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Phys 4250
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Lorenz Strange Attractor

Abstract

Non-linear systems with chaotic properties are deterministic systems with high sensitivity to small changes in initial conditions and parameters which lead to completely different solution. Logistic map and Lorenz strange attractor are ways to look at simple non-linear systems which lead to complex, chaotic behaviors. In the result, we study these two systems with different initial parameters. In the Lorenz case, we compute the dispersion rate for two different initial setups, with the initial difference of 10^{-12}. As well, we study the bifurcation graph of Lorenz system by collecting points crossing $z - r + 1 = 0$.

Theory

The logistic map is a one-dimensional discrete time mapping of degree 2, developed by Robert May, defined as

$$x_{n+1} = r x_n (1 - x_n)$$

where x_n remains bounded on $[0,1]$ while r arranges from $[0,4]$. In a biological sense, logistic map captures the relation between the growth rate r and the population x_n.

The Lorenz strange attractor is a system of three ordinary differential equations, invented by Edward Lorenz and Ellen Fetter, was first used as a simplified model for atmospheric convection, defined as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z$$

MATLAB interpretation

parameters

Logistic map

The parameter r for Logistic map is between $[0,4]$, with step size of 0.001. Our initial choice of x_0 is 0.5.

Lorenz attractor

The initial value of x_0, y_0, z_0 are chosen to be 3 3 3. The value of σ and β are fixed to be 10 and $\frac{8}{3}$, respectively. We study how the value of ρ affects our system, with the value of 14 and 28. As well, we study how slightly change of our initial condition, in the order of magnitude 10^{-12}, affect our final state of the system. For part c, in order to resemble the bifurcation diagram of Lorenz attractor, our initial choice of x_0, y_0, z_0 depends on the last value of the previous step ρ, for example, the x, y and z value for the last event $z - r + 1 = 0$ occurs.

Results

Logistic map

Below is the plot of logistic map with the parameters discussed above (transient state not shown). We observe different behaviors of the system with different value of r.

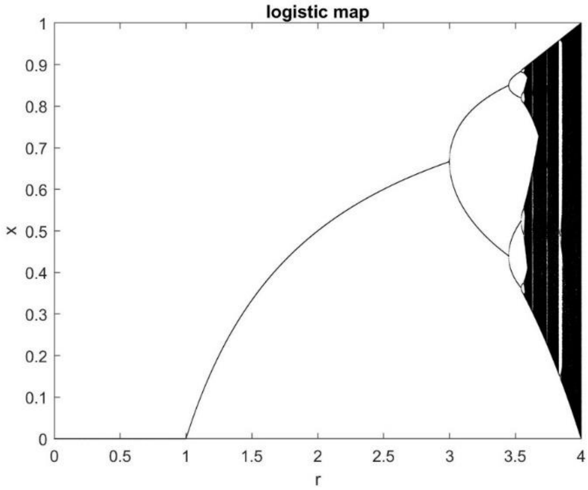

Figure 1 Logistic Map

As we can see that for r = [0,1], x_n will eventually approach 0, independent our choice of x_0. When r is between 1 and 2, x_n approaches the value of $\frac{r-1}{r}$[1].

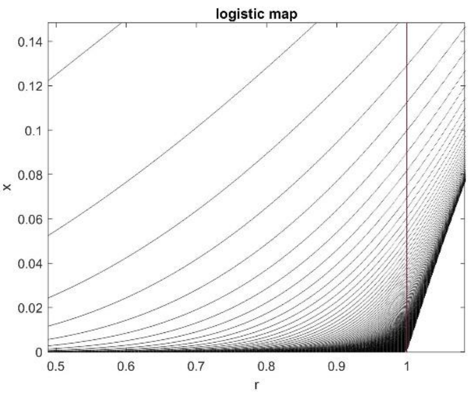

Figure 2 Logistic map, r = [0,1]

