Econ 702 Midterm Spring 2020 (Solutions)

Please write your answers on the exam. Provide enough explanation that I know that you got the answers correct because you knew how to solve the problem rather than by guessing.

There are four questions. You have one hour and 15 minutes. Good luck!

1. (a) (4 points) Write down the definition of a pure strategy Nash equilibrium.

   A profile a is a Nash equilibrium if $u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$ for all $a'_i \in A_i$, for all players $i \in N$

   (b) (4 points)

   i. What does it mean for strategy s_i to weakly dominate strategy s'_i?

   $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ *for all* $s_{-i} \in S_{-i}$

   ii. Write down the definition of a weakly dominant strategy.

   A strategy is a weakly dominant if it weakly dominates all other strategies.

   Consider a voting game with three voters Ann, Bob and Carol, and two candidates X and Y. Ann and Bob prefer X to Y and Carol prefers Y to X.

   Each voter must vote for one of the two candidates, and the candidate with the most votes wins.

   (c) (4 points) Is any action weakly dominant for Carol? If so, which actions is weakly dominant for Carol? If not, explain why not.

   Yes, voting for Y is weakly dominant for Carol.

   (d) (8 points) Find the set of all *pure strategy* Nash equilibria in this game.

   {*(X,X,X), (X,X,Y), (Y,Y,Y)*}

   Now suppose that there are three candidates, X, Y and Z. Z is the preferred candidate for all players. So Ann and Bob prefer Z to X to Y and Carol prefers Z to Y to X. Again each voter votes for one candidate. The candidate who gets the most votes wins. If X, Y, and Z each get one vote, then X wins.

   (e) (6 points) In this version of the game, is any action weakly dominant for Carol? If so, which action is weakly dominant for Carol? If not, explain why not.

   No action is weakly dominant for Carol, Z is not weakly dominat *since (X,Y,Z) is worse than (X,Y,Y). Y is not weakly dominant since (Z,Y,Y) is worse than (Z,Y,Z). X is not weakly dominant since (Z,Y,X) is worse than (Z,Y,Z).*

2. Consider a game between a goalie and a kicker. The kicker either kicks to the left or kicks to the right, and the goalie either jumps to the left or jumps to the right. The following table shows the probability of a successful goal for every possibility.

	Kick Left	Kick Right
Jump Left	.6	.9
Jump Right	.8	.7

Suppose that the kicker would like to maximize the probability that a goal is scored and the goalie would like to minimize that probability

(a) (3 points) Is there a pure strategy equilibrium in this game?
NO. For example, at (Left, Left) the kicker would choose Right, at (Left, Right) the goalie would choose Right and so on.

(b) (5 points) If the goalie jumps left with probability p and jumps right with probability $1-p$, what is the probability that the kicker scores if the kicker kicks left? What is the probability that the kicker scores if the kicker kicks right? (Your answers should be a function of p)
Left: $0.6p + 0.8(1-p) = 0.8 - 0.2p$
Right: $0.9p + 0.7(1-p) = 0.7 + 0.2p$

(c) (5 points) If the kicker kicks left with probability q and kicks right with probability $1-q$, what is the probability that the kicker scores if the goalie jumps left? What is the probability that the kicker scores if the goalie jumps right? (Your answers should be a function of p)
Left: $0.6q + 0.9(1-q) = 0.9 - 0.3q$
Right: $0.8q + 0.7(1-q) = 0.7 + 0.1q$

(d) (8 points) Find the mixed strategy equilibrium of this game.

Suppose that [(p,1-p),(q,1-q)] is a mixed strategy equlibirum of this game. Then we have

$$0.7 + 0.2p = 0.8 - 0.2p \Rightarrow p = \frac{1}{4}$$
$$0.7 + 0.1q = 0.9 - 0.3q \Rightarrow q = \frac{1}{2}$$

Therefore, the mixed strategy equilibrium of this game is [(0.25, 0.75), (0.5,0.5)].

(e) (3 points) Write down the definition of a zero sum game.

A two-player game is zero-sum if for all strategy profile s and s', $s \succeq_1 s' \Leftrightarrow s' \succeq_2 s$.

(f) (3 points) Is this a zero sum game?

Yes.

