

Algorithms

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1 Linked List

We can use array to initialize the linked list as the following code. While building up the linked list from the given array, we use the **two-pointer technique** to maintain the linking between two nodes.

```

1  #Definition for singly-linked list.
2  class ListNode(object):
3      def __init__(self, x):
4          self.val = x
5          self.next = None
6
7
8  class List(object):
9      def __init__(self, array):
10         if(array):
11             self.head = ListNode(array[0])
12             prev=self.head
13             cur = None
14             for i in range(1,len(array)):
15                 cur=ListNode(array[i])
16                 prev.next=cur
17                 prev=cur
18         else:

```

```

19         self.head = None
20
21     def __str__(self):
22         if(not self.head):
23             return "[]"
24         else:
25             s="["
26             cur=self.head
27             while(cur):
28                 s=s+str(cur.val)+" "
29                 cur=cur.next
30             s=s.strip()+"]"
31             return s
32
33
34 array1=[1,4,5]
35 list1=List(array1)
36 print(list1)

```

2 Two Pointers Technique

2.1 Container With Most Water (LeetCode 11)

Given n non-negative integers a_1, a_2, \dots, a_n , where each represents a point at coordinate (i, a_i) . n vertical lines are drawn such that the two endpoints of line i is at (i, a_i) and $(i, 0)$. Find two lines, which together with x-axis forms a container, such that the container contains the most water.

Note: You may not slant the container and n is at least 2.

Example:

Input: [1,8,6,2,5,4,8,3,7]
Output: 49

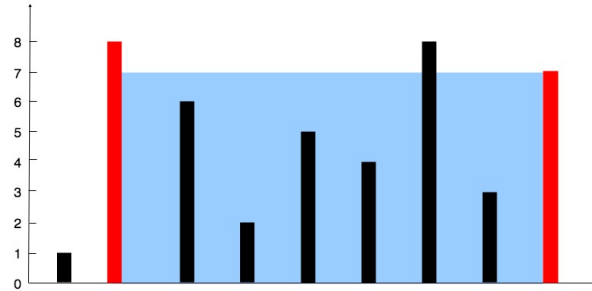


Figure 1: The above vertical lines are represented by array $[1, 8, 6, 2, 5, 4, 8, 3, 7]$. In this case, the max area of water (blue section) the container can contain is 49.

3 Sliding Window Technique

3.1 Count distinct elements in every window of size k

Tag: Sliding Window Technique, Hashtable. See ¹.

Input: `arr[] = {1, 2, 1, 3, 4, 2, 3}`, `k = 4`
 Output: `[3, 4, 4, 3]`

We use the sliding window to update a hashtable, which maintains the distinct elements. And the time complexity is $O(n)$.

```
class Solution():
    '''2018-12-21
    '''
    def distinct(self, nums, k):
        if(not nums or len(nums)<k):
            return []
        d=dict()
        res=[]
        #init the first window
        for j in range(0,k):
            if(nums[j] not in d):
                d[nums[j]]=1
            else:
                d[nums[j]]+=1
```

¹<https://www.geeksforgeeks.org/count-distinct-elements-in-every-window-of-size-k/>

```

res.append(len(d))

#update the remaining windows
for i in range(1,len(nums)-k+1):
    #remove the first in the window, and add the last
    #to the window.
    first=nums[i-1]
    last=nums[i+k-1]
    if(d[first]==1):
        d.pop(first)
    else:
        d[first]-=1

    if(last not in d):
        d[last]=1
    else:
        d[last]+=1

    res.append(len(d))

return res

def testAll(self):
    testcase1={"nums":[1, 2, 1, 3, 4, 2, 3],"k":4,"expected":[3,4,4,3]}
    testcase2={"nums":[1, 2, 1],"k":4,"expected":[]}
    testcase3={"nums":[1, 2, 1, 3, 4, 2, 3, 5],"k":4,"expected":[3,4,4,3,4]}
    testcases=[testcase1,testcase2,testcase3]
    for testcase in testcases:
        self.test(testcase["nums"],testcase["k"],testcase["expected"])

def test(self,nums,k,expected):
    res=self.distinct(nums,k)
    print("Test on nums={0}, k={1}. And {2} is expected, and {3} is got."\
        .format(nums,k,expected,res))

a=Solution()
a.testAll()

```

3.2 Sliding Window Maximum (Maximum of all subarrays of size k)

See ².

```
Input :
arr[] = {1, 2, 3, 1, 4, 5, 2, 3, 6}
k = 3
Output :
3 3 4 5 5 5 6

Input :
arr[] = {8, 5, 10, 7, 9, 4, 15, 12, 90, 13}
k = 4
Output :
10 10 10 15 15 90 90
```

We use the priority queue to .

```
class Solution():
    """2018-12-21
    """
    def maxSlidingWindow(self, nums, k):
        pass
```

4 Heap

Heap can be viewed as a complete tree, but stored as the array. Suppose the current node's index is idx , then the left child's index is $2 * idx + 1$, and the right child $2 * idx + 2$, while the parent $\text{floor}((idx - 1)/2)$.

We take the binary max heap as an example. The basic external function is **insert** and **extractMax**, which is implemented by **siftup** and **siftdown**. The **siftup** function check the current node's value with its parent's value, then swap them if the current node's value is bigger than the parent's, and do the check-swap operation recursively to meet the guarantee of the binary max heap.

The python source code is as following.

