Econ 700 Problem Set 3

Due: Nov 1, 2019

1 *Heterogeneous Labor*

Suppose that there are two types of workers: Good (low disutility of labor) and Bad (high disutility of labor), and a worker's type is common knowledge. Describe a competitiive equilibrium in which both types are hired by a given firm and show that in equilibrium, the fallback position of the good workers must exceed that of the bad workders.

2 *Credit Contracts*

A borrower, with no wealth and zero reservation income, seeks a loan of $1 from a lender to finance a machine, which produces an output of goods with a value of μf, where μ is a positive constant, when it is run at speed f and if it does not fail; but it fails with probability f yielding no goods. The lender chooses an interest factor δ, but has no means to observe the speed at which the machine is run and as a result f is not subject to contract. Both lender and borrower are risk neutral.

1. *Robinson Crusoe*. Suppose that the lender were to operate the machine as an owner–operator, what speed f would he operate the machine? Also give the surplus gained by the lender (as owner–operator) as a result.

2. *Enforceable Repay*. If the promise to repay is enforceable, show that the agent will set f as in that in Robinson Crusoe case, even if f is not contractible. Explain why this is true.

Answer.

1. The owner–operator's problem is to $\max_f \mu f(1-f)$. The first order condition is

$$\mu(1-2f) = 0 \Rightarrow f = \frac{1}{2}$$

   Therefore, the surplus gained by the lender as owner-operator is

$$T = \frac{1}{2}(1-\frac{1}{2})\mu = \frac{1}{4}\mu.$$

2. When repay is enforceable, the borrower's problem becomes

$$\max_f \quad U = \mu f(1-f) - \delta$$

The first order condition is

$$U_f = \mu(1-2f) = 0 \Rightarrow f = \frac{1}{2}$$

Therefore, when the promise to repay is enforceable, the borrower will set $f = \frac{1}{2}$ duplicating Robinson Crusoe case, even if f is not contractible. This is because the agent will have to repay with certainty, so she has no incentive to set a higher f to make the machine break in order to escape the repayment.

Since the borrower's reservation income is zero, the lender would set the interest factor δ such $U = 0$. Therefore,

$$V = \delta = \frac{1}{4}\mu = T$$

□

3 *Why Nobody Wants to Do Business with the Poor*

The observable output of a project depends on the agent's effort because it influences whether a "good" or "bad" state occurs. For example, the crop may fail or it may grow, and this is influenced (but not uniquely determined) by the agent's actions. The agent selects an (onerous and unobservable) effort level $e \in [0,1]$, which influences whether the good or bad state occurs, the former happening with probability $\pi(e)$ with $\pi' > 0$. Total revenues of the project in the good and bad state respectively are Y and y with $0 < Y - y \leq 2$. The disutility of effort is e^2, and to simplify things by a harmless normalization, let's say that $\pi(e) = e$. Because the agent is risk neutral, she maximizes expected income minus the disutility of effort.

1. If the agent were the owner of the output of the project (meaning she owned the revenues y or Y, whichever occurred), how would she select her level of effort? Give the first order conditions and the level of effort she would choose (write e in terms of Y and y, and call this $e^{\max}$).

Suppose the principal owns the project (and hence receives the income Y or y) and seeks to maximize expected profits by devising a payment scheme whereby the agent gets w in the bad state and W in the good state. The agent's fallback utility is zero, but she starts the interaction with wealth z. The wage offered in the bad outcome cannot be less than $-z$ (in the bad outcome, the most the principal can take from the agent is everything the agent has). It may help to think of the agent's wealth as the maximum collateral the agent can put up: by transacting with the principal, the agent stands to receive W and stands to lose some amount not to exceed z. The agent's utility in this period is expected pay minus the disutility of effort plus the consumption of the asset z. In making sense of this problem, you might want to devise a graph in wage-effort space, with (on the horizontal axis) the two wages, one of them possibly negative, and (on the vertical axis) effort. The three things you want to put in this space are: (i) the agent's participation constraint, (ii) the agent's best-response function, and (iii) the principal's iso-expected-profit loci.