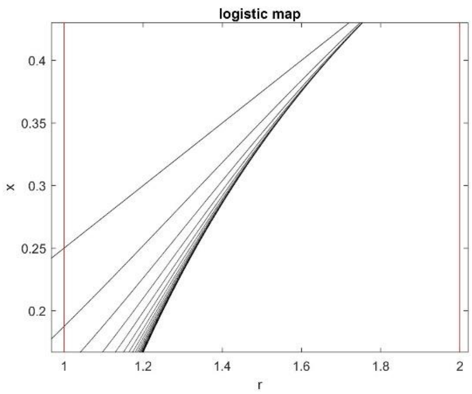

Figure 3 Logistic map, r = [1,2]

When r is between 2 and 3, x_n will fluctuate around $\frac{r-1}{r}$ and finally converge to this value. Interesting behavior arises when r goes beyond 3. When r is between 3 and 3.569, x_n will oscillate between 2, 4, 8 and 16 values, etc.

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[12] Logistic map. https://en.wikipedia.org/wiki/Logistic_map

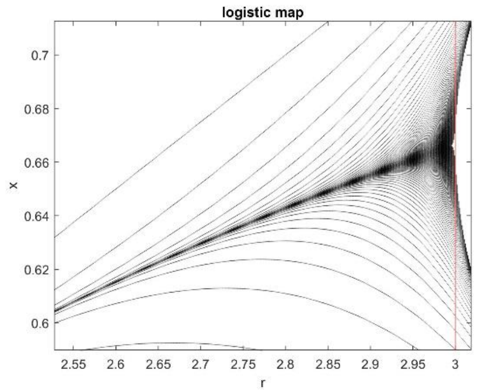


Figure 4 Logistic map. r = [2,3].

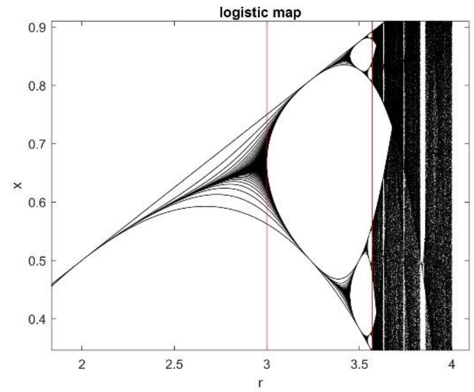


Figure 5 Logistic map. r= [3,3.569]

When r goes beyond 3.569, we no longer see finite period of oscillations, this is where the system shows the characteristic of chaos, where slight change of initial x_0 will result in a completely different x_n. However, there are still some regions that are stable. As shown in the graph below, we see when r is approximately between 3.82 and 3.85, we observe that x_n oscillate between 3, 6, and 12 values.

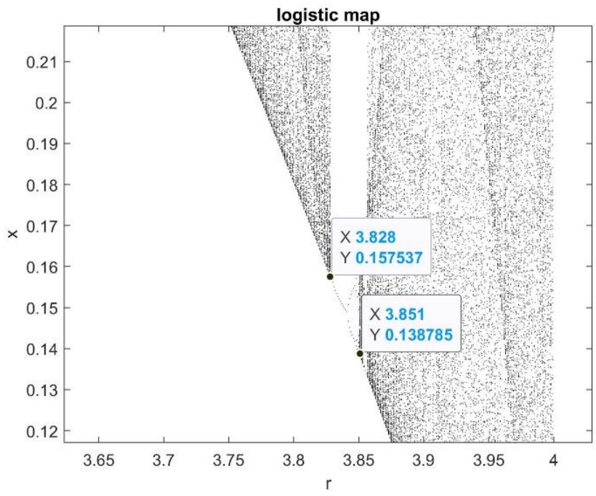


Figure 6 Logistic map. r= [3.828,3.851]

Lorenz attractor

Null test

Before proceeding, we shall see our system doesn't diverge from each other without introducing small perturbation. Our initial value for σ, β and ρ are 10, $\frac{8}{3}$ and 28, respectively. We ran through 25 tests under the same initial conditions.

