# Some symbolic dynamics in real quadratic fields with applications to inhomogeneous minima

Nick Ramsey\*

**DePaul University** 

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#### **Abstract**

Let K be a real quadratic field. We use a symbolic coding of the action of a fundamental unit on the torus  $\mathbb{T}_K = (K \otimes_{\mathbb{Q}} \mathbb{R})/\mathfrak{O}_K$  to study the family of subsets  $X_t \subseteq \mathbb{T}_K$  of norm distance  $\geq t$  from the origin. As an application, we prove that inhomogeneous spectrum of K contains a dense set of elements of K, and conclude that all isolated inhomogeneous minima lie in K.

#### 1 Introduction

Let D > 1 be a square-free positive integer and let  $K = \mathbb{Q}(\sqrt{D})$  be the associated real quadratic field with ring of integers  $\mathcal{O}_K$ . Let  $\mathbf{N} : K \longrightarrow \mathbb{Q}$  denote the absolute norm  $\mathbf{N}(a) = |\mathrm{Nm}_{K/\mathbb{Q}}(a)| = |a\overline{a}|$ , where  $a \mapsto \overline{a}$  is Galois conjugation, and recall that the ring  $\mathcal{O}_K$  is called norm-Euclidean if for all  $a \in K$  there exists  $q \in \mathcal{O}_K$  such that  $\mathbf{N}(a-q) < 1$ . The ring of integers  $\mathcal{O}_K$  embeds as a lattice in the two-dimensional real vector space  $V_K = K \otimes_{\mathbb{Q}} \mathbb{R}$ , and we denote the quotient torus by  $\mathbb{T}_K = V_K/\mathcal{O}_K$ . Galois conjugation extends linearly to  $V_K$ , and the absolute norm extends accordingly to an indefinite quadratic form on  $V_K$  that we also denote by  $\mathbf{N}$ . The norm is not  $\mathcal{O}_K$ -invariant, but the function defined by

$$M(P) = \inf_{Q \in \mathcal{O}_K} \mathbf{N}(P - Q)$$

is, and descends to a function on the torus  $\mathbb{T}_K$  which we also denote by M. The function M is upper-semicontinuous ([2], Theorem F).

The Euclidean minimum of K is defined by  $M_1(K) = \sup_{P \in K} M(P)$ . In particular,  $M_1(K) < 1$  implies that  $\mathcal{O}_K$  is norm-Euclidean, while  $M_1(K) > 1$  implies that it is not. The second Euclidean minimum is defined by

$$M_2(K) = \sup_{\substack{P \in K \\ M(P) < M_1(K)}} M(P)$$

and  $M_1(K)$  is said to be *isolated* if  $M_2(K) < M_1(K)$ . We may proceed in this fashion producing Euclidean minima  $M_i(K)$  until we find a non-isolated one. Note that upper-semicontinuity ensures that each of these suprema is actually achieved by some collection of points on the torus. The points P in the above suprema are constrained to K, but we may remove that restriction and define the *inhomogeneous minimum* of K

<sup>\*</sup>nramsey@depaul.edu

by  $M_1(\overline{K}) = \sup_{P \in \mathbb{T}_K} M(P)$  and proceed as above to define the inhomogeneous minima  $M_i(\overline{K})$ . The inhomogeneous spectrum of K is simply the image  $M(\mathbb{T}_K)$ , and the Euclidean spectrum of K is its subset M(K).

The inhomogeneous minima demonstrate a variety of behavior, in some cases producing an infinite sequence of isolated minima while in others we find that  $M_2(\overline{K})$  already fails to be isolated - see [11] for an overview of results. Barnes and Swinnerton-Dyer proved in [3] that  $M_1(K) = M_1(\overline{K})$  and conjectured that  $M_1(\overline{K})$  is always isolated and rational, and that  $M_2(\overline{K})$  is taken at a point with coordinates in K. Numerous computations by other authors (e.g. [6], [7], [10], [8], [9], [15]) suggest further that all inhomogeneous minima lie in K.

Much is known about these minima and spectra in higher degree when the unit group furnishes more automorphisms. Cerri showed in [5] that if K is has unit rank at least two, then  $M_1(K)$  is taken at rational point, and hence rational. He showed further that if such a K is not CM, then  $M_1(K)$  is attained and isolated, and the Euclidean and inhomogeneous spectra coincide and consist of a sequence of rational numbers converging to 0. Building on Cerri's work, Shapira and Wang proved in [13] that if K has unit rank at least three then  $M_1(K)$  is isolated and attained.

Returning to real quadratic K, the following is our main result.

