

**Problem Set 1 will be posted
today and due next Friday, Jan 24
(10:00pm)**

**PRR 1 will be posted (in Canvas) after
class, and due next Tuesday 12:00pm**



Class 2: Defining Definitions

University of Virginia
CS3120: DMT2

<https://weikailin.github.io/cs3120-toc>
Wei-Kai Lin

Questions from Survey

Q: What's the difference between this course and Floryan's?

A: Different textbook and outline. Circuit vs context-free grammar.

Q: Can we discuss with students in the other course?

A: Discuss with other students, people, or LLM is encouraged (unless explicitly disallowed, eg, in exams).

Copying in any format is cheating

(Diff: You understand your answer, or not.)

Write on your own.

HW release and due dates?

Problem Set (PS), total 10 (nearly every week):

- Release: Thursdays (after class, before midnight)
- Due: 1 week later Friday 10pm
- Late submission may be allowed

Pre-reading and reflect (PRR):

- Release: Thursdays weekly (ideally right after class)
- Due: next Monday 10pm
- No late submission

Post slides before class?

Probably not. Reasons:

- Last minute edits
- Taking notes is a (probably important) skill
- I need interaction and eye contact

(Actually I prefer whiteboard over slides)

Question from Last Class

How did
Babylonian number
system work?

Class 1

Place-value number system

- Babylonian (2000BC)

1	11	21	31	41	51
2	12	22	32	42	52
3	13	23	33	43	53
4	14	24	34	44	54
5	15	25	35	45	55
6	16	26	36	46	56
7	17	27	37	47	57
8	18	28	38	48	58
9	19	29	39	49	59
10	20	30	40	50	

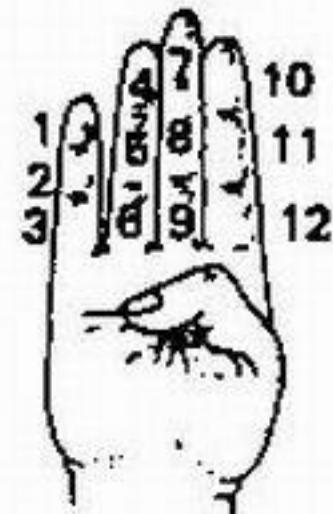
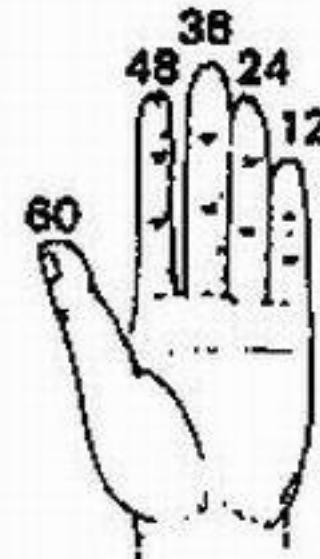
- The number: $x_k x_{k-1} x_{k-2} \dots x_2 x_1 x_0$

Means: $\sum x_i \cdot b^i$

- Extremely important development in history!

Babylonian Counting

𒐧 1	𒐧 11	𒐧 21	𒐧 31	𒐧 41	𒐧 51
𒐧 2	𒐧 12	𒐧 22	𒐧 32	𒐧 42	𒐧 52
𒐧 3	𒐧 13	𒐧 23	𒐧 33	𒐧 43	𒐧 53
𒐧 4	𒐧 14	𒐧 24	𒐧 34	𒐧 44	𒐧 54
𒐧 5	𒐧 15	𒐧 25	𒐧 35	𒐧 45	𒐧 55
𒐧 6	𒐧 16	𒐧 26	𒐧 36	𒐧 46	𒐧 56
𒐧 7	𒐧 17	𒐧 27	𒐧 37	𒐧 47	𒐧 57
𒐧 8	𒐧 18	𒐧 28	𒐧 38	𒐧 48	𒐧 58
𒐧 9	𒐧 19	𒐧 29	𒐧 39	𒐧 49	𒐧 59
𒐧 10	𒐧 20	𒐧 30	𒐧 40	𒐧 50	



