AUDIOVISUAL COMMUNICATIONS LCAV

Mathematical Foundations of Signal Processing

Bases and Least Squares Approximation

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Previous lecture

- Geometry is key to intuition!
  - From finite to infinite dimensional spaces
- Hilbert spaces: Complete inner product spaces
- Linear operators and their adjoints
  - Bounded linear operators, eigenvalues and eigenvectors
- Projection: Oblique and orthogonal
  - Idempotency and self-adjointness
  - Projection theorem
- Approximation
  - Best/least squares approximation
- Direct sums and decompositions

Today

1. Basis representation and expansions
   - Riesz bases
   - Orthonormal bases
   - Biorthogonal bases
2. Frame representation and expansions
   - Definition
   - Tight frames
   - General frames
3. Matrix representation of linear operators
   - Change of basis
   - Matrices
4. Gram-Schmidt orthonormalization and polynomial approximation

Readings:

- Section 2.5 of Chapter 2, "From Euclid to Hilbert", of *Foundations of Signal Processing*

Bases

Definition (Basis)

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset V$ is a basis when
  1. Φ is linearly independent and
  2. Φ is complete in V: $V = \overline{span}\{\Phi\}$

- Expansion formula: any $x \in V, x = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$

$$\begin{array}{rcl} \{\alpha_k\}_{k\in\mathcal{K}} & : & \text{is unique} \\ \alpha_k & : & \text{expansion coefficients} \end{array}$$

Example

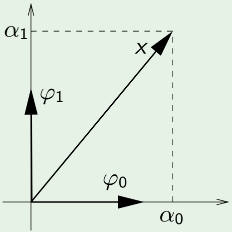
- The standard basis for $\mathbb{R}^N$

$$e_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \; k = 0, \ldots, N-1$$

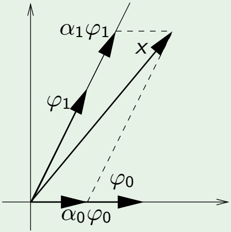
any $v \in \mathbb{R}^N : v = \sum_{k=0}^{N-1} v_k e_k$

Bases

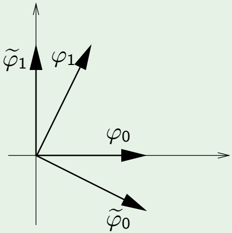
Examples



(a) Orthonormal basis



(b) Biorthogonal basis expansion



(c) and dual basis

Riesz bases

Definition (Riesz Basis)

- In a Hilbert space H, $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ is a Riesz basis when
  1. Φ is a basis
  2. There exists $0 < \lambda_{min} \leq \lambda_{max} < \infty$ s.t.

     for any $x = \sum_{k \in \mathcal{K}} \alpha_k \varphi_k \in H$

     $$\lambda_{min} \|x\|^2 \leqslant \sum_{k \in \mathcal{K}} | \alpha_k |^2 \leqslant \lambda_{max} \|x\|^2$$
- Numerical stability when $\lambda_{min} \approx \lambda_{max}$

Example

In $\mathbb{R}^N$

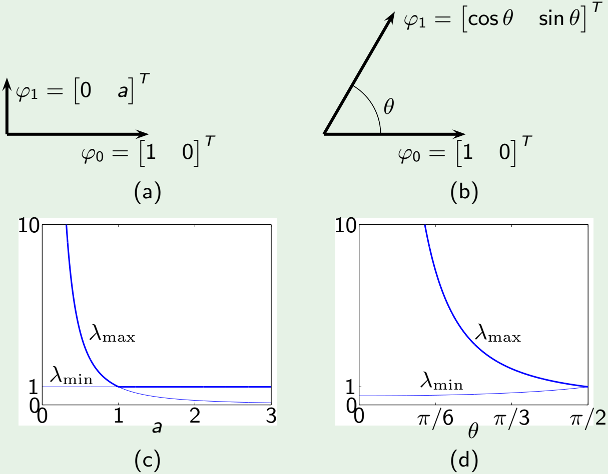
- Let $\Phi = \begin{bmatrix} \varphi_0 & \varphi_1 & \dots & \varphi_{N-1} \end{bmatrix}$ be a Riesz basis and define $G = \Phi^* \Phi$. Then $1/\lambda_{max} =$ minimum eigenvalue of G

  $1/\lambda_{min} =$ maximum eigenvalue of G

Riesz bases

Example

Eigenvalues of $\Phi_a^* \Phi_a$



Operators associated with Riesz bases

Definition (Basis synthesis operator)

- Synthesis operator
  - $\Phi : \ell^2(\mathcal{K}) \to H$, s.t. $\Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$
  - $\|\Phi\| \leqslant 1/\sqrt{\lambda_{min}}$ (follows from definition)
  - Adjoint: $y \in H$

    $$\langle \Phi\alpha, y\rangle = \langle \sum_{k\in\mathcal{K}} \alpha_k \varphi_k, y\rangle = \sum_{k\in\mathcal{K}} \alpha_k \langle y, \varphi_k\rangle^*$$

Definition (Basis analysis operator)

- Analysis operator
  - $\Phi^* : H \to \ell^2(\mathcal{K})$ $\qquad (\Phi^* x)_k = \langle x, \varphi_k\rangle, \quad k \in \mathcal{K}$
  - $\|\Phi^*\| \leqslant 1/\sqrt{\lambda_{min}}$

- Note that the analysis operator is the adjoint of the synthesis operator

Orthonormal bases

Definition (Orthonormal basis)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$ is an orthonormal basis for H when
  1. Φ is a basis for H and
  2. Φ is an orthonormal set

     $\langle \varphi_i, \varphi_k \rangle = \delta_{i-k}$ for all $i, k \in \mathcal{K}$

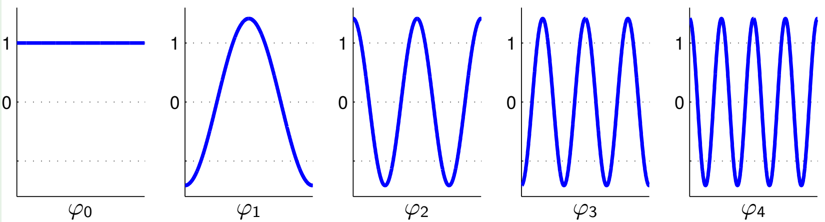
- If Φ is an orthogonal set, then it is linearly independent
- If $\overline{span}\{\Phi\} = H$ and Φ is an orthogonal set, then Φ is an orthogonal basis for H

  If we also have $\|\varphi_k\| = 1$, then Φ is an orthonormal basis

Orthonormal basis

Example

Consider the Hilbert space $H = \{f \in \mathcal{L}^2([-\frac{1}{2}, \frac{1}{2}]) : f(-x) = f(x)\}$



