Algebraic equations.

Algebraic equations of degree 1 with one unknown have the form

$$ax + b = c.$$

Algebraic equations of degree 1 with one unknown have the form

$$ax + b = c.$$

The solution formula for such equations is of course easy, *given that we write the equation in symbolic notation*, i. e. using letters. This is however a fairly recent invention. The first systematic treatment of algebra was the text by al-Khwarizmi, written approximately 800 AD.

Algebraic equations of degree 1 with one unknown have the form

$$ax + b = c.$$

The solution formula for such equations is of course easy, *given that we write the equation in symbolic notation*, i. e. using letters. This is however a fairly recent invention. The first systematic treatment of algebra was the text by al-Khwarizmi, written approximately 800 AD. The word 'algebra' comes from that book. The origin is the word 'al-jabr', which stands for the technique to remove equal quantities from both sides of an equation:

$$x + a = y + a, \text{implies} \quad x = y.$$

Algebraic equations of degree 1 with one unknown have the form

$$ax + b = c.$$

The solution formula for such equations is of course easy, *given that we write the equation in symbolic notation*, i. e. using letters. This is however a fairly recent invention. The first systematic treatment of algebra was the text by al-Khwarizmi, written approximately 800 AD. The word 'algebra' comes from that book. The origin is the word 'al-jabr', which stands for the technique to remove equal quantities from both sides of an equation:

$$x + a = y + a, \text{implies} \quad x = y.$$

Al-Khwarizmi apparently did not use symbolic notation, but formulated his problems in words like:

If some one says: "You divide ten into two parts: multiply the one by itself; it will be equal to the other taken eighty-one times."Computation: You say, ten less a thing, multiplied by itself, is a hundred plus a square less twenty things, and this is equal to eighty-one things. Separate the twenty things from a hundred and a square, and add them to eighty-one. It will then be a hundred plus a square, which is equal to a hundred and one roots. Halve the roots; the moiety is fifty and a half. Multiply this by itself, it is two thousand five hundred and fifty and a quarter. Subtract from this one hundred; the remainder is two thousand four hundred and fifty and a quarter. Extract the root from this; it is forty-nine and a half. Subtract this from the moiety of the roots, which is fifty and a half. There remains one, and this is one of the two parts.

If some one says: "You divide ten into two parts: multiply the one by itself; it will be equal to the other taken eighty-one times."Computation: You say, ten less a thing, multiplied by itself, is a hundred plus a square less twenty things, and this is equal to eighty-one things. Separate the twenty things from a hundred and a square, and add them to eighty-one. It will then be a hundred plus a square, which is equal to a hundred and one roots. Halve the roots; the moiety is fifty and a half. Multiply this by itself, it is two thousand five hundred and fifty and a quarter. Subtract from this one hundred; the remainder is two thousand four hundred and fifty and a quarter. Extract the root from this; it is forty-nine and a half. Subtract this from the moiety of the roots, which is fifty and a half. There remains one, and this is one of the two parts.

In modern notation:

$$(10 - x)^2 = 81x$$

$$100 + x^2 - 20x = 81x$$

$$100 + x^2 = 101x$$

Then he goes on to solve this equation by introducing $(101x/2)^2$ in more or less the usual way of solving quadratic equations. Therefore al-Khwarizmi is credited with giving the first systematic treatment of first and second order equations.

Then he goes on to solve this equation by introducing $(101x/2)^2$ in more or less the usual way of solving quadratic equations. Therefore al-Khwarizmi is credited with giving the first systematic treatment of first and second order equations. The first to introduce symbolic notation in full generality seems to have been Vieta (1540-1603). (More about him later.) However, the method of solving quadratic equations was known already to the Babylonians, 2000 years BC.

The solvability of cubic equations is of course much more difficult. The general method was first found by the Italian Del Ferro, in the beginning of the 16:th century. Del Ferro did not publish his method but communicated it to his students. After his death his students challenged the great mathematician Tartaglia (1500-1557) to solve a number of third degree equations. Tartaglia accepted the challenge and found a way to solve an arbitrary equation of third degree.

The solvability of cubic equations is of course much more difficult. The general method was first found by the Italian Del Ferro, in the beginning of the 16:th century. Del Ferro did not publish his method but communicated it to his students. After his death his students challenged the great mathematician Tartaglia (1500-1557) to solve a number of third degree equations. Tartaglia accepted the challenge and found a way to solve an arbitrary equation of third degree.