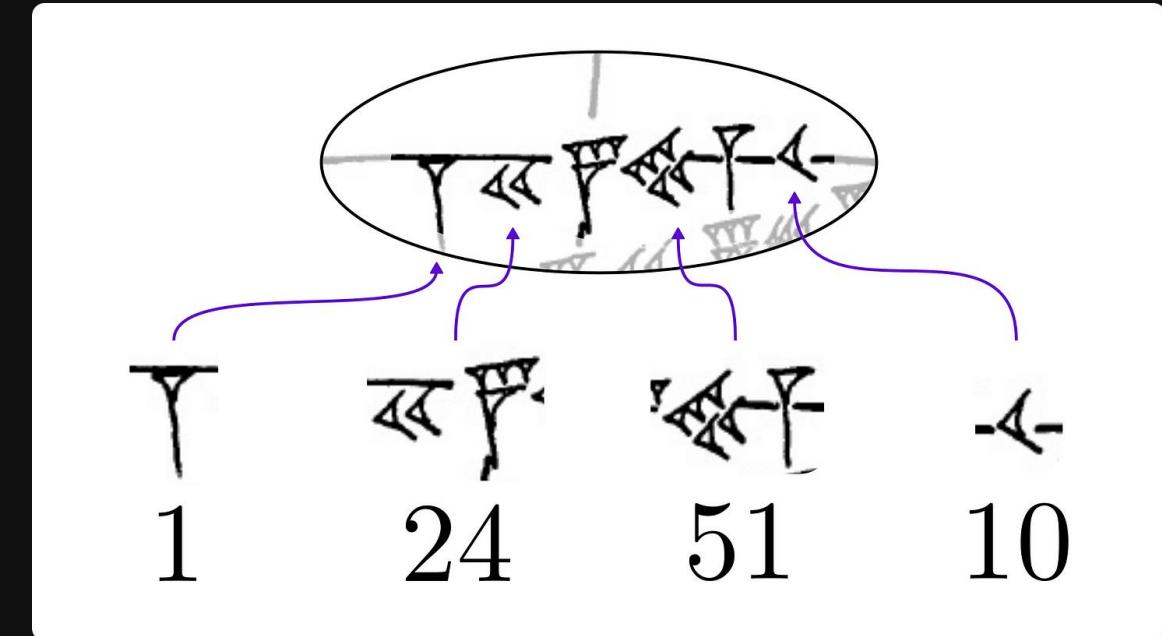
YBC 7289

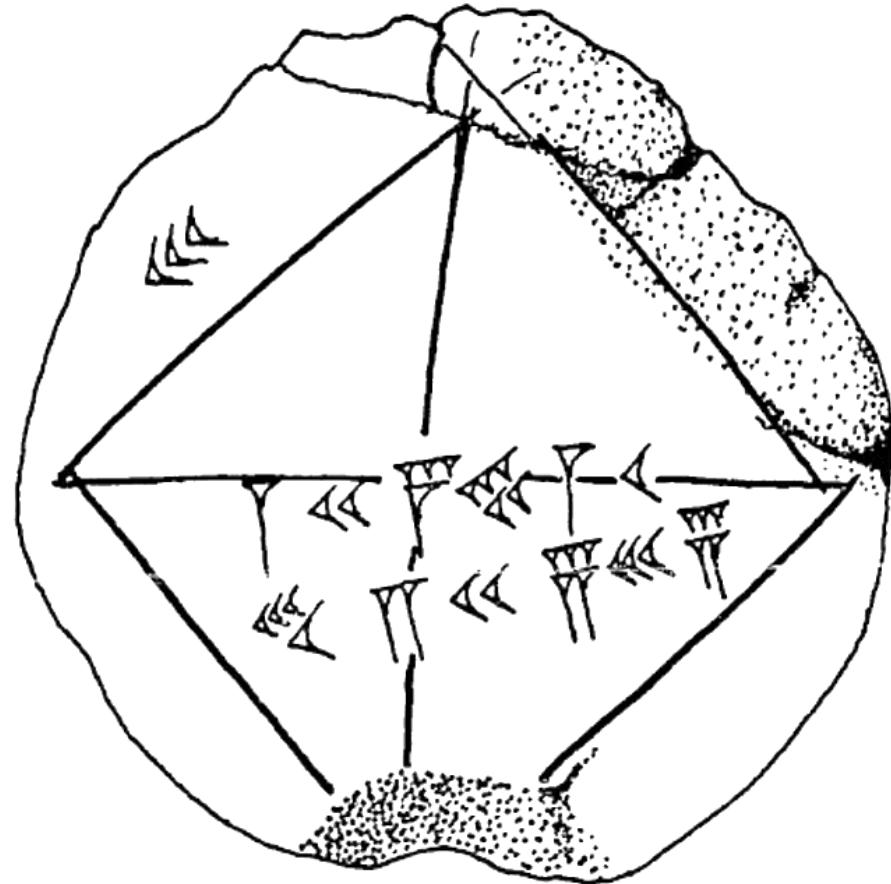
Babylonian clay tablet from
~1700 BCE





YBC 7289
Babylonian clay tablet from
~1700 BCE





YBC 7289
Babylonian clay tablet from
~1700 BCE

1 24 51 10

$$= 1 \cdot 60^0 + 24 \cdot 60^{-1} + 51 \cdot 60^{-2} + 10 \cdot 60^{-3}$$

What is the number

$$1 \cdot 60^0 + 24 \cdot 60^{-1} + 51 \cdot 60^{-2} + 10 \cdot 60^{-3}$$

Let's ask LLM/AI

- <https://copilot.cloud.microsoft>
- <https://gemini.google.com>

Plan

Mathematical Definitions

What makes a good definition?

Today: Chapter 0 and 1 in the TCS book
Next week: start Chapter 2

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due next Tuesday before 12pm**

Defining the Natural Numbers

Induction Principle

Sets

**Problem Set 1 will be
posted today and due
next Friday , Jan 24
(10:00pm)**

PRR 1 (Pre-Reading and Reflection)

- Posted (in Canvas). Due next Tuesday before 12pm.
- Read textbook sections and watch two videos, and
- Answer multiple-choice questions (I will tell you two of them the answer today!)

Quick check your understanding, and independent reading

Problem Set 1

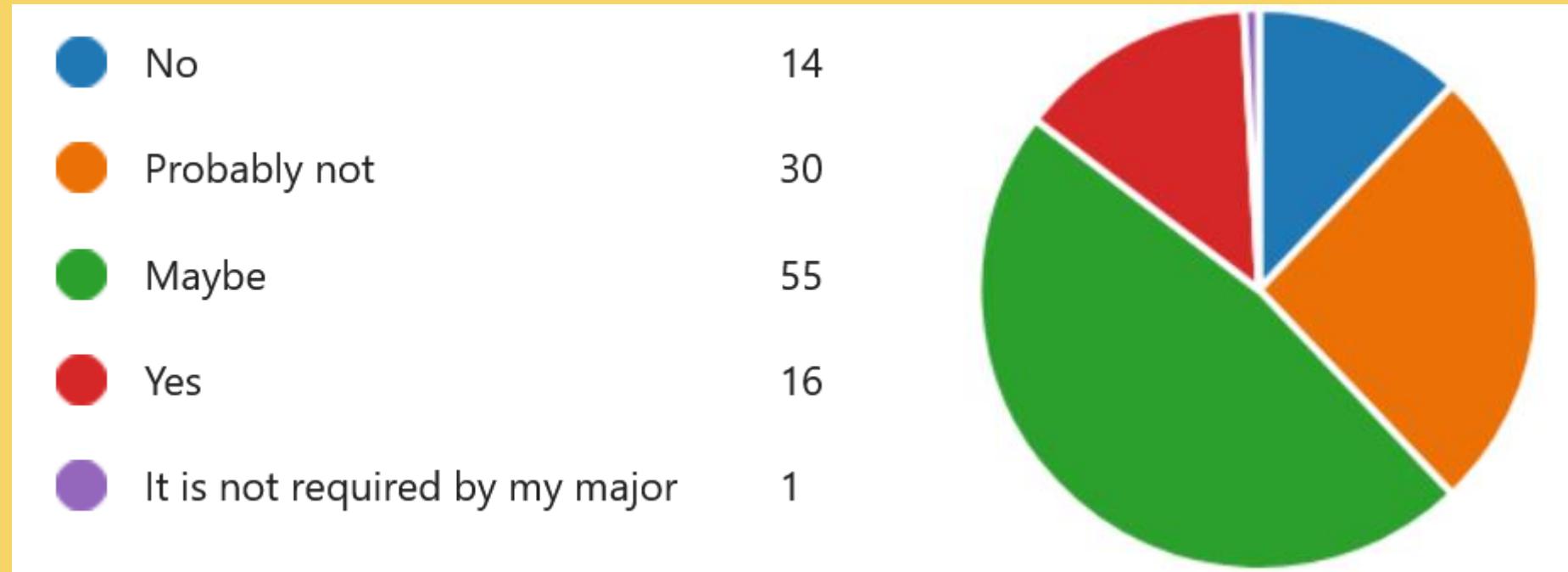
- You can use any resources you want
- You can discuss problems with anyone you want, but must write up solutions yourself and understand what you write
- You must credit the resources and people

Problem Sets are about understanding the concepts and learning how to solve problems – you will need to demonstrate this on your own on the exams

Office Hours Start Next Tuesday!

