Assignment1MA101

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1 Exercise1

To find the C(θ) we can integral the function in a) equals to 1, and integral over all x in [0, ∞]

$$\int_0^\infty \pi(x|\theta)dx = \int_0^\infty C(\theta)x^2 exp(-\theta x^3)dx = 1$$

$$C(\theta)\int_0^\infty x^2 exp(-\theta x^3)dx = 1$$

substitute $u = -\theta x^3$ then we have $du/dx = -3\theta x^2$ then $dx = -\frac{1}{3\theta x^2}du$.

$$\int x^2 exp(-\theta x^3)dx = -\frac{1}{3\theta}\int exp(u)du$$

And

$$\int exp(u)du = exp(u)$$

Finally we have

$$\int x^2 exp(-\theta x^3)dx = -\frac{exp(u)}{3\theta} = -\frac{exp(-\theta x^3)}{3\theta}$$

Now we can solve that

$$\int_0^\infty \pi(x|\theta)dx = \frac{C(\theta)}{3\theta} = 1$$

$$C(\theta) = 3\theta.$$

Assume $\pi(\theta|x)$ is a family of conjugate priors to $\pi(x|\theta)$. To show that the posterior is proportional to a Gamma, we just simply compute $\pi(\theta|x)$ (proportional to w.r.t θ) $\pi(x|\theta)\pi(\theta\ )$

$$\pi(\theta|x) = \frac{\pi(x|\theta)\pi(\theta)}{\pi(x)}.$$

And

$$\pi(\theta|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}exp(-\beta\theta)$$

Specifically we have the posterior

$$\pi(\theta|x) = Gamma(\theta;\alpha+x,\beta+1)$$

$$\pi(\theta|x) = \frac{C(\theta)x^2exp(-\theta x^3)\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}exp(-\beta\theta)}{\pi(x)}$$

For the posterior predictive density

$$\pi(x) = \frac{\pi(x|\theta)\pi(\theta)}{\pi(\theta|x)}$$

$$\pi(x) = \frac{3\theta x^2exp(-\theta x^3)\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}exp(-\beta\theta)}{Gamma(\theta;\alpha+x,+1)}$$

2 Exercise2

For problem a), it made up of three normal distributions with the weights, means and sds. See figure 1.

For problem b) the formula of prior predictive density is

$$\pi(x) = \frac{\pi(x|\theta)\pi(\theta)}{\pi(\theta|x)}$$

$$\pi(x|\theta) = Normal(x;\theta,0.0001^2)$$

$$\pi(\theta|\mu,\gamma) = Normal(\theta;\mu,1/\gamma)$$

then we have the posterior:

$$\pi(\theta|x) = Normal(\theta;\frac{\gamma_0 x+\gamma\mu}{\gamma_0+\gamma},\frac{1}{\gamma_0+\gamma})$$

then we have the density:

$$\pi(x) = \frac{Normal(x;\theta,0.0001^2)Normal(\theta;\mu,1/\gamma)}{Normal(\theta;\frac{0.0001^2x+\gamma\mu}{0.0001^2+\gamma},\frac{1}{0.0001^2+\gamma})}$$

Then we put the value of μ_i, γ_i from the table.

For problem c), prior predictive density is :

$$\pi(x) = \frac{\pi(x|\theta)\pi(\theta)}{\pi(\theta|x)}$$

$$\pi(\theta, x, x_{new}) = \pi(\theta)\pi(x|\theta)\pi(x_{new}|\theta)$$

$$\pi(x_{new}|x) = \int_\theta \pi(x_{new}|\theta)\pi(\theta|x)d\theta$$

We put the x_{new}=1.52083

For problem d), For the posterior predictive density of a new measurement x_{new}, prior predictive density is :

$$\pi(x_{new}|x) = \frac{\pi(x_{new}|\theta)\pi(\theta|x)}{\pi(\theta|x_{new}, x)}$$

x=1.52083

For problem e), the likelihood of a single observation is the product of $\pi(x|\theta)$ multiplied by $\pi(\theta)$ and integrated over θ.
The difference between the two hypotheses is that when they are independent, you take the project of two separate integrals.
When they are not independent you take the product with the dame integral (i.e over the same prior θ distribution).

For problem f), the likelihood of the data means the prior likelihood of observing both x_s and x_c.

