2

Stability of Linear Systems

2.1 Definition and representation

In the next two chapters, we consider linear systems described by

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t) \tag{2.1}$$
$$y(t) = Cx(t) + Du(t) \tag{2.2}$$

for continuous-time systems, and by

$$x(t+1) = Ax(t) + Bu(t) \tag{2.3}$$
$$y(t) = Cx(t) + Du(t) \tag{2.4}$$

for discrete-time systems.

In various field of engineering, and in particular in control theory, it is convenient to think of such systems under the representation as input-output system, with a block diagram between the input signals $u(t)$ and the output $y(t)$. The above system can be represented under that framework, with the diagram shown in Figure 2.1. This representation is sometimes called the “ABCD” representation of a linear system.

The representation of a discrete time linear system is the same, with the integrator replaced by a unit-delay element.

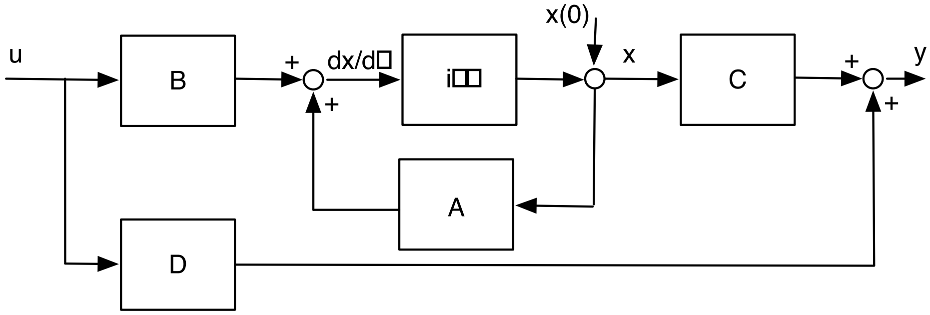

Figure 2.1: The block diagram of a continuous-time linear system

2.2 Solution of a Linear Dynamical System

In the Bachelor courses on Circuits and Systems, these equations were solved in the frequency domain by using Laplace transforms for continuous time systems and z-transforms for discrete time systems. This approach is limited to linear systems. Since we are eventually interested in nonlinear systems, we will perform the analysis in the time domain. The solution of (2.1), (2.2) can be explicitly computed (Lagrange formula) as

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{2.5}$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t), \tag{2.6}$$

whereas the solution of (2.3), (2.4) is

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-\tau-1}Bu(\tau) \tag{2.7}$$

$$y(t) = CA^t x(0) + C\sum_{\tau=0}^{t-1} A^{t-\tau-1}Bu(\tau) + Du(t). \tag{2.8}$$

The free solution (when $u(t) = 0$) can be represented as

$$x(t) = \phi(t)x(0) \tag{2.9}$$

$$y(t) = C\phi(t)x(0), \tag{2.10}$$

where

$$\phi(t) = e^{At} \tag{2.11}$$

for continuous-time systems and

$$\phi(t) = A^t \tag{2.12}$$

for discrete-time systems. In general, x and/or u are vectors, and therefore A, B, C and D are matrices. In this chapter, they are always time-invariant.

Because of the linearity of the system, the superposition principle applies and enables us to express the solution (2.5) or (2.7) as the sum of the free or zero-input solution (2.9) (i.e when $u(t) = 0$) and of the zero-state solution (i.e when $x(0) = 0$). Likewise, the output response (2.6) or (2.8) can be written as the sum of the free or zero-input response (i.e when $u(t) = 0$) and of the zero-state response (i.e when $x(0) = 0$).

2.3 Stability: Definitions

We introduce the following notions of stability of a linear system described by (2.1) or (2.3).

Definition 2.1 (Stability of a linear system)**.** *(i) A linear system is* asymptotically stable *if for all* $x_0 \in \Omega$ *we have*

$$\|\phi(t)x_0\| \xrightarrow[t\to\infty]{} 0. \tag{2.13}$$

(ii) A linear system is stable *if for all* $x_0 \in \Omega$ *there is a constant* $C > 0$ *such that for all* $t \in \mathcal{T}$

$$\|\phi(t)x_0\| \leq C. \tag{2.14}$$

(iii) A linear system is weakly unstable *if it not stable and if for all* $x_0 \in \Omega$ *there exist* $C > 0$ *and* $n > 0$ *such that for all* $t \in \mathcal{T}$

$$\|\phi(t)x_0\| \leq Ct^n. \tag{2.15}$$

(iv) A linear system is strongly unstable *if it neither stable nor weakly unstable.*

2.3.1 Examples

Example 1: Discharging a Capacitor

We consider a classical RC circuit, where a loaded capacitor $C > 0$ with initial voltage x_0 is discharged at time $t = 0$ through a resistor R, as shown in Figure 2.2.

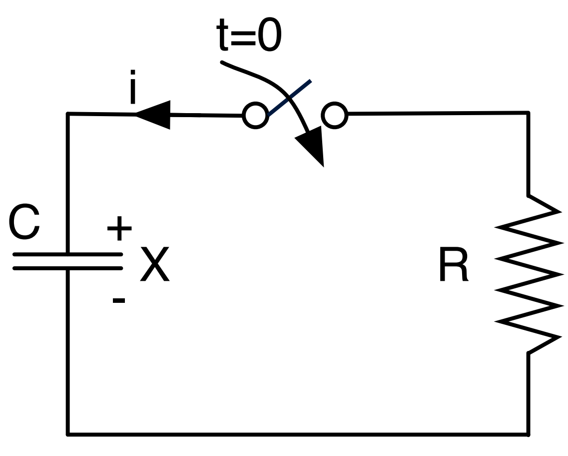

Figure 2.2: RC circuit

With the state x representing the voltage through the capacitor, the state equation reads

$$\frac{dx}{dt}(t) = -\frac{1}{RC}x(t),$$

whose solution is, with $x(0) = x_0$,

$$x(t) = e^{-t/RC}x_0.$$

This system is asymptotically stable if $0 \leq R < \infty$ (passive resistor), is stable if $R = \infty$ (open circuit for all time $t \geq 0$) and is strongly unstable if $R < 0$ (active resistor). The system is never weakly unstable.

Example 2: Free Frictionless Motion

Consider next a mass that is moved at constant speed v_0 over a frictionless plane. The state equations, with the state x_1 the position of the mass and x_2 being the speed, and with x_0 denoting its initial position is

$$\begin{aligned}\frac{dx_1}{dt}(t) &= x_2(t)\\ \frac{dx_2}{dt}(t) &= 0\end{aligned}$$

whose solution is

$$\begin{aligned}x_1(t) &= x_0 + v_0 t\\ x_2(t) &= v_0\end{aligned}$$

The system is weakly unstable (with $n = 1$).

