Euclidean Geometry

The Elements

The Elements

The Elements of Euclid (from around 300 BC) were based on a number of *axioms* and *postulates*.

The Elements

The Elements of Euclid (from around 300 BC) were based on a number of *axioms* and *postulates*.

The axioms are considered to be evidently true.

The Elements

The Elements of Euclid (from around 300 BC) were based on a number of *axioms* and *postulates*.

The axioms are considered to be evidently true. Examples are

(1) Things that are equal to the same thing are equal.

The Elements

The Elements of Euclid (from around 300 BC) were based on a number of *axioms* and *postulates*.

The axioms are considered to be evidently true. Examples are

(1) Things that are equal to the same thing are equal.
or
(2) The whole is greater than the part.

The Elements

The Elements of Euclid (from around 300 BC) were based on a number of *axioms* and *postulates*.

The axioms are considered to be evidently true. Examples are

(1) Things that are equal to the same thing are equal.
or
(2) The whole is greater than the part.
or even the somewhat more obscure
(3) Things which coincide with each other are equal to each other.

The Elements

The Elements of Euclid (from around 300 BC) were based on a number of *axioms* and *postulates*.

The axioms are considered to be evidently true. Examples are

(1) Things that are equal to the same thing are equal.
or
(2) The whole is greater than the part.
or even the somewhat more obscure
(3) Things which coincide with each other are equal to each other.

We will focus on the postulates which with modern eyes should not be considered as evident, but rather as assumptions:

Postulates

(1) We can draw a line segment between any two points.

Postulates

(1) We can draw a line segment between any two points.

(2) Any line segment can be extended to a straight line.

Postulates

(1) We can draw a line segment between any two points.

(2) Any line segment can be extended to a straight line.

(3) Given a point and a line segment, there is a circle with the given point as center and the line segment as a radius.

Postulates

(1) We can draw a line segment between any two points.

(2) Any line segment can be extended to a straight line.

(3) Given a point and a line segment, there is a circle with the given point as center and the line segment as a radius.

(4) All right angles are equal to each other.

Postulates

(1) We can draw a line segment between any two points.

(2) Any line segment can be extended to a straight line.

(3) Given a point and a line segment, there is a circle with the given point as center and the line segment as a radius.

(4) All right angles are equal to each other.

(5) Through any point outside a given line there is exactly one line which does not intersect the given line. (PA)

Given the axioms and postulates, all the other claims should be obtained as logical consequences.

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this.

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this. Example:

The sum of angles in any triangle is equal to $180^\circ = \pi$.

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this. Example:

The sum of angles in any triangle is equal to $180^\circ = \pi$. We will give two proofs of this.

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this. Example:

The sum of angles in any triangle is equal to $180^\circ = \pi$. We will give two proofs of this.

Other highlights are *Pythagora's theorem*, *The inscribed angles theorem*, and various constructions with ruler and compass.

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this. Example:

The sum of angles in any triangle is equal to $180^\circ = \pi$. We will give two proofs of this.

Other highlights are *Pythagora's theorem*, *The inscribed angles theorem*, and various constructions with ruler and compass.
It also also contains theorems on the length of a circle – it is proportional to the radius – and even the volume of e g cones – they are proportional to the cube of the lengths.

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this. Example:

The sum of angles in any triangle is equal to $180^\circ = \pi$. We will give two proofs of this.

Other highlights are *Pythagora's theorem*, *The inscribed angles theorem*, and various constructions with ruler and compass.
It also also contains theorems on the length of a circle – it is proportional to the radius – and even the volume of e g cones – they are proportional to the cube of the lengths.

These theorems use the method of *exhaustion* by Eudoxus (390-327 BC), and are obviously of a different nature than the other.

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this. Example:

The sum of angles in any triangle is equal to $180^\circ = \pi$. We will give two proofs of this.

Other highlights are *Pythagora's theorem*, *The inscribed angles theorem*, and various constructions with ruler and compass.
It also also contains theorems on the length of a circle – it is proportional to the radius – and even the volume of e g cones – they are proportional to the cube of the lengths.

These theorems use the method of *exhaustion* by Eudoxus (390-327 BC), and are obviously of a different nature than the other. E g , what is the length of a circle?

Given the axioms and postulates, all the other claims should be obtained as logical consequences. It is however dubious if Euclid really achieved this. Example:

The sum of angles in any triangle is equal to $180^\circ = \pi$. We will give two proofs of this.