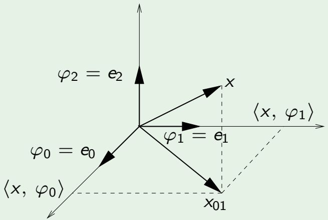
Orthonormal basis expansions

Definition (Orthonormal basis expansions)

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ orthonormal basis for H, then for any $x \in H$:

  $\alpha_k = \langle x, \varphi_k \rangle$ for $k \in \mathcal{K}$, or $\alpha = \Phi^* x$, α is unique
- Synthesis: $$x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k \rangle \varphi_k = \Phi\alpha = \Phi\Phi^* x$$

Example



Orthonormal basis: Parseval's equality

Theorem (Parseval's equalites)

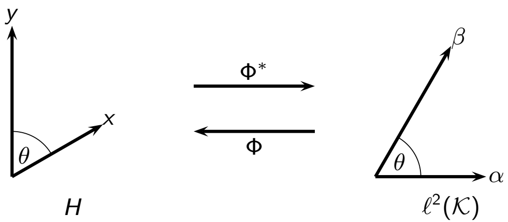
- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ *orthonormal basis for* H

$$\|x\|^2 = \sum_{k\in\mathcal{K}} |\langle x, \varphi_k\rangle|^2 = \|\Phi^* x\|^2 = \|\alpha\|^2$$

- *In general:*

$$\langle x, y\rangle = \langle \Phi^* x, \Phi^* y\rangle = \langle \alpha, \beta\rangle$$

$$\textit{with} \qquad \alpha_k = \langle x, \varphi_k\rangle,\ \beta_k = \langle y, \varphi_k\rangle$$



Orthonormal bases

- On $\ell^2(\mathcal{K}) : \Phi^*\Phi = I$
- On H : $\Phi\Phi^* = I$
- Analysis and synthesis operators associated with an orthonormal basis are unitary
- Isometry between any separable Hilbert space H and $\ell^2(\mathbb{Z})$
- How about approximation?

Orthogonal projection and decomposition

Theorem

- $\Phi = \{\varphi_k\}_{k\in\mathcal{I}} \subset H, \quad \mathcal{I} \subset \mathcal{K}$

  $$P_{\mathcal{I}}x = \sum_{k\in\mathcal{I}} \langle x, \varphi_k\rangle \varphi_k = \Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^* x$$

  is the orthogonal projection of x onto $S_{\mathcal{I}} = \overline{span}(\{\varphi_k\}_{k\in\mathcal{I}})$

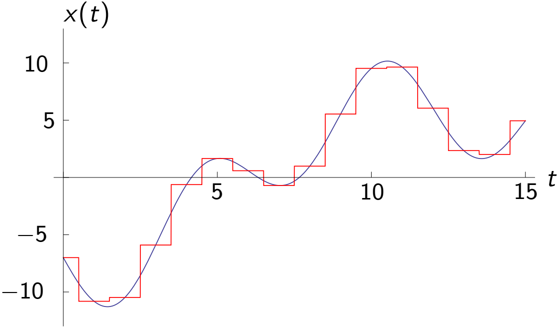
- Φ induces an orthogonal decomposition

$$H = \bigoplus_{k\in\mathcal{K}} S_{\{k\}} \quad \text{where } S_{\{k\}} = span(\varphi_k)$$

Example

$$\begin{aligned} S &= \left\{\begin{bmatrix} x_0 & x_1 & x_2\end{bmatrix}^T\right\} \in \mathbb{C}^3 | x_1 = x_0 + x_2\} \\ &= span(\{\tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0\end{bmatrix}^T, \tfrac{1}{\sqrt{6}}\begin{bmatrix}-1 & 1 & 2\end{bmatrix}^T\}) \\ P_S x &= \sum_{k=0}^{1} \langle x, \varphi_k\rangle\varphi_k = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2\end{bmatrix} x \end{aligned}$$

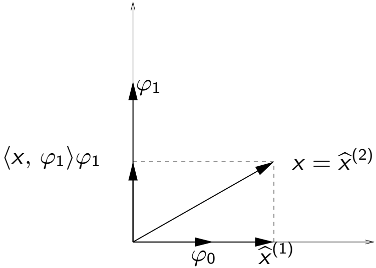
Orthogonal projection on $\mathcal{L}^2(\mathbb{R})$



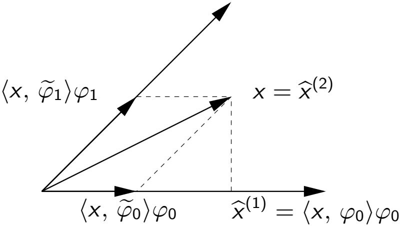
The local averaging operator A and its adjoint A^* are such that $AA^* = I$, so $P = A^*A$ is an orthogonal projection

Best approximation in orthogonal bases

- $\{\varphi_k\}_{k\in\mathcal{K}}$ orthogonal basis
- $\widehat{x}^{(k)}$ = best approximation of x in $\{\varphi_0, \ldots, \varphi_{k-1}\}$
- $\widehat{x}^{(0)} = 0$ and $\widehat{x}^{(k+1)} = \widehat{x}^{(k)} + \langle x, \varphi_k\rangle\varphi_k$ for $k = 0, 1, \ldots$
  successive approximation



(a)



(b)

Bessel's inequality

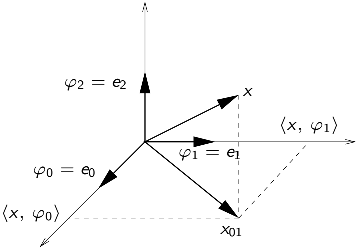
Theorem (Bessel's inequality)

- $\Phi = \{\varphi_k\}_{k\in\mathcal{I}} \subset H$, Φ *orthonormal set*

$$\begin{aligned}\|x\|^2 &\geqslant \sum_{k\in\mathcal{I}} |\langle x, \varphi_k\rangle|^2 \\ &= \|\Phi_{\mathcal{I}}^* x\|^2\end{aligned}$$

- *Equality when* Φ *is complete, i.e. an orthonormal basis*

- Example: Note that $\|x_{01}\| \leq \|x\|$



Biorthogonal pairs of bases

Definition

Biorthogonal pairs of bases

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset H$, and $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}} \subset H$ is a biorthogonal pair of bases when
  1. Φ and $\widetilde{\Phi}$ are both bases for H
  2. Φ and $\widetilde{\Phi}$ are biorthogonal

$$\langle \varphi_i, \widetilde{\varphi}_k \rangle = \delta_{i-k} \text{ for all } i, k \in \mathcal{K}$$

- Roles of Φ and $\widetilde{\Phi}$ are interchangeable

Example

$$\varphi_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \varphi_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\widetilde{\varphi}_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \widetilde{\varphi}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \widetilde{\varphi}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \Phi^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Biorthogonal basis expansion

