Proof of the Analytical Solution for UVA 11150

1. Some students give examples to show the bottles of cola one drinks is $n+\lfloor n/2 \rfloor$. However, examples is not a proof.
2. Many students show the sum of geometrical series of $(1/3)^k$ to be 1/2 and conclude the bottles of cola one drinks is $n+\lfloor n/2 \rfloor$. However, $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$ does not give the reason why $n+\lfloor n/2 \rfloor$ is the solution.

Use mathematical induction to prove the solution.

Proof:

Let the number of bottles in n, where $n \geq 1$, at the beginning. Assume when n is even, we will borrow an empty bottle. We will show function $f(n)=n+\lfloor n/2 \rfloor$ is the bottles of cola one drinks. Also, at the end there is one empty bottle left.

Base Case: When n=1, the person drinks only 1 bottle of cola and there is one empty bottle left. Hence, we have $f(1)=1+\lfloor 1/2 \rfloor=1$. When n=2, the person drinks two bottles at the beginning, and then use the two empty bottles and one borrowed empty bottle to exchange another cola. When he/she drinks the exchanged cola, he/she drinks 3 bottles of cola in total and there is one empty bottle which is returned to the lender in the final. Hence, we have $f(2)=2+\lfloor 2/2 \rfloor=3$.

Induction Case: For the induction hypothesis, $\forall$ $k \geq 1$, assume $f(2k-1)=2k-1+\lfloor (2k-1)/2 \rfloor$, for the odd case, and $f(2k)=2k+\lfloor 2k/2 \rfloor$, for the even case. In addition, there is one empty bottle left he/she drinks all bottles of cola.

Suppose there are 2(k+1)-1 bottles of cola at the beginning, for the odd case, or 2(k+1), for the even case, at the beginning. Cola drinking will be carried in two steps as the following. For the odd case, process 2k-1 bottles at the step and then 2 bottles the second step, i.e., we will show $f(2(k+1)-1)= f(2k-1)+f(2)$. Recall that, after finishing f(2k-1), there is one empty bottle left.

$f(2k-1)+f(2) = 2k-1+\lfloor (2k-1)/2 \rfloor+3$
$= 2k+2-1+\lfloor (2k+2-1)/2 \rfloor$
$= 2(k+1)-1+\lfloor (2(k+1)-1)/2 \rfloor$
$= f(2(k+1)-1)$.

For the even case, process 2k bottles at the step and then 2 bottles the second step, i.e., we will show $f(2(k+1))= f(2k)+f(2)$. Recall that, after finishing f(2k), there is one empty bottle left.

$f(2k)+f(2) = 2k+\lfloor 2k/2 \rfloor+3$
$= 2k+2+\lfloor (2k+2)/2 \rfloor$
$= 2(k+1)+\lfloor 2(k+1)/2 \rfloor$
$= f(2(k+1))$.

Hence, we have proved, $\forall$ $n \geq 1$, if there are n bottles at the beginning, he/she will drinks $f(n)=n+\lfloor n/2 \rfloor$, finally. **Q.E.D.**