Other highlights are *Pythagora's theorem*, *The inscribed angles theorem*, and various constructions with ruler and compass.
It also also contains theorems on the length of a circle – it is proportional to the radius – and even the volume of e g cones – they are proportional to the cube of the lengths.

These theorems use the method of *exhaustion* by Eudoxus (390-327 BC), and are obviously of a different nature than the other. E g , what is the length of a circle? These results are precursors of calculus.

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes.

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:
Assume there are only finitely many primes, say $p_1, p_2, ...p_N$.

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:

Assume there are only finitely many primes, say $p_1, p_2, ...p_N$. Then any number is divisible by one of these numbers (why?).

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:
Assume there are only finitely many primes, say $p_1, p_2, ...p_N$. Then any number is divisible by one of these numbers (why?). Let $x = p_1 p_2 ... p_n + 1$. Then x is not divisible by any p_j. This contradiction proves the theorem.

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:
Assume there are only finitely many primes, say $p_1, p_2, ...p_N$. Then any number is divisible by one of these numbers (why?). Let $x = p_1 p_2 ... p_n + 1$. Then x is not divisible by any p_j. This contradiction proves the theorem.

The second proof is by Euler, roughly 2000 years later.

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:
Assume there are only finitely many primes, say $p_1, p_2, ...p_N$. Then any number is divisible by one of these numbers (why?). Let $x = p_1 p_2 ... p_n + 1$. Then x is not divisible by any p_j. This contradiction proves the theorem.

The second proof is by Euler, roughly 2000 years later. Consider the infinite product

$$\prod_p (1 - 1/p)^{-1}.$$

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:
Assume there are only finitely many primes, say $p_1, p_2, ...p_N$. Then any number is divisible by one of these numbers (why?). Let $x = p_1 p_2 ... p_n + 1$. Then x is not divisible by any p_j. This contradiction proves the theorem.

The second proof is by Euler, roughly 2000 years later. Consider the infinite product

$$\prod_p (1 - 1/p)^{-1}.$$

We have

$$(1 - 1/p)^{-1} = \sum_j 1/p^j.$$

Euclid's Elements also contain results from arithmetics, like the Euclidean algorithm and the infinitude of the number of primes. We will give two proofs of the latter fact:
Assume there are only finitely many primes, say $p_1, p_2, ...p_N$. Then any number is divisible by one of these numbers (why?). Let $x = p_1 p_2 ... p_n + 1$. Then x is not divisible by any p_j. This contradiction proves the theorem.

The second proof is by Euler, roughly 2000 years later. Consider the infinite product

$$\prod_p (1 - 1/p)^{-1}.$$

We have

$$(1 - 1/p)^{-1} = \sum_j 1/p^j.$$

Hence

$$\prod_p (1 - 1/p)^{-1} = \prod_p \sum_j 1/p^j = \sum_{n \in N} 1/n = \infty.$$

The second proof is more powerful.

The second proof is more powerful.

Euclid’s book was used as a standard textbook until around 1900, for more than 2000 years. But it was not the end.

The second proof is more powerful.

Euclid's book was used as a standard textbook until around 1900, for more than 2000 years. But it was not the end.

Greek geometry developed e g with the work of Archimedes (287-212 BC), who computed the value of π with high accuracy, found the area of the circular disk and the surface area of the sphere – among many other things.

The second proof is more powerful.

Euclid's book was used as a standard textbook until around 1900, for more than 2000 years. But it was not the end.

Greek geometry developed e g with the work of Archimedes (287-212 BC), who computed the value of π with high accuracy, found the area of the circular disk and the surface area of the sphere – among many other things.
Here is how he found the area of a sphere:

The second proof is more powerful.

Euclid's book was used as a standard textbook until around 1900, for more than 2000 years. But it was not the end.

Greek geometry developed e g with the work of Archimedes (287-212 BC), who computed the value of π with high accuracy, found the area of the circular disk and the surface area of the sphere – among many other things.
Here is how he found the area of a sphere:

Erathostenes (276-195 BC) computed the circumference of the earth.

The second proof is more powerful.

Euclid's book was used as a standard textbook until around 1900, for more than 2000 years. But it was not the end.

