A Comparative Analysis of Correlation Approaches in Finance

(Claudio Albanese[1], David Li[2], Edgar Lobachevskiy[3], Gunter Meissner[4])

"…correlation, while being one of the most ubiquitous concepts in modern finance and insurance, is also one of the most misunderstood concepts." (Embrechts et al. 1999)

Abstract

This paper analyses the most widely applied correlation approaches in finance. We first discuss primarily bottom-up approaches, such as correlating Brownian motions (Heston 1993), the binomial correlation approach (Lucas 1995), Copulas (Sklar 1959, Li 2000), lattice models with dynamic copulas (Albanese et al. 2005, 2007, 2010), conditionally independent models (Vasicek 1987 and extensions), and the contagion correlation approach (Davis and Lo 1999, Jarrow and Yu 2001). New insights on the Gaussian copula, partly blamed for the global financial crisis 2007/2008 are given. We then discuss top-down approaches such as Vasicek's 1987 large homogeneous portfolio (LHP), inducing correlation in Markov chain models (Hurd and Kuznetsov 2006a, 2006b, Schönbucher 2006), and the top-down contagion model of Giesecke et al 2009. This paper addresses the mathematical properties of the correlation approaches. A follow up paper will empirically test the correlation approaches using default data of the US banking sector before and during the 2007/2008 crisis.

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Introduction and Motivation

Financial Correlations, heuristically characterized as the co-movement of asset prices in time, play a key role in finance. In portfolio theory, we remember from our studies of the Nobel Prize rewarded CAPM (Markowitz (1952), Sharpe (1964)), that an increase in diversification increases the return/risk ratio. The concepts of diversification and correlation are related. The lower, preferable negative the correlation, the higher is the benefit of diversification. Correlations are also critical in risk measurement and management. The lower the correlation of the assets in a portfolio, the lower is the risk, derived by any risk measure as VAR (Value at risk), SRM (Spectral risk measures) or ERM (enterprise risk management).

The practical relevance of financial correlations was highlighted recently during the global financial crisis of 2007/2008. Inefficient correlation modeling, especially the static Gaussian Copula model was partially blamed for the crisis. For a discussion see section 2.3, sub header 'Gaussian copula and the 2007/2008 financial crisis'. Consequently, richer correlation approaches were introduced as dynamic copula models, which are discussed in section 2.4. In addition, recent, complex correlation models as contagion correlation measures (section 2.6), or dynamic Markov chain models (section 3.2) may serve better in modeling the complex credit portfolios, which many investment banks manage.

As often in finance, financial practitioners have been at the forefront of financial innovations. 'Correlation desks' exist at every major investment bank since the mid-1990s. However, the theory of correlation modeling has been slow to respond. Nevertheless, in the recent past, several new correlation approaches have been derived. The objective of this paper is to provide an overview and a comparative analysis of the most popular correlation approaches in financial theory and practice.

Analytically, it is quite difficult to conceptualize the notion of correlation. When modeling volatility, one considers a single time series. The statistical notion of realized variance maps directly into a second moment for return distributions over short time horizons. Correlation instead is a notion that involves two or more processes. Depending on how the marginal single factor processes are modeled and the method chosen to mathematically induce a correlation, one arrives at possibly entirely different specifications. In the credit arena, matters are further complicated by the numerous observables of interest. When we discuss credit correlation, we may refer to correlations between

(i) default arrival times,

(ii) CDS spread returns,

(iii) equity returns,

(iv) interest rate and foreign exchange rate shocks

or any cross correlation between any of these. These effects are all arguably relevant and are characterized by different time scales. When the modeler decides to focus on one aspect or the other, she consciously invents a model where the rich variety of correlation effects are coalesced into the modeling of a single observable.

A note on terminology. In trading practice, we find the terms 'correlation desks' or 'correlation trading.' These terms are typically applied quite broadly, referring to any co-movement of asset prices in time. In financial theory however, the term 'correlation' is typically defined narrower, only referring to the linear Pearson correlation coefficient, as in Cherubini et al (2004), Nelsen (2006) or Gregory (2010). These authors refer to other than Pearson correlation coefficients as dependence measures or measures of association. In this paper, we will refer to all methodologies that measure some form of dependency as correlation approaches. In accordance with most literature, we will refer to the Pearson coefficient, discussed in section 1.1, as correlation coefficient. Ordinal dependence measures, discussed in section 1.2 and 1.3, which are related to the Pearson correlation approach, will be termed rank correlation measures. Parameters derived by other methodologies will be referred to as dependence coefficients or dependence measures.

The remainder of the paper is structured as follows. In section A we will discuss primarily bottom – up approaches. Here we start with section 1, which addresses statistical correlation measures and discuss their application to finance. In section 2, we discuss specific financial correlation measures. They include 2.1 correlating Brownian motions, 2.2 the binomial correlation coefficient, 2.3 static copulas and 2.4 dynamic copulas, 2.5 conditionally independent

(CID) correlation measures, and 2.6 contagion correlation measures. Section B addresses primarily top-down models, in particular the large homogeneous model of Vasicek (1987), the Markov chain models of Hurd and Kuznetsov (2006a), (2006b) and Schönbucher (2006), and the top-down contagion model of Giesecke et al (2009).

A. Primarily bottom-up correlation approaches

1. Statistical correlation measures

1.1 From our statistics 101 class we all remember the Pearson product moment correlation coefficient or Pearson correlation coefficient ρ_1. It is a function $f:[X \in \Re_X, Y \in \Re_Y] \rightarrow \rho_1 \in [0,1]$, explicitly defined as

$$\rho_1(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma(X)\sigma(Y)} \tag{1}$$

X and Y are sets of variate pairs with ranges $\Re_X$ and $\Re_Y$, respectively; cov(X,Y) is the covariance of X and Y and σ(X) and σ(Y) are the finite standard deviations of X and Y respectively. In terms of uncentered moments, the covariance is E[(X-E(X))-(Y-E(Y))] = E(XY)-E(X)E(Y). Also, the variances of X and Y are $\sigma_X^2 = E(X^2) - E(X)^2$ and $\sigma_Y^2 = E(Y^2) - E(Y)^2$ respectively. Hence equation (1) can assume the equivalent form

$$\rho_1(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2}\sqrt{E(Y^2) - (E(Y))^2}} \tag{2}$$

The application of the Pearson correlation coefficient and the related least square linear regression analysis is a standard statistical tool in finance, see for example Fitch (2006), who regress correlations between asset returns with sector specific regional factor loadings. Das et al 2006 linearly regress mean probability of default with market volatility and debt to asset ratio. In addition, Altman et al 2005 apply the Pearson correlation approach and its extensions to verify the negative correlation between default rates and recovery rates.

However, the limitations of Pearson correlation approach in finance are evident. First, the Pearson approaches only evaluates linear dependences. Second, linear correlation measures are only natural dependence measures if the joint distribution of the variables is elliptical. However, only few distributions such as the multivariate normal distribution and the multivariate student-t distribution are special cases of elliptical distributions, for which linear correlation measure can be meaningfully interpreted.[5] Third, zero correlation derived in equations (1) and (2) does not necessarily mean independence. This is because only the two first moments are considered in (1) and (2). For example, $Y = X^2$ $\{y \neq 0\}$ will lead to $p_1 = 0$, which is arguably misleading. Fourth, the variances of the variates X and Y have to be finite. However, for distributions with strong kurtosis, for example the student-t distribution with $v \leq 2$, the variance is infinite. Finally, in contrast to the Copula approach discussed below, which is invariant to strictly increasing transformations, the Pearson correlation approach is typically not invariant to transformations. For example, the Pearson correlation between pairs X and Y is in general different than the Pearson correlation between the pairs ln(X) and ln(Y). Hence the information value of the Pearson correlation coefficient after data transformation is limited. For these reasons, quants have developed specific financial correlation measures, which are discussed in section 2 below.

1.2 Ordinal correlation measures such as Spearman's rank correlation and Kendall's τ have gained popularity in finance in the recent past. Spearman's rank correlation coefficient is defined

$$\rho_2 = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)} \tag{3}$$

with i=1,…,n pairs x_i and y_i, numerically ranked by x_i, and $d_i = x_i - y_i$, (for more details see Defusco et al 2007)

[5] See Embrechts et al (1999) and Bingham and Kiesel (2001) for details

Spearman's correlation coefficient is sometimes referred to as the Pearson correlation coefficient for ranked variables. It measures the dependence of two variables via a monotonic function, hence a perfect Spearman dependence (Spearman correlation coefficient +1 or -1) results if the relationship between the variables is a perfectly monotone function, which does not have to be linear. The Spearman correlation approach is non-parametric in the sense that it can be applied without requiring knowledge of the joint distribution of the variables.

1.3 Kendall's τ is a further, fairly popular ordinal correlation measure applied in finance. It is defined as

$$\rho_3 = \frac{n_c - n_d}{0.5n(n-1)} \tag{4}$$

where n_c is the number of concordant data pairs and n_d is the number of discordant pairs. As the Spearman's correlation coefficient, Kendall's τ is non-parametric. Spearman's rank correlation and Kendall's τ will result in a perfect correlation coefficient of 1, if an increase in the variable x is always accompanied by an increase in y, regardless of the numerical increase, vice versa. In most other cases, the two rank correlation measures are not equal.

Rank correlation measures have been popular in analyzing rating categories (i.e. the categories AAA, AA, A,…, to D), since these are ordinal. Cherubini and Luciano (2002) apply Spearman's rank correlation and Kendall's τ to analyze the dependence of market prices and counterparty risk measured by rating categories in a copula setting. Burtschell et al (2008) compare Kendall's τ to various copulas and find significant difference in the correlation approaches when inferring CDO tranche spreads. Anderson (2010) analyses CDS correlations and finds that Spearman's rank correlations for CDS spreads have more than doubled during crisis from July 2007 to March 2009.

