Stability of N-Body Choreographies

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Abstract

The N-body problem refers to finding out the subsequent motions of multiple gravitationally attracted objects with known initial positions, velocities and masses in the context of classical mechanics. In general, for $n > 2$, the n-body problem is chaotic. However, in some special cases, all masses move in a same closed curve with repeat pattern. The term *n-body choreography* was first used by Chenciner and Montgomery to describe periodic solution to the n-body problem. The very first of such orbit was found by Lagrange in the year 1772, where a system of 3 equal mass in an equilateral triangle, traced out a circular path. The next solution was found 2 centuries later by Cristopher Moore, with 3 equal mass tracing out a figure-8 orbit. Many others were numerically found after Moore. Several new solutions were found by Carlés Simó in the year 2000.

In this report we use the initial conditions provided by Carlés Simó, and Joel Dyck to analyze the stability of these orbits. We use symplectic integrator in our numeric integration. A symplectic integrator is a numerical integration scheme for Hamiltonian systems, which allows the simulation to obey Hamiltonian equations to high accuracy and the energy of the system is conserved.

We analyzed 13 different choreographies by using three different orders of symplectic integrators. In our results, we found that only the 3-body choreography is stable within the parameters of our simulation. The other choreographies were all unstable.

Theory

The general Symplectic method(q, p) has the form:

$$\vec{q}(t + \mathrm{d}t) = \vec{q}_{N-1} + Q_N\left(\vec{\nabla}_p H\right)(q_{N-1}, p_{N-1})\mathrm{d}t$$

$$\vec{p}(t + \mathrm{d}t) = \vec{p}_{N-1} - P_N\left(\vec{\nabla}_q H\right)(q(t + \mathrm{d}t), p_{N-1})\mathrm{d}t$$

More specifically, the Hamiltonian in n body problem (G=1) is:

$$H = \frac{1}{2}\Sigma\frac{|p_i'|^2}{m_i} - \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{m_i m_j}{|q_i' - q_j'|}$$

Thus, the symplectic scheme becomes(m=1):

$$\vec{q}(t + \mathrm{d}t) = \vec{q}_{N-1} + Q_N\vec{p}(t)\mathrm{d}t$$

$$\vec{p}(t + \mathrm{d}t) = \vec{p}_{N-1} + P_N\vec{F}(\vec{q}(t + \mathrm{d}t))\mathrm{d}t$$

We have the second order symplectic method:

$$\vec{p}_1 = \vec{p}(t) + \left(\frac{1}{2}\right)\vec{F}(\vec{q}(t))\mathrm{d}t$$

$$\vec{q}_1 = \vec{q}(t) + (1)\vec{p}_1\mathrm{d}t$$

$$\vec{p}(t + \mathrm{d}t) = \vec{p}_1 + \left(\frac{1}{2}\right)\vec{F}(\vec{q}_1)\mathrm{d}t$$

$$\vec{q}(t + \mathrm{d}t) = \vec{q}_1 + (0)\vec{p}(t + \mathrm{d}t)\mathrm{d}t$$

which leads to the Verlet integration method:

$$\vec{q}(t+\mathrm{d}t)=\vec{q}(t)+\vec{p}(t)\mathrm{d}t+\left(\frac{1}{2}\right)\vec{F}(\vec{q}(t))\mathrm{d}t^2$$

$$\vec{p}(t+\mathrm{d}t)=\vec{p}(t)+\left(\frac{1}{2}\right)(\vec{F}(\vec{q}(t))+\vec{F}(\vec{q}(t+\mathrm{d}t)))\mathrm{d}t$$

the third order integrator used here was discovered by Ruth in 1983:

$$\vec{q}_1(t)=\vec{q}(t)+\left(\frac{2}{3}\right)\vec{p}(t)\mathrm{d}t;\ \vec{p}_1(t)=\vec{p}(t)+\left(\frac{7}{24}\right)\vec{F}(\vec{q}_1(t))\mathrm{d}t$$

$$\vec{q}_2(t)=\vec{q}_1(t)-\left(\frac{2}{3}\right)\vec{p}_1(t)\mathrm{d}t;\ \vec{p}_2(t)=\vec{p}_1(t)+\left(\frac{3}{4}\right)\vec{F}(\vec{q}_2(t))\mathrm{d}t$$

$$\vec{q}(t+\mathrm{d}t)=\vec{q}_2+(1)\vec{p}_2(t)\mathrm{d}t$$

$$\vec{p}(t+\mathrm{d}t)=\vec{p}_2(t)-\left(\frac{1}{24}\right)\vec{F}(\vec{q}(t+\mathrm{d}t))\mathrm{d}t$$

and fourth order (also by Ruth):

$$\vec{q}_1(t)=\vec{q}(t)+\frac{1}{2\left(2-2^{1/3}\right)}\vec{p}(t)\mathrm{d}t;\ \vec{p}_1(t)=\vec{p}(t)+\frac{1}{2-2^{\frac{1}{3}}}\vec{F}(\vec{q}_1(t))\mathrm{d}t$$