Greek geometry developed e g with the work of Archimedes (287-212 BC), who computed the value of π with high accuracy, found the area of the circular disk and the surface area of the sphere – among many other things.
Here is how he found the area of a sphere:

Erathostenes (276-195 BC) computed the circumference of the earth. When Columbus set out to sail to India it had become a matter of debate whether the earth was round.

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold?

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold? Does it follow from the other postulates?

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold? Does it follow from the other postulates? This question led to non Euclidean geometry.

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold? Does it follow from the other postulates? This question led to non Euclidean geometry.

Compare to the axioms (Peano's axioms) for the natural numbers.

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold? Does it follow from the other postulates? This question led to non Euclidean geometry.

Compare to the axioms (Peano's axioms) for the natural numbers.

Compare also to set theory: Does the axiom of choice hold? Does the continuum hypothesis hold? Can we prove it?

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold? Does it follow from the other postulates? This question led to non Euclidean geometry.

Compare to the axioms (Peano's axioms) for the natural numbers.

Compare also to set theory: Does the axiom of choice hold? Does the continuum hypothesis hold? Can we prove it? How does one prove that something can not be proved?

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold? Does it follow from the other postulates? This question led to non Euclidean geometry.

Compare to the axioms (Peano's axioms) for the natural numbers.

Compare also to set theory: Does the axiom of choice hold? Does the continuum hypothesis hold? Can we prove it? How does one prove that something can not be proved?

At any rate: The Elements became a model for how a scientific theory should look: A set of assumptions (preferably small), and deductions and predictions from them.

The Elements raise a philosophical question: How can we deduce properties of the outside world by pure thinking?

Answer (?): The axioms are laws of logic. The postulates are hypotheses that may or may not be true in our world.

Case in point: Does (PA) hold? Does it follow from the other postulates? This question led to non Euclidean geometry.

Compare to the axioms (Peano's axioms) for the natural numbers.

Compare also to set theory: Does the axiom of choice hold? Does the continuum hypothesis hold? Can we prove it? How does one prove that something can not be proved?

At any rate: The Elements became a model for how a scientific theory should look: A set of assumptions (preferably small), and deductions and predictions from them. This is perhaps the most important role of the Elements.

Non euclidean Geometry

Non euclidean geometry was introduced around 1830 by J Bolyai and N I Lobachevsky (although Gauss apparently already knew about it).

Non euclidean geometry was introduced around 1830 by J Bolyai and N I Lobachevsky (although Gauss apparently already knew about it). The way to prove that something (the parallell axiom) can not be proved from the other postulates was to construct a *model* where the other postulates hold, but the parallell axiom does not hold.

Non euclidean geometry was introduced around 1830 by J Bolyai and N I Lobachevsky (although Gauss apparently already knew about it). The way to prove that something (the parallell axiom) can not be proved from the other postulates was to construct a *model* where the other postulates hold, but the parallell axiom does not hold.

Let us first consider how we can construct other models, i e other systems of geometry. Here is a simple (but non historical) way:

Non euclidean geometry was introduced around 1830 by J Bolyai and N I Lobachevsky (although Gauss apparently already knew about it). The way to prove that something (the parallell axiom) can not be proved from the other postulates was to construct a *model* where the other postulates hold, but the parallell axiom does not hold.

Let us first consider how we can construct other models, i e other systems of geometry. Here is a simple (but non historical) way: Geometry is based on the measurement of distance, which in turn leads to straigh lines as the shortest curves between two points. So let us change the notion of distance:

Non euclidean geometry was introduced around 1830 by J Bolyai and N I Lobachevsky (although Gauss apparently already knew about it). The way to prove that something (the parallell axiom) can not be proved from the other postulates was to construct a *model* where the other postulates hold, but the parallell axiom does not hold.

Let us first consider how we can construct other models, i e other systems of geometry. Here is a simple (but non historical) way: Geometry is based on the measurement of distance, which in turn leads to straigh lines as the shortest curves between two points. So let us change the notion of distance:

Divide the (x, y)-plane into the upper half plane ($y > 0$) and the lower half plane ($y < 0$). Let us say that the distance between two points in the upper half plane is the usual distance, whereas the distance between two points in the lower half plane is twice the usual one.

Non euclidean geometry was introduced around 1830 by J Bolyai and N I Lobachevsky (although Gauss apparently already knew about it). The way to prove that something (the parallell axiom) can not be proved from the other postulates was to construct a *model* where the other postulates hold, but the parallell axiom does not hold.

