AUDIOVISUAL COMMUNICATIONS LCAV

Mathematical Foundations of Signal Processing

Module 5: Sequences and Discrete-Time Systems

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Discrete Time Signals

1. Modeling Discrete Time Signals
   - Sequences
   - Discrete-time Systems

2. Analyzing Discrete Time Signals
   - Why do we need Transforms?
   - Discrete Time Fourier Transform - DTFT
   - Z Transform
   - Discrete Fourier Transform - DFT

Modeling Discrete Time Signals

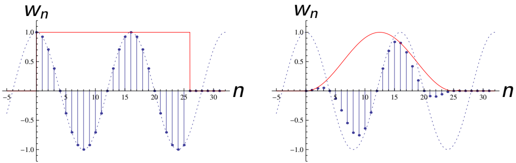
Sequences

Discrete-time signals are represented by sequences

- *Infinite-length* sequences: $[\dots\ x_{-2}\ x_{-1}\ \boxed{x_0}\ x_1\ x_2\ \dots]$
  Signal observed for an infinite lapse of time (theoretic case)
- *Finite-length* sequences
  - Truly finite $[x_0\ x_1\ x_2\ \dots\ x_{N-1}]$
  - A period of a periodic sequence $[\dots\ x_{N-1}\ \boxed{x_0}\ x_1\ x_2\ \dots\ x_{N-1}\ x_0\ x_1\ \dots]$
  - A window view (observation) of an infinite sequence

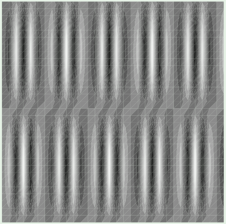
    Box window $w_n = \begin{cases} 1 & 0 \le n \le n_0 - 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad w = [\dots\ 0\ \underbrace{\boxed{1}\ \dots\ 1}_{n_0}\ 0\ \dots]^T$

    Raised cosine window $w_n = \begin{cases} \frac{1}{2}\left(1 - \cos\frac{2\pi n}{n_0 - 1}\right) & 0 \le n \le n_0 - 1 \\ 0 & \text{otherwise} \end{cases}$

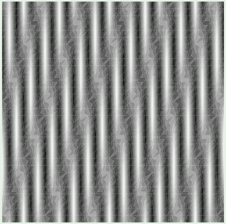


Sequences

Multidimensional sequences



(a)



(b)

(c)

(a) Two-dimensional separable sinusoidal sequence $x_{n,m} = \sin(\frac{\pi}{16}n)\sin(\frac{5\pi}{6}m)$.

(b) Two-dimensional nonseparable sinusoidal sequence $x_{n,m} = \sin(\frac{\pi}{16}n + \frac{5\pi}{6}m)$.

(c) Earth visible above the lunar surface, taken by Apollo 8 crew member Bill Anders on December 24, 1968. This could be considered a two-dimensional sequence if the image were gray scale representing the intensity, or a higher-dimensional sequence depending on how color is represented.

Sequence Spaces

Discrete-time sequences are classified into specific spaces

- $\ell^1(\mathbb{Z})$: absolutely summable sequences $\{x : \|x\|_1 = \sum_{n\in\mathbb{Z}} |x_n| < \infty\}$
- $\ell^2(\mathbb{Z})$: square summable sequences $\{x : \|x\|_2^2 = \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty\}$
- $\ell^\infty(\mathbb{Z})$: bounded sequences $\{x : \max_{n\in\mathbb{Z}} |x_n| < \infty\}$

with the following properties

- $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$
- $\ell^P(\mathbb{Z}), P \geq 1$ are all $\underbrace{\text{complete}}_{\text{every Cauchy sequence is convergent}}$ normed vector spaces $\longrightarrow$ Banach spaces
- $\ell^2(\mathbb{Z})$ is also a Hilbert space $\longrightarrow$ An inner product induces the norm

Remark: *Here too, sequence space definitions extend to the K-dimensional case.*

Special sequences

Kronecker delta

The Kronecker delta is defined as

$$\delta_n = \begin{cases} 1 & \text{for } n = 0\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

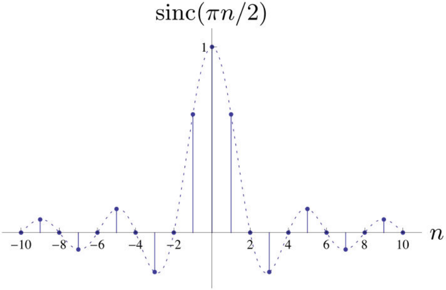
The set of Kronecker deltas $\{\delta_{n-k}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for $\ell^2(\mathbb{Z})$

Special sequences

Sinc function and sequences

Defined as

$$x_n = \frac{1}{\sqrt{T}}\,\mathrm{sinc}(\pi n/T) = \frac{1}{\sqrt{T}}\frac{\sin(\pi n/T)}{\pi n/T}\,.$$



The sinc sequence is in $\ell^\infty(\mathbb{Z})$, $\ell^2(\mathbb{Z})$, but not in $\ell^1(\mathbb{Z})$.

