AUDIOVISUAL COMMUNICATIONS LCAV

Mathematical Foundations of Signal Processing

Mathematical Foundations of Signal Processing

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Where are we now?

1. Geometrical Tools
   - Hilbert spaces, projections etc.

2. Modeling and Analysis
   - Transforms, DT and CT systems, etc.

3. Measuring and Processing
   - *Sampling and Interpolation*
   - Approximation and Compression
   - Localization and Uncertainty
   - Compressed Sensing

4. Applications

Sampling and Interpolation

1. Why Sampling?

2. Sampling and Interpolation as operators in a Hilbert space

3. Sampling and Interpolation of finite-dim vectors

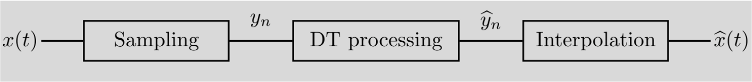
4. Sampling and Interpolation of sequences in $\ell^2(\mathbb{Z})$

5. Sampling and Interpolation of functions

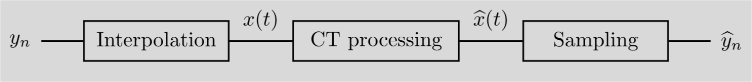
Why Sampling?

Why Sampling?

- World is analog (ch 4). But storing and processing more convenient digitally (ch 3).
- Sampling is the bridge: Given a signal (function) we record its values only at certain instants of time. Trading continuous time description of signal (function) with description based on countable set of values (sequence).
- Convenient but *often lossy*

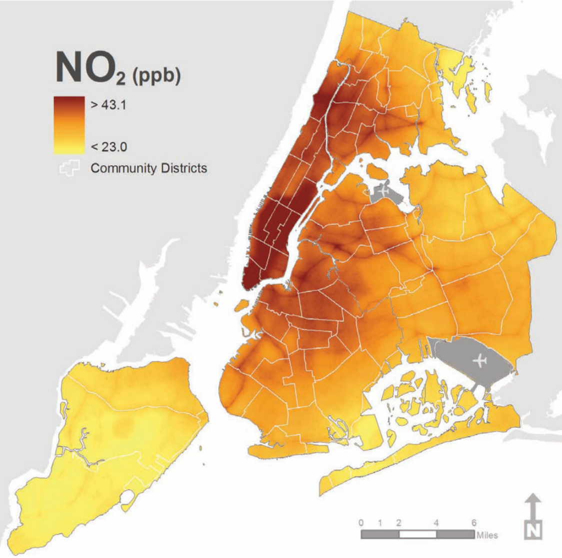


(a) Digital signal processing.

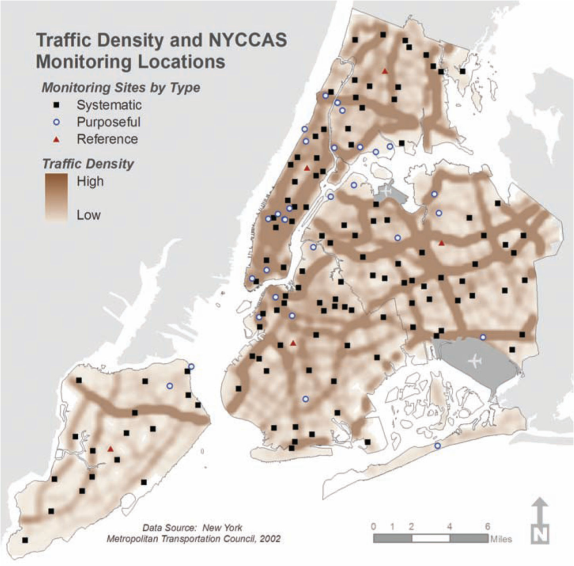


(b) Digital communications.

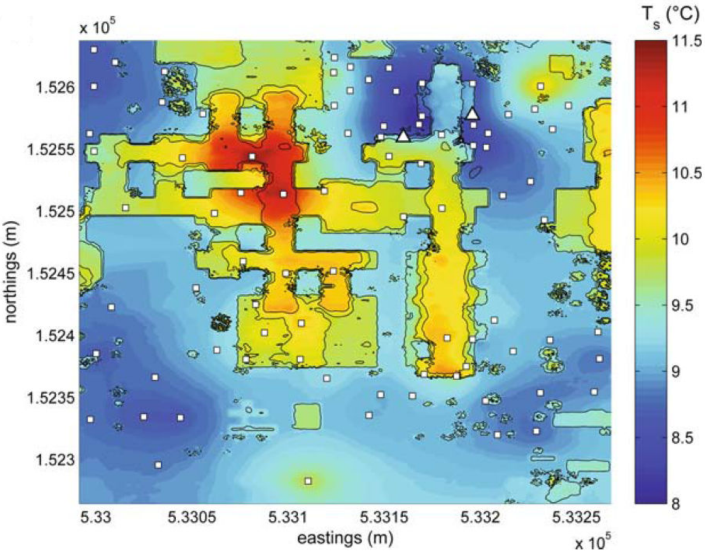
Example: Pollution concentration measurement



Example: Traffic density measurement

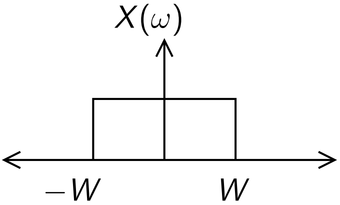


Example: Temperature distribution on campus

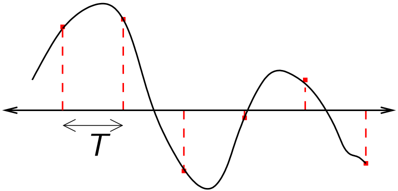


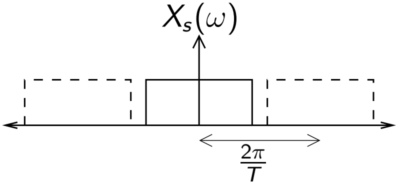
Classical sampling

- Given 1-D bandlimited signal



- Perfect recovery via uniform sampling provided $T \leq \frac{\pi}{W}$

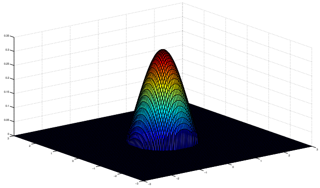




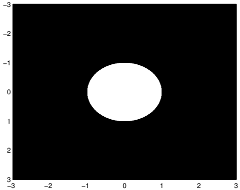
Classical sampling in higher dimensions

- Given: spatially bandlimited field $f : \mathbb{R}^d \mapsto \mathbb{C}$

$$\mathcal{F}(\omega) := \int f(r) e^{-j\langle \omega, r \rangle} dr = 0 \text{ for } \omega \notin \Omega$$



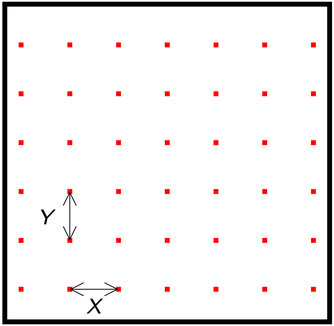
Spectrum $|\mathcal{F}(\omega_x, \omega_y)|$