<https://weikailin.github.io/cs3120-toc/calendar/>

Would you be taking this class if it was not required for your major (or desired major)?



Would you be
for your major?

Maybe 16 people actually
want to be in this class!

as not required

But only 1 for sure!

No

14

Probably not

30

Maybe

55

Yes

16

It is not required by my major

1



99/116 (>84%) are
probably here
under duress!

44/116 (>37%) are
here under duress!

What would you like to know about the course, instructors, or computing?

How it can help me outside of a career in computer science.

How can I learn to appreciate the theoretical realm of computing if I have always worked with concrete examples and systems?

Why study theory?

My Answer

It is fun! Like LEGO



E.g., Karatsuba built the same multiply but taking fewer bricks

(Maybe) Your Answer

1. Think Better and Bigger

Karatsuba's multiplication

$$\begin{array}{r} \bar{x} \quad x \\ \times \quad \bar{y} \quad y \\ \hline (\bar{x}+x)(\bar{y}+y) \quad xy \\ + \quad \bar{x}\bar{y} \quad -\bar{x}\bar{y}-xy \\ \hline \bar{x}\bar{y} \quad (\bar{y}+y) \quad xy \end{array}$$

What Chris Newcombe, senior engineer at Oracle, wrote.

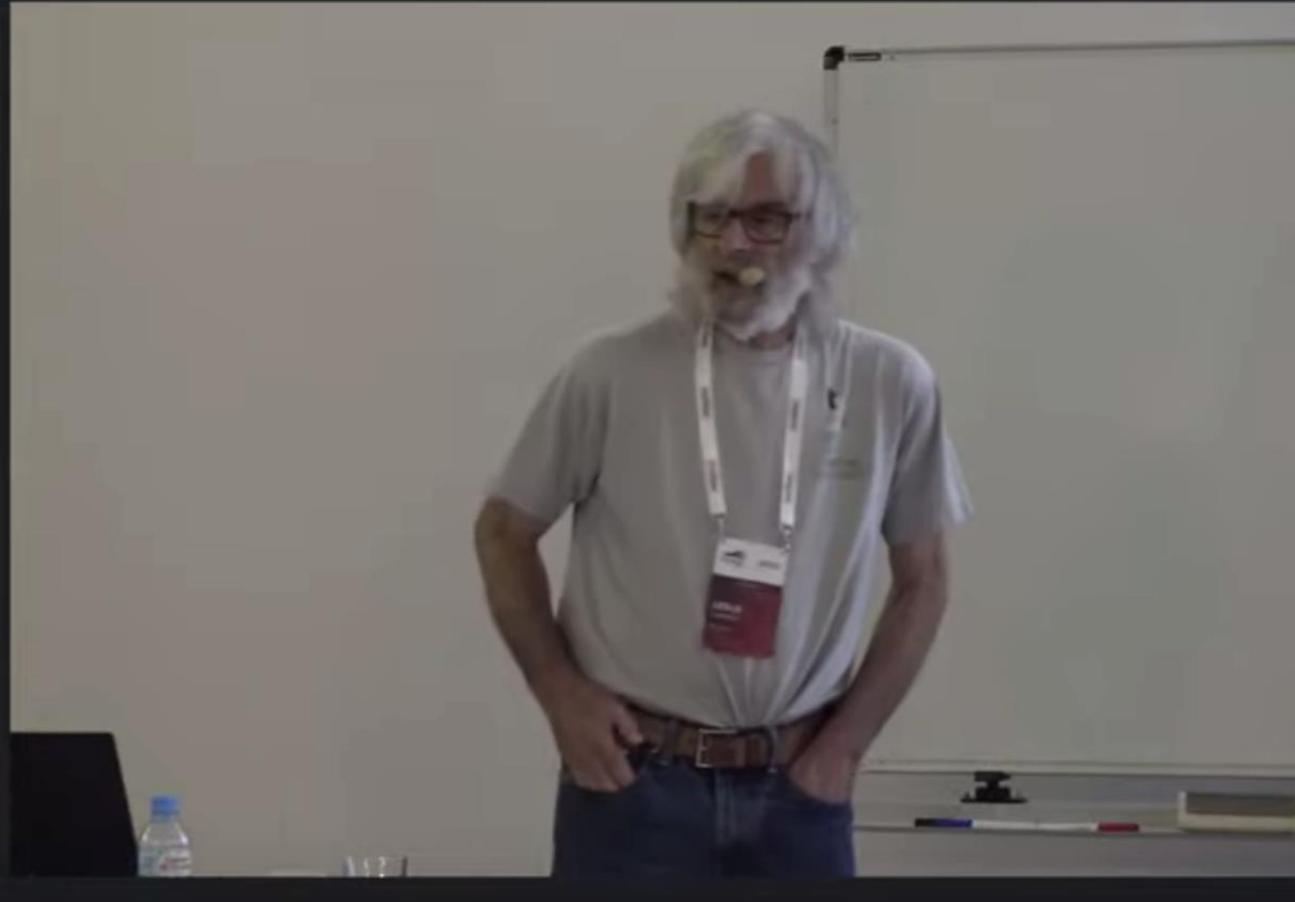
Mathematical thinking

TLA+ is the most valuable thing that I've learned in my professional career.