2. Financial Correlation Modeling

2.1 Correlating Brownian motions

One of the most widely applied correlation approaches in finance was generated by Steven Heston in 1993. Heston applied the approach to negatively correlate stochastic stock returns $dS(t)/S(t)$ and stochastic volatility $\sigma(t)$. The core equations of the original Heston model are the two SDEs

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma(t) dz_1(t) \tag{5}$$

and

$$d\sigma^2(t) = g[\sigma_L^2 - \sigma^2(t)]dt + \xi\, \sigma(t)\, dz_2(t) \tag{6}$$

where S is the underlying stock, μ is the expected growth rate of S, and $\sigma(t)$ is the stochastic volatility of S at time t. In equation (6), g is the mean reversion rate (gravity), which pulls the variance $\sigma^2(t)$ to its long term mean σ_L^2, and ξ is the volatility of the volatility $\sigma(t)$. dz(t) is the standard Brownian motion, i.e. $dz(t) = \varepsilon_t \sqrt{dt}$, ε_t is i.i.d., in particular ε_t is a random drawing from a standardized normal distribution n~(0,1). In equation (5), the underlying S follows the standard geometric Brownian motion, which is also applied in Black-Scholes-Merton, which however assumes constant volatility. Equation (6) models the stochastic volatility with a mean reverting CIR process, see Cox et al (1985).

The correlation between the stochastic processes (5) and (6) is introduced by correlating the two Brownian motions dz_1 and dz_2. The instantaneous correlation between the Brownian motions is

$$\mathrm{Corr}[dz_1(t), dz_2(t)] = \rho_4\, dt \tag{7}$$

The definition (7) can be conveniently modeled with the identity

$$dz_1(t) = \sqrt{\rho_4}\, dz_2(t) + \sqrt{1-\rho_4}\, dz_3(t) \tag{8}$$

where $dz_2(t)$ and $dz_3(t)$ are independent, and $dz(t)$ and $dz(t')$ are independent, $t \neq t'$.

The Heston correlation approach is a dynamic, versatile, and mathematically rigorous correlation model. It allows to positively or negatively correlate stochastic processes and permits dynamic risk management. Hence it is not surprising that the approach is an integral part of correlation modeling in finance.

One prominent application is in the SABR model of Hagan et al (2002), where stochastic interest rates and stochastic volatility are correlated to derive realistic volatility smiles and skews. For extensions of the SABR model see West (2005), and Henry-Labordere (2007), Kahl and Jaeckel (2009), and Benhamou et al (2009). Huang and Yildirim (2008) use the Heston approach to correlate the volatility of the inflation process and the volatility of the nominal discount bond process to value Tips (Treasury inflation protected) futures. Langnau (2009) combines the Heston approach with the local volatility model of Dupire (1994). The result is a dynamic 'local correlation model' (LCM), which matches the implied volatility skew of equity index options well.

In credit risk modeling Zhou (2001) derives analytical equations for joint default probabilities in a Black-Cox first time passage framework applying Heston correlations. His equations help to explain empirical default properties as 1) Default correlation and asset price correlation are positively related, and 2) Default correlations are small over short time horizons. They typically first increase in time, then plateau out and then gradually decline, as found by Lucas (1995). Brigo and Pallacinini (2008) apply two Heston correlations. The first correlates two factors driving the interest rate process, the second correlates the interest rate process with the default intensity process. Meissner et al 2009 apply the Heston approach in a reduced form framework. They correlate the Brownian motion of a LMM modeled reference asset and a LMM modeled counterparty and investigate the impact on the CDS spread. They find that just correlating the LMM processes results in a rather low impact on the CDS spread, i.e. leads to higher CDS spreads than correlating the default processes directly.
A variation of the Heston approach will be discussed in section 2.5, CID models.

2.2 The binomial correlation measure

A further popular correlation measure, mainly applied to default correlation, is the binomial correlation approach of Lucas (1995). Within the default probability sample space Ω, we define the binomial events $1_X = 1_{\{\tau_X \leq T\}}$ and $1_Y = 1_{\{\tau_Y \leq T\}}$ where τ_X is the default time of entity X and τ_Y is the default time of entity Y. Hence if entity X defaults before or at time T, the random indicator variable 1_X will take the value in 1, and 0 otherwise. The same applies to Y. Furthermore, let P(X) and P(Y) be the default probability of X and Y respectively, and P(XY) is the joint probability of default. The standard deviation of a one-trial binomial event is $\sqrt{P(X) - (P(X))^2}$, where P is the probability of outcome X. Hence, modifying equation (2), we derive the joint default dependence coefficient of the binomial events $1_{\{\tau_X \leq T\}}$ and $1_{\{\tau_Y \leq T\}}$ as

$$\rho_5(1_{\{\tau_X \leq T\}}, 1_{\{\tau_Y \leq T\}}) = \frac{P(XY) - P(X)P(Y)}{\sqrt{(P(X) - (P(X))^2}\sqrt{(P(Y) - (P(Y))^2}} \qquad (9)$$

By construction, equation (9) can only model binomial events, for example default and no default. With respect to equation (2), we observe that in equation (2) X and Y are sets of i=1,..,n variates, with $i \in \Re$. P(X) and P(Y) in equation (9) however are scalars, i.e. the default probabilities of entities X and Y for a certain time period T respectively, $0 \leq P(X) \leq 1$, and $0 \leq P(Y) \leq 1$. Hence the binomial correlation approach of equation (9) is a limiting case of the Pearson correlation approach of equation (2). As a consequence, the significant shortcomings of the Pearson correlation approach for financial modeling apply also to the binomial correlation model.

The binomial correlation approach (9) had been applied by rating agencies to value CDOs, for a discussion see BIS (2004) and Schönbucher (2005). However, the rating agencies have replaced the binomial correlation approach by a structural Merton based model in combination with Monte Carlo (see Meissner et al 2008c). Hull and White (2001) apply the binomial correlation measure to price CDSs with counterparty risk. They find that the impact of the counterparty risk on the CDS is small if the binomial correlation between the reference asset and the counterparty is small. The impact increases if the binomial correlation increases and the creditworthiness of the counterparty declines.

Numerous studies have applied the binomial correlation measure to analyze historical default correlations. Most of the studies show little statistical evidence of default correlation. Erturk (2000) finds no statistically significant evidence of default correlation for lower than one-year intervals for 1,500 investment grade entities in the US. Similarly, Nagpal and Bahar find low binomial correlation coefficients within 11 sectors in the US from 1981 and 1999. Li and Meissner study intra-sector and inter-sector default correlations of 10,348 US companies from 1981 to 2003. Inter-sector default correlations show 80.76% positive default dependencies. However, only 8.97% of these were statistically significant at a 5% level. Inter-sector default correlations increased to 100% positive in recessionary periods. Of these, again 8.97% were statistically significant at the 5% level.

2.3 Copula correlations

A fairly recent, famous as well as infamous correlation approach applied in finance is the copula approach. Copulas go back to Sklar (1959). Extensions are provided by Schweizer and Wolff (1981) and Schweizer and Sklar (1983). Copulas were introduced to finance by Vasicek (1987) and Li (2000).

Copulas simplify statistical problems. They allow the joining of multiple univariate distributions to a single multivariate distribution. Formally, a copula function C transforms an n-dimensional function on the interval [0,1] into a unit-dimensional one:

$$C : [0,1]^n \rightarrow [0,1] \tag{10}$$

More explicitly, let u_i be a uniform random vector with $u_i = u_1,\ldots,u_n$, $u_i \in [0,1]$ and $i \in N$. Then there exists a copula function C such that

$$C(u_1,\ldots,u_n) = F\ [F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)] \tag{11}$$

where F is the joint cumulative distribution function and F_i, $i=1,\ldots,n$ $i \in N$ are the univariate marginal distributions. F_i^{-1} is the inverse of F_i. If the marginal distributions $F_i^{-1}(u_i)$ are continuous, it follows that C is unique. For properties and proofs of equation (11), see Sklar (1959) and Nelsen (2006).

Numerous types of copula functions exist. They can be broadly categorized in one-parameter copulas as the Gaussian copula[6], and the Archimedean copula, the most popular being Gumbel, Clayton and Frank copulas. Often cited two-parameter copulas are student-t, Frechet, and Marshall-Olkin. For an overview of these copulas, see Cherubini et al 2004.

The Gaussian copula

Due to its convenient properties, the Gaussian copula C_G is among the most applied copulas in finance. In the n-variate case, it is defined

$$C_G(u_1,\ldots,u_n) = M_{n,R}\,[N_1^{-1}(u_1),\ldots,N_n^{-1}(u_n)] \qquad (12)$$

where $M_{n,R}$ is the joint, n-variate cumulative standard normal distribution with R, the n x n symmetric, positive-definite correlation matrix of the n-variate normal distribution. N^{-1} is the inverse of a univariate standard normal distribution.

It can be easily shown that if random vector u_i is uniform, then the marginal distributions $N^{-1}(u_i)$ are standard normal and $M_{n,R}$ is standard multivariate normal. For a proof, see Cherubini et al 2005.

It was Li (2000), who applied the copula approach of equation (12) to finance. He defined the uniform margins u_i as cumulative default probabilities Q for entity i at a fixed time t, $Q_i(t)$:

$$u_i = Q_i(t) \qquad (13)$$

[6] Strictly speaking, only the *bivariate* Gaussian copula is a one-parameter copula, the parameter being the dependence coefficient. A multivariate Gaussian copula may incorporate a correlation matrix, containing various dependence coefficients.

Hence, from equation (12) we derive the Gaussian default time copula C_{GD},

$$C_{GD}(u_1,\ldots,u_n) = M_{n,R}\,[N_1^{-1}(Q_1(t)),\ldots,N_n^{-1}(Q_n(t))] \qquad (14)$$

In equation (14) the terms N_i^{-1} map the cumulative default probabilities Q of asset i for time t, $Q_i(t)$, percentile to percentile to standard normal. The mapped standard normal marginal distributions $N_i^{-1}(Q_i(t))$ are then joined to a single n-variate distribution $M_{n,R}$ by applying the correlation structure of the multivariate normal distribution with correlation matrix R. The probability of n correlated defaults at time t is given by $M_{n,R}$.

