Dynamical Systems
for Engineers: Exercise Set 9, Solutions

Exercise 1

1. Because of Newton's law (mass m times acceleration a equals the resulting forces F applied to the mass), the behavior of the pendulum is given by the differential equation

$$ma = ml\frac{d^2\varphi}{dt^2} = -mg\sin\varphi. \tag{1}$$

This second-order differential equation can be written as

$$\dot{x_1} = x_2 \tag{2}$$
$$\dot{x_2} = -\omega^2 \sin x_1 \tag{3}$$

where $\omega = \sqrt{\frac{g}{l}}$, $x_1 = \varphi$ and $x_2 = \dot{\varphi}$.

2. The equilibria of the system are

$$\overline{x} = (\overline{x}_1, \overline{x}_2) = (k\pi, 0)\,.$$

for all $k \in \mathbb{Z}$. Because of the periodicity of $x_1 = \varphi$, we can restrict the subsequent analysis to the two equilbrium ponts within one period, e.g. $(0,0)$ and $(\pi, 0)$

3. The Jacobian matrix of the system is

$$J(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -\omega^2\cos x_1 & 0 \end{bmatrix}.$$

In $(0,0)$, its eigenvalues are $\pm j\omega$, and we cannot conclude that it is a stable equilibrium from the linearization. In $(\pi, 0)$, its eigenvalues are $\pm\omega$, and this corresponds to an unstable equilibrium.

4. For many mechanical systems, the Hamiltonian takes the form $H(q,p) = E_k(q,p) + E_p(q)$, where $E_k(q,p)$ is the kinetic energy, and $E_p(q)$ is the potential energy of the system. Such systems are called natural Hamiltonian systems. Here, since the speed is $l\dot{\varphi}$, the kinetic energy is

$$E_k = \frac{1}{2}m\,(l\dot{\varphi})^2 = \frac{1}{2}ml^2x_2^2 \tag{4}$$

whereas the potential energy is

$$E_p = mgl(1 - \cos\varphi) = mgl(1 - \cos x_1). \tag{5}$$

For the ideal pendulum, we can write:

$$q = \varphi = x_1$$
$$p = ml^2\dot{\varphi} = ml^2x_2$$

so that

$$H(q,p) = H(x_1, ml^2x_2) = \frac{1}{2}mlx_2^2 + mgl(1 - \cos x_1)$$

and it is obvious that:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q}. \end{aligned}$$

The system is therefore Hamitonian, and has therefore bounded solutions.

We can add the following discussion about the non dissipative pendulum. The law of conservation of energy says that the total energy of points moving according to equation (1) is conserved (which explains why the system is Hamiltonian), i.e., the total energy function $E(\varphi(t), \dot{\varphi}(t)) = E(x_1(t), x_2(t)) = E_p + E_k$ is independent of t.

Let φ_0 be the highest point of motion; then it is obvious that: $\dot{\varphi}|_{\varphi_0} = 0$. Thus, $E(\varphi, \dot{\varphi}dt|_{\varphi_0}) = mgl(1 - \cos\varphi_0)$. We can write:

$$E_k = E - E_p \Rightarrow \frac{1}{2}ml^2\dot{\varphi}^2 = mgl(1 - \cos\varphi_0) - mgl(1 - \cos\varphi).$$

Since $1 - \cos\varphi = 2\sin^2\frac{\varphi}{2}$, we have:

$$\dot{\varphi}^2 = 4\frac{g}{l}\left(\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi}{2}\right) \tag{6}$$

For small φ_0: If $\varphi_0 \ll 1$ then $\sin\frac{\varphi}{2} \approx \frac{\varphi}{2}$ and Eq. (6) can be written as:

$$\dot{\varphi}^2 \approx 4\frac{g}{l}\left(\left(\frac{\varphi_0}{2}\right)^2 - \left(\frac{\varphi}{2}\right)^2\right)$$

or

$$\left(\dot{\varphi}\sqrt{\frac{l}{g}}\right)^2 + \varphi^2 \approx \varphi_0^2,$$

or equivalently in the variables (x_1, x_2),

$$\left(x_2\sqrt{\frac{l}{g}}\right)^2 + x_1^2 \approx \varphi_0^2$$

which is the equation of an ellipse.