It has changed how I work

It has changed how I think

43



<https://lamport.azurewebsites.net/tla/paxos-algorithm.html>
<https://www.youtube.com/watch?v=tw3gsBms-f8>

Leslie Lamport
(2013 Turing Awardee)

(Maybe) Your Answer

1. Think Better and Bigger

2. Intrinsic Truth and Beauty

x	$\eta(x)$	$g(0)$	$g(1)$	$g(2)$	$g(3)$	$g(4)$	$g(5)$	\dots
11	1	$StF("")\{0\}$	$StF("")\{1\}$	$StF("")\{2\}$	$StF("")\{3\}$	$StF("")\{4\}$	$StF("")\{5\}$	\dots
0	2	$StF(0)\{0\}$	$StF(0)\{1\}$	$StF(0)\{2\}$	$StF(0)\{3\}$	$StF(0)\{4\}$	$StF(0)\{5\}$	\dots
1	3	$StF(1)\{0\}$	$StF(1)\{1\}$	$StF(1)\{2\}$	$StF(1)\{3\}$	$StF(1)\{4\}$	$StF(1)\{5\}$	\dots
00	4	$StF(00)\{0\}$	$StF(00)\{1\}$	$StF(00)\{2\}$	$StF(00)\{3\}$	$StF(00)\{4\}$	$StF(00)\{5\}$	\dots
10	5	$StF(10)\{0\}$	$StF(10)\{1\}$	$StF(10)\{2\}$	$StF(10)\{3\}$	$StF(10)\{4\}$	$StF(10)\{5\}$	\dots
01	6	$StF(01)\{0\}$	$StF(01)\{1\}$	$StF(01)\{2\}$	$StF(01)\{3\}$	$StF(01)\{4\}$	$StF(01)\{5\}$	\dots
\dots	\dots							

Cantor's Diagonalization Proof
(Chapter 2 and upcoming class)

(Maybe) Your Answer

- 1. Think Better and Bigger**
- 2. Intrinsic Truth and Beauty**
- 3. Track record for Impact and Success**

Definition 1 Let $E(u)$ be some vector over the Web pages that corresponds to a source of rank. Then, the PageRank of a set of Web pages is an assignment, R' , to the Web pages which satisfies

$$R'(u) = c \sum_{v \in B_u} \frac{R'(v)}{N_v} + cE(u) \quad (1)$$

such that c is maximized and $\|R'\|_1 = 1$ ($\|R'\|_1$ denotes the L_1 norm of R').

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The PageRank Citation Ranking: Bringing Order to the Web

Lawrence Page, Sergey Brin,
Rajeev Motwani, Terry Winograd

January 29, 1998

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No theorems or proofs in this paper! (but makes claims about convergence)

6.1 Manipulation by Commercial Interests

These types of personalized PageRanks are virtually immune to manipulation by commercial interests. For a page to get a high PageRank, it must convince an important page, or a lot of non-important pages to link to it. At worst, you can have manipulation in the form of buying advertisements(links) on important sites. But, this seems well under control since it costs money. This immunity to manipulation is an extremely important property. This kind of commercial manipulation is causing search engines a great deal of trouble, and making features that would be great to have very difficult to implement. For example fast updating of documents is a very desirable feature, but it is abused by people who want to manipulate the results of the search engine.

Dynamic Itemset Counting and Implication Rules for Market Basket Data

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Stanford University

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Shalom Tsur

R&D Division, Hitachi America Ltd.

tsur@hitachi.com

Copy Detection Mechanisms for Digital Documents *

Sergey Brin, James Davis, Hector Garcia-Molina

Department of Computer Science

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Stanford, CA 94305-2140

e-mail: sergey@cs.stanford.edu

Near Neighbor Search in Large Metric Spaces

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sergey@cs.stanford.edu

BOUNDS FOR SORTING BY PREFIX REVERSAL

William H. GATES

Microsoft, Albuquerque, New Mexico

Christos H. PAPADIMITRIOU*†

Department of Electrical Engineering, University of California, Berkeley, CA 94720, U.S.A.

Received 18 January 1978

Revised 28 August 1978

For a permutation σ of the integers from 1 to n , let $f(\sigma)$ be the smallest number of prefix reversals that will transform σ to the identity permutation, and let $f(n)$ be the largest such $f(\sigma)$ for all σ in (the symmetric group) S_n . We show that $f(n) \leq (5n + 5)/3$, and that $f(n) \geq 17n/16$ for n a multiple of 16. If, furthermore, each integer is required to participate in an even number of reversed prefixes, the corresponding function $g(n)$ is shown to obey $3n/2 - 1 \leq g(n) \leq 2n + 3$.

Theorem 1. Algorithm \mathcal{A} creates a permutation with $n - 1$ adjacencies by at most $(5n - 7)/3$ moves.

Proof. First, it is clear that if we have a permutation σ with less than $n - 1$ adjacencies, one of the cases 1 through 7 is applicable. Hence, the algorithm does not halt unless $n - 1$ adjacencies have been created. Obviously the algorithm will eventually halt, since at each execution of the main loop at least one new

Lemma 1. Let A_i be an (a, b) cut.

- (1) If $a, b > 1$, then both A_i, A_{i+1} are wastes.
- (2) If the only one of $a, b > 1$, then either A_i or A_{i+1} is a waste.

Proof. An easy case-by-case analysis.

Claim 1. There are exactly m events.

To prove Claim 1, we notice that χ_0 is k -stable for $k = 1, \dots, m$, and $\chi_{f(\chi)}$ is not k -stable for any k . Furthermore, no permutation can cease being k_1 -stable and k_2 -stable, $k_1 \neq k_2$, in only one move.

Theorem 4. $g(e_n^R) \geq \frac{3}{2}n - 1$.

Theorem 2. $19n/16 \geq f(\chi) \geq 17n/16$.

Proof. To show the upper bound, we first do the following sequences of moves

$$x \rightarrow \tau_2 \tau_1^R \tau_3 \cdots \rightarrow \tau_2^R \tau_1^R \tau_3 \cdots \rightarrow \tau_1 \tau_2 \tau_3 \cdots$$

and so on, bringing the even-indexed τ 's in front and then back with the reversal cancelled in three moves. Thus, in $3n/16$ moves we obtain $\chi' = \tau_1 \tau_2 \tau_3 \cdots \tau_m$. Then, for each copy of τ in χ' we repeat the following sequence of eight moves (among a number of possibilities)

Theorem 3. $g(n) \leq 2n + 3$.