$$\vec{q}_2(t)=\vec{q}_1(t)\frac{1-2^{\frac{1}{3}}}{2\left(2-2^{\frac{1}{3}}\right)}\vec{p}_1(t)\mathrm{d}t;\ \vec{p}_2(t)=\vec{p}_1(t)-\frac{2^{\frac{1}{3}}}{2-2^{\frac{1}{3}}}\vec{F}(\vec{q}_2(t))\mathrm{d}t$$

$$\vec{q}_3(t)=\vec{q}_2(t)\frac{1-2^{\frac{1}{3}}}{2\left(2-2^{\frac{1}{3}}\right)}\vec{p}_2(t)\mathrm{d}t;\ \vec{p}_3(t)=\vec{p}_1(t)+\frac{1}{2-2^{\frac{1}{3}}}\vec{F}(\vec{q}_3(t))\mathrm{d}t$$

$$\vec{q}(t+\mathrm{d}t)=\vec{q}_3+\frac{1}{2\left(2-2^{1/3}\right)}\vec{p}_3(t)\mathrm{d}t$$

$$\vec{p}(t+\mathrm{d}t)=\vec{p}_3(t)+(0)\vec{F}(\vec{q}(t+\mathrm{d}t))\mathrm{d}t$$

MATLAB implementation

The gravitational constant G is set to 1, as well the mass of each body is chosen to be 1. The period of each of the choreographies studied was mathematically shown to be 2π. Since the problem we study is highly dependent on size of time step, a relative small time step is needed. However, a step size of 10^{-n} takes 6.2832 *10^{n} iterations to complete one period. Considering we have limited computational resources, as well experimenting with different values, we choose our step size to be 10^{-4},which will be discussed further in our result. The system is solved using 2nd, 3rd and 4th order symplectic integrator with coefficients discussed above. It is also solved using ode45 as a comparison in later discussion. In general n-body problem, one might concern the likelihood of encountering $r \to 0$ as two objects get too close. However, in n-body choreographies, the n objects in the orbit are designed so that they are equally spaced, hence this case is out of our consideration.

Results

Below are the resulting plots using Simo's provided data, with parameters discussed above. We also interested in how fast the orbit diverge after one period. We compare the positions of each object at every 2π time interval and calculate the variance of the position. Since each object has a period of 2π, ideally you will find the object at the same position every 2π interval and therefore have a dispersion of zero.

3 body figure 8 (t= 8π)

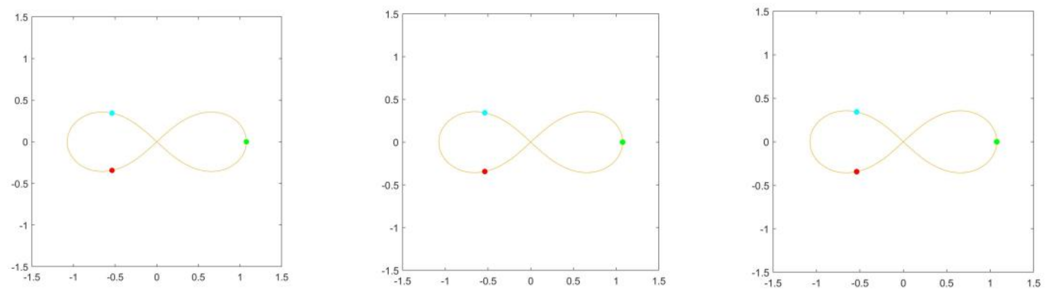


Figure: 1, 3-body figure 8 with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

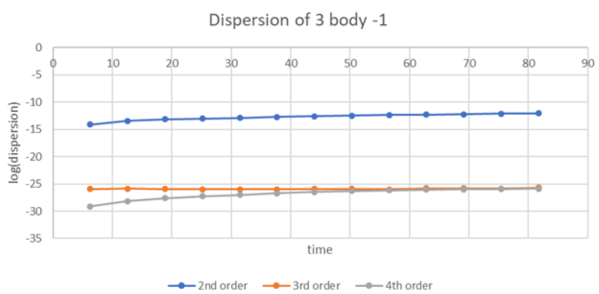


Figure: 2, Dispersion of 3-body with 2nd, 3rd and 4th order symplectic integrator

4 body 1 (t= 8π)

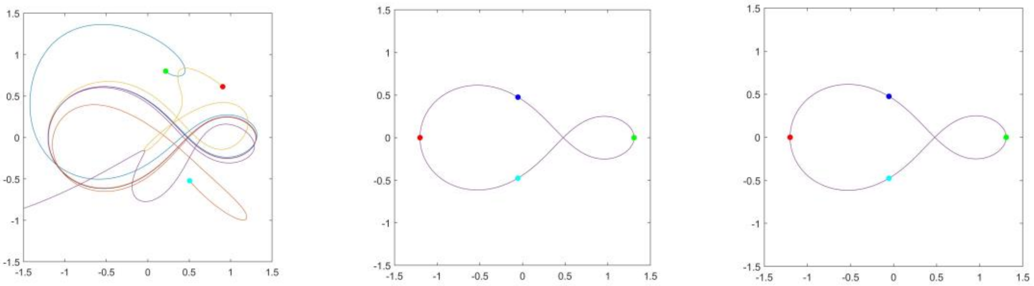


Figure: 3, 4-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

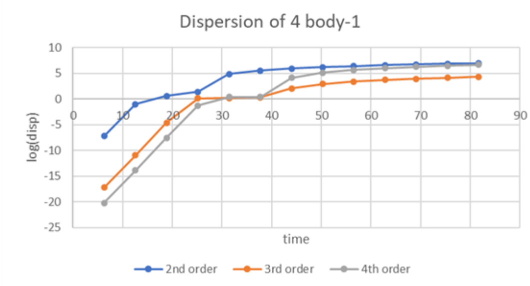


Figure:4, Dispersion of 4-body with 2nd, 3rd and 4th order symplectic integrator

4 body 2 (t= 8π)