Support of spectrum Ω

Classical sampling in higher dimensions

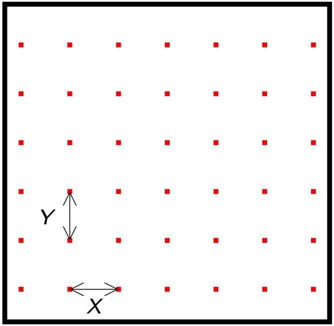
- Sampling on a lattice



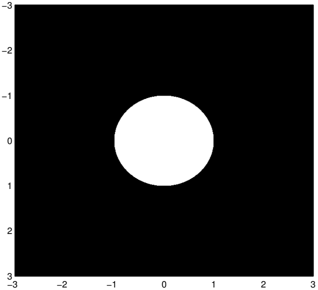
Sampling lattice

Classical sampling in higher dimensions

- Sampling on a lattice



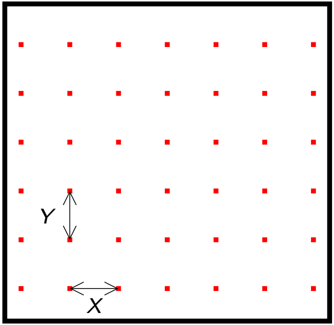
Sampling lattice



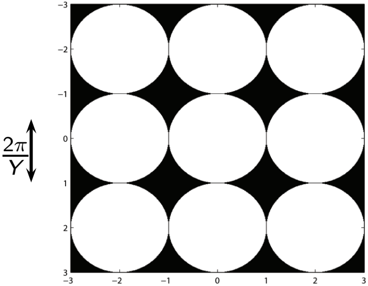
Original spectrum

Classical sampling in higher dimensions

- Sampling on a lattice



Sampling lattice

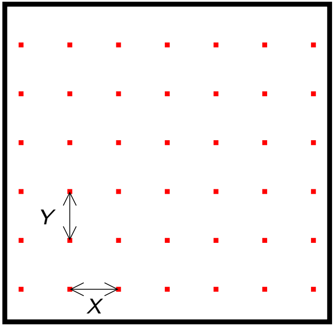


No aliasing in sampled spectrum for

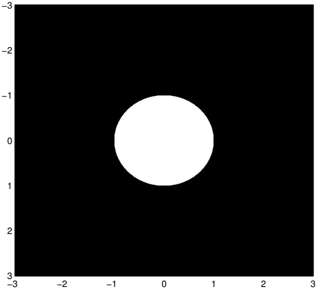
$$X = Y \leq \frac{\pi}{R}$$

Classical sampling in higher dimensions

- Sampling on a lattice



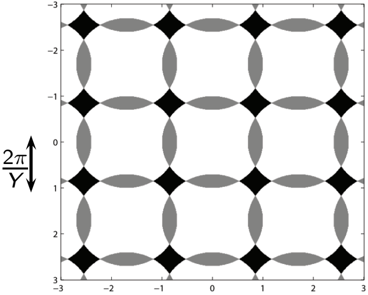
Sampling lattice



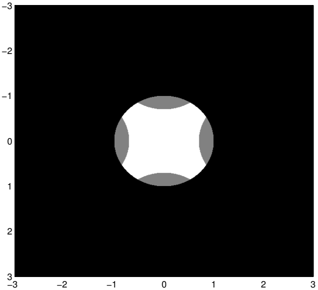
Perfect recovery of original spectrum

Classical sampling in higher dimensions

- Sampling on a lattice



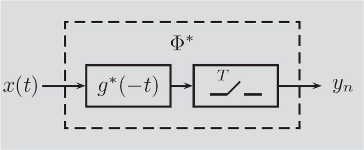
Aliased sampled spectrum



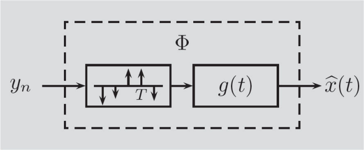
Perfect recovery impossible

- Lattice should be fine enough $\equiv$ Nyquist criterion in $\mathbb{R}^d$

Sampling and Interpolation as Operators



(a) Sampling.



(b) Interpolation.

- Two questions arise:
    1) *How much information* about the signal is contained in the samples?
    2) To what extent can we *recover* the signal from the sequence of samples?

- Classical sampling theorem: If x is in $\mathrm{BL}[-\frac{\pi}{T}, \frac{\pi}{T}]$ and $g(t) = \operatorname{sinc}\left(\frac{\pi t}{T}\right)$ then $\hat{x}(t) = x(t)$

Classical sampling theorem

Theorem (Sampling theorem)

If function x is in $\mathrm{BL}[-\pi/T, \pi/T]$,

$$x(t) \;=\; \sum_{n\in\mathbb{Z}} x(nT)\,\mathrm{sinc}\left(\tfrac{\pi}{T}(t-nT)\right), \qquad t\in\mathbb{R}.$$

- We will see:
  1) Why is this true?
  2) What are the properties of Φ that make this true?
  3) What happens when x is not in $\mathrm{BL}[-\frac{\pi}{T}, \frac{\pi}{T}]$?
  4) Can we use different filters in place of g^* and g?
  5) What properties do $\hat{x}$ have in such a case?

- All answers provided via *Hilbert space interpretation*

Sampling and Interpolation as operators in a Hilbert space

If you think about it...

- Classical sampling is a *linear transform* from *Hilbert space* $\mathcal{L}^2(\mathbb{R})$ to *Hilbert space* $\ell^2(\mathbb{Z})$ that admits a more compact representation
  - *Potentially lossy*: Only bandlimited signals can be recovered from the samples
- Classical interpolation is a *linear transform* from *Hilbert space* $\ell^2(\mathbb{Z})$ to *Hilbert space* $\mathcal{L}^2(\mathbb{R})$
  - Embeds information within the bandlimited subspace of $\mathcal{L}^2(\mathbb{R})$

Other kinds of Sampling and Interpolation

- Typical definition of sampling and interpolation:

$$\text{discrete-time signal } (\ell^2(\mathbb{Z})) \quad \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} \quad \text{continuous-time signal } (\mathcal{L}^2(\mathbb{R}))$$

- It could also be

$$\text{low-rate sequence } (\ell^2(\mathbb{Z})) \quad \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} \quad \text{high-rate sequence } (\ell^2(\mathbb{Z}))$$

- Can be extended to

$$\text{shorter finite-length vector } \mathbb{C}^N \quad \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} \quad \text{longer finite-length vector } \mathbb{C}^M$$

- All the above can be interpreted as *linear operators between two Hilbert spaces*

Sampling and Interpolation Operators

We shall discuss sampling and interpolation in the following cases:

- Finite dimensional vectors
- Sequences in $\ell^2(\mathbb{Z})$
- Functions in $\mathcal{L}^2(\mathbb{R})$

Sampling and Interpolation of finite-dim vectors

Sampling and Interpolating Finite dimensional vectors

- Sampling and interpolation are linear operators between finite dimensional subspaces, *for example,* , $\mathbb{R}^N$ and $\mathbb{R}^M$ (or $\mathbb{C}^N$ and $\mathbb{C}^M$) with $N < M$.
  - Represented by *matrix multiplication*
- Sampling will take M values and produce $N < M$ values
  - Sampling matrix is *fat* - i.e., has more columns than rows
- Interpolation will take N values and produce $M > N$ values
  - Interpolation matrix is *tall* - i.e., has more rows than columns

Sampling and interpolation with orthonormal vectors

Sampling

$$y = \begin{bmatrix} \text{---} & \varphi_0^* & \text{---} \\ \text{---} & \varphi_1^* & \text{---} \\ & \vdots & \\ \text{---} & \varphi_{N-1}^* & \text{---} \end{bmatrix}_{N\times M} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}_{M\times 1} = \Phi^* x\,.$$

where φ_n^* is the n-th row of Φ^*.

Equivalently, $y_n = \langle x, \varphi_n \rangle$.

Here Φ^* is a $N \times M$ (*fat*) matrix, or equivalently an operator:

$$\Phi^* : \mathbb{C}^M \mapsto \mathbb{C}^N$$

We assume φ_n, $n = 0, \ldots, N-1$ to be *orthonormal*

$$\langle \varphi_n, \varphi_k \rangle = \delta_{n-k} \quad \Leftrightarrow \quad \Phi^*\Phi = I\,,\ \text{where } N < M\,.$$

Sampling and interpolation with orthonormal vectors

Sampling

$$y = \begin{bmatrix} \text{---} & \varphi_0^* & \text{---} \\ \text{---} & \varphi_1^* & \text{---} \\ & \vdots & \\ \text{---} & \varphi_{N-1}^* & \text{---} \end{bmatrix}_{N\times M} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}_{M\times 1} = \Phi^* x\,.$$

Since $N < M$ sampling is a lossy operation

Sampling operator Φ^* has max rank N and $M - N$ dimensional null space $\mathcal{N}(\Phi^*)$ with orthogonal complement $S = \mathcal{N}(\Phi^*)^\perp = \text{span}(\{\varphi_n\}_{n=0,\ldots,N-1})$.

