}{w}\right)Y^{n-2}+\cdots+\left(-\frac{\zeta}{w}\right)^{n-2}Y+\left(-\frac{\zeta}{w}\right)^{n-1}+(-1)^nq_2=0.$$

Proof. We list n equations (58) explicitly that we have to solve:

$$\prod_{i=1}^{n-1}(x-y_i)=q_1,\qquad \begin{aligned} y_1^n &= (-1)^nq_2(x-y_1),\\ &\ \vdots \qquad\qquad \vdots \end{aligned}$$

$$y_{n-2}^n = (-1)^n q_2(x - y_{n-2}),$$
$$y_{n-1}^n = (-1)^n q_2(x - y_{n-1}).$$

Suppose $(\zeta, \eta_1, \dots, \eta_{n-1})$ is a non-degenerate solution to the above system of equations. Multiplying all the equations above, we observe that

$$(\eta_1\eta_2\cdots\eta_{n-1})^n = q_1 q_2^{n-1}.$$

Consider the polynomial

$$R_2(Y) = Y^n + (-1)^n q_2(Y - \zeta).$$

Note that $\eta_1, \dots, \eta_{n-1}$ are roots of R_2. We denote η_n for the missing root of $P_2(Y)$, that is $\{\eta_1, \eta_2, \dots, \eta_n\}$ is the complete list of roots.

Evaluating the polynomial at $Y = \zeta$, we obtain

$$\zeta^n = R_2(\zeta) = (\zeta - \eta_1)\cdots(\zeta - \eta_{n-1})(\zeta - \eta_n) = q_1(\zeta - \eta_n). \tag{59}$$

On the other hand, evaluating the polynomial at $Y = 0$ and exponentiating to the n-th exponent, we have the identity

$$(q_2\zeta)^n = R_2^n(0) = ((-1)^n\eta_1\eta_2\cdots\eta_n)^n = (-1)^n q_1 q_2^{n-1}\eta_n^n$$

which implies

$$\eta_n = -\zeta/w \text{ with } w \text{ an } n\text{-th root of } q_1/q_2. \tag{60}$$

Substituting in (59), we obtain the identity

$$\zeta^{n-1} = q_1(1 + w^{-1}). \tag{61}$$

Finally, we express elementary symmetric polynomials in $\eta_1, \eta_2, \dots, \eta_{n-1}$ in terms of q_2, w and ζ. Analyzing the coefficients of the polynomial $R_2(Y)$, we have

$$e_n(\eta_1, \eta_2, \dots, \eta_n) = -\zeta q_2, \quad e_{n-1}(\eta_1, \eta_2, \dots, \eta_n) = -q_2,$$

and $e_i(\eta_1, \eta_2, \dots, \eta_n) = 0$ for all $1 \leqslant i \leqslant n-2$. Using (60) and (61), we obtain

$$\begin{aligned}
e_{n-1}(\eta_1, \eta_2, \dots, \eta_{n-1}) &= & q_2 w \quad &= \left(\frac{\zeta}{w}\right)^{n-1} - q_2\\
e_{n-2}(\eta_1, \eta_2, \dots, \eta_{n-1}) &= & \frac{w}{\zeta}\cdot q_2(1+w) \quad &= \left(\frac{\zeta}{w}\right)^{n-2}\\
e_{n-3}(\eta_1, \eta_2, \dots, \eta_{n-1}) &= & \left(\frac{w}{\zeta}\right)^2\cdot q_2(1+w) \quad &= \left(\frac{\zeta}{w}\right)^{n-3}\\
&\vdots\\
e_1(\eta_1, \eta_2, \dots, \eta_{n-1}) &= & \left(\frac{w}{\zeta}\right)^{n-2}\cdot q_2(1+w) \quad &= \left(\frac{\zeta}{w}\right).
\end{aligned} \tag{62}$$

Note that we have obtained that there are at most $n\cdot(n-1)\cdot(n-1)!$ tuples $(\zeta, \eta_1, \dots, \eta_{n-1})$ satisfying (i) and (ii) in the statement of Lemma 7.1. It follows from Proposition 4.3 that they are all solutions to (58). $\square$

Let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_{n-1})$.