Simulating the correlated default arrival time

To derive the default time τ of asset i, τ_i, which is correlated to other assets i=1,…,n, we first derive a sample $M_{n,R}(\cdot)$ from a multivariate copula (r.h.s. of equation (12) in the Gaussian case), $M_{n,R}(\cdot) \in [0,1]$.This is done via Cholesky decomposition. The sample includes the default correlation via the default correlation matrix R. We equate the sample with the cumulative individual default probability Q of asset i at time τ, $Q_i(\tau_i) \in [0,1]$. Hence

$$M_{n,R}(\cdot) = Q_i(\tau_i) \quad \text{or}$$

$$\tau_i = Q_i^{-1}(M_{n,R}(\cdot)) \qquad (15)$$

In the idealized intensity model framework, we admit a continuous exponential default intensity function $\lambda_i(t)$. Hence, the probability of survival of asset i until t, $\Pr[\tau_i>t]$, is given by discounting with the default intensity to τ_i,

$$\Pr[\tau_i > t] = \exp\left\{-\int_0^{\tau_i} \lambda_i(t)\,dt\right\} \qquad (16)$$

The default time τ_i is found as the smallest t, at which the survival probability breaches a barrier, the drawing from a copula function $M_{n,R}(\cdot)$. Formally,

$$\tau_i = \inf\left\{ t : \int_0^{\tau_i} \lambda_i(t)\, dt \geq M_{n,R}(\cdot) \right\} \qquad t \geq 0 \tag{17}$$

Once $(M_{n,R}(\cdot))$ is drawn, we equate the survival probability with the barrier, i.e.

$$\exp\left\{ -\int_0^{\tau_i} \lambda_i(t)\, dt \right\} = M_{n,R}(\cdot) \tag{18}$$

or

$$\int_0^{\tau_i} \lambda_i(t)\, dt = -\ln[M_{n,R}(\cdot)] \tag{19}$$

and solve numerically for τ_i. In the case of a constant default intensity λ_i, equation (19) simplifies and we find the correlated default time closed form as

$$\tau_i = \frac{-\ln[M_{n,R}(\cdot)]}{\lambda_i} \tag{20}$$

The correlated default times are a core input when valuing structured products such as CDOs. In a CDO the tranche spread s is typically paid on the outstanding notional ON. To derive the fair tranche spread, we equate the present value of expected outstanding notional and the present value of the expected losses L. Hence we have

$$s\ PV(ON) = PV(L) \tag{21}$$

If an entity defaults at time τ, the loss is generated by N(1-R), where N is the notional amount and R is the recovery rate. These losses are mapped to CDO tranches (lower tranches first and subsequently higher tranches) and discounted from their respective default time τ. From the losses L, we find the outstanding notional ON and solve equation s PV(ON) = PV(L) for s. For details see Meissner et al 2008c.

Copula Applications

There are numerous applications of copula functions in finance. Due to its convenient properties, the multivariate Gaussian copula is one of the most popular. All three major rating agencies, Moody's, Standard and Poors, and Fitch apply the Gaussian default time copula in a structural Merton framework to value CDOs. They sample from the n-dimensional Gaussian copula with correlation matrix R (r.h.s. of equation (12)). The sample from $M_{n,R}(\cdot)$ for time t, is then compared with the asset i's individual default probability at time t, $\lambda_{i,t}$ which is mapped to standard normal via $N^{-1}(\lambda_{i,t})$. If the sample barrier is breached, default of asset i occurs, vice versa. Formally,

$$\tau_{i,t} = 1_{\{M_{n,R}(\cdot) \le N^{-1}(\lambda_{i,t})\}} \qquad (22)$$

where $\tau_{i,t}$ is the default time of asset i and time t. Monte Carlo simulation is applied to derive a default distribution for all assets i = 1,…,n at time t. The default distribution can then be mapped to tranches of a CDO to derive the tranche spread. For a comparative analysis of the rating agencies' approaches to value CDOs, see Meissner et al (2008c).

A further prominent application of the multivariate Gaussian copula is the modeling of credit rating changes by CreditMetrics. First, a copula dependence coefficient is derived for all asset pairs. This is often derived from equity correlation. A correlated sample from the bivariate copula (equation (12) with n=2) is then derived. The sample is then compared to the historical rating percentile to determine whether a rating change occurs. Monte Carlo simulation is conducted to derive the entire rating distribution. This approach has to be applied to all company pairs in question. Hence it is computationally quite intensive. For details see Finger (2009).

Copulas are also popular a tools to model CDSs with counterparty risk. Typically the bivariate Gaussian copula is applied to model the default correlation between the CDS seller and the reference asset, see Kim and Kim (2003), Hamp et al (2007) and Brigo and Chourdakis (2008).

Gaussian copula and the 2007/2008 financial crisis

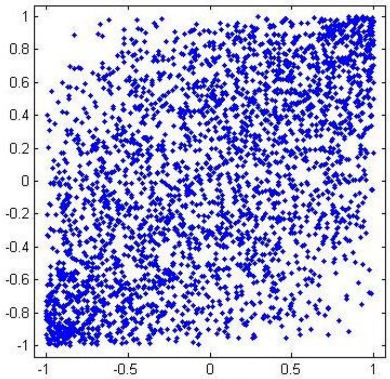
Numerous non-academic articles have been written, which demonize the copula approach and blame it for the 2007/2008 global financial crisis, see for example Salmon (2009), Jones (2009), and Lohr (2009). Due to their heuristic nature, the articles offer little insight. In the following, we will discuss several properties of the Gaussian copula, which relate to the crisis.

a) Tail dependence

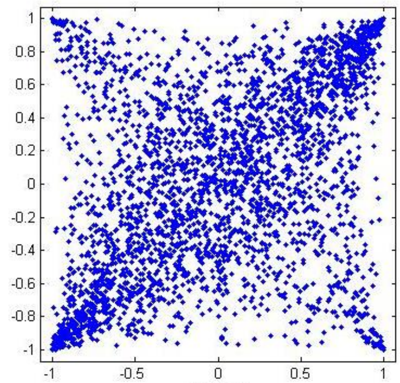
In a crisis, correlations typically increase, see studies by Das et al (2007) and Duffie et al (2009) and references therein. Hence it would be desirable to apply a correlation model with high co-movements in the lower tail of the joint distribution. Following the tail dependence definition of Joe (1999), a bivariate copula has lower tail dependence if

$$\lim_{u_1 \downarrow 0, u_2 \downarrow 0} P[(\tau_1 < N_1^{-1}(u_1) \,|\, (\tau_2 < N_2^{-1}(u_2)] > 0 \tag{23}$$

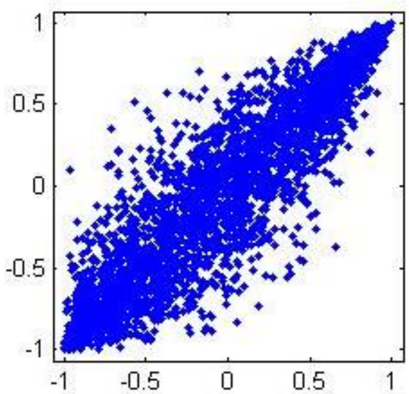
with variables as defined in equations (12) and (13). However, it can be easily shown that the Gaussian copula has no tail dependence for any correlation parameter ρ: $\lim_{u_1 \downarrow 0, u_2 \downarrow 0} P[(\tau_1 < N_1^{-1}(u_1) \,|\, (\tau_2 < N_2^{-1}(u_2)] = 0,\ \rho \in \{-1,1\}$. In contrast, the student-t copula satisfies equation (23) for any $\rho \in \{-1,1\}$. Hence it may be more desirable to apply the student-t copula in financial crisis modeling. Exhibits 1a) to 1d) show several copulas scatterplots.



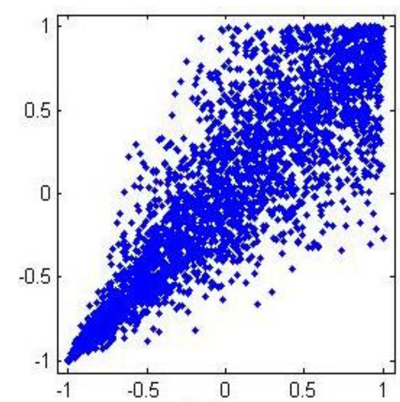
1a) Bivariate Gaussian copula with ρ=0.5



1b) Bivariate Student-t copula with ρ=0.5 and dof =1



1c) Bivariate Gumbel copula with α=4



1d) Bivariate Clayton copula with α=5

Exhibit 1: Scatter plots of different copula models

As seen in Exhibit 1c), the Gumbel copula displays high tail dependence especially for negative co-movements. Assuming that correlations increase when asset prices decrease, the Gumbel copula might also be a good correlation approach for financial modeling.

Before the global 2007/2008 financial crisis, numerous market participants trusted the copula model uncritically and naively. However, the 2007/2008 crisis was less a matter of a particular correlation model, but rather an issue of 'irrational complacency'. In the extremely benign time period from 2003 to 2006, proper hedging, proper risk management and stress test results were largely ignored. The prime example is AIG's London subsidiary, which had sold CDSs and CDOs in an amount of close to \$500 billion without conducting any major hedging.

For an insightful paper on inadequate risk management leading up to the crisis, see “A personal view of the crisis - Confessions of a Risk Manager” (The Economist 2008). In particular, if any credit correlation model is fed with benign input data as low default intensities and low default correlation, the risk output figures will be benign, ‘garbage in garbage out’ in modeling terminology.

b) Calibration

A further criticism of the Gaussian copula is the difficulty to calibrate it to market prices. In practice, typically a single correlation parameter (not a correlation matrix) is used to model the default correlation between any two entities in a CDO, see section 2.5 for details. Conceptually this correlation parameter should be the same for the entire CDO portfolio. However, traders randomly alter the correlation parameter for different tranches, in order to derive desired tranche spreads. Traders increase the correlation for ‘extreme’ tranches as the equity tranche or senior tranches, referred to as the correlation smile. This is similar to the often cited implied volatility smile in the Black-Scholes-Merton model. Here traders increase the implied volatility especially for out-of-the money puts, but also for out-of-the money calls to increase the option price.

There is a critical difference though between the correlation smile and the implied volatility smile. An increase in correlation increases the spread for senior tranches. However, an increase in correlation for the equity tranche *de*creases the equity tranche spread. This is referred to as being ‘long correlation’, since the PV of the investor i.e. protection seller, increases if correlation increases. For an interesting paper challenging the CDO equity tranche property of long correlation see Jarrow and van Deventer 2008.

