Econ 700 Problem Set 2 (Solution)

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1 Bad Chemistry

Consider the generic coordination problem given by

$$\begin{aligned} u &= \alpha + \beta a + \gamma A + \delta a A - \lambda a^2 \\ U &= \alpha + \beta A + \gamma a + \delta a A - \lambda A^2 \end{aligned} \tag{1}$$

where a and A are the strategies of the two players respectively, $\alpha, \beta, \delta, \lambda$ are constant with $\beta > 0$, $\lambda > 0$ and $\delta < \lambda$.

1. Find the Nash equilibrium (i.e., a^* and A^*), and give conditions under which the external effect is positive or negative and the two strategies a and A are substitutes or complements.

2. What is the first order condition for a symmetric Pareto-efficient allocation? Using this first order condition and your expression for the Nash equilibrium above to show that a^* and A^* exceed the Pareto-efficient levels if and only if the external effect is negative. Explain why this is so.

3. Assuming that the Nash equilibrium is in pure strategies, show that there will always be a first mover advantage, and that the second mover will do worse (than in the Nash equilibrium) if strategies are substitutes and better if strategies are complements. Explain why this is so.

4. Two adjacent farmers (Lower and Upper) choose whether to use a chemical intensive anti-pest strategy or a less chemical-intensive approach that uses natural predators to control the pests which threaten their crops (integrated pest management or IPM). The use of chemicals generates negative external effects (the chemicals kill the natural predators as well), while IPM generates positive external effects (the natural predators do not respect the farmer's property boundaries and prey on the pests throughout the area). Specifically, increased use of chemicals by one raises output of the user and lowers the output and raises the marginal productivity of chemical use in the other farm for any given level of other inputs. Letting a and A be the level of chemical use by the two, give the values of the parameters of the above utility functions that describe this interaction.

Answer.

1. The first order condition to maximize u is [1]

$$\frac{\partial u}{\partial a} = \beta + \delta A - 2\lambda a = 0 \Rightarrow a = \frac{\beta + \delta A}{2\lambda}$$

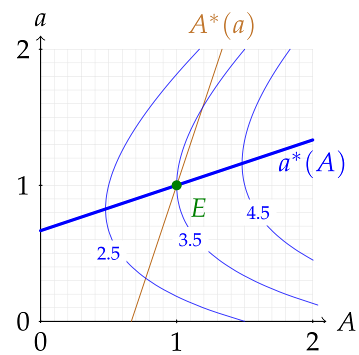
Then the Nash equilibrium is

$$a^* = A^* = \frac{\beta}{2\lambda - \delta}$$

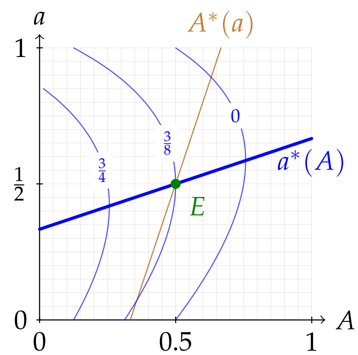
At Nash equilibrium (a^*, A^*), we have the external effect

$$\frac{\partial U}{\partial a} = \frac{\partial u}{\partial A} = \gamma + \delta a^* = \gamma + \frac{\delta\beta}{2\lambda - \delta} = \frac{2\gamma\lambda - \gamma\delta + \delta\beta}{2\lambda - \delta}$$

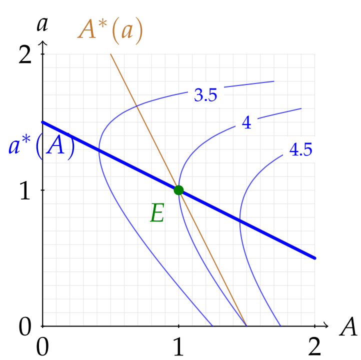
By $\lambda > 0$ and $\delta < \lambda$ we have $2\lambda - \delta > 0$. Therefore, if



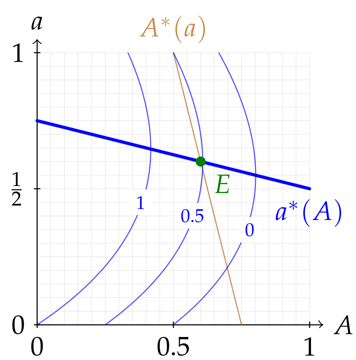
(a) Positive external effect and strategies complements with $u = 1 + 2a + A + aA - 1.5a^2$



(b) Negative external effect and strategies complements with $u = 1 + a - 2A + aA - 1.5a^2$



(c) Positive external effect and strategies substitutes with $u = 1 + 3a + 2A - aA - a^2$



(d) Negative external effect and strategies substitutes with $u = 1 + 3a - 2A - aA - 2a^2$

Figure 1: The indifference curves and best response function with

$$u = \alpha + \beta a + \gamma A + \delta a A - \lambda a^2$$

Figure (a) and (b) show strategies complements with

$$u_{aA} = \delta > 0$$

while (c) and (d) show strategies substitutes with

$$u_{aA} = \delta < 0.$$

In (a) and (c) we have positive external effect by

$$u_A > 0$$

while in (b) and (d) we have negative external effect by

$$u_A < 0.$$

By symmetry, we have the best response function of the Upper $A^*(a)$. In all the cases, the intersection of $A^*(a)$ and $a^*(A)$ is the Nash equilibrium E,

$$a^* = A^* = \frac{\beta}{2\lambda - \delta}$$

$$2\gamma\lambda - \gamma\delta + \delta\beta > 0 \Leftrightarrow \frac{\partial U}{\partial a} = \frac{\partial u}{\partial A} > 0 \tag{2}$$

the external effect is positive, and vice versa. From

$$\frac{\partial^2 u}{\partial A \partial a} = \frac{\partial^2 U}{\partial a \partial A} = \delta$$

[1] Similarly, we have

$$A = \frac{\beta + \delta a}{2\lambda}$$

we have a and A are strategic complements if $\delta > 0$, and strategic substitutes if $\delta < 0$, as shown in Figure 1.

2. Given any symmetric allocation $(a^*, A^*) = (x, x)$, it is Pareto efficient if and only if it maximizes

$$u = \alpha + (\beta + \gamma)x + (\delta - \lambda)x^2$$

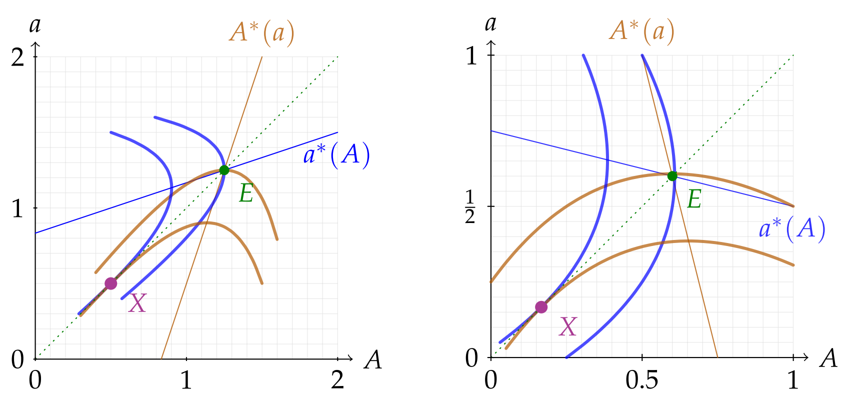
The first order condition is

$$\beta + \gamma + 2(\delta - \lambda)x = 0 \Rightarrow x^* = \frac{\beta + \gamma}{2(\lambda - \delta)}$$

Therefore,

$$\begin{aligned} a^* - x^* &= \frac{\beta}{2\lambda - \delta} - \frac{\beta + \gamma}{2(\lambda - \delta)} \\ &= -\frac{2\gamma\lambda - \gamma\delta + \delta\beta}{2(\lambda - \delta)(2\lambda - \delta)} > 0 \end{aligned}$$

if and only if $2\gamma\lambda - \gamma\delta + \delta\beta < 0$, i.e., the external effect is negative by (2). This is so because by negative external effect, i.e., $\frac{\partial U}{\partial a}, \frac{\partial u}{\partial A} < 0$, at Nash equilibrium there exists a common decrease in the action that will benefit both players, as shown in Figure 2.