Example 3: Population Growth

Here the state equation is

$$x(t+1) = ax(t)$$

with $a > 0$ and $x(0) = x_0$, whose solution is

$$x(t) = a^t x_0.$$

This system is asymptotically stable if $0 < a < 1$, is stable if $a = 1$ and is strongly unstable if $a > 1$. The system is never weakly unstable.

2.4 Stability: Conditions for Continuous-time Systems

Because the solution (2.9) of the free system is expressed as a function of the exponential (2.11) of the matrix A, the analysis of the stability of the system depends on A and its eigenvalues. We can represent the matrix exponential $\exp(At)$ as

$$e^{tA} = I_n + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots \tag{2.16}$$

where I_n is the $n \times n$ identity matrix.

2.4.1 Diagonalizable Matrix A

If A is diagonalizable, then there is a matrix S such that

$$A = S\Lambda S^{-1}, \tag{2.17}$$

where Λ is the diagonal matrix with all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, i.e.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0\\ 0 & \lambda_2 & 0 & \ldots & 0\\ \vdots & & \ddots & & \vdots\\ 0 & \ldots & & \lambda_{n-1} & 0\\ 0 & \ldots & & 0 & \lambda_n \end{bmatrix}.$$

Inverting (2.17), we get

$$\Lambda = S^{-1}AS. \tag{2.18}$$

Multiplying (2.16) by S^{-1} and S and inserting (2.18), we find

$$\begin{aligned} S^{-1}e^{tA}S &= S^{-1}S + tS^{-1}AS + \frac{t^2}{2!}S^{-1}ASS^{-1}AS + \frac{t^3}{3!}S^{-1}ASS^{-1}ASS^{-1}AS + \ldots \\ &= I_n + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \frac{t^3}{3!}\Lambda^3 + \ldots \\ &= e^{t\Lambda} \\ &= \begin{bmatrix} e^{t\lambda_1} & 0 & 0 & \ldots & 0 \\ 0 & e^{t\lambda_2} & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & & e^{t\lambda_{n-1}} & 0 \\ 0 & \ldots & & 0 & e^{t\lambda_n} \end{bmatrix}. \end{aligned}$$

Hence the solution becomes

$$x(t) = e^{tA}x_0 = Se^{t\Lambda}S^{-1}x_0. \tag{2.19}$$

If we change coordinates by setting $z = S^{-1}x$, (2.19) becomes

$$z(t) = e^{t\Lambda}z_0 = \begin{bmatrix} e^{t\lambda_1}z_{01} \\ e^{t\lambda_2}z_{02} \\ \vdots \\ e^{t\lambda_n}z_{0n} \end{bmatrix}.$$

This leads directly to the following result.

Theorem 2.1. *If the state matrix A is a diagonalizable matrix, then the system is*

- *asymptotically stable if all eigenvalues λ_i satisfy $\mathfrak{Re}(\lambda_i) < 0$ for $1 \leq i \leq n$,*
- *stable if all eigenvalues λ_i satisfy $\mathfrak{Re}(\lambda_i) \leq 0$ for $1 \leq i \leq n$,*
- *strongly unstable if $\mathfrak{Re}(\lambda_i) > 0$ for at least one eigenvalue λ_i, $1 \leq i \leq n$.*

Note that if the system has a diagonalizable matrix A it cannot be weakly unstable.

2.4.2 Non-diagonalizable Matrix A

In some cases however, the matrix A is not diagonalizable, and we cannot use Theorem 2.1. If we take Example 2, the matrix A is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and cannot be diagonalized. Otherwise, it would have two linearly independent eigenvectors with eigenvalue 0 (since it is a triangular matrix, all eigenvalues are diagonal elements). Indeed, any eigenvector $[x_1\ x_2]^T$ of this matrix must have $x_2 = 0$, and thus all eigenvectors lie in the 1-dimensional subspace $x_2 = 0$, i.e. there are no two linearly independent eigenvectors.

The general factorization of a matrix A is its reduction to the Jordan canonical form, which is recalled in Section 2.7.2. From (A result of linear algebra shows that for any matrix A, there is a matrix S such that

$$A = SJS^{-1} \tag{2.20}$$

where J is the block diagonal matrix given by (2.29).

Multiplying (2.16) by S^{-1} and S and inserting $J = S^{-1}AS$ because of (2.20), we find

$$\begin{aligned} S^{-1}e^{tA}S &= I_n + tJ + \frac{t^2}{2!}J^2 + \frac{t^3}{3!}J^3 + \dots \\ &= e^{tJ} \\ &= \begin{bmatrix} e^{tJ_1} & 0 & 0 & \dots & 0 \\ 0 & e^{tJ_2} & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & \ddots & 0 \\ 0 & \dots & & 0 & e^{tJ_q} \end{bmatrix}. \end{aligned}$$

where the last matrix is a block diagonal matrix, with Jordan blocks computed from (2.33) as

$$e^{tJ_i} = \begin{bmatrix} e^{t\lambda_i} & te^{t\lambda_i} & \frac{t^2}{2}e^{t\lambda_i} & \dots & \frac{t^{n_i-1}}{(n_i-1)!}e^{t\lambda_i} \\ 0 & e^{t\lambda_i} & te^{t\lambda_i} & \frac{t^2}{2}e^{t\lambda_i} & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & 0 & e^{t\lambda_i} & te^{t\lambda_i} \\ 0 & \dots & & 0 & e^{t\lambda_i} \end{bmatrix} \tag{2.21}$$

Hence the solution becomes

$$x(t) = e^{tA}x_0 = Se^{tJ}S^{-1}x_0, \tag{2.22}$$

which is a linear combination of the terms in this matrix, i.e. of terms of the form $e^{\lambda_i t}, te^{\lambda_i t}, t^2e^{\lambda_i t}, \dots, t^{r_i}e^{\lambda_i t}$. This leads directly to the following result.

Theorem 2.2. *If A is its state matrix, then the system is*

- *asymptotically stable if all eigenvalues λ_i satisfy $\mathfrak{Re}(\lambda_i) < 0$ for $1 \le i \le n$,*
- *stable if all eigenvalues λ_i, for $1 \le i \le n$, satisfy either $\mathfrak{Re}(\lambda_i) < 0$, or $\mathfrak{Re}(\lambda_i) = 0$ and the corresponding Jordan block J_i is of dimension 1 ($r_i = 1$, $J_i = \lambda_i$),*
- *weakly unstable if all eigenvalues λ_i satisfy $\mathfrak{Re}(\lambda_i) \le 0$ for $1 \le i \le n$, and if there is at least one eigenvalue λ_i with $\mathfrak{Re}(\lambda_i) = 0$ such that the corresponding Jordan block J_i is of dimension higher than 1 ($r_i > 1$),*
- *strongly unstable if $\mathfrak{Re}(\lambda_i) > 0$ for at least one eigenvalue λ_i, $1 \le i \le n$.*

A system that has no eigenvalue with $\mathfrak{Re}(\lambda_i) = 0$ is called *hyperbolic*. In terms of stability, hyperbolic systems are robust against parameter changes, and non-hyperbolic systems are fragile.

