Econ 702 Game Theory Problem Set 3 Solution

Weikai Chen

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Exercise 0. The following question is due to Steve Landsburg. I quote him verbatim:

> The five Dukes of Earl are scheduled to arrive at the royal palace on each of the first five days of May. Duke One is scheduled to arrive on the first day of May, Duke Two on the second, etc. Each Duke, upon arrival, can either kill the king or support the king. If he kills the king, he takes the king's place, becomes the new king, and awaits the next Duke's arrival. If he supports the king, all subsequent Dukes cancel their visits. A Duke's first priority is to remain alive, and his second priority is to become king. Who is king on May 6?

Solution. As shown in Figure 1, Duke One is king on May 6. □

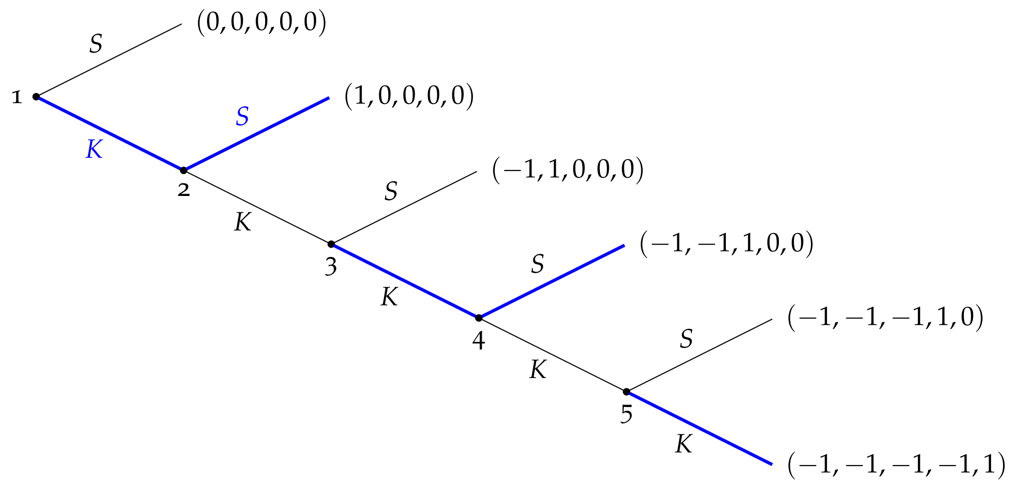

Figure 1: The game tree.

Exercise 1. Prove that the following conditios are sufficient for a downward sloping best reply

1. P is decreasing and concave
2. More generally, 2's marginal revenue is decreasing in q_1

Note that either assumption implies

$$\frac{\partial^2 \Pi_2}{\partial q_1 \partial q_2}(q_1, q_2) = P'(q_1 + q_2) + q_2 P''(q_1 + q_2) < 0 \tag{1}$$

Proof. For player 2, the best reply $q_2 = b_2(q_1)$ is given by the first order condition

$$\frac{\partial \Pi_2}{\partial q_2}(q_1, q_2) = 0$$

Then
$$\frac{\partial^2 \Pi_2}{\partial q_1 \partial q_2}(q_1, q_2)dq_1 + \frac{\partial^2 \Pi_2}{\partial^2 q_2^2}(q_1, q_2)dq_2 = 0$$
and thus
$$b_2'(q_1) = \frac{dq_2}{dq_1} = -\frac{\partial^2 \Pi_2/\partial q_1 \partial q_2}{\partial^2 \Pi_2/\partial^2 q_2^2} < 0$$
by (1) and the second order condition
$$\frac{\partial^2 \Pi_2}{\partial q_2^2} < 0$$
□

Exercise 168.1 (Osborne). Which of the Nash equlibria of the game in Figure 2 are subgame perfect?

Proof. As shown in Figure 2, ((D,G),E) is a subgame perfect Nash equilibrium. □

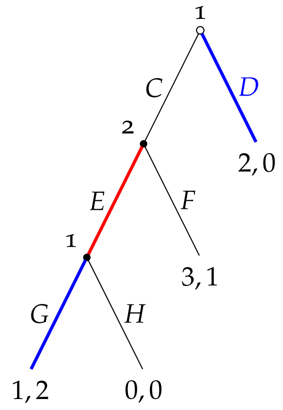

Figure 2: An extensive game in which player 1 moves both before and after player 2.