²<https://www.geeksforgeeks.org/sliding-window-maximum-maximum-of-all-subarrays-of-size-k/>

```

1 class MaxHeap():
2     '''
3     2018-12-24
4     The root is bigger than its left child and right child.
5     '''
6     def __init__(self):
7         self.array=[]
8
9     def insert(self,num):
10        if(not self.array):
11            self.array.append(num)
12        else:
13            self.array.append(num)
14            self.siftup(len(self.array)-1)
15
16
17    def siftup(self,idx):
18        if(idx==0):
19            return
20        parentIdx=(idx - 1) // 2
21        if(self.array[parentIdx]<self.array[idx]):
22            self.array[parentIdx], self.array[idx] = \
23                self.array[idx], self.array[parentIdx]
24            return self.siftup(parentIdx)
25
26    def extractMax(self):
27        #swap the head (max) and the last one in self.array,
28        ↪ then pop out the max
29        if(not self.array):
30            raise ValueError("pop out from an empty heap")
31
32        ↪ self.array[0],self.array[-1]=self.array[-1],self.array[0]
33        max=self.array.pop()
34        self.siftdown(0)
35        return max
36
37    def siftdown(self,idx):
38        '''

```

```

39         move the current node down
40         '''
41         if(not self.array or len(self.array)==1):
42             return
43         left=idx*2+1
44         right=idx*2+2
45         maxIdx=idx
46         if(left<len(self.array) and
47            ↪ self.array[maxIdx]<self.array[left]):
48             maxIdx=left
49         if(right<len(self.array) and
50            ↪ self.array[maxIdx]<self.array[right]):
51             maxIdx=right
52
53         self.array[idx], self.array[maxIdx] =
54            ↪ self.array[maxIdx], self.array[idx]
55         #sift down the smaller number recursively
56         if(idx!=maxIdx):
57             self.siftdown(maxIdx)
58
59     def __str__(self):
60         s=""
61         for i in range(len(self.array)):
62             if(i!=len(self.array)-1):
63                 s+=str(self.array[i])+ " "
64             else:
65                 s+=str(self.array[i])
66         return s
67
68 heap=MaxHeap()
69 for i in range(1,10):
70     heap.insert(i)
71 print("After inserting 1,2,3,4,5,6,7,8,9, the array of the
72 ↪ heap is {0}.".format(heap))
73
74 maxInHeap=heap.extractMax()
75 print("Pop out from the heap, we'll get the maximum number
76 ↪ {0}, "

```



```
74 |         "and the array of the heap becomes  
    |         ↪ {1}.".format(maxInHeap,heap))
```

4.1 Python's heapq

We can use the library **heapq** in python. Since the default **heapq** is the min heap, so we need a trick to reimplement **MaxHeap** by overriding the comparison function.

```
1 | import heapq
2 | '''2018-12-24
3 | Use python's heapq to implement a binary max heap.
4 | '''
5 |
6 | class MaxHeapObj(object):
7 |     def __init__(self, val):
8 |         self.val = val
9 |
10 |    def __lt__(self, other):
11 |        return self.val > other.val
12 |
13 |    def __eq__(self, other):
14 |        return self.val == other.val
15 |
16 |    def __str__(self):
17 |        return str(self.val)
18 |
19 |
20 | class MaxHeap(object):
21 |     def __init__(self):
22 |         self.h = []
23 |
24 |     def heappush(self,x):
25 |         heapq.heappush(self.h,MaxHeapObj(x))
26 |
27 |     def heappop(self):
28 |         return heapq.heappop(self.h).val
29 |
30 |     def __getitem__(self,i):
31 |         return self.h[i].val
32 |
```

```

33     def __str__(self):
34         s=""
35         for e in self.h:
36             s=s+str(e)+" "
37         return s
38
39
40 heap=MaxHeap()
41
42 for i in range(1,10):
43     heap.heappush(i)
44
45 print("After inserting 1,2,3,4,5,6,7,8,9, the array of the
↪ heap is {0}.".format(heap))
46
47 maxInHeap=heap.heappop()
48 print("Pop out from the heap, we'll get the maximum number
↪ {0}, "
49       "and the array of the heap becomes
↪ {1}.".format(maxInHeap,heap))

```

Or we can implement **MaxHeap** by multiplying -1 to each item in an array directly when using **heapq**.

```

1  import heapq
2  '''2018-12-24
3  Use python's heapq to implement a binary max heap.
4  Since heapq is the min heap, so we need to reimplement
5  MaxHeap by multiplying -1 to the item in min heap, and
6  multiply -1 as well when using heappop() function.
7  '''
8
9  class MaxHeap(object):
10     def __init__(self):
11         self.h = []
12
13     def heappush(self,x):
14         heapq.heappush(self.h,-x)
15
16     def heappop(self):
17         return heapq.heappop(self.h)*(-1)

```

```

18
19     def __getitem__(self, i):
20         return -1*self.h[i]
21
22     def __str__(self):
23         s=""
24         for i in range(len(self.h)):
25             s=s+str(self[i])+" "
26         return s
27
28
29 heap=MaxHeap()
30
31 for i in range(1,10):
32     heap.heappush(i)
33
34 print("After inserting 1,2,3,4,5,6,7,8,9, the array of the
    ↪ heap is {0}.".format(heap))
35
36 maxInHeap=heap.heappop()
37 print("Pop out from the heap, we'll get the maximum number
    ↪ {0}, "
38       "and the array of the heap becomes
    ↪ {1}.".format(maxInHeap,heap))

```

All the above three heap codes generate the following output.

```

After inserting 1,2,3,4,5,6,7,8,9, the array of the heap
↪ is 9 8 6 7 3 2 5 1 4 .
Pop out from the heap, we'll get the maximum number 9,
↪ and the array of the heap becomes 8 7 6 4 3 2 5 1 .

```

4.2 The application of heap

4.2.1 Merge k Sorted Lists (LeetCode 23)

Merge k sorted linked lists and return it as one sorted list. Analyze and describe its complexity³.