2.5 Stability: Conditions for Discrete-time Systems

The conditions are similar for discrete-time systems, with e^{At} replaced by A^t since the free solution is given by $x(t) = \Phi(t)x_0$ with $\Phi(t)$ given by (2.12) instead of (2.11). By applying (2.17) if A is diagonalizable or (2.20) in the general case, we get that

$$x(t) = A^t x_0 = SJ^tS^{-1}x_0$$

with J^t given by (2.32). A solution is therefore a linear combination of terms of the form $\lambda_i^t, t\lambda_i^t, t^2\lambda_i^t, \dots, t^{r_i}\lambda_i^t$. This leads directly to the following result.

Theorem 2.3. *If A is its state matrix, then the discrete-time system is*

- *asymptotically stable if all eigenvalues λ_i satisfy $|\lambda_i| < 1$ for $1 \leq i \leq n$,*
- *stable if all eigenvalues λ_i, for $1 \leq i \leq n$, satisfy either $|\lambda_i| < 1$, or $|\lambda_i| = 1$ and the corresponding Jordan block J_i is of dimension 1 ($r_i = 1$, $J_i = \lambda_i$),*
- *weakly unstable if all eigenvalues λ_i satisfy $|\lambda_i| \leq 1$ for $1 \leq i \leq n$, and if there is at least one eigenvalue λ_i with $|\lambda_i| = 1$ such that the corresponding Jordan block J_i is of dimension higher than 1 ($r_i > 1$),*
- *strongly unstable if $|\lambda_i| > 1$ for at least one eigenvalue λ_i, $1 \leq i \leq n$.*

A system that has no eigenvalue with $|\lambda_i| = 1$is called hyperbolic.

2.6 Stability: Connection with Frequency Domain Analysis

2.6.1 Continuous-time Systems

If we take the Laplace transform of (2.1), we get

$$X(s) = (sI_n - A)^{-1}x(0) + (sI_n - A)^{-1}BU(s) \tag{2.23}$$

where

$$U(s) = \int_0^{\infty} u(t)e^{-st}dt$$

is the Laplace transform of the input signal $u(t)$ (and similarly, $X(s)$ is the Laplace transform of $x(t)$). The expression (2.23) is valid for all $s \in \mathbb{C}$ which is not an eigenvalue of A.

Similarly, the Laplace transform of (2.2) becomes

$$Y(s) = C(sI_n - A)^{-1}x(0) + (C(sI_n - A)^{-1}B + D)U(s). \tag{2.24}$$

In classes of Circuits and Systems, or of Control Theory, the "internal" description of the system is often neglected, because $x(0) = 0$. Only the zero-state response (i.e. with $x(0) = 0$) is considered in the "external" (or input-output) of the system by the *transfer matrix*

$$H(s) = C(sI_n - A)^{-1}B + D.$$

When the dimensions of the input and output spaces are equal to 1, one speaks of a Single Input, Single Output (SISO) system, and its transfer matrix becomes the (scalar) transfer function $H(s) = \frac{Y(s)}{U(s)}$, which is the Laplace transform of the impulse response $h(t)$ in the convolution

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau.$$

The "internal" description of the system (2.23), (2.24) is more complete, because it accounts for both the initial internal state and the external output signal(s).

Definition 2.1 depends on the free solution (zero-input solution), which amounts to solve the system of n linear equations

$$(sI_n - A)X(s) = x(0)$$

for the components of $X(s)$. Applying Cramer's rule to solve systems of linear equations, we get that ith component of $X(s)$ is

$$X_i(s) = \frac{\det(sI_n - A)_i}{\det(sI_n - A)} \tag{2.25}$$

where $(sI_n - A)_i$ is the same matrix as $(sI_n - A)$ except that ith column is replaced by the independent term $x(0)$. Both the numerator and the denominator in (2.25) are polynomials in s. The roots of the denominator are the natural frequencies of the system. At the same time, they are the eigenvalues of the matrix A, which condition the stability of the system because of Theorems 2.1 and 2.2.

2.6.2 Discrete-time Systems

The analysis for discrete-time systems is similar, with the Laplace transforms replaced by z-transforms. The z-transform of (2.3) and (2.4) are

$$\begin{aligned} X(z) &= (zI_n - A)^{-1}x(0) + (zI_n - A)^{-1}BU(z) && (2.26)\\ Y(z) &= C(zI_n - A)^{-1}x(0) + (C(zI_n - A)^{-1}B + D)U(z) && (2.27) \end{aligned}$$

where

$$U(z) = \sum_{n=0}^{\infty} u(n)z^{-n}$$

is the z-transform of the input signal $u(n)$ (and similarly, $X(z), Y(z)$ are the z-transforms of $x(n), y(n)$ respectively).

Again, we need to solve

$$(zI_n - A)X(z) = x(0)$$

to find the free state solution, for the components of $X(z)$. Applying Cramer's rule to solve systems of linear equations, we get that ith component of $X(z)$ is

$$X_i(z) = \frac{\det(zI_n - A)_i}{\det(zI_n - A)} \tag{2.28}$$

Again, both the numerator and the denominator in (2.28) are polynomials in z. The roots of the denominator are the natural frequencies of the system, and are called in control theory the *poles* of the system.

2.7 Appendix: Some Useful Results of Linear Algebra

2.7.1 Diagonalizable Square Matrices

A $n \times n$ matrix A is said to be diagonalizable if and only if it is *similar* to a diagonal matrix, i.e. if and only if there is a $n \times n$ invertible matrix S such that $\Lambda = S^{-1}AS$ where Λ is a $n \times n$ diagonal matrix, or equivalently

$$A = S\Lambda S^{-1}.$$

A is diagonalizable *if and only if* the sum of the dimensions of its eigenspaces is equal to n, i.e. if and only if it has n linearly independent eigenvectors.

A *sufficient condition* for A to be diagonalizable is that it has n distinct eigenvalues.