**Theorem 1.** The set of  $M(\mathbb{T}_K) \cap K$  is dense in the inhomogeneous spectrum  $M(\mathbb{T}_K)$ .

This theorem is proven by introducing the intermediate collection of points

$$K/\mathfrak{O}_K \subseteq \widetilde{K}/\mathfrak{O}_K \subseteq \mathbb{T}_K$$

whose coordinates belong to K, establishing that the minima of such points lie in K, and finally proving that the associated spectrum  $M(\widetilde{K})$  is dense in the inhomogeneous spectrum. In the isolated case, this establishes the following extension of the conjecture of Barnes and Swinnerton-Dyer above.

Corollary 1. If  $M_i(\overline{K})$  is isolated, then it is taken at point with coefficients in K and we have  $M_i(\overline{K}) \in K$ .

The method of proof also demonstrates that  $M_1(\overline{K}) = M_1(K)$  is rational if it is isolated, but this was known already to Barnes and Swinnerton-Dyer ([3], Theorem M).

## 2 The dynamical systems $X_t$

By Dirichlet's unit theorem, we have  $\mathcal{O}_K^{\times} = \pm \varepsilon^{\mathbb{Z}}$  for some fundamental unit  $\varepsilon$  of infinite order. We will later fix an embedding of K into  $\mathbb{R}$  and assume that  $\varepsilon$  is chosen so that  $\varepsilon > 1$ . Multiplication by  $\varepsilon$  is absolute norm-preserving and extends by linearity to an endomorphism  $\phi$  of  $V_K$  that is also absolute norm-preserving. Since  $\phi$  preserves the lattice  $\mathcal{O}_K$ , it descends to an endomorphism of the torus  $\mathbb{T}_K$  with the property that  $M(\phi(P)) = M(P)$  for all  $P \in \mathbb{T}_K$ . The eigenvalues of  $\phi$  are the embeddings of  $\varepsilon$  into  $\mathbb{R}$  and hence not roots of unity, so  $\phi$  is an ergodic transformation of  $\mathbb{T}_K$ . This dynamical system, and a symbolic coding of it obtained from a Markov partition of the torus, is our main resource. Note that the subset  $K/\mathcal{O}_K$ , which coincides with the set of periodic points for  $\phi$ , is traditionally referred to as the rational points since they have rational (x,y) coordinates (see Section 3).

For t > 0, the  $\phi$ -invariant set  $X_t = \{P \in \mathbb{T}_K \mid M(P) \geq t\}$  is closed by upper semicontinuity. We can describe  $X_t$  alternatively by first noting that the open set

$$\mathfrak{U}(t) = \bigcup_{Q \in \mathcal{O}_K} \{ P \in V_K \mid \mathbf{N}(P - Q) < t \}$$

is translation-invariant and descends to an open subset of  $\mathbb{T}_K$ , and then observing that  $X_t$  is its complement. The sets  $X_t$  have Lebesgue measure zero for t > 0 since they are proper, closed, and  $\phi$ -invariant. It is natural to ask how the Hausdorff dimension  $\dim(X_t)$  varies with t. That  $\dim(X_t) \to 2$  as  $t \to 0$  is a simple consequence of Theorem 2.3 of [4]. We prove in Corollary 3 that  $\dim(X_t)$  is left-continuous everywhere. Right-continuity remains an open question.

We illustrate in the case  $K = \mathbb{Q}(\sqrt{5})$ . Davenport computed the Euclidean minima for this field in [6] and [7], finding the infinite decreasing sequence of minima  $M_1 = 1/4$ , and for  $i \ge 1$ ,

$$M_{i+1} = \frac{F_{6i-2} + F_{6i-4}}{4(F_{6i-1} + F_{6i-3} - 2)}$$

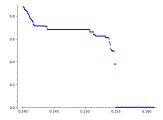
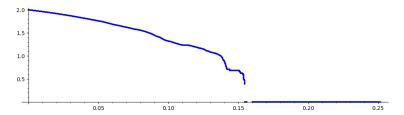


Fig. 1: Detail near t = .15

where  $F_k$  denotes the kth Fibonacci number. Each of these minima is obtained at a finite collection of elements of  $K/\mathcal{O}_K$ , and we have  $M_i \longrightarrow t_\infty = (-1+\sqrt{5})/8 \approx .1545$ . A plot of  $\dim(X_t)$  in this case is given below. The zero-dimensional region necessarily covers  $t > t_\infty$ , since the collection of points giving rise to the Euclidean minima is countable. We prove in [12] that  $\dim(X_t) > 0$  for all  $t < t_\infty$ , while  $\dim(X_{t_\infty}) = 0$ . In particular,  $\dim(X_t)$  is continuous at  $t_\infty$ .



The evident plateaus on this graph and its detail in Figure 1 have dynamical significance. The dimensions plotted here are actually upper bounds obtained by symbolically coding the torus dynamical system with a Markov partition and finding subshifts of finite type (SFTs) that contain the coding of  $X_t$ , as in Section 5. As we will make precise in Proposition 3, a plateau will occur wherever it is possible to make such a bound tight and  $X_t$  can be described directly by an SFT. The longest such plateau in the positive-dimensional region occurs around t = .15 (see Figure 1 for a detail), and we determine its endpoints and give an explicit symbolic coding of  $X_t$  on this plateau in [12].