Let us first consider how we can construct other models, i e other systems of geometry. Here is a simple (but non historical) way: Geometry is based on the measurement of distance, which in turn leads to straigh lines as the shortest curves between two points. So let us change the notion of distance:

Divide the (x, y)-plane into the upper half plane ($y > 0$) and the lower half plane ($y < 0$). Let us say that the distance between two points in the upper half plane is the usual distance, whereas the distance between two points in the lower half plane is twice the usual one. What is the distance between a point in the upper half plane and a point in the lower half plane?

Say that P lies in the upper half plane and Q in the lower half plane. A curve γ between P and Q has to intersect the x-axis in some point M (say only one).

Say that P lies in the upper half plane and Q in the lower half plane. A curve γ between P and Q has to intersect the x-axis in some point M (say only one). If γ minimizes distance the part in the upper plane must be a straight line, and the part in the lower plane is also a straight line. So, γ is a broken line.

Say that P lies in the upper half plane and Q in the lower half plane. A curve γ between P and Q has to intersect the x-axis in some point M (say only one). If γ minimizes distance the part in the upper plane must be a straight line, and the part in the lower plane is also a straight line. So, γ is a broken line. A moments reflection shows that if the segment in the upper half plane is not perpendicular to the x-axis, then the direction of the segment in the lower half plane will form a smaller angle with the normal than the upper segment.

Say that P lies in the upper half plane and Q in the lower half plane. A curve γ between P and Q has to intersect the x-axis in some point M (say only one). If γ minimizes distance the part in the upper plane must be a straight line, and the part in the lower plane is also a straight line. So, γ is a broken line. A moments reflection shows that if the segment in the upper half plane is not perpendicular to the x-axis, then the direction of the segment in the lower half plane will form a smaller angle with the normal than the upper segment.

This is of course the standard model for the refraction of light between a denser and a lighter medium. (Exercise: Show that $\sin \alpha / \sin \beta = a$ if the distances in the lower half plane are a times the distances in the upper half plane.)

We can generalize this model. Recall that if $\gamma(t), t \in [a, b]$ is a (piecewise) smooth curve its length is

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt.$$

We can generalize this model. Recall that if $\gamma(t), t \in [a, b]$ is a (piecewise) smooth curve its length is

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt.$$

Now let $\rho(x, y) > 0$ be a function in the plane, and let

$$L_\rho(\gamma) = \int_a^b \rho(\gamma(t)) |\dot{\gamma}(t)| dt.$$

We can generalize this model. Recall that if $\gamma(t), t \in [a, b]$ is a (piecewise) smooth curve its length is

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt.$$

Now let $\rho(x, y) > 0$ be a function in the plane, and let

$$L_\rho(\gamma) = \int_a^b \rho(\gamma(t)) |\dot{\gamma}(t)| dt.$$

This is a new way to measure lengths. If $\rho = 1$ in the upper half plane and 2 (or a) in the lower half plane, it is the distance that gives refraction of light.

We can generalize this model. Recall that if $\gamma(t), t \in [a, b]$ is a (piecewise) smooth curve its length is

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt.$$

Now let $\rho(x, y) > 0$ be a function in the plane, and let

$$L_\rho(\gamma) = \int_a^b \rho(\gamma(t)) |\dot{\gamma}(t)| dt.$$

This is a new way to measure lengths. If $\rho = 1$ in the upper half plane and 2 (or a) in the lower half plane, it is the distance that gives refraction of light. A curve of shortest length is called a *geodesic*, and the geodesics are our new straight lines.

We can generalize this model. Recall that if $\gamma(t), t \in [a, b]$ is a (piecewise) smooth curve its length is

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt.$$

Now let $\rho(x, y) > 0$ be a function in the plane, and let

$$L_\rho(\gamma) = \int_a^b \rho(\gamma(t))|\dot{\gamma}(t)| dt.$$

This is a new way to measure lengths. If $\rho = 1$ in the upper half plane and 2 (or a) in the lower half plane, it is the distance that gives refraction of light. A curve of shortest length is called a *geodesic*, and the geodesics are our new straight lines. Here is an interesting choice: Poincarè's model of hyperbolic geometry.

Instead of looking at the whole (complex) plane, we look at a part of it, the unit disk ($|z| < 1$).