2.7.2 Non Diagonalizable Square Matrices

Any $n \times n$ matrix with $1 \leq q \leq n$ linearly independent eigenvectors is similar to a matrix J in *Jordan (canonical) form* with q square blocks on its diagonal, which reads

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_q \end{bmatrix} \tag{2.29}$$

This means that there exists a $n \times n$ invertible matrix S such that

$$A = SJS^{-1}.$$

Each Jordan block J_i corresponds to one eigenvalue λ_i and to one (unit-norm) eigenvector, and has the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix} \tag{2.30}$$

In other words, the elements on the main diagonal are all equal to λ_i, the elements on the first upper diagonal are all equal to 1 and all the other entries of the Jordan block are 0.

The *geometric multiplicity* of an eigenvalue λ_k is equal to the dimension of its eigenspace, i.e. the number of linearly independent eignevectors with eigenvalue λ_i, and thus to the number of Jordan blocks corresponding to this eigenvalue.

The *algebraic multiplicity* of an eigenvalue λ_i is the number of times it is a root of the characteristic polynomial of A, i.e. the number of times it appears on the diagonal of J. It is equal to the sum of the dimensions of all Jordan blocks corresponding to the eigenvalue λ_i.

The canonical Jordan form of A allows to express the tth power of A as

$$A^t = SJ^tS^{-1} \tag{2.31}$$

where

$$J^t = \begin{bmatrix} J_1^t & 0 & \dots & 0 \\ 0 & J_2^t & \ddots & \vdots \\ \vdots & \ddots & J_{q-1}^t & 0 \\ 0 & \dots & 0 & J_q^t \end{bmatrix}, \tag{2.32}$$

and where one can compute that

$$J_i^t = \begin{bmatrix} \lambda_i^t & t\lambda_i^{t-1} & \frac{t(t-1)}{2}\lambda_i^{t-2} & \dots & \binom{t}{n_i}\lambda_i^{t-n_i} \\ 0 & \lambda_i^t & t\lambda_i^{t-1} & \frac{t(t-1)}{2}\lambda_i^{t-2} & \dots \\ \vdots & & \ddots & \ddots & \\ & & & & \\ 0 & & 0 & \lambda_i^t & t\lambda_i^{t-1} \\ 0 & \dots & & 0 & \lambda_i^t \end{bmatrix} \tag{2.33}$$

The matrix $S = [S_1, S_2, \ldots, S_q]$ has q (rectangular $n \times n_i$) blocks $S_i \in \mathbb{C}^{n \times n_i}$, each of which contains the columns of S associated with the Jordan block J_i. Let $S_i = [v_{i,1}\ v_{i,2}\ \ldots\ v_{i,n_i}]$. We can find the columns of S_i iteratively in the following way:

$$\begin{aligned}(A - \lambda_i I_n)v_{i,1} &= 0\\ (A - \lambda_i I_n)v_{i,j} &= v_{i,(j-1)}.\end{aligned}$$

The first column $v_{i,1}$ is the eigenvector of A associated to the eigenvalue λ_i. The other $n_1 - 1$ columns are called the generalized eigenvectors associated to the eigenvalue λ_i.

2.8 BIBO Stability

The previous sections analyze the stability of autonomous linear systems according to Definition 2.1, which is given in terms of the free or zero-input solution of the system. We note that if the free system is stable (respectively, asymptotically stable), its state remains bounded (respectively, tends asymptotically to zero) and consequently its output, which is a linear combination of the states, is bounded (respectively, tends to zero) as well.

What happens now if we apply a nonzero input u to the system?

In this section, we now address the stability of the zero-state response. This leads to the definition of BIBO (Bounded Input, Bounded Output) stability. A linear system is said to be BIBO stable if and only if any bounded input u produces a bounded output y. Although this definition can be extended to encompass non-zero initial states (by considering them as an additional particular input of the system), we will restrict ourselves to zero-state solutions and responses in this section.

2.8.1 Definition and Conditions

In control theory or signal processing, a linear (time-invariant) system is often described by the convolution

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$

in continuous time, or by

$$y(t) = \sum_{\tau=0}^{t-1} h(t-\tau)u(\tau)$$

rather than by the state and output equations (2.1) until (2.4). The *impulse response* $h(t)$ is obtained as the output $y(t)$ of the zero-state response of the system when the input is $u(t) = \delta(t)$, where δ is the Dirac impulse. For the linear systems described by (2.1)-(2.4), the Laplace transform $H(s)$ is given by

$$H(s) = C(sI_n - A)^{-1}B + D, \tag{2.34}$$

or the z-transform is given by

$$H(s) = C(zI_n - A)^{-1}B + D, \tag{2.35}$$

as we have seen in the previous sections.

Definition 2.2 (B.I.B.O. stability of a linear system)**.** *A linear system is* B.I.B.O. stable *if and only if for all $u \in \Gamma$ such that*

$$\|u(t)\| \le u_M$$

for some finite u_M and for all $t \in \mathcal{T}$, there is a finite constant K such that

$$\|y(t)\| \le Ku_M$$

for all $t \in \mathcal{T}$.

We then have the following result, which we state only in continuous time (the extension to discrete-time systems is similar):

Theorem 2.4. *A linear time-invariant system with impulse response $h(t)$ is BIBO stable if and only if there is some finite h_M such that*

$$\int_0^\infty \|h(\tau)\|d\tau \le h_M. \tag{2.36}$$

In the previous expressions, the norm is the infinite norm. In the multi-dimensional case, if h_{ij} denotes the (i,j)th entry of h, (2.36) can therefore be replaced by

$$\int_0^\infty \max_{ij} |h_{ij}(\tau)| d\tau \leq h_M.$$

If the transfer function $H(s)$ can be expressed as a rational fraction in s (this is the case for (2.34), then the roots of its denominator are called in control theory the *poles* of the system, they are the values of s for which $H(s)$ is infinite. Poles are always natural frequencies of the systems. Then one can show that BIBO stability amounts to require that all poles have a negative real part.

Theorem 2.5. *A linear time-invariant system with a transfer function $H(s)$ which is a rational function of s is BIBO stable if and only if all the poles of every entry of $H(s)$ have strictly negative real parts.*

Note that if a linear system described by (2.1) and (2.2) is asymptotically stable, then it is also BIBO stable, because all the poles of $H(s)$ are eigenvalues of A. The converse is however not true, as some eigenvalues of A may not appear in the denominator of $H(s)$ because of a cancellation between numerator and denominator.

2.8.2 Examples

Example 1: Non autonomous RC circuit

Consider again the a classical RC circuit of Figure 2.3, where the switch is replaced by an external voltage source delivering a voltage $u(t)$. The capacitor $C > 0$ is loaded with initial voltage $x_0 = 0$.