When a vector $x \in \mathbb{R}^M$ is sampled *information about the component of x in S is preserved* and is captured by $\Phi^* x$, while the *component in the null space $\mathcal{N}(\Phi^*)$ is lost*. I.e., $y = \Phi^* x_S$.

Sampling and interpolation with orthonormal vectors

Interpolation

$$\widehat{x} = \begin{bmatrix} | & | & & | \\ \varphi_0 & \varphi_1 & \cdots & \varphi_{N-1} \\ | & | & & | \end{bmatrix}_{M \times N} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}_{N \times 1} = \Phi y = \sum_{n=0}^{N-1} y_k \varphi_k \,,$$

where φ_n is the n-th column of Φ.

Since $N < M$, Φ is a *tall matrix*.

As was true for Φ^*, Φ has maximum rank N and the interpolation operator has an N dimensional range $S = \text{span}(\{\varphi_n\}_{n=0,\ldots,N-1})$. This subspace is the same as the orthogonal complement of the null space of the sampling operator,

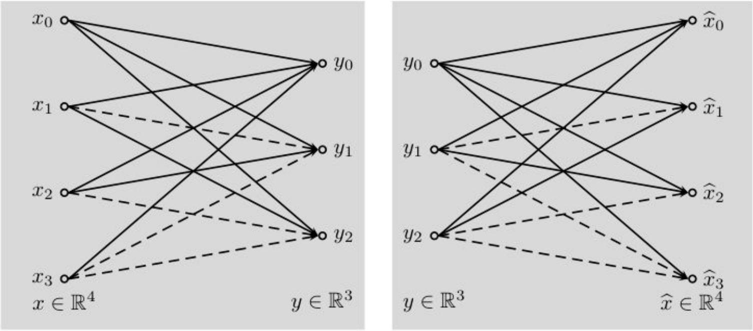
$$\mathcal{R}(\Phi) = S = \mathcal{N}(\Phi^*)^{\perp}.$$

Sampling and Interpolation Operators

Finite dimensional vectors

Sampling and Interpolation in $\mathbb{R}^4$

Let us define sampling of $x \in \mathbb{R}^4$ to obtain three samples $y \in \mathbb{R}^3$, where solid lines have weight $1/2$, while dashed lines have weight $-1/2$; for example, $y_1 = (x_0 - x_1 + x_2 - x_3)/2$.



(a) Sampling. (b) Interpolation.

An example

Sampling and Interpolation in $\mathbb{R}^4$

Consider sampling matrix with orthonormal rows

$$\Phi^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}_{3\times 4},$$

with

$$\mathcal{N}(\Phi^*) = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}, \text{ and } S = \left\{ \alpha_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

For instance

$$\Phi^* \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \Phi^* \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right) = \Phi^* \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \Phi^* \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

An example

Sampling and Interpolation in $\mathbb{R}^4$

Now the interpolator operator can be written as

$$\Phi = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}_{4\times 3},$$

The range of Φ is given by

$$S = \left\{ \alpha_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \;\middle|\; \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C} \right\}.$$

Can verify

$$\mathcal{R}(\Phi) = S = \mathcal{N}(\Phi^*)^\perp.$$

Interpolation followed by sampling

- Since sampling vectors are orthonormal we have

$$\Phi^*\Phi = I\,.$$

- This means that

$$\Phi^*\Phi y = y\,,\ \text{for all } y \in \mathbb{C}^N$$

  i.e., any vector y in the smaller space can be recovered perfectly by interpolating followed by sampling

- In this case, we say sampling and interpolation operators are *consistent*

Sampling followed by interpolation

- Sampling followed by interpolation

$$\Phi\Phi^* = P$$

Here P is an *orthogonal projection operator* because P is

idempotent:

$$P^2 = \Phi\Phi^*\Phi\Phi^* = \Phi(\Phi^*\Phi)\Phi^* = \Phi\Phi^* = P$$

and *self-adjoint*

$$P^* = (\Phi\Phi^*)^* = \Phi\Phi^* = P$$

- Therefore, $\hat{x} = Px$ is the *best least square approximation* of x in $S = (\mathcal{N}(\Phi^*))^\perp = \mathcal{R}(\Phi)$.

$$\hat{x} \;=\; \arg\min_{x_S \in S} \|x - x_S\|, \qquad x - \hat{x} \;\perp\; S.$$

In particular, $\hat{x} = x$ *when* $x \in S$

Aside: In general Hilbert spaces

- Same ideas extend to general Hilbert spaces, e.g., sequences ($\ell^2(\mathbb{Z})$) or functions ($\mathcal{L}^2(\mathbb{R})$)

- Sampling using *orthonormal vectors* in finite-dimensional vector spaces is analogous to classical sampling in $\mathcal{L}^2(\mathbb{R})$ with *sinc-filter* for filtering and reconstructing
  - There subspace $S = \mathrm{BL}[\frac{-\pi}{T}, \frac{\pi}{T}]$
  - To be discussed later

Sampling and interpolation with non-orthonormal vectors

Sampling

$$y = \begin{bmatrix} \text{———} & \widetilde{\varphi}_0^* & \text{———} \\ \text{———} & \widetilde{\varphi}_1^* & \text{———} \\ & \vdots & \\ \text{———} & \widetilde{\varphi}_{N-1}^* & \text{———} \end{bmatrix}_{N\times M} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}_{M\times 1} = \widetilde{\Phi}^* x\,.$$

where $\widetilde{\varphi}_n^*$ is the n-th row of $\widetilde{\Phi}^*$.

As before assume $\widetilde{\Phi}^*$ has full rank N and $M - N$ dimensional null space $\mathcal{N}(\Phi^*)$ with orthogonal complement $\widetilde{S} = \mathcal{N}(\widetilde{\Phi}^*)^\perp = \text{span}(\{\widetilde{\varphi}_n\}_{n=0,\ldots,N-1})$.

When a vector $x \in \mathbb{R}^M$ is sampled *information about the component of x in $\widetilde{S}$ is preserved* and is captured by $\widetilde{\Phi}^* x$, while the *component in the null space $\mathcal{N}(\widetilde{\Phi}^*)$ is lost*.

Sampling with non-orthonormal vectors: Example

Sampling and Interpolation in $\mathbb{R}^4$

Consider sampling matrix with non-orthonormal rows

$$\widetilde{\Phi}^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4},$$

with

$$\mathcal{N}(\widetilde{\Phi}^*) = \left\{ \beta \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}, \text{ and } \widetilde{S} = \left\{ \beta_0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Sampling and interpolation with non-orthonormal vectors

Interpolation

Interpolation represented by $M \times N$ matrix Φ, but is not the adjoint of $\widetilde{\Phi}^*$

Interpolation output lies in

$$S \;=\; \mathcal{R}(\Phi) \;=\; \left\{ \sum_{k=0}^{N-1} \alpha_k \varphi_k \;\middle|\; \alpha \in \mathbb{C}^N \right\}.$$

A possible choice of Φ is the *pseudoinverse of $\widetilde{\Phi}^*$*:

$$\Phi \;=\; \widetilde{\Phi}(\widetilde{\Phi}^* \widetilde{\Phi})^{-1}$$

In such a case $S = \widetilde{S}$

Sampling and interpolation with non-orthonormal vectors

Interpolation followed by sampling

Interpolation followed by sampling is defined by $\widetilde{\Phi}^*\Phi$

We say sampling and interpolation operators are *consistent* when Φ is a right inverse of $\widetilde{\Phi}^*$:

$$\widetilde{\Phi}^*\Phi \;=\; I \qquad \Leftrightarrow \qquad \langle \varphi_n, \widetilde{\varphi}_k \rangle \;=\; \delta_{n-k}.$$

In this case, the vectors are biorthogonal

They form a *biorthogonal pair* of bases for S when $S = \widetilde{S}$, e.g., when Φ is the pseudoinverse of $\widetilde{\Phi}^*$. In this case they are called *ideally matched*.