³<https://leetcode.com/problems/merge-k-sorted-lists/description/>

Input:

```
[
    1->4->5,
    1->3->4,
    2->6
]
```

Output: 1->1->2->3->4->4->5->6

```

1 import heapq
2
3 # Definition for singly-linked list.
4 class ListNode(object):
5     def __init__(self, x):
6         self.val = x
7         self.next = None
8
9 class List(object):
10     def __init__(self, array):
11         if array:
12             self.head = ListNode(array[0])
13             prev = self.head
14             cur = None
15             for i in range(1, len(array)):
16                 cur = ListNode(array[i])
17                 prev.next = cur
18                 prev = cur
19         else:
20             self.head = None
21
22 class Solution(object):
23     def mergeKLists(self, lists):
24         """
25         :type lists: List[ListNode]
26         :rtype: ListNode
27         """
28         heap = []
29         #init heap, which contains the tuple of head.val and
30         ↪ head of each list
31         for list in lists:

```

```

31         if(list):
32             heapq.heappush(heap, (list.val, list))
33
34         dummy=ListNode(0)
35         tail_new_list=dummy
36         #update
37         while(heap):
38             _,head=heapq.heappop(heap)
39             if(head.next):
40                 heapq.heappush(heap, (head.next.val, head.next))
41
42             tail_new_list.next=head
43             tail_new_list=tail_new_list.next
44         return dummy.next
45
46 def display(head):
47     s=""
48     cur=head
49     while(cur):
50         s+=str(cur.val)+" "
51         cur=cur.next
52     return s
53
54
55 array1=[1,4,5]
56 list1=List(array1)
57 print(list1)
58
59 array2=[1,3,4]
60 list2=List(array2)
61 print(list2)
62
63 array3=[2,6]
64 list3=List(array3)
65 print(list3)
66
67
68 solution=Solution()
69 mergedList=solution.mergeKLists([list1.head, list2.head, list3.head])
70 print(display(mergedList))

```

4.2.2 Kth Largest Element in a Stream (LeetCode 703)

Design a class to find the kth largest element in a stream. Note that it is the kth largest element in the sorted order, not the kth distinct element.

Your KthLargest class will have a constructor which accepts an integer k and an integer array nums, which contains initial elements from the stream. For each call to the method KthLargest.add, return the element representing the kth largest element in the stream.

Example:

```
int k = 3;
int[] arr = [4,5,8,2];
KthLargest kthLargest = new KthLargest(3, arr);
kthLargest.add(3); // returns 4
kthLargest.add(5); // returns 5
kthLargest.add(10); // returns 5
kthLargest.add(9); // returns 8
kthLargest.add(4); // returns 8
```

Note:

1. You may assume that nums' length $\geq k - 1$ and $k \geq 1$.

We use the min heap with the fixed size k to maintain the largest k elements in the stream. The minimum element in the min heap with size k will be the k-th largest element in a stream. The initialization of the heap is to heapify the given array. Since the time complexity of the operation of "heapify" is $O(n)$ ⁴, so the initialization is very efficient. And the add operation in the following code is to pop out the minimum element in the array which costs $O(\log k)$. In general, the time complexity is $O(n \log k)$, where n is the size of the stream, and the space complexity is $O(\log k)$.

```
1 import heapq
2
3 class KthLargest(object):
4     '''
5     2018-12-26
6     '''
7     def __init__(self, k, nums):
```

⁴<https://www.growingwiththeweb.com/data-structures/binary-heap/build-heap-proof/>

```

8         """
9         :type k: int
10        :type nums: List[int]
11        """
12        #self.array is the min heap
13        self.heap=nums
14        heapq.heapify(self.heap)
15        self.k=k
16        while(len(self.heap)>k):
17            heapq.heappop(self.heap)
18
19
20    def add(self, val):
21        """
22        :type val: int
23        :rtype: int
24        """
25        if(len(self.heap)<self.k):
26            heapq.heappush(self.heap, val)
27        else:
28            if(val>self.heap[0]):
29                #update min heap
30                heapq.heappop(self.heap)
31                heapq.heappush(self.heap, val)
32        return self.heap[0]
33
34    # Your KthLargest object will be instantiated and called as
35    ↪ such:
36    # obj = KthLargest(k, nums)
37    # param_1 = obj.add(val)

```

4.2.3 Top K Frequent Elements (LeetCode 347)

Given a non-empty array of integers, return the k most frequent elements.

Example 1:

Input: `nums = [1,1,1,2,2,3]`, `k = 2`
Output: `[1,2]`

Example 2:

Input: `nums = [1], k = 1`
Output: `[1]`

Note:

1. You may assume k is always valid, $1 \leq k \leq$ number of unique elements.
2. Your algorithm's time complexity must be better than $O(n \log n)$, where n is the array's size.

There are two solutions for this problem: **min heap** and **max heap**. As the first solution, we use the **min heap** with the fixed size k to maintain the top k frequent elements, which is inspired by the problem **Kth Largest Element in an Array**. Taking the counting frequent, heapifying the frequent-num array, and popping out the results into consideration, the time complexity is $O(n) + O(k + (n - k) \log k) + O(k \log k)$. So if $n \gg k$, the overall time complexity is $O(n \log k)$, but if $n \approx k$, the overall time complexity is $O(n \log n)$.

```
1 import heapq
2 import collections
3
4 class Solution(object):
5     '''
6     2018-12-27
7     min heap method
8     '''
9     def topKFrequent(self, nums, k):
10        """
11        :type nums: List[int]
12        :type k: int
13        :rtype: List[int]
14        """
15        #firstly, count the frequent.
16        #The time complexity is O(n).
17        numfreq=collections.defaultdict(int)
18        for num in nums:
19            numfreq[num]+=1
20
21        #secondly, use the min heap with
22        #the fixed size k to get the top k frequent
```



```

23         #numbers. The time complexity is O(n * log k)
24         heap=[]
25         for num in numfreq:
26             freq=numfreq[num]
27             if(len(heap)<k):
28                 heapq.heappush(heap, (freq,num))
29             else:
30                 if(freq>heap[0][0]):
31                     heapq.heappop(heap)
32                     heapq.heappush(heap, (freq,num))
33
34         #thirdly, output the result from the min heap.
35         #The time complexity is O(k * log k).
36         res=[]
37         while(heap):
38             num=heapq.heappop(heap)[1]
39             res.append(num)
40         res.reverse()
41         return res