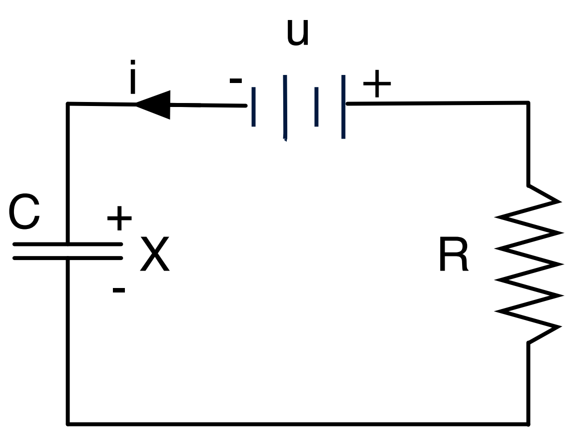

Figure 2.3: RC circuit with an external input

With the state x representing the voltage through the capacitor, the state equation reads

$$\frac{dx}{dt}(t) = -\frac{1}{RC}(x(t) + u(t)).$$

Let the output of the circuit be the voltage accross the capacitor: $y(t) = x(t)$. The corresponding transfer function is

$$H(s) = \frac{1}{1 + RCs}.$$

We see that the pole of this system is $s = -1/RC$ and is negative. Hence this system is BIBO stable. We also see that the system is asymptotically stable.

Example 2: A BIBO stable system with an unstable free system

Consider a system whose state and output equations are

$$\frac{dx_1}{dt}(t) = x_1(t) \tag{2.37}$$

$$\frac{dx_2}{dt}(t) = x_1(t) - x_2(t) + u(t) \tag{2.38}$$

$$y(t) = x_1(t) + x_2(t). \tag{2.39}$$

The corresponding matrices A, B, C and D are

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

The transfer function is, from (2.34),

$$H(s) = \frac{1}{s+1},$$

and since the only pole of this system is $s = -1$, Theorem 2.5 implies that this system is BIBO stable. However, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 1$, which shows that the system is (strongly) unstable. (In fact, the positive eigenvalue 1 that should appear in the denominator of $H(s)$ got cancelled by a zero in 1 at the numerator of $H(s)$, one speaks of a pole-zero cancellation).

Example 3: Integrator

Finally, let is consider a simple system whose state equation is With the state x representing the voltage through the capacitor and u being an external source of current applied, the state equation reads

$$\frac{dx}{dt}(t) = u(t)$$

with the output equation

$$y(t) = x(t).$$

The output $y(t)$ is the integral of the input signal $u(t)$ The corresponding transfer function is

$$H(s) = \frac{1}{s}.$$

We see that the pole of this system is $s = 0$ and is not negative. Hence this system is not BIBO stable. We also see that the (free) system is not asymptotically stable, but only simply stable.

2.9 Asymptotic Behavior of 2-dim Systems: Geometric Analysis

The criteria given in Theorem 2.2 and Theorem 2.3 for assessing the stability of a linear dynamical system are all expressed as functions of the eigenvalues of the state matrix A. In this section, we continue our study of stability, by analyzing the geometry of the phase portraits of continuous linear systems in 2 dimensions. The extension to 2-dim. discrete dynamical systems is left as an exercise.

With $n = 2$, we consider therefore in this section a dynamical system whose free state equation is

$$\frac{dx}{dt}(t) = Ax(t) \tag{2.40}$$

with initial condition $x(0) = x_0 \in \mathbb{R}^2$. Let λ_1, λ_2 be the two eigenvalues of the state matrix A.

2.9.1 Planar Hyperbolic Systems

We start first with hyperbolic systems, where $\mathfrak{Re}(\lambda_i) \neq 0$ for $i = 1, 2$. Note that this implies that $\det A \neq 0$, hence the origin is the only point such that the system remains at equilibrium (i.e. such that $dx/dt = 0$). It is also the only possible attractor.

Node

(i) If λ_1 and λ_2 are real and of the same sign, then the origin is a node. If $\lambda_1, \lambda_2 < 0$ the origin is a *stable node* (or sink), and conversely, if $\lambda_1, \lambda_2 > 0$, it is an *unstable node*.

Let us first consider the case where $\lambda_1 \neq \lambda_2$. The matrix A is diagonalizable, and the solution is given by (2.19) where the matrix S contains the two eigenvectors corresponding the eigenvalues λ_1 and λ_2, which are, respectively, $[s_{11}\ s_{21}]^T$ and $[s_{12}\ s_{22}]^T$, with T denoting transposition.

Let us change coordinates by setting $z = S^{-1}x$, (2.19) becomes

$$z(t) = e^{t\Lambda} z_0 = \begin{bmatrix} e^{t\lambda_1} z_{01} \\ e^{t\lambda_2} z_{02} \end{bmatrix}.$$

By eliminating t from these equations, we find

$$\frac{z_2}{z_{02}} = \left(\frac{z_1}{z_{01}}\right)^{\lambda_2/\lambda_1}. \tag{2.41}$$

This equation describes the orbits of the solution in the phase plane $\mathbb{R}^2$. Some are represented in Figure 2.4.

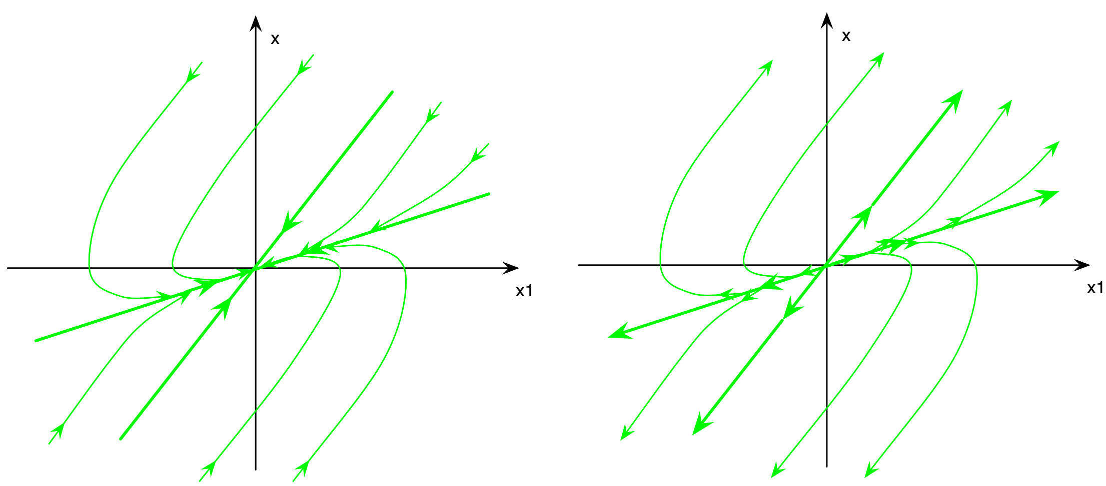

Figure 2.4: Phase portraits of a stable node (left) and of an unstable node (right).