Sampling and interpolation with non-orthonormal vectors

Interpolation followed by sampling

Consider sampling operator

$$\widetilde{\Phi}^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4}.$$

Two possible consistent interpolators (i.e., right-inverses):

$$\Phi_1 = \frac{1}{2}\begin{bmatrix} 3 & -2 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \\ 1 & -2 & 3 \end{bmatrix}$$

Pseudoinverse. Ideally matched

$$\Phi_2 = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

Not pseudoinverse. Not ideally matched

Sampling and interpolation with non-orthonormal vectors

Sampling followed by interpolation

Sampling followed by interpolation is defined by $P = \Phi\widetilde{\Phi}^*$

When sampling and interpolation operators are *consistent* P is a *projection operator*

$$P^2 \;=\; (\Phi\widetilde{\Phi}^*)\,(\Phi\widetilde{\Phi}^*) \;=\; \Phi\,(\widetilde{\Phi}^*\Phi)\,\widetilde{\Phi}^* \;=\; \Phi\, I\,\widetilde{\Phi}^* \;=\; \Phi\widetilde{\Phi}^* \;=\; P$$

If Φ is the *pseudoinverse* of $\widetilde{\Phi}^*$, then P is self-adjoint and hence is an *orthogonal projection operator*

$$\begin{aligned} P^* \quad &= \quad (\Phi\widetilde{\Phi}^*)^* \;=\; (\widetilde{\Phi}(\widetilde{\Phi}^*\widetilde{\Phi})^{-1}\widetilde{\Phi}^*)^* \;=\; \widetilde{\Phi}((\widetilde{\Phi}^*\widetilde{\Phi})^{-1})^*\widetilde{\Phi}^* \\ &= \quad \widetilde{\Phi}(\widetilde{\Phi}^*\widetilde{\Phi})^{-1}\widetilde{\Phi}^* \;=\; \Phi\widetilde{\Phi}^* \;=\; P \end{aligned}$$

In this case $S = \widetilde{S}$ and sampling and interpolation operators are called *ideally matched*.

Sampling and interpolation with non-orthonormal vectors

Sampling followed by interpolation

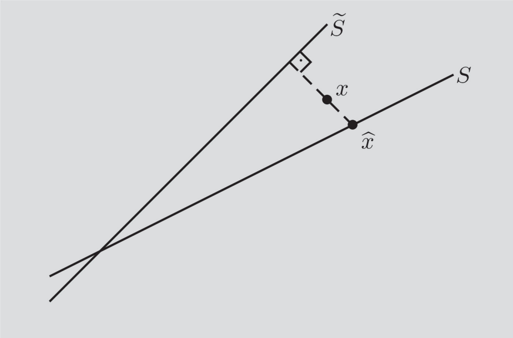
Theorem (Recovery for vectors, nonorthogonal)

Let sampling operator $\widetilde{\Phi}^* : \mathbb{C}^M \to \mathbb{C}^N$ *and interpolation operator* $\Phi : \mathbb{C}^N \to \mathbb{C}^M$ *satisfy consistency condition* $\widetilde{\Phi}^* \Phi = I$. *Then, with* $S = \mathcal{R}(\Phi)$, $\widetilde{S} = \mathcal{N}(\widetilde{\Phi}^*)^\perp$, $P = \Phi \widetilde{\Phi}^*$, *and* $\hat{x} = P\,x$:

1. P *is a projection operator with range* S, *and* $x - \hat{x} \perp \widetilde{S}$. *In particular,* $\hat{x} = x$ *when* $x \in S$.
2. *If* Φ *is the pseudoinverse of* $\widetilde{\Phi}^*$, *then* $S = \widetilde{S}$ *and* P *is an orthogonal projection operator onto* S. *Hence* Px *gives best approximation of* x *in* S.

Sampling and interpolation with non-orthonormal vectors

Subspaces defined in sampling and interpolation



$\widetilde{S}$ represents what can be measured; it is the orthogonal complement of the null space of the sampling operator $\widetilde{\Phi}^*$. S represents what can be reproduced; it is the range of the interpolation operator Φ. When sampling and interpolation are *consistent*, $\Phi\widetilde{\Phi}^*$ is a projection and $x - \widehat{x}$ is orthogonal to $\widetilde{S}$. When furthermore $S = \widetilde{S}$, the projection becomes an orthogonal projection and the sampling and interpolation are *ideally matched*.

Recap

- Sampling and interpolation as *linear operators* between Hilbert spaces
  - Simplest example: Finite dimensional vector spaces

- Sampling matrix Φ^* is *fat* and interpolation matrix Φ is *tall*

- Case 1: Orthogonal sampling vectors (columns of Φ). Then:

$$\Phi^*\Phi = I \quad \text{and} \quad \Phi\Phi^* \text{ is an orthogonal projection operator}$$

- Case 2: Non-orthogonal sampling vectors
  - Sampling $\widetilde{\Phi}^*$ and interpolation Φ are *consistent* when

$$\widetilde{\Phi}^*\Phi \ = \ I$$

  - If Φ is pseudoinverse of $\widetilde{\Phi}^*$ then sampling and interpolation operators are *ideally matched* and $\Phi\widetilde{\Phi}^*$ forms an orthogonal projection

- Read: Chapter 5, sections 5.1-5.2

Sampling and Interpolation of sequences in $\ell^2(\mathbb{Z})$

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

A different Hilbert Space: Sequences in ℓ^2

We will study downsampling and upsampling of sequences in $\ell^2(\mathbb{Z})$ using Hilbert space framework

- *Shift invariant subspaces of ℓ^2*
  A subspace $S \in \ell^2$ is a shift-invariant subspace with respect to shift $L \in \mathbb{Z}^+$ when $x_n \in S$ implies $x_{n-kL} \in S$ for every integer k.
- *Subspace of bandlimited sequences*
  A sequence $x_n \in \ell^2(\mathbb{Z})$ is said to have bandwidth $\omega_0 \in (0, 2\pi]$ if the discrete time Fourier transform $X(e^{j\omega})$ satisfies

  $$X(e^{j\omega}) = 0 \text{ for all } |\omega| > \frac{\omega_0}{2} .$$

  We define then the subspace of ω_0 bandlimited sequences as

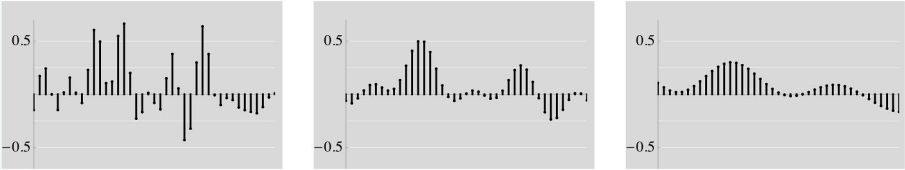
  $$BL[-\omega_0/2, \omega_0/2] = \{x_n \mid x_n \text{ has bandwidth at most } \omega_0\} .$$

  Remark: *Subspace of bandlimited sequences is shift invariant (prove it!)*

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sequences in ℓ^2

Bandlimited sequences

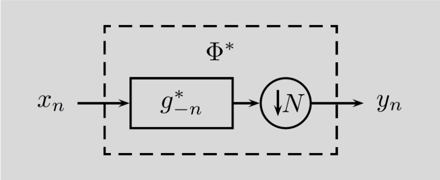


(a) $\mathrm{BL}[-\pi/2, \pi/2]$. (b) $\mathrm{BL}[-\pi/4, \pi/4]$. (c) $\mathrm{BL}[-\pi/8, \pi/8]$.