Special sequences

Heaviside sequence

The Heaviside or unit-step sequence is defined as

$$u_n = \begin{cases} 1 & \text{for } n \geq 0\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

It belongs to ℓ^∞ but not in ℓ^2 or ℓ^1.

The Kronecker delta and Heaviside sequences are related via

$$u_n = \sum_{k=-\infty}^{n} \delta_k\,.$$

Deterministic correlation

- *Deterministic autocorrelation*

$$a_n \;=\; \sum_{k\in\mathbb{Z}} x_k\, x^*_{k-n} \;=\; \langle x_k,\, x_{k-n}\rangle_k$$

- Properties

$$a_n = a^*_{-n}$$

$$a_0 = \sum_{k\in\mathbb{Z}} |x_k|^2 = \|x\|^2$$

- *Deterministic crosscorrelation*

$$c_n \;=\; \sum_{k\in\mathbb{Z}} x_k\, y^*_{k-n} \;=\; \langle x_k,\, y_{k-n}\rangle_k$$

-
- Properties

$$c_{x,y,n} = c^*_{y,x,-n}$$

Special finite-length sequences

Periodic Kronecker delta sequences

A periodic Kronecker delta obtained by adding all shifts of δ_n by integer multiples of N:

$$\varphi_n = \sum_{l \in \mathbb{Z}} \delta_{n-lN}\,, \quad n \in \mathbb{Z}$$

The resulting sequence is

$$\varphi = [\ldots\ 0\ \underbrace{\boxed{1}\ 0\ \ldots\ 0}_{N}\ 1\ 0\ \ldots]^T$$

The set of N sequences generated from φ by shifts of $\{0, 1, \ldots, N-1\}$ spans the space of N-periodic sequences

Special finite-length sequences

Complex exponential sequence

Complex exponential sequences form a natural basis for N-periodic sequences:

$$\varphi_{k,n} = \frac{1}{\sqrt{N}}\, e^{j(2\pi/N)kn}, \quad k \in \{0,\ 1,\ \ldots,\ N-1\}, \quad n \in \mathbb{Z}$$

Discrete-time systems

$$[\dots x_{-1}\ \boxed{x_0}\ x_1\ \dots] \longrightarrow \boxed{\quad T \quad} \longrightarrow [\dots y_{-1}\ \boxed{y_0}\ y_1\ \dots]$$

A discrete-time system is an operator T that maps an input sequence $x \in V$ into an output sequence $y \in V$

$$y = T(x)$$

Definition (Linear system)

A discrete-time system T is called *linear* when, for any inputs x and y and any $\alpha, \beta \in \mathbb{C}$,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Discrete-time systems

- A linear operator has a unique matrix representation once bases have been chosen for the domain and codomain of the operator
- Matrix representations of linear systems will be with respect to the standard basis $\{\delta_{n-k}\}_{k\in\mathbb{Z}}$ for both the inputs and outputs
- The matrix representation has as column k the output that results from taking the shifted Kronecker delta sequence δ_{n-k} as the input
- For each $k \in \mathbb{Z}$, let input $x^{(k)}$ result in output $y^{(k)}$, where

$$x_n^{(k)} = \delta_{n-k}, \qquad n \in \mathbb{Z}.$$

- The matrix representation of the system is

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & y_{-1}^{(-1)} & y_{-1}^{(0)} & y_{-1}^{(1)} & \cdots \\ \cdots & y_{0}^{(-1)} & \boxed{y_{0}^{(0)}} & y_{0}^{(1)} & \cdots \\ \cdots & y_{1}^{(-1)} & y_{1}^{(0)} & y_{1}^{(1)} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

System types

Definition (Memoryless system)

A discrete-time system T is called *memoryless* when, for any integer k and inputs x and x',

$$1_{\{k\}}\, x \;=\; 1_{\{k\}}\, x' \quad\Rightarrow\quad 1_{\{k\}}\, T(x) \;=\; 1_{\{k\}}\, T(x')$$

Definition (Causal system)

A discrete-time system T is called *causal* when, for any integer k and inputs x and x',

$$1_{\{-\infty,\dots,k\}}\, x \;=\; 1_{\{-\infty,\dots,k\}}\, x' \quad\Rightarrow\quad 1_{\{-\infty,\dots,k\}}\, T(x) \;=\; 1_{\{-\infty,\dots,k\}}\, T(x')$$

Definition (Shift-invariant system)

A discrete-time system T is called *shift invariant* when, for any integer k and input x,

$$y \;=\; T(x) \quad\Rightarrow\quad y' \;=\; T(x'), \qquad \text{where } x'_n = x_{n-k} \text{ and } y'_n = y_{n-k}.$$

Stable systems

Definition (BIBO stable system)

A discrete-time system T is called *bounded-input bounded-output stable* when a bounded input x produces a bounded output $y = T(x)$:

$$x \in \ell^\infty(\mathbb{Z}) \quad \Rightarrow \quad y \in \ell^\infty(\mathbb{Z})$$

- In a matrix representation of a linear and BIBO-stable system, every row of the matrix will be absolutely summable
- A linear and BIBO stable system is a bounded linear operator from $\ell^\infty(\mathbb{Z})$ to $\ell^\infty(\mathbb{Z})$
- Absolute-summability also insures that the system is a bounded linear operator from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$

Impulse response

Definition (Impulse response)

A sequence h is called the *impulse response* of LSI discrete-time system T when input δ produces output h.