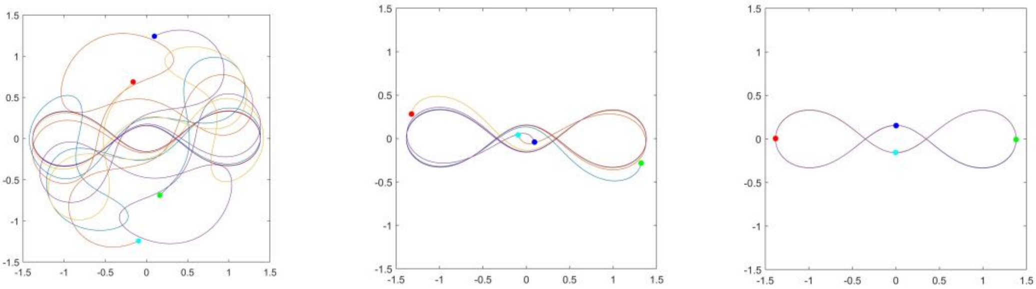


Figure: 5, 4-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

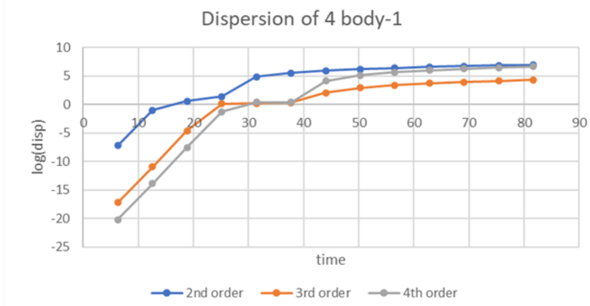


Figure: 6, Dispersion of 4-body with 2nd, 3rd and 4th order symplectic integrator

4 body 3 (t= 4π)

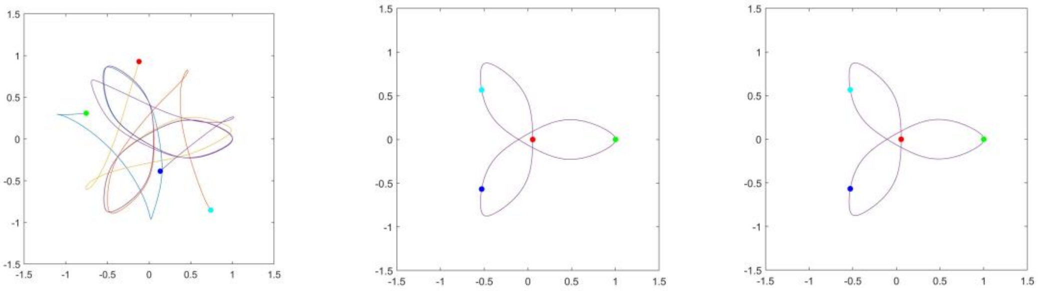


Figure: 7, 4-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

Figure: 8, Dispersion of 4-body with 2nd, 3rd and 4th order symplectic integrator

4 body 4 (t= 4π)

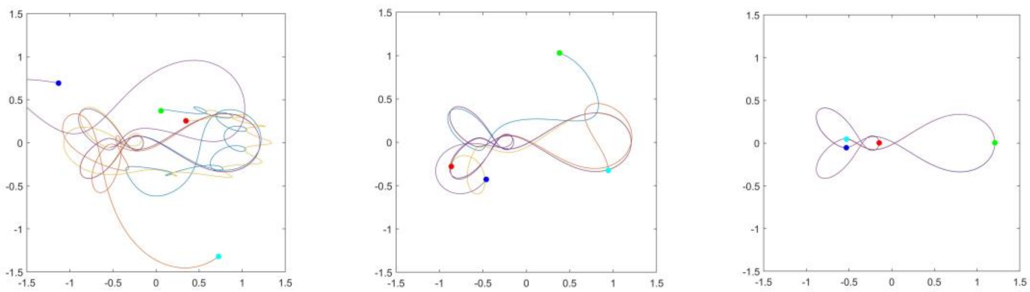


Figure: 9, 4-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

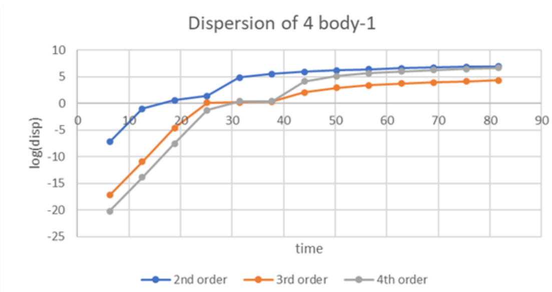


Figure: 10, Dispersion of 4-body with 2nd, 3rd and 4th order symplectic integrator