Proof. First observe that $g(\sigma)$ is not greater than $f(\sigma')$ where $\sigma' \in S_{2n}$ is defined as follows, for each $\sigma \in S_n$: $\sigma'(2i-1) = 2\sigma(i)-1$ and $\sigma'(2i) = 2\sigma(i)$ for all $i = 1, \dots, n$. The complexity of sorting σ' without the restriction can now be bounded from above by the algorithm \mathcal{A} of Section 2. The equations governing the complexity of \mathcal{A} when applied to σ' are (1), (2), and (3) of Section 2 with n replaced by $2n$, $b = a = n$, and also noting that only x_5 can be nonzero, since all other actions are possible only in the presence of free elements. The maximum is therefore $2n - 2$. Allowing five more moves to sort the resulting permutation, we get the claimed bound.

Claim 3. For all j , $1 \leq j \leq m - 1$, there exists a waste χ_l with $i_j \leq l \leq i_{j+1}$.

To prove Claim 3, suppose that it fails. In other words, suppose that there is an event i_j other than the last one, such that all moves χ_l , $i_j \leq l \leq i_{j+1}$ construct a new adjacency without destroying an existing adjacency. Suppose that k is the

BOUNDS FOR SORTING

William H. GATES

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"Two years later, I called to tell him our paper had been accepted to a fine math journal. He sounded eminently disinterested. He had moved to Albuquerque, New Mexico to run a small company writing code for microprocessors, of all things. I remember thinking:

'Such a brilliant kid. What a waste.'

BOUNDS FOR SORTING BY PREFIX REVERSAL

William H. GATES

Microsoft, Albuquerque, New Mexico

1. Introduction

We introduce our problem by the following quotation from [1]

The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function $f(n)$ of n) that I will ever have to use to rearrange them?

largest at the bottom) by
this (varying the number
is the maximum number
rearrange them?

References

- [1] Amer. Math. Monthly 82 (1) (1975) 1010.

(Maybe) Your Answer

1. Think Better and Bigger
2. Intrinsic Truth and Beauty
3. Track record for Impact and Success

Ask LLM:

- Largest companies by market cap
- How about other CEOs?

(Maybe) Your Answer

- 1. Think Better and Bigger**
- 2. Intrinsic Truth and Beauty**
- 3. Track record for Impact and Success**
- 4. Being wrong often and admitting it**

Survey Question on Course Pledge

According to the Course Pledge, which of the following is expected of students in this class? (check all that apply)

- I should not lie, cheat, or steal. 114
- I will be open minded and curio... 112
- I will provide useful feedback. 113
- I will make sure to suffer before ... 13

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We do not want you to lie,
cheat, or steal!

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We do not want you to lie,
cheat, or steal!

We do not want you to suffer!

This two questions **will be repeated on PRR1** just we make sure everyone understands this!

How are we doing on “I will be open minded and curious” and “I will provide useful feedback”?

-  I should not lie, cheat, or steal. 114
-  I will be open minded and curio... 112
-  I will provide useful feedback. 113
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How are we doing on “I will be open minded and curious” and “I will provide useful feedback”?

- I should not lie, cheat, or steal. 114
- I will be open minded and curio... 112
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- I will make sure to suffer before ... 13

20. What would you like to know about the course, instructors, or computing?

[More Details](#)

 Insights

35

Responses

We will (eventually!) answer the questions on the survey
(more questions are always welcome)

Is 0 a Natural Number?

8. Is 0 a Natural Number?

[More Details](#)

 Insights

 Yes

 No

 What's that?

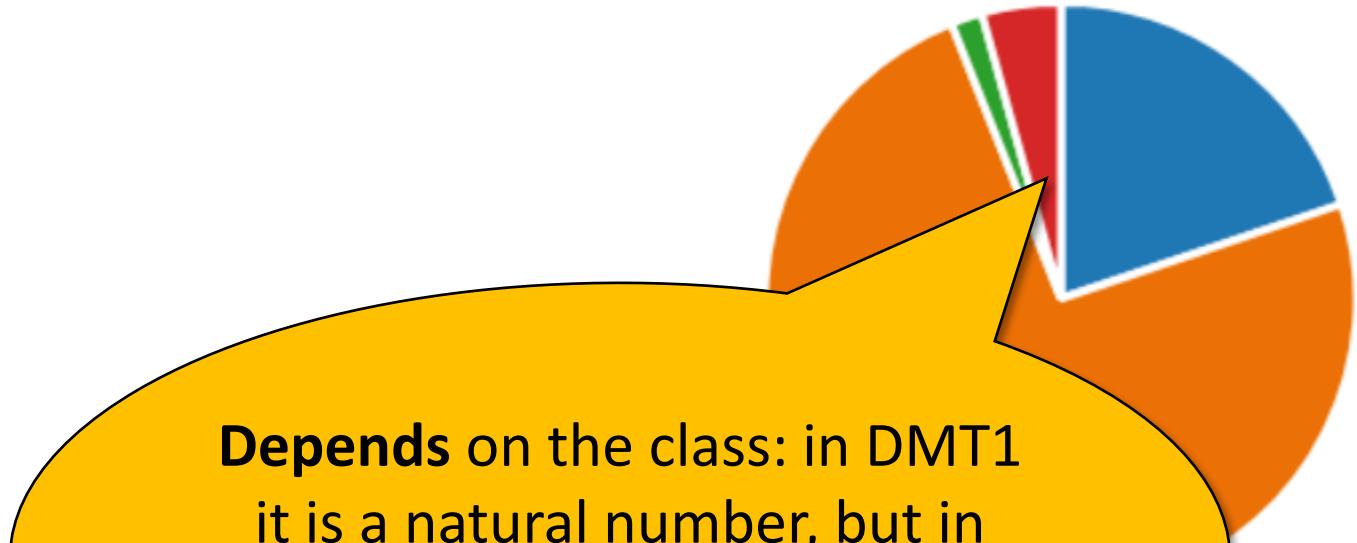
 Other

23

86

2

5



Depends on the class: in DMT1 it is a natural number, but in Basic Real Analysis it is not. It is preference for tradition or not.

What makes a good definition?