```

As the second solution, we use the **max heap** to heapify the frequent-num tuples to a heap with the size n . Then we pop out the first k elements in the max heap, which will be the result. Similar as the analysis on the min heap method, taking the counting frequent, heapifying the frequent-num array, and popping out the results into consideration, the time complexity is $O(n) + O(n) + O(k \log n)$. So if $n \gg k$, the overall time complexity is $O(n + k \log n)$ ($\approx O(n)$), but if $n \approx k$, the overall time complexity is $O(n \log n)$.

```

1  import heapq
2  import collections
3
4  class Solution(object):
5      '''
6      2018-12-27
7      max heap method
8      '''
9      def topKFrequent(self, nums, k):
10         """
11         :type nums: List[int]

```

```

12         :type k: int
13         :rtype: List[int]
14         """
15         #firstly, count the frequent.
16         #The time complexity is O(n).
17         numfreq=collections.defaultdict(int)
18         for num in nums:
19             numfreq[num]+=1
20
21         #secondly, use the max heap with
22         #the size n to heapify the frequent-number tuples
23         #The time complexity is O(n)
24         heap=[(-1*freq,num) for num,freq in numfreq.items()]
25         heapq.heapify(heap)
26
27         #thirdly, pop out the top k frequent elements.
28         #The time complexity is O(k*log n)
29         res=[]
30         while(len(res)<k):
31             num=heapq.heappop(heap)[1]
32             res.append(num)
33         return res

```

5 Tree

5.1 Univalued Binary Tree (LeetCode 965)

A binary tree is univalued if every node in the tree has the same value.

Return true if and only if the given tree is univalued.

Example 1:

Input: [1,1,1,1,1,null,1]
Output: true

Example 2:

Input: [2,2,2,5,2]
Output: false

Note:

1. The number of nodes in the given tree will be in the range [1, 100].
2. Each node's value will be an integer in the range [0, 99].

It's an easy problem. We use the recursion to get the result.

```

1  # Definition for a binary tree node.
2  # class TreeNode(object):
3  #     def __init__(self, x):
4  #         self.val = x
5  #         self.left = None
6  #         self.right = None
7
8  class Solution(object):
9      def isUnivalTree(self, root):
10         """
11         :type root: TreeNode
12         :rtype: bool
13         """
14         if(not root):
15             return True
16
17         if(root.right!=None and root.right.val!=root.val):
18             return False
19
20         if(root.left!=None and root.left.val!=root.val):
21             return False
22
23         return self.isUnivalTree(root.right) and
           ↪ self.isUnivalTree(root.left)

```

6 Sorting

6.1 Sort Characters By Frequency (LeetCode 451)

Given a string, sort it in decreasing order based on the frequency of characters.

Example 1:

Input:

"tree"

Output:

"eert"

Explanation:

'e' appears twice while 'r' and 't' both appear once.
So 'e' must appear before both 'r' and 't'. Therefore
↪ "eetr" is also a valid answer.

Example 2:

Input:

"cccaa"

Output:

"cccaa"

Explanation:

Both 'c' and 'a' appear three times, so "aaacc" is
↪ also a valid answer.

Note that "cacaca" is incorrect, as the same
↪ characters must be together.

Example 3:

Input:

"Aabb"

Output:

"bbAa"

Explanation:

"bbaA" is also a valid answer, but "Aabb" is
↪ incorrect.

Note that 'A' and 'a' are treated as two different
↪ characters.

This problem is not difficult. But we still should pay attention to the potential TLE problem may caused by the combination of string in python. We cannot use "s=s+c" which time complexity is $O(n^2)$ and space complexity is also $O(n^2)$ since it needs deep copy for every adding operation.

```
1 import collections
2
3 class Solution(object):
4     '''
5     2018-12-28
6     '''
7     def frequencySort(self, s):
8         """
9         :type s: str
10        :rtype: str
11        """
12        # count
13        charFreq = collections.defaultdict(int)
14        for c in s:
15            charFreq[c] += 1
16
17        # sort
18        tuples = [(freq, c) for c, freq in charFreq.items()]
19        tuples.sort(reverse=True)
20
21        res = []
22        for freq, c in tuples:
23            res += [c]*freq
24
25        return "".join(res)
```

7 Dynamic Programming

7.1 Decode Ways (LeetCode 91)

A message containing letters from A-Z is being encoded to numbers using the following mapping:

```
'A' -> 1
'B' -> 2
...
'Z' -> 26
```

Given a non-empty string containing only digits, determine the total number of ways to decode it.

Example 1:

```
Input: "12"
Output: 2
Explanation: It could be decoded as "AB" (1 2) or "L"
             ↪ (12).
```

Example 2:

```
Input: "226"
Output: 3
Explanation: It could be decoded as "BZ" (2 26), "VF"
             ↪ (22 6), or "BBF" (2 2 6).
```

Firstly we use the Backtracking to search all the possible decoding ways, but failed to solve the problem with the **Time Limited Error**. The backtracking method is to decode the sequence like "1226..." in two ways: "1/226..." and "12/26...". Suppose the length of the input is n , the backtracking method's time complexity is $O(2^n)$ in worst case. A sequence of all one "1...1" or all two "2...2" are the worst cases.

```
1 class Solution(object):
2     '''
3     2018-12-29
4     Backtracking method, but TLE, the time complexity is
    ↪  $O(2^n)$  for
5     the worst case.
6     '''
7     def dfs(self,s,solution,res):
8         if(len(s)==0):
9             if(0<int(solution[-1])<=26):
10                res[0]+=1
```

```

11         elif(len(s)==1):
12             if(s[0]!="0"):
13                 self.dfs("",solution+[s[0]],res)
14         else:
15             if(s[0]!="0"):
16                 self.dfs(s[1:],solution+[s[0]],res)
17             if(1<=int(s[0:2])<=26):
18                 self.dfs(s[2:],solution+[s[0:2]],res)
19
20
21     def numDecodings(self, s):
22         """
23         :type s: str
24         :rtype: int
25         """
26         res=[0] #the number of decoding ways
27         solution=[]
28         self.dfs(s,solution,res)
29         return res[0]