Sequences in ℓ^2: Sampling

We define as sampling of a sequence $x_n \in \ell^2$ the operation of filtering by g^*_{-n} and downsampling by integer $N > 1$ and we denote it with the operator Φ^*

$$y_n = (\Phi^* x)_n$$



$$\begin{aligned} y_k &= (\Phi^* x)_k = \left(g^*_{-n} *_n x_n\right)\Big|_{n=kN} = \left(\sum_{m \in \mathbb{Z}} x_m \, g^*_{m-n}\right)\Bigg|_{n=kN} \\ &= \sum_{m \in \mathbb{Z}} x_m \, g^*_{m-kN} = \langle x_m, \, g_{m-kN} \rangle_m = \langle x, \, \varphi_k \rangle, \end{aligned}$$

where $\varphi_{k,n} = g_{n-kN}, \qquad n \in \mathbb{Z}.$

Sequences in ℓ^2: Sampling

The sampling operator Φ^* is now an infinite matrix with rows equal to φ^* and its shifts by integer multiples of N.

We assume these rows to be *orthonormal*,

$$\langle \varphi_k, \varphi_\ell \rangle \ = \ \delta_{k-\ell} \qquad \Leftrightarrow \qquad \langle g_{n-kN}, g_{n-\ell N} \rangle_n \ = \ \delta_{k-\ell}$$

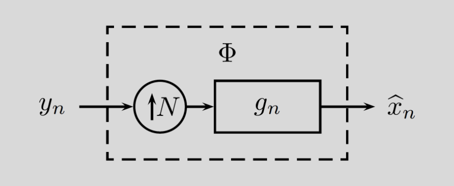
or equivalently, $\Phi^* \Phi = I$.

The sampling operator Φ^* has a nontrivial null space $\mathcal{N}(\Phi^*)$ and $S = \mathcal{N}(\Phi^*)^\perp = \operatorname{span}(\{\varphi_n\}_{n \in \mathbb{Z}})$.

Sequences in ℓ^2: Interpolation

We define as interpolation of a sequence $y_n \in \ell^2$ the operation of upsampling by integer $N > 1$ and filtering by g_n, and we denote it with the operator Φ

$$\widehat{x}_n = (\Phi y)_n$$



$$\widehat{x}_n \;=\; (\Phi y)_n \;=\; \sum_{k \in \mathbb{Z}} y_k \, g_{n-kN} \;=\; \left(\textstyle\sum_{k \in \mathbb{Z}} y_k \varphi_k\right)_n,$$

The interpolation operator Φ is now an infinite matrix with columns equal to φ and its shifts by integer multiples of N.

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sampling and Interpolation in ℓ^2

Set $N = 2$ and choose

$$g_{-n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cdots & 0 & 1 & \boxed{1} & 0 & 0 & \cdots \end{bmatrix} .$$

Then the sampling reads

$$\begin{bmatrix} \vdots \\ \boxed{y_0} \\ y_1 \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \boxed{1} & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \boxed{x_0} \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \Phi^* x \, .$$

For every two inputs samples x_{2k} and x_{2k+1}, we get one output sample $y_k = (x_{2k} + x_{2k+1})\sqrt{(}2)$, and we have

$$\mathcal{N}(\Phi^*) = \left\{ x \in \ell^2(\mathbb{Z}) \mid x_{2k} = -x_{2k+1} \, , k \in \mathbb{Z} \right\} .$$

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sampling and Interpolation in ℓ^2

$$S = \mathcal{N}(\Phi^*)^\perp = \{x \in \ell^2(\mathbb{Z}) \mid x_{2k} = x_{2k+1} \text{ for all } k \in \mathbb{Z}\}$$

$$= \left\{ \cdots + \alpha_{-1} \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ \boxed{0} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \alpha_0 \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \boxed{1} \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \alpha_1 \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \boxed{0} \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \cdots \;\middle|\; \alpha \in \ell^2(\mathbb{Z}) \right\}.$$

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sampling and Interpolation in ℓ^2

We have $g = \frac{1}{\sqrt{2}} \begin{bmatrix} \cdots & 0 & 0 & \boxed{1} & 1 & 0 & \cdots \end{bmatrix}^T$.

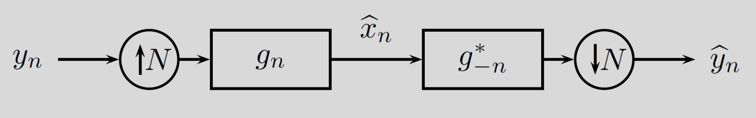
The output of interpolation with $N = 2$ and postfilter g is

$$\begin{bmatrix} \vdots \\ \boxed{\widehat{x}_0} \\ \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} & \vdots & \vdots & \\ \cdots & \boxed{1} & 0 & \cdots \\ \cdots & 1 & 0 & \cdots \\ \cdots & 0 & 1 & \cdots \\ \cdots & 0 & 1 & \cdots \\ & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ \boxed{y_0} \\ y_1 \\ \vdots \end{bmatrix} = \Phi\, y.$$

For every input sample y_k, we get two output samples $x_{2k} = x_{2k+1} = y_k/\sqrt{2}$.

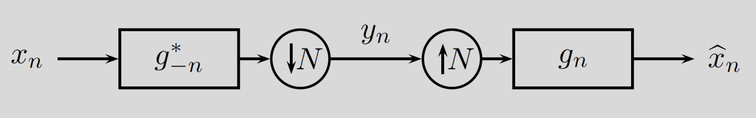
Sampling and Interpolation in $\ell^2(\mathbb{Z})$

- Interpolation followed by sampling



$$\text{Since } \Phi^*\Phi = I \text{ we have } \hat{y}_n = y_n.$$

- Sampling followed by interpolation



$$\Phi\Phi^* = P.$$

As before P is an orthogonal projection operator. Therefore, Px is the best least square approximation of x in S.

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Theorem (Recovery for sequences, orthogonal)

Assume filter g is such that,

$$\langle \varphi_k, \varphi_\ell \rangle \;=\; \delta_{k-\ell} \qquad \Leftrightarrow \qquad \langle g_{n-kN}, g_{n-\ell N} \rangle_n \;=\; \delta_{k-\ell}.$$

Then,

$$\widehat{x}_n \;=\; \sum_{k\in\mathbb{Z}} y_k\, g_{n-kN}, \qquad n \in \mathbb{Z},$$

where

$$y_k \;=\; \sum_{m\in\mathbb{Z}} x_m\, g^*_{m-kN}, \qquad k \in \mathbb{Z},$$

is the best approximation of x in $S = \mathcal{R}(\Phi)$:

$$\widehat{x} \;=\; \arg\min_{x_S \in S} \|x - x_S\|, \qquad x - \widehat{x} \;\perp\; S.$$

In particular, $\widehat{x} = x$ when $x \in S$.

Sequences in $BL[-\omega_0/2, \omega_0/2] \subset \ell^2(\mathbb{Z})$

$$g_n \;=\; \frac{1}{\sqrt{N}}\,\mathrm{sinc}\left(\frac{\pi n}{N}\right), \quad n \in \mathbb{Z}, \qquad \overset{\mathrm{DTFT}}{\longleftrightarrow} \qquad G(e^{j\omega}) \;=\; \begin{cases} \sqrt{N}, & |\omega| \le \pi/N; \\ 0, & \text{otherwise}, \end{cases}$$

Like in continuous time, we have that g is a generator with shift N of $\mathrm{BL}[-\pi/N,\, \pi/N]$ (Prove it!). Moreover, as before, shifted versions are orthonormal,

$$\begin{aligned} \langle g_{n-kN},\, g_{n-\ell N}\rangle_n &= \frac{1}{2\pi}\langle e^{-j\omega k N} G(e^{j\omega}),\, e^{-j\omega \ell N} G(e^{j\omega})\rangle_\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(k-\ell)N} |G(e^{j\omega})|^2\, d\omega \\ &= \frac{N}{2\pi}\int_{-\pi/N}^{\pi/N} e^{-j\omega(k-\ell)N}\, d\omega \;=\; \delta_{k-\ell}. \end{aligned}$$

Sequences in $BL[-\omega_0/2, \omega_0/2] \subset \ell^2(\mathbb{Z})$

Theorem (Projection to bandlimited subspace)

Using sinc filter for g we have

$$\widehat{x}_n \;=\; \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}} y_k \operatorname{sinc}\left(\frac{\pi}{N}(n - kN)\right), \qquad n \in \mathbb{Z},$$

where

$$y_k \;=\; \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} x_n \operatorname{sinc}\left(\frac{\pi}{N}(n - kN)\right), \qquad k \in \mathbb{Z},$$

is the best approximation of x in $\mathrm{BL}[-\pi/N, \pi/N]$:

$$\widehat{x} \;=\; \underset{x_{\mathrm{BL}} \in \mathrm{BL}[-\pi/N, \pi/N]}{\arg\min} \|x - x_{\mathrm{BL}}\|, \qquad x - \widehat{x} \;\perp\; \mathrm{BL}[-\pi/N, \pi/N].$$

In particular, $\widehat{x} = x$ when $x \in \mathrm{BL}[-\pi/N, \pi/N]$.