### **3** Coordinates and *K*-points

Let us now take K to be a subset of  $\mathbb{R}$  by fixing an embedding, and take  $\varepsilon$  to be a fundmental unit with  $\varepsilon > 1$ . Recall that  $\{1, \alpha_K\}$  is a  $\mathbb{Z}$ -basis of  $\mathcal{O}_K$ , where

$$\alpha_K = \begin{cases} \sqrt{D} & D \equiv 2, 3 \pmod{4} \\ \frac{1+\sqrt{D}}{2} & D \equiv 1 \pmod{4} \end{cases}$$

Coordinates with respect to this basis will be denoted (x, y). The choice of embedding gives an isomorphism

$$V_K = K \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}$$
$$a \otimes 1 \longmapsto (\overline{a}, a)$$

of  $\mathbb{R}$ -algebras, and thus another coordinate system. Multiplication by  $\varepsilon$  has the effect of multiplying by  $\overline{\varepsilon} = \pm \varepsilon^{-1}$  in the first coordinate and  $\varepsilon$  in the second coordinate. Accordingly, these are known as the *stable* and *unstable* coordinates and denoted (s, u). Note that the absolute norm is simply  $\mathbf{N}(s, u) = |su|$  in these coordinates, and that the coordinate transformations between (x, y) and (s, u) coordinates are K-linear.

A point  $P \in \mathbb{T}_K$  is called determinate if it has a representative  $Q \in V_K$  with  $\mathbf{N}(Q) = M(P)$ . It is shown in [4] (Theorem 2.6) that the set of determinate points is a meagre  $F_{\sigma}$  set of measure zero and Hausdorff dimension 2. For a general point  $P \in \mathbb{T}_K$ , the following two lemmas help relate the value M(P) to the more concrete values  $\mathbf{N}(Q)$  for  $Q \in V_K$ .

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**Lemma 1** ([4], Lemma 4.2). Suppose that  $P \in \mathbb{T}_K$  satisfies M(P) < t. There exists a point  $Q = (s, u) \in V_K$  representing an element of the orbit of P satisfying

$$|s|, |u| < \sqrt{\varepsilon t}$$

such that  $\mathbf{N}(Q) = |su| < t$ .

**Lemma 2.** Let  $P \in \mathbb{T}_K$ . There exists  $Q \in V_K$  representing an element of the orbit closure of P satisfying

$$\mathbf{N}(Q) = M(Q) = M(P)$$

*Proof.* Let  $R_{\text{big}}$  denote the rectangle in  $V_K$  given by  $|s|, |u| < \sqrt{\varepsilon(M_1(K) + 1)}$ . By Lemma 1, there is for each  $n \in \mathbb{N}$  a point  $Q_n \in R_{\text{big}}$  representing an element of orbit of P with

$$\mathbf{N}(Q_n) < M(P) + \frac{1}{n} \tag{1}$$

Since  $R_{\text{big}}$  is bounded, there exists a subsequence  $Q_{k_n}$  converging to some point Q. Observe that

$$M(Q) \le \mathbf{N}(Q) = \lim_{k \to \infty} \mathbf{N}(Q_{k_n}) \le M(P)$$

where the last inequality follows from (1). The definition of Q ensures that it represents an element of the orbit closure of P. But this implies that  $M(Q) \geq M(P)$  by upper-semicontinuity, since the value M(P) is common to the entire orbit of P, and the result follows.

Let us call a point in  $V_K$  with rational (x,y) coordinates a  $\mathbb{Q}$ -point. Similarly a K-point is one whose (x,y) coordinates lie in K, or equivalently whose (s,u) coordinates lie in K. The set of  $\mathbb{Q}$ -points coincides with  $K/\mathbb{O}_K$ , which is also the set of periodic points for  $\phi$ . In particular, if P is a  $\mathbb{Q}$ -point then the previous lemma immediately implies that P is determinate and  $M(P) \in \mathbb{Q}$ .

**Proposition 1.** Let  $P \in \mathbb{T}_K$  be a K-point.

1. There exists  $N \in \mathbb{N}$  such that

$$\phi^k(NP) \longrightarrow 0$$
 as  $|k| \to \infty$ 

 $2. M(P) \in K$ 

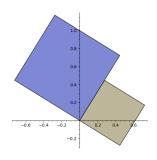
Proof.

