Calculus.

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The calculus of integrals and derivatives is often presented as having started with Newton and Leibniz, but of course there were many predecessors. We have already seen how versions of of integration was already used in antiquity, by Eudoxus and Archimedes. Another early theorem that today would be interpreted as a theorem about integrals is *Cavalieri's principle* (1635).

It says that if we have two regions in the plane, both defined by the graph of functions

$$R_i = \{(x, y); g_i(x) < y < f_i(x)\},$$

and if

$$f_1(x) - g_1(x) = f_2(x) - g_2(x)$$

for all x then the areas of the two regions are equal.

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$$p(x) = 2x^3 - 9x^2 + 12x.$$

If a is a local maximum there will be two distinct points x_1 and x_2 close to a such that $p(x_1) = p(x_2)$ (one on each side of a).

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$$0 = \frac{p(x_1) - p(x_2)}{x_1 - x_2} = 2(x_1^2 + x_1 x_2 + x_2^2) - 9(x_1 + x_2) + 12.$$

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Since this holds for points arbitrarily close to a it must hold for $x_1 = x_2 = a$. We get

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Apparently this was first noted my Newton's teacher, Barrow. (Barrow was reputedly a 'wild character', sent off to academic studies by his wealthy father who did not want him involved in the family business. As subject of study he choose – theology. Theology lead to chronology and attempts to reconcile the age of the earth according to the bible with known historical records. Chronology in turn lead to astronomy and, then, mathematics.)

Barrow's results were however not as clearly formulated as in the succinct equation above. The honor of having discovered the fundamental theorem of calculus is instead ascribed to Newton and Leibniz. The story is complicated by the fact that Newton did not publish his work on derivatives until fairly late, in 1693. By that time, Leibniz had already published his version of the theory, in 1684, which lead to a long controversy between the two.

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Newton is said to have stated that any person in science must make a choice: Either to publish nothing, or to devote all his life to a struggle for priority. According to the russian mathematician Arnold, a great admirer of Newton's, Newton made the worst of these alternatives; he published almost nothing – *and* was constantly struggling for priority.

Most of Newton's most well known work was carried out between 1665 -1667, during the plague years. (He was born in 1642). This includes, probably, his work on the method of derivatives and also the Newtonian theory of classical mechanics that was not published until 1687, in his Principia Mathematica.

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In principia Newton formulated what is now known as Newton's laws, essentially the law of acceleration

$$\vec{F} = m\vec{a}$$

and the law of gravitation

$$F = \frac{mM}{r^2}$$

or rather

$$\vec{F} = -mM\frac{\vec{r}}{r^3},$$

(the inverse square law).

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Wikipedia on Principia

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It shows:

how astronomical observations prove the inverse square law of gravitation (to an accuracy that was high by the standards of Newton's time);

offers estimates of relative masses for the known giant planets and for the Earth and the Sun;

defines the very slow motion of the Sun relative to the solar-system barycenter;

shows how the theory of gravity can account for irregularities in the motion of the Moon;

identifies the oblateness of the figure of the Earth;

accounts approximately for marine tides including phenomena of spring and neap tides by the perturbing (and varying) gravitational attractions of the Sun and Moon on the Earth's waters;

explains the precession of the equinoxes as an effect of the gravitational attraction of the Moon on the Earth's equatorial bulge; and

gives theoretical basis for numerous phenomena about comets and their elongated, near-parabolic orbits

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Let $x(t)$ where t runs from a to b be a curve, such that it is the shortest curve between $A := x(a)$ and $B = x(b)$. We may assume that x is parameterized by arc length. Let

$$L(s) = \int_a^b |\dot{x}(t) + s\dot{y}(t)|dt,$$

where $y(a) = y(b) = 0$.

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$$L(s) = \int_a^b |\dot{x}(t) + s\dot{y}(t)| dt,$$

where $y(a) = y(b) = 0$. Then $L'(0) = 0$.

But

$$L'(0) = \int_a^b \frac{\dot{x} \cdot \dot{y}}{|\dot{x}|} dt = \int_a^b \dot{x} \cdot \dot{y} \, dt,$$

since $|\dot{x}| = 1$ when the curve is parametrized by arc length.

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Similarily one can show that a circle is the curve of a given length that encompasses the greates area. (Much more difficult though.) But all these methods presuppose that *there exists* a curve that gives the minimum.

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Similarily one can show that a circle is the curve of a given length that encompasses the greates area. (Much more difficult though.) But all these methods presuppose that *there exists* a curve that gives the minimum. Such problems were not solved until much later, after the rigorous introduction of the real number system, limits and the supremum axiom, by Cauchy, Weierstrass and Dedekind.

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$$L(x, \dot{x}) := \frac{m\dot{x}^2}{2} - V(x),$$

where V is the *potential energy*. Thus, the Lagrangian is the *difference* between the kinetic energy and the potential energy, as opposed to the total energy which is the *sum* of kinetic and potential energy.

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Newton's laws can be written

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This can be written elegantly in terms of the action:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

So far this is just a rewrite. Now introduce the total *action* of a curve $\gamma = x(t), a < t < b$:

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Assume that γ minimizes the action among all curves with the same end points $x(a), x(b)$. Then

$$0 = (d/ds)|_{s=0} S(\gamma + sy(t)) = \int_a^b y\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial \dot{x}} dt,$$

for all $y(t)$ that vanish at the end points.

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After an integration by parts in the second term this means precisely that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x},$$

i e Newton's equations!

So, a curve $x(t)$ satisfies Newton's equations when it minimizes the action integral (or, more precisely, is stationary for the action integral). This is called the principle of least action.

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This simplifies a lot! No need to write down all the forces and, more importantly, it does not depend on the choice of coordinates.

This is very important in modern physics where a new physical law is not defined in terms of forces, but given as a new Lagrangian.

Noether's theorem

Since

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} = \frac{\partial L}{\partial x_j},$$

it also shows that if

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is conserved (i e constant). This is called *Noether's principle*, after Emmy Noether (1882-1935), and has been called the most important theorem in physics.

A trivial example

Say $V(x) = 0$, so the Lagrangian is just

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On the next slide we give a less trivial example.

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$$\dot{x} = \dot{r}(\cos\theta, \sin\theta) + r\dot{\theta}(-\sin\theta, \cos\theta).$$

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Since $\partial L/\partial\theta = 0$, Noether's principle gives

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This is Kepler's law. It says that the *angular momentum* is conserved.

(Challenging) Exercise:

Look at a pendelum of length r hanging from the origin in $\mathbb{R}^2$.

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Look at a pendelum of length r hanging from the origin in $\mathbb{R}^2$. Let θ be the angle the pendelum makes with the negative half axis. Show that the Lagrangian is

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Look at a pendelum of length r hanging from the origin in $\mathbb{R}^2$. Let θ be the angle the pendelum makes with the negative half axis. Show that the Lagrangian is

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Can you solve the equation? (Answer: No.)

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This was a few years after the births of Fourier and Gauss, whose work would mark a new era in mathematics.