Elementary row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operation 1: switch 2 rows

E.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Operation 2: scale one row by a nonzero real/complex number

E.g. $\begin{bmatrix} 1+i & 1-i \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow{\times(2+2i)} \begin{bmatrix} (1+i)(2+2i) & (1-i)(2+2i) \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4i & 4 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= \begin{bmatrix} 2+2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Operation 3: add a nonzero $\mathbb{R}/\mathbb{C}$-multiple of one row to another

E.g. $\begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 & 4 \end{bmatrix} + (-2+2i)\cdot \text{row } 1 \rightsquigarrow \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4+(-2+2i)(1+i) & 4+(-2+2i)2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2+2i & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 0 & 4i \end{bmatrix}$

Elementary lumn operations:

Operation 1: switch 2 columns

E.g. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 3 & 1 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Operation 2: scale one column by a nonzero real/complex number

E.g. $\begin{bmatrix} 1+i & 3 & 5 \\ 1-i & 4 & 6 \end{bmatrix} \underset{\times(2+2i)}{\rightsquigarrow} \begin{bmatrix} (1+i)(2+2i) & 3 & 5 \\ (1-i)(2+2i) & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4i & 3 & 5 \\ 4 & 4 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1+i & 3 & 5 \\ 1-i & 4 & 6 \end{bmatrix} \begin{bmatrix} 2+2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operation 3: add a nonzero $\mathbb{R}/\mathbb{C}$-multiple of one column to another

E.g. $\begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1+i & 3 & 4+(-2+2i)(1+i) \\ 2 & 4 & 4+(-2+2i)2 \end{bmatrix} = \begin{bmatrix} 1+i & 3 & 0 \\ 2 & 4 & 4i \end{bmatrix}$

$+(-2+2i)\cdot$ column 1

$$= \begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2+2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Def. The matrices by which multiplications give elementary row/column operations are called elementary matrices

☆ Note: Elementary matrices are exactly the matrices you get by performing an elementary row/column operation to I_n

E.g. Let $T: \mathbb{R}^5 \to \mathbb{R}^3$ be given by the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

under standard bases. Compute $K(T), R(T)$.

$\mathcal{X} = \{(1,0,0), (0,1,0), (0,0,1)\}$ standard basis for $\mathbb{R}^3$

$\mathcal{Y} = \{(1,0,0,0,0), \cdots, (0,0,0,0,1)\}$ standard basis for $\mathbb{R}^5$

$[T]_{\mathcal{X}\mathcal{Y}}$ $v=(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$, $T(v)=$

$$\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \longrightarrow \boxed{\begin{matrix} y_1 \cdot (1,0,0) + \\ y_2 \cdot (0,1,0) + \\ y_3 (0,0,1). \end{matrix}} = (y_1, y_2, y_3) \text{ in } \mathbb{R}^3$$

$K(T)$ = the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ in $\mathbb{R}^5$ such that

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e., $K(T)$ is given by the set of real solutions to

$$\begin{cases} 1x_1 + (-2)x_2 + 0x_3 + 2x_4 + (-3)x_5 = 0 \\ 2x_1 + (-4)x_2 + 2x_3 + 0x_4 + 8x_5 = 0 \\ 1x_1 + (-2)x_2 + 3x_3 + (-3)x_4 + 16x_5 = 0 \end{cases}$$

The 3 elementary row operations on the matrix $\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$ correspond to switching 2 equations, scaling one equation by a real number, and adding a real-multiple of one equation to another. Therefore, they do not change the set of solutions.

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \begin{matrix} \\ + (-2)\cdot \text{row } 1 \\ + (-1)\cdot \text{row } 1 \end{matrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix} \begin{matrix} \\ \times \frac{1}{2} \\ \\ \end{matrix}$$

$$\Downarrow$$

this process is called Gaussian Elimination

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix} \begin{matrix} \\ \\ + (-3)\cdot \text{row } 2 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} + (-2)\cdot \text{row } 3 \\ + 2\cdot \text{row } 3 \\ \\ \end{matrix}$$

$$\downarrow$$

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \\ \begin{matrix} \quad & x_2 & \quad & \quad & x_5 \end{matrix} \end{array}$$

← reduced row echelon form of the matrix

x_2 x_5 are free variables → can take any real value

$$1x_4 + (-2)x_5 = 0 \Rightarrow x_4 = 2x_5$$

$$1x_3 + 3x_5 = 0 \Rightarrow x_3 = -3x_5$$

$$1x_1 + (-2)x_2 + 1x_5 = 0 \Rightarrow x_1 = 2x_2 - x_5$$

, x_2, x_5 any real value

general solution.

$$K(T) = \left\{ \begin{bmatrix} 2x_2 - x_5 \\ x_2 \\ -3x_5 \\ 2x_5 \\ x_5 \end{bmatrix}, \; x_2, x_5 \in \mathbb{R} \right\}$$

☆ Note that $R(T)$ is exactly the span of the column vectors of the matrix.

Dimension Theorem: V, W vector spaces over $\mathbb{R}/\mathbb{C}$, $T: V \to W$ linear transformation, then $\dim V = \underset{\text{nullity of } T}{\underset{\shortparallel}{\dim K(T)}} + \underset{\text{rank of } T}{\underset{\shortparallel}{\dim R(T)}}$

$\shortparallel$

rank of the matrix = dimension of column span of the matrix = dimension of row span of the matrix

Definitions A matrix is said to be in **row echelon form** if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.
2. The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.[5]

If a matrix also satisfies the following two additional conditions, we say that it is in **reduced row echelon form**.[6]

4. If a column contains the leading entry of some row, then all the other entries of that column are 0.
5. The leading entry of each nonzero row is 1.

* It can be shown that the reduced row echelon form of a matrix is unique.