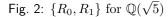
- 1. Since P has (s, u) coordinates in K, there exists  $N \in \mathbb{N}$  such that NP has (s, u) coordinates in  $\mathcal{O}_K$ . In (s, u) coordinates, the lattice  $\mathcal{O}_K \subseteq V_K$  is given by the set of pairs  $(\overline{a}, a)$  for  $a \in \mathcal{O}_K$ . It follows by subtracting such elements that NP has a representative whose stable coordinate vanishes, as well as a representative whose unstable coordinate vanishes. Now  $\phi^k(NP) \to 0$  as  $|k| \to \infty$  follows immediately.
- 2. By the previous part, the orbit closure of the K-point P consists of the orbit of P together with a finite collection of N-torsion points on the torus. By Lemma 2, there exists  $Q \in V_K$  representing an element of this orbit closure with  $\mathbf{N}(Q) = M(P)$ . Should Q represent an N-torsion point, then  $M(P) \in \mathbb{Q}$  since torsion points are  $\mathbb{Q}$ -points. On the other hand, if Q = (s, u) represents an element of the orbit of P then Q is also a K-point, so we have  $M(P) = \mathbf{N}(Q) = |su| \in K$ .

#### 4 Markov partitions

For each K, the dynamical system  $(\mathbb{T}_K, \phi)$  admits a Markov partition consisting of two open rectangles. Such a partition  $\{R_0, R_1\}$  for  $K = \mathbb{Q}(\sqrt{5})$  is pictured in Figure 2 in (x, y) coordinates. Figure 3 furnishes a uniform description in (s, u) coordinates of a two-rectangle Markov partition for any K. This description is simply the one provided by Adler in [1] translated into (s, u) coordinates. See also [14], where the construction may originate.

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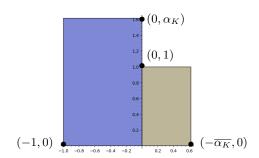


Fig. 3: Original partition in (s, u) coordinates (scale shown for  $\mathbb{Q}(\sqrt{5})$ )

These two-rectangle partitions are typically not generators essentially because the intersections  $R \cap \phi(S)$  are generally disconnected. In the case of  $\mathbb{Q}(\sqrt{5})$  however, the original partition  $\mathcal{P}_0 = \{R_0, R_1\}$  is a generator. Moreover,  $R_0 \cap \phi(R_0) = \emptyset$ , while the remaining intersections consist of a single nonempty rectangle each. Let  $\Sigma$  denote the subset of  $\{0,1\}^{\mathbb{Z}}$  that avoids the string 00 and let  $\sigma: \Sigma \to \Sigma$  be the shift operator  $\sigma(s)_i = s_{i+1}$ . The Markov generator property furnishes a map

$$\pi:\Sigma\longrightarrow\mathbb{T}_K$$

intertwining  $\phi$  and the shift operator on  $\Sigma$  that sends each string of coordinates to the unique point in  $\mathbb{T}_K$  whose orbit has these coordinates:

$$\pi(s) = \bigcap_{n \in \mathbb{N}} \bigcap_{i=-n}^{n} \phi^{-i}(s_i) = \bigcap_{i \in \mathbb{Z}} \phi^{-i}(\overline{s_i})$$

**Remark 1.** This construction ensures that  $\phi^k \pi(s) \in \overline{s(k)}$  for all k. It follows that if the coordinate word of  $A \in \mathcal{P}_n$  occurs in  $s \in \Sigma$ , then  $\phi^k \pi(s) \in \overline{A}$  for a suitable  $k \in \mathbb{Z}$ .

The map  $\pi$  is continuous, surjective, bounded-to-one, and essentially one-to-one. Moreover, if  $X \subseteq \mathbb{T}_K$  is a closed, invariant subset then  $\pi$  restricts to a map

$$\pi^{-1}(X) \longrightarrow X$$

with the same properties, from which it follows that the entropy of  $\phi|_X$  coincides with the entropy of the shift restricted to  $\pi^{-1}(X)$ . In the case of  $X = X_t$ , this entropy can be approximated by approximating the set  $\pi^{-1}(X_t)$  by subshifts obtained by refining the partition  $\mathcal{P}_0$  and omitting some rectangles. The refinements are defined by taking  $\mathcal{P}_n$  to consist of all nonempty intersections of the form

$$\phi^n(A_{-n}) \cap \dots \cap \phi(A_{-1}) \cap A_0 \cap \phi^{-1}(A_1) \cap \dots \cap \phi^{-n}(A_n), \quad A_i \in \mathcal{P}_0, \tag{2}$$

and we say that this particular rectangle has *coordinate word*  $A_{-n} \cdots A_0 \cdots A_n$ . When a representative rectangle in the plane  $V_K$  is needed for a member of  $\mathcal{P}_n$ , we take the one contained in the original footprint  $R_0 \cup R_1$ .