```

Still consider the worst case " $1 \cdots 1$ ", which length is n . Suppose we have got the number of decoding ways for the subproblem " $1 \cdots 1(n-1 \text{ ones})$ " as $F(n-1)$ and the subproblem " $1 \cdots 1(n-2 \text{ ones})$ " as $F(n-2)$. Then the recursive formula is $F(n) = F(n-1) + F(n-2)$ for the worst case.

For the general cases " $c_1 c_2 c_3 \cdots c_{n-2} c_{n-1} c_n$ ", we should pay attention for the impossible decoding ways. Denote the number of decoding ways for the subproblem $c_1 \cdots c_{n-2}$ and the subproblem $c_1 \cdots c_{n-1}$. We discuss the recursive formula in different conditions as follows.

1. $c_{n-1} = 0$ and $c_n = 0$. The decoding way $c_1 c_2 c_3 \cdots c_{n-2} 0|0$ is not available, so $F(n) = 0$.
2. $c_{n-1} = 0$ and $c_n \neq 0$. The decoding way $c_1 c_2 c_3 \cdots c_{n-2} 0|c_n$ not available, but $c_1 c_2 c_3 \cdots c_{n-2} 0|c_n$ is available, so $F(n) = F(n-1)$.
3. $c_{n-1} = 1$ or 2 , and $c_n = 0$. The decoding way is $c_1 c_2 c_3 \cdots c_{n-2} |c_{n-1} 0$, so $F(n) = F(n-2)$.
4. $c_{n-1} > 2$, and $c_n = 0$. The decoding way is not available, so $F(n) = 0$.
5. $c_{n-1} \neq 0$ and $c_n \neq 0$, and $\text{int}(c_{n-1} c_n) \leq 26$. Both two decoding ways are available, so $F(n) = F(n-1) + F(n-2)$.

6. $c_{n-1} \neq 0$ and $c_n \neq 0$, and $\text{int}(c_{n-1}c_n) > 26$. Only the decoding way $c_1c_2c_3 \cdots c_{n-2}c_{n-1}|c_n$ can be applied, so $F(n) = F(n-1)$.

```

1 class Solution(object):
2     '''
3     2018-12-29
4     tabulation dynamic programming
5     '''
6     def numDecodings(self, s):
7         """
8         :type s: str
9         :rtype: int
10        """
11        F=[None]*(len(s)+1)
12        F[0]=1
13        if(int(s[0])==0):
14            F[1]=0
15        else:
16            F[1]=1
17
18        for i in range(1,len(s)):
19            currentDigit=int(s[i])
20            previousDigit=int(s[i-1])
21            if(previousDigit==0):
22                if(currentDigit==0):
23                    #...00
24                    F[i+1]=0
25                else:
26                    #...0/2
27                    F[i+1]=F[i]
28            else:
29                if(currentDigit==0):
30                    if (previousDigit==1 or previousDigit==2):
31                        #.../20
32                        F[i+1]=F[i-1]
33                    else:
34                        #.../30
35                        F[i+1]=0
36                else:
37                    tailTwoDigits=int(s[i-1:i+1])

```



```

38         if(tailTwoDigits<=26):
39             #.../12 or ...1/2
40             F[i+1]=F[i-1]+F[i]
41         else:
42             #...3/2
43             F[i+1]=F[i]
44     return F[len(s)]

```

7.2 Decode Ways II (LeetCode 639)

A message containing letters from A-Z is being encoded to numbers using the following mapping way:

```

'A' -> 1
'B' -> 2
...
'Z' -> 26

```

Beyond that, now the encoded string can also contain the character '*', which can be treated as one of the numbers from 1 to 9.

Given the encoded message containing digits and the character '*', return the total number of ways to decode it.

Also, since the answer may be very large, you should return the output mod $10^9 + 7$.

Example 1:

```

Input: "*"
Output: 9
Explanation: The encoded message can be decoded to
↳ the string: "A", "B", "C", "D", "E", "F", "G",
↳ "H", "I".

```

Example 2:

```

Input: "1*"
Output: 9 + 9 = 18

```

Note:

1. The length of the input string will fit in range $[1, 10^5]$.

2. The input string will only contain the character '*' and digits '0' - '9'.

We start the analysis from the worst case "***...(n times)", and from its simplest situation. "*" represents 9 digits. And in "***", if the first star represents 1 or 2, then there are 32(=2*9+8+6) decoding ways; if the first star represents 3 to 9, then there are 63(=7*9) decoding ways. And in "***".

TODO

1. $c_{n-1} = 0$ and $c_n = 0$. The decoding way $c_1c_2c_3 \cdots c_{n-2}0|0$ is not available, so $F(n) = 0$.
2. $c_{n-1} = 0$ and $c_n \neq 0$. The decoding way $c_1c_2c_3 \cdots c_{n-2}0|c_n$ not available, but $c_1c_2c_3 \cdots c_{n-2}0|c_n$ is available, so $F(n) = F(n-1)$.
3. $c_{n-1} = 1$ or 2 , and $c_n = 0$. The decoding way is $c_1c_2c_3 \cdots c_{n-2}|c_{n-1}0$, so $F(n) = F(n-2)$.
4. $c_{n-1} > 2$, and $c_n = 0$. The decoding way is not available, so $F(n) = 0$.
5. $c_{n-1} \neq 0$ and $c_n \neq 0$, and $\text{int}(c_{n-1}c_n) \leq 26$. Both two decoding ways are available, so $F(n) = F(n-1) + F(n-2)$.
6. $c_{n-1} \neq 0$ and $c_n \neq 0$, and $\text{int}(c_{n-1}c_n) > 26$. Only the decoding way $c_1c_2c_3 \cdots c_{n-2}c_{n-1}|c_n$ can be applied, so $F(n) = F(n-1)$.