For large φ_0: For φ_0 near π, $\sin\varphi/2 \approx 1$ and we have:

$$\dot{\varphi}^2 \approx 4\frac{g}{l}(1 - \sin^2\varphi/2) = 4\frac{g}{l}(\cos^2\varphi/2).$$

Thus for $\varphi_0 \approx \pi$, the curves are the cosines:

$$\dot{\varphi} = \pm 2\sqrt{\frac{g}{l}}\cos\frac{\varphi}{2},$$

or equivalently in the variables (x_1, x_2),

$$x_2 = \pm 2\sqrt{\frac{g}{l}}\cos(x_1/2).$$

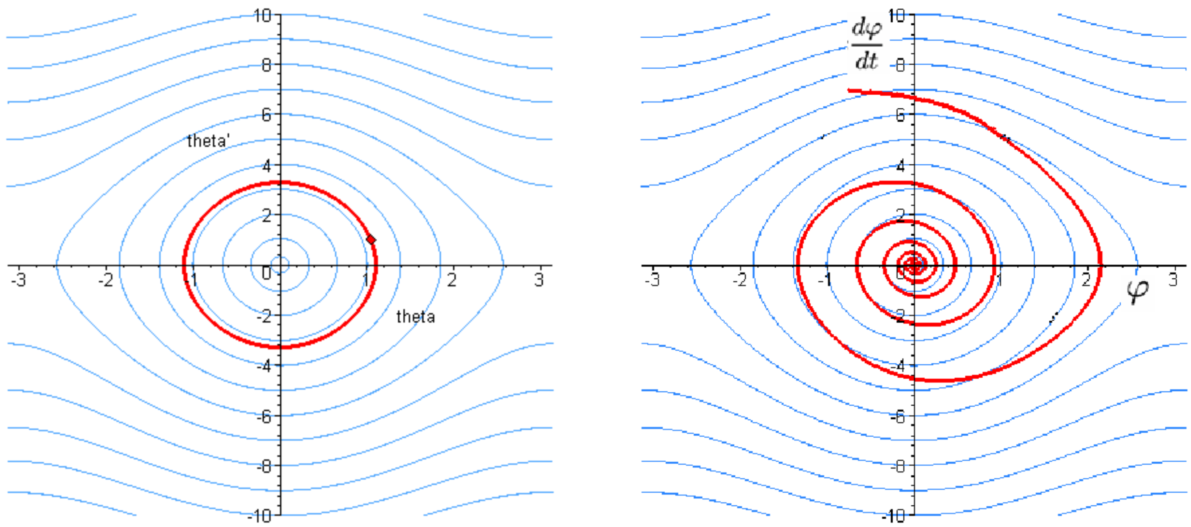

Figure 1: Phase portrait of the pendulum system: friction-less (left) and with friction (right).

In fact, the curve related to $\varphi = \pi$ (i.e. $x_1 = \pi$) separates oscillatory motion ($E < 2mgl$) from rotary motions ($E > 2mgl$). Figure 1 (left) shows a typical phase portrait for this non-dissipative system.

Exercise 2

1. Because of Newton's law, the behavior of the pendulum is now given by the following differential equation:

$$ml\frac{d^2\varphi}{dt^2} = -mg\sin\varphi - mk\frac{d\varphi}{dt}. \tag{7}$$

This second-order differential equation can be written as

$$\dot{x_1} = x_2 \tag{8}$$

$$\dot{x_2} = -\frac{g}{l}\sin x_1 - \frac{k}{l}x_2. \tag{9}$$

where $x_1 = \varphi$, $x_2 = \dot{\varphi}$.