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sequences in $BL[-\omega_0/2, \omega_0/2] \subset \ell^2(\mathbb{Z})$

Other results from sampling of functions can be generalized:

- Sampling without prefilter
- Sampling with non-orthogonal functions

Summary

- Sampling and Interpolation as linear operators between Hilbert spaces
  - Intuition from finite dimensional Euclidean spaces ($\mathbb{C}^M \rightleftarrows \mathbb{C}^N$)
  - Generalizes to sampling of sequences ($\ell^2(\mathbb{Z}) \rightleftarrows \ell^2(\mathbb{Z})$)

- *Consistency*: Interpolation followed by Sampling is identity

- *Ideally matched*: Sampling followed by Interpolation is orthogonal projection onto $S = \mathcal{R}(\Phi) = \mathcal{N}(\widetilde{\Phi}^*)^\perp$
  - Ideally matched interpolator: Pseudoinverse of Sampling operator
  - For orthonormal vectors, pseudoinverse is the adjoint!

- Reading: Sections 5.1, 5.2, 5.3.1 and parts of 5.3.2 up to Theorem 5.7. Shannon's original paper sections I and II.

Sampling and Interpolation of functions

Shift-Invariant Subspaces of Functions

Definition (Shift-invariant subspace of $\mathcal{L}^2(\mathbb{R})$)

A subspace $S \subset \mathcal{L}^2(\mathbb{R})$ is a *shift-invariant subspace* with respect to shift $T \in \mathbb{R}^+$ when $x(t) \in S$ implies $x(t - kT) \in S$ for every integer k. In addition, $s \in \mathcal{L}^2(\mathbb{R})$ is called a *generator* of S when $S = \overline{\operatorname{span}}(\{s(t - kT)\}_{k \in \mathbb{Z}})$.

Why should you care?
Because bandlimited functions with a given bandwidth form a shift invariant space for all shifts!
Later we will see *splines* which also form shift-invariant spaces

Sampling with Orthonormal Functions

Sampling operator $\Phi^* : \mathcal{L}^2(\mathbb{R}) \mapsto \ell^2(\mathbb{Z})$

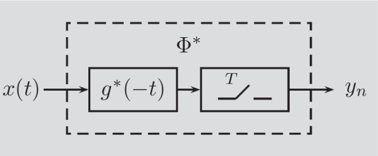


Figure : Sampling $x(t)$ at time instants nT using prefilter $g^*(-t)$

$$y = \Phi^* x \in \ell^2(\mathbb{Z})$$

$$\text{with } y_k = \int_{-\infty}^{\infty} x(\tau)\, g^*(\tau - kT)\, d\tau = \langle x(\tau), g(\tau - kT)\rangle_\tau = \langle x, \varphi_k\rangle$$

where $\varphi_k(t) = g(t - kT)$ and $\varphi_k \in \mathcal{L}^2(\mathbb{R})$

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Sampling with Orthonormal Functions

First assume φ_k are orthonormal in $\mathcal{L}^2(\mathbb{R})$:

$$\langle \varphi_n, \varphi_k \rangle = \delta_{n-k} \qquad \Leftrightarrow \qquad \langle g(t-nT), g(t-kT) \rangle_t = \delta_{n-k}.$$

Sampling operator Φ^* gives inner products with all functions in $\{\varphi_k\}_{k\in\mathbb{Z}}$.

$\mathcal{N}(\Phi^*)$ is null space of Φ^*; the set $\{\varphi_k\}_{k\in\mathbb{Z}}$ spans its orthogonal complement, $S = \mathcal{N}(\Phi^*)^\perp = \overline{\text{span}}(\{\varphi_k\}_{k\in\mathbb{Z}})$,a shift-invariant space

When a function $x \in \mathcal{L}^2(\mathbb{R})$ is sampled, its component in the null space $S^\perp$ has no effect on the output y and is thus completely lost; its component in S is captured by $\Phi^* x$.

Interpolation with Orthonormal Functions

Interpolation: $\hat{x} = \Phi y$

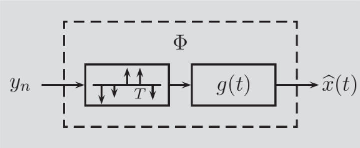


Figure : Interpolating using postfilter $g(t)$

$$\widehat{x}(t) \;=\; \sum_{k\in\mathbb{Z}} y_k\, g(t-kT) \;=\; \left(\textstyle\sum_{k\in\mathbb{Z}} y_k\, \varphi_k\right)(t),$$

Choosing pre- and postfilters related through *time-reversed conjugation* makes the sampling and interpolation operators *adjoints* of each other: for any $x \in \mathcal{L}^2(\mathbb{R})$ and $y \in \ell^2(\mathbb{Z})$,

$$\langle \Phi^* x, y\rangle_{\ell^2} = \langle x, \Phi\, y\rangle_{\mathcal{L}^2}$$

Interpolation followed by Sampling

Interpolation followed by Sampling is represented by $\Phi^*\Phi$

Since functions $\{g(t-kT)\}_{k\in\mathbb{Z}}$ is an *orthonormal set* we have

$$\begin{aligned}
\widehat{y}_n &= \int_{-\infty}^{\infty} \widehat{x}(\tau)\, g^*(\tau - nT)\, d\tau \;=\; \int_{-\infty}^{\infty} \left(\sum_{k\in\mathbb{Z}} y_k\, g(\tau - kT) \right) g^*(\tau - nT)\, d\tau \\
&= \sum_{k\in\mathbb{Z}} y_k \int_{-\infty}^{\infty} g(\tau - kT)\, g^*(\tau - nT)\, d\tau \;=\; \sum_{k\in\mathbb{Z}} y_k\, \delta_{n-k} \;=\; y_n,
\end{aligned}$$

Or in other words

$$\Phi^*\Phi = I$$

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Sampling followed by Interpolation

Sampling followed by Interpolation is represented by $P = \Phi\Phi^*$

Since orthonormality is satisfied P is an orthogonal projection operator

Theorem (Recovery for functions, orthogonal)

If

$$y_k \; = \; \int_{-\infty}^{\infty} x(\tau)\, g^*(\tau - kT)\, d\tau, \qquad k \in \mathbb{Z},$$

then

$$\widehat{x}(t) \; = \; \sum_{k \in \mathbb{Z}} y_k\, g(t - kT), \qquad t \in \mathbb{R},$$

is the best approximation of x *in* $S = \mathcal{R}(\Phi)$*:*

$$\widehat{x} \; = \; \arg\min_{x_S \in S} \|x - x_S\|, \qquad x - \widehat{x} \;\perp\; S.$$

In particular, $\widehat{x} = x$ *when* $x \in S$.

Definition (Bandwidth of function)

A function x is called *bandlimited* when there exists $\omega_0 \in [0, \infty)$ such that

$$X(\omega) = 0 \qquad \text{for all } \omega \text{ with } |\omega| \in (\omega_0/2, \infty).$$

The smallest such ω_0 is called the *bandwidth* of x.

Note: BL functions are *smooth*!