Linear Shift-Invariant Systems

Convolution

- The impulse response and its shifts form the columns of the matrix representation of an LSI system
- Since a general input x to LSI system T can be written as $x_n = \sum_{k\in\mathbb{Z}} x_k \delta_{n-k}$ for any $n \in \mathbb{Z}$, we can express the output as

$$y \;=\; T\,x \;=\; T\sum_{k\in\mathbb{Z}} x_k\delta_{n-k} \;=\; \sum_{k\in\mathbb{Z}} x_k\,T\delta_{n-k} \;=\; \sum_{k\in\mathbb{Z}} x_k h_{n-k} \;=\; h * x,$$

Definition (Convolution)

The *convolution* between sequences h and x is defined as

$$(H\,x)_n \;=\; (h*x)_n \;=\; \sum_{k\in\mathbb{Z}} x_k h_{n-k} \;=\; \sum_{k\in\mathbb{Z}} x_{n-k} h_k,$$

where H is called the *convolution operator* associated with h.

Linear Shift-Invariant Systems

Convolution properties

The convolution satisfies

- *Connection to the inner product*

$$(h * x)_n \;=\; \sum_{k\in\mathbb{Z}} x_k h_{n-k} \;=\; \langle x_k,\, h^*_{n-k}\rangle_k$$

- *Commutativity*

$$h * x \;=\; x * h$$

- *Associativity*

$$g * (h * x) \;=\; g * h * x \;=\; (g * h) * x$$

- *Deterministic autocorrelation*

$$a_n \;=\; \sum_{k\in\mathbb{Z}} x_k\, x^*_{k-n}$$

Linear Shift-Invariant Systems

Filters

The impulse response is often called a *filter* and the convolution is called *filtering*. Here are some basic classes of filters:

- *Causal filters* are such that $h_n = 0$ for all $n < 0$.
- *Anticausal filters* are such that $h_n = 0$ for all $n > 0$.
- *Two-sided filters* are neither causal nor anticausal.
- *Finite impulse response (FIR) filters* have only a finite number of coefficients h_n different from zero.
- *Infinite impulse response (IIR) filters* have infinitely many nonzero terms.

Linear Shift-Invariant Systems

Stability

Theorem (BIBO stability)

An LSI system is BIBO stable if and only if its impulse response is absolutely summable

Theorem (Filtering with BIBO stable filter)

When $h \in \ell^1(\mathbb{Z})$ *and* $x \in \ell^p(\mathbb{Z})$ *for any* $p \in [1, \infty]$*, the result of* $h * x$ *is in* $\ell^p(\mathbb{Z})$ *as well*

Linear Shift-Invariant Systems

Matrix view

Any linear operator can be expressed in matrix form

$$
y \;=\; \begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & h_0 & h_{-1} & h_{-2} & h_{-3} & h_{-4} & \cdots \\ \cdots & h_1 & h_0 & h_{-1} & h_{-2} & h_{-3} & \cdots \\ \cdots & h_2 & h_1 & \boxed{h_0} & h_{-1} & h_{-2} & \cdots \\ \cdots & h_3 & h_2 & h_1 & h_0 & h_{-1} & \cdots \\ \cdots & h_4 & h_3 & h_2 & h_1 & h_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}}_{H} \begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \;=\; Hx
$$

- An LSI discrete-time system, linear operator (on sequences), filter and (doubly-infinite) matrix are all synonyms
- The filtering matrix has the Toeplitz structure
- Adjoint H^* of the convolution matrix given by the Hermitian transposition

$$
H^* \, x \;=\; \sum_{n \in \mathbb{Z}} h^*_{n-k} \, x_n
$$

Linear Shift-Invariant Systems

Convolution with circularly-extended signal

- x a periodic sequence with period N
- Filter h in $\ell^1(\mathbb{Z})$
- Periodized version of h

$$h_{N,n} \;=\; \sum_{k\in\mathbb{Z}} h_{n-kN}$$

- Circular convolution

$$\begin{aligned}
(h * x)_n \quad &= \quad \sum_{k\in\mathbb{Z}} h_k\, x_{n-k} \;=\; \sum_{\ell\in\mathbb{Z}} \sum_{k=\ell N}^{(\ell+1)N-1} h_k\, x_{n-k} \\
&= \quad \sum_{\ell\in\mathbb{Z}} \sum_{k'=0}^{N-1} h_{k'+\ell N}\, x_{n-k'-\ell N} \;=\; \sum_{\ell\in\mathbb{Z}} \sum_{k=0}^{N-1} h_{k+\ell N}\, x_{n-k} \\
&= \quad \sum_{k=0}^{N-1} \underbrace{\sum_{\ell\in\mathbb{Z}} h_{k+\ell N}}_{h_{N,k}}\, x_{n-k} \;=\; \sum_{k=0}^{N-1} h_{N,k}\, x_{n-k} \\
&= \quad \sum_{k=0}^{N-1} h_{N,k}\, x_{(n-k) \bmod N} \;=\; (h_N \circledast x)_n
\end{aligned}$$