Exercise 176.1 (Osborne).(Dollar auction) An object that two people each value at v (a positive integer) is sold in an auction. In the auction, the people alternately have the opportunity to bid; a bid must be a *positive integer greater than the previous bid*. (In the situation that gives the game its name, v is 100 cents.) On her turn, a player may pass rather than bid, in which case the game ends and the other player receives the object; both players pay their last bids (if any). (If player 1 passes initially, for example, player 2 receives the object and makes no payment; if player 1 bids 1, player 2 bids 3, and then player 1 passes, player 2 obtains the object and pays 3, and player 1 pays 1.) Each person's wealth is w, which exceeds v; neither player may bid more than her wealth. For $v = 2$ and $w = 3$ model the auction as an extensive game and find its subgame perfect equilibria. (A much more ambitious project is to find all subgame perfect equilibria for arbitrary values of v and w.)

Solution. The extensive game can be represented in Figure 3. It has four subgame perfect equilibria. In all the equilibria player 2 passes after player 1 bids \$2 (orange in the figure). The four equilibria are as follows.

1. ((1,3),(P,P)): player 1 bids 1 at the beginning, and 3 at the history (1,2); player 2 bids P at history 1 and 2. (shown in red)

2. ((P,P),(2,P)): player 1 bids P at the beginning and history (1,2); player 2 bids 2 at hisotry 1 and P at hisotry 2. (shown in blue)

3. ((2,P),(2,P)): player 1 bids 2 at the beginning and P at history (1,2); player 2 bids 2 at history 1 and P at history 2. (shown with double arrows)

4. ((1,P),(P,P)): player 1 bids 1 at the beginning and P at history (1,2); player 2 bids P at history 1 and history 2. (shown in dotted)

□

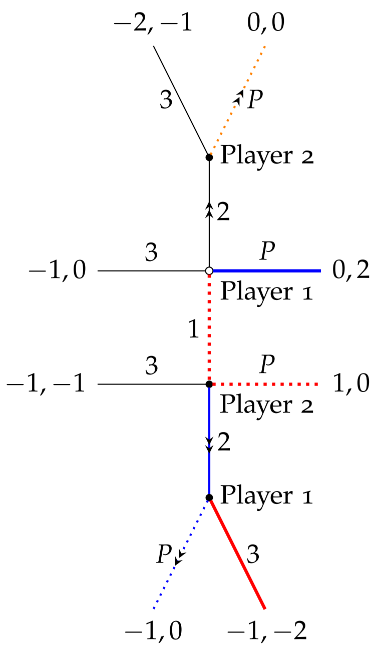

Figure 3: The game tree of the dollar auction with $v = 2$ and $w = 3$, A pass is denoted by P.

Exercise 177.3 (Osborne). (Comparing simultaneous and sequential games) The set of ac tions available to player 1 is A_1; the set available to player 2 is A_2. Player 1's preferences over pairs (a_1, a_2) are represented by the payoff $u_1(a_1, a_2)$, and player 2' preferences are represented by the payoff $u_2(a_1, a_2)$. Compare the Nash equilibria (in pure strategies) of the strategic game in which the players choose actions simultaneously with the subgame perfect equilibria of the extensive game in which player 1 chooses an action, then player 2 does so. (For each history a_1 in the extensive game, the set of actions available to player 2 is A_2.)

a. Show that if, for every value of a_1, there is a unique member of A_2 that maximizes $u_2(a_1, a_2)$, then in every subgame perfect equilibrium of the extensive game, player 1's payoff is at least equal to her highest payoff in any Nash equilibrium of the strategic game.

b. Show that player 2's payoff in every subgame perfect equilibrium of the extensive game may be higher than her highest payoff in any Nash equilibrium of the strategic game.

c. Show that if for some values of a_1 more than one member of A_2 maximizes $u_2(a_1, a_2)$, then the extensive game may have a subgame perfect equilibrium in which player 1's payoff is less than her payoff in all Nash equilibria of the strategic game.

(For parts b and c you can give examples in which both A_1 and A_2 contain two actions.)