The refinement  $\mathcal{P}_n$  is also a Markov generator, and we have a refined coding  $\pi_n: \Sigma_n \longrightarrow \mathbb{T}_K$  by the set of admissible strings in the alphabet  $\mathcal{P}_n$ . Note that  $\Sigma_n$  is simply a "block form" of  $\Sigma$  and there is a canonical bijection  $\Sigma \cong \Sigma_n$  compatible with the shift operator and the two codings of  $\mathbb{T}_K$ . While for general K, the partition  $\{R_0, R_1\}$  is not a generator, in all cases the connected components of  $A_0 \cap \phi^{-1}(A_1)$  for  $A_i \in \{R_0, R_1\}$  do comprise a Markov generator (see the proof of Theorem 8.4 of [1]). Thus for any K other than  $\mathbb{Q}(\sqrt{5})$  we may let  $\mathcal{P}_0$  denote this generator and then proceed as in the previous paragraph to produce refinements  $\mathcal{P}_n$ . In all cases, the diameter of  $\mathcal{P}_n$  tends to zero as  $n \to \infty$ .

The following explicit construction of  $\pi$  will be useful below. Here,  $\mathcal{P}$  can be any Markov generator on  $\mathbb{T}_K$  arising from a collection of rectangles in the plane  $V_K$  with sides parallel to the stable and unstable axes. In particular we suppose we have a chosen representative in the plane for each member of  $\mathcal{P}$ , or equivalently

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a choice of stable and unstable interval of which this member is the product. Let  $s \in \Sigma$ , the set of all admissible bi-infinite strings in the alphabet  $\mathcal{P}$ . First we show how to compute the unstable coordinate of  $\pi(s)$ . The intersections

$$r_0 = s_0$$

$$r_1 = s_0 \cap \phi^{-1}(s_1)$$

$$r_2 = s_0 \cap \phi^{-1}(s_1) \cap \phi^{-2}(s_2)$$
:

on the torus can be viewed in the plane as a sequence of rectangles within  $s_0$  whose stable interval is constant (and equal to that of  $s_0$ ) and whose unstable interval is shrinking. Up to similarity, the footprint of the unstable interval of  $r_{i+1}$  inside that of  $r_i$  depends only on the rectangles  $s_i$  and  $s_{i+1}$  and is independent of i. This is because  $\phi$  simply scales by the positive number  $\varepsilon$  in the unstable direction, preserving similarity.

Given a rectangle in the plane with sides parallel to the stable and unstable axes, let us denote its stable and unstable intervals by  $[\alpha_s(A), \beta_s(A)]$  and  $[\alpha_u(A), \beta_u(A)]$ , and let  $\ell_*(A) = \beta_*(A) - \alpha_*(A)$  denote the corresponding lengths. For each pair  $A, B \in \mathcal{P}$  with AB admissible, we define

$$\rho_u(A, B) = \frac{\alpha_u(A \cap \phi^{-1}(B)) - \alpha_u(A)}{\ell_u(A)}$$

Pictured in the (s, u) plane, this is the height of the bottom of the subrectangle  $A \cap \phi^{-1}(B)$  inside A, expressed as a fraction of the total height of A, and is a measure of the footprint of this subrectangle in A alluded to above. The left endpoint of the unstable interval of  $r_i$  is then equal to

$$\alpha_u(s_0) + \rho_u(s_0, s_1)\ell_u(s_0) + \rho_u(s_1, s_2)\frac{\ell_u(s_1)}{\varepsilon} + \dots + \rho_u(s_{i-1}, s_i)\frac{\ell_u(s_{i-1})}{\varepsilon^{i-1}},$$

so the unstable coordinate of  $\pi(s)$  is given by the series

$$\alpha_u(s_0) + \rho_u(s_0, s_1)\ell_u(s_0) + \rho_u(s_1, s_2)\frac{\ell_u(s_1)}{\varepsilon} + \rho_u(s_2, s_3)\frac{\ell_u(s_2)}{\varepsilon^2} + \cdots$$
 (3)

The stable coordinate works the same way if  $\bar{\varepsilon} > 0$ . Some additional care must be taken if  $\bar{\varepsilon} < 0$ , since then  $\phi$  is orientation-reversing in the stable direction and the footprints alternate with their mirror images up to similarity instead of being independent of i. In that case we define coefficients

$$\rho_s^+(A,B) = \frac{\alpha_s(A \cap \phi(B)) - \alpha_s(A)}{\ell_s(A)}$$

and

$$\rho_s^-(A,B) = \frac{\beta_s(A) - \beta_s(A \cap \phi(B))}{\ell_s(A)},$$

and the stable coordinate alternates between these:

$$\alpha_s(s_0) + \rho_s^+(s_0, s_{-1})\ell_s(s_0) + \rho_s^-(s_{-1}, s_{-2})\frac{\ell_s(s_{-1})}{\varepsilon} + \rho_s^+(s_{-2}, s_{-3})\frac{\ell_s(s_{-2})}{\varepsilon^2} + \cdots$$
 (4)

Let us now return to the partitions  $\mathcal{P}_n$  derived from the two-rectangle partition above. If  $s \in \Sigma$  is periodic, then the image  $\pi(s) \in \mathbb{T}_K$  has periodic orbit, and hence is a  $\mathbb{Q}$ -point. The following lemma furnishes a similar description of some K-pts.