(a) $u = 1 + 2.5a - 2A + aA - 1.5a^2$

(b) $u = 1 + 3a - 2A - aA - 2a^2$

Figure 2: Given the utility function

$$u = \alpha + \beta a + \gamma A + \delta a A - \lambda a^2$$

and

$$U = \alpha + \beta A + \gamma a + \delta A a - \lambda A^2$$

The Nash equilibrium levels

$$a^* = A^* = \frac{\beta}{2\lambda - \delta}$$

exceed the symmetric Pareto-efficient action levels

$$x^* = \frac{\beta + \gamma}{2(\lambda - \delta)}$$

if and only if the external effect is negative, regardless of (a) strategies complements or (b) strategies substitutes.

Mathematically, at NE, for $da, dA < 0$, we have

$$du = \frac{\partial u}{\partial A} dA > 0, \quad dU = \frac{\partial U}{\partial a} da > 0$$

3. Intuitively, there will always be a first mover advantage because the first mover would at least choose the Nash equilibrium which is not Pareto efficient. Therefore, the first mover will always change her action in the direction that make her better off. If strategies are substitutes, then the second mover will change the action in the opposite direction and do worse (than in the Nash

equilibrium). If strategies are complements, then the second mover will change the action in the same direction and do better (than in the Nash equilibrium).

Mathematically, suppose the lower is the first mover, the problem is

$$\begin{aligned} \max_a \quad & u(a, A) \\ \text{s.t.} \quad & A = A(a) \equiv \arg\max_A U(A, a) \end{aligned}$$

Denote the solution by a^F and $A^F = A(a^F)$. We first show that there will always be a first mover advantage, i.e.,

$$u(a^F, A^F) > u(a^*, A^*)$$

where (a^*, A^*) is the NE [2]. At the NE (a^*, A^*), we have $u_a = 0$. Therefore, there exists a Δa such that

$$\Delta u = \frac{\partial u}{\partial A}\frac{dA}{da}\Delta a > 0 \tag{3}$$

by $\frac{\partial u}{\partial A}\frac{dA}{da} \neq 0$ [3]. Then, from (3), there exists $a' = a^* + \Delta a$ such that $u(a', A(a')) > u(a^*, A^*)$. Thus,

$$u(a^F, A^F) \geq u(a', A(a')) > u(a^*, A^*).$$

Now consider the second mover, along the path $(a, A(a))$, we have

$$\frac{\partial U}{\partial A} = 0$$

then

$$\frac{dU(A(a), a)}{da} = \frac{\partial U}{\partial A}\frac{dA}{da} + \frac{\partial U}{\partial a}da = \frac{\partial U}{\partial a}da$$

Thus by the Newton-Leibniz formula[4]

$$U(A^F, a^F) - U(A^*, a^*) = \int_{a^*}^{a^F} \frac{dU(A(a), a)}{da}da = \int_{a*u}^{a^F} \frac{\partial U}{\partial a}da$$

From (3) we know that

$$a^F > a^* \Leftrightarrow \frac{\partial u}{\partial A}\frac{dA}{da} > 0.$$

Therefore, if strategies are substitutes, $\frac{dA}{da} < 0$, then

$$a^F > a^* \Leftrightarrow \frac{\partial u}{\partial A} < 0 \Leftrightarrow \frac{\partial U}{\partial a} < 0.$$

Thus

$$U(A^F, a^F) - U(A^*, a^*) = \int_{a^*}^{a^F} \frac{\partial U}{\partial a}da < 0.$$

The second mover will do worse than in the Nash equilibrium.

[2] Since (a^*, A^*) satisfies the constraint, i.e., $A^* = A(a^*)$, it is easy to see that

$$u(a^F, A^F) \geq u(a^*, A^*)$$

otherwise, the lower would just choose the NE.

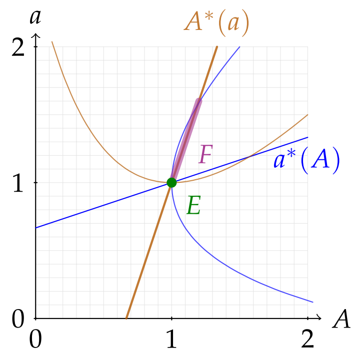
[3] It can be seen that $\Delta a > 0$ if and only if

$$\frac{\partial u}{\partial A}\frac{dA}{da} > 0.$$

That is the external effect is positive and the two strategies are complements, or the external effect is negative and the two strategies are substitutes.

[4] Suppose $F'(x) = f(x)$, we have

$$F(b) - F(a) = \int_a^b f(x)dx$$



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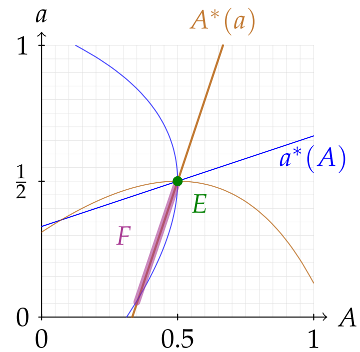
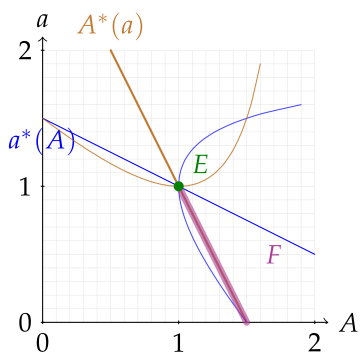


(b) Negative external effect and strategies complements with $u = 1 + a - 2A + aA - 1.5a^2$



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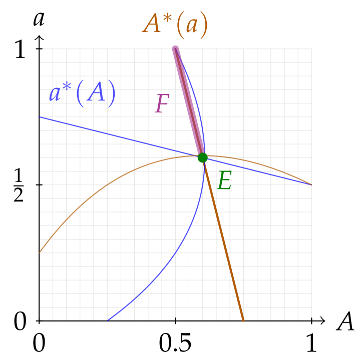

(d) Negative external effect and strategies substitutes with $u = 1 + 3a - 2A - aA - 2a^2$

Figure 3: The programming problem of the lower as the first mover is to maximize

$$u = \alpha + \beta a + \gamma A + \delta a A - \lambda a^2$$

subject to the Upper's best response $A^*(a)$. The solution must locate in the segment on the best reponse function intersected by the indifference curve through the Nash equilibrium E,

$$F = \{(A^*(a), a) \mid u(A, a) > u(E)\}$$

In all the four cases, there are always a first mover advantage.

In figure (a) and (c) with strategies complements the second mover will do better, while in figure (b) and (d) with strategies substitutes the second mover will do worse.