4 body 5 (t= 8π)

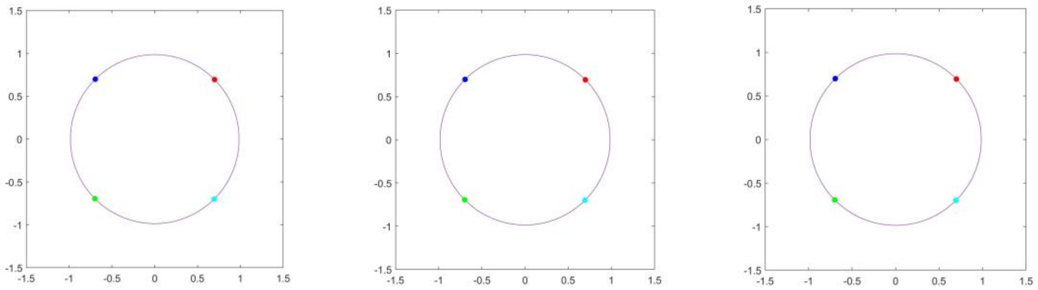


Figure: 11, 4-body circle with 2^{nd}, 3^{rd} and 4^{th} order symplectic integrator (from left to right in respective order)

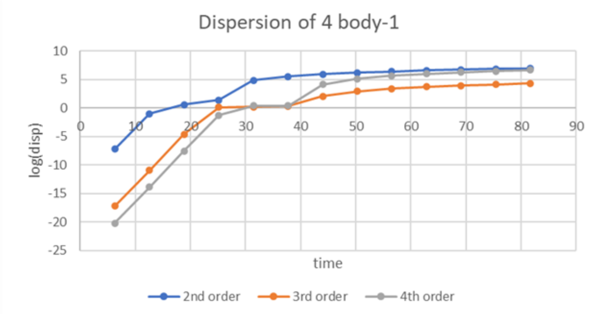


Figure: 12, Dispersion of 3-body with 2^{nd}, 3^{rd} and 4^{th} order symplectic integrator

5 body 1 (t= 4π)

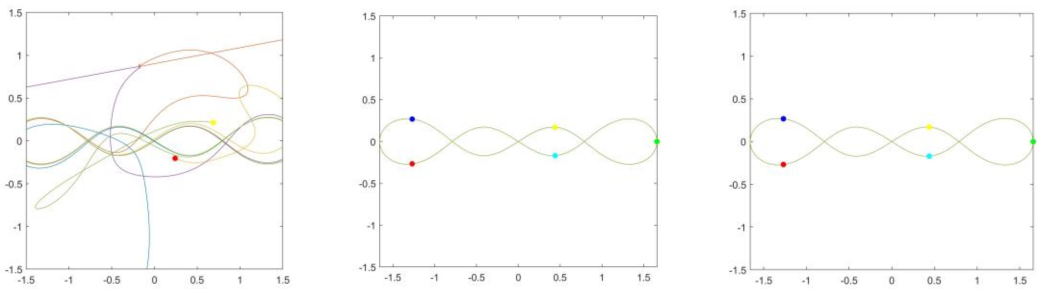


Figure: 13, 5-body with 2^{nd}, 3^{rd} and 4^{th} order symplectic integrator (from left to right in respective order)

Figure: 14, Dispersion of 3-body with 2^{nd}, 3^{rd} and 4^{th} order symplectic integrator

5 body 2 (t= 2π)

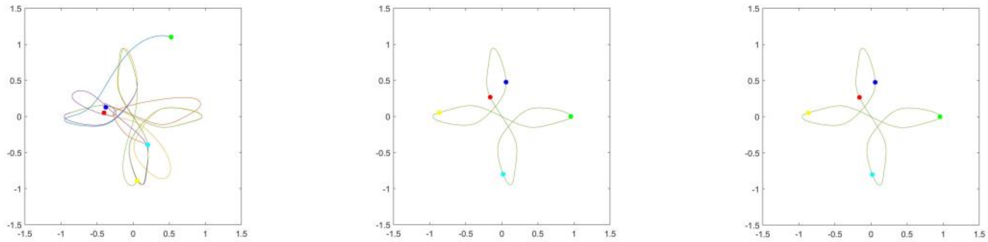


Figure: 15, 5-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

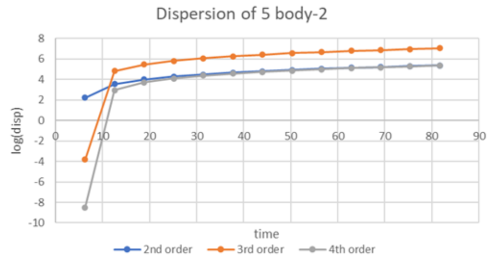


Figure: 16, Dispersion of 3-body with 2nd, 3rd and 4th order symplectic integrator

