MCEN 5023/ASEN 5012

Chapter 2

Fundamentals of Tensor Analysis

Fall, 2006

Fundamentals of Tensor Analysis

Concepts of Scalar, Vector, and Tensor

Scalar α A physical quantity that can be completely described by a real number.

Example: *Temperature; Mass; Density; Potential*….

The expression of its component is independent of the choice of the coordinate system.

Vector **a** A physical quantity that has both direction and length.

Example: *Displacement; Velocity; Force; Heat flow;* ….

The expression of its components is dependent of the choice of the coordinate system.

Tensor **A** A 2^{nd} order tensor defines an operation that transforms a vector to another vector

A tensor contains the information about the directions and the magnitudes in those directions.

In general, Scalar is a 0^{th} order tensor; Vector is A 1^{st} order tensor; 2^{nd} order tensor; 3^{rd} order tensor…

Fundamentals of Tensor Analysis

Vectors and Vector Algebra

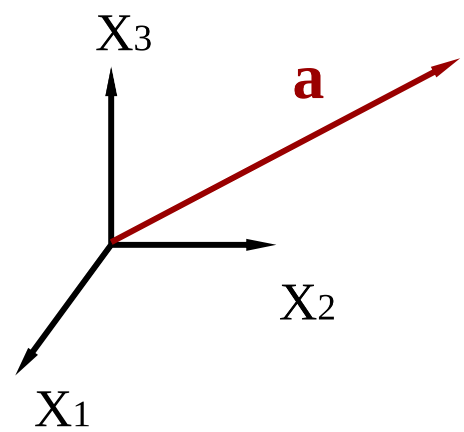
A vector is a physical quantity that has both direction and length

What do we mean the two vectors are equal?

The two vectors have the same length and direction

What is a unit vector?

The length of a unit vector is one



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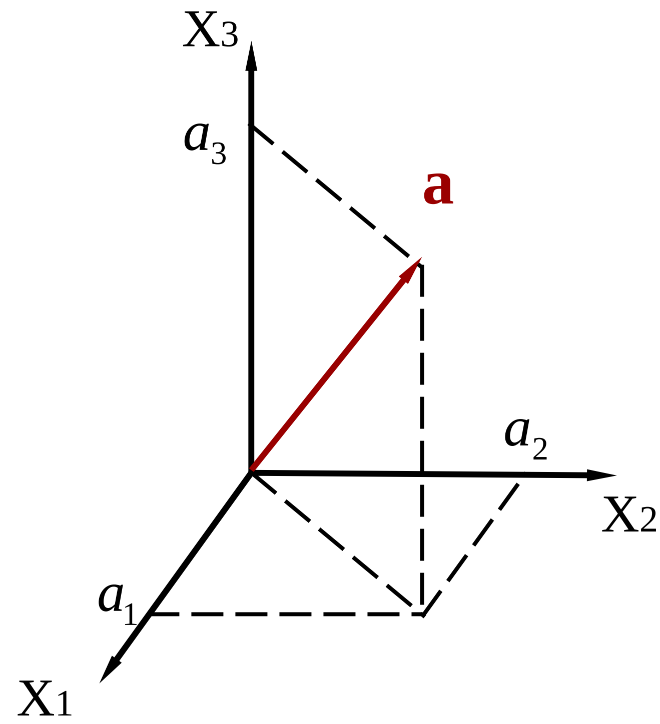
Vectors and Vector Algebra

In a Cartesian coordinate, a vector can be expressed by three ordered scalars

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i$$

Summation convention $\mathbf{a} = a_i\mathbf{e}_i$



Dummy index and free index

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Vector Algebra

Sum: $$\mathbf{a} + \mathbf{b} = (a_i + b_i)\mathbf{e}_i$$

Scalar Multiplication $$\alpha\,\mathbf{a} = \alpha\, a_i \mathbf{e}_i$$

Dot Product $$\mathbf{a} \bullet \mathbf{b} = |a||b| \cos\theta(\mathbf{a}, \mathbf{b})$$