Tartaglia in turn did also not publish, but explained his method to Cardano, in exchange for a promise of silence. When Cardano later learnt about Del Ferro's work, he broke his promise and published. The formula is therefore now known as Cardano's formula.

How did Tartaglia solve it?

Let the equation be

$$x^3 + ax^2 + bx + c = 0.$$

How did Tartaglia solve it?

Let the equation be

$$x^3 + ax^2 + bx + c = 0.$$

First substitute $x = y - a/3$. Then the equation becomes

$$y^3 + py + q = 0.$$

How did Tartaglia solve it?

Let the equation be

$$x^3 + ax^2 + bx + c = 0.$$

First substitute $x = y - a/3$. Then the equation becomes

$$y^3 + py + q = 0.$$

Now write $y = u + v$, so that we replace the one unknown y by two unknowns. The result is

$$u^3 + v^3 + 3u^2v + 3uv^2 + pu + pv + q = 0.$$

Thus we arrive at the equation

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

Thus we arrive at the equation

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

Choose u, v so that $3uv + p =$, i. e. $uv = -p/3$.

Thus we arrive at the equation

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

Choose u, v so that $3uv + p =$, i. e. $uv = -p/3$. We get two equations

$$u^3 + v^3 = -q, \quad uv = -p/3.$$

Next, put $t = u^3, s = v^3$ and get

$$t + s = -q, ts = -(p/3)^3.$$

Thus we arrive at the equation

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

Choose u, v so that $3uv + p =$, i. e. $uv = -p/3$. We get two equations

$$u^3 + v^3 = -q, \quad uv = -p/3.$$

Next, put $t = u^3, s = v^3$ and get

$$t + s = -q, ts = -(p/3)^3.$$

Solving $s = -(p/3)^3/t$ we get finally

$$t + q - (p/3)^3/t = 0, \quad t^2 + qt - (p/3)^3 = 0,$$

which is a second degre equation that we know how to solve.

Exercise:

Solve the equation

$$x^3 - 7x - 6 = 0$$

using Tartaglia's method!

Exercise:

Solve the equation

$$x^3 - 7x - 6 = 0$$

using Tartaglia's method! (Not by guessing a root!)

The next step was taken by Vieta. It is essentially the same method as tartaglia's, but more concise.

The next step was taken by Vieta. It is essentially the same method as tartaglia's, but more concise.

Recall that in Tartaglia's method $y = u + v, uv = -p/3$. This means that

$$y = u - p/3u.$$

The next step was taken by Vieta. It is essentially the same method as tartaglia's, but more concise.

Recall that in Tartaglia's method $y = u + v, uv = -p/3$. This means that

$$y = u - p/3u.$$

This is called Vieta's substitution. Plug into the equation

$$y^3 + py + q = 0,$$

and get

$$u^6 + qu^3 - \frac{p^3}{27} = 0.$$

This is a second degree equation in u^3 which we can solve.

Vieta (1540-1603) was a French mathematician working for the Kings Henri III and Henri IV, partly as a code breaker. He is also famous for solving “Adriaan van Roomens problem”, a polynomial equation of degree 45.

Vieta (1540-1603) was a French mathematician working for the Kings Henri III and Henri IV, partly as a code breaker. He is also famous for solving “Adriaan van Roomens problem”, a polynomial equation of degree 45. When Vieat heard the problem, he ’leaned on a window’ for few minutes and solved the problem, which was a challenge from the Dutch ambassador to the King of France.

Vieta (1540-1603) was a French mathematician working for the Kings Henri III and Henri IV, partly as a code breaker. He is also famous for solving “Adriaan van Roomens problem”, a polynomial equation of degree 45. When Vieat heard the problem, he ’leaned on a window’ for few minutes and solved the problem, which was a challenge from the Dutch ambassador to the King of France.

The solution of the cubic equation was important partly because it was the first time that ’modern’ science surpassed the classics. This led to a new attitude; it was possible to find new knowledge and science should no longer be restricted to commentaries on older work.