Lemma 7.2. *Continuing with the notation in Lemma 7.1, for any choice of w and ζ, we have*

$$J(\zeta,\boldsymbol{\eta}) = n(n-1)q_2 w\zeta^{n-2}.$$

Proof. Suppose $(\zeta,\eta_1,\dots,\eta_{n-1})$ is a non-degenerate solution. The term $J(\zeta,\boldsymbol{\eta})$ in Theorem 1.1 in our case is explicitly given by

$$\left(\det\begin{bmatrix} \frac{\partial P_1(x)}{\partial x} & \frac{\partial P_1(x)}{\partial y_1} & \frac{\partial P_1(x)}{\partial y_2} & \cdots & \frac{\partial P_1(x)}{\partial y_{n-1}} \\ (-1)^{n-1}q_2 & \frac{\partial P_2(y_1)}{\partial y_1} & 0 & \cdots & 0 \\ (-1)^{n-1}q_2 & 0 & \frac{\partial P_2(y_2)}{\partial y_2} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ (-1)^{n-1}q_2 & 0 & 0 & \cdots & \frac{\partial P_2(y_{n-1})}{\partial y_{n-1}} \end{bmatrix} \cdot \prod_{1\leqslant i\neq j\leqslant n-1}(y_i-y_j)^{-1}\right)_{x=\zeta,\ \mathbf{y}=\boldsymbol{\eta}}$$

Recall

$$R_2(Y) = Y^n + (-1)^n q_2(Y-\zeta) = (Y-\eta_1)(Y-\eta_2)\cdots(Y-\eta_n),$$

where η_n is the missing root as in the proof of Lemma 7.1. Note that

$$\prod_{1\leqslant i\neq j\leqslant n-1}(\eta_i-\eta_j) = \frac{(-1)^{n-1}}{R_2'(\eta_n)}\prod_{i=1}^{n-1}R_2'(\eta_i)$$

and

$$\left.\frac{\partial P_2(y_i)}{\partial y_i}\right|_{x=\zeta,\ \mathbf{y}=\boldsymbol{\eta}} = R_2'(y_i)|_{\mathbf{y}=\boldsymbol{\eta}} = R_2'(\eta_i).$$

Using the identity

$$\frac{\partial P_1(x)}{\partial x} = -\sum_{i=1}^{n-1}\frac{\partial P_1(x)}{\partial y_i},$$

we expand the determinant along the first row and obtain

$$\begin{aligned} J(\zeta,\boldsymbol{\eta}) &= (-1)^{n-1}R_2'(\eta_n)\left[\sum_{i=1}^{n-1}\left(-1+(-1)^n\frac{q_2}{R_2'(\eta_i)}\right)\cdot\left.\frac{\partial P_1(x)}{\partial y_i}\right|_{x=\zeta,\ \mathbf{y}=\boldsymbol{\eta}}\right] \\ &= (-1)^{n-1}R_2'(\eta_n)\cdot\left[\sum_{i=1}^{n-1}\frac{-R_2'(\eta_i)+(-1)^n q_2}{(\zeta-\eta_i)R_2'(\eta_i)}\right]\cdot\prod_{j=1}^{n-1}(\zeta-\eta_j) \\ &= (-1)^{n-1}R_2'(\eta_n)\cdot\left[\sum_{i=1}^{n-1}\frac{n\eta_i^{n-1}}{(\zeta-\eta_i)R_2'(\eta_i)}\right]\cdot\prod_{j=1}^{n-1}(\zeta-\eta_j) \\ &= (-1)^{n-1}R_2'(\eta_n)\cdot\left[\sum_{i=1}^{n-1}\frac{n\cdot(-1)^n q_2}{\eta_i R_2'(\eta_i)}\right]\cdot q_1, \end{aligned}$$

where in the last equality, we used $q_1 = \prod_{j=1}^{n-1}(\zeta-\eta_j)$ and $\eta_i^n = (-1)^n q_2(\zeta-\eta_i)$. We note that

$$\eta_i R_2'(\eta_i) = n\eta_i^n + (-1)^n q_2\eta_i = (-1)^n q_2(n-1)\left(\frac{n\zeta}{n-1}-\eta_i\right),$$

and hence summing over all roots of R_2 and using the identity $\zeta^{n-1} = q_1(1+w^{-1})$, we have

$$\sum_{i=1}^{n} \frac{n\cdot(-1)^n q_2}{\eta_i R_2'(\eta_i)} = \frac{n}{n-1}\cdot\frac{R_2'\left(\frac{n\zeta}{n-1}\right)}{R_2\left(\frac{n\zeta}{n-1}\right)} = \frac{n}{\zeta}. \tag{63}$$