Cross Product

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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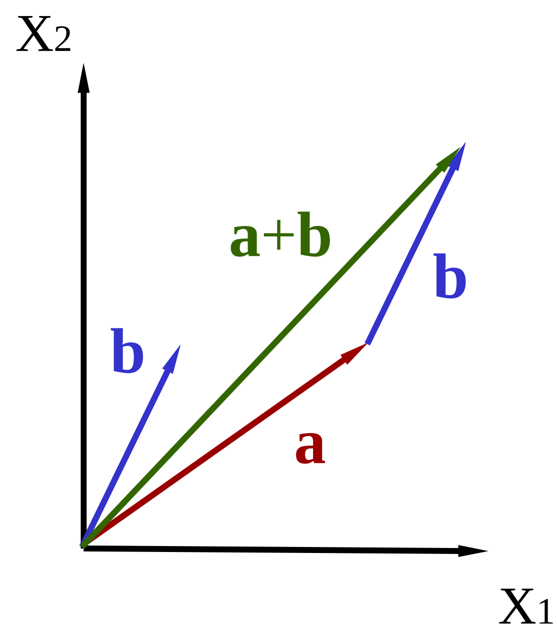
Vector Algebra

Sum: $$\mathbf{a}+\mathbf{b}=\left(a_i+b_i\right)\mathbf{e}_i$$



Parallelogram law of addition

<u>Properties of Sum</u>

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$

$$\left(\mathbf{a}+\mathbf{b}\right)+\mathbf{c}=\mathbf{a}+\left(\mathbf{b}+\mathbf{c}\right)$$

$$\mathbf{a}+\left(-\mathbf{a}\right)=\mathbf{o}$$

$$\mathbf{a}+\mathbf{o}=\mathbf{a}$$

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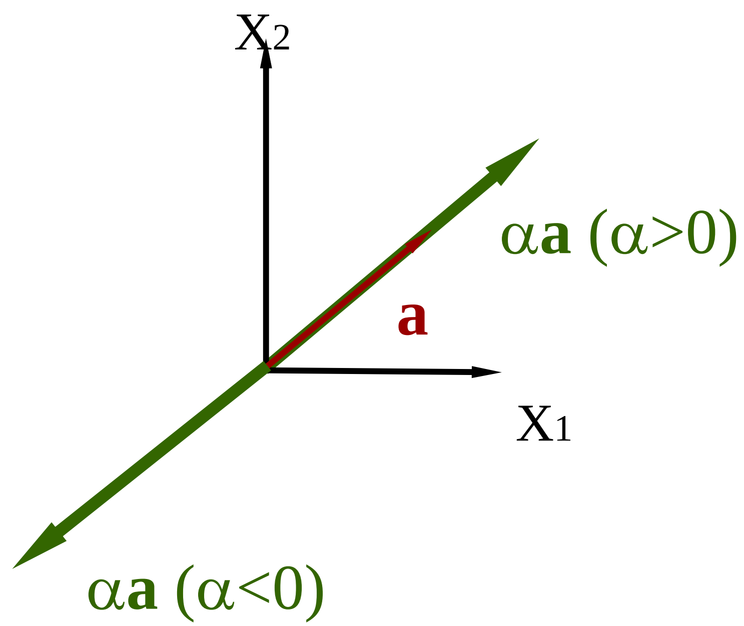
Vector Algebra

Scalar Multiplication $\alpha \mathbf{a} = \alpha a_i \mathbf{e}_i$



Properties of Scalar Multiplication

$$(\alpha\beta)\mathbf{a} = \alpha(\beta\,\mathbf{a})$$

$$(\alpha+\beta)\mathbf{a} = \alpha\,\mathbf{a} + \beta\,\mathbf{a}$$

$$\alpha(\mathbf{a}+\mathbf{b}) = \alpha\,\mathbf{a} + \alpha\,\mathbf{b}$$

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Vector Algebra

Dot Product $\qquad \mathbf{a} \bullet \mathbf{b} = |a||b| \cos\theta(\mathbf{a}, \mathbf{b})$

Properties of Dot Product

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$$

$$\mathbf{a} \bullet \mathbf{o} = \mathbf{o}$$

$$\mathbf{a} \bullet (\alpha\,\mathbf{b} + \beta\,\mathbf{c}) = \alpha\,(\mathbf{a} \bullet \mathbf{b}) + \beta\,(\mathbf{a} \bullet \mathbf{c})$$

$\mathbf{a} \bullet \mathbf{a} > 0 \quad \Leftrightarrow \quad \mathbf{a} \neq \mathbf{o}$ and $\mathbf{a} \bullet \mathbf{a} = 0 \quad \Leftrightarrow \quad \mathbf{a} = \mathbf{o}$