For problem g)

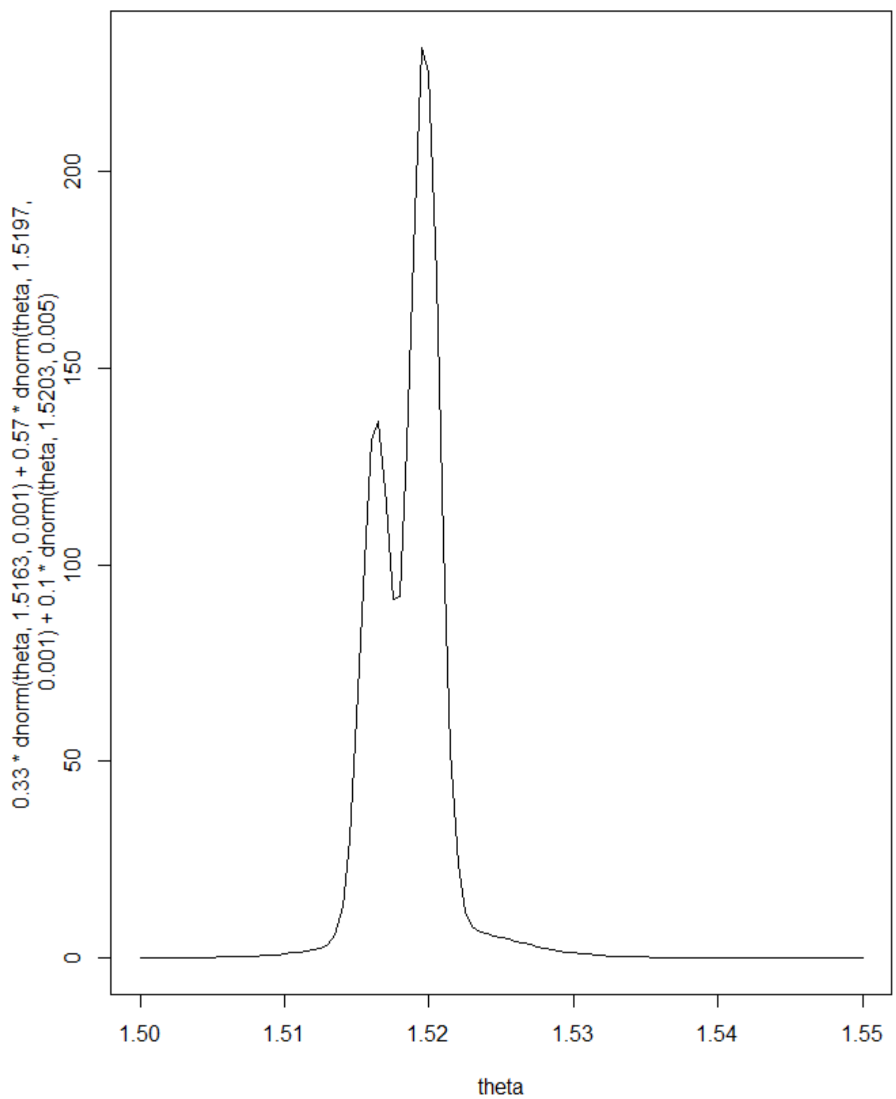

Figure 1: the density in a)

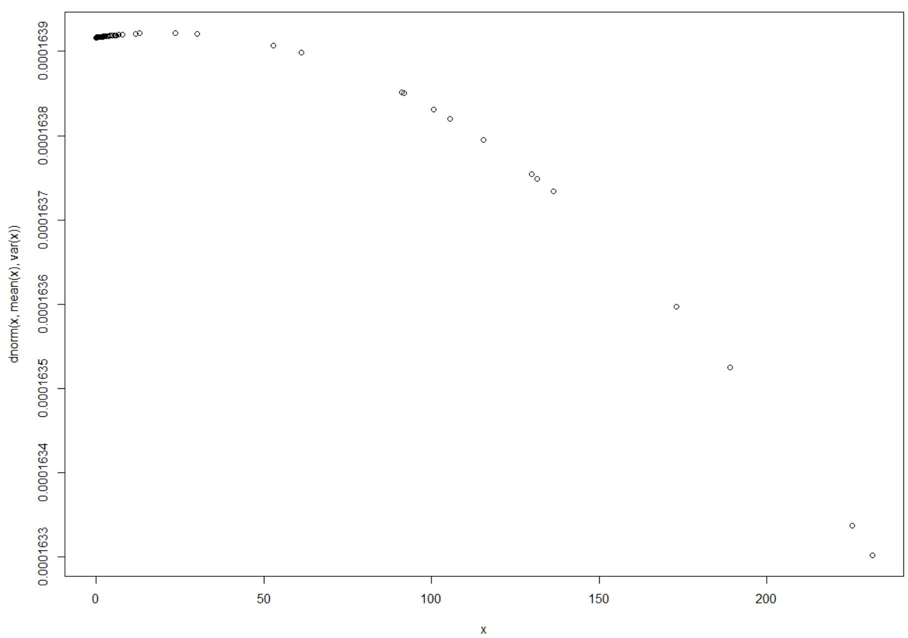
0.0001639
0.0001638
0.0001637
0.0001636
0.0001635
0.0001634
0.0001633
dnorm(x, mean(x), var(x))
0
50
100
150
200
x

Figure 2: b