Instead of looking at the whole (complex) plane, we look at a part of it, the unit disk ($|z| < 1$). There we take

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

What are the geodesics?

Instead of looking at the whole (complex) plane, we look at a part of it, the unit disk ($|z| < 1$). There we take

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

What are the geodesics? It is fairly easy to see that if $-1 < P, Q < 1$, then the shortest curve between P and Q is the line segment beween them.

Instead of looking at the whole (complex) plane, we look at a part of it, the unit disk ($|z| < 1$). There we take

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

What are the geodesics? It is fairly easy to see that if $-1 < P, Q < 1$, then the shortest curve between P and Q is the line segment beween them. An important property of our new geometry is that any Möbius transformation

$$w = M(z) = e^{i\theta} \frac{z - \alpha}{1 - z\bar{\alpha}}$$

preserves length. This gives (using some basic facts about Möbius transformations) that the geodesics are the half circles (or line segments) that intersect $|z| = 1$ in two right angles:

Instead of looking at the whole (complex) plane, we look at a part of it, the unit disk ($|z| < 1$). There we take

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

What are the geodesics? It is fairly easy to see that if $-1 < P, Q < 1$, then the shortest curve between P and Q is the line segment beween them. An important property of our new geometry is that any Möbius transformation

$$w = M(z) = e^{i\theta} \frac{z - \alpha}{1 - z\bar{\alpha}}$$

preserves length. This gives (using some basic facts about Möbius transformations) that the geodesics are the half circles (or line segments) that intersect $|z| = 1$ in two right angles:

The line segment $[-1, 1]$ is a geodesic. It is a straight line that intersects the unit circle at right angles.

Instead of looking at the whole (complex) plane, we look at a part of it, the unit disk ($|z| < 1$). There we take

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

What are the geodesics? It is fairly easy to see that if $-1 < P, Q < 1$, then the shortest curve between P and Q is the line segment beween them. An important property of our new geometry is that any Möbius transformation

$$w = M(z) = e^{i\theta} \frac{z - \alpha}{1 - z\bar{\alpha}}$$

preserves length. This gives (using some basic facts about Möbius transformations) that the geodesics are the half circles (or line segments) that intersect $|z| = 1$ in two right angles:

The line segment $[-1, 1]$ is a geodesic. It is a straight line that intersects the unit circle at right angles. Therefore, by a result from complex analysis, its image under M is either a line or a circle, intersecting the image of the unit circle at right angles. But, the image of the unit circle is the unit circle.

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'.

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'. Now look at the origin. This is our point outside the 'straight line'. Clearly there are many diameters of the circle that do not intersect C. Thus, PA does not hold, but one can check that the other postulates do hold.

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'. Now look at the origin. This is our point outside the 'straight line'. Clearly there are many diameters of the circle that do not intersect C. Thus, PA does not hold, but one can check that the other postulates do hold.

2. *The sum of angles in a triangle is smaller than* π.

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'. Now look at the origin. This is our point outside the 'straight line'. Clearly there are many diameters of the circle that do not intersect C. Thus, PA does not hold, but one can check that the other postulates do hold.

2. *The sum of angles in a triangle is smaller than* π.

To see this, take two half lines through the origin. This gives us two sides of a triangle.

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'. Now look at the origin. This is our point outside the 'straight line'. Clearly there are many diameters of the circle that do not intersect C. Thus, PA does not hold, but one can check that the other postulates do hold.

2. *The sum of angles in a triangle is smaller than* π.

To see this, take two half lines through the origin. This gives us two sides of a triangle. Let the third side be a half circle intersecting the unit circle at right angles outside the sector formed by the two half lines. Inspection of the figure shows that the sum of angles is smaller than π.

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'. Now look at the origin. This is our point outside the 'straight line'. Clearly there are many diameters of the circle that do not intersect C. Thus, PA does not hold, but one can check that the other postulates do hold.

2. *The sum of angles in a triangle is smaller than π.*

To see this, take two half lines through the origin. This gives us two sides of a triangle. Let the third side be a half circle intersecting the unit circle at right angles outside the sector formed by the two half lines. Inspection of the figure shows that the sum of angles is smaller than π. Note however that if the half circle is close to a diameter, so that our triangle is very small, then the sum of angles is *almost* π.