$|\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}}$ norm of **a**

$$\mathbf{a}^2 = \mathbf{a} \bullet \mathbf{a}$$

$\mathbf{a} \bullet \mathbf{b} = \mathbf{o}$ **a** is orthogonal to **b**

Direction of a vector:

$$\mathbf{n} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{\sqrt{\mathbf{a} \bullet \mathbf{a}}}$$

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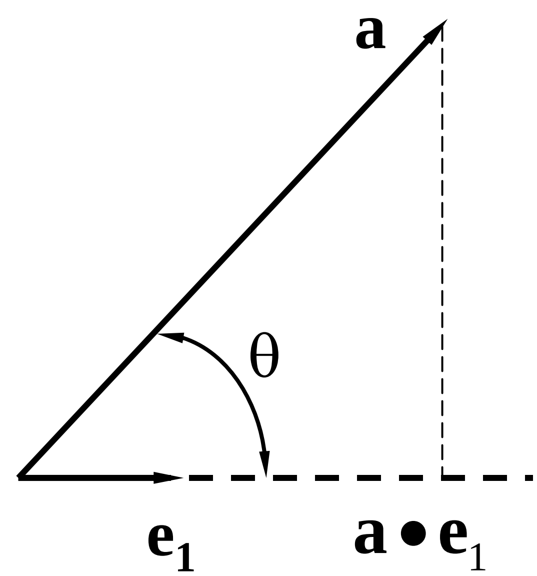
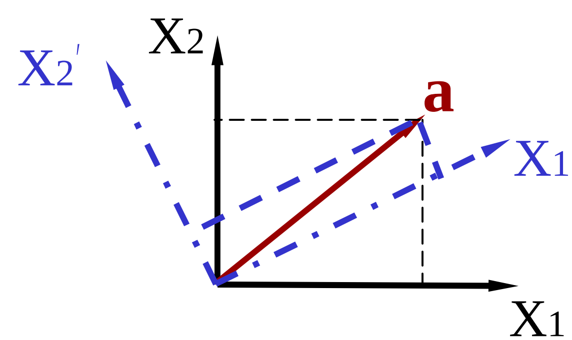
Vector Algebra

Components of a vector



$\mathbf{a}$ Symbolic expression

$a_i \mathbf{e}_i$ Component expression



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Vector Algebra

Dot Product $$\mathbf{a} \bullet \mathbf{b} = |a||b| \cos\theta(\mathbf{a}, \mathbf{b})$$

Three basis vectors of a Cartesian coordinate $\mathbf{e}_i$ i=1,2,3

$$\mathbf{e}_i \bullet \mathbf{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$ Kronecker delta

Properties of δ_{ij}

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Vector Algebra

Cross Product
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Permutation symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \textit{even permutation} \\ -1 & \textit{odd permutation} \\ 0 & \textit{repeated index} \end{cases}$$

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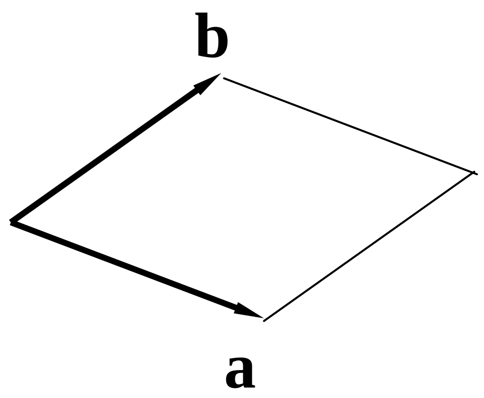
Vector Algebra

Physical Meaning of Cross Product



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Vector Algebra

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Properties of Cross Product</u>

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

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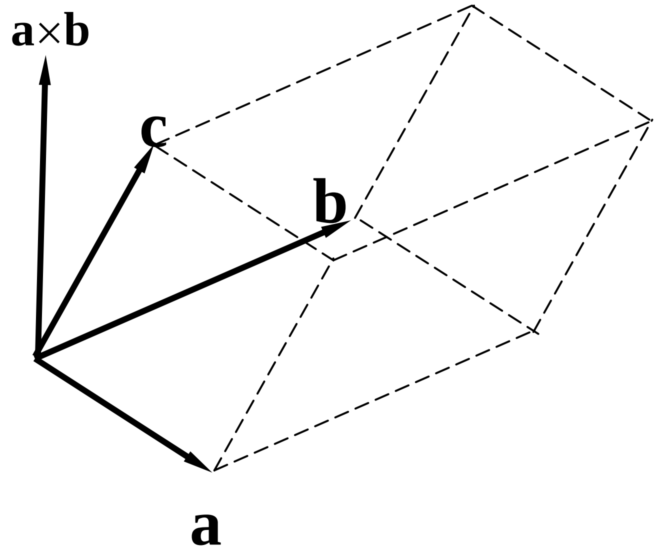
Vector Algebra