While if strategies are complements, $\frac{dA}{da} > 0$, then

$$a^F > a^* \Leftrightarrow \frac{\partial u}{\partial A} > 0 \Leftrightarrow \frac{\partial U}{\partial a} > 0.$$

Thus

$$U(A^F, a^F) - U(A^*, a^*) = \int_{a^*}^{a^F} \frac{\partial U}{\partial a} da > 0,$$

that is, the second mover will do better than in the Nash equilibrium, as shown in Figrue 3.

4. Let a and A be the level of chemical use by each farmer then the utility function (1) describe the interaction of the two adjacent farmers, with the following interpretation of the parameters:

- $\alpha > 0$ represents the farmer's utility from using no fertilizer and only IPM when the other farmer uses no fertilizer as well.
- $\beta > 0$ captures the positive effect of the framer's own pesticide use on productivity.

- $\gamma < 0$ captures the negative effect of runoff from the other farmer's pesticide use on lower's utility.
- $\delta > 0$ captures the strategy of pesticide use as being a strategic complement to the other farmer using pesticide. That is, if the other farmer decides to use more pesticide then you should as well because increased pesticide use lowers the effectiveness of IPM, making it a good idea for you to shift your use in concert with the other farmer.
- $\lambda > 0$ represents the decreasing marginal productivity of one's own fertilizer use.

□

2 *The Tragedy of Fishers Revisited*

We return to the two fishers, now called Upper and Lower for ease of notation, who fish in the same lake, using their labor and their nets. They consume their catch and do not engage in any kind of exchange, nor do they make any agreements about how to pursue their economic activities. Yet the activities of each affect the well-being of the other: the more Upper fishes, the harder it is for Lower to catch fish, and conversely. To be specific (using lower case letters for Lower, upper case for Upper):

$$\begin{aligned} y &= [\alpha - \beta(e+E)]e \\ Y &= [\alpha - \beta(e+E)]E \end{aligned} \tag{4}$$

where y,Y are the amount of fish caught by Lower, Upper over some given period; α and β are positive constants; and e, E are the amount of time (fraction of a twenty-four-hour day) that Lower, Upper each spend fishing. Each derives well-being from eating fish and experiences a loss of well being with additional effort, according to the utility functions:

$$\begin{aligned} u &= y - \frac{1}{2}e^2 \\ U &= Y - \frac{1}{2}E^2 \end{aligned} \tag{5}$$

1. Find the symmetric Nash equilibrium (e^N, E^N) and the soical optimum $(e^\sim, E^\sim)$ that maximizes the joint surplus $\omega = u + U$.

2. Say how you would determine the maximum Lower would be willing to pay Upper to purchase ownership rights in the lake, assuming that ownership would allow Lower to regulate Upper's access to the lake and that without this assignment of rights the two would fish at the Nash equilibrium.

3. Consider the allocations at the (i) Nash equilibrium, (ii) the social welfare optimum as well as the allocations resulting when (iii) both are altruistic with

$$v = y - \frac{1}{2}e^2 + aU$$
$$V = Y - \frac{1}{2}E^2 + au \tag{6}$$

where $a \in (0,1)$ and the Nash equilibrium obtains, and when Lower (iv) is first mover and (v) makes a take-it-or-leave-it offer. Which pairs can you Pareto rank?

4. How can being second mover be advantageous (relative to the Nash allocation)? Hint: turn the Fishers Tragedy into a Stag Hunt by assuming $\beta < 0$ (fishing is a group activity and one's catch varies positively with the effort level of the other).

5. Assume that the two fishers have utility functions expressing the fact that their concern about the other's well-being is conditional on the other's behavior. So Lower's modified utility function, w, is

$$w = u + U\frac{b + \lambda(1 - E)}{1 + \lambda} \tag{7}$$

where u and U are given by equations (5), $\lambda \in [0,1]$, and Upper's modified utility function W is analogous. Derive Lower's best response function and show that there exists no level of λ (in the unit interval) that will result in the social optimum $(e^{\sim}, E^{\sim})$ being implemented if $b = 0$, while $b = 1$ (with $\lambda = 0$) implements the social optimum.

Answer.

1. To maximize $u = y - e^2 = [\alpha - \beta(e+E)]e - \frac{1}{2}e^2$ we have

$$u_e = (\alpha - \beta E) - (1 + 2\beta)e = 0 \Rightarrow e = \frac{\alpha - \beta E}{1 + 2\beta}$$

Similarly, we have

$$U_E = 0 \Rightarrow E = \frac{\alpha - \beta e}{1 + 2\beta}.$$

For symmetric Nash Equilibrium $e^N = E^N$ we have

$$\alpha - \beta e^N - (1 + 2\beta)e^N = 0 \Rightarrow e^N = \frac{\alpha}{1 + 3\beta} = E^N$$

Denote the utilities at Nash equilibrium by $u^N = U^N$.

To maximize the joint surplus

$$\omega = u + U = [\alpha - \beta(e+E)]e - \frac{1}{2}e^2 + [\alpha - \beta(e+E)]E - \frac{1}{2}E^2$$