Because

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = S \cdot \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \cdot \begin{bmatrix} e^{t\lambda_1} z_{01} \\ e^{t\lambda_2} z_{02} \end{bmatrix} = \begin{bmatrix} e^{t\lambda_1} z_{01} s_{11} + e^{t\lambda_2} z_{02} s_{12} \\ e^{t\lambda_1} z_{01} s_{21} + e^{t\lambda_2} z_{02} s_{22} \end{bmatrix}$$

any solution $x(t)$ is a linear combination of the two eigenvectors of A. In particular, we observe that the two eigenspaces of A (the axis representing the z_1 and z_2 axes in the original plane $\{(x_1, x_2)\}$) are invariant.

When $\lambda_1 < \lambda_2 < 0$, and with initial conditions x_0 that are not eigenvectors of A, the trajectories approach the origin tangentially to the "slow" eigenvector (the one corresponding to the dominant eigenvalue, which is here λ_2), and arrive from initial conditions following parallels to the "fast" eigenvector (the one corresponding to the non-dominant eigenvalue, which is here λ_1).

(ii) If $\lambda_1 = \lambda_2 = \lambda$, some special cases occur. Let us consider the case where $\lambda < 0$, the case $\lambda > 0$ being similar.

If A admits two linearly independent eigenvectors, then one can easily show that $A = \lambda I_2$ and therefore any vector of $\mathbb{R}^2$ is an eigenvector of A. The orbits of all solutions are therefore straight lines starting in x_0 and arriving in 0, as shown on the left of Figure 2.5.

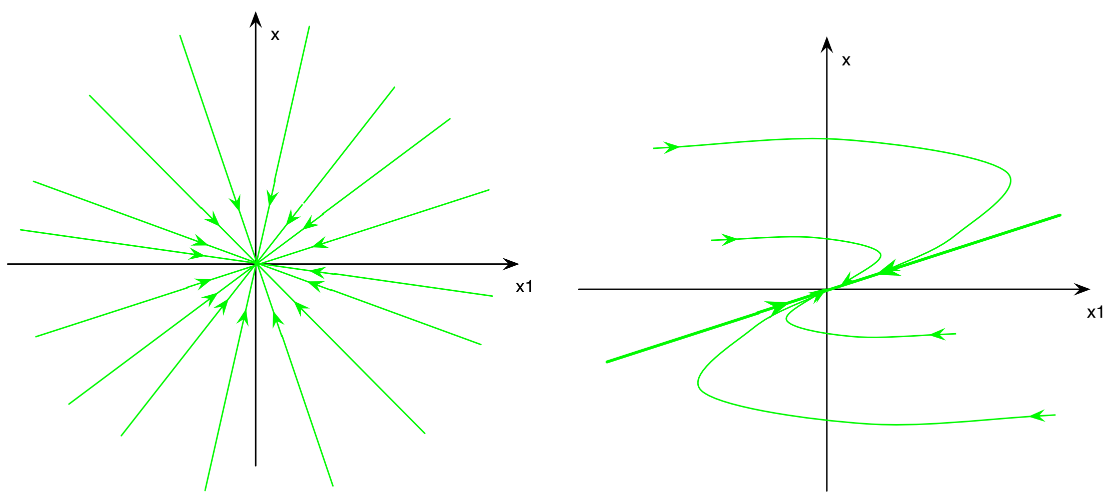

Figure 2.5: Phase portraits of a stable node with repeated eigenvalues, and with either two independent eigenvectors (left) or one (stable improper node) (right).

If A admits only one eigenvector (up to a multiplicative constant), then A is non diagonalizable but is similar to a Jordan form

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

i.e. there is an invertible matrix S such that $J = S^{-1}AS$. Setting again $z = S^{-1}x$, (2.40) becomes

$$\frac{dz}{dt}(t) = Az(t)$$

with $z(0) = S^{-1}x_0 = z_0$, and whose solution is $z(t) = e^{tJ}z_0$, which can be explanded as

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{bmatrix} \cdot \begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix} = \begin{bmatrix} e^{t\lambda}z_{01} + te^{t\lambda}z_{02} \\ e^{t\lambda}z_{02} \end{bmatrix}.$$

By eliminating t, we get

$$\frac{z_1}{z_{01}} = \frac{z_2}{z_{02}} + \frac{z_2}{z_{01}\lambda} \ln\left(\frac{z_2}{z_{02}}\right),$$

which describes the orbits of the system, also represented on the right part of Figure 2.5. There is only one eigenvector, which is the first column of S and is associated with the z_1-axis. The z_1-axis is invariant, but the z_2-axis is not.

Saddle

If λ_1 and λ_2 are real and of opposite signs, then the origin is a *saddle*. The change of variables $z = S^{-1}x$ yields the same equation (2.41) to describe the orbits of this system, but observe now that the exponent in the right hand side is negative, and therefore that the orbits are hyperbola. The only initial conditions that are attracted to the origin are those on the eigenspace corresponding to the negative eigenvalue, all the others diverge along the hyperbolas described by (2.41) and represented in Figure 2.6.

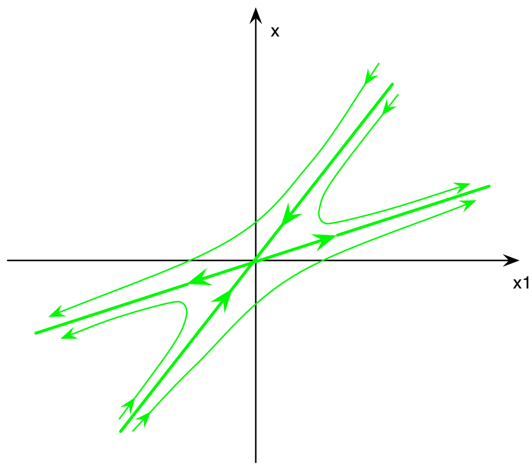

Figure 2.6: Phase portrait of a saddle

Focus

Finally, when the eigenvalues are complex, we can still use the change of variables $z = S^{-1}x$ to diagonalize A, but S will be a complex matrix. To visualize the phase portrait, it is more convenient to adopt the change of variables $z = T^{-1}x$, where T is a real matrix.