Triple scalar product $\left(\mathbf{a}\times\mathbf{b}\right)\bullet\mathbf{c}$



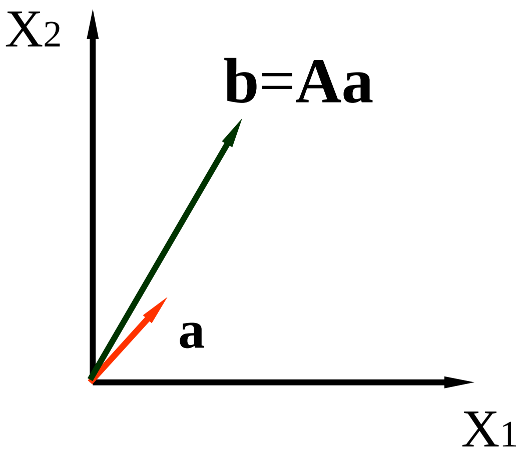
$$\left(\mathbf{a}\times\mathbf{b}\right)\bullet\mathbf{c}=\left(\mathbf{b}\times\mathbf{c}\right)\bullet\mathbf{a}=\left(\mathbf{c}\times\mathbf{a}\right)\bullet\mathbf{b}$$

Fundamentals of Tensor Analysis

Concept of Tensor

A 2nd order tensor is a linear operator that transforms a vector **a** into another vector **b** through a dot product.

$\mathbf{A}: \mathbf{a} \longrightarrow \mathbf{b}$ or $\mathbf{b}=\mathbf{A}\mathbf{a}$



Properties due to linear operation

$$\mathbf{A}(\alpha\mathbf{a}+\mathbf{b}) = \alpha\mathbf{A}\mathbf{a} + \mathbf{A}\mathbf{b}$$

$$(\mathbf{A} \pm \mathbf{B})\mathbf{a} = \mathbf{A}\mathbf{a} \pm \mathbf{B}\mathbf{a}$$

Fundamentals of Tensor Analysis

Anatomy of a Tensor: Concepts of Dyad and Dyadic

Dyad $\mathbf{a} \otimes \mathbf{b}$ $\quad (\mathbf{ab})$

A dyad is a tensor. It transforms a vector by

A dyadic is also a tensor. It is a linear combination of dyads with scalar coefficients.

$$\mathbf{B} = \mathbf{a} \otimes \mathbf{b} + \mathbf{c} \otimes \mathbf{d}$$

Fundamentals of Tensor Analysis

Concepts of Dyad and Dyadic

Now, consider a special dyad $\mathbf{e}_i \otimes \mathbf{e}_j$

Fundamentals of Tensor Analysis

Concepts of Dyad and Dyadic

Now, consider a special dyad $\mathbf{e}_i \otimes \mathbf{e}_j$

Fundamentals of Tensor Analysis

Properties of Dyad and Dyadic

$$(\mathbf{a}\otimes\mathbf{b})(\alpha\,\mathbf{c}+\mathbf{d})=\alpha(\mathbf{a}\otimes\mathbf{b})\mathbf{c}+(\mathbf{a}\otimes\mathbf{b})\mathbf{d}$$

$$(\alpha\,\mathbf{a}+\beta\,\mathbf{b})\otimes\mathbf{c}=\alpha(\mathbf{a}\otimes\mathbf{c})+\beta(\mathbf{b}\otimes\mathbf{c})$$

$$\mathbf{A}(\mathbf{a}\otimes\mathbf{b})=(\mathbf{A}\mathbf{a})\otimes\mathbf{b}$$

Fundamentals of Tensor Analysis

Special Tensors

Positive semi-definite tensor: $\mathbf{a} \bullet \mathbf{A}\mathbf{a} \geq 0$ for any $\mathbf{a} \neq \mathbf{0}$

Positive definite tensor: $\mathbf{a} \bullet \mathbf{A}\mathbf{a} > 0$ for any $\mathbf{a} \neq \mathbf{0}$