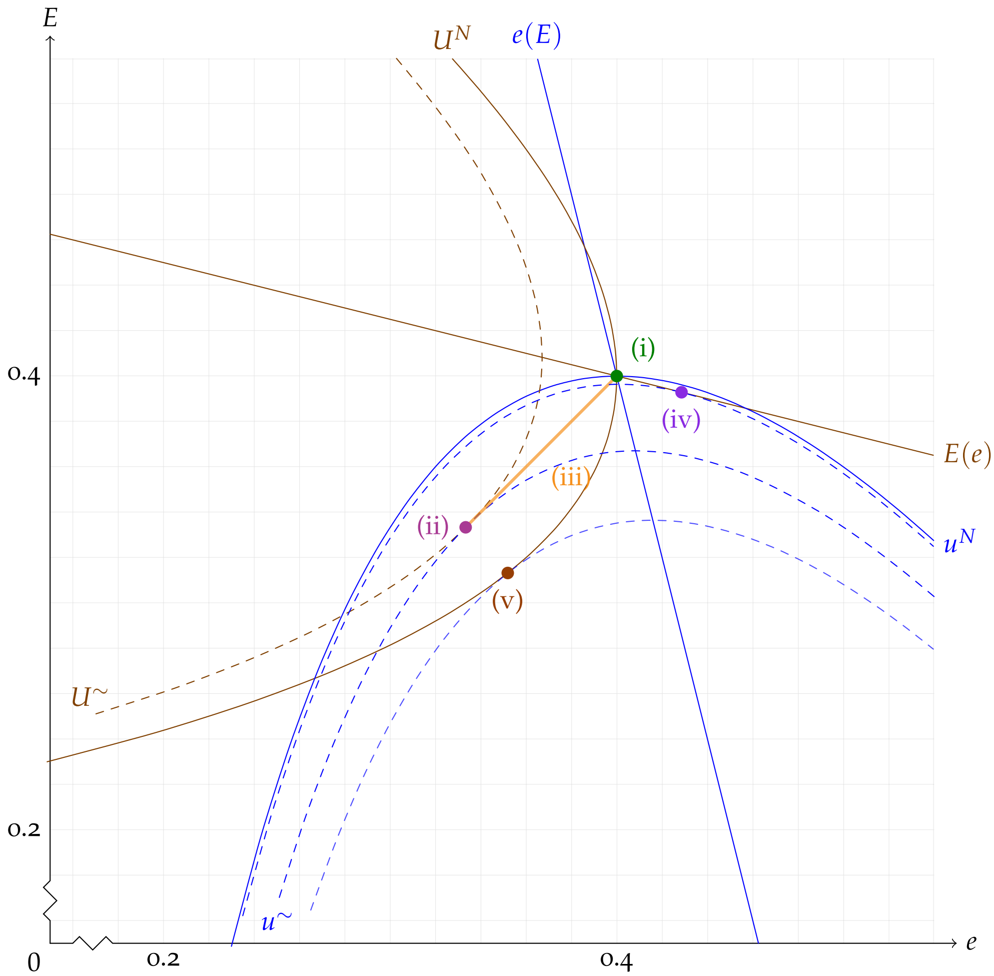

Figure 4: Assuming $\alpha = 1$, $\beta = 0.5$, the five allocations are

(i) the Nash equilibrium

$$e^N = \frac{\alpha}{1+3\beta} = E^N$$

(ii) the social optimum

$$e^\sim = \frac{\alpha}{1+4\beta} = E^\sim$$

(iii) the altruistic allocation

$$e^a = \frac{\alpha}{1+(3+a)\beta} = E^a$$

(iv) the first-mover allocation

(v) the Take-it-or-leave-it allocation

we have

$$\omega_e = (\alpha - \beta E) - (1+2\beta)e - \beta E = 0$$
$$\omega_E = (\alpha - \beta e) - (1+2\beta)E - \beta e = 0$$

Then

$$e^\sim = \frac{\alpha}{1+4\beta} = E^\sim$$

at which the utilities are denoted by $u^\sim = U^\sim$, as shown in Figure 4.

2. Both the Upper and the Lower should be at least not worse off after the ownership transaction, compared with the situation at Nash equilibrium. Suppose the payment is P. For the Lower, the

optimal problem is to

$$\max_{e,E} \quad v = [\alpha - \beta(e+E)]e - \frac{1}{2}e^2 - P$$
$$\text{s.t.} \quad V = [\alpha - \beta(e+E)E - \frac{1}{2}E^2 + P \geq U^N$$

It is easy to see that the participating constraint must be binding. Therefore, the problem is to maximize

$$v = [\alpha - \beta(e+E)]e - \frac{1}{2}^2 + [\alpha - \beta(e+E)]E - \frac{1}{2}E^2 - U^N = \omega - U^N$$

Then the solution is the social optimal $(e^{\sim}, E^{\sim})$, and $v = u^{\sim} - P$ and $V = U^{\sim} + P$. Then we should have

$$v = u^{\sim} - P \geq u^N, \quad V = U^{\sim} + P \geq U^N$$

Thus, $P \leq u^{\sim} - u^N$ otherwise, the Lower has no incentive to buy the ownership.

3. Consider case (iii) both are altruistic with $a \in (0,1)$. To maximize the utilities in (6), we have

$$(\alpha - \beta E) - (1+2\beta)e - a\beta E = 0$$
$$(\alpha - \beta e) - (1+2\beta)E - a\beta e = 0$$

Then the Nash equilibrium in this case is

$$e^a = \frac{\alpha}{1+(3+a)\beta} = E^a$$

It can be seen that (e^a, E^a) is equal to (e^N, e^N) when $a = 0$, and the social optimal $(e^{\sim}, E^{\sim})$ when $a = 1$, as shown is Figure 4. As shown is Figure 4, we have

$$\text{(ii)} \succ \text{(iii)} \succ \text{(i)}; \quad \text{(v)} \succ \text{(i)}$$

where $\succ$ means Pareto superior using the preferences (5). We can not rank (v) with (ii) or (iii), neither the pair of (i) and (iv).

4. To make the second mover be advantageous, we need strategic complements. That is

$$u_{eE} = U_{eE} = -\beta > 0$$

which hold if we assume $\beta < 0$.

5. To maximize (7), we have

$$w_e = u_e + U_e \frac{b + \lambda(1-E)}{1+\lambda}$$

To implement the social optimum, we need

$$\frac{b + \lambda(1 - E)}{1 + \lambda} = 1$$

which is impossible when $b = 0$ since $E, \lambda \in [0, 1]$. When $b = 1$, we have

$$\frac{1 + \lambda(1 - E)}{1 + \lambda} = 1 \Rightarrow \lambda = 0$$

Therefore, there exists no level of λ (in the unit interval) that will result in the social optimum $(e^{\sim}, E^{\sim})$ being implemented if $b = 0$, while $b = 1$ (with $\lambda = 0$) implements the social optimum.

$\square$

3 *Footloose Jobs and Fiscal Competition*

The day after the a cut in the U.S. corporate tax rate from 35 percent to 21 percent was signed into law, the *New York Times* reported:

> To President Trump [. . .] the overhaul of the tax code that became law on Friday will make the U.S. a better place to do business. To the rest of the world, it has the potential to challenge the global economic order [. . .] setting off a race among countries to cut corporate taxes.
>
> "It's a huge incentive to governments around the world who want to see more investment to be part of that," said Andrew Mackenzie, the chief executive of the mining giant BHP, which has its headquarters in Australia and major operations in North and South America. "They will have to follow suit." (*New York Times*, Dec 22, 2017.[5])

5

Consider two nations, Here and There, whose governments each select a rate of taxation to provide an unconditional income grant to all members of the population of each, choosing the level of the tax which maximizes the grant. Population size is fixed. (Lower case letters refer to Here, while upper case letters refer to There.) The problem facing each government is that capital is mobile between countries and the level of employment depends on the size of the capital stock, which, due to capital mobility, varies inversely with the tax rate. The tax rates in each country, t and T, are levied as a fraction of income produced in each country and vary between 0 and 1. The income produced in each country (y and Y) is the product of the exogenously given level of productivity (q and Q) and the number of people employed (n and N), that is,

$$Y = QN \text{ and } y = qn$$

so the total payments for the grant in each country are

$$g = tqn \text{ and } G = TQN$$

The dependence of the level of employment on the tax rates of the two countries is expressed by

$$n = \underline{n}(1 + m(T - t) - rt)$$

where n, r, and m are positive constants, the latter reflecting the degree of openness of the economy, and the consequent loss of producers associated with tax rates higher than the other country. (A closed economy is one for which $m = 0$ and a completely open economy is one for which $m = +\infty$.) The employment equation for There is analogous.

1. Assuming that neither country is either completely closed or open ($0 < m < +\infty$), derive the two countries' best-response functions and graph them. Give an explicit expression for the effect on t^* of variations in T, sign this term (if possible), and explain what it means.

2. Do you have enough information to determine if an increase in the openness of one economy will increase, leave unchanged, or decrease the responsiveness of its own optimal tax rate to variations in the tax rate of the other country? If you have enough information, derive the appropriate expression and explain what it means. If not, explain why not.

3. What is the Nash equilibrium if $m = 0.75$ and $r = 0.75$ for both countries?

4. Using the first order conditions defining the two best-response functions, show why it must be that at the Nash equilibrium there is some increase in both tax rates that is Pareto improving.

5. What would be the (numerical value of the) optimal tax rate if the two nations agreed to adopt a common tax rate (assuming as above $m = r = 0.75$, and ignoring any costs of negotiating the agreement)? Compare your answer to the optimal tax rate for a closed economy and explain why they are similar or different.

6. *An "imperial" solution*. Imagine that Here (a powerful country) dictates tax policy to There, and that There complies because There believes Here's threat to adopt the Nash equilibrium strategy if There does not comply. What optimizing problem would Here solve to determine the tax rate to impose on There and to adopt for itself? Are the two tax rates imposed by Here (i.e., the solution to the above optimizing problem) Pareto optimal? Explain why, why not, or why you cannot say.