8 Prefix Sum

8.1 What's Prefix Sum

Given an array `arr[]` of size `n`, its prefix sum array `prefixSum[]` is the same size array such that `prefixSum[i]=arr[0]+arr[1]+arr[2]+...+arr[i]`.

8.2 Find if there is a subarray with 0 sum

Input: {4, 2, -3, 1, 6}
Output: **true**
There is a subarray with zero sum from index 1 to 3.

Input: {4, 2, 0, 1, 6}
Output: **true**
There is a subarray with zero sum from index 2 to 2.

Input: {-3, 2, 3, 1, 6}
Output: **false**
There is no subarray with zero sum.

Solution: Use Prefix Sum method and HashMap used in Two Sum problem.

```
1 class Solution(object):
2     def findSubarray(self, nums):
3         prefixSum = [0] * len(nums)
4         # sum(nums[i+1:j+1]) = prefixSum[j] - prefixSum[i]
5         # sum(nums[i+1:j+1]) = 0 is equal to find two index i
6         # such that prefixSum[i] = prefixSum[j] and i != j
7         prefixSum[0] = nums[0]
8         d = {}
9         d[prefixSum[0]] = 0
10        for i in range(1, len(nums)):
11            prefixSum[i] = prefixSum[i-1] + nums[i]
12            if prefixSum[i] in d:
13                return True
14            else:
15                d[prefixSum[i]] = i
16        return False
17
18
19 a = Solution()
20 print("There is a subarray in %s such that its sum is 0,"
21       " True or False? %s" % ([4, 2, -3, 1, 6],
22                                a.findSubarray([4, 2, -3, 1, 6])))
```

```

23 print("There is a subarray in %s such that its sum is 0,"
24       " True or False? %s" % ([4, 2, 0, 1, 6],
25                               a.findSubarray([4, 2, 0, 1, 6])))
26 print("There is a subarray in %s such that its sum is 0,"
27       " True or False? %s" % ([-3, 2, 3, 1, 6],
28                               a.findSubarray([-3, 2, 3, 1, 6])))

```

The time complexity is $O(n)$. The output of the running above code is as follows. Example:

```

There are subarray in [4, 2, -3, 1, 6] such that its
↪ sum is 0, True or False? True
There are subarray in [4, 2, 0, 1, 6] such that its
↪ sum is 0, True or False? True
There are subarray in [-3, 2, 3, 1, 6] such that its
↪ sum is 0, True or False? False

```

8.3 Find the length of largest subarray with 0 sum

Given an array of integers, find length of the largest subarray with sum equals to 0.

```

Input: arr[] = {15, -2, 2, -8, 1, 7, 10, 23};
Output: 5
The largest subarray with 0 sum is -2, 2, -8, 1, 7

Input: arr[] = {1, 2, 3}
Output: 0
There is no subarray with 0 sum

Input: arr[] = {1, 0, 3}
Output: 1

```

```

1 class Solution(object):
2     def findLongestSubarraySum0(self, nums):
3         prefixSum = [0] * len(nums)
4         # sum(nums[i+1:j+1]) = prefixSum[j] - prefixSum[i]
5         # sum(nums[i+1:j+1]) = 0 is equal to find two index i
           ↪ and j

```

```

6      # such that prefixSum[i]=prefixSum[j] and i!=j
7      prefixSum[0]=nums[0]
8      d={}
9      d[prefixSum[0]]=0
10
11     maxlen=0
12     for i in range(1,len(nums)):
13         prefixSum[i]=prefixSum[i-1]+nums[i]
14         if(prefixSum[i] in d):
15             if(i-d[prefixSum[i]]>maxlen):
16                 maxlen=i-d[prefixSum[i]]
17         else:
18             d[prefixSum[i]]=i
19     return maxlen
20
21
22 a=Solution()
23 print("Length of the longest 0 sum subarray in %s is %s"
24       %([-3, 2, 3, 1, 6],
25         a.findSubarray([-3, 2, 3, 1, 6])))
26 print("Length of the longest 0 sum subarray in %s is %s"
27       %([15, -2, 2, -8, 1, 7, 10, 13],
28         a.findSubarray([15, -2, 2, -8, 1, 7, 10, 13])))

```

The time complexity is $O(n)$. The output of the running above code is as follows. Example:

```

Length of the longest 0 sum subarray in [-3, 2, 3, 1,
↪ 6] is 0.
Length of the longest 0 sum subarray in [15, -2, 2,
↪ -8, 1, 7, 10, 13] is 5.

```

8.4 Subarray Sum Equals K (LeetCode 560)

Given an array of integers and an integer k, you need to find the total number of continuous subarrays whose sum equals to k.

Example 1:

```
Input: nums = [1,1,1], k = 2
Output: 2
```

1. The length of the array is in range [1, 20,000].
2. The range of numbers in the array is [-1000, 1000] and the range of the integer k is [-1e7, 1e7].

```
1 class Solution(object):
2     def subarraySum(self, nums, k):
3         """
4         :type nums: List[int]
5         :type k: int
6         :rtype: int
7         """
8         presum=0 #presum(i)=sum_{l=0}^{i-1}nums[i]
9
10        # key is presum, value is the number of current
11        ↪ indices making
12        # the presum.
13        d = {0:1}
14
15        res=0
16        for num in nums:
17            presum+=num
18            if(presum-k in d):
19                res+=d[presum-k]
20            #update d
21            if(presum not in d):
22                d[presum]=1
23            else:
24                d[presum]+=1
25
26        return res
```

8.5 Subarray Sums Divisible by K (LeetCode 974)

Given an array A of integers, return the number of (contiguous, non-empty) subarrays that have a sum divisible by K.