Unit tensor: $\mathbf{I}$ $\quad \mathbf{I} = \delta_{ij}\mathbf{e}_i\mathbf{e}_j \qquad \mathbf{A}\mathbf{I} = \mathbf{A}$

Transpose of a tensor $\quad \mathbf{A}^T$

Fundamentals of Tensor Analysis

Dot Product $\mathbf{AB}$

Properties of dot product

$$\mathbf{AB} \neq \mathbf{BA}$$

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$$

$$\mathbf{A}^2 = \mathbf{AA}$$

$$\mathbf{A}^n = \mathbf{AA}\ldots\mathbf{A}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Fundamentals of Tensor Analysis

Trace and Contraction

Trace $tr(\mathbf{A}) = A_{ii}$

$$tr(\mathbf{A}) = tr(\mathbf{A}^T)$$

$$tr(\mathbf{AB}) = tr(\mathbf{BA})$$

$$tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$$

$$tr(\alpha\mathbf{A}) = \alpha\, tr(\mathbf{A})$$

Contraction (double dot) $\mathbf{A} : \mathbf{B}$

Fundamentals of Tensor Analysis

Tensor Review

$$\otimes$$

$$\mathbf{a} \otimes \mathbf{b}$$

Dayd: vector **a** is sitting in front vector **b**

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \bullet \mathbf{c})$$

$$\mathbf{c}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b}$$

$$\times$$

Cross Product

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Fundamentals of Tensor Analysis

Rule of Thumb:

For algebra on vectors and tensors, an index must show up twice and only twice.

If an index shows up once on the left hand side (LHS) of “ = ” sign, it must show up once and only once on the right hand side (RHS) of “ = ” sign. This index is free index.

If an index shows up twice on either LHS or RHS of “ = ”, it does not have to show up on the other side of “ = ”. This index is dummy index.

You are free to change the letters that represent a “free” index or a “dummy” index. But you have to change it in pair.

Fundamentals of Tensor Analysis

Determinant and inverse of a tensor

Determinant

$$\det \mathbf{A} = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$$

$$\det \mathbf{A}^T = \det \mathbf{A}$$

A is singular if and only if det**A**=0

Inverse

if det**A** is not zero $\quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$

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Properties of inverse tensor

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(\alpha\,\mathbf{A})^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{-1}\right)^{T} = \left(\mathbf{A}^{T}\right)^{-1} \qquad \left(\mathbf{A}^{-1}\right)^{T} = \mathbf{A}^{-T}$$

$$\mathbf{A}^{-2} = \mathbf{A}^{-1}\mathbf{A}^{-1} \qquad \mathbf{A}^{-n} = \mathbf{A}^{-1}\mathbf{A}^{-1}\ldots\mathbf{A}^{-1}$$

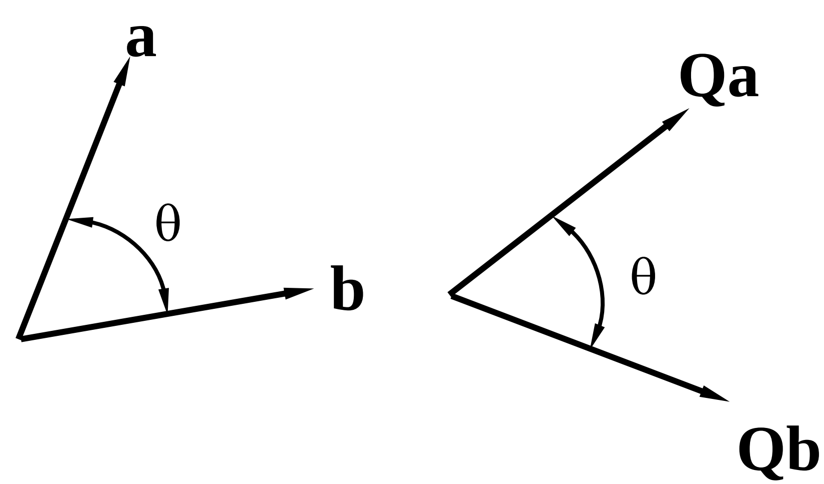
$$\det\left(\mathbf{A}^{-1}\right) = (\det\mathbf{A})^{-1}$$