Lagrange's work

In 1770-1771 Lagrange published an extensive study of algebraic equations, where he introduced the *Lagrange resolvents*. If $p(x)$ is a polynomial with rational coeffcients, let $x_0, ...x_{n-1}$ be its roots. The Lagrange resolvents are

$$L(m) = \sum_{0}^{n-1} x_k e^{2\pi ikm/n}.$$

(Compare to Fourier series.) Here is one example how Lagrange applied these to solving 4:th degree equations, without using any magical substitutions:

Lagrange's work

In 1770-1771 Lagrange published an extensive study of algebraic equations, where he introduced the *Lagrange resolvents*. If $p(x)$ is a polynomial with rational coeffcients, let $x_0, ...x_{n-1}$ be its roots. The Lagrange resolvents are

$$L(m) = \sum_{0}^{n-1} x_k e^{2\pi ikm/n}.$$

(Compare to Fourier series.) Here is one example how Lagrange applied these to solving 4:th degree equations, without using any magical substitutions: Take $n = 4, k = 2$ and let the roots be $x_0, ...x_3$. Then

$$L(2) = x_0 - x_1 + x_2 - x_3.$$

Here we have assumed an ordering of roots. Now permute the roots in all possible ways. We get 24 new numbers of which 6 are different, say $L_1, ... L_6$.

Here we have assumed an ordering of roots. Now permute the roots in all possible ways. We get 24 new numbers of which 6 are different, say $L_1, ...L_6$. Let $Q(y)$ be the monic polynomial of degree 6 with L_j as roots. Its coefficients are functions of $x_0, ...x_3$ that are invariant under a permutation of these numbers.

Here we have assumed an ordering of roots. Now permute the roots in all possible ways. We get 24 new numbers of which 6 are different, say $L_1, ...L_6$. Let $Q(y)$ be the monic polynomial of degree 6 with L_j as roots. Its coefficients are functions of $x_0, ...x_3$ that are invariant under a permutation of these numbers. Therefore they are polynomials in the coeffcients of $p(x)$, and they can be computed explicitly.

Here we have assumed an ordering of roots. Now permute the roots in all possible ways. We get 24 new numbers of which 6 are different, say $L_1, ...L_6$. Let $Q(y)$ be the monic polynomial of degree 6 with L_j as roots. Its coefficients are functions of $x_0, ...x_3$ that are invariant under a permutation of these numbers. Therefore they are polynomials in the coeffcients of $p(x)$, and they can be computed explicitly. Moreover, Q is an even polynomial so it is a polynomial of degree 3 in y^2.

Here we have assumed an ordering of roots. Now permute the roots in all possible ways. We get 24 new numbers of which 6 are different, say $L_1, ...L_6$. Let $Q(y)$ be the monic polynomial of degree 6 with L_j as roots. Its coefficients are functions of $x_0, ...x_3$ that are invariant under a permutation of these numbers. Therefore they are polynomials in the coeffcients of $p(x)$, and they can be computed explicitly. Moreover, Q is an even polynomial so it is a polynomial of degree 3 in y^2. Therefore we can solve $P(y) = 0$, find all L_j and the all the roots x_j of the original equation.

Lagrange was however fully aware that his methods did not give solution formulas for higher degree equations, and wrote

" The problem of solving (by radicals) equations whose degree is higher than four is one of those problems which have not been solved, although nothing proves the impossibility of solving them.

Nevertheless he was convinced that his method, based on permutations of the roots was important, and that *[the theory of permutations] was the true philosophy of the whole question*.
The next big step was the paper by Ruffini in 1799, which claimed to prove that it was impossible to solve fifth degree equations by radicals. His paper however contained a gap, and the first complete proof of the *Abel-Ruffini theorem* was given by Abel in 1824.

The situation was further clarified by the work of Galois in 1830, who systematically described the relation between the permutation groups of the roots of an equation and the solvability by radicals. This explained the difference between the general equation of degree four and lower and those of higher degree, and also could be applied to specific equations to study their solvability.

The situation was further clarified by the work of Galois in 1830, who systematically described the relation between the permutation groups of the roots of an equation and the solvability by radicals. This explained the difference between the general equation of degree four and lower and those of higher degree, and also could be applied to specific equations to study their solvability.

Galois wrote his paper in 1830 when he was 18 years old. It was rejected by the Academy of Science one year later, and in 1832 Galois was killed in a duel. It took a long time for Galois' work to be fully understood. Now it is considered a landmark in the history of mathematics, and its later developments is one of the most vigorous parts of research in mathematics today.