Example 1:

Input: $A = [4, 5, 0, -2, -3, 1]$, $K = 5$
Output: 7
Explanation: There are 7 subarrays with a sum
↪ divisible by $K = 5$:
 $[4, 5, 0, -2, -3, 1]$, $[5]$, $[5, 0]$, $[5, 0, -2, -3]$,
↪ $[0]$, $[0, -2, -3]$, $[-2, -3]$

Note:

1. $1 \leq A.length \leq 30000$
2. $-10000 \leq A[i] \leq 10000$
3. $2 \leq K \leq 10000$

```
1 class Solution(object):
2     def subarraysDivByK(self, A, K):
3         """
4         :type A: List[int]
5         :type K: int
6         :rtype: int
7         """
8         #presum(i)=sum_{l=0}^{i-1}A[l]
9         presum=0
10
11         #In d, the key is the presum, and the
12         #value is the number of indices making the presum
13         d={0:1}
14
15         res=0
16         for num in A:
17             presum=(presum+num)%K
18
19             ↪ #sum(j)-sum(i)=sum_{l=0}^{j-1}A[l]-sum_{l=0}^{i-1}A[l]
20             #=sum_{l=i}^{j-1}A[l]
21             if((presum-K)%K in d):
22                 res+=d[(presum-K)%K]
23
```

```

24         #update d
25         if(presum not in d):
26             d[presum]=1
27         else:
28             d[presum]+=1
29
30     return res

```

Or use the second solution as follows.

```

1  import collections
2
3  class Solution(object):
4      def subarraysDivByK(self, A, K):
5          """
6              :type A: List[int]
7              :type K: int
8              :rtype: int
9          """
10         presum=[0]
11
12         for num in A:
13             presum.append((presum[-1]+num)%K)
14
15         counter=collections.Counter(presum)
16         return sum(v*(v-1)//2 for v in counter.values())

```

9 Maths

9.1 Prime numbers

9.1.1 Count Primes (LeetCode 204)

Count the number of prime numbers **less than** a non-negative number, n.
Example:

```

Input: 10
Output: 4
Explanation: There are 4 prime numbers less than 10,
↳ they are 2, 3, 5, 7.

```


Tag: Primes.

We use the **Sieve of Eratosthenes** to label each number within the array of $[1, \dots, n]$ is a prime or not.

```
1 import math
2
3 class Solution(object):
4     '''2018-12-15
5     This is an  $O(n \log n)$  solution.
6     '''
7     def countPrimes(self, n):
8         """
9         :type n: int
10        :rtype: int
11        """
12        if(n<=1):
13            return 0
14        #primeFlags[i]=True means the number i is a prime
15        ↪ number.
16        primeFlags=[True]*(n+1)
17        primeFlags[0]=False
18        primeFlags[1]=False
19
20        for p in range(2,int(math.sqrt(n))+1):
21            if(primeFlags[p]==True):
22                for multiplier in range(p,n//p+1):
23                    primeFlags[p*multiplier]=False
24        return sum(primeFlags[:-1])
25
26    def test(self):
27        print("n={0}, output={1},
28        ↪ expected={2}".format(5,self.countPrimes(5),2))
29        print("n={0}, output={1},
30        ↪ expected={2}".format(10,self.countPrimes(10),4))
31
32 a=Solution()
33 a.test()
```

The line 18 to line 19 in the code can be optimized as the following code without the time consumption on the loop.

```
primeFlags[p*p:n:p] =  
    ↪ [False]*len(primeFlags[p*p:n:p])
```

By eliminating the inner loop, the time consumption is reduced from 860ms to 304ms.

9.1.2 Ugly Number (LeetCode 263)

Write a program to check whether a given number is an ugly number.

Ugly numbers are positive numbers whose prime factors only include 2, 3, 5.

Example 1:

```
Input: 6  
Output: true  
Explanation: 6 = 2 * 3
```

Example 2:

```
Input: 8  
Output: true  
Explanation: 8 = 2 * 2 * 2
```

Example 3:

```
Input: 14  
Output: false  
Explanation: 14 is not ugly since it includes another  
    ↪ prime factor 7.
```

Note:

1. 1 is typically treated as an ugly number.
2. Input is within the 32-bit signed integer range: $[-2^{31}, 2^{31} - 1]$.

We use the while loop to do the check and the decomposition for a given number. Since the given number is within the range $[-2^{31}, 2^{31} - 1]$, so we can do the check and the decomposition by recursion without worrying about the stack overflow (exceeding the maximum recursion depth).

```

1 class Solution(object):
2     '''2018-12-25
3     '''
4     def isUgly(self, num):
5         """
6         :type num: int
7         :rtype: bool
8         """
9         if (num<1):
10             return False
11
12         if (num == 1):
13             return True
14
15         while (num > 1):
16             ugly = False
17             for p in [2, 3, 5]:
18                 if (num % p == 0):
19                     num = num / p
20                     ugly = True
21                     break
22             if (ugly == False):
23                 return False
24         return True
25
26
27 a=Solution()
28 print(a.isUgly(2147483648))

```

The recursion version is as following.

```

1 class Solution(object):
2     '''2018-12-25
3     '''
4     def isUgly(self, num):
5         """
6         :type num: int
7         :rtype: bool
8         """
9         def helper(num):
10             if(num<=0):

```

```

11         return False
12     if(num==1):
13         return True
14
15     if(num%2==0):
16         return helper(num//2)
17     if(num%3==0):
18         return helper(num//3)
19     if(num%5==0):
20         return helper(num//5)
21     return False
22
23     return helper(num)

```

9.1.3 Ugly Number II (LeetCode 264)

Write a program to find the n-th ugly number. Ugly numbers are positive numbers whose prime factors only include 2, 3, 5. Example:

Input: `n = 10`
Output: `12`
Explanation: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 is the
↪ sequence of the first 10 ugly numbers.