(g) (8 points) Now suppose that the kicker's brother is watching the game, and tells the kicker publicly that he would prefer that the kicker kicked left. Suppose that the kicker would both like to score and would also prefer to kick to the left to please his brother. The kicker trades off these two goals. To be concrete you can imagine that it is as if scoring is worth \$$x$ to the agent and kicking to the left is worth \$$y$, where y is assumed to be small in comparison to x. You can think of the situation as if it is one in which the kicker tries to maximize his expected winnings. Suppose the goalie knows all this, and that the goalie's preferences are unchanged–the goalie only wants to minimize the probability that the kicker scores. Suppose moreover that there is still a mixed strategy equilibrium in this new version of the game. Does the kicker's probability of kicking to the left go up, go down or stay the same in the mixed strategy equilibrium of this new game relative to the game without the brother? Does the goalie's probability of jumping to the left go up, go down or stay the same in the mixed strategy equilibrium of this new game relative to the game without the brother?

Hint: Observe that if the probability of scoring by kicking to the left and right were equal, then the kicker would prefer to kick to the left, but if the probability of scoring by kicking to the right were sufficiently larger than the probability of scoring by kicking to the left, then the kicker will prefer to kick to the right.

At the mixed strategy equilibrium, both the kicker and the goalie are indifferent between Left and Right. To the make the goalie indifferent, the kicker's probability of kicking to the left stay the same. However, since the kicker would prefer to kick to the left if the probability of score by kicking to the left and right were equal, to make her indifferent the probability of score by kicking to the left should be higher, which requires that goalie's probability of jumping to the left goes up.

3. (a) (5 points) State the independence axiom.

   For all $\ell, \ell', \ell'' \in \Delta$ and for all $\alpha \in (0, 1)$,

$$\ell \preceq \ell' \Leftrightarrow \alpha\ell + (1-\alpha)\ell'' \preceq \alpha\ell' + (1-\alpha)\ell''$$

   (b) (5 points) State the continuity axiom.

   If $\ell \prec \ell' \prec \ell''$, then there exists $\alpha \in (0, 1)$ such that

$$\ell' \sim \alpha\ell + (1-\alpha)\ell''$$

Suppose that there are three outcomes $X = \{1, 2, 3\}$ and let Δ be the set of lotteries. Suppose that for all lotteries $\ell = (p_1, p_2, p_3), \ell' = (p_1', p_2', p_3')$,

$$\ell \precsim \ell' \Leftrightarrow p_2 + 2p_3 \leq p_2' + 2p_3'$$

   (c) (8 points) Does $\precsim$ satisfy the independence axiom? If so, prove it. If not prove that it doesn't.

Yes. For any $\ell = (p_1, p_2, p_3), \ell' = (p'_1, p'_2, p'_3), \ell'' = (p''_1, p''_2, p''_3)$,

$$
\begin{aligned}
& \ell \precsim \ell' \\
\Leftrightarrow & p_2 + 2p_3 \le p'_2 + 2p'_3 \\
\Leftrightarrow & \alpha(p_2 + 2p_3) + (1-\alpha)(p''_2 + 2p''_3) \le \alpha(p'_2 + 2p'_3) + (1-\alpha)(p''_2 + 2p''_3) \\
\Leftrightarrow & [\alpha p_2 + (1-\alpha)p''_2] + 2[\alpha p_3 + (1-\alpha)p''_3] \le [\alpha p'_2 + (1-\alpha)p''_2] + 2[\alpha p'_3 + (1-\alpha)p''_3] \\
\Leftrightarrow & \alpha\ell + (1-\alpha)\ell' \precsim \alpha\ell' + (1-\alpha)\ell''
\end{aligned}
$$

4. Consider a two player game in which each player each player must select a strategy $x_i \in [0,1]$ and the players' payoffs are:

$$u_i(x_i, x_{-i}) = x_i(1 - x_i - x_{-i})$$

   (a) (5 points) Provide an interpretation of this game, that is, a story of what the game is about.
   The game of two fishermen, with the fishing time x_i*, output* $y_i = x_i(1 - x_{-i})$*, and utility* $u_i = y_i - x_i^2$.

   (b) (8 points) Find a pure strategy Nash equilibrium to this game.
   The best reply:

$$1 - x_{-i} - 2x_i = 0 \Rightarrow x_i = \frac{1 - x_{-i}}{2}$$

   Suppose that (x^*, x^*) *is a pure strategy Nash equilibrium, then we have*

$$1 - 3x^* = 0 \Rightarrow x^* = \frac{1}{3}$$

   (c) (8 points) Is the Nash equilibrium Pareto optimal? If so, prove it. If not, find a strategy profile that Pareto dominates the equilibrium strategy profile.
   No. The profile $(0.25, 0.25)$ *Pareto dominates the equilibrium pro-*

file since

$$u_i(\frac{1}{4}, \frac{1}{4}) = \frac{1}{8} > u_i(\frac{1}{3}, \frac{1}{3}) = \frac{1}{9}$$