**Lemma 3.** Suppose that s is eventually periodic in both directions. Then  $\pi(s)$  is a K-point.

*Proof.* First observe that all members of our partitions  $\mathcal{P}_n$  have coordinates in the field K. If s is eventually periodic in both directions, then the series (3) and (4) (and its analog in case  $\bar{\varepsilon} > 0$ ) decompose into finitely many geometric series with all terms and coefficients expressible in terms of these coordinates, and the result follows.

**Lemma 4.** If t' < t and  $X_t \subsetneq X_{t'}$ , then there exists a finite word occurring in  $\pi^{-1}(X_{t'})$  that does not occur in  $\pi^{-1}(X_t)$ .

Proof. Suppose to the contrary that every word appearing in  $\pi^{-1}(X_{t'})$  also occurs in  $\pi^{-1}(X_t)$ . We claim this forces  $\pi^{-1}(X_{t'})$  to be contained in the closure of  $\pi^{-1}(X_t)$ , which is a contradiction since the latter is closed assumed distinct from the former. Let  $s \in \pi^{-1}(X_{t'})$ , and for  $k \in \mathbb{N}$  let  $w_k$  be the word  $s(-k) \cdots s(0) \cdots s(k)$ . By hypothesis, this word occurs in  $\pi^{-1}(X_t)$ , and by applying  $\phi$  we may assume that it occurs centrally in some element  $x_k \in \pi^{-1}(X_t)$ . In particular,  $x_k$  and s agree on the index interval [-k, k], and it follows that  $x_k \to s$  as  $k \to \infty$ , so s lies in the closure of  $\pi^{-1}(X_t)$ .

# 5 Upper bounds via trapping rectangles

Given a collection of rectangles  $\mathfrak{C} \subseteq \bigcup_n \mathfrak{P}_n$ , we denote by  $\Sigma \langle \mathfrak{C} \rangle$  the subshift of  $\Sigma$  that avoids the coordinate words of elements of  $\mathfrak{C}$ . If  $\mathfrak{C}$  is finite, then there is a largest n for which  $\mathfrak{P}_n$  contains an element of  $\mathfrak{C}$ . Now every element of  $\mathfrak{C}$  breaks up into rectangles in  $\mathfrak{P}_n$ , and we let  $\mathfrak{C}' \subseteq \mathfrak{P}_n$  denote the collection of rectangles occurring in this fashion. Under the identification  $\Sigma \cong \Sigma_n$ , the subshift  $\Sigma \langle \mathfrak{C} \rangle$  can alternately be described as the collection of  $s \in \Sigma_n$  for which  $s(k) \notin \mathfrak{C}'$  for all  $k \in \mathbb{Z}$ .

Let  $I \subseteq \mathcal{O}_K$  be a finite set of lattice points and let

$$U(t,I) = \bigcup_{Q \in I} \{ P \in V_K \mid N(P - Q) < t \}$$

and let

$$\mathfrak{I}_n(t,I) = \{ A \in \mathfrak{P}_n \mid \overline{A} \subseteq \mathfrak{U}(t,I) \}$$

be the collection of rectangles in  $\mathcal{P}_n$  whose closures are trapped within the norm-distance t "neighborhood" of some lattice point in I. The following lemma says that  $\Sigma \langle \mathfrak{T}_n(t,I) \rangle$  is an upper bound not only for  $X_t$  but for  $X_{t-\eta}$  for some  $\eta > 0$ .

**Lemma 5.** There exists  $\eta > 0$  such that  $\pi^{-1}(X_{t-\eta}) \subseteq \Sigma \langle \mathfrak{T}_n(t,I) \rangle$ .

Proof. The elements of  $\mathfrak{T}_n(t,I)$  have closures contained in the  $\mathfrak{U}(t,I)$  and thus in  $\mathfrak{U}(t-\eta,I)$  for some  $\eta>0$  since I is finite. If  $s\in\Sigma$  contains the coordinates of  $A\in\mathfrak{T}_n(t,I)$ , then  $\phi^k\pi(s)$  lies in  $\overline{A}$  for some k, by Remark 1. But then  $M(\pi(s))=M(\phi^k\pi(s))< t-\eta$ . Thus  $s\notin\pi_n^{-1}(X_{t-\eta})$ .

The entropy of  $\phi$  on  $X_t$  is thus bounded above by the shift entropy of  $\Sigma\langle \mathfrak{T}_n(t,I)\rangle$ , which is computable by Perron-Frobenius theory. These upper bounds depend on the set  $I\subseteq \mathfrak{O}_K$  and improve as I grows. The following proposition and its corollary ensure that it is possible to choose I so that the bounds are tight in the limit as  $n\to\infty$ .