2. Write down the agent's participation constraint. Hint: start with what you know about the relationship of w to z; using this, eliminate w; and then write the constraint in terms of e and W.

3. The agent varies e to maximize utility. What is her best-response function?

4. Knowing the agent's best-response function, the principal varies W to maximize his expected profits. Give the relevant first order conditions and indicate what W the principal will choose.

5. If the principal implements his expected-profits-maximizing pay scheme (w^*, W^*), what level of e, call it e^*, will the agent choose? Check to see that the resulting pay package (w^*, W^*) satisfies the agent's participation constraint.

6. Why does e^* differ from $e^{\max}$, the surplus-maximizing level of effort that occurs when the agent is also residual claimant?

7. Suppose an amount of wealth Δz is transferred to the agent. Assuming that Δz is not so large that the agent can undertake the project as an owner-operator rather than as an agent, what effect does this have on e^*, the agent's utility, and the principal's profits?

8. Why would the principal prefer to transact with wealthier agents (assuming wealthier agents have the same fallback positions as the less wealthy, namely, zero)?

9. Short of simply giving the project to the agent, is there a contract governing the relationship of P to A in this case that would assure the Pareto-efficient level of e, assuming as before that e is not observable? Say what it is and explain why P would not offer this contract.

Answer.

1. The utility of the agent is

$$u = Y\pi(e) + y(1 - \pi(e)) - e^2 = Ye + y(1 - e) - e^2$$

The first order condition is

$$u_e = Y - y - 2e = 0 \Rightarrow e^{\max} = \frac{1}{2}(Y - y)$$

2. The participation constraint is

$$\begin{aligned} u &= W\pi(e) + w(1-\pi(e)) - e^2 + z \\ &= We + w(1-e) - e^2 + z \geq 0 \end{aligned}$$

Let $w = -z$, we have

$$(W+z)e \geq e \Rightarrow W + z \geq e$$

3. To maximize the utility of the agent

$$u = We - z(1-e) - e^2 + z = (W+z)e - e^2$$

the first order condition is

$$u_e = W + z - 2e = 0 \Rightarrow e = \frac{W+z}{2}$$

which is her best response function.

4. The principal's utility function is

$$U = (Y-W)e + (y+z)(1-e) = y + z + (Y - W - y - z)e$$

Knowing the agent's best response function $e = (W+z)/2$ we have

$$U = y + z + \frac{1}{2}(Y - W - y - z)(W+z)$$

To maximize U by varying W, the first order condition is

$$U_W = \frac{1}{2}(Y - y - 2z - 2W) = 0 \Rightarrow W^* = \frac{1}{2}(Y-y) - z$$

5. Given $W^* = \frac{1}{2}(Y-y) - z$, we have

$$e^* = \frac{1}{2}(W^* + z) = \frac{1}{4}(Y-y)$$

and

$$W^* + z = \frac{Y-y}{2} > e^*$$

Therefore, the package $(w, W) = (-z, W^*)$ satisfies the agent's participation constraint.

6. The marginal benifit is $Y - y$ when the agent is the residual claimant, and $W^* + z = (Y-y)/2$ when she is not. e^* differs from $e^{\max}$ because when the agent is not residual claimant her incentives are diluted.

7. When $z' = z + \Delta z$, $e^* = \frac{1}{4}(Y-y)$ remain the same, so as the agent's utility since

$$u^* = (W^* + z)e^* - e^{*2} = \frac{1}{2}(Y-y)e^* - e^{*2}$$

Given $W^* = \frac{1}{2}(Y - y) - z$, we have

$$\begin{aligned} U^* &= y + z + \frac{1}{2}(Y - y - W^* - z)(W^* + z) \\ &= y + z + \frac{1}{8}(Y - y)^2 \end{aligned}$$

Then when z becomes $z + \Delta z$, the change in the principal's utility is

$$\Delta U^* = \Delta z$$

8. The principal would prefer to transact with wealthier agents because she will get higher expected profits by taking more from the workers if the project fails, with the same effort level provided by the agents, as illurstrated by $\frac{dU^*}{dz} = 1 > 0$.

