Econ 702 Game Theory Problem Set 1 Solution

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February 5, 2020

Exercise 2. Prove that completeness implies reflexivity.

Proof. For all $x \in X$, either $x \succeq x$ or $x \preceq x$ by completeness. Then $x \succeq x$. □

Exercise 4.

1. Prove that if there exists a utility function u that represents $\succeq$, then $\succeq$ is complete and transitive.
2. Does this part of the theorem depend on the assumption that X is finite?

Solution.

1. By the completeness of $\geq$ on $\mathbb{R}$, for any $x, y \in X$, we have either

$$u(x) \geq u(y) \text{ or } u(y) \geq u(x)$$

then either $x \succeq y$ or $y \succeq x$, since u represents $\succeq$. Thus, the $\succeq$ is complete.

Suppose $x \succeq y$ and $y \succeq z$, then we have $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Therefore, by the transitivity of $\geq$ on $\mathbb{R}$ we have $u(x) \geq u(z)$ and then $x \succeq z$. Thus, $\succ$ is transitive.

2. No.

□

Exercise 8. Prove that if $\preceq$ has a representation of the expected utility form, then $\preceq$ satisfies the four axioms.

Proof. Suppose that $\preceq$ has a representation of the expected utility form $U(l) = U(p_1, \ldots, p_n) = \sum_k p_k u_k$. Then

$$U(\alpha l + (1-\alpha) l'') = \sum_k [\alpha p_k + (1-\alpha) p''_k] u_k = \alpha U(l) + (1-\alpha) U(l'') \tag{1}$$

Continuity. If $l \prec l' \prec l''$, let $U(l) = a, U(l') = b, U(l'') = c$, then $a < b < c$. Let

$$\alpha = \frac{c-b}{c-a}$$

then $\alpha \in (0,1)$ and

$$\begin{aligned} U(\alpha l + (1-\alpha)l'') &= \alpha U(l) + (1-\alpha)U(l'') \\ &= \frac{c-b}{c-a}a + \frac{c-b}{c-a}c \\ &= \frac{cb - ac + ac - ba}{c-a} \\ &= b = U(l') \end{aligned}$$

Therefore, $\alpha l + (1-\alpha)l'' \sim l'$.

Independence. For all l, l', l'' and all $\alpha \in (0,1)$, by (1) we have

$$\begin{aligned} l \preceq l' &\Leftrightarrow U(l) \leq U(l') \\ &\Leftrightarrow \alpha U(l) + (1-\alpha)U(l'') \leq \alpha U(l') + (1-\alpha)U(l'') \\ &\Leftrightarrow U(\alpha l + (1-\alpha)l'') \leq U(\alpha l' + (1-\alpha)l'') \\ &\Leftrightarrow \alpha l + (1-\alpha)l'' \preceq \alpha l' + (1-\alpha)l'' \end{aligned}$$

For completeness and transitivity, see the proof in **Exercise 4.** □

Exercise 9. Show that for all lotteries l in Δ, $\delta_1 \preceq l \preceq \delta_n$ when $n > 3$.

Proof. Note that $\delta_1 = p_1\delta_1 + p_2\delta_1 + \cdots + p_n\delta_1$, $\delta_n = p_1\delta_n + \cdots + p_n\delta_n$ and

$$l = (p_1, \ldots, p_n) = p_1\delta_1 + p_2\delta_2 + \cdots + p_n\delta_n$$

Then by independence and transitivity we have $\delta_1 \preceq l \preceq \delta_n$ when $n = 2$. Suppose that it holds when $n = k-1$, then for $n = k$, we have

$$\begin{aligned} \delta_1 &= (p_1\delta_1 + p_2\delta_1 + \cdots + p_{k-2}\delta_1 + p_{k-1}\delta_1) + p_k\delta_1 \\ &\preceq (1-p_k)\frac{1}{1-p_k}(p_1\delta_1 + p_2\delta_1 + \cdots + p_{k-2}\delta_1 + p_{k-1}\delta_1) + p_k\delta_k \text{ (independence)} \\ &\preceq (1-p_k)l + p_k\delta_k = l \text{ (induction hypothesis)} \\ &\preceq p_1\delta_k + (p_2\delta_2 + \cdots + p_{k-1}\delta_{k-1} + p_k\delta_k) \\ &\preceq p_1\delta_k + (1-p_1)\delta_k = \delta_k \end{aligned}$$

□

Exercise 10. Suppose that we have

$$f(\alpha l + (1-\alpha)l') = \alpha f(l) + (1-\alpha)f(l')$$

Use the above lemma to derive the following lemma when $n > 3$: for any $l = (p_1, \ldots, p_n)$, $f(l) = \sum_k p_k u_k$ where $u_k = f(\delta_k)$.

