Notes on Coordination Problem

Weikai Chen

October 7, 2019

For simplicity, we consider a symmetric game with continuous strategic spaces and well-behaved utility function $u(a, A)$ and $U(A, a)$.

1 *What information can we get from a set of indifference curves?*

First, recall the following conditions about the social interaction at Nash equilibrium mathematically:

1. Positive Externality: $u_A > 0$
2. Negative Externality: $u_A < 0$
3. Strategic Complements: $u_{aA} > 0$
4. Strategic Substitutes: $u_{aA} < 0$

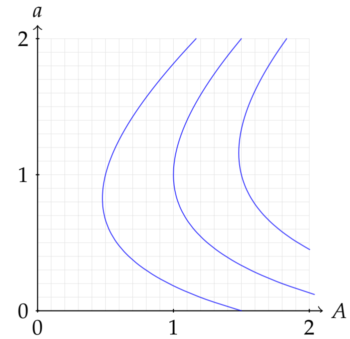

Figure 1: Given a set indifference curves $u(a, A) = \bar{u}_i$, what can you tell about the structure of interaction? Do we have externality? If so, is that positive or negative? How about strategic complements or substitutes?

Now suppose that we have a set of indifference curves as shown in Figure 1, what can you tell about the externality and strategic complements/substitutes?

Step 1 How does the utility change if my opponent increases her action?

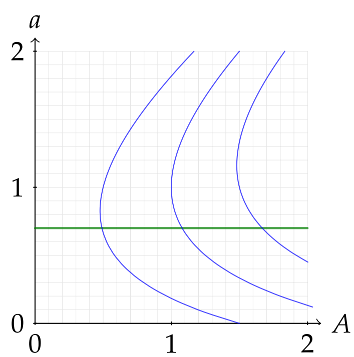

Figure 2: Given a set indifference curves $u(a, A) = \bar{u}_i$, fix a and increase A, we know that u increases. Therefore, there is positive externality.

Step 2 Find the best response function in the figure. As shown in Figure 3, the best response function is upward-sloping, then we could conclude that it is strategic complements.

Indeed, we can show the following statement:

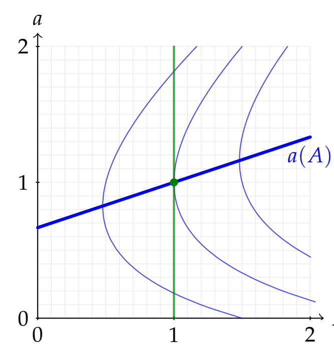

Figure 3: Given a set indifference curves $u(a, A) = \bar{u}_i$, the tangency of any verticle line and the indifference curve is the best response. Why?

Proposition 1.1. *At Nash equilibirum* (a^*, A^*),

$$\frac{da(A^*)}{dA} > 0 \Leftrightarrow u_{aA} > 0$$

Proof. At Nash equilibirum, we have $u_a(a^*, A^*) = 0$, then

$$u_{aa}da + u_{aA}dA = 0 \Rightarrow \frac{da(A^*)}{dA} = -\frac{u_{aA}}{u_{aa}}$$

by implicit function theorem. Assume that the second order condition holds, that is, $u_{aa} < 0$, then $\frac{da(A^*)}{dA}$ and u_{aA} have the same sign. $\square$

Exercise 1.1. How about the set of indifference curves shown in Figure 4?

Exercise 1.2. Draw a figure for the case with negative externality and strategic substitutes.

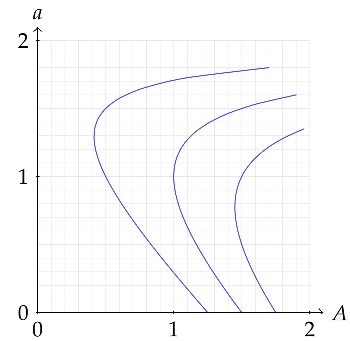

Figure 4: Given a set indifference curves $u(a, A) = \bar{u}_i$, what can you tell about the structure of interaction? Do we have externality? If so, is that positive or negative? How about strategic complements or substitutes?

2 *Stability: what happen if one deviates a little bit?*

Now assume that we have the best response function f for both players, i.e., $A(a) = f(a)$ and $a(A) = f(A)$, as shown in Figure 5. Then the Nash equlibrium is just a fixed point of the function f, i.e., $a^* = A^* = x^*$ such that

$$x^* = f(x^*)$$

Suppose that one has a small deviation from the Nash equilibrium, say $a_1 = x^* + \Delta x$ for some small Δx, what will happen?