1. *In this hyperbolic geometry there are infinitely many geodesics through a point outside a given geodesic, that do not intersect the given geodesic. Hence, PA does not hold.*

To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'. Now look at the origin. This is our point outside the 'straight line'. Clearly there are many diameters of the circle that do not intersect C. Thus, PA does not hold, but one can check that the other postulates do hold.

2. *The sum of angles in a triangle is smaller than π.*

To see this, take two half lines through the origin. This gives us two sides of a triangle. Let the third side be a half circle intersecting the unit circle at right angles outside the sector formed by the two half lines. Inspection of the figure shows that the sum of angles is smaller than π. Note however that if the half circle is close to a diameter, so that our triangle is very small, then the sum of angles is *almost* π. So, on small scales, the new geometry is almost Euclidean.

More generally still we can define lengths by

$$L_g(\gamma) = \int_a^b \sqrt{\sum g_{ij}(x(t))\dot{x}_i(t)\dot{x}_j(t)}dt,$$

where $g(x) = (g_{ij}(x)$ is a matrix valued function that is positive definite at any point.

More generally still we can define lengths by

$$L_g(\gamma) = \int_a^b \sqrt{\sum g_{ij}(x(t))\dot{x}_i(t)\dot{x}_j(t)}dt,$$

where $g(x) = (g_{ij}(x)$ is a matrix valued function that is positive definite at any point.

A particular case is when we measure lengths of curves on a surface in $\mathbb{R}^3$, using a parametrization with a domain in the plane.

Why is this a special case?

Why is this a special case?

If $\gamma(t) = x(t)$ is a curve in the surface S in $\mathbb{R}^3$, its length is

$$L(\gamma) = \int_a^b |\dot{x}(t)|^2 dt,$$

where $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$, and $|\dot{x}|^2 = (\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2$.

Why is this a special case?

If $\gamma(t) = x(t)$ is a curve in the surface S in $\mathbb{R}^3$, its length is

$$L(\gamma) = \int_a^b |\dot{x}(t)|^2 dt,$$

where $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$, and $|\dot{x}|^2 = (\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2$. Now parametrize the surface by $x = x(u_1, u_2)$.

Why is this a special case?

If $\gamma(t) = x(t)$ is a curve in the surface S in $\mathbb{R}^3$, its length is

$$L(\gamma) = \int_a^b |\dot{x}(t)|^2 dt,$$

where $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$, and $|\dot{x}|^2 = (\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2$. Now parametrize the surface by $x = x(u_1, u_2)$.

Then γ becomes a curve $(u_1(t), u_2(t))$ in the (u_1, u_2)-plane and

$$\dot{x}_i = \frac{\partial x_i}{\partial u_1}\dot{u}_1 + \frac{\partial x_i}{\partial u_2}\dot{u}_2.$$

Why is this a special case?

If $\gamma(t) = x(t)$ is a curve in the surface S in $\mathbb{R}^3$, its length is

$$L(\gamma) = \int_a^b |\dot{x}(t)|^2 dt,$$

where $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$, and $|\dot{x}|^2 = (\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2$. Now parametrize the surface by $x = x(u_1, u_2)$.

Then γ becomes a curve $(u_1(t), u_2(t))$ in the (u_1, u_2)-plane and

$$\dot{x}_i = \frac{\partial x_i}{\partial u_1}\dot{u}_1 + \frac{\partial x_i}{\partial u_2}\dot{u}_2.$$

Hence $|\dot{x}|^2$ becomes a quadratic expression in $\dot{u}_i$

$$|\dot{x}|^2 = \sum_1^2 g_{ij}\dot{u}_i\dot{u}_j.$$

Why is this a special case?

If $\gamma(t) = x(t)$ is a curve in the surface S in $\mathbb{R}^3$, its length is

$$L(\gamma) = \int_a^b |\dot{x}(t)|^2 dt,$$

where $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$, and $|\dot{x}|^2 = (\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2$. Now parametrize the surface by $x = x(u_1, u_2)$.