Exhibit 2 shows CDO tranche spread for different correlation parameters

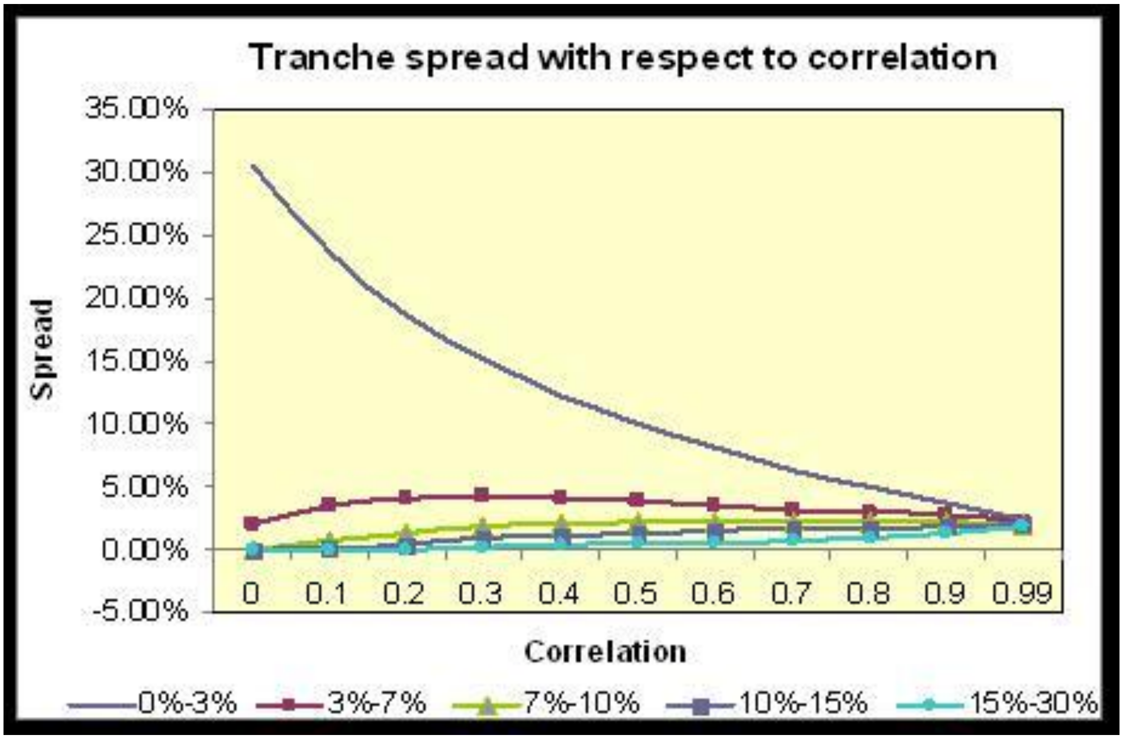


Exhibit 2: Tranche spread derived from the Gaussian copula model with respect to correlation for different CDX tranches; 125 credits, 1% hazard rate, 5 year maturity, 30,000 MC simulations

Exhibit 2 highlights the correlation crisis of May 2005. Hedge funds had invested in the equity tranche (i.e. sold protection) and had ‘hedged’ with going long protection the mezzanine tranche. After a downgrade of Ford and General Motors to BB+, implied correlation decreased sharply from 24% to 12%. As a result, the equity tranche spread increased sharply. At the same time, following Exhibit 2, the mezzanine tranche spread decreased. Hence the hedge funds lost on both legs of their position. As a consequence, several hedge funds such as Marin Capital, Aman Capital and Baily Coates filed for bankruptcy. Importantly, the correlation crisis was not model-inherent, but due to a lack of understanding of the correlation properties of the model.

Another criticism of the Gaussian copula is that for certain parameter constellations it may not be possible to imply a market CDO tranche spread for a correlation parameter between 0 and 1. Kherraz (2006) tests the LHP version of the Gaussian copula (see sections 2.5 and 3.1) and finds that the lowest 40% and highest 20% of losses of the equity tranche cannot be explained by the model. However, Kherraz uses a fairly high default probability of 40% in his study and does not mention the frequency or timing of the occurrence. Finger (2009) tests the calibration of the LHP model with base correlation, a correlation with zero attachment point, which is bootstrapped from the implied correlation, (see JP Morgan 2004 for details on base correlation). Finger finds calibration failures for just 20 days for the iTraxx and 21 days for the CDX indices before July 2007. He finds no calibration failures after July 2007.

Several other studies, such as Hull and White (2004), Andersen and Sidenius (2004) and Burtschell et al (2008) test the one-factor Gaussian copula as well as other copulas such as Marshall-Olkin, Clayton or double-t. None of the studies finds any calibration failures for these copulas.

c) Risk management

A further criticism of the Copula approach is that the copula model is static and consequently allows only limited risk management, see Finger (2009) or Donnelly and Embrechts (2010). The original copulas models of Vasicek (1987) and Li (2000) and several extensions of the model as Hull and White (2004) or Gregory and Laurent (2004) do have a one period time horizon, i.e. are static. In particular, there is no stochastic process for the critical underlying variables default intensity and default correlation. However, even in these early copula formulations, back testing and stress testing the variables for different time horizons can give valuable sensitivities, see Whetten and Adelson (2004) and Meissner et al (2008b).

In addition, the copula variables can be made a function of time as in Hull el at (2005), see equation (28) below. This still does not create a fully dynamic stochastic process with drift and noise, which allows flexible hedging and risk management. The best solutions are truly dynamic copula frameworks, will be addressed now.

2.4 Dynamic Copulas

The copulas discussed so far are principally static, i.e. have a one period fixed time horizon. In the following we will discuss dynamic copulas. They apply to models where each reference name is modeled by either a distance-to-default process or a defaultable equity process. In either case, the state variables for each name take up a fairly large number of values, typically ranging between 512 and 1024 in implementations. A model for default arrival times instead is based on processes which can take two values only: 0 to represent default and 1 to represent the opposite. By augmenting the number of state variables, one can model the volatility process and address the problem of vega hedging. The method is entirely algebraic in nature and works well without any assumption of analytic tractability.

Dynamic copulas are formulated on time horizons typically longer than the elementary time step over which the underlying process varies. In (Albanese et al. 2005, 2007) quarterly time intervals are used over a total time period of 10 years. In a dynamic copula scheme, at each time step one correlates the returns of the process for each name. Two types of dynamic Gaussian copulas have been considered: binomial and Gaussian.

Dynamic binomial copulas and dynamic conditioning

In (Albanese et al. 2005, 2007) binomial Gaussian copulas are introduced for the sake of avoiding a Monte Carlo simulation. Dynamic conditioning is a combinatorial scheme based on super-lattices which solves the pricing problem for synthetic CDOs by explicitly correlating as many processes as there are underlyings in a multifactor setting, exactly and without resorting to simulation.

The method involves the following stages:

(i) Group credit reference names by industry sector and jurisdiction and associate to each family a parameterized model for a distance to default or a defaultable equity process.

(ii) Calibrate each model using cross-sectional information such as the CDS spread curve form factors (ratios between 10 and 2 year or 5 and 2 year CDS spreads), CDS spread volatilities as a function of spread level, etc..

(iii) Adjust each model to individual reference name by adding jump to default terms in such a way to fit the term structure of CDS spreads.

(iv) Devise a correlation super-lattice to condition each name. In the preprocessing phase, find a family of conditional probabilities of default as described in (Albanese and Vidler, 2007)

(v) For any given correlation structure, assemble the building blocks obtained in (iv) to obtain the term structure of tranche spreads, quarter by quarter, reaching out to ten years.

(vi) Calibrate globally to the main synthetic CDO indices simultaneously, including CDX and iTraxx investment grade and high yield indices.

Global calibration sets the basis for pricing bespoke CDOs of varying concentration, consistently. The technology underpinning of the methodology is the use of high-throughput microchip coprocessors that enable high performance matrix algebra.

Although global calibration to synthetic CDOs works remarkably well, the method is limited to unmanaged, synthetic CDOs. However, more complex structures such as cash CDOs, managed CDOs or even complex derivative portfolios corresponding to the netting agreement with a given counterparty remain out of reach. Furthermore, dynamic copulas based on binomial correlations are less desirable than the smoother alternative of dynamic Gaussian copulas, which however cannot be tackled with combinatorial methods. The Monte Carlo method in the next subsection addresses both issues.

Dynamic Gaussian copulas and global Monte Carlo simulations

Dynamic Gaussian copulas remedy the shortcomings of dynamic binomial copulas by relying on global Monte Carlo simulation as opposed to combinatorial methods. The single factor marginals are instead estimated in a similar way. The technology is described in Albanese et al. (2010) in the context of the securitization problem for a global portfolio of netting sets subject to counterparty credit risk.

The method leverages on the emerging technology of high density MIMD boards with GPU co-processors. On suitable equipment, full simulations to value CDOs can be carried out to generate up to 20M elementary factor returns per second, a speed sufficient to calibrate the term structure of the three leading index CDOs simultaneously through an optimizer. Counterparty credit risk of portfolios of multi-currency portfolios of netting sets with underlying fixed income derivatives presents further challenges as this task also involves modeling interest rate and foreign exchange factors consistently. The method involves splitting the positions in a netting agreement into single-factor sub-portfolios and a multi-factor remainder. For the former, one precomputes valuation tables using backward induction and then runs a simple Monte Carlo simulation. For the remainder instead, a nested simulation is required. Nested simulations are needed for instance for portfolios of CDO tranches. The nested simulation is based on generating scenarios based on globally calibrated processes and then pricing all CDO positions with respect to the same scenarios.

2.5 Conditionally independent default (CID) correlation modeling

In order to avoid specifying the default correlation between each entity pair in equation (12) or (14), a factorization is often applied. This leads to conditionally independent default (CID) modeling. The most widely applied CID model is the one-factor Gaussian copula (OFGC) model. It was the de-facto market model for pricing CDOs before the 2007/2008 global financial crisis. The core equation of the OFGC model is

$$x_i = \sqrt{\rho_6}\, M + \sqrt{1-\rho_6}\, Z_i \tag{24}$$

where M and Z_i are random drawings from n~(0,1) and $0 \le \rho_6 \le 1$. As a result, the latent variable x_i, sometimes interpreted as the asset value of entity i, see Turc et al (2005), is also n~(0,1). The common factor M can be interpreted as the economic environment, possibly represented by the return of the S&P 500. Z_i is the idiosyncratic component, the 'strength' of entity i, possibly measured by entity i's stock price return. From equation (24) we see, that the correlation between entities i is modeled indirectly by conditioning the latent variable x_i on the common factor M. For example, for p_6 =1, the latent variables of all entities i=1,…n, x_i = M, so the x_i are identical in each simulation. For p_6 = 0, all latent variable for all entities i=1,…n, x_i = Z_i, hence the x_i are independent. Importantly, once we fix the value of M, the defaults of the n entities are (conditionally on M) mutually independent.