Solution.

a. Let (a_1^*, a_2^*) be the Nash equilibrium of the strategic game such that player 1 achieve the highest payoff in any NE. In the extensive game, at history a_1^*, player 2 would choose a_2^* at any subgame perfect equilibrium since $u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2)$ for any $a_2 \in A_2$ by the definition of NE. Then at any subgame perfect equilibrium, $u_1 \geq u_1(a_1^*, a_2^*)$.

b. Suppose that $A_1 = \{U, D\}$, $A_2 = \{L, R\}$ and the payoff is given by Table 1. There exists a unique NE at strategic game (U, L) in which player 2's payoff is 1. In the extensive game, the unique perfect subgame equilbrium is (U, LR) in which player 2's payoff is 3.

c. Suppose that $A_1 = \{U, D\}$, $A_2 = \{L, R\}$ and the payoff is given by Table 2. The strategic game has a unique NE (U, L) in which player 1's payoff is 2. In one of the subgame perfect equilibria, (D, RL), player 1's payoff is 1.

□

	L	R
U	1,1	3, 0
D	0, 0	2, 3

Table 1: The payoff matrix in Exercise 177.3(b).

	L	R
U	2,2	0, 2
D	1, 1	4, 0

Table 2: The payoff matrix in Exercise 177.3(c).

Exercise 183.4 (Osborne). (Variants of ultimatum game and impunity game with equity – conscious players) Consider variants of the ultimatum game and impunity game in which each person cares not only about the amount of money she receives, but also about the equity of the allocation. Specifically, suppose that person i's preferences are represented by the payoff function given by $u_i(x_1, x_2) = x_i - \beta_i |x_1 - x_2|$, where x_i is the amount of money person i receives, $\beta_i > 0$. Find the set of subgame perfect equilibria of each game and compare them. Are there any values of β_1 and β_2 for which an offer is rejected in equilibrium?

Solution. I find the subgame perfect equlibrium in the ultimatum game. The same method can be applied to the impunity game. For player 2, given the offer x, she would accept it only if

$$x - \beta_2|1-2| \geq 0$$

Let

$$a = \frac{\beta_2}{1+2\beta_2}, b = \frac{\beta_2}{2\beta_2 - 1}$$

then we have the set of offers that player 2 could accept is given by

$$a \leq x \leq 1 \text{ if } \beta_2 \leq 1$$
$$a \leq x \leq b \text{ if } \beta_2 > 1$$

as shown in Figure 4.

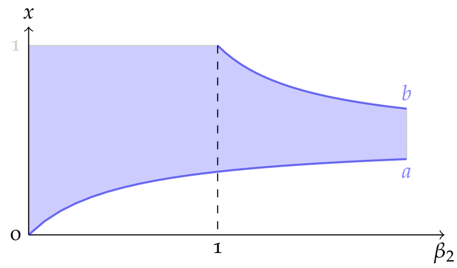

Figure 4: The set of offers x that person 2 accepts.

For player 1, her payoff is 0 if her offer is rejected and if it is accepted

$$u_1(x) = 1 - x - \beta_1|1-2x| = \begin{cases} 1 - \beta_1 + (2\beta_1 - 1)x & x \leq \frac{1}{2} \\ 1 + \beta_1 - (2\beta_1 + 1)x & x > \frac{1}{2} \end{cases}$$

If $\beta_1 < \frac{1}{2}$, $u_2(x)$ is deacreasing in x and thus player 1 would choose the smallest one that person 2 accepts. If player 2's strategy rejects the offer a, then player 1 has no optimal response. Thus in any subgame perfect equilibrium player 1 offers a, and player 2 accepts any offer $x \in [a, b)$, rejects any offer $x \in [0, a) \cup (b, 1]$, either accepts or rejects the offer b.

If $\beta_1 = \frac{1}{2}$, then $u_1(x)$ is constant in $[0, \frac{1}{2}]$ and decreasing when $x > \frac{1}{2}$. Therefore, in any subgame perfect equilibrium, player 1 could offer $x \in [a, 1/2]$ and player 2 accepts $x \in (a, b)$, rejects $x \in [0, a) \cup (b, 1]$, either accepts or rejects b, either accepts or rejects a unless it is offered, in which case she would accept it.