Then γ becomes a curve $(u_1(t), u_2(t))$ in the (u_1, u_2)-plane and

$$\dot{x}_i = \frac{\partial x_i}{\partial u_1}\dot{u}_1 + \frac{\partial x_i}{\partial u_2}\dot{u}_2.$$

Hence $|\dot{x}|^2$ becomes a quadratic expression in $\dot{u}_i$

$$|\dot{x}|^2 = \sum_1^2 g_{ij}\dot{u}_i\dot{u}_j.$$

Hence

$$L_g(\gamma) = \int_a^b \sqrt{\sum g_{ij}(u(t))\dot{u}_i(t)\dot{u}_j(t)}dt.$$

Curvature

Gauss defined the curvature of S at a point in the following way: Say the point is the origin and write S as a graph

$$x_3 = f(x_1, x_2),$$

where $f(0) = 0$, $df(0) = 0$.

Curvature

Gauss defined the curvature of S at a point in the following way: Say the point is the origin and write S as a graph

$$x_3 = f(x_1, x_2),$$

where $f(0) = 0$, $df(0) = 0$. Then the curvature at 0 is

$$K_0 = \det(\frac{\partial^2 f}{\partial x_i \partial x_j}).$$

Curvature

Gauss defined the curvature of S at a point in the following way: Say the point is the origin and write S as a graph

$$x_3 = f(x_1, x_2),$$

where $f(0) = 0$, $df(0) = 0$. Then the curvature at 0 is

$$K_0 = \det(\frac{\partial^2 f}{\partial x_i \partial x_j}).$$

It seems that this depends on f (the embedding of the surface in 3-space), but it does not!

Curvature

Gauss defined the curvature of S at a point in the following way: Say the point is the origin and write S as a graph

$$x_3 = f(x_1, x_2),$$

where $f(0) = 0$, $df(0) = 0$. Then the curvature at 0 is

$$K_0 = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

It seems that this depends on f (the embedding of the surface in 3-space), but it does not!

It can be computed from the metric g_{ij} in a parametrization. So, the curvature can be detected by beings that live only on the surface and don't know about the ambient 3-space!

Curvature

Gauss defined the curvature of S at a point in the following way: Say the point is the origin and write S as a graph

$$x_3 = f(x_1, x_2),$$

where $f(0) = 0$, $df(0) = 0$. Then the curvature at 0 is

$$K_0 = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

It seems that this depends on f (the embedding of the surface in 3-space), but it does not!

It can be computed from the metric g_{ij} in a parametrization. So, the curvature can be detected by beings that live only on the surface and don't know about the ambient 3-space! This is Gauss' *Theorema Egregium*.

For instance, if we look at the surface

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\},$$

it has Gauss curvature zero.

For instance, if we look at the surface

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\},$$

it has Gauss curvature zero. But, the sphere

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$$

has Gauss curvature greater than zero.

For instance, if we look at the surface

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\},$$

it has Gauss curvature zero. But, the sphere

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$$

has Gauss curvature greater than zero.

Riemann generalized Gauss's theory to 'surfaces' (manifolds) of arbitrary dimension.

For instance, if we look at the surface

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\},$$

it has Gauss curvature zero. But, the sphere

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$$

has Gauss curvature greater than zero.

Riemann generalized Gauss's theory to 'surfaces' (manifolds) of arbitrary dimension. He defined the *Riemann curvature tensor*, $R = (R_{ijkl})$, which vanishes precisely when the geometry is Euclidean.

For instance, if we look at the surface

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\},$$

it has Gauss curvature zero. But, the sphere

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$$

has Gauss curvature greater than zero.

Riemann generalized Gauss's theory to 'surfaces' (manifolds) of arbitrary dimension. He defined the *Riemann curvature tensor*, $R = (R_{ijkl})$, which vanishes precisely when the geometry is Euclidean. R is then not just a function, but a *tensor* with four indices.

For instance, if we look at the surface

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\},$$

it has Gauss curvature zero. But, the sphere

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$$

has Gauss curvature greater than zero.

Riemann generalized Gauss's theory to 'surfaces' (manifolds) of arbitrary dimension. He defined the *Riemann curvature tensor*, $R = (R_{ijkl})$, which vanishes precisely when the geometry is Euclidean. R is then not just a function, but a *tensor* with four indices. (Whatever that means!)

