

Elementary row operations:

Operation 1: switch 2 rows

E.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

nonzero

Operation 2: scale one row by a ^{real/complex} number

E.g.

$$\begin{bmatrix} 1+i & 1-i \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4i & 4 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 1-i \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2+2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

nonzero

Operation 3: add a ^{real/complex}-multiple of one row to another

E.g.

$$\begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\rightsquigarrow$$

$$\begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 & 4 \end{bmatrix} + (2+2i) \cdot \text{row 1} \quad \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4+6(2+2i) & 4+6(2+2i) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2+2i & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 0 & 4i \end{bmatrix}$$

Elementary column operations:

Operation 1: switch 2 columns

E.g.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 5 & 3 & 1 \\ 6 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Operation 2: scale one column by a ^{nonzero} real/complex number

E.g. $\begin{bmatrix} 1+i & 3 & 5 \\ -i & 4 & 6 \end{bmatrix} \xrightarrow{\times(2+2i)} \begin{bmatrix} (1+i)(2+2i) & 3 & 5 \\ (-i)(2+2i) & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4i & 3 & 5 \\ 4 & 4 & 6 \end{bmatrix}$

$$\begin{bmatrix} 1+i & 3 & 5 \\ -i & 4 & 6 \end{bmatrix} \left[\begin{array}{ccc|ccc} 2+2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \equiv$$

Operation 3: add a ^{nonzero} R/C - multiple of one column to another

E.g. $\begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \xrightarrow{+(-2+2i) \cdot \text{Column 1}} \begin{bmatrix} 1+i & 3 & 4 + (-2+2i)(1+i) \\ 2 & 4 & 4 + (-2+2i)2 \end{bmatrix} = \begin{bmatrix} 1+i & 3 & 0 \\ 2 & 4 & 4i \end{bmatrix}$

$$\begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \left[\begin{array}{ccc|ccc} 1 & 0 & -2+2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \equiv$$

Def. The matrices by which multiplications give elementary row/column operations are called **elementary matrices**

* Note: Elementary matrices are exactly the matrices you get by performing an elementary row/column operation to I_n

E.g. Let $T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be given by the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \text{ under standard bases. Compute } K(T), R(T).$$

$$x = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \text{ standard basis for } \mathbb{R}^3$$

$$y = \{(1, 0, 0, 0, 0), \dots, (0, 0, 0, 0, 1)\} \text{ standard basis for } \mathbb{R}^5$$

$$[T]_{xy} v = (x_1, x_2, x_3, x_4, x_5) \text{ in } \mathbb{R}^5, T(v) =$$

$$\left[\begin{array}{c} \\ \\ \\ \\ \end{array} \right] \cdot \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right] = \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right] \rightarrow \left\{ \begin{array}{l} y_1 \cdot (1, 0, 0) + \\ y_2 \cdot (0, 1, 0) + \\ y_3 \cdot (0, 0, 1). \end{array} \right\} = (y_1, y_2, y_3) \text{ in } \mathbb{R}^3$$

$K(T)$ = the set of vectors in \mathbb{R}^5 such that

$$\left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right], \text{ i.e., } K(T) \text{ is given by the}$$

set of real solutions to

$$\left\{ \begin{array}{l} 1x_1 + (-2)x_2 + 0x_3 + 2x_4 + (-3)x_5 = 0 \\ 2x_1 + (-4)x_2 + 2x_3 + 0x_4 + 8x_5 = 0 \\ 1x_1 + (-2)x_2 + 3x_3 + (-3)x_4 + 16x_5 = 0 \end{array} \right.$$

The 3 elementary row operations on the matrix

$\left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{array} \right]$ correspond to switching 2 equations, scaling one equation by a real number, and adding a real-multiple of one

equation to another. Therefore, they do not change the set of solutions.

$$\left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{array} \right] \xrightarrow{\text{②} \rightarrow \text{①} + (-2) \cdot \text{row 1}} \left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 1 & -2 & 3 & -3 & 16 \end{array} \right] \xrightarrow{\text{③} \rightarrow \text{③} + (-1) \cdot \text{row 1}} \left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 0 & 0 & 3 & -5 & 19 \end{array} \right]$$

$$\left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 0 & 0 & 3 & -5 & 19 \end{array} \right] \xrightarrow{\text{③} \rightarrow \frac{1}{2} \cdot \text{row 3}} \left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 0 & 0 & 1.5 & -2.5 & 9.5 \end{array} \right]$$

This process is called Gaussian Elimination.

$$\left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 3 & -5 & 19 \end{array} \right] + (-3) \cdot \text{row 2}$$

$$\left[\begin{array}{ccccc} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right] + (-2) \cdot \text{row 3} + 2 \cdot \text{row 3}$$

$$\left[\begin{array}{ccccc} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right] \leftarrow \text{reduced row echelon form of the matrix}$$

x_2 x_5 are free variables can take any real value

$$1x_4 + (-2)x_5 = 0 \Rightarrow x_4 = 2x_5$$

$$1x_3 + 3x_5 = 0 \Rightarrow x_3 = -3x_5 , x_2, x_5 \text{ any real value}$$

$$1x_1 + (-2)x_2 + 1x_5 = 0 \Rightarrow x_1 = 2x_2 - x_5$$

$$K(T) = \left\{ \begin{bmatrix} 2x_2 - x_5 \\ x_2 \\ -3x_5 \\ 2x_5 \\ x_5 \end{bmatrix}, x_2, x_5 \in \mathbb{R} \right\}, \text{ general solution.}$$

* Note that $R(T)$ is exactly the span of the column vectors of the matrix.

Dimension Theorem: V, W vector spaces over \mathbb{R}/\mathbb{C} , $T: V \rightarrow W$ linear transformation, then $\dim V = \underset{\text{nullity of } T}{\underset{\parallel}{\dim K(T)}} + \underset{\text{rank of } T}{\underset{\parallel}{\dim R(T)}}$

Rank of the matrix =

dimension of
column span
of the matrix
||
dimension of
row span of
the matrix

Definitions A matrix is said to be in **row echelon form** if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.
2. The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.⁵

If a matrix also satisfies the following two additional conditions, we say that it is in **reduced row echelon form**.⁶

4. If a column contains the leading entry of some row, then all the other entries of that column are 0.
 5. The leading entry of each nonzero row is 1.
-

* It can be shown that the reduced row echelon form of a matrix is unique.