Theorem

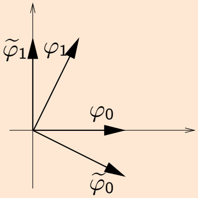
- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}, \widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *biorthogonal pair of bases for* H
- *Any* $x \in H$ *has expansion coefficients*

$$\alpha_k = \langle x, \widetilde{\varphi}_k\rangle,\ k \in \mathcal{K},\ \text{or } \alpha = \widetilde{\Phi}^* x$$

- *Synthesis:*

$$\begin{aligned} x &= \sum_{k\in\mathcal{K}} \langle x, \widetilde{\varphi}_k\rangle \varphi_k \\ &= \Phi\alpha = \Phi\widetilde{\Phi}^* x \end{aligned}$$

- *Also* $x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \widetilde{\varphi}_k$



Biorthogonal bases: Parseval's equality

Theorem

Parseval's equalities for biorthogonal pairs of bases

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}, \widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *biorthogonal pair of bases for* H
- $$\begin{aligned} \|x\|^2 &= \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \langle x, \widetilde{\varphi}_k\rangle^* \\ &= \langle \Phi^* x, \widetilde{\Phi}^* x\rangle = \langle \widetilde{\alpha}, \alpha\rangle \end{aligned}$$
- *In general* $$\begin{aligned} \langle x, y\rangle &= \sum \langle x, \varphi_k\rangle \langle y, \widetilde{\varphi}_k\rangle^* \\ &= \langle \Phi^* x, \widetilde{\Phi}^* y\rangle = \langle \widetilde{\alpha}, \beta\rangle \end{aligned}$$

- $\widetilde{\Phi}^*$ is the inverse of Φ: $$\begin{aligned} \Phi\widetilde{\Phi}^* &= I \quad \text{on } H \text{ and} \\ \widetilde{\Phi}^*\Phi &= I \quad \text{on } \ell^2(\mathcal{K}) \end{aligned}$$

Gram matrix

- $G \;=\; \Phi^*\Phi$ is the Gram matrix

  $G_{ik} \;=\; \langle \varphi_k, \varphi_i \rangle \qquad$ for every i, $k \in \mathcal{K}$,

  $$G \;=\; \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \varphi_{-1} \rangle & \langle \varphi_0, \varphi_{-1} \rangle & \langle \varphi_1, \varphi_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \varphi_0 \rangle & \boxed{\langle \varphi_0, \varphi_0 \rangle} & \langle \varphi_1, \varphi_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \varphi_1 \rangle & \langle \varphi_0, \varphi_1 \rangle & \langle \varphi_1, \varphi_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

- Assume $x = \Phi\alpha$, $y = \Phi\beta$ then
  $\langle x, y \rangle \;=\; \langle \Phi\alpha, \Phi\beta \rangle \;=\; \langle \Phi^*\Phi\alpha, \beta \rangle \;=\; \langle G\alpha, \beta \rangle \;=\; \beta^* \, G \, \alpha$

  The inner product in H becomes an inner product in $\ell^2(\mathcal{K})$!

Dual basis

- How to compute the dual basis $\widetilde{\Phi}$?

Theorem (Dual basis)

- *Let* $A = (\Phi^*\Phi)^{-1}$ *be the inverse of the Gram matrix*
- *Given a Riesz basis* $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ *for Hilbert space* H, *the set* $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *defined via*

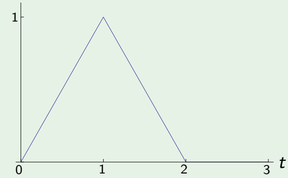
$$\widetilde{\varphi}_k = \sum_{\ell\in\mathcal{K}} a_{\ell,k}\varphi_\ell, \qquad \text{for each } k \in \mathcal{K},$$

$$\widetilde{\Phi} = \Phi A = \Phi(\Phi^*\Phi)^{-1},$$

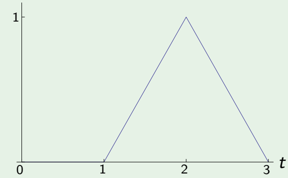
is a basis for H, *called the dual basis, and the sets* Φ *and* $\widetilde{\Phi}$ *are a biorthogonal pair of bases*

Dual basis of periodic hat functions

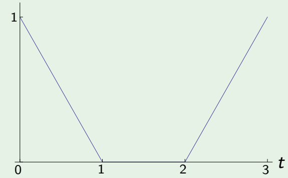
Example



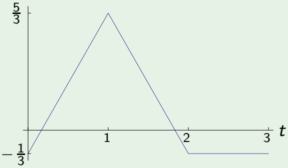
$\varphi_1(t)$



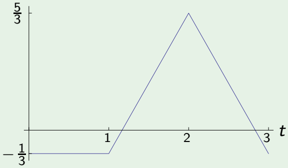
$\varphi_2(t)$



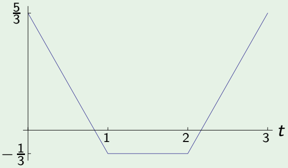
$\varphi_3(t)$



$\widetilde{\varphi}_1(t)$



$\widetilde{\varphi}_2(t)$



$\widetilde{\varphi}_3(t)$

Dual basis of periodic hat functions: Computing the dual basis

- Consider the functions in $\mathcal{L}^2([0,3])$ such that $\varphi_0 = \begin{cases} t, & \text{for } t \in [0,1); \\ 2-t & \text{for } t \in (1,2]; \\ 0 & \text{for } t \in (2,3] \end{cases}$

  and their circular shifts by 1.
- Let $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ be the basis for $span\{\varphi_1, \varphi_2, \varphi_3\}$ = subspace of functions x satisfying $x(0) = x(3)$ and are piecewise linear on [0, 3] with breakpoints at 1 and 2
- $G = \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{bmatrix}$
- We find a dual basis for Φ by using $\widetilde{\Phi} = \Phi A = \Phi G^{-1}$ where

  $$G^{-1} = \begin{bmatrix} 5/3 & -1/3 & -1/3 \\ -1/3 & 5/3 & -1/3 \\ -1/3 & -1/3 & 5/3 \end{bmatrix}$$

Dual bases and projection

- How to compute the projection?

Theorem

- *Given sets* $\Phi_{\mathcal{I}} = \{\varphi_k\}_{k \in \mathcal{I}} \subset H$ *and* $\widetilde{\Phi}_{\mathcal{I}} = \{\widetilde{\varphi}_k\}_{k \in \mathcal{I}} \subset H$ *satisfying*

$$\langle \varphi_i, \widetilde{\varphi}_k \rangle \; = \; \delta_{i-k} \qquad \text{for every } i, \, k \in \mathcal{I},$$

- *Then, for any* x *in* H*,*

$$P_{\mathcal{I}} \, x \; = \; \sum_{k \in \mathcal{I}} \langle x, \, \widetilde{\varphi}_k \rangle \varphi_k \; = \; \Phi_{\mathcal{I}} \, \widetilde{\Phi}_{\mathcal{I}}^* \, x$$

  is an oblique projection of x *onto* $S_{\mathcal{I}} = \overline{\operatorname{span}}(\{\varphi_k\}_{k \in \mathcal{I}})$.