The Riemann Curvature tensor

The Riemann Curvature tensor

If (g_{ij}) is the metric, we first define the Christoffel symbols

$$\Gamma_{ij}^{m} = 1/2 \sum_{k} g^{mk} \left(\frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right).$$

The Riemann Curvature tensor

If (g_{ij}) is the metric, we first define the Christoffel symbols

$$\Gamma_{ij}^{m} = 1/2 \sum_k g^{mk} \left(\frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right).$$

Then we simply have

$$R_{ijk}^{l} = \frac{\partial \Gamma_{ik}^{l}}{\partial x_j} - \frac{\partial \Gamma_{ij}^{l}}{\partial x_k} + \sum_s \Gamma_{js}^{l} \Gamma_{ik}^{s} - \Gamma_{ks}^{l} \Gamma_{ij}^{s}.$$

Theory of relativity

Theory of relativity

The Riemann geometry is the mathematical framework for Einstein's theory of general relativity.

Theory of relativity

The Riemann geometry is the mathematical framework for Einstein's theory of general relativity.
There the universe is modeled as a 4-dimensional Riemannian manifold, where the metric g is determined by the mass distribution.

Theory of relativity

The Riemann geometry is the mathematical framework for Einstein's theory of general relativity.
There the universe is modeled as a 4-dimensional Riemannian manifold, where the metric g is determined by the mass distribution.
Particles that are influenced only by gravity move along geodesics.

Theory of relativity

The Riemann geometry is the mathematical framework for Einstein's theory of general relativity.
There the universe is modeled as a 4-dimensional Riemannian manifold, where the metric g is determined by the mass distribution.
Particles that are influenced only by gravity move along geodesics.

The metric is not positive definite, but Lorentzian – it has one positive and three negative eigenvalues at any point.

Theory of relativity

The Riemann geometry is the mathematical framework for Einstein's theory of general relativity.
There the universe is modeled as a 4-dimensional Riemannian manifold, where the metric g is determined by the mass distribution.
Particles that are influenced only by gravity move along geodesics.

The metric is not positive definite, but Lorentzian – it has one positive and three negative eigenvalues at any point. This leads to a surprising consequence: A geodesic straight line is the *longest* curve between two points.

Theory of relativity

The Riemann geometry is the mathematical framework for Einstein's theory of general relativity.
There the universe is modeled as a 4-dimensional Riemannian manifold, where the metric g is determined by the mass distribution.
Particles that are influenced only by gravity move along geodesics.

The metric is not positive definite, but Lorentzian – it has one positive and three negative eigenvalues at any point. This leads to a surprising consequence: A geodesic straight line is the *longest* curve between two points. For instance, the manifold could be $\mathbb{R}^4$ with metric

$$\sum_{1}^{4} g_{ij}\dot{x}_i\dot{x}_j = (\dot{x}_0)^2 - (\dot{x}_1)^2 - (\dot{x}_2)^2 - (\dot{x}_3)^2.$$

Theory of relativity

The Riemann geometry is the mathematical framework for Einstein's theory of general relativity.
There the universe is modeled as a 4-dimensional Riemannian manifold, where the metric g is determined by the mass distribution. Particles that are influenced only by gravity move along geodesics.

The metric is not positive definite, but Lorentzian – it has one positive and three negative eigenvalues at any point. This leads to a surprising consequence: A geodesic straight line is the *longest* curve between two points. For instance, the manifold could be $\mathbb{R}^4$ with metric

$$\sum_1^4 g_{ij}\dot{x}_i\dot{x}_j = (\dot{x}_0)^2 - (\dot{x}_1)^2 - (\dot{x}_2)^2 - (\dot{x}_3)^2.$$

Geodesics are ordinary straight lines and they are the longest path between two points. This is *Minkowski space*.

Time

A curve in Minkowski space models the history of an 'observer'.

Time

A curve in Minkowski space models the history of an 'observer'.

The time the observer measures between two points is the length of the curve between these two points.

Time

A curve in Minkowski space models the history of an 'observer'.

The time the observer measures between two points is the length of the curve between these two points.

So, if two observers travel from A to B in space time, and one follows a straight line, (s)he measures the longest time.

Time

A curve in Minkowski space models the history of an 'observer'.

The time the observer measures between two points is the length of the curve between these two points.

So, if two observers travel from A to B in space time, and one follows a straight line, (s)he measures the longest time.

This the *twin 'paradox'*.

Time

A curve in Minkowski space models the history of an 'observer'.

The time the observer measures between two points is the length of the curve between these two points.

So, if two observers travel from A to B in space time, and one follows a straight line, (s)he measures the longest time.

This the *twin 'paradox'*. Is it a paradox?