2. The equilibria of the system are

$$\overline{x} = (\overline{x}_1, \overline{x}_2) = (k\pi, 0)\,.$$

for all $k \in \mathbb{Z}$. Because of the periodicity of $x_1 = \varphi$, we can restrict the subsequent analysis to the two equilbrium ponts within one period, e.g. $(0,0)$ and $(\pi, 0)$

3. The Jacobian matrix of the system is

$$J(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -(g/l)\cos x_1 & -k/l \end{bmatrix}.$$

In $(0,0)$, its eigenvalues are

$$\frac{-k \pm \sqrt{k^2 - 4lg}}{2l},$$

which shows that $(0,0)$ is asymptotically stable (it is a stable focus node in the linearized system). In $(\pi, 0)$, its eigenvalues are

$$\frac{-k \pm \sqrt{k^2 + 4lg}}{2l},$$

and the largest one is always strictly positive, hence $(0, \pi)$ is an unstable equilibrium.

4. The total energy of the system can be considered as a natural Lyapunov function of the system:

$$W(x_1, x_2) = ml^2 \frac{x_2^2}{2} + mgl(1 - \cos x_1).$$

Clearly, W is continuously differentiable, and $W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\overline{x}\}$, with $W(\overline{x}) = 0$. Because of the periodicity of x_1, we can restrict $-\pi < x_1 < \pi$.

Now, $W(x_1, x_2) < E$ implies that

$$\frac{l}{g}\frac{x_2^2}{2} + (1 - \cos x_1) < \frac{E}{lmg}.$$

If we pick $E > 0$ small enough, i.e. $0 < E \ll 1/lmg$, then all terms in the above equations must be small enough and we can approximate the left hand side by

$$\frac{l}{g}\frac{x_2^2}{2} + \frac{x_1^2}{2} < \frac{E}{lmg},$$

which shows that $\mathcal{U}_E = \{x \in \mathbb{R}^n \mid W(x) < E\}$ is bounded.

One can check that

$$\begin{aligned}
\dot{W}(x_1, x_2) &= ml^2 x_2 \dot{x}_2 + mgl \sin x_1 \dot{x}_1 \\
&= ml\left(-g x_2 \sin x_1 - k x_2^2 + g \sin x_1 x_2\right) = -kml x_2^2.
\end{aligned}$$

Since $F(x_1, 0) = (0, -(g/l)x_1)$, we see that any solution starting at a initial state $(x_1(0), 0)$ on the x_1-axis leaves this axis immediately, unless $x_1 = 0$. Therefore W is non-increasing along trajectories as long as $W(x(t)) < E$, snf W is decreasing along trajectories as long as $W(x(t)) < E$ and $x(t) \neq \overline{x}$.

Level sets of the function W are shown in figure 1 (right). For the stable equilibrium point at the origin (and its periodic replica at $(2k\pi, 0)$), there is a basin of attraction and the Lyapunov function above can approximate this basin of attraction (although the formal computation of E is difficult, one can only do it for "E small enough").

Exercise 3

We consider the system

$$\begin{aligned}
\dot{x_1} &= -x_1 + 2x_2^3 - 2x_2^4 \\
\dot{x_2} &= -x_1 - x_2 + x_1 x_2
\end{aligned}$$

1. The Jacobian matrix at the origin is

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

and has one repeated eigenvalue -1 (with multiplicity 2), which shows that the origin is asymptotically stable.

2. For this system, the standard sum of squares does not work because of the terms of order higher than two. We generalize the Lyapunov candidate as follows:

$$W(x_1, x_2) = a x_1^m + b x_2^l, \quad a, b > 0 \text{ and } m, l \text{ even.} \tag{10}$$

We see that

- W is continuously differentiable,
- $W(0,0) = 0$ and $W(x_1, x_2) > 0$ for all $(x_1, x_2) \in \mathbb{R}^2$ with $(x_1, x_2) \neq (0,0)$,
- the strict level sets $\mathcal{U}_K = \{(x_1, x_2) \in \mathbb{R}^2 \mid W(x_1, x_2) < K\}$ are bounded for all $K > 0$