- *The residual satisfies* $x - P_{\mathcal{I}} \, x \perp \widetilde{S}_{\mathcal{I}}$*, where* $\widetilde{S}_{\mathcal{I}} = \overline{\operatorname{span}}(\{\widetilde{\varphi}_k\}_{k \in \mathcal{I}})$.

Projection onto a subspace

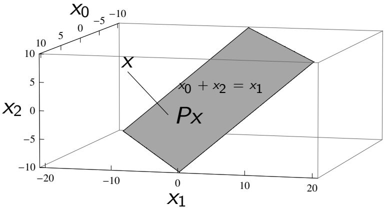
Example (Projection onto a subspace)

Let $$A = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

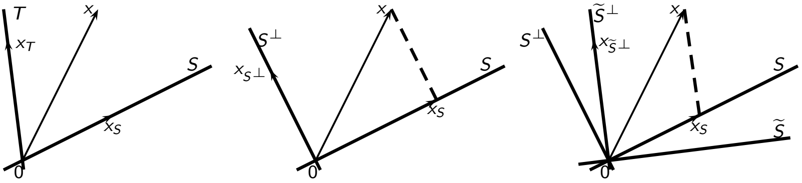
Since A is a left inverse of B, then

$$P = BA = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \text{ is a projection operator.}$$

We can verify $P^2 = P$. Does $P^* = P$?



Direct sum decomposition



(a)
Direct sum in
a vector space
$V = S \oplus T$

(b)
Orthogonal projection
on the subspace S
$x = x_S + x_{S^\perp}$

(c)
Oblique projection
on the subspace S
$x = x_S + x_{\widetilde{S}^\perp}$

Best approximation & the normal equations

Theorem (Normal equations)

- *$x \in H$ and $\{\phi_k\}_{k \in \mathcal{I}}$ a Riesz basis for a closed subspace S*
- *The closest vector to x in S is*

$$\widehat{x} \;=\; \sum_{k \in \mathcal{I}} \beta_k \phi_k = \Phi\beta$$

  where β is the unique solution to:

$$\Phi^*\Phi\beta = \Phi^* x \qquad \text{or}$$

$$\sum_{k \in \mathcal{I}} \beta_k \langle \phi_k, \phi_i \rangle = \langle x, \phi_i \rangle \text{ for all } i \in \mathcal{I}$$

  Normal equations

- $\widehat{x} = \Phi(\Phi^*\Phi)^{-1}\Phi^* x = Px$

  P is an orthogonal projection

- We can verify that $P^2 = \Phi \underbrace{(\Phi^*\Phi)^{-1}\Phi^*\Phi}_{I}(\Phi^*\Phi)^{-1}\Phi^* = P$ and that $P^* = P$

Frames: Overcomplete representations

Definition

Frame

- $\Phi = \{\varphi_k\}_{k\in\mathcal{J}} \subset H$ is a frame for the Hilbert space H when there exist $0 < \lambda_{min} \leq \lambda_{max} < \infty$ s.t. for any $x \in H$

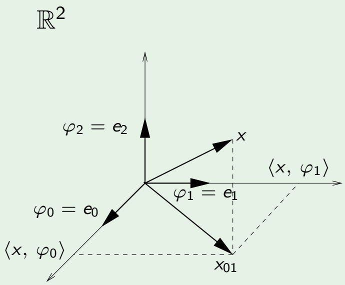
$$\lambda_{min}\|x\|^2 \leqslant \sum_{k\in\mathcal{J}} |\langle x, \varphi_k\rangle|^2 \leqslant \lambda_{max}\|x\|^2$$

- $\lambda_{min}, \lambda_{max}$: frame bounds

- Synthesis operator: $\Phi : \ell^2(\mathcal{J}) \to H, \quad \Phi\alpha = \sum_{k\in\mathcal{J}} \alpha_k \varphi_k$
- Analysis operator: $\Phi^* : H \to \ell^2(\mathcal{J}), \quad (\Phi^* x)_k = \langle x, \varphi_k\rangle, \ k \in \mathcal{J}$
- $\lambda_{min} I \leqslant \Phi\Phi^* \leqslant \lambda_{max} I$

Frames

Example

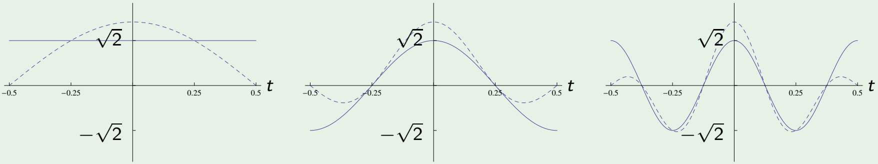


- Given $\Phi = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$
- $\{\varphi_0, \varphi_1, \varphi_2\}$ is a frame for $\mathbb{R}^2$
- λ_{min} and λ_{max} computed as smallest and largest eighenvalues of $\Phi\Phi^*$

- $$\Phi\Phi^* = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{\Lambda} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{V^{-1}}$$
- $\lambda_{min} = 1,\ \lambda_{max} = 3$

Example: Frame of cosine functions

Example



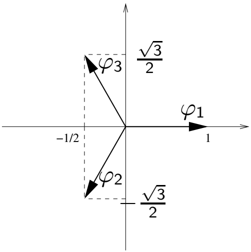
(a) $\varphi_0(t)$ and $\psi_0(t)$ (b) $\varphi_1(t)$ and $\psi_1(t)$ (c) $\varphi_2(t)$ and $\psi_2(t)$

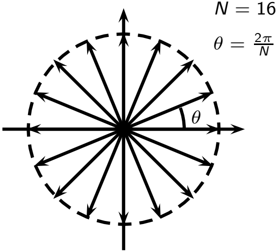
Tight frames

Definition

- $\Phi = \{\varphi_k\}_{k \in \mathcal{J}} \subset H$ is a tight frame or λ-tight frame for H when $\lambda_{min} = \lambda_{max} = \lambda$

- Counterpart of orthonormal basis
- When $\lambda = 1$: Parseval tight frame





Tight frames expansions

Theorem (Tight frame expansions)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{J}}$ *a 1-tight frame for* H, $x \in H$
- *Expansion coefficients of* x *w.r.t.* Φ*:*

$$\alpha_k = \langle x, \varphi_k \rangle \quad k \in \mathcal{J}, \text{ or } \alpha = \Phi^* x$$

- *Expansion* $x = \sum_{k \in \mathcal{J}} \langle x, \varphi_k \rangle \varphi_k = \Phi\Phi^* x$

- $\Phi\Phi^* = I$ but (in general) $\Phi^*\Phi \neq I$
- Note: similar to orthonormal basis, but $\mathcal{J}$ is overcomplete and vectors Φ are linearly dependent