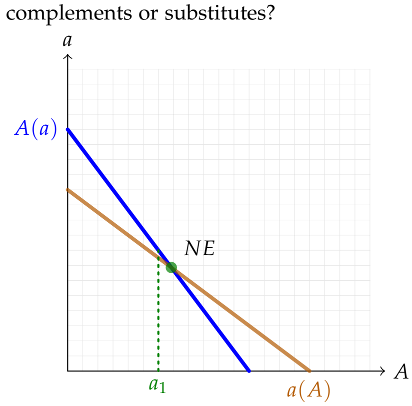

Figure 5: Given the best response functions, suppsoe that $a_1 = x^* + \Delta x$.

- In words, both will move back to the equlibrium if they won't overreact. In such a case, we say the equilibrium is (asymptotically) stable.
- Graphically, we show the dynamics in Figure 6.

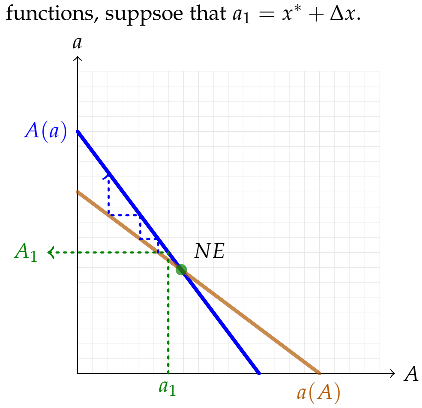

Figure 6: Unstable Equilibrium: A dynamic path.

- Mathematically, we have the equilibrium is stable if $|f'| < 1$, and unstable if $|f'| > 1$. Why?

  Here is a simple "proof". Consider the following difference equation

$$x_{n+1} = f(x_n) \tag{1}$$

  and let $x_1 = a_1 = x^* + \Delta x$, then the dynamics can be characterized by the difference equation (1). For example, $A_1 = f(a_1) = f(x_1) = x_2$ and so on. Then use the first-order approximation at x^*, we have the following:

$$\begin{aligned}
x_2 &= f(x^* + \Delta x) = f(x^*) + f'(x^*)\Delta x = x^* + f'(x^*)\Delta x \\
x_3 &= f[x^* + f'(x^*)\Delta x] = x^* + [f'(x^*)]^2 \Delta x \\
&\dots \\
x_n &= x^* + [f'(x^*)]^{n-1}\Delta x
\end{aligned}$$

  Then if $|f'| < 1$, we have $[f'(x^*)]^n \to 0$ as $n \to \infty$, and thus $x_n \to x^*$.

Exercise 2.1. Draw a figure showing the case of stable equlibrium.

3 Social Multiplier: the accumulated effect of the change in some exogenous parameters

Now suppose that utlity function contains some exogenous parameter s, i.e., $u(a, A; s)$ and $U(A, a; s)$, and the derived best response functions become $a(A; s)$ and $A(a; s)$. Denote the Nash equilibrium by a^*, A^* where $a^* = A^* = a^N(s)$.

Then the partial effect of the change in s on action is $\frac{\partial a}{\partial s}$, while the total effect of the change in s is $\frac{da^N}{ds}$. Define the social multiplier m by

$$\frac{\partial a}{\partial s}(1 + m) = \frac{da^N}{ds} \tag{2}$$

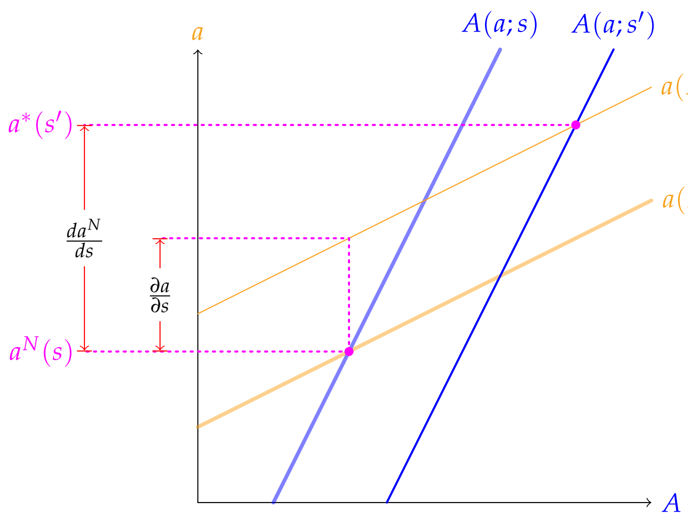

Figure 7: The Social Multiplier: strategic complements and positve social multiplier.

Exercise 3.1. Draw a graph to show multiplier with strategic substitutes.

Exercise 3.2. Draw a graph to show the case with $m = 0$.