The screenshot shows the Merriam-Webster Dictionary homepage with the search term 'definition' entered. The results page displays the following content:

Full Definition of DEFINITION

1 : an act of **determining**; specifically : the formal proclamation of a Roman Catholic **dogma**

2 **a** : a statement expressing the essential nature of something
b : a statement of the meaning of a word or word group or a sign or symbol <dictionary *definitions*>
c : a product of **defining**

3 : the action or process of stating the meaning of a word or word group

4 **a** : the action or the power of describing, explaining, or making **definite** and clear <the *definition* of a telescope> <her comic genius is beyond *definition*>
b (1) : clarity of visual presentation : **distinctness** of outline or detail <improve the *definition* of an image> (2) : clarity especially of musical sound in reproduction
c : sharp **demarcation** of outlines or limits <a jacket with distinct waist *definition*>

What makes a *good definition*?

Not many room Interpretation

Specific (No ambiguity)

Widely agreed by many (Communication)

Not- Circular

What makes a good definition?

What makes a good definition?



Copilot

A good definition has several key characteristics:

1. **Clarity:** It should be clear and easy to understand, avoiding ambiguous or vague language.
2. **Precision:** It should be precise, providing an exact explanation of the term without unnecessary details.
3. **Conciseness:** It should be concise, using as few words as possible while still conveying the full meaning.
4. **Comprehensiveness:** It should cover all essential aspects of the term, ensuring that the definition is complete.
5. **Non-circularity:** It should avoid using the term being defined within the definition itself.
6. **Contextual Relevance:** It should be relevant to the context in which the term is used, providing appropriate examples if necessary.

Would you like an example of a well-defined term?

Toy Example: Define “Bread”

- Ask LLM



Copilot

can you give me a definition of "bread"

Certainly! Here is a definition of "bread":

Bread (noun):

1. A food made from flour, water, and usually yeast, mixed together and baked. It is a staple food in many cultures and comes in various forms such as loaves, rolls, and flatbreads ^①.
2. In slang, it can also refer to money ^①.

For example:

- "She bakes bread every day."
- "I need to earn some bread to pay the bills."

If you have any other questions or need more information, feel free to ask!

Toy Example: Define “Bread”

- Dictionary: made from flour, water, and usually yeast, mixed together and baked



Descriptive Definitions

Basic Analysis I

Introduction to Real Analysis. Volume I

Example 0.3.3: The following are sets including the standard notations.

- (i) The set of *natural numbers*, $\mathbb{N} := \{1, 2, 3, \dots\}$.

by Jiří Lebl

June 8, 2021
(version 5.4)

Descriptive Definitions

Basic Analysis I

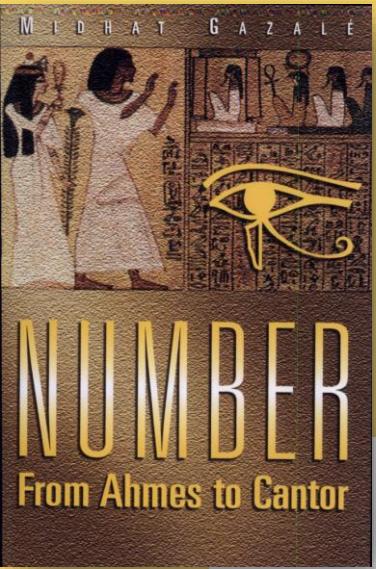
Introduction to Real Analysis, Volume I

Example 0.3.3: The following are sets including the standard notations.

- (i) The set of *natural numbers*, $\mathbb{N} := \{1, 2, 3, \dots\}$.

That the notation $|A| = n$ is justified we leave as an exercise. That is, for each nonempty finite set A , there exists a unique *natural number* n such that there exists a bijection from A to $\{1, 2, 3, \dots, n\}$.

So-called *counting numbers* have been used, with varied symbolic representation, since prehistoric times. Notation for these has stabilized to the standard Arabic numerals 1, 2, 3, . . . for the past millennium or so, and classical usage has referred to these numbers as “natural.” The concept of zero achieved acceptance as a number much more recently—in roughly 300 B.C.E.—and its common denotation as 0 occurred well after the early usage of 1, 2, 3 So, it may be deemed to be slightly less natural than the others. Nevertheless, we (and most contemporary mathematicians) adjoin 0 to the counting numbers, call the totality $\{0, 1, 2, 3, \dots\}$ the *set of natural numbers*, and denote it by the letter \mathbb{N} .



Definition-by-Example

If you are asked the question “What is a natural number?” you will probably reply, “1, 2, 3, . . . ,” and not be aware that your reply was not to the question originally put to you, but to the instruction “Recite the natural numbers.” In so doing, you have, unbeknownst to you, revealed your affiliation with the intuitionist school of the ancient Greeks, who believed that numbers, which were the essence of everything, just were! To the ancient Greeks, numbers were god-given, and to recite them was to define them.

Definition-by-Example

1.4.2 Special sets

TCS Book

There are several sets that we will use in this book time and again. The set

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

contains all *natural numbers*, i.e., non-negative integers. For any natural number $n \in \mathbb{N}$, we define the

The symbol \mathbb{N} stands for the set of *nonnegative integers*: 0, 1, 2, 3, ... (ask your instructor for the complete list).

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1.4.2 Special sets

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Definition. A *Natural Number* is a non-negative integer.

Definition. A *Natural Number* is a non-negative integer.

We will also occasionally use the set $\mathbb{Z} = \{\dots, -2, -1, 0, +1, +2, \dots\}$ of (negative and non-negative) *integers*,¹ as well as the set \mathbb{R} of *real*

¹ The letter Z stands for the German word “Zahlen”, which means *numbers*. TCS

The **set** of integers consists of zero (**0**), the positive **natural numbers** (**1, 2, 3, ...**), also called *whole numbers* or *counting numbers*,^{[2][3]} and their **additive inverses** (the **negative integers**, i.e., **-1, -2, -3, ...**).

[Wikipedia](#)

Natural number

文 A 143 languages ▾

Article Talk

Read Edit View history

From Wikipedia, the free encyclopedia

(Redirected from [Natural numbers](#))

This article is about "positive integers" and "non-negative integers". For all the numbers ..., −2, −1, 0, 1, 2, ..., see [Integer](#).

In [mathematics](#), the **natural numbers** are those [numbers](#) used for counting (as in "there are *six* coins on the table") and ordering (as in "this is the *third* largest city in the country"). Numbers used for counting are called [cardinal numbers](#), and numbers used for ordering are called [ordinal numbers](#).