Linear Shift-Invariant Systems

Circular convolution

Definition (Circular convolution)

The *circular convolution* between length-N sequences h and x is defined as

$$(H\,x)_n \;=\; (h \circledast x)_n \;=\; \sum_{k=0}^{N-1} x_k\, h_{(n-k) \bmod N} \;=\; \sum_{k=0}^{N-1} x_{(n-k) \bmod N}\, h_k,$$

where H is called the *circular convolution operator* associated with h

Theorem (Equivalence of circular and linear convolutions)

Linear and circular convolutions between a length-M sequence x and a length-L sequence h are equivalent when the period of the circular convolution N satisfies

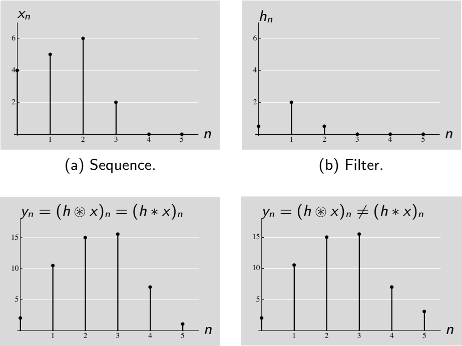
$$N \;\geq\; L + M - 1.$$

Linear Shift-Invariant Systems

Circular convolution

Linear and circular convolution

- Sequence x of length $M = 4$
- Filter h of length $L = 3$
- Linear convolution results in a sequence of length $L + M - 1 = 6$, the same as a circular convolution with a period $N \geq L + M - 1$, $N = 6$ in this case
- Circular convolution with a smaller period, $N = 5$, does not lead to the same result



(a) Sequence. (b) Filter.

(c) Linear convolution. (d) Circular convolution.

Linear Shift-Invariant Systems

Circular convolution: Matrix view

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ h_1 & h_0 & h_{N-1} & \cdots & h_2 \\ h_2 & h_1 & h_0 & \cdots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \end{bmatrix}}_{H} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = Hx.$$

Analyzing Discrete Time Signals

Why do we need Transforms?

Here again the same two motivations (one or both)

- View from a different perspective the characteristics of the function
- Move to a space where certain computations are simplified

  Notice that here we need the transform to be *invertible* (a one to one application)!

Discrete Time Fourier Transform - DTFT

Eigensequences of the convolution operator

Consider a complex exponential sequence

$$v_n = e^{j\omega n}, \qquad n \in \mathbb{Z}$$

- v is bounded since $|v_n| = 1$ for all $n \in \mathbb{Z}$
- If the impulse response h is in $\ell^1(\mathbb{Z})$, the output $h * v$ is bounded as well and has the form

$$\begin{aligned}(H\,v)_n &= (h * v)_n = \sum_{k\in\mathbb{Z}} v_{n-k}\,h_k = \sum_{k\in\mathbb{Z}} e^{j\omega(n-k)}\,h_k \\ &= \underbrace{\sum_{k\in\mathbb{Z}} h_k e^{-j\omega k}}_{\lambda_\omega}\ \underbrace{e^{j\omega n}}_{v_n}\end{aligned}$$

- v is an eigensequence of H
- The eigenvalue λ_ω is called the *frequency response* of the system $H(e^{j\omega})$

Discrete Time Fourier Transform - DTFT

Definition

Finding the appropriate Fourier transform now amounts to projecting onto the subspaces generated by each of the eigensequences

Definition (Discrete-time Fourier transform)

The *discrete-time Fourier transform* of a sequence x is

$$X(e^{j\omega}) = \sum_{n\in\mathbb{Z}} x_n e^{-j\omega n}, \qquad \omega \in \mathbb{R}.$$

It exists when this sum converges for all $\omega \in \mathbb{R}$; we then call it the *spectrum* of x. The *inverse DTFT* of a 2π-periodic function $X(e^{j\omega})$ is

$$x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega, \qquad n \in \mathbb{Z}.$$

When the DTFT exists, we denote the DTFT pair as

$$x_n \quad \overset{\mathrm{DTFT}}{\longleftrightarrow} \quad X(e^{j\omega})$$

- DTFT is always a 2π-periodic function, which is emphasized by the notation $X(e^{j\omega})$

Sequences in ℓ^1

- *Existence*:
  Existence straightforwardly follows from the existence (or convergence) of the sum for absolutely summable sequences

$$\sum_{n\in\mathbb{Z}} |x_n e^{-j\omega n}| = \sum_{n\in\mathbb{Z}} |x_n| < \infty$$

  Moreover, as a consequence of absolute convergence for all ω, the limit $X(e^{j\omega})$ is a continuous function of ω

- *Inverse transform*:
  The inverse always exists and it corresponds to the sequence itself

$$x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega\,.$$

  Indeed one can simply check that $x_n \in \ell^1$ guarantees that $X(e^{j\omega}) \in \mathcal{L}^1([-\pi,\pi])$.