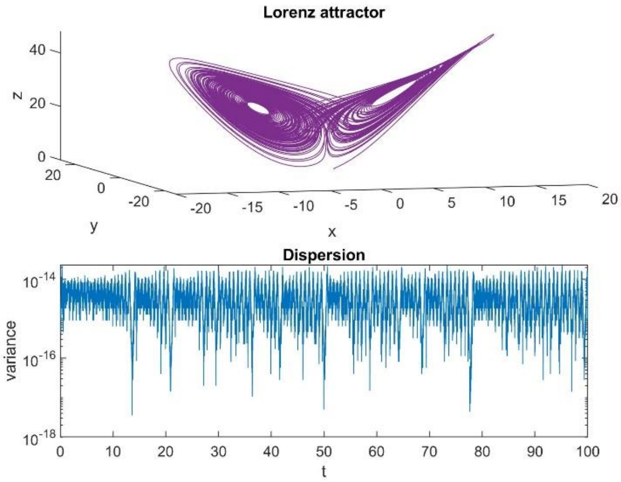


Figure 7 Lorenz attractor. 25 sets of same initial condition with sigma, beta, rho = 10, 8/3, 28.

As we can see above, without small perturbation in initial conditions, all 25 tests remain the same, with the non-increasing dispersion in the magnitude of 10^{-14}. Theoretically there should be no dispersion, this error could be due to the error when rounding off.

Introduce small perturbation

Now we test using the value of 10, $\frac{8}{3}$ and 14 for σ, β and ρ. We perturbate our initial values of x_0, y_0, z_0 by some random amount in the magnitude of 10^{-12}. For our initial choice of any non-zero values, the system does not diverge from each other as we expected, for value of ρ smaller than 24.74[2]. We can see the trajectory going around one fixed point attractor and converges to the fix point.

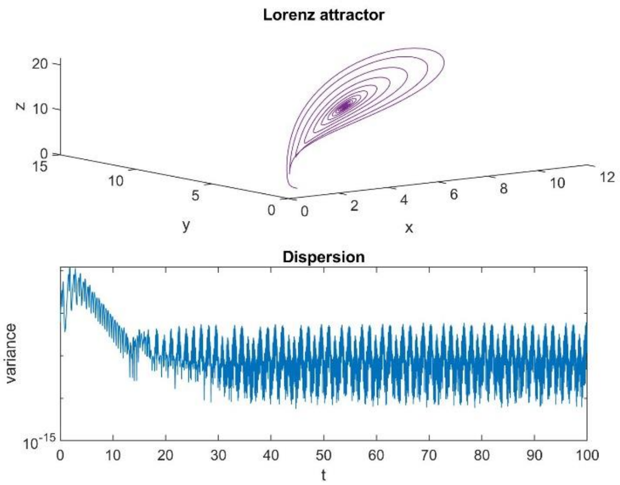


Figure 8 Lorenz attractor. 25 sets of initial condition with small perturbation, with sigma, beta, rho = 10, 8/3, 14.

We see interesting phenomenon when [σ β ρ] = [10, $\frac{8}{3}$, 28]. The trajectory goes around two fixed point repulsors, and getting further away from those two points. We

[2] Lorenz system. https://en.wikipedia.org/wiki/Lorenz_system

observe that the system has the characteristic of chaos, as 25 sets of initial value [1 1 1] with small difference of magnitude of 10^{-12} between them, diverge into totally different values in their end state.