Tight frames Parseval's equality

Theorem (Parseval's equalities for 1-tight frames)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{J}}$ *a 1-tight frame for H*
- $\|x\|^2 = \sum_{k \in \mathcal{J}} |\langle x, \varphi_k \rangle|^2 = \|\Phi^* x\|^2 = \|x\|^2$
- *In general:* $$\begin{aligned} \langle x, y \rangle &= \sum_{k \in \mathcal{J}} \langle x, \varphi_k \rangle \langle y, \varphi_k \rangle^* \\ &= \langle \Phi^* x, \Phi^* y \rangle = \langle \alpha, \beta \rangle \end{aligned}$$

General frames I

Definition (Dual pair of frames)

- Dual frame pairs and expansion

  $\Phi = \{\varphi_k\}_{k\in\mathcal{J}} \in H$, $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{J}} \in H$ form a dual pair of frames when:

  1. Each is a frame for H
  2. for any x in H,

  $$x \;=\; \sum_{k\in\mathcal{K}} \langle x, \widetilde{\varphi}_k\rangle \varphi_k \;=\; \Phi\,\widetilde{\Phi}^*\,x$$

General frames II

Example

- Let $\Phi = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ be the frame synthesis operator for $\mathbb{R}^2$ of rank 2
- Φ has infinitely many right inverses
- Examples: $\widetilde{\Phi} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, $\widetilde{\Phi} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ and $\widetilde{\Phi} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$
- Check that $\Phi\widetilde{\Phi}^* = I_2$

General frames: Operators I

- Inner product: Assume $x = \Phi\alpha, y = \Phi\beta, \quad \langle x, y \rangle_H \; = \; \beta^* \, G \, \alpha$
- Oblique projection: $P = \widetilde{\Phi}^* \Phi$

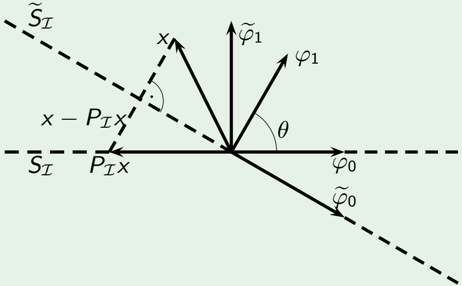
  $$P^2 = (\widetilde{\Phi}^* \Phi)(\widetilde{\Phi}^* \Phi) = \widetilde{\Phi}^* \underbrace{(\Phi \widetilde{\Phi}^*)}_{I} \Phi = \widetilde{\Phi}^* \Phi = P$$

  Does $P^* = P$?
- Canonical dual: $\widetilde{\Phi} = (\Phi\Phi^*)^{-1}\Phi$

  $P = \widetilde{\Phi}^* \Phi$ is an orthogonal projection

General frames: Operators II

Example



- $\Phi_{\mathcal{I}} \;=\; \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \qquad \widetilde{\Phi}_{\mathcal{I}} \;=\; \begin{bmatrix} 1 & -\cot\theta \end{bmatrix}^T$
- $P_{\mathcal{I}}\, x \;=\; \Phi_{\mathcal{I}}\, \widetilde{\Phi}^*_{\mathcal{I}}\, x$

Change of basis: Orthonormal basis

- How are the expansion coefficients in two orthonormal bases related?
- Assume $x = \Phi\alpha = \Psi\beta$. Then $\Psi^*\Phi\alpha = \Psi^*\Psi\beta = \beta$
- The change of basis from Φ to Ψ that maps α to β is the operator

$$C_{\Phi,\Psi} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \text{ s.t. } C_{\Phi,\Psi} = \Psi^*\Phi \text{ (or } \beta = \Psi^*\Phi\alpha)$$

- As a matrix

$$C_{\Phi,\Psi} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \psi_{-1}\rangle & \langle \varphi_0, \psi_{-1}\rangle & \langle \varphi_1, \psi_{-1}\rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_0\rangle & \boxed{\langle \varphi_0, \psi_0\rangle} & \langle \varphi_1, \psi_0\rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_1\rangle & \langle \varphi_0, \psi_1\rangle & \langle \varphi_1, \psi_1\rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

Change of basis: Biorthogonal basis

- How are the expansion coefficients in two biorthogonal bases related?
- Assume $x = \Phi\alpha = \Psi\beta$. Then $\Psi^{-1}\Phi\alpha = \Psi^{-1}\Psi\beta = \beta$ and $\beta = \Psi^{-1}\Phi\alpha$
- The change of basis from Φ to Ψ that maps α to β is the operator

$$C_{\Phi,\Psi} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \text{ s.t. } C_{\Phi,\Psi} = \Psi^{-1}\Phi$$

- As a matrix

$$C_{\Phi,\Psi} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \widetilde{\psi}_{-1} \rangle & \langle \varphi_0, \widetilde{\psi}_{-1} \rangle & \langle \varphi_1, \widetilde{\psi}_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \widetilde{\psi}_0 \rangle & \boxed{\langle \varphi_0, \widetilde{\psi}_0 \rangle} & \langle \varphi_1, \widetilde{\psi}_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \widetilde{\psi}_1 \rangle & \langle \varphi_0, \widetilde{\psi}_1 \rangle & \langle \varphi_1, \widetilde{\psi}_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

Matrix representation: Orthonormal basis

- How are the expansion coefficients of x and y related when $A : H \to H$, s.t. $y = Ax$?
  - $\{\varphi_k\}_{k \in \mathcal{K}}$ orthonormal basis of H
  - $x = \Phi\alpha, y = \Phi\beta$
- A matrix representation allows A to be computed directly on coefficient sequences

$$\Gamma : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \text{ s.t. } \beta = \Gamma\alpha$$

- As a matrix

$$\Gamma = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle A\varphi_{-1}, \varphi_{-1}\rangle & \langle A\varphi_0, \varphi_{-1}\rangle & \langle A\varphi_1, \varphi_{-1}\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \varphi_0\rangle & \boxed{\langle A\varphi_0, \varphi_0\rangle} & \langle A\varphi_1, \varphi_0\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \varphi_1\rangle & \langle A\varphi_0, \varphi_1\rangle & \langle A\varphi_1, \varphi_1\rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

Matrix representation: Orthonormal basis

$$
\begin{tikzcd}
H & & H \\
x \arrow[rr, "A"] \arrow[d, bend right, "\Phi^*"'] & & y \arrow[d, bend right, "\Phi^*"'] \\
\alpha \arrow[rr, "\Gamma"'] \arrow[u, bend right, "\Phi"'] & & \beta \arrow[u, bend right, "\Phi"'] \\
\ell^2(\mathcal{K}) & & \ell^2(\mathcal{K})
\end{tikzcd}
$$