Note:

1. 1 is typically treated as an ugly number.
2. n does not exceed 1690.

Tag: Maths, Primes, Tricky.

This problem's solution is very very tricky. Although the tag on this problem includes dynamic programming. But I don't think it's a good example of the dynamic programming technique, because it can not get a clear **recursive formula**. Instead, I would rather call it a tricky solution only using a tabulation.

Suppose the resulted ugly number list is F . Since F includes the numbers whose factors only include 2, 3, and 5. So we build up three lists $l_2 = 2 * F$, $l_3 = 3 * F$, and $l_5 = 5 * F$. Therefore the ugly list F has the property that $F = [1] + \text{merge}(l_2, l_3, l_5)$. Based on this property, what we should do is merging l_2 , l_3 , and l_5 . The **tricky part** is that F should be merged from

l_2 , l_3 , and l_5 , while these three lists are also need to be built from F . We handle it by updating them simultaneously as the following code.

```

1 class Solution(object):
2     '''
3     2018-12-25
4     '''
5     def nthUglyNumber(self, n):
6         """
7         :type n: int
8         :rtype: int
9         """
10        F=[0]*(n+1)#F is the list of ugly numbers
11        F[0]=0
12        F[1]=1
13        l2=[1*2]#l2 is the list of 2*F
14        l3=[1*3]#l3 is the list of 3*F
15        l5=[1*5]#l5 is the list of 5*F
16
17        cur2=0
18        cur3=0
19        cur5=0
20        for i in range(2,n+1):
21            #update F, since F is the merge of l2, l3, and l5
22            ↪ except for 0, and 1.
23            #So we apply the merging method to update F.
24            #And l2, l3, and l5 is based on F, so the update
25            ↪ is simultaneously,
26            #very tricky.
27            F[i]=min(l2[cur2],l3[cur3],l5[cur5])
28
29            #update l2, l3, and l5
30            l2.append(F[i] * 2)
31            l3.append(F[i] * 3)
32            l5.append(F[i] * 5)
33
34            #update the pointers to l2, l3, and l5
35            if(F[i]==l2[cur2]):
36                cur2+=1
37            if(F[i]==l3[cur3]):

```

```

36         cur3+=1
37         if(F[i]==l5[cur5]):
38             cur5+=1
39     return F[n]

```

If we want to save the space of l_2 , l_3 , and l_5 , then we'll do not claim space for them but use the space of F only by maintaining the pointers in these three lists. The code is as following.

```

1  class Solution(object):
2      '''
3      2018-12-25
4      '''
5      def nthUglyNumber(self, n):
6          """
7          :type n: int
8          :rtype: int
9          """
10         F=[0]*n
11         F[0]=1
12         n2=2 #the value of current node in l2
13         n3=3 #the value of current node in l3
14         n5=5 #the value of current node in l5
15
16         cur2=0 #the pointer to the current node in l2
17         cur3=0 #the pointer to the current node in l3
18         cur5=0 #the pointer to the current node in l5
19         for i in range(1,n):
20             #update F
21             F[i]=min(n2,n3,n5)
22
23             #update the pointers to l2, l3, and l5
24             if(F[i]==n2):
25                 cur2+=1
26                 n2=F[cur2]*2
27             if(F[i]==n3):
28                 cur3+=1
29                 n3=F[cur3]*3
30             if(F[i]==n5):
31                 cur5+=1

```

```

32         n5=F[cur5]*5
33     return F[-1]

```

9.1.4 Super Ugly Number (LeetCode 313)

Write a program to find the nth super ugly number.

Super ugly numbers are positive numbers whose all prime factors are in the given prime list primes of size k.

Example:

```

Input: n = 12, primes = [2,7,13,19]
Output: 32
Explanation: [1,2,4,7,8,13,14,16,19,26,28,32] is the
↪ sequence of the first 12
super ugly numbers given primes =
↪ [2,7,13,19] of size 4.

```

Note:

1. 1 is a super ugly number for any given primes.
2. The given numbers in primes are in ascending order.
3. $0 < k \leq 100$, $0 < n \leq 10^6$, $0 < \text{primes}[i] < 1000$.
4. The nth super ugly number is guaranteed to fit in a 32-bit signed integer.

Our solution is treat this problem as the extension of the problem **Ugly Number II**. This method's time complexity is $O(nk)$, where k is the length of the array primes. But this method is not optimized, which should be speeded up to $O(n \log k)$. Think about the problem **Merging k Sorted Lists**, which is optimized by using the data structure heap.

```

1 class Solution(object):
2     '''
3     2018-12-25
4     '''
5     def nthSuperUglyNumber(self, n, primes):
6         """
7         :type n: int
8         :type primes: List[int]

```

```

9      :rtype: int
10     """
11     F=[1]*n
12     pointers=[0]*len(primes)
13     for i in range(1,n):
14         #merge the lists
15         F[i]=min([primes[j]*F[pointers[j]] for j in
16                  ↪ range(len(primes))])
17         #update pointers
18         for j in range(len(primes)):
19             if(F[i]==primes[j]*F[pointers[j]]):
20                 pointers[j]+=1
21     return F[-1]

```

9.2 Permutation

9.2.1 Next Permutation (LeetCode 31)

Implement next permutation, which rearranges numbers into the lexicographically next greater permutation of numbers.

If such arrangement is not possible, it must rearrange it as the lowest possible order (ie, sorted in ascending order).

The replacement must be in-place and use only constant extra memory.

Here are some examples. Inputs are in the left-hand column and its corresponding outputs are in the right-hand column.

1,2,3	->	1,3,2
3,2,1	->	1,2,3
1,1,5	->	1,5,1