9. Consider the contract that the agent will pay the principal a fixed rent R to operate the project and get the revenue. For the agent, the problem is

$$\max_e \quad u = Ye + y(1 - e) - e^2 + z - R$$

The first order condition is

$$u_e = Y - y - 2e = 0 \Rightarrow e^* = \frac{1}{2}(Y - y) = e^{\max}$$

Given that the fallback position of agent is zero, to maximize the rent R subject to the participation constraint, the principal will get

$$\begin{aligned} R^* &= Ye^{\max} + y(1 - e^{\max}) - (e^{\max})^2 + z \\ &= y + z + \frac{1}{4}(Y - y)^2 > U^* \end{aligned}$$

P would not offer this contract if A has wealth constraint and can not pay the rent.

□

4 *Landowning and its Discontents*

Institutions are often described as integrated and organic wholes, more or less like a member of a species. Everyone can tell an elephant from a dog, and similarly capitalism, feudalism and socialism are not likely to be mixed up. But when one studies institutions empirically, one is impressed by the diversity of often highly local arrangements. Individual farmers often work under as many as three distinct contracts, working one's own land, hiring out as wage labor, and renting land (possibly under a fixed rent, a crop share, or other distinct contract).This question concerns the mix of contracts which may exist in equilibrium.

Consider a landowner with ten units of land that she does not farm herself. She can offer access to her land under two types of contracts, sharecropping and wage labor. Prospective farmers and the landlord alike have identical utility functions $U = V = y - e^2$, where y is income (in units of agricultural output), and e effort over a given period. Each farmer, when working full-time, farms exactly one acre of land, and cannot, we will assume, farm more or less. (The farmers can, of course, split their time between wage work and sharecropping.) The production function on each acre of the land is simply $q = e$, where q is the level of output.

There are neighboring landlords identical to this one offering sharecropping contracts, but because these are absentee owners they cannot oversee wage work and hence do not offer wage contracts. If wage labor is used, the monitoring is done by the landlord, who experiences a disutility occasioned by the associated effort. Sufficient monitoring is done to extract $e = 1/2$ from each worker hired for wages, and the amount of the landlord's effort needed to perform the monitoring to enforce this level of worker effort is $e = 1/8$. (Wages are not used to induce higher levels of effort, so the wage is simply the minimum necessary to secure the supply of labor time, namely, the wage that gives the worker the utility attainable in the neighboring share contracts.)

The landlord is trying to decide how much land to rent to sharecroppers, how much to farm using wage labor, and what contracts to offer each. Local traditions preclude very complex contracts, so she simply wants to know what the landlord's share, s, should be in the sharecropping contracts, what the wage w should be in the wage labor contracts, and how many acres of land should be devoted to cultivation by wage labor, n. (An amount equal to $10 - n$ will be cultivated by sharecroppers.) She asks you for advice. You instruct the landlord to first determine how the tenants' effort levels will be affected by s, the share claimed by the landlord.

1. What is the sharecropper's best-response function: $e^* = e^*(s)$? What share will the landlord offer (she sets $s = s^*$ to maximize her utility)?

2. Turning now to the possibility of hiring wage labor, and assuming that all landlords in the area are offering s^* contracts, indicate the wage the landlord will offer, w^*.

3. Given $s^*, e^*(s^*)$, and w^*, determine the landlord's utility maximizing level of n, call this n^*.

4. At the equilibrium e^*, s^*, w^*, n^*, what is the equilibrium level of utility of the three types of agents: landlord, worker, and sharecropper? Is the result given by (e^*, s^*, w^*, n^*) Pareto optimal? If you think not, indicate an offer that one or more of the (noncolluding) agents might make which would result in a Pareto improvement and explain why the Pareto improvement was possible. Hint: begin by indicating how much (per period) the farmer would be willing to pay to acquire ownership of an unit of land (or the least amount the landloard would be willing to receive to give up an acre). Then indicate any Pareto-improving offers.