Sequences in $\ell^2(\mathbb{Z})$

- For sequences not in $\ell^1(\mathbb{Z})$, the DTFT series might fail to converge for some values of ω
- To extend beyond $\ell^1(\mathbb{Z})$, we consider the limit as $N \to \infty$ of the *partial sums*

$$X_N(e^{j\omega}) \;=\; \sum_{n=-N}^{N} x_n e^{-j\omega n}.$$

- If $x \in \ell^2(\mathbb{Z})$, the partial sum $X_N(e^{j\omega})$ converges to a function $X(e^{j\omega}) \in \mathcal{L}^2([-\pi, \pi))$ in the sense that

$$\lim_{N\to\infty} \|X(e^{j\omega}) - X_N(e^{j\omega})\| \;=\; 0$$

  This convergence in $\mathcal{L}^2([-\pi, \pi))$ norm implies convergence for almost all values of ω, but there is no guarantee of the convergence being uniform or the limit function $X(e^{j\omega})$ being continuous

Discrete Time Fourier Transform - DTFT

Gibbs Phenomenon

The sequence

$$x_n = \frac{1}{\sqrt{2}} \frac{sin(\pi n/2)}{\pi n/2} \,,$$

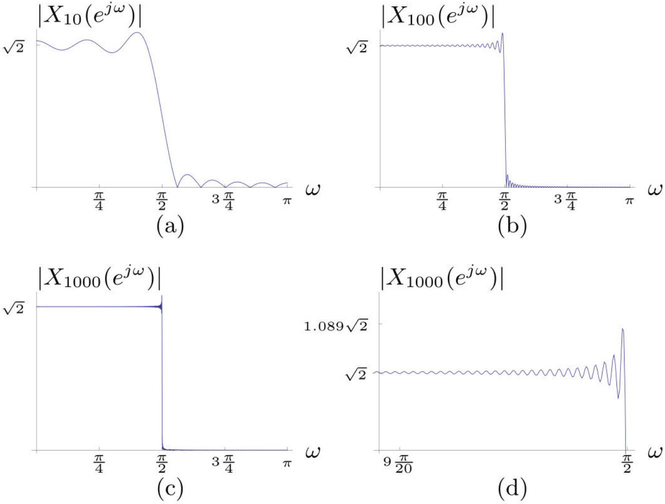
is not in $\ell^1(\mathbb{Z})$ but in $\ell^2(\mathbb{Z})$ and in particular the corresponding DTFT is not continuous Consequently, the truncated DTFT

$$X_N(e^{j\omega}) = \sum_{n=-N}^{N} \frac{1}{\sqrt{2}} \frac{sin(\pi n/2)}{\pi n/2} \,,$$

will converge in norm $\mathcal{L}^2([-\pi, \pi))$ but not uniformly The values of the truncated DTFT oscillates around the points of discontinuities of the DTFT

Discrete Time Fourier Transform - DTFT

Gibbs Phenomenon



(a) N = 10, (b) N = 100 and (c) N = 1000, with (d) a detail from (c).

Discrete Time Fourier Transform - DTFT

DTFT properties

- *Shift in time* $x_{n-n_0} \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$
- *Shift in frequency* $e^{j\omega_0 n} x_n \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$
- *Time reversal* $x_{-n} \overset{\text{DTFT}}{\longleftrightarrow} X(e^{-j\omega})$
- *Convolution in time* $(h * x)_n \overset{\text{DTFT}}{\longleftrightarrow} H(e^{j\omega}) X(e^{j\omega})$
- *Circular convolution in frequency* $h_n x_n \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2\pi}(H \circledast X)(e^{j\omega})$ with $(H \circledast X)(e^{j\omega}) = \int_{-\pi}^{\pi} X(e^{j\theta}) H(e^{j(\omega-\theta)}) \, d\theta$
- *Deterministic autocorrelation*
  $a_n = \sum_{k\in\mathbb{Z}} x_k x_{k-n}^* \overset{\text{DTFT}}{\longleftrightarrow} A(e^{j\omega}) = |X(e^{j\omega})|^2$