If $\beta_1 > \frac{1}{2}$, then $u_1(x)$ is increasing up to $x = \frac{1}{2}$ and then decreasing and thus reaches its maximum at $x = \frac{1}{2}$. Therefore, in any subgame perfect equilibrium, player 1 offers $x = \frac{1}{2}$, player 2 accepts any offer $x \in (a, b)$, rejects $x \in [0, a) \cup (b, 1]$, either accepts or rejects $x = a, b$. □

Exercise 191.1 (Osborne). (Stackelberg's duopoly game with fixed costs) Suppose that the inverse demand function is given by

$$P_d(Q) = \begin{cases} \alpha - Q & Q \leq \alpha \\ 0 & Q > \alpha \end{cases}$$

and the cost function of each firm i is given by

$$C_i(q_i) = \begin{cases} 0 & q_i = 0 \\ f + cq_i & q_i > 0, \end{cases}$$

where $c \geq 0$, $f > 0$ and $c < \alpha$. Show that if $c = 0, \alpha = 12$, and $f = 4$, Stackelberg's game has a unique subgame perfect equilibrium, in which firm 1's output is 8 and firm 2's output is zero.

Solution. Given any history q_1, let

$$b(q_1) = \arg\max_{q_2} q_2 P_d(q_1 + q_2) - C_2(q_2)$$

When $q_1 \geq \alpha = 12$, it is clear that $b(q_1) = 0$. For $q_1 < \alpha = 12$, the first ordr condition is

$$\alpha - q_1 - 2q_2 = 0 \Rightarrow q_2 = \frac{\alpha - q_1}{2} = 6 - \frac{q_1}{2} > 0$$

Then firm 2's payoff becomes

$$u_2 = \left(6 - \frac{q_1}{2}\right)^2 - 4$$

and it would rather set $q_2 = 0$ if $u_2 < 0$, i.e., $q_1 > 8$. Therefore, we have

$$b(q_1) = \begin{cases} 6 - \frac{q_1}{2} & q_1 < 8 \\ \{0, 2\} & q_1 = 8 \\ 0 & q_1 > 8 \end{cases}$$

For firm 1, if it sets $q_1 < 8$, then

$$u_1 = \left(6 - \frac{q_1}{2}\right) q_1 - 4 = -\frac{1}{2}(6 - q_1)^2 + 14$$

attains the maximum 14, which is higher than the payoff when $q_1 = 8, q_2 = 2$.

However, for $q_1 \geq 8$ and $q_2 = 0$, $u_1 = (12 - q_1)q_1 - 4$ is decreasing and thus attains the maximum 28 at $q_1 = 8$, which is higher than 14. Thus, there eixsts a unique subgame perfect equilibrium $q_1^* = 8$ and

$$q_2^* = \begin{cases} 6 - \frac{q_1}{2} & q_1 < 8 \\ 0 & q_1 \geq 8 \end{cases}$$

at which firm 1's output is 8 and firm 2's output is zero. □

Exercise 196.2 (Osborne). (Interest groups buying votes under supermajority rule) Consider an alternative variant of the model in which a supermajority is required to pass a bill. There are two bills, X and Y, and a "default outcome". A bill passes if and only if it receives at least $k^* > \frac{1}{2}(k+1)$ votes; if neither bill passes the default outcome occurs. There are two interest groups. Both groups attach value 0 to the default outcome. Find the bill that is passed in any subgame perfect equilibrium when $k = 7$, $k^* = 5$, and (a) $V_X = V_Y = 700$ and (b) $V_X = 750$, $V_Y = 400$. In each case, would the legislators be better off or worse off if a simple majority of votes were required to pass a bill?

Solution.

(a) However group X allocates payments summing to 700, group Y can buy off five legislators for at most 500. Thus in any subgame perfect equilibrium neither group makes any payment, and bill Y is passed.

(b) If group X pays each legislator 80 then group Y is indifferent between buying off five legislators, in which case bill Y is passed, and in making no payments, in which case bill X is passed. If group Y makes no payments then X is selected, and group X is better off than it is if it makes no payments. There is no subgame perfect equilibrium in which group Y buys off five legislators, because if it were to do so group X could pay each legislator slightly more than 80 to ensure the passage of bill X. Thus in every subgame perfect equi- librium group X pays each legislator 80, group Y makes no payments, and bill X is passed.

If only a simple majority is required to pass a bill, in case a the outcome under majority rule is the same as it is when five votes are required. In case b, group X needs to pay each legislator 100 in order to prevent group Y from winning. If it does so, its total payments are less than V_X , so doing so is optimal. Thus in this case the payment to each legislator is higher under majority rule.

□