Imagine now that the ten cultivators (sharecroppers and wage workers alike), angry at what they consider to be their exploitation, meet to plan a collective strategy. Before long they have succeeded in securing a binding agreement of all cultivators in the area to refuse any contract with $s > 0.4$. As a result, all sharecropping contracts in the area are now revised so that $s = 0.4$. All other parameters remain unchanged.

5. Indicate the resulting new equilibrium values: e', w', and n'. (The value of n' need not be an integer.) Why does the change in s alter the wage rate? Compare the levels of utility gained by the three types of agents in the new equilibrium with their utility levels before the collective action.

One of the ten cultivators suggests that they simply occupy the landlord's land forcibly and farm it themselves as owners on individual plots. The revolutionary cultivator claims that it will be possible to pay the (ex) landlord an amount sufficient so that the landlord's utility is no less after the revolution than under the collective action case with $s = 0.4$, thereby securing the landlord's support or at least attenuating her opposition. The other cultivators are skeptical. They ask for your advice.

6. If no compensation were paid to the landlord, what would be the effort levels and resulting utility levels of the cultivators?

7. If each of the cultivators paid an equal lump sum tax (per period) to provide the minimal compensation to the landlord necessary to allow her to attain a level of utility not less than in the previous equilibrium (i.e., the collective action case), how much would each pay?

8. If the compensation outlined above is possible, why did the cultivators not simply purchase the land?

Answer.

1. For the sharecropper, $y = (1-s)q = (1-s)e$ so the problem is

$$\max_{e} \quad U = (1-s)e - e^2$$

The first order condition is

$$U_e = 1 - s - 2e = 0 \Rightarrow e = \frac{1}{2}(1-s)$$

Therefore, the sharecropper's best response function is

$$e^*(s) = \frac{1}{2}(1-s) \tag{1}$$

For the landlord, the problem is to maximize the utility $V = se$ subject to the sharecropper's best response function. That is

$$\max_{s} \quad V = se(s) = \frac{1}{2}(1-s)s$$

Then we have $s^* = \frac{1}{2}$ by the first order condition $V_s = 0$.

With $s^* = \frac{1}{2}$, $e^* = \frac{1}{4}$, we have $U^* = (1-s)e - e^2 = \frac{1}{16}$ and $V^s = se = \frac{1}{8}$.

2. The workers' fallback position is the utility of the sharecroppers $U^* = \frac{1}{16}$. For the landlord to hire workers, she needs to perform the monitoring with $e = \frac{1}{8}$ to extract $e = 1/2$ from each worker hired for wages, then

$$V = \frac{1}{2} - w - (\frac{1}{8})^2$$

which is decreasing in w. To maximize V subject to the worker's participating constraint

$$w - \frac{1}{4} \geq \frac{1}{16}$$

the landlord will offer $w^* = \frac{5}{16}$.

3. Suppose the landlord decides n acres of land to be cultivated by wage labors and the rest $10 - n$ by sharecroppers, her utility is

$$\begin{aligned} V &= (\frac{1}{2} - w^*)n - (\frac{1}{8}n)^2 + (10-n)V^s \\ &= \frac{1}{64}(80 + 4n - n^2) \end{aligned}$$

Therefore, to maximize V, we have $n^* = 2$ by $V_n = 0$. Then $V^* = \frac{21}{16}$.

4. At the equilibrium, the utility of the worker and sharecropper is $U^* = \frac{1}{16}$, and that of the landlord is $V^* = \frac{21}{16}$.

The result is not Pareto-optimal. Consider the contract that the farmer pays a price p per period to acquire the ownership of an acre of land.