7. *Evaluation*. Using whatever graphs, numerical calculations, or other reasoning you have presented above, rank the outcomes resulting from the three solution concepts (Nash, cooperative with identical tax rates, and "imperial") for each country. (For Here indicate which solution gives the highest level of total tax revenues, the next highest, and so on, and then do the same for There.) Where possible, Pareto rank the outcomes.

Answer.

1. For Here, to maximize

$$g = tq\underline{n}[1 + m(T - t) - rt]$$

the first order condition is

$$g_t = q\underline{n}(1 + mT - 2mt - 2rt) = 0 \Rightarrow t^* = \frac{1 + mT}{2(m + r)}$$

which is the best-response function.

Similarly, There's best-response function is

$$T^* = \frac{1 + mt}{2(m + r)}$$

as shown in Figure 5.

The effect on t^* of variations in T is

$$\frac{dt^*}{dT} = \frac{m}{2(m + r)} > 0$$

which means that the two counties have strategic complements. An increase in the rate of tax in There will stimulate Here to increase tax rate.

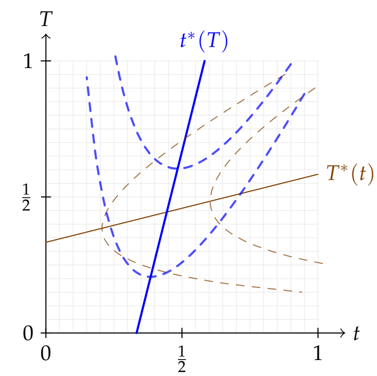

Figure 5: The best response functions

$$t^*(T) = \frac{1 + mT}{2(m + r)}$$

and

$$T^*(t) = \frac{1 + mt}{2(m + r)}$$

assuming $m = r = 0.75$.

2. The responsiveness of one country's own optimal tax rate to variations in the tax rate of the other country is the elasticity

$$\varepsilon = \frac{dt^*}{dT}\frac{T}{t^*} = \frac{m}{2(m + r)}\frac{T}{\frac{1+mT}{2(m+r)}} = 1 - \frac{1}{1 + mT}$$

Then we have

$$\frac{d\varepsilon}{dm} = \frac{T}{(1 + mT)^2} > 0,$$

which means that an increase in the openness of one country will increase the responsiveness of its own optimal tax rate to variations in the tax rate of the other country.

3. If $m = r = 0.75$ then

$$\begin{cases} t = 1/3 + T/4 \\ T = 1/3 + t/4 \end{cases} \Rightarrow t^N = T^N = \frac{4}{9}$$

Therefore, the Nash equilibrium for both countries is

$$(t^N, T^N) = (\frac{4}{9}, \frac{4}{9})$$

as shown in Figure 6.

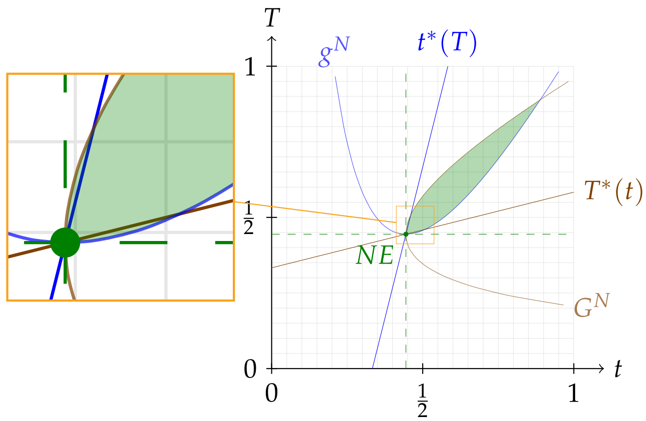

Figure 6: Assuming $m = r = 0.75$, the Nash equilibrium NE is

$$(t^N, T^N) = (\frac{4}{9}, \frac{4}{9})$$

which is not Pareto-efficient since any point in the shadow region is an improvement. Please note that the best response function $t^*(T)$ is not tangent with the indifference curve through the Nash equilibrium G^N.

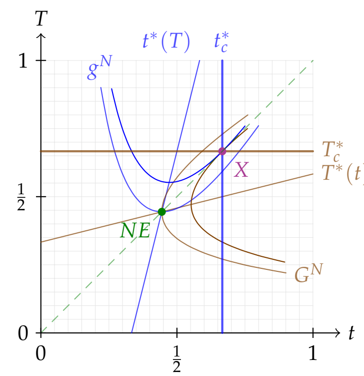

Figure 7: The common tax rates X

$$(x, x) = (\frac{1}{2r}, \frac{1}{2r})$$

is the same as the Nash equilibrium tax rates in close economy, which is the intersection of the best response functions

$$t_c^* = \frac{1}{2r}$$

and

$$T_c^* = \frac{1}{2r}$$

assuming $r = 0.75$.

4. At the Nash equilibrium, we have

$$g_t = G_T = 0$$

and

$$g_T = t^N q\underline{n}m > 0, \quad G_t = T^N Q\underline{N}m > 0$$

Therefore,

$$dg = g_t dt + g_T dT = g_T dT > 0$$
$$dG = G_t dt + G_T dT = G_t dt > 0$$

for $dt, dT > 0$. Therefore, at the Nash equilibrium there is some increase in both tax rates that is Pareto improving, as shown in Figure 6.

5. Let $t = T = x$, then $g = xq\underline{n}(1 - rx)$, $G = xQ\underline{N}(1 - rx)$. From $g_x = G_x = 0$, we have

$$x = \frac{1}{2r} = \frac{2}{3}$$

then both countries will adopt a common tax rate

$$x = \frac{2}{3}.$$

In close economy, $m = 0$, from the best response functions, we have

$$t_c^* = T_c^* = \frac{1}{2r} = \frac{2}{3}$$

Therefore, the result of common tax rate is the same as that of close economy, as shown in Figure 7.This is because the agreement of common tax rate eliminates the possible incentives for capital to move, and thus the externalities.

6. Here's optimal problem is

$$\max_{t,T} \quad g = tq\underline{n}(1-rt)$$
$$\text{s.t.} \quad G = TQ\underline{N}(1-rT) \geq G^N$$

where G^N is the grant at Nash equilibrium. Since Here maximizes its payoff subject to There's utility function, the two indifference curves are mutually tangent with each other at the solution (t^i, T^i), implying that (t^i, T^i) is Pareto optimal.

7. As shown in Figure 8, for Here,

$$g(t^N, T^N) < g(t_c^*, T_c^*) < g(t^i, T^i)$$

and for There,

$$G(t^N, T^N) = G(t^i, T^i) < G(t_c^*, T_c^*).$$

Both (t_c^*, T_c^*) and (t^i, T^i) are Pareto superior to (t^N, T^N), while we cannot rank (t_c^*, T_c^*) and (t^i, T^i).

□

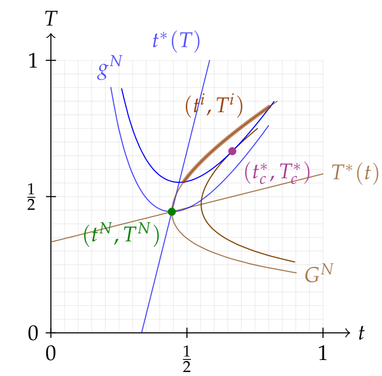

Figure 8: The three allocations are: (i) the Nash equilibrium (t^N, T^N), (ii) the common tax rates (t_c^*, T_c^*) and (iii) the imperical solution (t^i, T^i).