If $\lambda_{1,2} = \alpha \pm j\beta$ with $\alpha, \beta \in \mathbb{R}$, one can show in linear algebra that a possible factorization of A is

$$A = TKT^{-1}$$

where

$$K = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},$$

and T is a real matrix. In the coordinates $z = T^{-1}x$, the system (2.40) becomes

$$\frac{dz}{dt}(t) = Kz(t). \tag{2.42}$$

Now, let us do a second change of variables, by introducing the polar coordinates

$$r = \left(z_1^2 + z_2^2\right)^{1/2} \tag{2.43}$$

$$\varphi = \arctan\left(\frac{z_2}{z_1}\right). \tag{2.44}$$

The system (2.42) then becomes

$$\begin{aligned}\frac{dr}{dt}(t) &= \alpha r(t)\\ \frac{d\varphi}{dt}(t) &= -\beta\end{aligned}$$

whose solution is

$$r(t) = r(0)\exp(\alpha t) \tag{2.45}$$
$$\varphi(t) = \varphi(0) - \beta t. \tag{2.46}$$

The angular variable $\varphi(t)$ increases at a fixed rate β, spinning around the origin in a clockwise direction if $\beta > 0$ and in a counterclockwise direction if $\beta < 0$. The amplitude variable $r(t)$ increases exponentially fast if $\alpha > 0$, and the origin is then an *unstable focus*, whereas $r(t)$ decreases exponentially fast if $\alpha < 0$, in which the origin is then a *stable focus*. The orbits spiral respectively away or towards the origin, as shown in Figure 2.7.

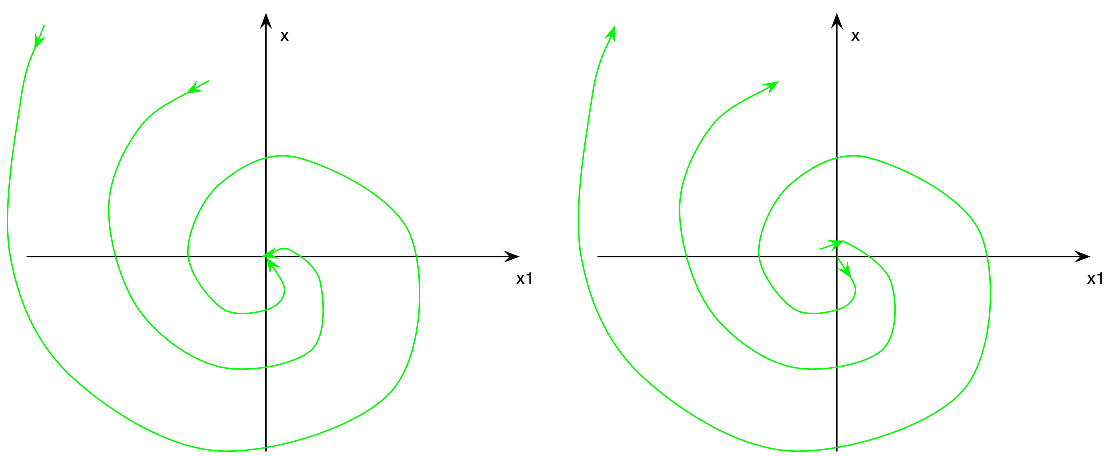

Figure 2.7: Phase portraits of a stable focus (left) and of an unstable focus (right).

2.9.2 Planar Non-hyperbolic Systems

We now consider non hyperbolic systems, where at least one of the two eigenvalues has a zero real part: $\mathfrak{Re}(\lambda_i) = 0$ for $i = 1$ and/or $i = 2$. In this case, A may be singular, and there may be other ω-limit sets than the origin.

Non-hyperbolic Node

If λ_1 and λ_2 are real and only one of them equal to zero, then the origin is a degenerate node. With $\lambda_1 = 0$, the origin is a *degenerate, non hyperbolic stable node* if $\lambda_2 < 0$ and conversely, is a *degenerate, non hyperbolic unstable node* if $\lambda_2 > 0$.

Applying the same change of coordinates $z = S^{-1}x$ as before, where S diagonalizes A (the matrix A is diagonalizable because it has two distinct eigenvalues), we get

$$z(t) = e^{t\Lambda}z_0 = \begin{bmatrix} e^{t\lambda_1}z_{01} \\ e^{t\lambda_2}z_{02}\end{bmatrix} = \begin{bmatrix} z_{01} \\ e^{t\lambda_2}z_{02}\end{bmatrix}$$

and therefore

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = S \cdot \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \cdot \begin{bmatrix} z_{01} \\ e^{t\lambda_2} z_{02} \end{bmatrix} = \begin{bmatrix} z_{01}s_{11} + e^{t\lambda_2} z_{02}s_{12} \\ z_{01}s_{21} + e^{t\lambda_2} z_{02}s_{22} \end{bmatrix}.$$

If $\lambda_2 < 0$, then $x(t)$ tends exponentially fast to the equilibrium $[z_{01}s_{11} \;\; z_{01}s_{21}]^T$. It does not however converges to the origin: the system is simply stable, but is not asymptotically stable. If $\lambda_2 > 0$, then all solutions except those for which $z_{02} = 0$ diverge, and the system is strongly unstable. Figure 2.8 shows two examples of non hyperbolic degenerate nodes.

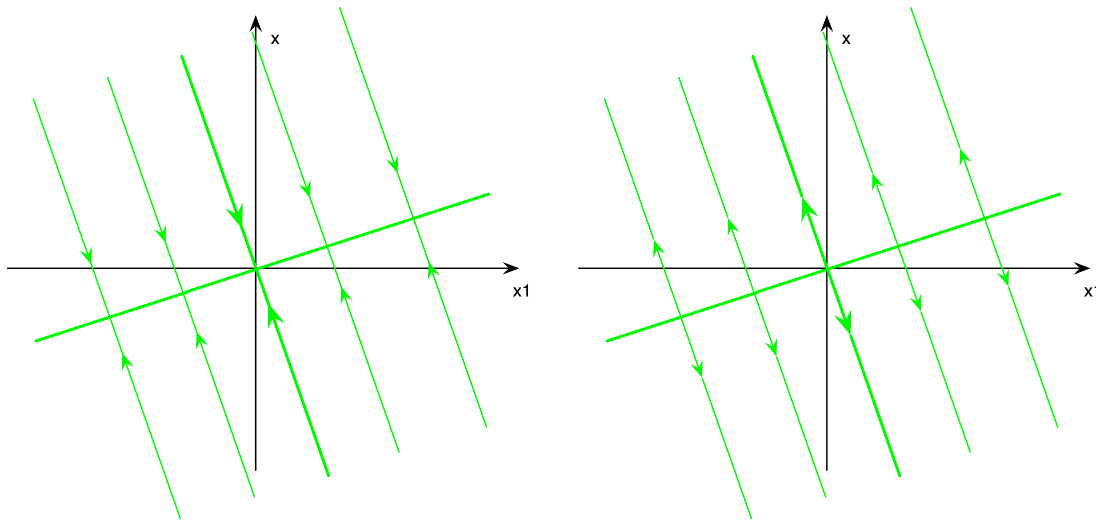

Figure 2.8: Phase portraits of a degenerate non hyperbolic (simply) stable (left) and (weakly) unstable nodes (right).