Definition (Subspace of bandlimited functions)

The set of functions in $\mathcal{L}^2(\mathbb{R})$ with bandwidth at most ω_0 is a closed subspace denoted $\mathrm{BL}[-\omega_0/2, \omega_0/2]$.

$$x(t - kT) \quad \overset{\mathrm{FT}}{\longleftrightarrow} \quad e^{-j\omega kT} X(\omega).$$

Hence $\mathrm{BL}[-\omega_0/2, \omega_0/2]$ forms a *shift-invariant subspace*

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Projection to Bandlimited Subspace

Suppose

$$g(t) \;=\; \frac{1}{\sqrt{T}}\,\mathrm{sinc}\left(\frac{\pi t}{T}\right), \quad t \in \mathbb{R}, \qquad \stackrel{\mathrm{FT}}{\longleftrightarrow} \qquad G(\omega) \;=\; \begin{cases} \sqrt{T}, & |\omega| \leq \pi/T \\ 0, & \text{otherwise} \end{cases}$$

Then *g is a generator with shift T of* $\mathrm{BL}[-\pi/T, \pi/T]$

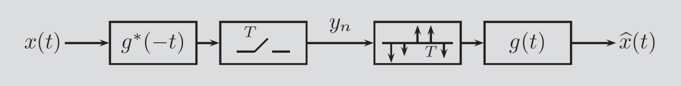
Moreover, shifted versions $\{g(t-kT)\}_k$ are orthonormal

$$\begin{aligned} \langle g(t-nT), g(t-kT)\rangle_t \;&=\; \frac{1}{2\pi}\langle e^{-j\omega nT}G(\omega), e^{-j\omega kT}G(\omega)\rangle_\omega \\ &=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(n-k)T}\,|G(\omega)|^2\,d\omega \\ &=\; \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} e^{-j\omega(n-k)T}\,d\omega \;=\; \delta_{n-k} \end{aligned}$$

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Projection to Bandlimited Subspace

Consider sampling and interpolating with sinc filter $g(t)$



In words, given any function $x(t) \in \mathcal{L}^2(\mathbb{R})$, we do the following:

- Filter using sinc filter $g(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)$
- Sample at time instants nT
- Reconstruct $\hat{x}(t)$ using filter $g(t)$

then $\hat{x}(t)$ is the function in $\mathrm{BL}[-\pi/T, \pi/T]$ that is *closest* to $x(t)$ in $\mathcal{L}^2$ norm

Equivalently *$\hat{x}$ is the orthogonal projection of x onto $\mathrm{BL}[-\pi/T, \pi/T]$*

Theorem (Projection to bandlimited subspace)

$$\widehat{x}(t) \;=\; \tfrac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}} y_k \,\mathrm{sinc}\left(\tfrac{\pi}{T}(t-kT)\right), \qquad t\in\mathbb{R},$$

where

$$y_k \;=\; \tfrac{1}{\sqrt{T}} \int_{-\infty}^{\infty} x(\tau)\,\mathrm{sinc}\left(\tfrac{\pi}{T}(\tau-kT)\right)\,d\tau, \qquad k\in\mathbb{Z},$$

is the best approximation of x in $\mathrm{BL}[-\pi/T, \pi/T]$:

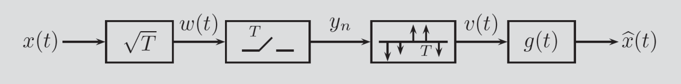
$$\widehat{x} \;=\; \operatorname*{arg\,min}_{x_{\mathrm{BL}}\in\mathrm{BL}[-\pi/T,\pi/T]} \|x - x_{\mathrm{BL}}\|, \qquad x-\widehat{x} \;\perp\; \mathrm{BL}[-\pi/T, \pi/T].$$

In particular, $\widehat{x} = x$ when $x\in\mathrm{BL}[-\pi/T, \pi/T]$.

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Sampling without a prefilter followed by interpolation

A simpler sampling setup: No prefilter



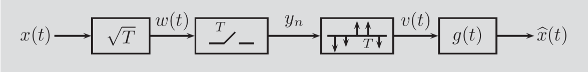
Caveat 1: For a general $x \in \mathcal{L}^2(\mathbb{R})$, we are not guaranteed to have $y \in \ell^2(\mathbb{Z})$. The exact conditions on x to ensure $y \in \ell^2(\mathbb{Z})$ are difficult to state exactly. A sufficient condition is that x is bandlimited with any bandwidth.

Caveat 2: We will use a Dirac comb function to obtain an intuitive understanding of the spectra of Y and V. But Dirac delta functions are not in $\mathcal{L}^2(\mathbb{R})$. An exact derivation can be performed using Poisson summation formula under strong assumptions on decay rates of $x(t)$ and $X(\omega)$.

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Sampling without a prefilter followed by interpolation

A more practical sampling setup: No prefilter



We have

$$v(t) \;=\; \sum_{n\in\mathbb{Z}} w(nT)\delta(t-nT) = s_T(t)\, w(t)$$

where $s_T(t)$ is the *Dirac comb*

$$s_T(t) \;=\; \sum_{n\in\mathbb{Z}} \delta(t-nT) \quad \overset{\text{FT}}{\longleftrightarrow} \quad S_T(\omega) \;=\; \frac{2\pi}{T}\sum_{k\in\mathbb{Z}} \delta\left(\omega - \frac{2\pi}{T}k\right)$$

$$\Rightarrow V(\omega) \;=\; \frac{1}{2\pi}\left(S_T * W\right)(\omega) \;=\; \frac{1}{T}\sum_{k\in\mathbb{Z}} W\left(\omega - \frac{2\pi}{T}k\right) \;=\; \frac{1}{\sqrt{T}}\sum_{k\in\mathbb{Z}} X\left(\omega - \frac{2\pi}{T}k\right)$$

$$\text{Moreover, } \; Y(e^{j\omega T}) = \sum_{n\in\mathbb{Z}} y_n e^{-j\omega nT} = \mathcal{F}\left(\sum_{n\in\mathbb{Z}} y_n \delta(t-nT)\right) = V(\omega)$$

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Sampling without a prefilter followed by interpolation

Now

$$\hat{X}(\omega) \;=\; G(\omega)V(\omega) \;=\; \frac{1}{\sqrt{T}}\sum_{k\in\mathbb{Z}} G(\omega)X\left(\omega - \frac{2\pi}{T}k\right)$$

No *spectral overlaps* if $x \in \mathrm{BL}[-\pi/T, \pi/T]$. Hence $\hat{x} = x$.

Theorem (Sampling theorem)

If function x is in $\mathrm{BL}[-\pi/T, \pi/T]$,

$$x(t) \;=\; \sum_{n\in\mathbb{Z}} x(nT)\,\mathrm{sinc}\left(\tfrac{\pi}{T}(t-nT)\right), \qquad t \in \mathbb{R}.$$

If x has bandwidth ω_0 (i.e., $x \in \mathrm{BL}[-\omega_0/2, \omega_0/2]$) then we need $T < 2\pi/\omega_0$ (*Nyquist interval*). The frequency $\omega_0/2\pi$ is called the *Nyquist rate*.

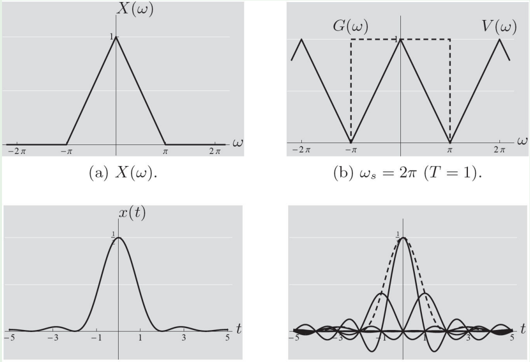
Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Sampling without a prefilter followed by interpolation

Sampling the sinc-squared function

$$x(t) \;=\; \tfrac{1}{2}\,\mathrm{sinc}^2\left(\tfrac{1}{2}\pi t\right)$$

Since $x \in \mathrm{BL}[-\pi,\,\pi]$ Nyquist rate is 2π rad/s



(a) $X(\omega)$. (b) $\omega_s = 2\pi$ $(T = 1)$.

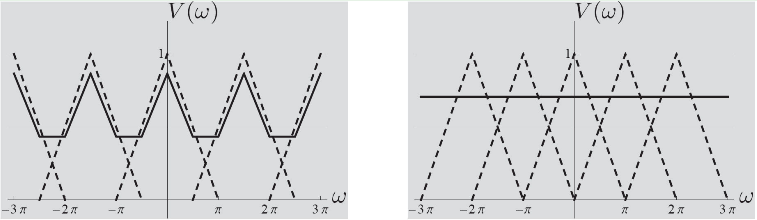
(c) $x(t)$. (d) Terms in sampling expansion.