5 body 3 (t= 4π)

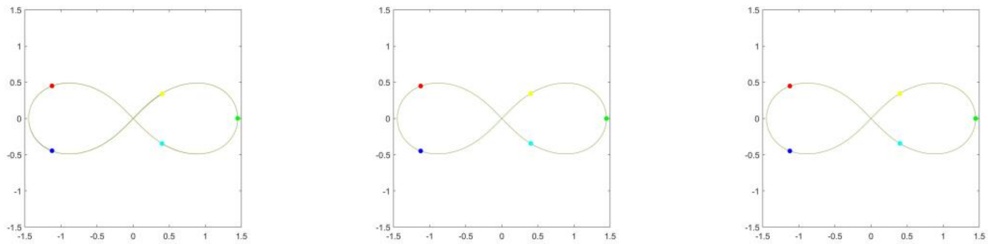


Figure: 17, 5-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

Figure: 18, Dispersion of 3-body with 2nd, 3rd and 4th order symplectic integrator

5 body 4 (t= 2π)

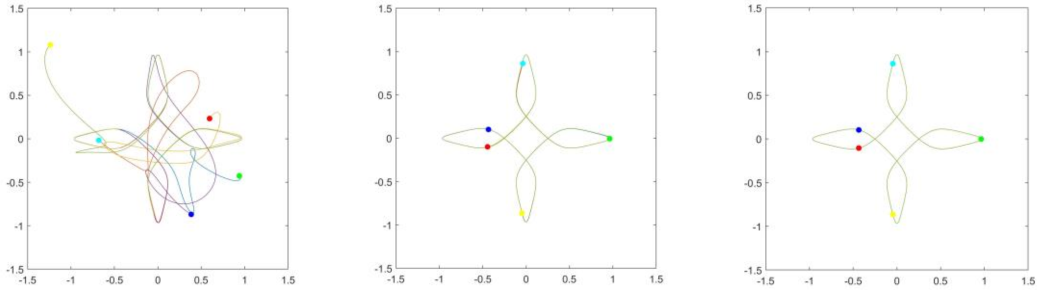


Figure: 19, 5-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

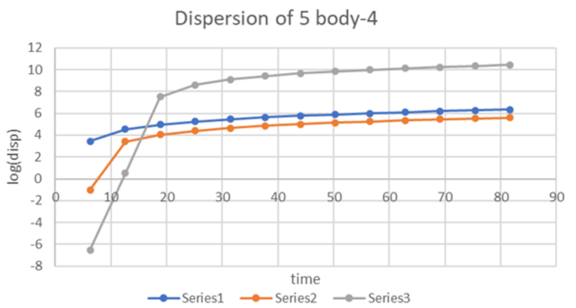


Figure: 20, Dispersion of 3-body with 2nd, 3rd and 4th order symplectic integrator

5 body 5 (t= 2π)

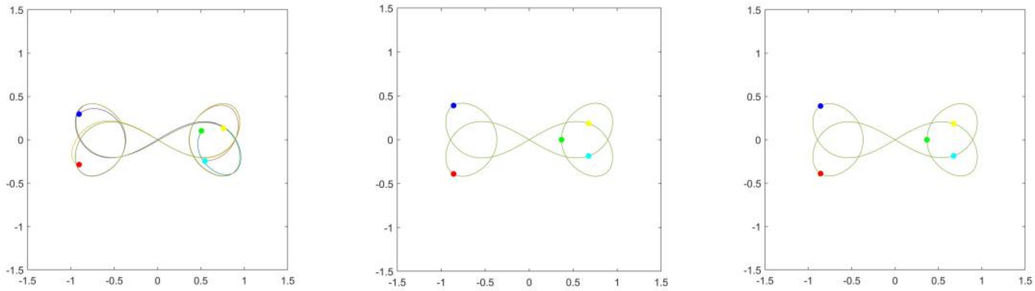


Figure: 21, 5-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

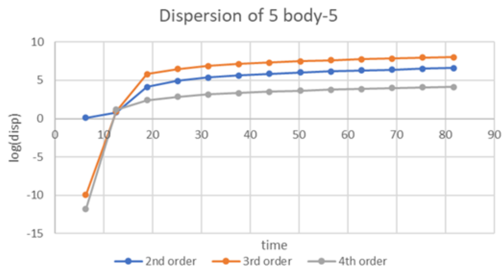


Figure: 22, Dispersion of 5-body with 2nd, 3rd and 4th order symplectic integrator

5 body 6 (t= 4π)

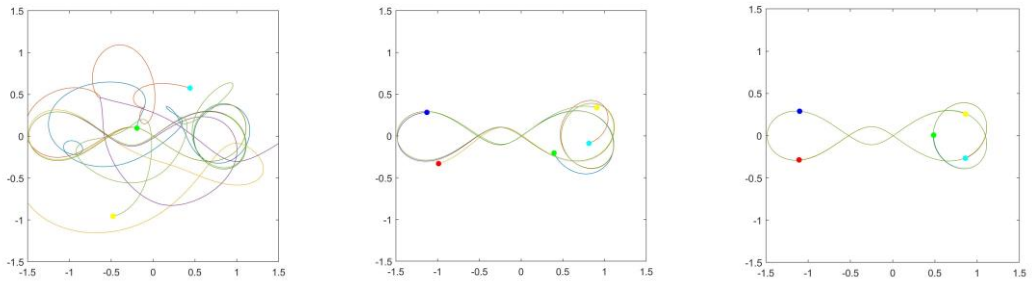


Figure: 23, 5-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

Figure: 24, Dispersion of 5-body with 2nd, 3rd and 4th order symplectic integrator

