Analytic Geometry and Linear Algebra.

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Now the basic objects of geometry can be described in terms of coordinates.

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5. The *angle* between two directions v and w is given by

$$\arccos(\frac{v \cdot w}{|v||w|}),$$

where $v \cdot w = \sum v_j w_j$.

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is an extra (unexpected ?) bonus with the translation to coordinates: We can do geometry in any dimension, and it is in principle as easy as in two dimensions. Here is an example of this:

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But this system is *overdetermined*; there are $n > 2$ equations but only 2 unknowns. Instead we try to minimize the error

$$\epsilon^2 = \sum_j (y_j - (ax_j + b))^2$$

over all choices of a and b.

Let $\mathbf{x} = (x_1, ...x_n)$, $\mathbf{1} = (1, 1...1)$ (**two points in $\mathbb{R}^n$!!**). We can write the system of equaltions that we are trying to solve as

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How do we find it?

It is clear from a figure that the minimum will occur in a point (a_0, b_0) such that $\mathbf{y} - (a_0\mathbf{x} + b_0\mathbf{1})$ is perpendicular to any vector in the plane. (Excercise: prove this!).

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$$[\mathbf{y} - (a_0\mathbf{x}+b_0\mathbf{1}] \cdot \mathbf{x} = 0, \quad [\mathbf{y} - (a_0\mathbf{x} + b_0\mathbf{1})] \cdot \mathbf{1} = 0.$$

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This is an inhomogenous system of *two equations* and two unknowns which always has a solution (why?). Observe that a_0 and b_0 are the unknowns, and $\mathbf{x}$, $\mathbf{y}$ are given!

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The problem has a unique solution when the vectors $\vec{f_k}$ are linearly independent.

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The method of least squares was probably first used by Gauss, who applied it to find a 'lost planet'.

The astronomers had found a dwarf planet Ceres and recorded its positions for some time, but suddenly they could not see it anymore, because it was close to aligned with the sun. Gauss extrapolated the position from the known trajectory and could tell the astronomers where to look.

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Matrices

One central topic in linear algebra is the solution of linear systems of equations

$$\begin{array}{c} a_{11}x_1 + ...a_{1n}x_n = y_1 \\ a_{21}x_1 + ...a_{2n}x_n = y_2 ... \\ a_{m1}x_1 + ...a_{mn}x_n = y_m \end{array},$$

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The main theorem on systems of equations

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Thus, the index of A does not depend on A. If $A : V \to V$ where V is a vector space of finite dimension, then the index is always zero. This is not always the case in infinite dimensions as we shall see later. The index is an important object of study in the theory of partial differential equations, when A is a differential operator.

Quadratic forms

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and we may assume that A is symmetric. If we change basis in $\mathbb{R}^n$, $x = My$, where M is an invertible matrix, we have

$$Q(x) = y^t M^t A M y = Q'(y).$$

We now have the second important theorem of linear algebra:

Theorem

We may find an (orthonormal) M such that

$$Q'(y) = \sum \lambda_j y_j^2.$$

This is the *Spectral Theorem*. If we interpret A as a linear operator, $A' = M^{-1}AM$ is the matrix for the same operator in the new basis, where y are coordinates. But, since M is orthonormal, $M^t = M^{-1}$. hence the theorem says that we change coordinates so that A' is the diagonal with eigenvalues λ_j.

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An application

Say that we have a system of ordinary differential equations

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Say that we have a system of ordinary differential equations

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where $\mathbf{x}$ is a *vector* and A a matrix. We can change coordinates so that A becomes diagonal. Then the system becomes

$$\begin{aligned} x_1' &= \lambda_1 x_1 \\ x_2' &= \lambda_2 x_2 \dots \\ x_n' &= \lambda_n x_n \end{aligned} ,$$

which is easy to solve, $x_i(t) = x_i(0)e^{\lambda_i t}$.

Infinite dimension and Hilbert space.

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In the case of *Hilbert spaces*, this way of measuring distances comes from a *scalar product*:

$$(u, v).$$

The study of $\mathbb{R}^2$ and $\mathbb{R}^n$ from a geometric viewpoint, led to the notion of an abstract linear space. Its elements can be added and multiplied by scalars according to certain rules. Nothing in this general picture presupposes that the space is finite dimensional, i e has a finite basis.

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exists.
In other words, there is an element u in the space such that

$$\|u - \sum^n u_j\| \to 0.$$

Example 1: Let
$V = \{u = (u_0, ...u_n, ...), u_k = 0 \quad \text{for k sufficiently large}\}$, with norm

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Example 2 is complete, Example 1 is not.

Theorem

Every Hilbert space V has an orthonormal basis, i e there is an orthonormal set of vectors $\{e_\alpha\}_{\alpha \in A}$ such that any vector in V can be written

$$x = \sum_A c_\alpha e_\alpha,$$

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Briefly, there is only one Hilbert space.

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Fourier series:

Let $L^2(T) = \{f; \int_0^1 |f(x)|^2 dx < \infty\}$.

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Note that

$$(e_k, e_m) = \int_0^1 e^{2\pi ikx}\overline{e^{2\pi imx}}dx = \int_0^1 e^{2\pi i(k-m)x}dx = \delta_{km},$$

so the system is orthonormal. It is also a basis for $L^2(T) \sim l^2$; i. e. any element in L^2 can be written

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This is the Fourier series of f.

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(a *Fredholm equation of the second kind*).
Notice that they are continuous analogs of the matrix equations

$$T(f) = g, \quad (I - \lambda T)(f) = g,$$

where

$$T(f)(y) = \int K(y,x)f(x)dx.$$

Fredholm determinants

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Let

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Having defined determinants he could 'solve' the equations by Cramer's rule!

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Then we can define Fredholm determinants by the same formula.

Theorem

*(**The Fredholm alternative**) Let*

$$Tf(x) = \int K(x, y)f(y)dy,$$

where K is continuous. Then, for any complex number λ, either the equation

$$(I - \lambda T)f = g$$

has a solution f for any choice of g, or the equation

$$(I - \lambda T)f = 0$$

has a non trivial solution.

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The next big step was John von Neumann's general theory of Hilbert spaces (he introduced that name) as a foundation of quantum mechanics in 1932 (when von Neumann was 29 years old).

The spectral Theorem

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The *Spectral Theorem* says that (under some conditions on T) we can choose a basis e_k so that if

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(In more generality one has to replace the sum by an integral.)

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where T is selfadjoint. After diagonalizing we get

$$x_i(t) = x_i(0)e^{-\sqrt{-1}\lambda_i t},$$

like we had before in finite dimension.

Quantum mechanics

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The operator H in Schrödinger's equation corresponds to the observable 'energy'. In the last slide we have seen that solving the Schrödinger equation is closely linked to the spectral theorem.

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Observable quantities are things like 'position', 'momentum' or 'energy'. When we measure a quantity in quantum mechanics, the outcome is a stochastic variable, that can take any of the values λ_k.

The probability that we will get λ_k is

$$\frac{|c_k|^2}{\sum |c_j|^2},$$

if the state is

$$f = \sum c_j e_j$$

in an orthonormal basis that diagonlizes the operator.

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In this way we can see Hilbert space as the mathematical theory of quantum mechanics, similarily to how Riemannian geometry is the mathematics of the theory general relativity.