Let us now compute $\dot{W}$:

$$\begin{aligned} \dot{W} &= a m x_1^{m-1} \dot{x_1} + b l x_2^{l-1} \dot{x_2} \\ &= a m(-x_1^m + 2x_1^{m-1} x_2^3 - 2x_1^{m-1} x_2^4) + b l(-x_1 x_2^{l-1} - x_2^l + x_1 x_2^l). \end{aligned}$$

The terms $-x_1^m$ and $-x_2^l$ are always negative if $x_1, x_2 \neq 0$ because m and l are even. To have $\dot{W} \leq 0$, We have to choose a, b, m and l such that the rest of the cross terms cancels. After some trial and error, we find that $a = 1$, $b = 1$, $m = 2$ and $l = 4$ works, as one can check that

- W is non-increasing along trajectories,
- W is decreasing along trajectories as long as $(x_1(t), x_2(t)) \neq (0,0)$.

This proves that the origin is a globally asymptotically stable equilibrium point.

Exercise 4

We consider the gradient system $\dot{x} = -\nabla_x V(x)$, where $x = (x_1, x_2)$.

1. If $V(x_1, x_2) = x_1^2 x_2^2$, the system becomes

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{\partial V}{\partial x_1} = -2x_1 x_2^2 \\ \frac{dx_2}{dt} &= -\frac{\partial V}{\partial x_2} = -2x_1^2 x_2 \end{aligned}$$

By setting the above derivatives to 0, we obtain the nullclines of this system (i.e. the curves along which one of the state variables remains constant), $x_1 = 0$ and $x_2 = 0$. There are an infinity of equilibrium points along the nullclines, i.e all points on both the x_1- and x_2-axes are equilibrium points.

The vector field is plotted in Fig. 2, together with the function $V(x_1, x_2)$. The lines $x_1 = x_2$ and $x_1 = -x_2$ are invariant sets, because the solutions starting on them stay along these lines forever (and converge to the equilibrium point $(0,0)$). All other solutions converge to an equilibrium point that has either $x_1 = 0$ or $x_2 = 0$.

Interestingly, the Jacobian matrix

$$J(x_1, x_2) = \begin{bmatrix} -2x_2^2 & -4x_1x_2 \\ -4x_1x_2 & -2x_1^2 \end{bmatrix}$$

has always at least one eigenvalue equal to zero at any equilibrium point, hence we cannot conclude much about the convergence of the solutions, using the linearization technique. However, the fact that the system is a gradient system enables us to conclude that all solutions converge to one of the equilibirum point.