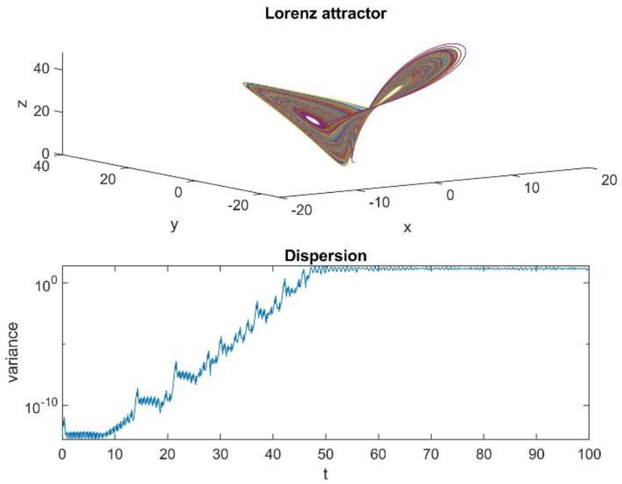


Figure 9 Lorenz attractor. 25 sets of initial condition with small perturbation, with sigma, beta, rho = 10, 8/3, 28.

As well, we can see how quickly they diverge from each other, by trimming the data right before it reaches the plateau. After taking the natural log value of the dispersion and use first order polynomial fit, we observe the slope to be 0.69.

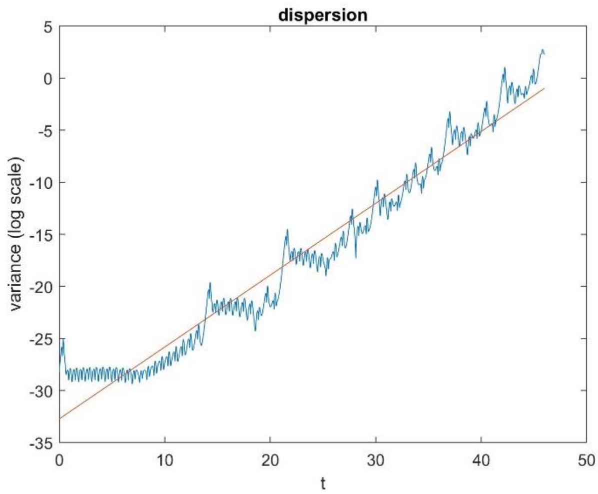
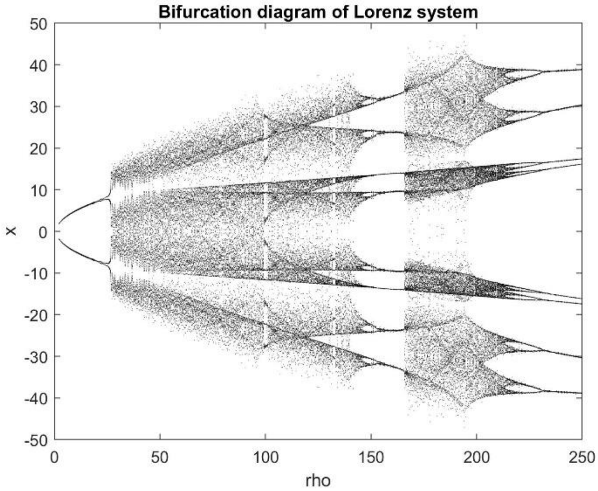


Figure 10 Dispersion between 25 sets of different initial value, with the difference of 1e-12.

Bifurcation diagram of Lorenz system

We generate the bifurcation diagram of Lorenz system by recording the coordinate of x when the trajectory crosses the plane $z - \rho + 1 = 0$, where ρ range from 0 to 250, with step of 0.25. In order to do this, we include the event function in the ode45 option. Moreover, the last set of coordinates when the event happens in the current value ρ, are being use as the initial condition to solve the ode45 as next value of ρ. This way we can resemble the plot in the reddit post.



We notice that the bifurcation diagram of Lorenz system shares some similarity of logistic map. For r between 0 to around 27.5, x appears in two ‘strings’ of values that densely stay together. Beyond that is where the system shows the characteristic of chaos. However, for ρ around 150 and after 220, the system shows non-chaotic behavior. We can also see the ‘strings’ in some shade(chaotic) region, where some x values appears more often.