Natural numbers are sometimes used as labels, known as [nominal numbers](#), having none of the properties of numbers in a mathematical sense (e.g. sports [jersey numbers](#)).^{[1][2]}

Some definitions, including the standard [ISO 80000-2](#),^{[3][a]} begin the natural numbers with 0, corresponding to the **non-negative integers** 0, 1, 2, 3, ..., whereas others start with 1, corresponding to the **positive integers** 1, 2, 3, ...^{[4][b]} Texts that exclude zero from the natural numbers sometimes refer to the natural numbers together with zero as the **whole numbers**, while in other writings, that term is used instead for the integers (including negative integers).^[5]

The natural numbers form a [set](#). Many other [number sets](#) are built by successively extending the set of natural numbers: the [integers](#), by including an [additive identity](#) 0 (if not yet in) and an [additive inverse](#) $-n$ for each nonzero natural number n ; the [rational numbers](#), by including a



The [double-struck](#) capital N symbol, often used to denote the set of all natural numbers (see [Glossary of mathematical symbols](#)). □

How can we define something new
(more complex than things we already know)?

Constructive Definitions

Two symbols:

0 “Zero”

S “Successor”

Constructive Definitions

Two symbols:

0 “Zero”

S “Successor”

Constructive definition of the ***Natural Numbers***:

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Constructive Definitions

Constructive definition of the Natural Numbers:

Base
Case

Rule 1: 0 is a Natural Number

Inductive
Clause

**Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.**

Recursive definition: we use Natural Number in our definition!

// ARITHMETICES PRINCIPIA //

NOVA METHODO EXPOSITA

A

888
6

JOSEPH PEANO

in R. Academia militari professore

Analysis infinitorum in R. Taurinensi Athenaeo docente.



AUGUSTAE TAURINORUM
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Via del Corso, 216-217.
FLORENTIAE
Via Cerretani, 8.
1889



Giuseppe Peano
1858-1932

ARITHMETICES PRINCPIA //

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ARITHMETICES PRINCPIA.

§ 1. De numeris et de additione.

Explicationes.

Signo N significatur numerus (*integer positivus*).

» 1 » unitas.

» $a + 1$ » sequens a , sive a plus 1.

» = » est aequalis. Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.

Is 0 a Natural Number?

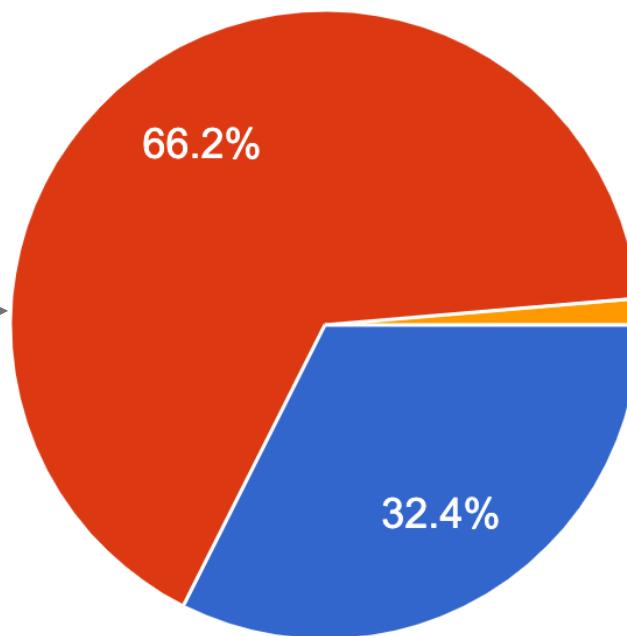


Peano in 1899

Is 0 a Natural Number?

74 responses

“No”



- Yes
- No
- Huh?

ARITHMETICES PRINCIPIA.

§ 1. De numeris et de additione.

Explicationes.

Signo N significatur *numerus (integer positivus)*.

- » 1 » *unitas.*
- » $a + 1$ » *sequens a, sive a plus 1.*
- » = » *est aequalis. Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.*

Axiomata.

1. $1 \in N.$
2. $a \in N \cdot \circlearrowleft a = a.$
3. $a, b, c \in N \cdot \circlearrowleft a = b \cdot \circlearrowleft b = a.$
4. $a, b \in N \cdot \circlearrowleft a = b \cdot b = c \cdot \circlearrowleft a = c.$
5. $a = b \cdot b \in N \cdot \circlearrowleft a \in N.$
6. $a \in N \cdot \circlearrowleft a + 1 \in N.$
7. $a, b \in N \cdot \circlearrowleft a = b \cdot \circlearrowleft a + 1 = b + 1.$
8. $a \in N \cdot \circlearrowleft a + 1 = 1.$
9. $k \in K \cdot \circlearrowleft 1 \in k \cdot \circlearrowleft x \in N \cdot x \in k \cdot \circlearrowleft x \cdot x + 1 \in k \cdot \circlearrowleft N \subset k.$

Definitiones.

10. $2 = 1 + 1; 3 = 2 + 1; 4 = 3 + 1; \text{ etc.}$

Constructive Definitions

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

How should we define *equality* for Natural Numbers?

Rule 1: 0 is a Natural Number
Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Defining Equality

Rule 1: $0 = 0$

Rule 2: If $n = S(p)$ $\notin \mathbb{N}$

Then $m \neq n$ if $m = 0$
 $\quad\quad\quad m = n$ if $\begin{cases} m = S(q) \\ \text{and} \\ p = q \end{cases}$

Rule 1: **0** is a Natural Number
Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Defining Equality

Definition. For any two Natural Numbers, n and m , we define equality ($n = m$) as:

- (1) if n is **0**: $n = m$ iff m is **0**.
- (2) otherwise, n is $S(p)$ for some Natural Number p .
 - (2a) If m is **0**, n is **not equal** to m .
 - (2b) Otherwise, m is $S(q)$ for some Natural Number q . Then, $n = m$ iff $p = q$.

$$\text{Prove } S(S(0)) = S(S(0)).$$

$n = m$

Definition. For any two natural numbers, n and m , we define equality ($=$) as:

- (1) if n is **0**: $n = m$ iff m is **0**.
- (2) otherwise, n is $S(p)$ for some natural number p .
 - (2a) If m is **0**, n is not equal to m .
 - (2b) Otherwise, m is $S(q)$ for some natural number q . $n = m$ iff $p = q$.

(2b) $P = S(0), \quad M \neq S(0)$

(2b) $P = 0 \quad Q = 0$

(1) $P = f$

Prove $S(S(0)) = S(S(0))$.