Proof. Let $K = \min\{k : \sum_{i=1}^k p_i = 1\}$ then for any $k \leq K$ we have $\sum_{i=k}^n p_i = 1 - \sum_{i=1}^{k-1} p_i > 0$ and $p_k = 0$ for all $k > K$. Then

$$l = p_1\delta_1 + \cdots + p_K\delta_K$$

and

$$
\begin{aligned}
f(l) &= f(p_1\delta_1 + (1-p_1)\sum_{i=2}^{K}\frac{p_i}{1-p_1}\delta_i) = p_1 f(\delta_1) + (1-p_1) f(\sum_{i=2}^{K}\frac{p_i}{1-p_1}\delta_i) \\
&= p_1 u_1 + (1-p_1) f(\frac{p_2}{1-p_1}\delta_2 + \frac{1-p_1-p_2}{1-p_1}\sum_{i=3}^{K}\frac{p_i}{1-p_1-p_2}\delta_i) \\
&= p_1 u_1 + p_2 u_2 + (1-p_1-p_2) f(\sum_{i=3}^{K}\frac{p_i}{1-p_1-p_2}\delta_i) \\
&\dots \\
&= p_1 u_1 + \cdots + p_{K-1}u_{K-1} + (1-p_1-\cdots-p_{K-1}) f(\frac{p_K}{1-p_1-\cdots-p_{K-1}}\delta_K) \\
&= \sum_{k=1}^{K} p_k u_k = \sum_{k=1}^{n} p_k u_k
\end{aligned}
$$

□

Exercise 11. Prove the following **Theorem** $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ represent the same vNM preference relation if and only if there exist numbers $a > 0$ and b such that for all k, $v_k = au_k + b$.

Proof. ($\Leftarrow$) For any $l = (p_1, \dots, p_n) \in \Delta$ we have

$$v(l) = \sum_k p_k v_k = \sum_k p_k(au_k + b) = a\sum_k p_k u_k + nb = au(l) + nb$$

Then for any $l, l' \in \Delta$,

$$u(l) \geq u(l') \Leftrightarrow au(l) + nb \geq au(l') + nb \Leftrightarrow v(l) \geq v(l')$$

($\Rightarrow$) Without loss of generality, assume that $\delta_1 \preceq \delta_2 \preceq \cdots \preceq \delta_n$ and $\delta_1 \prec \delta_n$. By continuity, for all i, there exist $\alpha_i \in [0,1]$ such that $\delta_i \sim \alpha_i\delta_1 + (1-\alpha_i)\delta_n$. Since u and v represent the $\preceq$, for all i

$$
\begin{aligned}
u_i &= \alpha_i u_1 + (1-\alpha_i)u_n = u_n - \alpha_i(u_n - u_1) \\
v_i &= \alpha_i v_1 + (1-\alpha_i)v_n = v_n - \alpha_i(v_n - v_1)
\end{aligned}
$$

Then

$$v_i = v_n - \frac{v_n - v_1}{u_n - u_1}(u_n - u_i)$$

Let

$$
\begin{aligned}
a &= \frac{v_n - v_1}{u_n - u_1} > 0 \\
b &= v_n - au_n
\end{aligned}
$$

then we have $v_i = au_i + b, \forall i$ □

Exercise 12. Prove the following Proposition.

1. An agent is risk neutral if for all $x, y \in \mathbb{R}$ and all $\lambda \in [0,1]$, $\lambda u(x) + (1-\lambda)u(y) = u(\lambda x + (1-\lambda)y)$
2. An agent is risk averse if for all $x, y \in \mathbb{R}$ and all $\lambda \in [0,1]$, $\lambda u(x) + (1-\lambda)u(y) \leq u(\lambda x + (1-\lambda)y)$
3. An agent is risk loving if for all $x, y \in \mathbb{R}$ and all $\lambda \in [0,1]$, $\lambda u(x) + (1-\lambda)u(y) \geq u(\lambda x + (1-\lambda)y)$

Proof. I prove the case of risk averse by induction. For any lottery $L = (p_1, x_1; \ldots, p_n, x_n)$, it is sufficient to show that

$$U(L) = \sum_{i=1}^{n} p_i u(x_i) \leq u(\sum_{i=1}^{n} p_i x_i) = u(\delta_{\mu_L})$$

by the concavity of u. It holds when $n = 2$. Suppose that it holds for $n = k-1$. When $n = k$, for some $p_j \neq 1$,

$$\begin{aligned} u(\sum_{i=1}^{k} p_i x_i) &= u(p_j x_j + (1-p_j)\sum_{i \neq j} \frac{p_i}{1-p_j} x_i) \\ &\geq p_j u(x_j) + (1-p_j) u(\sum_{i \neq j} \frac{p_i}{1-p_j} x_i) \text{ by concavity of } u \\ &\geq p_j u(x_j) + (1-p_j) \sum_{i \neq j} \frac{p_i}{1-p_j} u(x_i) \text{ by the induction hypothesis} \\ &= \sum_{i=1}^{k} p_i u(x_i) \end{aligned}$$

□