Discrete Time Fourier Transform - DTFT

Parseval's equality

- DTFT is a linear operator from the space of sequences to the space of 2π-periodic functions, $X = F\,x$
- $F : \ell^2(\mathbb{Z}) \to \mathcal{L}^2([-\pi,\pi))$ because $x \in \ell^2(\mathbb{Z})$ implies that $X(e^{j\omega})$ has finite $\mathcal{L}^2([-\pi,\pi))$ norm
$$\|X\|^2 = 2\pi\,\|x\|^2$$
- It is easily shown that $\frac{1}{\sqrt{2\pi}}F$ is a unitary operator:
$$\left\langle \frac{1}{\sqrt{2\pi}}Fx,\, \frac{1}{\sqrt{2\pi}}Fy \right\rangle \;=\; \langle x,\, y\rangle \qquad \text{for every } x \text{ and } y \text{ in } \ell^2(\mathbb{Z})$$
- Equivalently:
$$\langle x,\, y\rangle \;=\; \frac{1}{2\pi}\langle X,\, Y\rangle \qquad \text{for every } x \text{ and } y \text{ in } \ell^2(\mathbb{Z})$$

z-Transform

The z transform extends the DTFT from the unitary circle defined by $e^{j\omega}$ to the complex plane.

It can be defined over regions of convergence that might not contain the unitary circle $e^{j\omega}$ (this also might be considered as an extension), therefore enabling a different perspective (analysis) also for those sequences not admitting a DTFT.

z-Transform

Eigensequences of the convolution operator

Consider a complex exponential sequence with an arbitrary modulus

$$v_n \;=\; z^n \;=\; (re^{j\omega})^n, \qquad n \in \mathbb{Z}$$

- This is also an eigensequence of the convolution operator H associated with the LSI system with impulse response h since

$$\begin{aligned} (H\,v)_n \quad &= \quad (h * v)_n \;=\; \sum_{k\in\mathbb{Z}} v_{n-k}\,h_k \;=\; \sum_{k\in\mathbb{Z}} z^{n-k}\,h_k \\ &= \quad \underbrace{\sum_{k\in\mathbb{Z}} h_k z^{-k}}_{\lambda_z}\;\underbrace{z^n}_{v_n} \end{aligned}$$

- It can thus be written:

$$H\,z^n \;=\; h * z^n \;=\; H(z)\,z^n$$

- The set of impulse responses h for which the sum converges now depends on $|z|$

z-Transform

Definition

Definition (z-transform)

The *z-transform* of a sequence x is

$$X(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n}, \qquad z \in \mathbb{C}.$$

It exists when this sum converges absolutely for some values of z; these values of z are called the *region of convergence (ROC)*,

$$\mathrm{ROC} = \{z \mid |X(z)| < \infty\}.$$

When the z-transform exists, we denote the z-transform pair as

$$x_n \quad \stackrel{\mathrm{ZT}}{\longleftrightarrow} \quad X(z),$$

where the ROC is part of the specification of $X(z)$.

z-Transform

Convergence

- For the z-transform to exist and have $z = re^{j\omega}$ in its ROC, $X(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n}$ must converge absolutely
- Since
$$\sum_{n\in\mathbb{Z}} \left|x_n z^{-n}\right| = \sum_{n\in\mathbb{Z}} \left|x_n\, r^{-n}\right| \left|e^{-j\omega n}\right| = \sum_{n\in\mathbb{Z}} \left|x_n r^{-n}\right|,$$
absolute summability of $x_n r^{-n}$ is necessary and sufficient for the circle $|z| = r$ to be in the ROC of $X(z)$
- ROC is a ring of the form
$$\mathrm{ROC} = \{z \mid 0 \le r_1 < |z| < r_2 \le \infty\}.$$

z-Transform

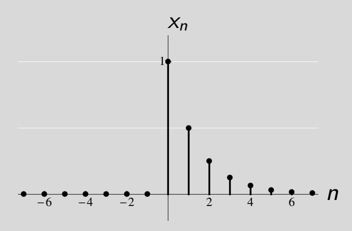
Properties

- The z-transform has the same properties as the DTFT, but for a larger class of sequences
- The main new twist is to properly account for ROCs
- Convolution of two sequences can be computed as a product in the transform domain even when the sequences do not have proper DTFTs, provided that the sequences have some part of their ROCs in common

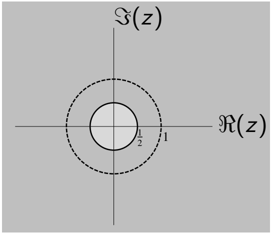
Geometric series

- Right-sided geometric series (causal)

$$x_n = \begin{cases} \alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}, \quad X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \mathrm{ROC} = \{z \mid |z| > |\alpha|\}$$



$x_n = (\frac{1}{2})^n u_n.$

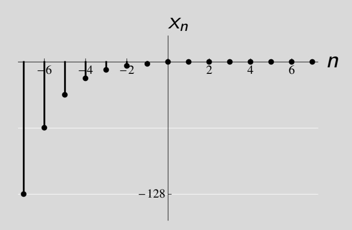


$\mathrm{ROC} = \{z \mid |z| > \frac{1}{2}\}.$

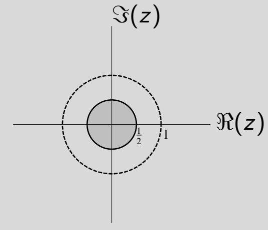
z-Transform

- Left-sided geometric series (anti-causal)

$$x_n = \begin{cases} \alpha^n & n \le 0 \\ 0 & n > 0 \end{cases}, \quad X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad ROC = \{z \mid |z| < |\alpha|\}$$



$x_n = -(\frac{1}{2})^n u_{-n-1}$.