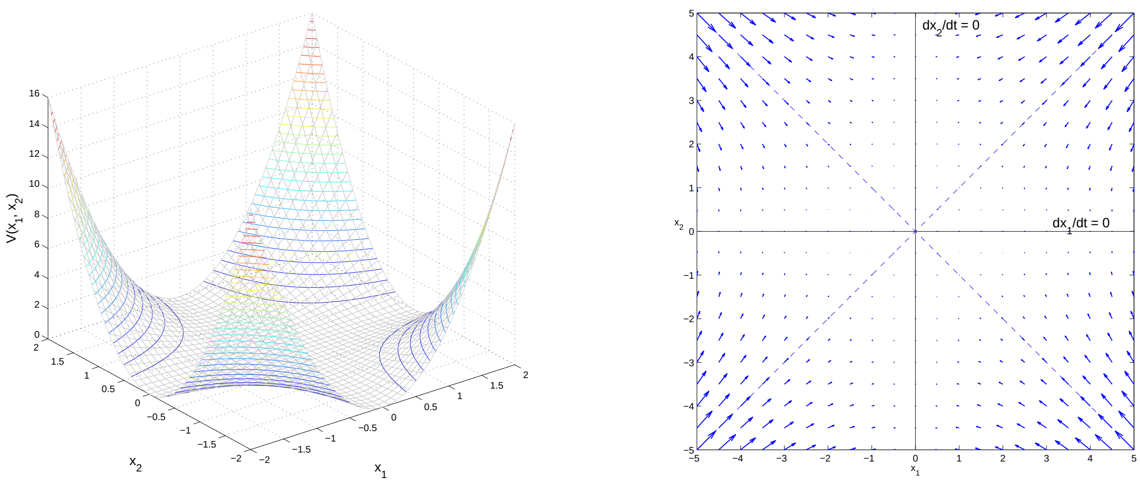

Figure 2: Plot of $V(x_1, x_2) = x_1^2 x_2^2$ (left). There are an infinity of equilibrium points of the gradient system. Vector field of the gradient system (right). The lines $x_1 = x_2$ and $x_1 = -x_2$ are invariant sets.

2. If $V(x_1, x_2) = x_1^4/4 + x_2^4/4 - x_1x_2$, the system becomes

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{\partial V}{\partial x_1} = -x_1^3 + x_2 \\ \frac{dx_2}{dt} &= -\frac{\partial V}{\partial x_2} = x_1 - x_2^3 \end{aligned}$$

By setting the above derivatives to 0, we can find the three equilibrium points of the gradient system, namely $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (1, 1)$ and $(x_1, x_2) = (-1, -1)$. We can also compute the Jacobian, which is

$$J(x_1, x_2) = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix},$$

and computing its eigenvalues, we find that $(0, 0)$ is a saddle point, wehereas the other two equilibrium points are stable nodes (and local minima of V, see Figure 3). There are no periodic solutions as in any gradient system and all solutions converge to one of the equilibrium points. The lines $x_1 = x_2$ and $x_1 = -x_2$ are again invariant sets.

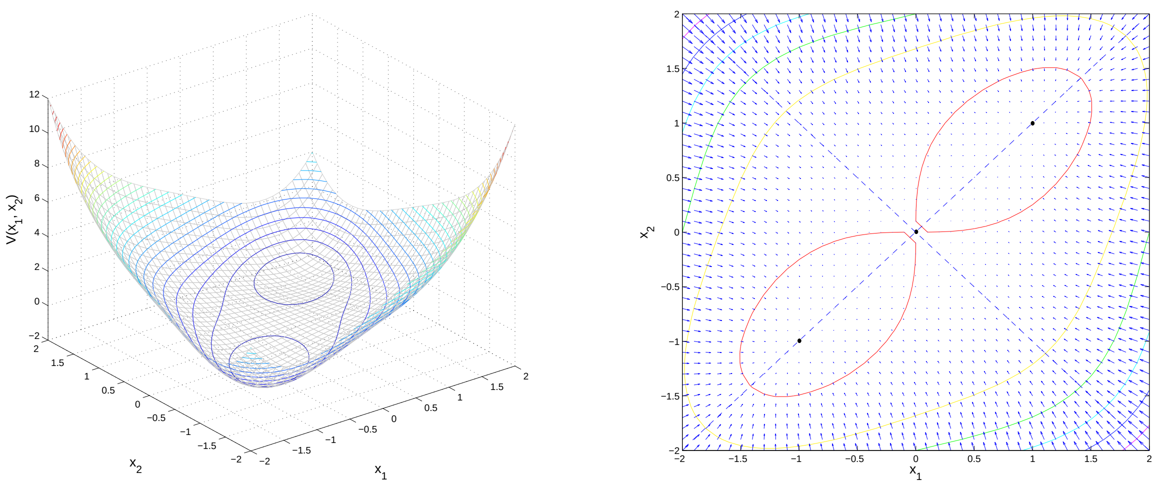

Figure 3: Plot of $V(x_1, x_2) = x_1^4/4 + x_2^4/4 - x_1 x_2$ (left). There are 3 equilibrium points of the gradient system. Vector field of the gradient system (right). The lines $x_1 = x_2$ and $x_1 = -x_2$ are invariant sets.