4 *The Tragedy of Fishers*

> Right now, my only incentive is to go out and kill as many fish as I can [...] any fish I leave is just going to be picked by the next guy.
> — John Sorlien, Rhode Island lobersterman

SUPPOSE the fishing technology and abundance of fish is such that, letting H_A and H_B be the number of hours worked by the Alfredo and Bob respectively, the number of kilos of fish caught respectively by Alfredo and Bob is

$$F_n = 100\frac{\sqrt{H_n}}{\sqrt{H_A + H_B}}, \quad n = A, B$$

Let the dis-utility of working H_n be δH_n^2, $n = A, B$. Taking both the value they place on the fish and the fact that effort is onerous into account we can write the utility of the two as the fish caught minus the subjective cost of effort. That is

$$U_n = 100\frac{\sqrt{H_n}}{\sqrt{H_A + H_B}} - \delta H_n^2, \quad n = A, B \tag{8}$$

Suppose $\delta = 0.1$ and there are only two choices Alfredo and Bob can make: fish 4 hours or fish 9 hours.

4. Suppose that Bob is first mover and can commit to fishing 9 hours some fraction p of the time and 4 hours the rest of the time (the World Cup is no longer on, so once again: $\delta = 0.1$), give the fraction p^* he will choose.

5. If Bob has take it or leave it (TIOLI) power to dictate the fraction of time that each fish 4 or 9 hours, what offer will he make to Alfredo?

6. Now suppose that as before Bob has TIOLI power but in addition to dictating the fraction of the time, he can demand a payment from Alfredo.
   (a) Rewrite their utility functions to reflect this fact.
   (b) What is the allocational aspect of this problem? What is the distributional aspect?
   (c) What offer will he make (hours of each, and any possible payment)?

7. If the resulting two allocations differ (the case where payments are possible and where they are impossible), explain why. If they do not, explain why not.

Table 1: The Game of Fishermen's Tragedy, with $\delta = 0.1$.

	Bob	
Alfredo	4 hours	9 hours
4 hours	69.1, 69.1	53.9, **75.1**
9 hours	**75.1**, 53.9	**62.6**, **62.6**

Answer.

4. Suppose Bob chose p, Alfredo's best response is to fish 9 hours all the time since Fish 9 hours is the dominant strategy as shown in Table 1. Then the payoff to Bob by choosing p is

$$\pi_B(p) = 53.9(1-p) + 62.6p = 8.7p + 53.9$$

Therefore, the Bob will choose to fish 9 hours, that is $p^* = 1$.

5. Suppose that Bob can offer (p, q) where p is the fraction of time to fish 9 hours himself and q is that for Alfredo. Then Given the payoff matrix in Table 1, the payoff to them is

$$\begin{aligned}
\pi_A(p,q) &= 69.1(1-p)(1-q) + 53.9p(1-q) + 75.1(1-p)q + 62.6pq \\
&= 2.7pq - 15.2p + 6q + 69.1 \\
\pi_B(p,q) &= 2.7pq + 6p - 15.2q + 69.1
\end{aligned}$$

Since Bob has the TIOLI power and Alfredo's fallback position is back to play the Nash equilibrium (Fish 9 hours, Fish 9 hours), Bob's programming problem is

$$\begin{aligned}
\max_{0 \le p,q \le 1} \quad & \pi_B = 2.7pq + 6p - 15.2q + 69.1 \\
\text{s.t.} \quad & \pi_A = 2.7pq - 15.2p + 6q + 69.1 \ge 62.6
\end{aligned}$$

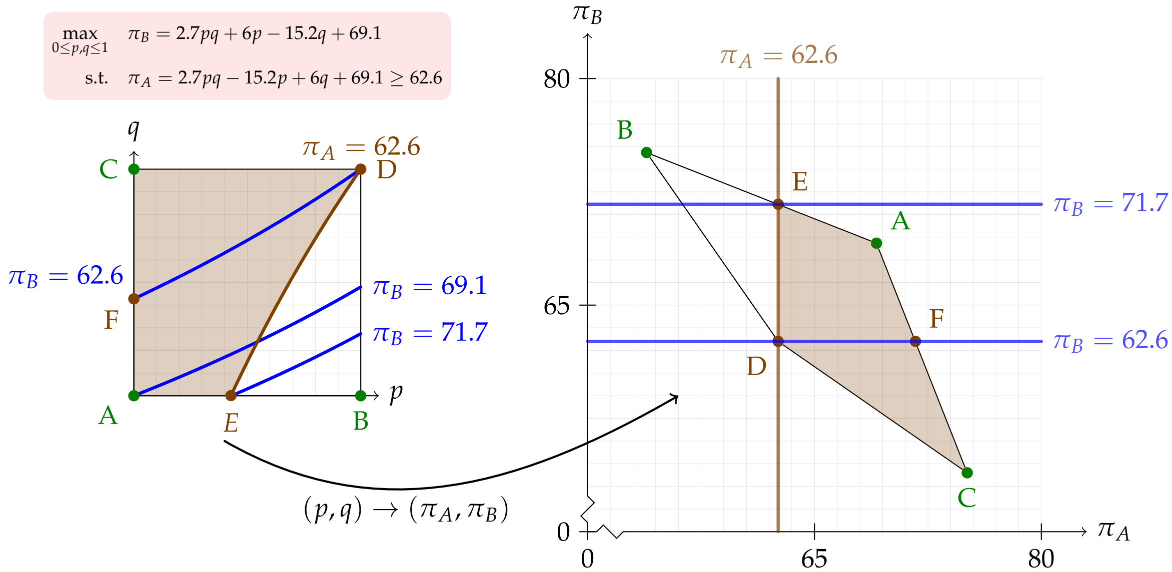

Figure 9: Denote the four ordered pairs of payoffs in the payoff matrix by A, B, C, and D. Bob has the TIOLI power and offers (p,q). The expected payoff function

$$(p,q) \to (\pi_A, \pi_B)$$

maps any point in left plane to the point in the right. $\pi_A = c$ and $\pi_B = c$ are indifferences curves, and the shadow region

$$\{(p,q) \mid \pi_A(p,q) \ge 62.6\}$$

is the feasible set. Point E maximizes π_B, representing the offer he will make $(p^*, q^*) = (0.43, 0)$.

Let

$$\mathcal{L} = \pi_B + \lambda(\pi_A - 62.6) + \mu(1-p) + \gamma(1-q)$$

The Kuhn-Tucker condition is

$$\mathcal{L}_p = 2.7q + 6 + \lambda(2.7q - 15.2) - \mu \le 0 \tag{9}$$
$$\mathcal{L}_q = 2.7p - 15.2 + \lambda(2.7p + 6) - \gamma \le 0 \tag{10}$$
$$p\mathcal{L}_p = q\mathcal{L}_p = \lambda(\pi_A - 62.6) = \mu(1-p) = \gamma(1-q) = 0 \tag{11}$$
$$p, q, \lambda, \mu, \gamma \ge 0, \quad \pi_A \ge 62.6 \tag{12}$$

We will show that $0 < p < 1$ and $q = 0$ by contradiction. Suppose $p = 1$, by $\pi_A(1,q) = 53.9(1-q) + 62.6q \ge 62.6$, we have $q = 1$. Then (9) and (10) become

$$\begin{cases} 8.7 - 12.5\lambda - \mu = 0 \\ -12.5 + 8.7\lambda - \gamma = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{8.7-\mu}{12.5} \le \frac{8.7}{12.5} \\ \lambda = \frac{12.5+\gamma}{12.5} \ge \frac{12.5}{8.7} \end{cases}$$

contradicted. Therefore, $p < 1$ and then $\mu = 0$.