After simulating x_i, the default arrival time of entity i is found similar in nature to the modeling in equations (18), (19) and (20). From equation (24), we derive $N(x_i)$, $N(x_i) \in [0,1]$, where N is the cumulative standard normal function. We interpret $1-N(x_i)$ as the market survival barrier (which includes the correlation of the entities i=1,…,n) and equate it with the entity's individual survival function $1-\lambda_i(t_i)$.

$$1-N(x_i) = 1- \lambda_i(t_i) \tag{25}$$

and solve numerically for t_i. In equation (25), the default barrier $1-N(x_i)$ is applied to derive the default arrival time t_i. Hence the OFGC - and its extensions, which are discussed below - , can be viewed a simplified structural models as Hull (2009) points out.

In case of constant (but individually different) default intensity λ_i of entity i, the survival function is $(1-\lambda_i)^{t_i}$. Hence we have $1-N(x_i) = (1- \lambda_i)^{t_i}$ and the default time t_i of entity i can be derived closed form as

$$t_i = \ln(1-N(x_i) / \ln(1- \lambda_i) \tag{26}$$

Equations (25) and (26) show why the auxiliary, latent variable x_i is also termed 'frailty variable'. The lower x_i, the higher the market survival barrier $1-N(x_i)$ and the higher the probability of default of asset i. The risk factors M and Z_i in equation (24) are generated independently from the default arrival time in equations (25) and (26). Hence the OFGC has the form of a doubly stochastic process or Cox process.

The OFGC was first introduced by Vasicek (1987) within a credit framework. Vasicek applied two simplifying assumptions in the model: a) a constant and identical default intensity of all entities in a portfolio and b) the same default correlation between the entities. These two conditions constitute a 'large homogeneous portfolio' (LHP). From the LHP we derive the credit at risk, CaR, with a confidence interval α, for the time horizon T, CaR(α,T) as

$$CAR(\alpha,T) = N\left(\frac{N^{-1}(Q(T)) + \rho_6 N^{-1}(\alpha)}{\sqrt{1-\rho_6^2}}\right) \tag{27}$$

see Vasicek (1987) or Hull and White (2006). Equation (27) is currently (year 2010) applied as the foundation of the credit risk capital adequacy ratio of Basel II.

In equation (27) Q(T) is the cumulative, identical default probability of *all* entities until T and ρ_6 is the identical default dependency coefficient between *all* entities. Hence there is no reference to an entity's individual default intensity as in equations (25) and (26), or an entity's specific default dependency with another entity. Hence the LHP falls primarily in the top-down modeling category, which we will discuss in section 3.1. We mention it briefly here since it is derived from equation (23).

It is worth mentioning, that the OFGC of equations (24) to (26) is equivalent to the bivariate case of the Gaussian copula of equation (12). Sampling from equation (12) is achieved by Cholesky decomposition. In the bivariate case, Cholesky sampling of two correlated variables x_1 and x_2 from equation (12) reduces to

$$x_1 = \varepsilon_1$$
$$x_2 = \sqrt{\rho_6}\,\varepsilon_1 + \sqrt{1-\rho_6}\,\varepsilon_2$$

where ε_1 and ε_2 are independent samples from n~(0,1). This is equivalent to samples i=1,2 from equation (24).

We also observe a structural similarity of the Heston approach in equation (8) and the OFGC in (24). Both approaches use the statistical property that two random variables, which are n~(0,1), when weighted with $\sqrt{\rho}$ and $\sqrt{(1-\rho}$, $\rho \in [0,1]$, and added, result in a new random variable, which is conveniently again n~(0,1). Contrary to equation (24), equation (8) however does not incorporate a factorization. Also, equation (24) is principally static, i.e. has a one-period fixed time horizon.

Extensions of the OFGC

However, the static property of the OFGC can be relaxed, as in Hull et al (2005), who apply a dynamic CID model. Hence they modify equation (24) and model

$$dz_i(t) = \sqrt{\rho_7(t)}\, dM(t) + \sqrt{1-\rho_7(t)}\, dZ_i(t) \qquad (28)$$

where dM(t) and $dZ_i(t)$ are n~(0,1) and independent. It follows from equation (28) that $dz_i(t)$ is also n~(0,1). The dependence on M(t) determines again indirectly the correlation between assets i. For example, if $\rho_7(t)$ =1, $dz_i(t)$ depends only one dM(t), hence all assets i have the same Brownian motion at time t. If $\rho_7(t)$=0, $dz_i(t)$ = $dZ_i(t)$, hence the Brownian motions of assets i are uncorrelated at time t.

Furthermore, m>1 common factors M can be modeled. In this case equation (24) generalizes to

$$x_i = \sum_{k=1}^{m} \sqrt{p_{i,k}}\, M_k + Z_i \sum_{k=1}^{m} \sqrt{p_{i,k}}$$

and the correlation between x_i and x_j is $\sum_{k=1}^{m} \sqrt{p_{i,k} p_{j,k}}$.

Numerous further extensions of the OFGC approach exist. One of the most popular is the one-factor student-t copula. It is equivalent to the bivariate case of the standard student-t copula of equation (11) for F being student-t distributed and F^{-1} being the student-t inverse. Hence we can write

$$\bar{x}_i = \sqrt{\rho_8}\, M + \sqrt{1-\rho_8}\, Z_i \qquad (29)$$

where M and Z_i are independent and n~(0,1). $x_i = \bar{x}_i \sqrt{W}$ where W follows an inverse Gamma distribution. It follows that the latent variable x_i is student-t distributed.

A further popular extension of the OFGC in equation (24) is the double-t copula. It is defined as

$$x_i = \sqrt{\rho_9}\, M_S + \sqrt{1-\rho_9}\, Z_{S,i} \tag{29}$$

where M_S and $Z_{i,S}$ are independent and follow a student-t distribution. Since the student-t distribution is not stable under convolution, the latent variable x_i in equation (29) is not student-t distributed.

A further extension of the OFGC is integrating a binomial representation of stochastic correlation. Burtschell et al (2007) model the latent variable x_i as

$$x_i = B_i(\sqrt{\rho_{10}}\, M + \sqrt{1-\rho_{10}}\, Z_i) + (1-B_i)(\sqrt{\rho_{11}}\, M + \sqrt{1-\rho_{11}}\, Z_i) \tag{30}$$

where B_i is a Bernoulli random variable. We define a Bernoulli cutoff $B^* \in [0,1]$ and model

$$B_i = \begin{cases} 0 \text{ if } r < B^* \\ 1 \text{ if } r \geq B^* \end{cases}$$

where r is a random drawing from a uniform distribution $n \in [0,1]$. If we set $p_{10} > p_{11}$, the cutoff level B* can be set low to model high correlation in distressed times.

A further extension of the OFGC is creating a local correlation model (LCM), see Turc et al (2005), where the correlation is state-dependent. In particular, Turc et al assume that the correlation ρ is dependent on the state of the economy M. Hence the OFGC changes to

$$x_i = -\sqrt{\rho_{11}(M)}\, M + \sqrt{1-\rho_{11}(M)}\, Z_i$$

The approach is similar in nature to the local volatility model of Dupire (1994), where volatility at time t, σ_t, is a function of the state of the underlying S at time t and t, $\sigma_t(S_t,t)$. Whereas Dupire is able to model the implied volatility skew and smile in the equity option market well, the local correlation model is able to reproduce the implied correlation smile of CDO tranches spreads accurately. As a result, the marked-to-market and hedge ratios of the local correlation model outperform those of the original OFGC.

Deriving the default arrival time in CID models

The correlated default arrival time in the CID framework is typically modeled with a Merton style barrier, similar to the Gaussian copula approach in equation (17). Mortensen (2006) proposes

$$\tau_i = \inf\left\{ t : \int_0^t \lambda_i(s)\,dt \geq E_i \right\} \qquad t \geq 0,\ \ i=1,\ldots,n \qquad (31)$$

where $E_i \sim \exp(1,\bullet)$, i.e. E_i is an i.i.d. random drawing from the exponential distribution with mean 1. The intensity of entity i, λ_i itself is modeled with the dynamic process

$$\lambda_i(t) = a_i\ M(t) + Z_i(t) \qquad (32)$$

Schönbucher (2007) models a stochastic speed of time change, an idea that goes back to Hurd and Kuznetsov (2006a and 2006b) which we will discuss in section 3.2, Joshi and Stacey (2005) and Giesecke and Tomecek (2005). The default time is modeled

$$\tau_i = \inf\left\{ t : \int_0^{T(t)} \lambda_i(s)\,ds \geq E_i \right\} \qquad t \geq 0, \quad i=1,\ldots,n \qquad (33)$$

where T(t) is given by

$$T(t) = \int_0^t \alpha(s)\,ds \qquad (34)$$

where $\alpha(s)$ is some stochastic process. For $\alpha(s) >> 0$, time speeds up, which means more migration and default events, hence correlation increases. For $\alpha(s) \sim 0$, time slows down, which means less migration and fewer default events, hence correlation decreases. We will discuss correlation structures, which are induced by stochastic time changes in more detail in section 3.2.

Kou and Peng (2008) suggest a ‘conditional survival’ (CS) approach, in which correlation is modeled by jumps in the common market factors M_j. The core equations are

$$\tau_i = \inf\{t : \Lambda_i(t) \geq E_i\} \qquad t \geq 0, \quad i=1,\ldots,n \tag{35}$$

where

$$\Lambda_i(t) = \sum_{j=1}^{J} \rho_{i,j} M_j(t) + Z_i(t) \tag{36}$$

The default intensity λ_i is modeled by a CIR process $d\lambda(t) = a(b - \lambda(t))dt + \sigma\sqrt{\lambda(t)}\,dz(t)$. One choice to model the common factors $M_j(t)$ is the integral of this CIR process, $M(t) = \int_0^t \lambda(s)ds$. Applying Laplace transform results in a closed form solution, which allows fast and exact simulation of the integral M(t). Additionally to the CIR process, Kou and Peng model the common factor M with a discrete jump process, namely a Polya process. As a result, the cumulative intensity $\Lambda_i(t)$ can have jumps, which is a unique feature of the model.

The Gaussian copula of equation (14) and CID equations (31) to (36) apply a Merton style barrier to derive the default arrival time. Conveniently, in the Gaussian copula the default correlation is incorporated in the barrier $M_{n,R}$. In the CID models of Mortensen, Schönbucher and Kou and Peng, the default correlation is modeled in a separate process, constituting a Cox or doubly stochastic process.

2.6 Contagion Default Modeling

Contagion default modeling can be viewed as a variation of CID modeling. As discussed in section 2.5, in the CID framework, correlation is modeled by conditioning on a common market factor M, which impacts all entities to the same degree. The lower the random drawing for M, the higher is the default intensity of all entities (unless $\rho = 0$). Hence CID modeling can elucidate default clustering. In contrast, contagion approaches model the default intensity of an entity as a function of the default of another entity. Hence contagion default modeling incorporates counterparty risk, i.e. the direct impact of a defaulting entity on the default intensity of another entity.