If $\lambda_1 = \lambda_2 = 0$, then either all the solutions are constant (the matrix A is the zero matrix), as in Figure 2.9(left), or the matrix is similar to the Jordan matrix

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

whose solution in the z-coordinates $z = S^{-1}x$ is

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} z_{01} + tz_{02} \\ z_{02} \end{bmatrix}.$$

In this case, the system is weakly unstable. This is the example of the free motionless movement (Example 2), with orbits shown in Figure 2.9(right).

Center

Finally, if the two eigenvalues of A are imaginary, then we can apply the same changes of polar coordinates as (2.43) and (2.44), where this time $\alpha = 0$ and $\lambda_{1,2} = \pm j\beta$. Then (2.45) yields that the amplitude variable remains constant $r(t) = r(0)$, whereas (2.45) indicates that the angular variable $\varphi(t)$ increases at a fixed rate β, spinning around the origin in a clockwise direction if $\beta > 0$ and in a counterclockwise direction if $\beta < 0$. The phase portrait consists of an infinite number of closed circular orbits, as in Figure 2.10. We have already encountered such a phase plot in Figure 1.3. The system has no attractor, and is simply stable.

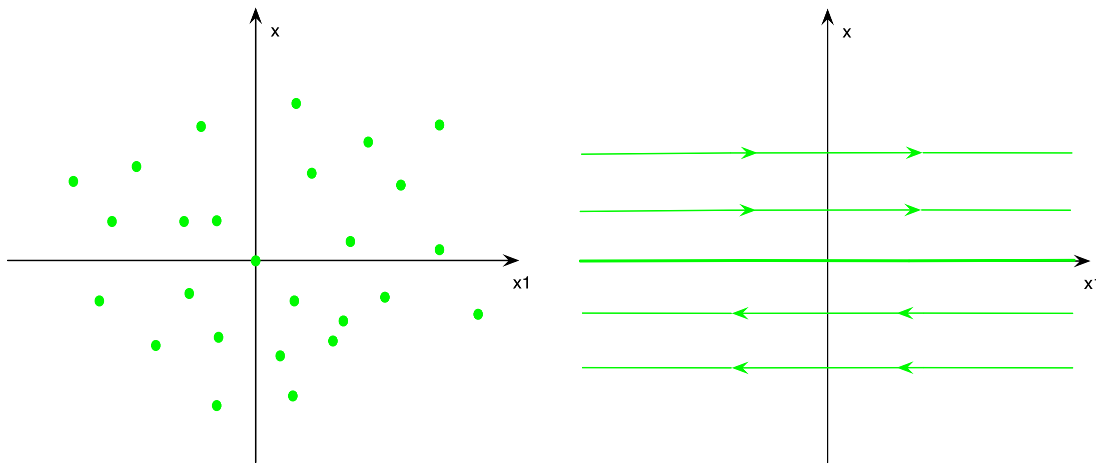

Figure 2.9: Phase portraits of a degenerate non hyperbolic (simply) stable (left) and (weakly) unstable nodes (right), for a repeated 0-eigenvalue.

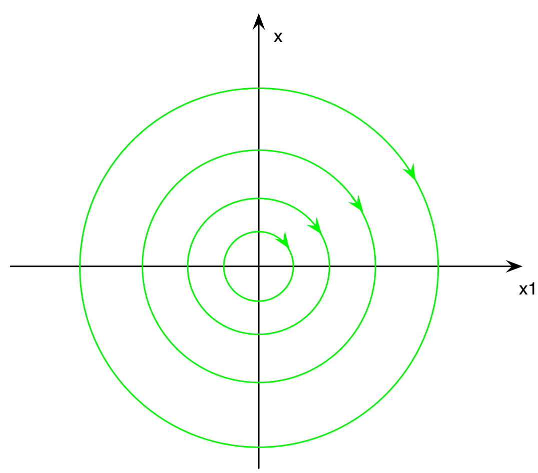

Figure 2.10: Phase portrait of a center

2.10 Appendix: Some Laplace and z-Transforms

Table 2.1 gives some of the most usual pairs of Laplace transforms (the function $f(t)$ is defined on $t \in \mathbb{R}_+$). Table 2.2 gives some of the most usual pairs of (unilateral) z-transforms (the function $f(t)$ is defined on $t \in \mathbb{N}$).

Table 2.1: Some Laplace transforms

$f(t)$	$F(s)$
$\delta(t)$	1
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!$	$1/s^n$
e^{at}	$1/(s-a)$
$t^{n-1}e^{at}/(n-1)!$	$1/(s-a)^n$
$\cos(at)$	$s/(s^2+a^2)$
$\sin(at)$	$a/(s^2+a^2)$
$\cos(at)e^{bt}$	$\frac{s-b}{(s-b)^2+a^2}$
$\sin(at)e^{bt}$	$\frac{a}{(s-b)^2+a^2}$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f(at)$	$F(s/a)/a$
$f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\frac{df}{dt}(t)$	$sF(s)-f(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$
$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$

Table 2.2: Some (unilateral) z-transforms

$f(t)$	$F(z)$
$\delta(t)$	1
1	$1/(1-z^{-1})$
t	$z^{-1}/(1-z^{-1})^2$
a^t	$1/(1-az^{-1})$
ta^t	$az^{-1}/(1-az^{-1})^2$
$\cos(at)$	$\frac{1-z^{-1}\cos(a)}{1-2z^{-1}\cos(a)+z^{-2}}$
$\sin(at)$	$\frac{1-z^{-1}\sin(a)}{1-2z^{-1}\cos(a)+z^{-2}}$
$\cos(at)b^t$	$\frac{1-bz^{-1}\cos(a)}{1-2bz^{-1}\cos(a)+b^2z^{-2}}$
$\sin(at)b^t$	$\frac{1-bz^{-1}\sin(a)}{1-2bz^{-1}\cos(a)+b^2z^{-2}}$
$af(t)+bg(t)$	$aF(z)+bG(z)$
$f(t-a)$	$z^{-a}F(z)$
$a^tf(t)$	$F(z/a)$
$(f\star g)(t)$	$F(z)G(z)$