$\mathrm{ROC} = \{z \mid |z| < \frac{1}{2}\}$.

Rational z-transform

- An important class of z-transforms are those that are rational functions, since transfer functions of most realizable systems (systems that can be built and used in practice) are rational
- Such transfer functions are of the form

$$H(z) \;=\; \frac{B(z)}{A(z)},$$

  where $A(z)$ and $B(z)$ are polynomials in z^{-1} with no common roots, of degree N and M, respectively, with $M \leq N$
- The zeros of the numerator $B(z)$ and denominator $A(z)$ are called the *zeros* and *poles* of the rational transfer function $H(z)$

$$H(z) \;=\; \frac{b_0 \prod_{k=1}^{M}(1 - z_k z^{-1})}{a_0 \prod_{k=1}^{N}(1 - p_k z^{-1})}$$

- A causal filter is BIBO stable if and only if the poles of its transfer function are inside the unit circle (ROC contains the unit circle)

Deterministic autocorrelation

- z-transform pair corresponding to the deterministic autocorrelation of a sequence x is

$$a_n \;=\; \sum_{k\in\mathbb{Z}} x_k\, x^*_{k-n} \quad \overset{\text{ZT}}{\longleftrightarrow} \quad A(z) \;=\; X(z)\, X_*(z^{-1}), \qquad \mathrm{ROC}_x \cap \frac{1}{\mathrm{ROC}_x}$$

- $X_*(z)$ denotes $X^*(z^*)$, which amounts to conjugating coefficients but not z
- $A(z)$ satisfies

$$A(z) \;=\; A_*(z^{-1})$$

- For a real x

$$A(z) \;=\; X(z)\, X(z^{-1}) \;=\; A(z^{-1}).$$

Rational autocorrelation

Theorem (Rational autocorrelation)

A rational function $A(z)$ is the z-transform of the deterministic autocorrelation of a stable real sequence x, if and only if

- *its complex poles and zeros appear in quadruples:*

$$\{z_i, z_i^*, z_i^{-1}, (z_i^{-1})^*\}, \qquad \{p_i, p_i^*, p_i^{-1}, (p_i^{-1})^*\};$$

- *its real poles and zeros appear in pairs:*

$$\{z_i, z_i^{-1}\}, \qquad \{p_i, p_i^{-1}\};$$

  and

- *its zeros on the unit circle are double zeros:*

$$\{z_i, z_i^*, z_i^{-1}, (z_i^{-1})^*\} \;=\; \{e^{j\omega_i}, e^{-j\omega_i}, e^{-j\omega_i}, e^{j\omega_i}\},$$

  with possibly double zeros at $z = \pm 1$. There are no poles on the unit circle

Corollary (Spectral factorization)

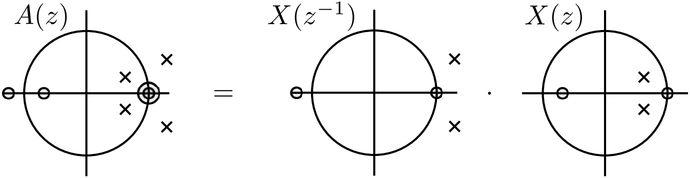
A rational z-transform $A(z)$ is the deterministic autocorrelation of a stable real sequence x_n if and only if it can be factored as $A(z) = X(z)X(z^{-1})$

- Spectral factorization amounts to assigning poles and zeros from quadruples and pairs $X(z)$ and $X(z^{-1})$
- For the poles, there is a unique rule: take all poles inside the unit circle and assign them to $X(z)$
  - Stability of x requires $X(z)$ to have only poles inside the unit circle
  - x real requires that conjugate pairs be kept together
- For the zeros, there is a choice, since we are not forced to assign only zeros inside the unit circle to $X(z)$. Doing so, however, creates a unique solution called the *minimum-phase solution*

- *Minimum phase sequences*: A sequence h_n is called minimum phase if it is stable, causal and all zeros are inside or on the unit circle (strictly minimum phase if zeros strictly inside the unit circle).

z-Transform

Spectral Factorization



Discrete Fourier Transform (DFT)

- DFT is a tool for fast computation of linear convolution
- DFT contains eigensequences of the circular convolution operator H

$$v_n = e^{j(2\pi/N)kn} = W_N^{-kn}, \qquad v = \begin{bmatrix} 1 & W_N^{-k} & \dots & W_N^{-(N-1)k} \end{bmatrix}^T$$

$$\begin{aligned} (H\,v)_n \quad &= \quad (h \circledast v)_n = \sum_{i=0}^{N-1} v_{(n-i) \bmod N}\, h_i = \sum_{i=0}^{N-1} W_N^{-k[(n-i) \bmod N]}\, h_i \\ &= \quad \sum_{i=0}^{N-1} W_N^{k(i-n)}\, h_i = \underbrace{\sum_{i=0}^{N-1} h_i W_N^{ki}}_{\lambda_k} \underbrace{W_N^{-kn}}_{v_n} \end{aligned}$$