Contagion default modeling was pioneered by Davis and Lo (1999 and 2001) and Jarrow and Yu (2001). Davis and Lo model the latent variable Z of entity i, Z_i with equation

$$Z_i = X_i + (1 - X_i)\left(1 - \prod_{\substack{i=1 \\ i \neq j}}^{n} (1 - X_j K_{ij})\right) \tag{37}$$

Z_i is a binomial default indicator variable of entity i, i.e. Z_i = 1 in case of default of entity i and Z_i = 0 in case of survival. Entity i can default 'directly' when the Bernoulli random variable X_i = 1. Entity i can also default indirectly, i.e. when it is infected by the default of entity j. In this case, the Bernoulli random variable X_j =1. The degree of infection is modeled with the Bernoulli random 'contagion variable' K_{ij}. Formally

$$\begin{aligned} &\Pr(X_i=1) = p \\ &\Pr(X_j=1) = q \\ &\Pr(K_{ij} = 1) = r \end{aligned} \tag{38}$$

where p, q and r and input parameters which are $\in [0,1]$. In a dynamic setting, the persistence of the contagion variable K_{ij} may be modeled as an exponentially decreasing function in time t. A parameter g(t) (gravity) determines the degree of decreasing contagion in t, i.e. $K_{ij}(t) = e^{-g(t)t}$ where g(t)>0 and $\partial g / \partial t < 0$.

Jarrow and Yu (2001) introduce default intensity contagion with a set of linear equations

$$\lambda_A(t) = a_1 + a_2 1_{\{\tau_B \le t\}} \tag{39}$$

$$\lambda_B(t) = b_1 + b_2 1_{\{\tau_A \le t\}} \tag{40}$$

where a_1, a_2, b_1 and b_2 are parameters, which are >0, and have to be calibrated.

Introducing symmetric contagion among all entities creates the problem of circularity, which Jarrow and Yu (2001) call 'looping defaults.' In this case, the construction of a joint distribution is rather complex. Jarrow and Yu solve the problem by introducing the concept of 'asymmetric dependence', i.e. the default of primary entities impacts the default of secondary entities, but not vice versa. In this case, the joint default distribution conveniently becomes the product of the individual primary default times.

Due to the structural similarity between CID models (see section 2.5) and contagion models, the default time in contagion models is typically modeled with a Merton style barrier as in equations (17) to (19).

Hybrid CID – contagion modeling

CID models as well as contagion models offer realistic correlation features such as default clustering and contagion to counterparty default. Hence it is not surprising that several models incorporate the CID common factor feature as well as contagion properties. In these models, typically a contagion term is simply added to the CID process. Schönbucher and Schubert (2001), Frey-Backhaus (2003), Giesecke and Weber (2004) and Yu (2007) propose

$$\lambda_i = \alpha_i M + Z_i + \sum_{j \neq i} \beta_{i,j} N_j \qquad (41)$$

where M and Z_i are defined as in equation (23). $\beta_{i,j}$ is a function that models the contagion of firm i to a default of firm j and N_j is a default counting process $N_j = \sum_{j \geq 1} 1_{\{Tj \leq t\}}$ where T_j is the stopping time of firm j. A special case of equation (41) is derived by Giesecke and Weber (2006) with $Z_i = 0$ and α_i M = c_i. Hence the deterministic function c_i, which models the 'base intensity' may not incorporate a systematic factor M.

Alternatively Schönbucher (2004), Giesecke and Goldberg (2004), and Duffie, Eckner, Horel and Saita (2009) suggest

$$\lambda_i = \alpha_i M + Z_i + \beta_i \hat{U}$$

where β_i is a deterministic function and $\bar{U}$ is a common factor, which is unobservable. However the factor $\bar{U}$ is transformed into an observable process U= E $(\bar{U}_t|\mathcal{F}_t)$. The filtered process U is updated with observable information, in particular information about default events, which constitutes the contagion of firm i on defaults of other firms j.

B: Primarily top-down correlation approaches

Correlated credit models can be characterized by the way the portfolio default intensity distribution is derived. In a bottom-up model, the distribution of the portfolio intensity is an aggregate of the individual entities' default intensity. In a top-down model the evolution of the portfolio intensity distribution is derived directly, i.e. abstracting from the individual entities' default intensities.

Top-down models are typically applied in practice if

- The default intensities of the individual entities are unavailable or unreliable.
- The default intensities of the individual entities are unnecessary. This may be the case when evaluating a homogeneous portfolio such as an index of homogeneous entities.
- The sheer size of a portfolio makes the modeling of individual default intensities problematic

Top-down models are typically more parsimonious, computationally efficient and can often be calibrated better to market prices than bottom-up models. Although seemingly important information such as the individual entities' default intensities is disregarded, a top-down model can typically capture properties of the portfolio such as volatility or correlation smiles better. In addition, the individual entities' default information can often be inferred by random thinning techniques.

In the following chapter we will analyze the correlation modeling of several top-down approaches. In particular we will discuss Vasicek's (1987) large homogeneous portfolio (LHP), the Markov chain models of Hurd and Kuznetsov 2006a, 2006b, and Schönbucher 2006; as well top-down contagion model of Giesecke et al 2009.

3.1 Vasicek's 1987 large homogeneous portfolio (LHP)

We already briefly mentioned Vasicek's LHP model is section 2.5, since it is a special case of the one-factor Gaussian copula (OFGC) model. In particular, Vasicek assumes a) a constant and identical default intensity of all entities in a portfolio and b) the same default correlation between the entities. These two conditions constitute a 'large homogeneous portfolio' (LHP).

The LHP model allows creating a loss distribution to find k=1,…,n defaults of a basket of n entities at time T. Solving the core equation (24) for Z_i and mapping the cumulative default probabilities Q(T) to standard normal via $x_i = N^{-1}(Q(T))$, where N^{-1} is the inverse of cumulative standard normal N, we derive

$$Z_i = \frac{N^{-1}(Q(T)) - \sqrt{p_6}M}{\sqrt{1-p_6}} \tag{42}$$

The correlation between the n entities is model indirectly by conditioning on M. Once we determine the value of M, it follows that defaults of the entities are mutually independent. In particular, the cumulative default probability of the idiosyncratic factor Z_i, $N(Z_i)$, can be expressed as the cumulative default probability dependent on M, Q(T|M). Hence we have

$$Q(T \mid M) = N\left(\frac{N^{-1}(Q(T)) - \sqrt{p_6}M}{\sqrt{1-p_6}}\right) \tag{43}$$

Equation (43) gives the cumulative default probability conditional on the market factor M. We now have to find the unconditional default probabilities. We do this by first discretely integrating over M. Since M is standard normal, this is computationally easy; we can use the discrete Gaussian quadrature (Norm (x) – Norm (x-1) in MatLab). We now have to derive all possible k = 0,..,n default combinations. We do this by applying the binomial distribution B, hence B(k; n, Q(T|M)) and weighing it with the piecewise integrated units of M. The result is a distribution of the number of defaults until T, as shown in Exhibit 3.

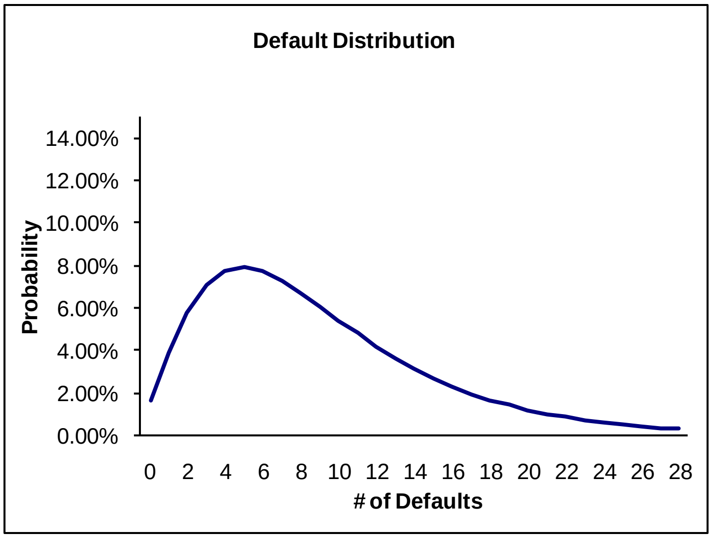


Exhibit 3: Unconditional default distribution derived from the LHP model with parameters Q(T) = 7.3%, ρ_6 = 10%, portfolio size 125 entities, recovery rate 40%

We can now find the CDO tranche spread similar to the procedure used in the full Copula in equations (17), (18) and (19). The main difference is that in the full copula we had derived the individual default time τ_i for each entity i by applying a unique default intensity function for each entity and a unique pair wise default correlations for each two entities. In the LHP the aggregate default distribution for a fixed time T is derived without reference to individual default entities and a constant pair wise correlation.

The unconditional default distribution in the LHP framework enables us to find the spread of a tranche of a CDO again via equation (21) s x PV(ON) = PV(EL). We first derive the expected losses of the tranche. This is done by assigning the unconditional default probabilities to the tranches. From the expected losses of a tranche we derive the expected outstanding notional and solve s x PV(ON) = PV(EL) for s. For details see Meissner (2006) or Hull (2009).

The strong assumptions of the LHP model have been relaxed in numerous extensions. The default intensity can be modeled for each entity i and as a function of time, $\lambda(i,t)$. In addition, the correlation between the entities can be modeled via a correlation matrix. These extensions lead to the full copula framework, for example the Gaussian copula in equation (14).

3.2 Markov Chain models

<u>Inducing correlation structures via transition rate volatilities</u>

Schönbucher (2006) generates different transition and default correlation properties via different transition rate volatilities in a time-inhomogeneous, finite-state Markov chain framework. The model is inspired by the Heath-Jarrow-Morton (HJM) (1992) interest rate model. While the HJM model generates an interest rate term structure at future times t, Schönbucher creates a stochastic evolution of transition rates to derive the loss distribution at future times t. In analogy to the HJM model, Schönbucher applies the current (time 0) term structure of transition rates as inputs. Hence the model does not require any calibration.

