Econ 702 Game Theory Midterm Practice Problems Solution

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Exercise 1. Prove that the two versions of the Characterization of Mixed Strategy Nash Equilibrium of Finite Game are equivalent.

(i) For each player i and action $a_i \in A_i$,

$$\sigma_i^*(a_i) > 0 \Rightarrow a_i \in P_i(\sigma_{-i}^*)$$

where $P_i = \{a_i \in A_i : U_i(a_i, \sigma_{-i}) \geq U_i(a_i', \sigma_{-i}) \forall a_i' \in A_i\}$.

(ii) For each player i and action $a_i, a_i' \in A_i$,

$$\sigma_i^*(a_i) > 0 \quad \& \quad \sigma_i^*(a_i') > 0 \Longrightarrow U_i(a_i, \sigma_{-i}^*) = U_i(a_i', \sigma_{-i}^*)$$
$$\sigma_i^*(a_i) > 0 \quad \& \quad \sigma_i^*(a_i') = 0 \Longrightarrow U_i(a_i, \sigma_{-i}^*) \geq U_i(a_i', \sigma_{-i}^*)$$

Proof. (ii) $\Rightarrow$ (i) is obvious from the defintions. Next I show (i) $\Rightarrow$ (ii). First, if $\sigma_i^*(a_i) > 0$ and then by (i) we have $a_i \in P_i(\sigma_{-i}^*)$. Therefore, $U_i(a_i, \sigma_{-i}^*) \geq U_i(a_i', \sigma_{-i}^*)$.

Moreover, if at the same time $\sigma_i^*(a_i') > 0$, then again by (i) we have $a_i' \in P_i(\sigma_{-i}^*)$ and $U_i(a_i', \sigma_{-i}^*) \geq U_i(a_i, \sigma_{-i}^*)$. So the equality holds. □

Exercise 3. The game Bach vs. Stravinsky is given by Table 1.

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

Table 1: The Back vs. Stravinsky Game

1. Find Ann's mixed strategy in the mixed strategy equilibrium strategy in Bach vs. Stravinsky.
2. Find all the pure and mixed strategy equilibira of this game.
3. Do any of the equilibria Pareto dominate any of the other equilibria? If so, what dominates what?

Solution. Suppose that $[(p, 1-p), (q, 1-q)]$ is a mixed strategy equilibrium, then we have

$$U_1(B, (q, 1-q)) = 2q = U_1(S, (q, 1-q)) = 1 - q \Rightarrow q = 1/3$$
$$U_2(B, (p, 1-p)) = p = U_2(S, (p, 1-p)) = 2(1-p) \Rightarrow p = 2/3$$

Two pure strategy equlibria are (B,B) and (S,S). At the mixed strategy equlibrium $[(2/3, 1/3), (1/3, 2/3)]$, $U_1 = U_2 = 2/3$. Therefore, the two pure strategy equlibria Pareto dominate the mixed strategy equlibirum. □

Exercise 4. Complete the argument on the slide in Figure 1. In particular, show that $(0, 1/3, 2/3)$ is a best reply for player 2 against player 1's strategy of $(3/4, 0, 1/4)$.

Solution. Player 2's payoffs are:

$$\begin{aligned} L &: \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{5}{2} \\ C &: \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{5}{2} \\ R &: \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 7 = \frac{5}{2} \end{aligned}$$

So any probability distribution is a best reply for player 2. □

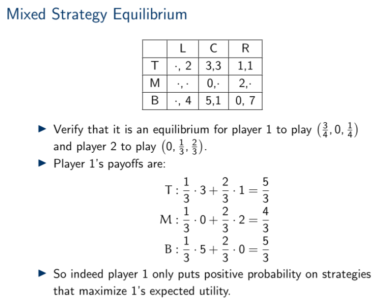

Figure 1: Exercise 4

Exercise 7(1). Prove: The strategy profile σ is an equilibrium if and only if $g_i^j(\sigma) = 0$ for each $i \in I$ and $j = 1, 2, \ldots, m_i$, where

$$g_i^j(\sigma) = \max\{0, U_i(a_i^j, \sigma_{-i}) - U_i(\sigma)\}$$

Proof. Suppose that $g_i^j(\sigma) = 0$ for each $i \in I$ and $j = 1, \ldots, m_i$, then by definition

$$U_i(a_i^j, \sigma_{-i}) - U_i(\sigma) \le 0 \Rightarrow U_i(a_i^j, \sigma_{-i}) \le U_i(\sigma), \forall i \forall j$$

Then for any mixed strategy $\sigma_i' = (p_1, \ldots, p_{m_i})$, we have

$$U_i(\sigma_i', \sigma_{-i}) = \sum_{j=1}^{m_i} p_j U_i(a_i^j, \sigma_{-i}) \le U_i(\sigma) \quad \forall i \forall j$$

Therefore, $\sigma_i \in B_i(\sigma_{-i})$ for any player i, and σ is a Nash equilibrium.

Suppose that σ is an equlibrium, then $\sigma_i \in B_i(\sigma_{-i})$ for all i. Then

$$U_i(\sigma) \ge U_i(a_i^j, \sigma_{-i}), \forall j$$

and thus $g_i^j(\sigma) = 0$. □

Exercise 9. A pure strategy a_i' is strictly dominated if there exists a mixed strategy $\sigma_i \in \Delta(A_i)$ such that

$$U_i(\sigma_i, \sigma_{-i}) > U_i(a_i', \sigma_{-i}) \quad \forall \sigma_{-i}$$

Prove that a_i' is strictly dominated if and only if there exists $\sigma_i \in \Delta(A_i)$ such that

$$U_i(\sigma_i, a_{-i}) > U_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i}$$

Proof. $(\Rightarrow)$ can be easily shown by choosing $\sigma_{-i} = a_{-i}$. On the other hand, suppose that there exists $\sigma_i \in \Delta(A_i)$ such that

$$U_i(\sigma_i, a_{-i}) > U_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i}$$

then for any σ_{-i}, we have

$$\begin{aligned} U_i(\sigma_i, \sigma_{-i}) &= \sum_{a_{-i}} \sigma_{-i}(a_{-i}) U_i(\sigma_i, a_{-i}) \\ &> \sum_{a_{-i}} \sigma_{-i}(a_{-i}) U_i(a_i', a_{-i}) \\ &= U_i(a_i', \sigma_{-i}) \end{aligned}$$

□