5 body 7 (t= 4π)

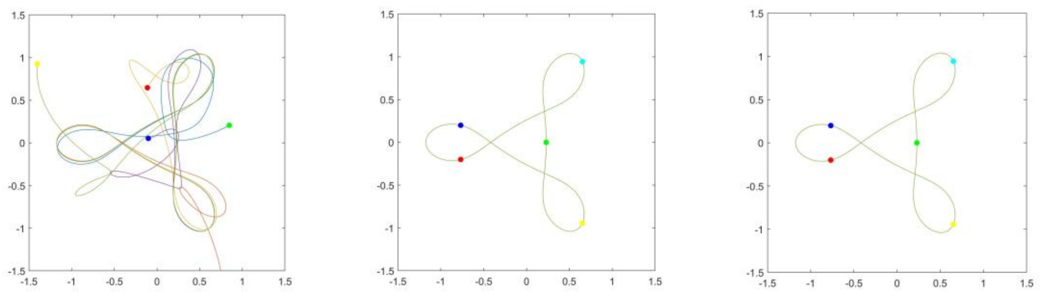


Figure: 26, 5-body with 2nd, 3rd and 4th order symplectic integrator (from left to right in respective order)

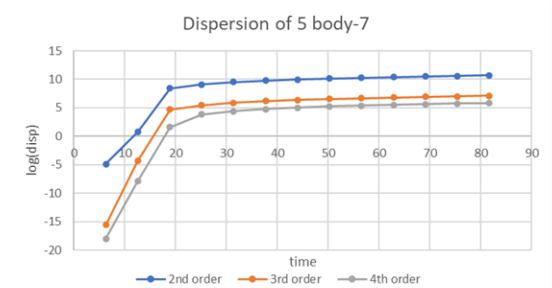


Figure: 27, Dispersion of 5-body with 2nd, 3rd and 4th order symplectic integrator

From the above plots we see that the 3-body choreography is the only orbit that is stable withing the parameters of our simulation. The rest of the choreographies show an exponential divergence during the first four revolutions. After the objects have been flung away from the gravitational pull of the rest of the objects, they continue their orbit in a straight line unimpeded, which can be see as a nearly linearly line on the plots. We also see that for a system of more bodies, the orbit becomes chaotic more quickly. As we can see that for most cases of 5 body choreography, the orbit diverges only after one period.

<u>Introducing a perturbation to a stable periodic orbit</u>

We study how a slight change of initial positions affect our orbit for the stable (3-body) case. For a given initial position of $R = (r_1, r_2, r_3)$, we introduce small parameter ε and we first perturbed only one body $R_1 = (\varepsilon r_1, r_2, r_3)$. Next, we perturbed bodies $R_2 = (\varepsilon r_1, \varepsilon r_2, r_3)$. We used the 4$^{\text{th}}$ order symplectic integrator and ran the simulation for ten revolutions.

Perturbing a single body

Starting with the initial conditions provided by Carlés Simo, we modify a single body in the system by multiplying small parameter ε. The value of ε and its corresponding plots are shown below.

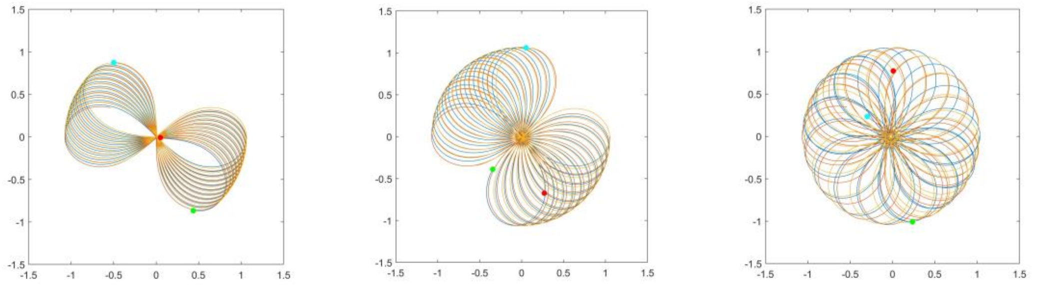


Figure: 28, ε=0.99, 0.98,.0.95 (from left to right in respective order)