Specifically, the model consists of a time-inhomogeneous, hypothetical Markov chain of cumulative losses L(t), $t \geq 0$ with discrete states {0,1,2…I} of the I entities of the portfolio. The generator matrix A(t,T), $t \leq T$, of transition probabilities satisfies the usual conditions, see Jarrow et al 1997 on deriving the risk-neutral generator matrix for continuous and discrete time. Integrating the Kolmogorov differential equations $\frac{dP(t,T)}{dT}\frac{1}{P(t,T)} = A(T)$, we find the transition probability matrix $\Lambda(t,T)$, which reproduces the loss distribution $\mathbf{p}(t,T) = (p_0(t,T),\ldots, p_I(t,T))$. The components of the loss distribution, the probabilities $p_n(t,T)$ are set so that

$$p_n(t,T) := P[L(T) = n \mid \mathcal{F}_t] \tag{44}$$

i.e. $p_n(t,T)$ represents the probability of exactly n losses in the portfolio, viewed at time t for maturity T.

In order to create a no-arbitrage framework and a unique correspondence of transition probabilities to the loss distribution, Schönbucher initially allows only one-step transitions (i.e. only to the next lower rating class). Hence the transition probability matrix at time t for maturity T, $\Lambda(t,T)$ contains only zero entries, except on the diagonal and directly adjacent higher nodes:

$$\Lambda(t,T) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & ..0 & 0 \\ 0 & a_{2,2} & a_{2,3} & ..0 & 0 \\ : & : & : & ..: & : \\ 0 & 0 & 0 & ..a_{I-1,I-1} & a_{I-1,I} \\ 0 & 0 & 0 & ..0 & 0 \end{pmatrix}$$

where $a_{i,i}$, $0 \leq i \leq I$ is the probability of staying in the same state and $a_{i,j}$, $0 \leq j \leq I$ is the transition probability of moving from state i to state j. The transition probabilities evolve stochastically in time to reproduce the arbitrage-free term structure of loss distributions **p**(t,T) at future times t with maturity T. In particular, the transition probability of entity n, seen at time t for maturity T, $a_n(t,T)$, $0 < n < I$, follows a standard generalized Wiener process, i.e.

$$da_n(t,T) = \mu_{a_n}(t,T)\,dt + \sigma_{a_n}(t,T)\,dz \qquad (45)$$

with variables as defined in equation (5). Equation (45) brings us to the correlation properties of the model. Default correlation is induced by the dynamics of the transition volatility $\sigma_{a_n}(t,T)$. Schönbucher specifies a parameter constellation in which an increase in the factor loading of the transition rates a_n increases the volatility of a_n, and vice versa. In this framework, a higher volatility of a_n means a higher transition rate of all entities n to a lower state, hence a higher default correlation; a lower volatility of a_n means a lower transition rate of all entities n to a lower state, hence lower default correlation. The model can also replicate 'local correlation' by specifying a higher volatility, hence higher correlation only for a short period of time i.e. $\frac{\partial \sigma_{a_n}(t,T)}{\partial t} > 0$ for the current time t, i.e. for n = L(t), and $\frac{\partial \sigma_{a_n}(t,T)}{\partial t} = 0$ for future times t, i.e. for $n > L(t)$.

Inducing correlation structures via stochastic time change

To the best of our knowledge it was Clark (1973) who first applied stochastic time processes to financial modeling. Clark proposes a stochastic time process T(t) with independent increments drawn from a lognormal distribution. T(t) is a ‘directing process’, a stochastic clock which determines the speed of the evolution of the stock price process S(t), forming the new process S(T(t)). This new process S(T(t)) serves as a ‘subordinator process’ for the stock price process S(t). Clark finds that the subordinated distributions can explain futures cotton prices better than alternative standard distributions.

The Variance Gamma model of Madan et al (1998) applied stochastic time change to option pricing, generalizing previous work by Madan and Seneta (1990) and Madan and Milne (1991). The model consists of a standard Brownian motion, whose drift μ however is evaluated at random time changes t, which are modeled by a gamma process. The model has the same subordinated structure as Clark (1973):

$$S(t; \mu, \sigma, \upsilon) = \mu\, \Gamma(t;1,\upsilon) + \sigma\, dz\, (\Gamma(t;1,\upsilon)) \tag{46}$$

with variables as defined in equation (5) and $\Gamma(t;1,\upsilon)$ is a gamma distribution with unit mean and variance υ. By controlling the skew via μ and the kurtosis via υ, the model is able to match volatility smiles in the market well. Further models that apply stochastic time change to option pricing are Geman et al (2001), Carr et al (2003) and Cont and Tankov (2004).

The stochastic time models above help to explain phenomena in financial practice. Regarding the modeling of correlation via stochastic time change, we already briefly discussed the frameworks of Joshi and Stacy (2005) and Schönbucher (2007) in section 2.5 in a bottom-up setting. In the following, we will discuss Hurd and Kuznetsov (2006a) and (2006b), who were the first to induce correlation via stochastic time change. Their time-homogeneous Markov chain model of K discrete rating classes $Y_t \in \{1,2\ldots,K)$ assumes that transition and default intensities are identical for entities in the same rating category. Hence the model does not directly reference individual transition and default intensities and therefore qualifies as primarily top-down.

At the core of the model is a continuous time, time-homogeneous Markov chain with time-constant generator matrix $\mathcal{L}_Y$

$$
\mathcal{L}_Y = \begin{pmatrix} l_{1,1} & l_{1,2} & l_{1,3} & .. & l_{1,K} \\ l_{2,1} & l_{2,2} & l_{2,3} & .. & l_{2,K} \\ : & : & : & ..: & : \\ l_{K\text{-}1,1} & l_{K\text{-}1,2} & l_{K\text{-}1,3} & .. & l_{K\text{-}1,K} \\ 0 & 0 & 0 & .. & 0 \end{pmatrix}
$$

where K is the absorbing bankruptcy state. $l_{i,j}$, $i \in \{1,2\ldots,K\text{-}1)$, $j = \in \{1,2\ldots,K)$, are the instantaneous transition intensities under the historical (real-world or reference) measure P of migrating from rating class i to j.

Hurd and Kuznetsov further introduce a vector-valued process

$$
\mathbf{X}_t = \{r_t, u_t, \lambda_t\} \qquad (47)
$$

where r_t is the risk-free interest rate, the recovery rate is $R_t = e^{-u_t}$ and, importantly, λ_t is the stochastic migration intensity process. The vector $\mathbf{X}_t$ captures macroeconomic data and represents a common factor, which affects all entities. The credit migration process of the rating classes $Y_t \in \{1,2\ldots,K\}$ is conditioned on the vector $\mathbf{X}_t$, hence the Y_t are conditionally independent, applying the CID property discussed in section 2.5. Hence $\mathbf{X}_t$ has a similar interpretation as the scalar M in equation (24) $x_i = \sqrt{\rho_6}M + \sqrt{1-\rho_6}Z_i$ .The main motivation for this approach is again to reduce complexity.

More specifically, the correlation dynamics of the model can be derived by a probability measure change. From the generator matrix $\mathcal{L}_Y$ we have

$$
E^P(Y_{t+dt} = j \mid Y_t = i) = l_{ij}\, dt \qquad (48)
$$

where, as mentioned above, P is the historical probability measure. We now introduce a time-changed process, a stochastic clock τ_t, which may have continuous and jump components. τ_t is a function of λ_t,

$$
\tau_t = \int_0^t \lambda_s\, ds \qquad (49)
$$

Under the Girsanov theorem (see Neftci 1996 for an intuitive discussion) we can define a new stochastic process under the risk-neutral measure Q with the changed drift (and jump) but constant volatility

$$E^Q(Y_{t+dt} = j \mid Y_t = i) = l_{ij}\,\lambda_t\,dt \tag{50}$$

Since λ_t is an element of the conditioning market factor X_t (see equation 47), the migration processes Y_t in the new process under Q are now dependent. Specifically, from equations (49) and (50) we observe that default correlation is induced by the speed of the stochastic clock τ_t. An increase in the speed of the clock increases the speed of migration of all entities and hence increases the probability of simultaneous defaults. If the stochastic clock jumps, the probability of simultaneous defaults is even higher.

We find that the induction of correlation via volatility changes (Schönbucher 2006) and the induction of correlation via stochastic time change have a similar interpretation. An increase in transition volatility as well as an increase in the stochastic clock both increase the migration within the transition matrix, hence increase the probability of simultaneous defaults, and vice versa.

3.3 Contagion default modeling in top-down models

In a popular credit risk model applied in financial practice, Giesecke et al (2009) derive a random thinning process, which allocates the portfolio intensity to the sum of the individual entities' intensities. Giesecke et al show that this process uniquely exists and can be realized analytically. More formally, the thinning process Z^k under the reference measure μ is predictable and has the form

$$Z^k = \frac{\lambda^k}{\lambda} \tag{51}$$

where λ is the portfolio default intensity and λ^k is the default intensity of entity k, k=1,…,m and

$\lambda = \lambda^1 + \ldots + \lambda^m$. The model has the property that the thinning processes add up to one, $\sum_{k=1}^{m} Z^k = 1$ and that immediately after entity k defaults, the thinning process of this entity k drops to zero, $Z^k_{\tau^{k+}} = 0$.

The thinning process and a resulting basic default dependence can be explained with a m=2 entity portfolio with an assumed loss distribution N of

$$P(N_T = n) = \begin{cases} (.) & \text{for } n = 0 \\ 1 - e^{-T} & \text{for } n = 1 \\ 1 - e^{-T} - Te^{-T} & \text{for } n = 2 \end{cases} \tag{52}$$

where P is an integrable probability measure, n is the number of entities defaulting with the associated probability in (52) and N is a Poisson process with stopping time T^2. The thinning process can be parameterized with a non-negative constant q^{k1}.

$$Z^k_t = \frac{\lambda^k_t}{\lambda_t} = \begin{cases} q^{k1} & \text{for } t \le T^1 \\ 1_{\tau^{k1} = T^1} & \text{for } T^1 < t \le T^2 \\ 0 & \text{for } T^2 < t \end{cases} \tag{53}$$

From (53) we observe that the thinning process Z^k_t equals q^{k1} before or at time T^1 and equals 1 if the first entity defaults before or at T^1 since $\sum_{k=1}^{m} Z^k = 1$ and $Z^k_{\tau^{k+}} = 0$. The parameter q^{k1} governs the joint default dependence structure via $P(\tau^1 \le T \cap \tau^2 \le T) = 1 - e^{-T} - (1 - q^{k1})Te^{-T}$. From the equation set (52), we see that the extreme values of $q^{k1} = 1$ and $q^{k1} = 0$ generate the probability of exactly one default or two defaults, respectively. The name of the entity that defaults is revealed at the default time, highlighting the fact that random thinning allocates the portfolio intensity to the individual entities.