**Proposition 2.** There exists a finite set  $I_K$  such that if  $I_K \subseteq I$  and t' < t, then for n sufficiently large we have

$$\pi^{-1}(X_t) \subseteq \Sigma \langle \mathfrak{T}_n(t,I) \rangle \subseteq \pi^{-1}(X_{t'})$$
 (5)

In particular, for such I we have

$$\pi^{-1}(X_t) = \bigcap_{n>0} \Sigma \langle \mathfrak{I}_n(t,I) \rangle$$

Proof. The second assertion here follows immediately from the first. Let  $R_{\text{big}}$  denote the rectangle in  $V_K$  given by  $|s|, |u| < \sqrt{\varepsilon(M_1(K)+1)}$  and let  $I_K$  be the set of all  $q \in \mathcal{O}_K$  such that R-q meets  $R_{\text{big}}$  for some  $R \in \mathcal{P}_0$ . The set  $I_K$  is finite and necessarily contains any q for which there exists some  $A \in \mathcal{P}_n$  such that A-q meets  $R_{\text{big}}$ . Since the diameter of  $\mathcal{P}_n$  tends to zero, there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies that every translate of A that meets the region defined by  $\mathbf{N} < t'$  in  $R_{\text{big}}$  must have closure entirely contained within the region  $\mathbf{N} < t$ .

The first containment in (5) is clear from the preceding lemma, and we prove the second by contrapositive. Suppose that  $s \in \Sigma$  is not in  $\pi^{-1}(X_{t'})$ . Then with  $P = \pi(s)$  we have M(P) < t', so

$$M(P) < t'' = \min(t', M_1(K) + 1)$$

Thus we may take Q = (s, u) as in Lemma 1 representing an element of the orbit of P with  $\mathbf{N}(Q) < t''$  and

$$|s|, |u| < \sqrt{\varepsilon t''}$$

In particular,  $Q \in R_{\text{big}}$ . For each n, the point Q lies in the  $\mathcal{O}_K$ -translates of the the closures of one or more members of the partition  $\mathcal{P}_n$ . Let  $A \in \mathcal{P}_n$  and  $q \in \mathcal{O}_K$  such that  $Q \in \overline{A} - q$ . Thus  $\overline{A} - q$  meets the region defined by  $\mathbf{N} < t'$  in  $R_{\text{big}}$ , which requires that A - q meet this region since A is open, and hence  $q \in I_K \subseteq I$ . Now if  $n \geq N$ , it follows that  $A \in \mathcal{T}_n(t, I)$ .

By Remark 1, the 0th symbolic coordinate of any element of  $\pi_n^{-1}(Q)$  must be a member of  $\mathfrak{T}_n(t,I)$ , which implies that each element of  $\pi_n^{-1}(P)$  has some symbolic coordinate in  $\mathfrak{T}_n(t,I)$ . This is to say that each element of  $\pi^{-1}(P)$ , including s, contains the coordinates of some element of  $\mathfrak{T}_n(t,I)$ , and thus  $s \notin \Sigma \langle \mathfrak{T}_n(t,I) \rangle$ .  $\square$ 

Corollary 2. If  $I_K \subseteq I$ , then

$$h(\phi|X_t) = \lim_{n \to \infty} h(\sigma|\Sigma\langle \mathfrak{T}_n(t,I)\rangle$$

*Proof.* Let  $\mu_n$  be a measure of maximal entropy for  $\Sigma \langle \mathfrak{T}_n(t,I) \rangle$ . Extended to  $\Sigma$ , this sequence of measures has some weak-\* limit point  $\mu$  in the convex, compact space of invariant probability measures on  $\Sigma$ . The measure  $\mu$  is supported on the intersection  $\pi^{-1}(X_t)$ , and by upper semi-continuity of entropy in subshifts we have

$$h(\sigma|\pi^{-1}(X_t)) \ge h_{\mu}(\sigma) \ge \limsup h_{\mu_n}(\sigma)$$

$$= \limsup h(\sigma|\Sigma\langle \mathfrak{I}_n(t,I)\rangle \ge \liminf h(\sigma|\Sigma\langle \mathfrak{I}_n(t,I)\rangle \ge h(\sigma|\pi^{-1}(X_t))$$

This implies that  $\mu$  is a measure of maximal entropy for  $\pi^{-1}(X_t)$ , as well as the claim.

Corollary 3. The function  $t \mapsto \dim(X_t)$  is left-continuous at each point.

