Complex analysis and Fourier analysis.

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Complex numbers appear already in Cardano's formula for the solution of third degree equations, where combinations of complex numbers could be used to get real solutions of an equation. They were also used in the middle 18:th century by de Moivre and Euler in their famous formulas

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It is unclear (to me) when complex numbers began to be considered as a well defined mathematical structure. Perhaps this did not happen until Gauss, who however had been reluctant to compute openly with complex numbers in his early work. Complex analysis, involving e g the Cauchy-Riemann equations goes back at least to d'Alembert, Euler and Riemann in the 18:th century, and became a fully developed theory with Cauchy's work.

The representation of complex numbers as points in $\mathbb{R}^2$ is usually ascribed to the Norwegian cartographer C Wessel who published his findings in 1797. His work went unnoticed for many years, but the idea was rediscovered by Argand in 1806 and later Gauss. Argand had a bookstore in Paris and was an amateur mathematician. Today he is also given credit for the first correct proof of the fundamental theorem of algebra in (1814). Previous proofs had been given by Euler, d'Alembert and Gauss, but none of them is considered complete.

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Let us start by giving an idea of Gauss' first proof. In modern language he considered an algebraic equation

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where a_j are *real*, and wanted to show that it has a complex solution. Taking real and imaginary parts, and writing $z = r(\cos\theta + i\sin\theta)$ we get two equations

$$u(z) = r^n \cos n\theta + a_1 r^{n-1} \cos n\theta + ... = 0$$

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He then argued that the solution set of the first equation is an algebraic curve C_1, the solution set of the second equation is also an algebraic curve, C_2, and we want to prove that these two curves intersect.

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"It seems to be well demonstrated that an algebraic curve neither ends abruptly (as it happens in the transcendental curve y = 1/ log x), nor loses itself after an infinite number of windings in a point (like a logarithmic spiral). As far as I know nobody has ever doubted this, but if anybody requires it, I take it on me to present, on another occasion, an indubitable proof."

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4. Then $p(z) = a_n + z^k(b_0 + zb_1 + ...)$. There are (small) values of z which makes this smaller than $|a_n|$.

5. This contradiction proves the theorem.

This was also the main scheme in Argand's proof. Both Argand and d'Alembert took 2. for granted: A continuous real valued function defined on a compact set has a minimum value. This was only proved rigorously much later!
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Exercise: Help him! In other words, prove that if $p(z) = a_1 + z^k(b_0 + b_1 z + ...)$, $k = 1, 2, ...$ and $a_1 \neq 0$, then $|p(0)| = |a_1|$ is *not* the minimum of $|p|$

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1. f has a series expansion $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots.$.

2. Any function that is holomorphic in all of $\mathbb{C}$ and bounded must be constant.

3. If two holomorphic functions are equal on $\mathbb{R}$ then they are equal everywhere. Example: $e^{x+y} = e^x e^y$ implies $e^{i\theta + i\phi} = e^{i\theta} e^{i\phi}$ implies

$$\sin(\theta + \phi) + i\cos(\theta + \phi) = (\sin\theta + i\cos\theta)(\sin\phi + i\cos\phi).$$

From a more modern view point the definition can be described as follows. Let $f : \mathbb{C} \to \mathbb{C}$ be a smooth function. We can view it as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then it has a differential at any point z

$$df|_z = f_x dx + f_y dy.$$

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The function is holomorphic if the differential is **linear as a map from $\mathbb{C}$ to $\mathbb{C}$**:

$$df(ib) = idf(b),$$

which means

$$f_y b = if_x b,$$

i e

$$f_x + if_y = 0.$$

This is called the *Cauchy-Riemann equations* and usually written $f_{\bar{z}} = 0$.

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But, how do we map the surface of the earth (a sphere) to the plane?

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Take the earth in form of a globe and put it on table (the complex plane), with the north pole facing up. For any other point p on the surface of the earth, draw a line through the north pole and p. That line (extended) intersects the table (complex plane) in one point, $s(p)$.

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Any other (orientable) closed surface without boundary (like the sphere) can also be looked at in a similar way, as a complex manifold. This leads to the concept of a Riemann surface.

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Because g has a singularity when $x = i$.

Fourier Analysis

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J-B Fourier was born in 1768. He studied mathematics at a military academy, run by the Benedictine monks, and planned to become a monk himself, stopped only by the dissolution of the religuous orders during the French revolution, two days before he were to make his oath. He was himself active in the revolution, and narrowly escaped to be executed during the reign of terror. He became a professor of mathematics at Ecole Polytechnique in 1797.

One year later he was sent by Napoleon with the expedition to Egypt, and became a prominent actor in the scientifique part of the expedition. His name is therfore also well known in Egyptology, and he wrote the preface to the treatise ‘Description de l’Egypte’, a general description of all the historical findings of the french in Egypt (many of which were exported to Paris).

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Then, in 1822, he published his ‘Theorie analytique de la chaleur’. Here he introduced the *heat equation* and Fourier analysis as a tool to solve it.

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If we are a bit more precise and assume that f lies in

$$L^2(T) = \{f; \int |f|^2 dt < \infty\},$$

we can give this representation a meaning:

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$$\langle e_k, e_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikt} e^{-imt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-m)t} dt = \delta_{k,m}.$$

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Fourier's theorem then says that there is an orthonormal basis of eigenvectors for A,. i e that A can be diagonalized.

The second version of Fourier analysis is for functions defined on $\mathbb{R}$, or why not $\mathbb{R}^n$. Let

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This is the Fourier inversion formula. It is less easy to interpret in the Hilbert space sense; we have now a 'continuous basis', and one needs von Neumann's general operator theory to discuss such matters: The operator id/dx has *continuous spectrum* on $\mathbb{R}$, i e a continuous family of eigenvalues, namely $\mathbb{R}$.

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To illustrate, say $x = (x_1, 0)$. Then

$$\hat{f}(x_1, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix_1 t_1} dt_1 \int_{-\infty}^{\infty} f(t_1, t_2) dt_2.$$

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The integral with respect to t_2 is the integral over vertical lines, which we assume we know. So, we know $\hat{f}$, so we know f by Fourier inversion!

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Our next example of applications of Fourier analysis is *the Shannon Sampling Theorem*:

Theorem

Let $f(t)$ be a function on $\mathbb{R}$ which is such that its Fourier transform $\hat{f}(x)$ vanishes outside the interval $[-\pi, \pi]$. Then f is uniquely determined by its values at the integers, and more precisely

$$f(t) = \sum f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}$$

We can also rescale the function and get equivalently

Theorem

Let $f(t)$ be a function on $\mathbb{R}$ which is such that its Fourier transform $\hat{f}(x)$ vanishes outside the interval $[-B\pi, B\pi]$. Then f is uniquely determined by its values at the points n/B,

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Think of $f(t)$ as a signal varying with time, like the sound of music. ($f(t)$ is then the air pressure at a given fixed point at time t.) The theorem says that if the signal does not 'contain frequences' higher than $B\pi$ then we can, from a sequence of discrete measurements with interval $1/B$, reconstruct the signal perfectly.

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