Suppose $p = 0$, then

$$\pi_A(0,q) = 69.1(1-q) + 75.1q > 62.6$$

so we have $\lambda = 0$. Then (9) becomes $2.7q + 6 \le 0$, contradicted. Therefore, we have $0 < p < 1$ and $\mathcal{L}_p = 0$. Therefore, $\lambda > 0$ otherwise $\mathcal{L}_p = 2.7q + 6 > 0$ contradicted.

Suppose $q = 1$, by $p < 1$ we have

$$\pi_A(p,1) = 75.1(1-p) + 62.6p > 62.6$$

contradicted with $\lambda > 0$. Therefore, we have $q < 1$ and then $\gamma = 0$. Suppose $q > 0$, then $\mathcal{L}_q = 0$, which is inconsistent with $\mathcal{L}_p = 0$[6]. Therefore, we have $q = 0$. Then by $\pi_A(p, 0) = 62.5$ we have

$$p = \frac{69.1 - 62.6}{15.2} \approx 0.43$$

Therefore, Bob will offer $(p^*, q^*) = (0.43, 0)$ if he has TIOLI power. As a result, $p_A = 62.5$ and $\pi_B = 71.7$.

[6] Since

$$\mathcal{L}_p = 0 \Rightarrow \lambda = \frac{2.7q + 6}{15.2 - 2.7q} < 8.7/12.5$$

while

$$\mathcal{L}_q = 0 \Rightarrow \lambda = \frac{15.2 - 2.7p}{2.7p + 6} > 12.5/8.7.$$

6. Denote Bob's offer by (p, q, y) where p and q are the fractions of time to fish 9 hours for Bob and Alfredo as defined above, and y is the payment from Alfredo to Bob.
   (a) The utility functions for Alfredo and Bob are
   $$\pi_A(p, q, y) = 2.7pq - 15.2p + 6q + 69.1 - y$$
   $$\pi_B(p, q, y) = 2.7pq + 6p - 15.2q + 69.1 + y$$
   (b) To find the efficient fraction of time p, q is the allocational aspect of the problem, while to determine the payment such that both parties would participate is the distributional aspect.
   (c) Bob's problem is
   $$\max_{0 \le p, q \le 1, y} \quad \pi_B = 2.7pq + 6p - 15.2q + 69.1 + y$$
   $$\text{s.t.} \quad \pi_A = 2.7pq - 15.2p + 6q + 69.1 - y \ge 62.6$$
   It is obvious that the constraint is binding, otherwise a slightly higher y with the same p, q would give Bob higher utility and still satisfies the constraint. Therefore,
   $$y = 2.7pq - 15.2p + 6q + 6.5$$
   The problem becomes
   $$\max_{0 \le p, q \le 1} \quad \pi_B = 5.4pq - 9.2p - 9.2q + 75.6$$
   Since $\frac{\partial \pi_B}{\partial p} = 5.4q - 9.2 < 0$ and $\frac{\partial \pi_B}{\partial q} = 5.4p - 9.2 < 0$, the optimal solution is $(p^*, q^*) = (0, 0)$. Then $y^* = 6.5$. Therefore, Bob will make the offer $(0, 0, 6.5)$, i.e., both fish 4 hours, and Alfredo pays Bob 6.5. As a result, $\pi_A = 62.5$ and $\pi_B = 75.6$.

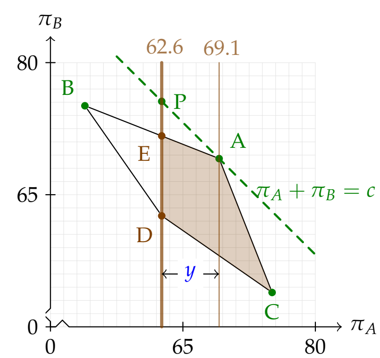

Figure 10: Denote the four ordered pairs of payoffs in the payoff matrix by A, B, C, and D. The allocational problem is to find the point miximizing the soical welfare.

$$\pi_A + \pi_B = c$$

is the level curve of the objective function, which obtains its maximum at point A, i.e., $p = q = 0$.

The distributional aspect of the problem is using the participating constraint $\pi_A = 62.6$ to determine the payment $y = 6.5$ and the final distribution P.

E is the outcome when payments are impossible, which is different from both A and P.

7. The two allocations differ as shown in Figure 10. When payments are impossible, allocation and distribution are inseparable. When payments are possible, they can be separated. In this case,the player with TIOLI power (Bob) has the incentive to find the social optimal outcome since the benefit can be transferred.

□

5 *Public Goods and the Social Multiplier*

Suppose each of n identical citizens may contribute to a public good and their identical utility functions are

$$u_i = f(\sum_{i=1}^{n} a_i) + sa_i - g(a_i) \tag{13}$$

where a_i, for $i = 1, \ldots, n$, is the citizen's non negative contribution to the public good, and the first term on the right hand side of the equation is the total benefit of the public good to each citizen with $f' > 0$ and $f'' < 0$, while the cost of contributing is $g(a_i)$ with $g' > 0$ and $g'' \geq 0$, and individual i may receive a subsidy introduced by a social planner that amounts to sa_i.

Consider the following two specific utility functions.

$$u_i = f \ln \sum_{i=1}^{n} a_i + sa_i - ga_i \tag{14}$$

where $f > 0$ and $g > 0$ are constants.

$$u_i = f(\sum_{i=1}^{n} a_i) + sa_i - \frac{1}{2}a_i^2 \tag{15}$$

where f is a increasing function with $f'' < 0$.

1. Give the first order condition indicating citizen i's utility maximizing contribution.

2. Say whether the citizen's contributions are strategic complements or substitutes (show why you answered this as you did)?

3. With the utility function (14), if $s = 0$, is mutual non-contribution ($a_i = 0$ for $i = 1, \ldots, n$) a Nash equilibrium? Explain why or why not.

4. What is the symmetric Nash equilibrium level of contributions, namely $a_i^N = a^N$ for all i?

5. Using the first order conditions that define the Nash equilibrium for this case prove that it is, or is not Pareto efficient (or explain why you cannot say).

6. Effect of variations in s:

   (a) What is the partial effect of variations in s on the citizen's contribution ("partial" here means holding constant the other citizens' contributions)?

   (b) What is the effect of variations in s on the symmetric Nash equilibrium level of contributions (a_i^N)?

(c) Why is the effect of the subsidy on the Nash equilibrium different from the partial effect?

(d) In the case of $n = 2$, explain why, if contributions to the public good are strategic substitutes, the total effect of variations in the subsidy will be less than the effect on an individual's contributions, holding the contributions of the other constant. (This is a negative social multiplier).

7. Suppose you are the social planner and you wish select a level of subsidy to maximize the sum of the citizen's benefits from the public good minus the cost of their contributions; the planner's objective function ignores the contribution of the subsidy to the citizens' utility (sa_i) as this is simply a transfer. Give the first order conditions for the contribution levels of each citizen that she would like to implement and the value of s (call it s^*) that implements this result.

8. Suppose you are the social planner, but while you can observe the total benefit of the public project for each citizen, that is, the first term on the right hand side of the utility function at the outset, you cannot observe individual contributions. So you cannot implement a subsidy based on the individual's contribution. Is there some other subsidy mechanism by which you could implement the socially optimal level of contributions (assuming no change in the in the individual's utility functions and other aspects of the set up of the problem, above)? If so explain what it is. If not say why it is impossible.

