Assignment 8

1. Show that the sliced score matching (SSM) loss can also be written as

$$L_{SSM} = \mathbb{E}_{x\sim p(x)}\mathbb{E}_{v\sim p(v)}\left[\|v^T S(x;\theta)\|^2 + 2v^T\nabla_x(v^T S(x;\theta))\right].$$

Let $v \in \mathbb{R}^d$ be a random vector s.t. $E_{v\sim p(v)}[vv^T] = I$,

then for any $a \in \mathbb{R}^d$, $E_{v\sim p(v)}[\|v^T a\|^2] = E_{v\sim p(v)}[(v^T a)^2]$

$$= a^T E_{v\sim p(v)}[vv^T]\, a = a^T I\, a = \|a\|^2 .$$

Let $a = S(x;\theta)$, we obtain $\|S(x;\theta)\|^2 = E_{v\sim p(v)}[\|v^T S(x;\theta)\|^2]$

Thus $L_{SSM}(\theta) = E_{x\sim p(x)}\|S(x;\theta)\|^2 + E_{x\sim p(x)} E_{v\sim p(v)}[2v^T \nabla (v^T S(x;\theta))]$

$$= E_{x\sim p(x)} E_{v\sim p(v)}[\|v^T S(x;\theta)\|^2 + 2v^T \nabla_x (v^T S(x;\theta))]$$

2. Briefly explain SDE.

stochastic differential equation (SDE)

An SDE describes a system that is influenced by both deterministic and random factor.

Definition: An Ito SDE in $\mathbb{R}^d$ is $\underset{\text{stochastic process}}{dX_t} = \underbrace{\overset{\text{(deterministic)}}{f(x,t)}}_{\text{drift}}\, dt + \underbrace{\overset{\text{(stochastic)}}{G(x,t)}}_{\text{diffusion}}\, \underset{\text{Brownian motion}}{dW_t}$,

$X(0) = X_0$.

Integral form: $X_t = X_0 + \int_0^t f(X_s, s)\, ds + \int_0^t G(X_s, s)\, dW_s$

Brownian motion: $W_t \in \mathbb{R}^d$ is a continuous stochastic process s.t.

1. $W_0 = 0$

2. $W_{t+u} - W_t \sim N(0, uI)$

3. $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$ are indep.

4. $t \mapsto W_t$ are continuous functions of t.

Properties: 1. $E[W(t)] = 0$, $Var[W(t)] = t$.

2. W_t is nowhere differentiable (with prob. 1)

White noise: $h(t) \in \mathbb{R}^d$ is a random function with:

1. $h(t)$ and $h(t')$ are indep. if $t \neq t'$

2. $E[h(t)] = 0$, $E[h(t) h^T(s)] = \delta(t-s) I$.

Euler-maruyama method: For small time step Δt, $t_k = k\Delta t$.

Set $X_0 = x_0$. $X_{n+1} = X_n + f(X_n, t_n)\Delta t + G(X_n, t_n)\Delta W(t_n)$

3. Unanswered Questions

There are unanswered questions from the lecture, and there are likely more questions we haven't covered.

In the equation $dx_t = f(x_t, t)\,dt + G(x_t, t)\,dW_t$, W_t is a Brownion motion that nowhere differentiable. So what is dW_t from a mathematical point of view?