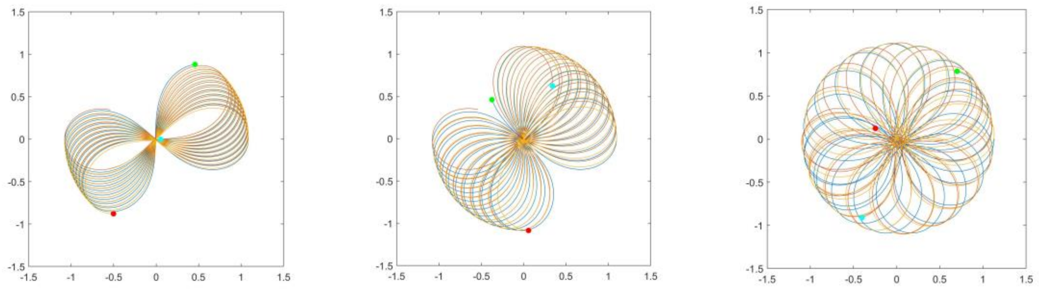


Figure: 29, ε=1.01, 1.02,.1.05 (from left to right in respective order)

Figure: 30, ε=0.75, 1.25 (from left to right in respective order)

We can see that for $\varepsilon = (0.75, 0.99)$, the orbit rotates clockwise at a constant rate. The rate of rotation is proportional to the value of ε. The rotation direction flips for $\varepsilon = (1.01, 1.25)$. It appears that the rotation of the orbit will continue indefinitely. For ε smaller than 0.75 or higher than 1.25 will introduce too much of a perturbance that results in chaotic motion.

Perturbing two bodies

Now we look at how perturbing two values of initial positions affects our orbit. Similar to the previous scenario, the initial position of the 2^{nd} and 3rd bodies are perturbate by an amount of ε. We run the simulation for ten revolutions.

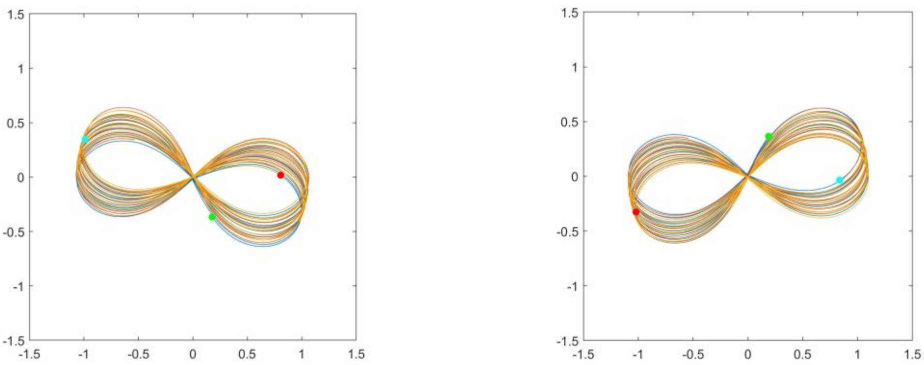


Figure: 31, ε=0.99, 1.01 (from left to right in respective order)

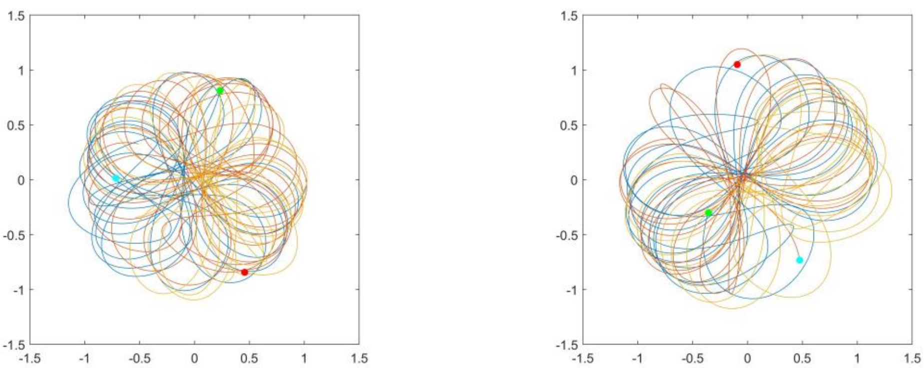


Figure: 32, ε=0.95, 1.05 (from left to right in respective order)

When perturbing 2 initial values, unlike the first case where it remains stable at ±20 %, a small shift of around ±5 % makes the orbit become unstable very quickly.

Discussion on time step

We now look into how our choice of time step affect our orbit. We again look at 3-body figure-8 case, using symplectic 2nd and 4th order with step size of $1e^{-4}$, $1e^{-5}$ and $1e^{-6}$. It is unexpected that the results seems to suggest that the choose of step size have no noticeable effects to our stability of the orbit. It is even stranger that for 4th order symplectic method, smaller step size leads to larger dispersion. The reasoning behind this is unknown, further discussion is needed.