Answer.

1. For the citizen i, the problem is

$$\max_{a_i} \quad u_i = f(\sum_{i=1} na_i) + sa_i - g(a_i)$$

The first order condition for the choice of a_i is

$$f'(\sum_{i=1}^{n} a_i) + s - g'(a_i) = 0 \Rightarrow f'(\sum_{i=1}^{n} a_i) + s = g'(a_i) \tag{16}$$

That is the marginal benefit $f' + s$ equals to the marginal cost g'.

Using the utility function (14), we have

$$\frac{f}{\sum_{i=1}^{n} a_i} + s = g \Rightarrow \sum_{i=1}^{n} a_i = \frac{f}{g - s} \tag{17}$$

Or

$$a_i = \frac{f}{g - s} - \sum_{j \neq i} a_j \tag{18}$$

Using the utility function (15), we have

$$f'(\sum_{i=1}^{n} a_i) + s = a_i \tag{19}$$

2. In general, the citizen's contributions are strategic substitutes since

$$\frac{\partial^2 u_i}{\partial a_i \partial a_j} = f'' < 0$$

With utility function (14), the citizen's contributions are strategic substitutes. From the first order condition (17) we know that the sum of the contributions is constant. [7]

[7] Or, from (18), we have

$$\frac{da_i}{da_j} = -1 < 0$$

With the utility function (15), from (19) we have

$$\frac{da_i}{da_j} = \frac{f''}{1 - f''} < 0$$

since $f'' < 0$ and $f'' - 1 < 0$ by the second order condition.

3. Given the utility function (14), if $s = 0$, mutual non-contribution ($a_i = 0, \forall i$) is not a Nash equilibrium. Since at this situation, for citizen i, the marginal utility of a_i is infinity, while the marginal cost is g. Therefore, she has the incentive to increase a_i. [8]

[8] Or, it is easy to seen that equation (17) does not hold for $s = 0$ and $a_i = 0, \forall i$.

4. The condition for the Nash equilibrium is

$$\sum_{i=1}^{n} a_i^N = \frac{f}{g - s}$$

By symmetry $a_i^N = a^N$, we have

$$na^N = \frac{f}{g - s} \Rightarrow a^N = \frac{f}{n(g - s)} \tag{20}$$

5. At the Nash equilibirum, we have

$$\frac{\partial u_i}{\partial a_i} = 0, \quad i = 1, \dots n$$

However,

$$\frac{\partial u_i}{\partial a_j} = \frac{f}{\sum_{i=1}^{n} a_i} > 0, \quad \forall j \neq i, i = 1, \dots n$$

Therefore, the mutual increase in contributions $da_i > 0, i = 1, \dots n$ would be a Pareto improvement, by

$$du_i = \frac{\partial u_i}{\partial a_i} da_i + \sum_{j \neq i} \frac{\partial u_i}{\partial a_j} da_j = \sum_{j \neq i} \frac{\partial u_i}{\partial a_j} da_j > 0, \quad i = 1, \dots n.$$

Thus, the Nash equilibirum is not Pareto efficient.

6.(a) The partial effect of variation in s on the citizen's contribution is

$$\frac{\partial a_i}{\partial s} = \frac{f}{(g-s)^2}$$

from the best response function (18).

(b) The effect of variations in s on the symmetric Nash equilibrium level of contributions (a^N) is

$$\frac{\partial a^N}{\partial s} = \frac{f}{n(g-s)^2}$$

from (20).

(c) The effect of the subsidy on the Nash equilibrium includes the effect of the change in one's contributions on the others, which is excluded in partial effect by holding constant other citizen's contributions.

(d) In the case of $n = 2$, we have the decomposition

$$\frac{da_i^N}{ds} = \frac{\partial a_i}{\partial s} + \frac{\partial a_i}{\partial a_j}\frac{da_j^N}{ds}$$

If contributions to the public good are strategic substitutes, i.e., $\frac{\partial a_i}{\partial a_j} < 0$, then we have

$$\frac{da_i^N}{ds} < \frac{\partial a_i}{\partial s}$$

Intuitively, an increase in subsidy would also induce the other citizen j to contribute more, which discourages citizen i's contribution by stragetic substitutes, as shown in Figure 11.

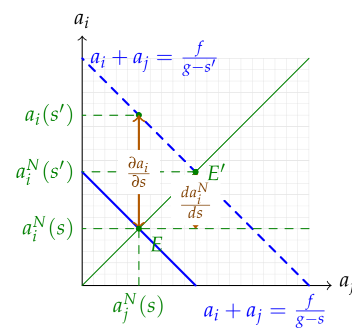

Figure 11: The curve

$$a_i + a_j = \frac{f}{g-s}$$

is the best response function and E the symmetric Nash equilibrium. Suppose the social planner increases the subsidy to s', then best response function shift upward to

$$a_i + a_j = \frac{f}{g-s'}$$

and the symmetric Nash equilibrium becomes E', with $f = 1, g = 3, s = 1$ and $s' = 2$. It can be seen that

$$a_i(s') - a_i^N(s) = \frac{\partial a_i}{\partial s}$$

is greater than

$$a_i^N(s') - a_i^N(s) = \frac{da_i^N}{ds}.$$

7. The social planner's object function is

$$\omega = \sum_{i=1}^{n}(f \ln \sum_k a_k - g a_i) = n[f \ln(na) - ga]$$

The first order condition for the contribution levels of each citizen that she would like to implement is

$$\omega_a = n(\frac{f}{a} - g) = 0 \Rightarrow a^* = \frac{f}{g}$$

The implement $a_i = a^*, i = 1, \ldots, n$, by the first order condition (17), we have

$$\frac{f}{g-s} = na^* = \frac{nf}{g} \Rightarrow s = g(1 - \frac{1}{n})$$

8. Observing the total benefit $B = f \ln \sum_{i=1}^{n} a_i$, the social planner can provide a subsidy $S = (n-1)B$ to implement the soically optimal level of contribution a^*.

The problem of the citizen becomes

$$\max_{a_i} \quad u_i = f \ln \sum_{i=1}^{n} a_i - g a_i + S = n f \ln \sum_{i=1}^{n} a_i - g a_i$$

The first order condition is

$$\frac{nf}{\sum_{i=1}^{n} a_i} - g = 0 \Rightarrow \sum_{i}^{n} a_i = \frac{nf}{g}$$

Then the symmetric Nash $a_i = a^S$ equilibrium in this case, we have

$$na^S = \frac{nf}{g} \Rightarrow a^S = \frac{f}{g} = a^*$$

AN ALTERNATIVE SOLUTION. The social planner could set $S = w \sum_{i=1}^{n} a_i$, then the problem of the citizen becomes

$$\max_{a_i} \quad u_i = f \ln \sum_{i=1}^{n} a_i - g a_i + S = f \ln \sum_{i=1}^{n} a_i - g a_i + w \sum_{i=1}^{n} a_i$$

The first order condition is

$$\frac{nf}{\sum_{i=1}^{n} a_i} - g = 0 \Rightarrow \sum_{i}^{n} a_i = \frac{f}{g - w}$$

Then the symmetric Nash $a_i = a^S$ equilibrium in this case, we have

$$na^S = \frac{f}{g - w} \Rightarrow a^S = \frac{f}{n(g - w)}$$

To implement $a^* = \frac{f}{g}$, let

$$a^S = a^* \Rightarrow \frac{f}{n(g - w)} = \frac{f}{g} \Rightarrow w = (1 - \frac{1}{n}) g$$

$\square$