Recall from (60) that $\eta_n = -\zeta/w$, and thus

$$R_2'(\eta_n) = n\eta_n^{n-1} + (-1)^n q_2 = (-1)^{n-1} q_2(n(1+w)-1) \tag{64}$$

Subtituting (63) and (64) back in the required expression, we have

$$\begin{aligned} J(\zeta,\boldsymbol{\eta}) &= q_1q_2(n(1+w)-1)\left[\frac{n}{\zeta} - \frac{nw}{\zeta(n(1+w)-1)}\right] \\ &= \frac{n(n-1)q_1q_2(1+w)}{\zeta}. \end{aligned}$$

We obtain the required expression using the identity $q_1(1+w) = w\zeta^{n-1}$. □

Proof of Corollary 1.2. Applying Theorem 1.1, and using Lemmas 7.1 and 7.2 (in particular, the identities in (62)), we obtain that

$$\begin{aligned} \mathbf{B}^{\ell,\mathbf{m}}_{g,\mathbf{r},n,0}(q_1,q_2) &= n^{\overline{g}}(n-1)^{\overline{g}}\sum_w\left(\sum_\zeta \zeta^\ell\cdot\prod_{i=1}^{n-2}\left(\frac{\zeta}{w}\right)^{im_i}\cdot(q_2w)^{m_{n-1}}\cdot(q_2w\zeta^{n-2})^{\overline{g}}\right) \\ &= n^{\overline{g}}(n-1)^{\overline{g}}q_2^{m_{n-1}+\overline{g}}\sum_w w^{m_{n-1}+\overline{g}-\sum_{i=1}^{n-2} im_i}\left(\sum_\zeta \zeta^{\ell+\sum_{i=1}^{n-2} im_i+(n-2)\overline{g}}\right). \end{aligned}$$

The sum is taken over all roots of the equation $w^n = q_1/q_2$ and subsequently all roots of $\zeta^{n-1} = (q_1(1+w^{-1}))$. Substitute

$$\sum_{i=1}^{n-2} im_i = (n-1)(d-m_{n-1}) - \ell - (2n-3)\overline{g}$$

to the above expression, we obtain that $\mathbf{B}^{\ell,\mathbf{m}}_{g,\mathbf{r},n,0}(q_1,q_2)$ equals

$$\begin{aligned} & n^{\overline{g}}(n-1)^{\overline{g}}q_2^{m_{n-1}+\overline{g}}\sum_w w^{nm_{n-1}+\ell-(n-1)d+(2n-2)\overline{g}}\left(\sum_\zeta \zeta^{(n-1)(d-m_{n-1}-\overline{g})}\right) \\ =& n^{\overline{g}}(n-1)^g q_1^{d-m_{n-1}-\overline{g}}q_2^{m_{n-1}+\overline{g}}\sum_w w^{nm_{n-1}+\ell-(n-1)d+(2n-2)\overline{g}}(1+w^{-1})^{(d-m_{n-1}-\overline{g})} \\ =& n^{\overline{g}}(n-1)^g q_1^{d-\overline{g}}q_2^{\overline{g}}\sum_w w^{\ell-(n-1)d+(2n-2)\overline{g}}(1+w^{-1})^{(d-m_{n-1}-\overline{g})} \\ =& n^{\overline{g}}(n-1)^g q_1^{d-\overline{g}}q_2^{\overline{g}}\sum_w w^{m_{n-1}+\ell+2n\overline{g}-\overline{g}-nd}(1+w)^{(d-m_{n-1}-\overline{g})} \\ =& n^{\overline{g}}(n-1)^g q_1^{\overline{g}}q_2^{d-\overline{g}}\sum_w w^{m_{n-1}+\ell-\overline{g}}(1+w)^{(d-m_{n-1}-\overline{g})}. \end{aligned}$$

To simplify the above, we note that

$$\sum_{w^n=q_1/q_2} w^k = \begin{cases} n(q_1/q_2)^{k/n} & n \mid k, \\ 0, & n \nmid k. \end{cases}$$

By the binomial theorem,

$$\sum_{w^n=q_1/q_2} w^{\ell+m_{n-1}-\overline{g}}(1+w)^{d-\overline{g}-m_{n-1}} = \sum_{\substack{j\in\mathbb{Z} \\ 0\leqslant jn-m_{n-1}-\ell+\overline{g}}} \binom{d-\overline{g}-m_{n-1}}{jn-\ell-m_{n-1}+\overline{g}} \cdot n\left(\frac{q_1}{q_2}\right)^j.$$

Substituting this identity into the previous expression completes the proof. □

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