Matrix representation: Orthonormal basis

- How are the expansion coefficients of x and y related when $A : H_0 \to H_1$, s.t. $y = Ax$?
  - Φ orthonormal basis of H_0
  - Ψ orthonormal basis of H_1
  - $x = \Phi\alpha, y = \Psi\beta$
- A matrix representation allows A to be computed directly on coefficient sequences

$$\Gamma : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \text{ s.t. } \beta = \Gamma\alpha$$

- As a matrix

$$\Gamma = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle A\varphi_{-1}, \psi_{-1}\rangle & \langle A\varphi_0, \psi_{-1}\rangle & \langle A\varphi_1, \psi_{-1}\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \psi_0\rangle & \boxed{\langle A\varphi_0, \psi_0\rangle} & \langle A\varphi_1, \psi_0\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \psi_1\rangle & \langle A\varphi_0, \psi_1\rangle & \langle A\varphi_1, \psi_1\rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

Example

- Let $A : H_0 \to H_1$, $$y(t) \;=\; Ax(t) \;=\; \tfrac{1}{2}\int_{2\ell}^{2(\ell+1)} x(\tau)\,d\tau$$ $$\text{for } 2\ell \le t < 2(\ell+1), \quad \ell \in \mathbb{Z}$$

  H_0 : space of piecewise-constant, finite-energy functions with breakpoints at integers
  H_1 : space of piecewise-constant, finite-energy functions with breakpoints at even integers.

- Given $\chi_I(t) \;=\; \begin{cases} 1, & \text{for } t \in I; \\ 0, & \text{otherwise} \end{cases}$,

  Let $$\begin{aligned} \Phi \;&=\; \{\varphi_k(t)\}_{k\in\mathbb{Z}} \;=\; \{\chi_{[k,k+1)}(t)\}_{k\in\mathbb{Z}}, \\ \Psi \;&=\; \{\psi_i(t)\}_{i\in\mathbb{Z}} \;=\; \{\tfrac{1}{\sqrt{2}}\chi_{[2i,2(i+1))}(t)\}_{i\in\mathbb{Z}}, \end{aligned}$$

  be orthogonal bases for H_0, H_1 respectively

Example: Averaging Operator II

Example (Cont.)

- $A\varphi_0(t) \;=\; \frac{1}{2}\;\chi_{[0,2)}(t) \quad \Rightarrow \quad \langle A\varphi_0,\, \psi_0\rangle \;=\; \int_0^2 \frac{1}{2}\frac{1}{\sqrt{2}}\,d\tau \;=\; \frac{1}{\sqrt{2}}$
- Then $\Gamma \;=\; \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{1} & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$

Matrix representation: Biorthogonal basis

- How are the expansion coefficients of x and y related when $A : H_0 \to H_1$, s.t. $y = Ax$?
  - $\Phi, \widetilde{\Phi}$ biorthogonal pair of bases of H_0
  - $\Psi, \widetilde{\Psi}$ biorthogonal pair of bases of H_1
- A matrix representation is similar to the orthogonal case, but involves the dual bases
- As a matrix

$$\Gamma = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle A\varphi_{-1}, \widetilde{\psi}_{-1}\rangle & \langle A\varphi_{0}, \widetilde{\psi}_{-1}\rangle & \langle A\varphi_{1}, \widetilde{\psi}_{-1}\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \widetilde{\psi}_{0}\rangle & \boxed{\langle A\varphi_{0}, \widetilde{\psi}_{0}\rangle} & \langle A\varphi_{1}, \widetilde{\psi}_{0}\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \widetilde{\psi}_{1}\rangle & \langle A\varphi_{0}, \widetilde{\psi}_{1}\rangle & \langle A\varphi_{1}, \widetilde{\psi}_{1}\rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

Matrix representation: Biorthogonal basis

$$
\begin{tikzcd}
H_0 & & H_1 \\
x \arrow[rr, "A"] \arrow[d, bend right, "\Phi^{-1}"'] & & y \arrow[d, bend right, "\Psi^{-1}"'] \\
\alpha \arrow[rr, "\Gamma"'] \arrow[u, bend right, "\Phi"'] & & \beta \arrow[u, bend right, "\Psi"'] \\
\ell^2(\mathcal{K}_0) & & \ell^2(\mathcal{K}_1)
\end{tikzcd}
$$

Example: Derivative Operator I

Example

- Let $A : H_0 \to H_1$ be the derivative operator

  H_0 : space of piecewise-linear, continuous, finite-energy functions with breakpoints at integers
  H_1 : space of piecewise-constant, finite-energy functions with breakpoints at integers.

- Let $\Phi = \{\varphi_k(t)\}_{k\in\mathbb{Z}} = \{\varphi(t-k)\}_{k\in\mathbb{Z}}, \quad \varphi(t) = \begin{cases} 1-|t|, & |t|<1; \\ 0, & \text{otherwise} \end{cases}$
- Let $\Psi = \{\psi_i(t)\}_{i\in\mathbb{Z}} = \{\chi_{[i,i+1)}(t)\}_{i\in\mathbb{Z}}$.

Example: Derivative Operator II

Example (Cont.)

- We evaluate $\langle A\varphi_k, \widetilde{\psi}_i\rangle$ for all k and i.

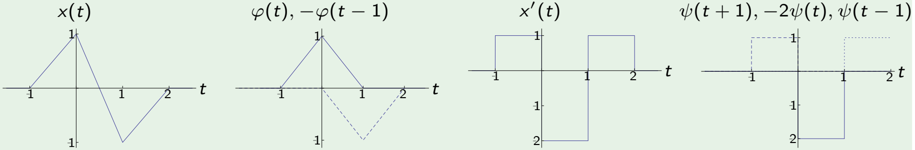
$$A\varphi(t) = \varphi'(t) = \begin{cases} 1, & \text{for } -1 < t < 0; \\ -1, & \text{for } 0 < t < 1; \\ 0, & \text{for } |t| > 1, \end{cases}$$

- Then $\langle A\varphi_0, \widetilde{\psi}_i\rangle = \begin{cases} 1, & \text{for } i = -1; \\ -1, & \text{for } i = 0; \\ 0, & \text{otherwise.} \end{cases}$ and $\langle A\varphi_k, \widetilde{\psi}_i\rangle = \begin{cases} 1, & \text{for } i = k-1; \\ -1, & \text{for } i = k; \\ 0, & \text{otherwise.} \end{cases}$

$$\text{and } \Gamma = \begin{bmatrix} \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & -1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{-1} & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & -1 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Example: Derivative Operator III

Example (Cont.)