By buying the land, the farmer's utility is $U = e - e^2$ as the owner. To maximize utility she will choose $e = \frac{1}{2}$ and get $U = \frac{1}{4}$. Therefore, the maximum price she would like to offer to get the ownership of a unit of land in one period is

$$p^f = \frac{1}{4} - U^* = \frac{3}{16}$$

That is, the transaction at price lower than p^f would make the farmer better off.

For the landloard, after selling one acre of land, her utility becomes

$$\begin{aligned} V &= (\frac{1}{2} - w^*)n - (\frac{1}{8}n)^2 + (9-n)V^s \\ &= \frac{1}{64}(72 + 4n - n^2) \end{aligned}$$

To maximize V she will again choose $n = 2$ and get

$$V = \frac{19}{16}$$

Therefore, the least amount the landloard would be willing to receive per period to give up an are is

$$p^l = V^* - \frac{19}{16} = \frac{1}{8}$$

That is, the transaction at price higher than p^f would make the landloard better off.

Therefore, the contract with price $p \in [p^l, p^f] = [\frac{1}{8}, \frac{3}{16}]$ is a Pareto-improvement offer.

5. Given $s = 0.4$, from the best response function (1), we have

$$e' = \frac{1}{2}(1 - 0.4) = 0.3$$

Then the fallback position of the workers become

$$U' = (1-s)e' - e'^2 = 0.09$$

and the utility of the landlord gained by sharecropping is

$$V^{s'} = se' = 0.12$$

For the landlord, she will offer the new wage to the worker

$$w' = \frac{1}{4} + U' = 0.34$$

because of the change in the worker's fallback position.

To maximize

$$\begin{aligned} V &= (\frac{1}{2} - w')n - \frac{n^2}{64} + (10-n)V^{s'} \\ &= 1.2 + 0.04n - \frac{1}{64}n^2 \end{aligned}$$

the landlord will choose $n' = 1.28$ by $V_n = 0$. She will get $V' = \frac{766}{625} = 1.2256$. The levels of utility gained by the three types of agents in the new equilibrium with their utility levels before the collective action are summarized in Table **??**.

6. If no compensation were paid, the farmers would choose e to maximize $U = e - e^2$, then $e = \frac{1}{2}$ and $U = \frac{1}{4}$ as a result.

7. Each of the cultivators would pay

$$T = \frac{V'}{10} = 0.12256$$

8. The cultivators can not simply purchase the land because the price of the land would increase and end up to be greater than the lump sum tax.

□

5 *Rental Housing Market*

Abby is spending the summer vacation working at a job in her home town and would like to rent an apartment for two months. She finds a place that is available and she would be willing to pay at most v dollar per month if no other place were available. She also has the option of living with her parents which she values per month at $z < v$.

She approaches the landlord, Bob, who offers to rent the apartment at a rent per month of r_1 and r_2 for the two months (no other contract, for example, including a security deposit, is possible). But he is concerned about the upkeep of the apartment and so adds that there will be an unannounced inspection sometime during the first month, the result of which will determine if Abby gets to rent the place for the second month. If she is evicted she will spend the second month with her parents (she could also stay there the first month, but would prefer to avoid staying there altogether). The subjective cost to Abby of devoting effort to maintaining the apartment the apartment is

$$\delta(e) = \frac{1}{2}\underline{\delta} e^2, \quad e \in [0,1]$$

Bob would like to maximize the total rent he receives minus the cost of the necessary repairs resulting from Abby's not taking proper care of the apartment, $c(e)$ which is decreasing and convex in its argument. He tells Abby that he will terminate here lease after the first month with probability $t(e) = 1 - e$ (Bob cannot observe e so his termination schedule is based on the condition of the apartment when he monitors it which will be better, the more effort that Abby has devoted). If Bob terminates her lease the apartment will remain unrented for the second month. Both Abby and Bob are entirely self-regarding; the passage of time is so limited (just two months) that neither time-discount the costs and benefits associated with their transaction.