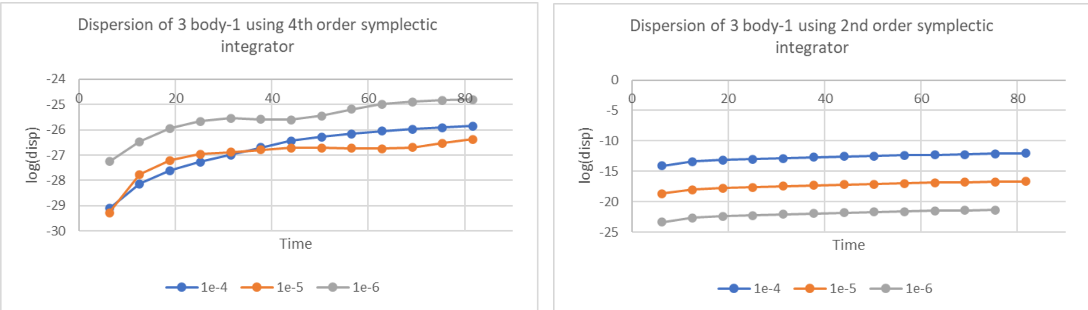
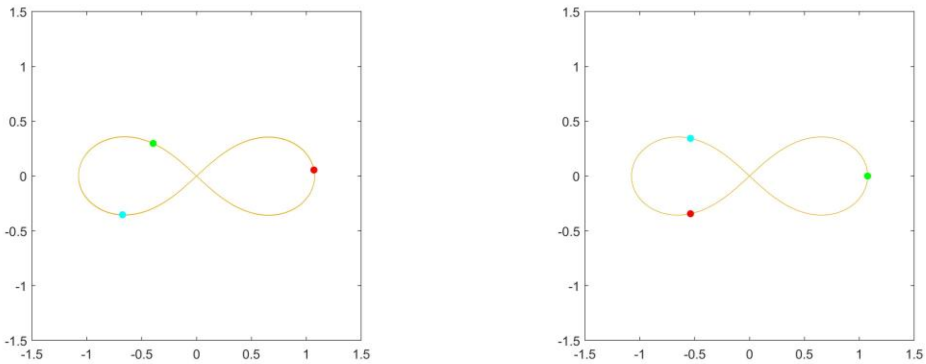


Figure: 33, Comparison of dispersion using various step size, with 2nd and 4th order symplectic method.

Comparison between ode45 and a symplectic integrator

For the stable orbit figure-8, we want to investigate how the two different types of integrators affect our orbit in long time behavior. We solve the system using ode45 and the 4th order symplectic integrator for a time span of 1280π (640 periods). To demonstrate that the symplectic method does a better job in conserving energy than an ordinary integrator, we also generate the phase-space plot of figure-8.



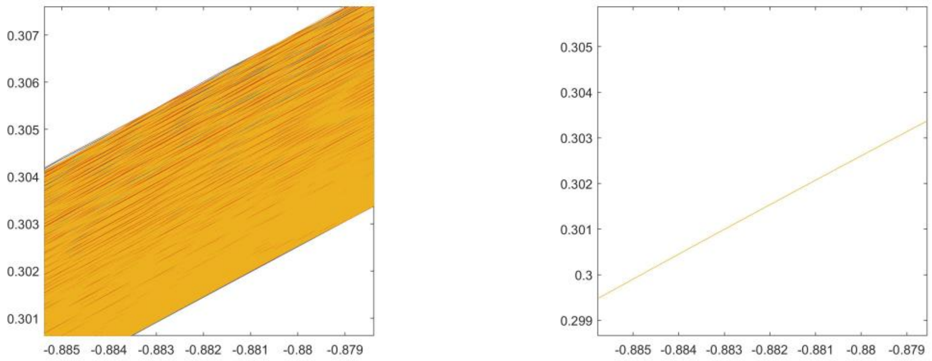


Figure: 34, solved using ode45 and symplectic integrator 4th order (from left to right in respective order) bottom images are zoomed into a section of the orbit

We see that the orbit changes slightly over a long period of time since the energy is not conserved when using ode45. The orbit of the 4th order symplectic integrator, under the same magnification as the ode45 orbit, shows no visible change at all. It is surprising that, even though ode45 dynamically selects the optimal step size, symplectic scheme still does a better job using fixed time step.

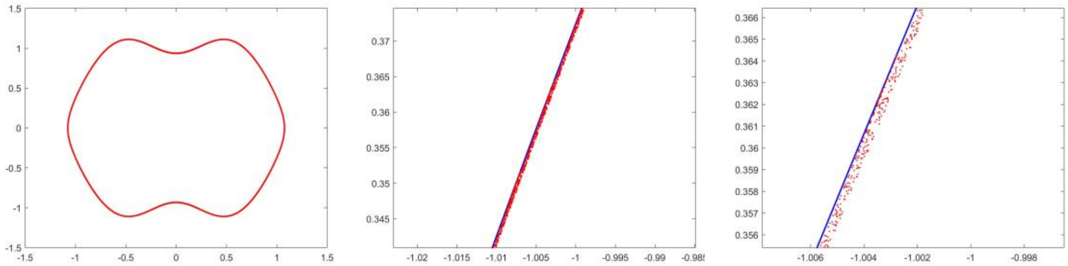


Figure: 35, A two dimensional slice of the four dimensional phase plot.

In terms of the result of phase plot, the trajectory of ode45 solution slowly merging towards the origin. The 4th order symplectic however, remains thin. What we can see is that symplectic integrators tend to capture the long-time patterns better than ode45 because of this lack of drift and this almost guarantee of periodicity. Hence symplectic integrators are good at long term integrations on problems that have symplectic property (Hamiltonian system).

References

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