(a) Original function $x(t)$

(b) its decomposition in the basis for H_0

(c) Derivative function $x'(t)$

(d) its decomposition in the basis for H_1

- Expansion in Φ:
$$\begin{aligned} x(t) &= \varphi(t) - \varphi(t-1) \\ \alpha &= \begin{bmatrix} \dots & 0 & \boxed{1} & -1 & 0 & 0 & \dots \end{bmatrix}^T \end{aligned}$$
- Expansion of the derivative in Ψ:
$$\begin{aligned} x'(t) &= \psi(t+1) - 2\psi(t) + \psi(t-1), \\ \beta &= \begin{bmatrix} \dots & 0 & 1 & -2 & 1 & 0 & \dots \end{bmatrix}^T \end{aligned}$$
- Check: $\beta = \Gamma\alpha$

Example: Polynomial approximation

- Hilbert space $P_N[-1,1] \subset \mathcal{L}^2[-1,1]$
- a self-evident, naive basis: $\mathbf{s}^{(k)} = t^k, \quad k = 0, 1, \ldots, N-1$
- naive basis is not orthonormal

- goal: approximate $\mathbf{x} = \sin t$ over $P_3[-1,1]$

Building an orthonormal basis

Gram-Schmidt orthonormalization procedure:

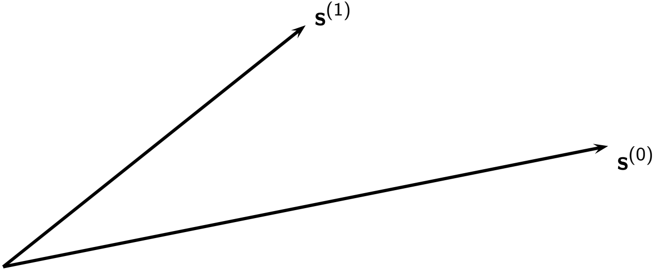
$$\{\mathbf{s}^{(k)}\} \longrightarrow \{\mathbf{u}^{(k)}\}$$

original set orthonormal set

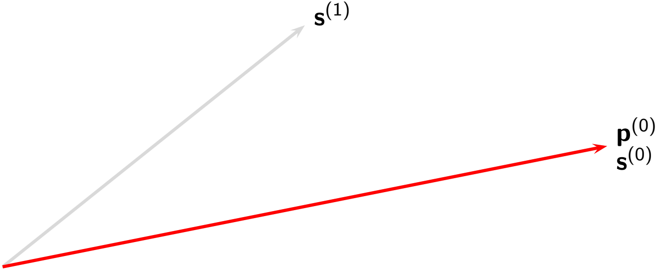
Algorithmic procedure: at each step k

1. $\mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \, \mathbf{u}^{(n)}$
2. $\mathbf{u}^{(k)} = \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$

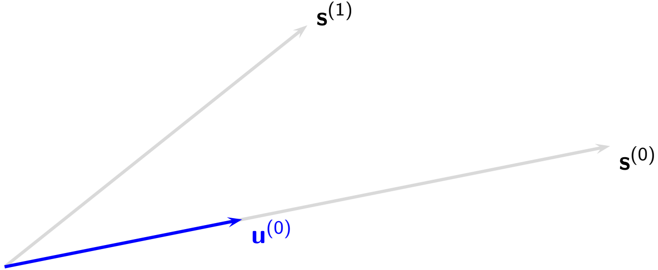
Least Squares Approximation



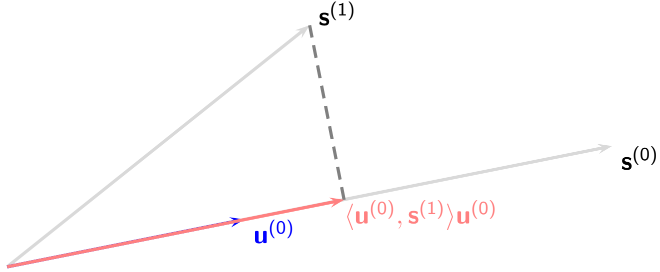
Least Squares Approximation



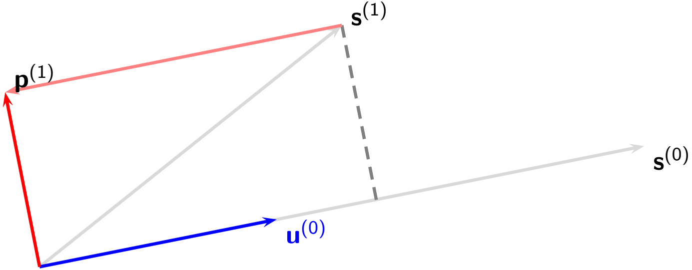
Least Squares Approximation



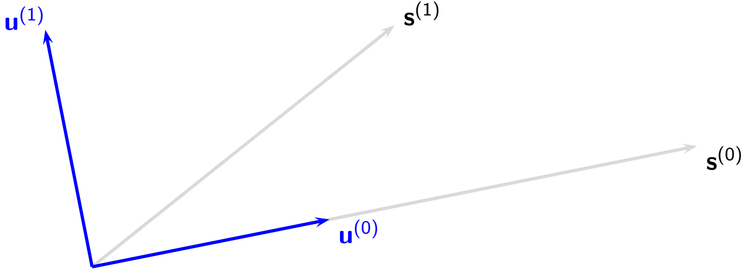
Least Squares Approximation



Least Squares Approximation



Least Squares Approximation



Building an orthonormal basis

Gram-Schmidt orthonormalization of the naive basis: $\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$

- $\mathbf{s}^{(0)} = 1$
  - $\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$
  - $\|\mathbf{p}^{(0)}\|^2 = 2$
  - $\mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2}$
- $\mathbf{s}^{(1)} = t$
  - $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t/\sqrt{2} = 0$
  - $\mathbf{p}^{(1)} = \mathbf{s}^{(1)} = t$
  - $\|\mathbf{p}^{(1)}\|^2 = 2/3$
  - $\mathbf{u}^{(1)} = \sqrt{3/2}\, t$
- $\mathbf{s}^{(2)} = t^2$
  - $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(2)} \rangle = \int_{-1}^{1} t^2/\sqrt{2} = 2/3\sqrt{2}$
  - $\langle \mathbf{u}^{(1)}, \mathbf{s}^{(2)} \rangle = \int_{-1}^{1} t^3/\sqrt{2} = 0$
  - $\mathbf{p}^{(2)} = \mathbf{s}^{(2)} - (2/3\sqrt{2})\mathbf{u}^{(0)} = t^2 - 1/3$
  - $\|\mathbf{p}^{(2)}\|^2 = 8/45$
  - $\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$

Legendre polynomials

An orthonormal basis for $P_N([-1,1])$

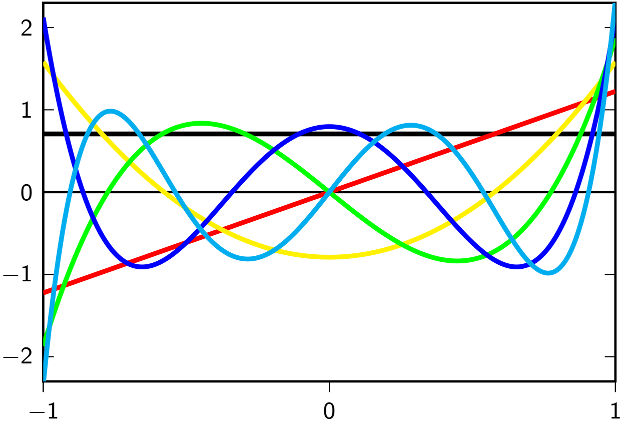
$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2}\, t$$