1. How much effort will Abby devote to maintaining the apartment in the second month?
2. Write down the first order condition of Abby's maximizing problem and the resulting best response function.
3. What is Bob's maximizing problem? What constraint(s) must he satisfy? (Show both his participation constraint and incentive compatibility constraint).
4. Suppose that
$$c(e) = \frac{1}{2}(e-1)^2$$
$z = \underline{\delta} = 1$ and $v = 2$, what rents will Bob charge?

Answer.

1. For Abby, given (r_1, r_2), the utility function is
$$u = v - r_1 - \delta(e_1) + [1 - t(e_1)][v - r_2 - \delta(e_2)] + t(e_1)z$$
Since
$$\frac{\partial u}{\partial e_2} = -e_1\delta'(e_2) = -\underline{\delta}e_1e_2 \le 0$$
Abby will set $e_2 = 0$.
2. Abby's maximizing problem is
$$\max_{e_1} \quad u = v - r_1 - \frac{1}{2}\underline{\delta}e_1^2 + e_1(v - r_2) + (1 - e_1)z$$
The first order condition is
$$\frac{\partial u}{\partial e_1} = -\underline{\delta}e_1 + (v - r_2 - z) = 0$$
Therefore, the best response function is
$$e_1 = \frac{v - r_2 - z}{\underline{\delta}}$$

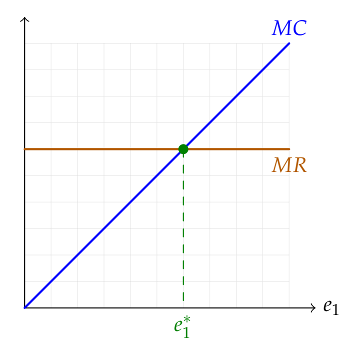

Figure 1: Abby's choice in the Rental Housing Market: she would choose an effort level e_1^* to equate marginal cost and marginal benefit, where
$$MC = \delta'(e_1) = \underline{\delta}e_1$$
and
$$MR = t'(e_1)(v - r_2 - z)$$

3. Bob's maximizing problem is

$$\begin{aligned}
\max_{r_1, r_2} \quad & \pi = r_1 - c(e_1) + e_1[r_2 - c(0)] \\
\text{s.t.} \quad & u = v - r_1 - \delta(e_1) + e_1(v - r_2) + (1 - e_1)z \geq 2z \qquad \text{(PC)} \\
& e_1 = \frac{v - r_2 - z}{\underline{\delta}} \qquad \text{(ICC)}
\end{aligned}$$

4. It is clear that PC must be binding, otherwise Bob can always rise the rent in the first month r_1. Therefore, we have

$$r_1 = v - \frac{1}{2}\underline{\delta}e_1^2 + e_1(v - r_2 - z) - z$$

By ICC, we have $\frac{de_1}{dr_2} = -\frac{1}{\underline{\delta}}$. Then

$$\frac{dr_1}{dr_2} = -\underline{\delta}e_1\frac{de_1}{dr_2} - e_1 + \frac{de_1}{dr_2}(v - r_2 - z) = -e_1$$

and therefore the first order condition is

$$\begin{aligned}
\frac{d\pi}{dr_2} &= \frac{dr_1}{dr_2} - c'(e_1)\frac{de_1}{dr_2} + \frac{de_1}{dr_2}[r_2 - c(0)] + e_1 \\
&= -e_1 + \frac{c'(e_1)}{\underline{\delta}} - \frac{r_2 - c(0)}{\underline{\delta}} + e_1 \\
&= \frac{1}{\underline{\delta}}[c'(e_1) - r_2 + c(0)] = 0
\end{aligned}$$

Since

$$c(e) = \frac{1}{2}(e-1)^2$$

we have $c'(e) = e - 1$ and then

$$\frac{v - r_2 - z}{\underline{\delta}} - 1 = r_2 - c(0)$$

With $z = \underline{\delta} = 1$ and $v = 2$,

$$1 - r_2 - 1 = r_2 - \frac{1}{2} \Rightarrow r_2 = \frac{1}{4}$$

and then $e_1 = \frac{3}{4}$, $r_1 = 41/32$

□