Fundamentals of Tensor Analysis

Orthogonal tensor

$$\mathbf{Qa} \bullet \mathbf{Qb} = \mathbf{a} \bullet \mathbf{b}$$

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$$



$$\det \mathbf{Q} = \pm 1$$

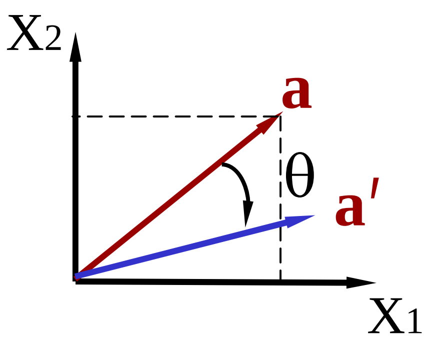
$\det \mathbf{Q} = 1$ **Q** is a proper orthogonal tensor and corresponds to a rotation

$\det \mathbf{Q} = -1$ **Q** is an improper orthogonal tensor and corresponds to a reflection

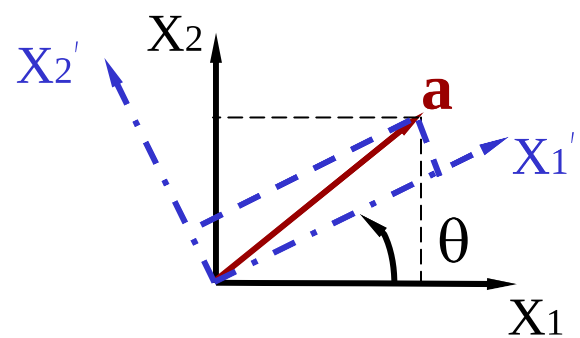
Fundamentals of Tensor Analysis

Rotation

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation of the vector



Rotation of the coordinate system

Fundamentals of Tensor Analysis

Symmetric and skew tensor

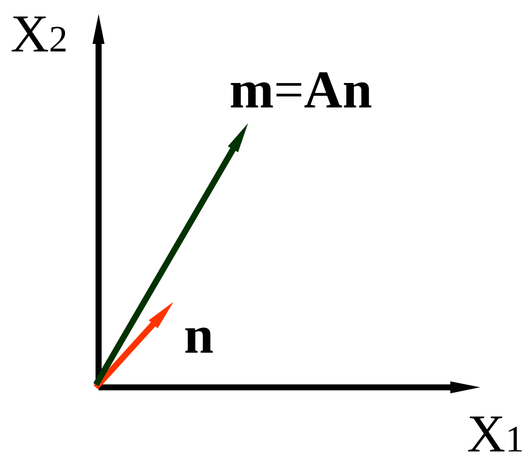
$$\mathbf{A}=\mathbf{S}+\mathbf{W} \qquad \mathbf{S}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{T}\right) \qquad \mathbf{W}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^{T}\right)$$

Fundamentals of Tensor Analysis

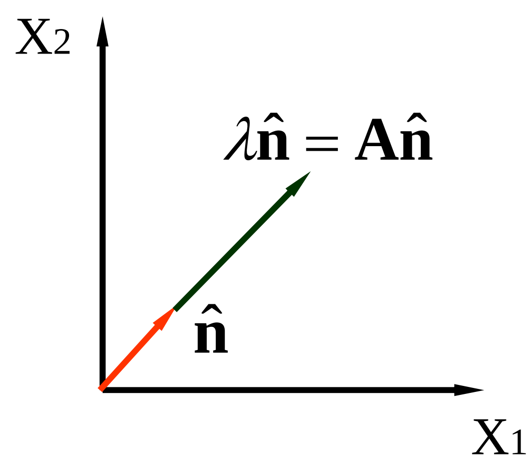
Eigenvectors and eigenvalues of a tensor **A**

The scalar λ is an eigenvalue of a tensor **A** if there is a non-zero vector unit eigenvector $\hat{\mathbf{n}}$ of **A** so that

$$\mathbf{A}\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}}$$



General case: **m**=**An**



Eigenvector: $\lambda\hat{\mathbf{n}} = \mathbf{A}\hat{\mathbf{n}}$

Fundamentals of Tensor Analysis

Eigenvectors and eigenvalues of a tensor **A**

$$\mathbf{A}\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}} \quad \Longrightarrow \quad \left(\mathbf{A} - \lambda\,\mathbf{I}\right)\hat{\mathbf{n}} = \mathbf{0}$$

$$\Longrightarrow \quad \det\left(\mathbf{A} - \lambda\,\mathbf{I}\right) = 0$$

Fundamentals of Tensor Analysis

Spectral Decomposition