$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \dots$$

Legendre Polynomials



$\mathbf{u}^{(0)}$, $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$, $\mathbf{u}^{(4)}$, $\mathbf{u}^{(5)}$

Orthogonal projection over $P_3[-1,1]$

$$\alpha_k = \langle \mathbf{u}^{(k)}, \mathbf{x} \rangle = \int_{-1}^{1} u_k(t)\, \sin t \, dt$$

- $\alpha_0 = \langle \sqrt{1/2}, \sin t \rangle = 0$
- $\alpha_1 = \langle \sqrt{3/2}\, t, \sin t \rangle \approx 0.7377$
- $\alpha_2 = \langle \sqrt{5/8}(3t^2 - 1), \sin t \rangle = 0$

Approximation

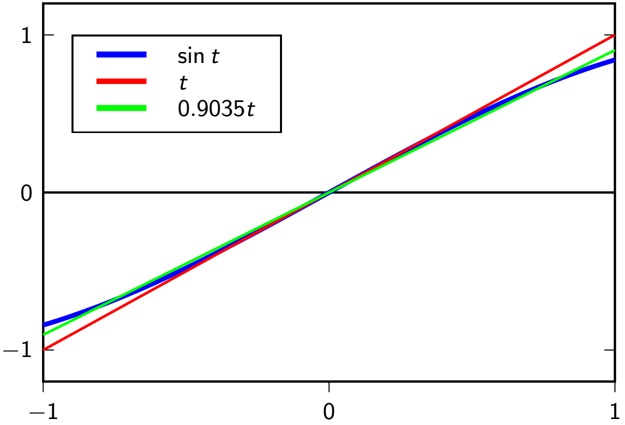
Using the orthogonal projection over $P_3[-1,1]$:

$$\sin t = \alpha_1 \mathbf{u}^{(1)} \approx 0.9035\, t$$

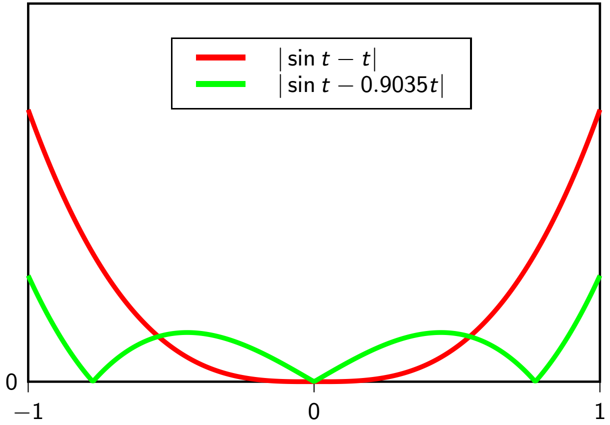
Using Taylor's series:

$$\sin t \approx t$$

Sine approximation



Approximation error



Error norm

Orthogonal projection over $P_3[-1,1]$:

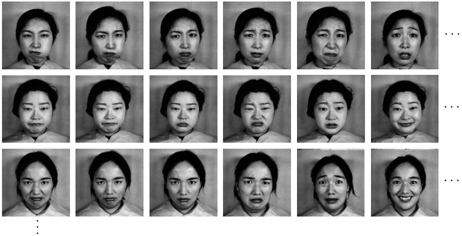
$$\|\sin t - \alpha_1\, t\| \approx 0.0337$$

Taylor series:

$$\|\sin t - t\| \approx 0.0857$$

Example: Eigenfaces

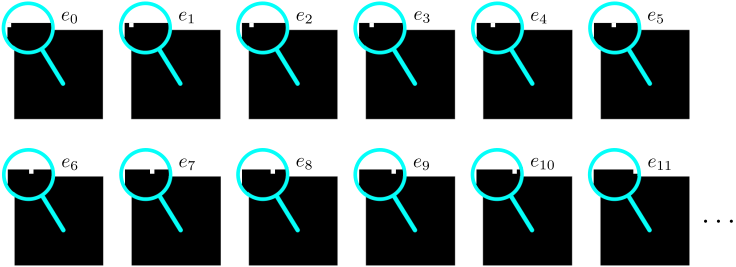
- Images of ten women with about 20 facial expressions (213 images in total)*
- Images of size 256×256: dimensionality *too high* (65536)
- Goal: represent images in *few dimensions*



*Michael J. Lyons, Miyuki Kamachi, Jiro Gyoba. Japanese Female Facial Expressions (JAFFE), Database of digital images (1997).
`http://www.kasrl.org/jaffe.html`

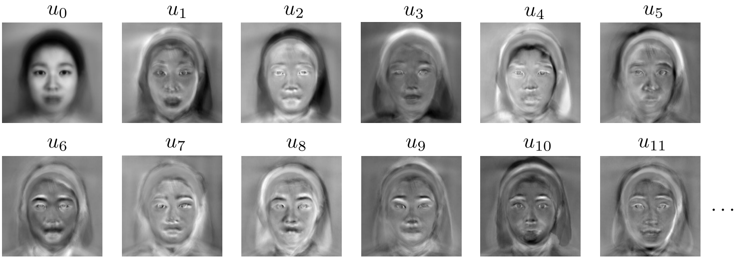
Example: Eigenfaces

Canonical (usual) basis



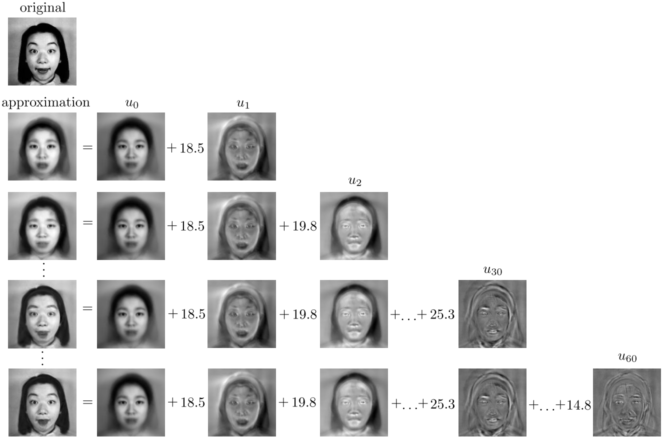
Example: Eigenfaces

Eigenface basis



Example: Eigenfaces

Reconstruction with eigenfaces



Summary

- Bases
  - Bases span a space S
  - Well behaved bases are Riesz bases (conditioning)
  - Best behaved bases are orthonormal bases
  - Biorthogonal bases have dual bases
  - Parseval's equalities
  - Approximations and normal equations
- Matrix representation of operators
  - Change of basis (orthonormal, biorthogonal)
  - Matrix representations of operators