To incorporate a more rigorous joint default dependency, Giesecke et al suggest that the joint default distribution is governed by the portfolio intensity λ. In particular, Giesecke et al suggest that the process of the portfolio default intensity λ has an exponentially mean reverting drift with a stochastic jump component, which models default contagion:

$$d\lambda_t = g(\lambda_L - \lambda_t) + \delta\, dJ_t \tag{54}$$

where J is the jump with magnitude $\delta \geq 0$ at default of an entity. The jump elevates the level of the portfolio default intensity λ, i.e. the default intensity of all entities. After the jump, the contagion reverts exponentially at rate $g \geq 0$ (gravity) to its long term non-contagious mean λ_L.

In section 2.6 we had discussed contagion default modeling in a bottom-up framework. In this framework the contagion is modeled at an individual entity level, i.e. the default of entity i directly impacts the default intensity of entity j. This had led to problems of 'circularity', which complicates the derivation of a joint default distribution significantly. In the top-down environment, the default contagion is modeled conveniently at a portfolio level, circumventing problem of circularity.

Calibrating the parameters in equation (54) and those of the thinning process to the CDX high yield index during the crisis in September 2008, Giesecke et al find that their model outperforms copula based hedges. In addition, the mean profit is higher than compared to the copula approach.

In an extension to the Giesecke et al (2009) model, Giesecke and Tomecek (2005)[7] incorporate a stochastic time change. However, in contrary to Hurd and Kuznetsov (2006a) and (2006b), where stochastic time change is applied to induce correlation, Giesecke and Tomecek 2009 utilize the stochastic time change to transform a standard Poisson into a counting process N of default arrival times T_n. The counting process is represented by a standard Poisson process of the form

$$N_t = \sum_{k=1}^{\infty} 1_{\{S_n \leq t\}} \tag{55}$$

where

$$S_n = \sum_{i=1}^{n} V_i \tag{56}$$

[7] The first version of Giesecke et al (2009) model was published in 2004.

and the V_i are i.i.d., in particular the V_i are exponential random variables.

The continuous process $G(t) = \int_0^t \lambda_s ds$ defines the time change. G is adapted to the filtration $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$, where $\mathcal{G}_t$ represents all information available at time t. Hence the process G(t) is predictable.

The Poisson process (55) is mapped to arrival times T_n by the inverse of the time change process G. Hence

$$T_n = G^{-1}(S_n) \tag{57}$$

For a rigorous proof see Giesecke et al (2009). Equation (57) implies that the Poisson arrivals S_n serve as a Merton style barrier to derive the arrival times T_n:

$$T_n = G^{-1}(S_n) = \inf\{t : G(t) \geq S_n\} = \inf\left\{t : \int_0^t \lambda_s ds \geq S_n\right\} \tag{58}$$

Since G(t) and S_n are generated independently, the model has the form of a doubly stochastic process.

We observe that deriving the default time in the Copula approach of equation (17) $\tau_i = \inf\left\{t : \int_0^{\tau_i} \lambda_i(t)\, dt \geq M_{n,R}(\cdot)\right\}$ is conceptually similar to equation (58). However, in equation (17), the default time is modeled individually for each entity i with respect to the entities' default intensity function λ_i. In the top-down approach (58), the intensity λ modeled at a portfolio level. A further difference is that in equation (17) the default correlation is elegantly incorporated in the barrier $M_{n,R}(\cdot)$. In the approach (58) the default correlation is modeled separately in the core equation (54). One benefit of the model (58) is that by construction the default times T_n are ordered, i.e. $T_1 = \min(T_k)$ and $T_m = \max(T_k)$, k=1,...,m. In the copula model the default distribution is built by numerical integration over unordered default times, see sections 2.3 and 3.1.

4. Concluding summary

In this paper we have categorized and analyzed the most popular correlation approaches in finance. We have first outlined the numerous shortcomings of the classical Pearson correlation coefficients for financial modeling. We then briefly discussed the ordinal correlation measures Spearman's rank correlation and Kendall's τ. These two correlation measures have been gaining popularity in analyzing rating categories in transition matrices, since these are ordinal.

As the broadest categorization with respect to specific financial correlation models, we differentiate bottom-up and top-down models. Within the bottom-up framework, the Heston 1993 approach of correlating the Brownian motions of stochastic processes is an integral part of financial correlation modeling. The approach is dynamic, versatile, and mathematically rigorous. It allows to positively or negatively correlate stochastic processes and permits dynamic risk management.

The binomial correlation approach of Lucas 1995 is by construction limited to modeling binomial events as for example default or no default. The random indicator variable 1_X takes the value 1 if entity X defaults before or at time T, and 0 otherwise. The same applies to entity Y. It follows that the joint probability of default can be expressed closed form. The binomial correlation approach is a special case of the Pearson correlation model. Hence it suffers the same restrictions for financial correlation modeling as the Pearson model.

Copula correlation models were created by Sklar in 1959. Copulas were introduced to finance by Vasicek (1987) and Li (2000). Numerous types of copulas exist. They can be broadly categorized in one-parameter copulas such as the Gaussian copula, and the Archimedean copula, which comprise Gumbel, Clayton and Frank copulas. Often cited two-parameter copulas are student-t, Frechet, and Marshall-Olkin.

The Gaussian default-time copula was the de-facto standard approach to value CDOs before the 2007/2008 crisis. The cumulative default times of each entity i, $Q_i(\tau_i) \in [0,1]$, which can be derived by historical data, are mapped to a single n-variate distribution $M_{n,R}(\cdot) \in [0,1]$, which incorporates a Gaussian correlation structure R. The correlated default times are found when the individual default times $Q_i(\tau_i)$ breach the Merton-style barrier $M_{n,R}(\cdot)$.

The Gaussian default-time copula, first enthusiastically embraced by financial markets, was later partially blamed for the global financial crisis 2007/2008. As any model, the Gaussian copula model has its limitations. Among them is the zero-tail dependence when following a definition of Joe (1999). In this sense it may be more desirable to apply the student-t or Gumbel copula, which exhibit higher tail-dependencies. In addition, Copula models are static, i.e. have a one-period fixed time horizon. However, back testing and stress testing the variables for different time horizons can give valuable sensitivities. Before the 2007/2008 financial crisis, numerous market participants trusted the copula model uncritically and naively. The foremost reason for the global financial crisis however was not a certain correlation model but rather excessive risk taking without conducting prudent risk management. The prime example is AIG's London subsidiary, which had sold CDSs and CDOs in an amount of close to $500 billion without conducting any major hedging.

A core enhancement of copula models are dynamic copulas, introduced by Albanese et al (2005, 2007, and 2010). The 'dynamic conditioning' approach models the evolution of multi-factor super-lattices, which correlate the return processes of each entity at each time step. Binomial dynamic copulas apply combinatorial methods to avoid Monte Carlo simulations. Richer dynamic Gaussian copulas apply Monte Carlo simulation and come at the cost of requiring powerful computer technology.

Conditionally independent (CID) correlation models apply a factorization to avoid specifically modeling the correlations between each entity. Correlations are modeled indirectly by conditioning auxiliary variables on a common factor, for example the current economic environment. CID models are typically parsimonious special cases of copulas and admit the same benefits and shortcomings as copulas.

Contagion default modeling can be viewed as a variation of CID modeling. While CID models condition on a common factor, contagion approaches model the default intensity of an entity as a function of the default of another entity. Hence contagion default modeling incorporates counterparty risk, i.e. the direct impact of a defaulting entity on the default intensity of another entity. After a default of a particular entity, the default intensity of all assets in the portfolio increases. This default contagion then typically fades exponentially to non-contagious default intensity levels. CID models and contagion models both offer attractive, realistic properties. Hence numerous authors have combined the two technologies to create practical correlation models, which often calibrate well to the market.

A fairly new class of correlation models are top-down approaches. In this framework the evolution of the portfolio intensity distribution is derived directly, i.e. abstracting from the individual entities' default intensities. Vasicek's large homogeneous portfolio (LHP) is a top-down model, since it assumes a) a constant and identical default intensity of all entities in a portfolio and b) the same default correlation between the entities. The model is a one-factor version of the Gaussian copula. The model is currently (year 2010) the basis for credit risk management in Basel II. The benefits of the model are simplicity and intuition. One of the main shortcomings of the model is that traders randomly alter the correlation parameter for different tranches to achieve desired tranche spreads. However conceptually, the correlation parameter should be identical for the whole portfolio.

Within the top-down framework, Schönbucher (2006) creates a time-inhomogeneous Markov-chain of transition rates. Default correlation is introduced by changes in the volatility of transition rates. For certain parameter constellations, higher volatility means faster transition to lower states as default, hence implies higher default correlation, and vice versa. Similarly, Hurd and Kuznetsov (2006a) and (2006b) induce correlation by a random change in the speed of time. A faster speed of time means faster transition to a lower state, possibly default, hence increases default correlation, and vice versa. These models are attractive, elegant and mathematically rigorous. The models do depend on reliable transition data as inputs and come at the cost of relatively high complexity.

Two key questions remain with respect to financial correlation modeling.

a) Should we model bottom-up or top-down? The answer depends on the nature of the underlying portfolio. Most financial institutions risk-manage complex, heterogeneous portfolios with highly different default intensities and different default correlations. For these portfolios, individual default intensities, specific pair-wise correlations and individual recovery rates should not be discarded. Hence dynamic, versatile bottom-up models should be the primary choice for modeling these heterogeneous portfolios. However, when evaluating portfolios with homogeneous default intensities and default correlation, parsimonious top-down models may be adequate. They typically calibrate well to market data. In addition, random thinning techniques can allocate the portfolio intensity to the sum of the individual entities' intensities.

b) Will there be a Black-Scholes-Merton correlation model, which dominates correlation modeling? The answer is 'probably not'. As discussed above, the choice of correlation model depends on the nature of the underlying portfolio. Complex portfolios require complex correlation models. For simpler, homogeneous portfolios, parsimonious top-down correlation models may be sufficient. For static portfolios with fixed time horizons, elegant copula models may be the right choice. For truly dynamic portfolios, correlating Brownian motions (Heston 1993), or complex dynamic copulas (Albanese et al 2007, 2010) may be suitable. Modeling volatility or stochastic time change in Markov chain models are a further attractive approach if transition rates are available and deemed reliable.

The authors do apologize for the length of this paper.

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