- λ_k is the eigenvalue called the frequency response H_k
- k is called the discrete frequency

Discrete Fourier Transform (DFT)

Definition (Discrete Fourier transform)

The *discrete Fourier transform* of a length-N sequence x is

$$X_k \;=\; (F\,x)_k \;=\; \sum_{n=0}^{N-1} x_n W_N^{kn}, \qquad k \in \{0, 1, \ldots, N-1\};$$

we call it the *spectrum* of x. The *inverse DFT* of a length-N sequence X is

$$x_n \;=\; \frac{1}{N}(F^*\,X)_n \;=\; \frac{1}{N}\sum_{k=0}^{N-1} X_k W_N^{-kn}, \qquad n \in \{0, 1, \ldots, N-1\}.$$

We denote the DFT pair as

$$x_n \quad \overset{\mathrm{DFT}}{\longleftrightarrow} \quad X_k.$$

DFT: Matrix View

$$F = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \dots & W_N^{(N-1)^2} \end{bmatrix},$$

$$F^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \dots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \dots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \dots & W_N^{-(N-1)^2} \end{bmatrix} = \frac{1}{N} F^*$$

- DFT is an orthogonal basis

$$FF^* = F^*F \;=\; NI_{N\times N}$$

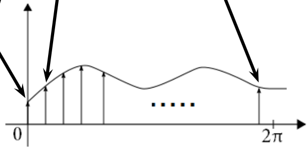
Relation between DFT & DTFT

- For a given length-N sequence, DFT is the sample of DTFT spectrum at $\omega = \dfrac{2\pi k}{N}$

$$\begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix} \xrightarrow{\quad DFT \quad} \begin{bmatrix} X_0 & X_1 & \dots & X_{N-1} \end{bmatrix}$$

$$\Big\downarrow \text{ Zero extension to } \pm\infty$$

$$\dots 0\,0\,0 \begin{bmatrix} \boxed{x_0} & x_1 & \dots & x_{N-1} \end{bmatrix} 0\,0\,0 \dots \xrightarrow{\quad DTFT \quad}$$



Sampling the spectrum

- Formally

$$\begin{aligned} X(e^{j\omega})\Big|_{\omega=(2\pi/N)k} &= X(e^{j(2\pi/N)k}) = \sum_{n\in\mathbb{Z}} x_n\, e^{-j(2\pi/N)kn} \\ &= \sum_{n=0}^{N-1} x_n\, e^{-j(2\pi/N)kn} = X_k \end{aligned}$$

Properties of DFT

Properties

- Linearity $\alpha x_n + \beta y_n \overset{\text{DFT}}{\longleftrightarrow} \alpha X_k + \beta Y_k$
- Shift in time $x_{(n-n_0) \bmod N} \overset{\text{DFT}}{\longleftrightarrow} W_N^{kn_0} X_k$
- Shift in frequency $W_N^{-k_0 n} x_n \overset{\text{DFT}}{\longleftrightarrow} X_{(k-k_0) \bmod N}$
- Convolution in time $(h \circledast x)_n \overset{\text{DFT}}{\longleftrightarrow} H_k \, X_k$
- Convolution in frequency $h_n \, x_n \overset{\text{DFT}}{\longleftrightarrow} \frac{1}{N} (H \circledast X)_k$
- Deterministic auto-correlation $a_n = \sum_{k=0}^{N-1} x_k \, x^*_{(k-n) \bmod N} \overset{\text{DFT}}{\longleftrightarrow} A_k = |X_k|^2$
- Deterministic cross-correlation $c_n = \sum_{k=0}^{N-1} x_k \, y^*_{(k-n) \bmod N} \overset{\text{DFT}}{\longleftrightarrow} C_k = X_k \, Y_k^*$

Diagonalization of the circular convolution operator

- H – the circular convolution operator associated with length-N filter h
- The DFT spectrum of h gives a diagonal form for H

$$\Lambda \;=\; \mathrm{diag}([H_0,\, H_1,\, \ldots,\, H_{N-1}]), \qquad \text{where } H_k = \sum_{n=0}^{N-1} h_n W_N^{kn}$$

- X_k – DFT spectrum of length-N sequence x_n
- The circular convolution property

$$(h \circledast x)_n \quad \overset{\mathrm{DFT}}{\longleftrightarrow} \quad H_k\, X_k$$

gives

$$F\,(H\,x) \;=\; \Lambda\, F\, x \quad \Rightarrow \quad H \;=\; F^{-1}\Lambda F$$

- DFT operator F diagonalizes the circular convolution operator H

$$F\,H\,F^{-1} \;=\; \Lambda$$