*Proof.* The dimension of a closed, invariant subset  $X \subseteq \mathbb{T}_K$  is related to the entropy of  $\phi$  on X via

$$\dim(X) = \frac{2h(\phi|X)}{\log(\varepsilon)},$$

so it suffices to prove that  $t \mapsto h(\phi|X_t)$  is left continuous. Since this function is decreasing, left-discontinuity at t would imply there exists B > 0 such that

$$h(\phi|X_{t-\eta}) - h(\phi|X_t) \ge B$$
 for all  $\eta > 0$ 

By the previous corollary we know there exists  $n \in \mathbb{N}$  with

$$h(\sigma|\Sigma\langle \mathfrak{I}_n(t,I_K)\rangle) - h(\phi|X_t) < B$$

Now Lemma 5 ensures that  $\Sigma \langle \mathfrak{I}_n(t, I_K) \rangle$  contains  $\pi^{-1}(X_{t-\eta})$  for some  $\eta > 0$ , which implies

$$h(\sigma|\Sigma\langle \mathfrak{T}_n(t,I_K)\rangle) \ge h(\sigma|\pi^{-1}(X_{t-\eta})) = h(\phi|X_{t-\eta}),$$

contradicting the inequalities above.

# 6 Applications to the inhomogeneous spectrum

The plot of dim $(X_t)$  contains a number of plateaus as illustrated in the case  $K = \mathbb{Q}(\sqrt{5})$  above. Sometimes these are actually set-theoretic plateaus, and the following proposition demonstrates that  $\pi^{-1}(X_t)$  is particularly simple in such cases.

**Proposition 3.** Suppose that  $X_t = X_{t-\eta}$  for some  $\eta > 0$ . Then  $\pi^{-1}(X_t)$  is a subshift of finite type.

*Proof.* By Proposition 2, we may choose  $n \in \mathbb{N}$  so that

$$\pi^{-1}(X_t) \subseteq \Sigma \langle \mathfrak{T}_n(t,I) \rangle \subseteq \pi^{-1}(X_{t-n}) = \pi^{-1}(X_t)$$

Thus  $\pi^{-1}(X_t) = \Sigma \langle \mathfrak{T}_n(t,I) \rangle$ , which is expressible directly as an SFT via a 0-1 matrix when viewed in block form in  $\Sigma_m$  for some m (namely, any  $m \geq n-1$ ).

Finally, we prove the main density result.

Proof of Theorem 1. First suppose that  $t \in M(\mathbb{T}_K)$  is an isolated point. By the previous proposition,  $\pi^{-1}(X_t)$  is a subshift of finite type, which is to say that it can be described by a 0-1 transition matrix when viewed in block form  $\Sigma_m$  for some m. Since t is isolated, we know by Lemma 4 that  $\pi^{-1}(X_t)$  contains a finite word w that does not occur in  $\pi^{-1}(X_{>t})$ . Let  $s = uwv \in \Sigma$  with  $M(\pi(s)) = t$ . Viewed in  $\Sigma_m$ , there is by the Pigeanhole Principle a repeated block in both u and v. We can then truncate u and v and loop the segment between these books indefinitely to produce an element  $s' \in \pi^{-1}(X_t)$  that contains w and is eventually periodic in both directions. Then  $\pi(s')$  is a K-point by Lemma 3, and  $M(\pi(s')) = t$  since s' contains w.

Now suppose that  $t \in M(\mathbb{T}_K)$  is not isolated, so there is a strictly monotone sequence  $(t_k)$  in  $M(\mathbb{T}_K)$  with  $t_k \to t$ . Fixing  $k \in \mathbb{N}$ , we will show that there is a K-point P with such that M(P) lies between t and  $t_k$ , which will finish the density claim. First suppose that  $(t_k)$  increases to t. Since  $t_{k+1} \in M(\mathbb{T}_K)$ , Lemma 4 ensures that there exists  $s \in \pi^{-1}(X_{t_{k+1}})$  containing a word w that does not occur in  $\pi^{-1}(X_t)$ . Now take n large enough so that

$$\pi^{-1}(X_{t_{k+1}}) \subseteq \Sigma \langle \mathfrak{I}_n(t_{k+1}, I) \rangle \subseteq \pi^{-1}(X_{t_k})$$

as in Proposition 2. Since s belongs to the SFT  $\Sigma \langle \mathfrak{T}_n(t_{k+1},I) \rangle$ , we can modify it by looping its ends as in the previous paragraph to obtain another element s' of this SFT that also contains w. But then we have  $t_k \leq M(\pi(s')) < t$ , so  $P = \pi(s')$  is the desired K-point.

Now suppose that  $(t_k)$  is decreasing. Since  $t_{k+1} \in M(\mathbb{T}_K)$ , Lemma 4 ensures there is word w occurring in  $\pi^{-1}(X_{t_{k+1}})$  that does not occur in  $\pi^{-1}(X_{t_k})$ . Now take n large enough so that

$$\pi^{-1}(X_{t_{k+1}}) \subseteq \Sigma \langle \mathfrak{T}_n(t_{k+1}, I) \rangle \subseteq \pi^{-1}(X_t)$$

and proceed as before to produce  $s' \in \Sigma \langle \mathfrak{T}_n(t_{k+1}, I) \rangle$  that contains w and is eventually periodic in both directions. We have  $t \leq M(\pi(s')) < t_k$ , and again  $P = \pi(s')$  is the desired K-point.

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