Dynamical Systems For Engineers

Test 1

School I&C, Master Course

NAME and First name:

If a page is unstapled, please mark your name on it. There is a total of 9 pages. Your answers must be clear, precise and complete.

The notation $\dot{x}$ stands for dx/dt.

Maximum: 20 points

Question 1 **(4 points)**

Consider a continuous-time dynamical system, with state $(x_1, x_2) \in \mathbb{R}^2$, whose state equation is the set of nonlinear ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= 1. \end{aligned}$$

1. (1pt) Find a solution $(x_1(t), x_2(t))$ with initial condition $(x_1(0), x_2(0)) = (1, 0)$.

2. (1pt) Is this solution unique? Justify your answer.

3. (1pt) Does the solution with initial condition $(x_1(0), x_2(0)) = (1, 0)$ have a non-empty ω-limit set? If so, which one; if not, justify your answer.

4. (1pt) Does this dynamical system have an attractor? If so, which one? Justify your answer.

Question 2 (8 points)

Consider the autonomous linear system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \alpha x_1(t),\end{aligned}$$

where $\alpha \in \mathbb{R}$ is a parameter.

1. (2 pts) Characterize the stability (i.e. asymptotic stable, stable, weakly unstable, strongly unstable) of the system, as a function of α (i.e., specify the corresponding range of values α for which your answer is valid). Justify rigorously your answer.

2. (3 pts) Characterize all the types of equilibrium (i.e. stable focus, saddle node, etc) that this system have, as a function of α (i.e. for each type of equilibrium, precise the corresponding range of values α for which your answer is valid). Draw the phase portrait in $\mathbb{R}^2$ of this system as accurately as possible, for each of the type of equilibrium that the system can have.

3. (3 pts) Let $(x_1(t), x_2(t)$ be the solution with initial condition $(x_1(0), x_2(0)) = (-\sqrt{|\alpha|}, 1)$. For which range(s) of values of $\alpha \in \mathbb{R}$, if any, does this solution have a non-empty ω-limit set? For the value(s) of of α such that your answer is positive, specify the non-empty ω-limit set of the solution. For the value(s) of of α such that your answer is negative, explain why the ω-limit set of the solution is empty.

<u>Question 3</u> (4 points)

We consider the third-order autonomous linear system

$$\begin{aligned}\dot{x}_1(t) &= -3x_1(t) - x_2(t)\\ \dot{x}_2(t) &= -x_1(t) - 3x_2(t)\\ \dot{x}_3(t) &= -2x_3(t).\end{aligned}$$

1. (1pt) Characterize the stability of the system.

2. (3pts) Find the solution $x(t) = (x_1(t), x_2(t), x_3(t))$ of the system subject to the initial condition $(x_1(0), x_2(0), x_3(0)) = (1, 1, 1)$.

Question 4 (4 points)

The state and output equations of a non-autonomous linear system are

$$\begin{aligned}\dot{x}_1(t) &= -4x_1(t) + \alpha x_2(t) \\ \dot{x}_2(t) &= -2x_1(t) + \beta x_2(t) + u(t) \\ y(t) &= x_2(t)\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are two parameters, where $u(t)$ is the input signal and $y(t)$ is the output response. The initial condition of the system is $(x_1(0), x_2(0)) = (0, 0)$. et $U(s)$ (respectively, $Y(s)$) denote the Laplace transforms of $u(t)$ (resp., of $y(t)$): you measure that the transfer function $H(s) = Y(s)/U(s)$ of this system is

$$H(s) = \frac{1}{s+1}. \tag{1}$$

1. (1pt) Does the knowledge of the transfer function $H(s)$ given by (1) give you an indication on the value(s) of the parameters $\alpha, \beta \in \mathbb{R}$? If so, give the (range of) values of $\alpha, \beta \in \mathbb{R}$ for which the system has indeed the transfer function $H(s)$ given by (1). If not, explain why.

2. (1pt) For the values of $\alpha, \beta \in \mathbb{R}$ obtained in question 3.1, is the system B.I.B.O. stable? Justify.

3. (1pt) For the values of $\alpha, \beta \in \mathbb{R}$ obtained in question 3.1, is the system observable? Justify.

4. (1pt) For the values of $\alpha, \beta \in \mathbb{R}$ obtained in question 3.1, is the system controllable? Justify.