Definition. For any two natural numbers, n and m , we define equality ($=$) as:

- (1) if n is **0**: $n = m$ iff m is **0**.
- (2) otherwise, n is $S(p)$ for some natural number p .
 - (2a) If m is **0**, n is not equal to m .
 - (2b) Otherwise, m is $S(q)$ for some natural number q , then $n = m$ iff $p = q$.

- 1) By definition of equality (2b), $S(S(0)) = S(S(0))$ iff $S(0) = S(0)$,
- 2) By definition of equality (2b), $S(0) = S(0)$ iff $0 = 0$.
- 3) By definition of equality (1), $0 = 0$.
- 4) Thus, $S(0) = S(0)$ (from step 2), and $S(S(0)) = S(S(0))$ (from step 1).

Quod Erat Demonstrandum

What kind of proof is this?

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Is there any Natural Number m such that $S(m) = 0$?

Rule 1: 0 is a Natural Number

**Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.**

Is there any Natural Number m such that $S(m) = 0$? **No, by axiom.**

Axiomata.

1. $1 \in \mathbb{N}$.
2. $a \in \mathbb{N} \cup a = a$.
3. $a, b, c \in \mathbb{N} \cup a = b \Rightarrow b = a$.
4. $a, b \in \mathbb{N} \cup a = b \wedge b = c \Rightarrow a = c$.
5. $a = b \wedge b \in \mathbb{N} \Rightarrow a \in \mathbb{N}$.
6. $a \in \mathbb{N} \cup a + 1 \in \mathbb{N}$.
7. $a, b \in \mathbb{N} \cup a - b = a + 1 = b + 1$.
8. $a \in \mathbb{N} \cup a + 1 = 1$.
9. $k \in \mathbb{K} \cup i \in k \cup a \in \mathbb{N} \cup a \in k \Rightarrow a + 1 \in k \cup a \in \mathbb{N} \cup k$.



In mathematics or logic, an *axiom* is an **unprovable** rule or first principle accepted as **true** because it is self-evident or particularly useful.

Using the Definition

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number,
 $S(n)$ is a Natural Number.

Notation: The set of Natural Numbers is denoted as \mathbb{N} .

Induction



Aqua Claudia, finished by Emperor Claudius (41–54 AD) in 52 AD. https://en.m.wikipedia.org/wiki/Aqua_Claudia

Principle of Induction

Suppose that X is a subset of \mathbb{N} that satisfies these two properties:

- (1) $0 \in X$
- (2) if $n \in X$, then $S(n) \in X$.

Then, $X = \mathbb{N}$.

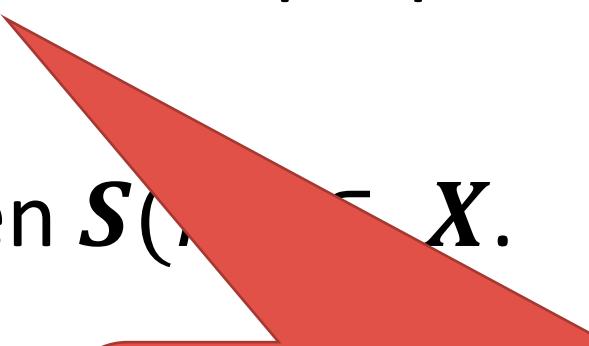
Is this an axiom? Logic rule? Theorem?

Principle of Induction

Suppose that X is a subset of \mathbb{N} that satisfies these two properties:

- (1) $0 \in X$
- (2) if $n \in X$, then $S(n) \subseteq X$.

Then, $X = \mathbb{N}$.



Whoa! We've just used sets, subset, element of, set equality, etc. all without defining them!

Suppose that X is a subset of \mathbb{N}
that satisfies these two properties:

- (1) $0 \in X$
- (2) if $n \in X$, then $S(n) \in X$.

Then, $X = \mathbb{N}$.

Principle of Induction

Principle of Induction: Suppose that X is a subset of \mathbb{N} that satisfies these two properties: (1) $0 \in X$ (2) if $n \in X$, then $S(n) \in X$. Then, $X = \mathbb{N}$.

Proof: 1. Define $Y = \mathbb{N} \setminus X$. (All elements of \mathbb{N} that are not in X .)

2. For contradiction, assume Y is non-empty.

3. Then, Y has some smallest element z .

(Note: this is non-obvious – using well-ordering principle.)

4. By property (1), z cannot be 0 .

5. Thus, $z = S(p)$ for some $p \in \mathbb{N}$.

6. Since p is smaller than z , it cannot be in Y , so must be in X .

7. But, this contradicts property (2), since if $p \in X$, then $S(p) \in X$.

8. Thus, the assumption that Y is non-empty is invalid.

9. Hence, Y is empty, and $\mathbb{N} = X$.

Proof by Induction

Principle of Induction: Suppose that X is a subset of \mathbb{N} that satisfies these two properties: (1) $0 \in X$ (2) if $n \in X$, then $S(n) \in X$. Then, $X = \mathbb{N}$.

Proof by Induction: For any predicate $P(\mathbb{N})$, showing (1) $P(0)$ and (2) if for any $n \in \mathbb{N}$ if $P(n)$ then $P(S(n))$ proves P holds for all \mathbb{N} .

Sets

Defining Sets

Q: Is \mathbb{N}
a set?

Rule 1

Base Clause:

The empty set (denoted as \emptyset) is a Set.

Rule 2

Inductive Clause:

If S is a Set and x is anything, $S \cup \{x\}$ is a Set.

$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots$

$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots, 1, 0, 2, \dots, N\}$

Constructive Definitions

Rule 1: 0 is a Natural Number

Rule 2: If n is a Natural Number, $S(n)$ is a Natural Number.

Rule 1: \emptyset is a Set.

Rule 2: If S is a set and x is anything, $S \cup \{x\}$ is a set.

0 , \emptyset , S , \cup , $+$, $\{\}$, etc. are all just symbols: their meaning comes from the definitions we agree to give them.

Recap

Mathematical Definitions

What makes a good definition?

Defining the Natural Numbers

Induction Principle

Sets

Today: Chapter 0 and 1 in the TCS book
Next week: start Chapter 2

**PRR 1 is posted (in Canvas), and
due next Tuesday before 12pm**

**Problem Set 1 will be
posted today and due
next Friday , Jan 24
(10:00pm)**