Sampling and Interpolating functions in $\mathcal{L}^2(\mathbb{R})$

Sampling without a prefilter followed by interpolation

Undersampling the sinc-squared function

Suppose we use a sampling rate $\omega_s < 2\pi$ rad/s we get *aliasing*



(a) $\omega_s = 3\pi/2$ $(T = 4/3)$. (b) $\omega_s = \pi$ $(T = 2)$.

When $T = 2$ the samples are given by

$$x(2n) \;=\; \tfrac{1}{2}\,\mathrm{sinc}^2\,(\pi n) = \tfrac{1}{2}\delta_n$$

Hence spectrum is flat!

Aliasing

Undersampling a sinusoid

Suppose $x(t) = \cos(\omega_0 t) \leftrightarrow \pi\left(\delta(\omega - \omega_0) + \delta(\omega - \omega_0)\right)$. *(Note: $x \notin \mathcal{L}^2(\mathbb{R})$)*
Nyquist rate: $\omega_s = 2\omega_0$. Suppose we sample at half the Nyquist rate. Then,

$$x(nT) = x\left(\frac{2\pi n}{\omega_s/2}\right) = \cos(2n\pi) = 1 \text{ for all } n$$

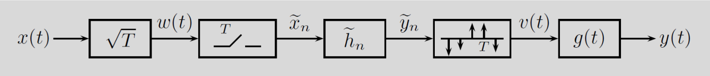
`http://en.wikipedia.org/wiki/Stroboscopic_effect`

Aliasing in images and audio

`http://en.wikipedia.org/wiki/Aliasing`
`http://en.wikipedia.org/wiki/File:Sawtooth-aliasingdemo.ogg`

Aliasing errors often lead to *more perceptible* distortions than errors due to noise even if the errors are of the same $\mathcal{L}^2$ norm (squared error)

CT processing via DSP



Theorem (CT convolution implemented using DT processing)

For $x \in \mathrm{BL}[-\pi/T, \pi/T]$, the continuous-time convolution $y = h * x$ can be computed as shown where postfilter g is the ideal lowpass filter (sinc) and the discrete-time LSI filter $\widetilde{h}$ is given by

$$\widetilde{h}_n = \langle h(t), \operatorname{sinc}\left(\tfrac{\pi}{T}(t - nT)\right)\rangle_t, \qquad n \in \mathbb{Z}.$$

The discrete-time filter input is

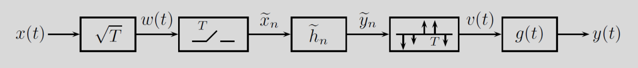
$$\widetilde{x}_n = \sqrt{T}\, x(nT), \qquad n \in \mathbb{Z},$$

and the system output in terms of the discrete-time filter output is

$$y(t) = \sqrt{T} \sum_{n \in \mathbb{Z}} \widetilde{y}_n \operatorname{sinc}\left(\tfrac{\pi}{T}(t - nT)\right), \qquad t \in \mathbb{R}.$$

CT processing via DSP

Proof



We have $$\widetilde{Y}(e^{j\omega}) = \widetilde{H}(e^{j\omega})\widetilde{X}(e^{j\omega}) = \widetilde{H}(e^{j\omega})\frac{1}{\sqrt{T}}\sum_{k\in\mathbb{Z}} X\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right).$$

Hence $$Y(\omega) = G(\omega)V(\omega) = G(\omega)\widetilde{Y}(e^{j\omega T}) = G(\omega)\widetilde{H}(e^{j\omega T})\frac{1}{\sqrt{T}}\sum_{k\in\mathbb{Z}} X\left(\omega - \frac{2\pi}{T}k\right).$$

Since $x \in \mathrm{BL}[-\pi/T, \pi/T]$ and $G(\omega)$ is ideal low pass

$$Y(\omega) = \frac{1}{\sqrt{T}}G(\omega)\widetilde{H}(e^{j\omega T})X(\omega).$$

From defn of $\widetilde{h}$ we have

$$\widetilde{H}(e^{j\omega}) = \frac{1}{T}\sum_{k\in\mathbb{Z}} \sqrt{T}\, G\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right) H\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right).$$

Substituting we get $$Y(\omega) = \frac{1}{T}G^2(\omega)H(\omega)X(\omega) = H(\omega)X(\omega)$$

Approximations to Ideal Filter

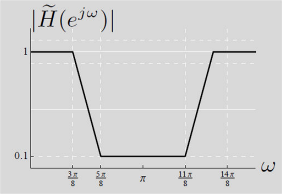
Speech processing in mobile phones

- Humans can't hear about 20kHz. CDs use 44 kHz sampling frequency.
- But passband from 0.3 to 3.4 kHz is sufficient for good quality speech signals
- In mobile phones: $f_s = 8$ kHz or $T = 0.125$ ms with pre and postfilter passband up to 3.4 kHz and high attenuation above 4 kHz
  - Implemented via a combination of analog and digital filters
  - Continuous-time LPF with cutoff at 4 kHz or 8π krad/s; and a discrete filter

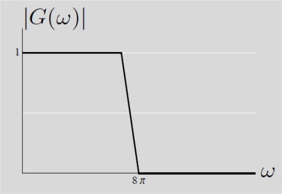
$$\widetilde{H}(e^{j\omega}) = \begin{cases} 1, & \text{for } |\omega| \leq 3\pi/8; \\ 10^{-1}, & \text{for } 5\pi/8 \leq |\omega| < \pi; \\ \text{unspecified}, & \text{else.} \end{cases}$$

Approximations to Ideal Filter

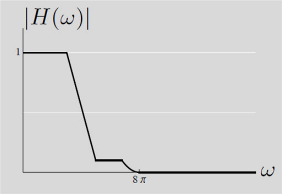
Speech processing in mobile phones



(a) Discrete-time filter.



(b) Continuous-time pre/postfilter.

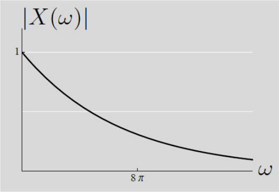


(c) Equivalent continuous-time filter.

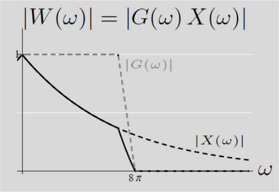
$$H(\omega) = \begin{cases} \widetilde{H}(e^{j\omega T})\, G^2(\omega), & \text{for } |\omega| \leq 8\pi \cdot 10^3; \\ 0, & \text{for } |\omega| > 8\pi \cdot 10^3. \end{cases}$$

Approximations to Ideal Filter

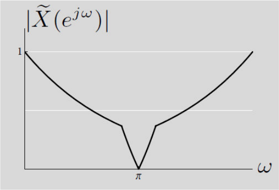
Speech processing in mobile phones



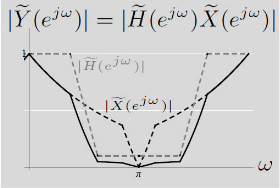
(a) Input spectrum.



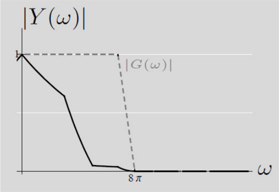
(b) Prefiltering.



(c) Sampling.



(d) Discrete-time filtering.



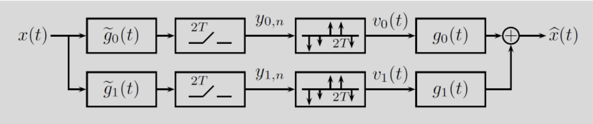
(e) Output spectrum.

Multichannel Sampling

- For $x \in \mathrm{BL}[-\pi/T, \pi/T]$ we need to sample at $\frac{1}{T}$ Hz
  - All DSP must work at $\frac{1}{T}$ Hz
  - May be difficult to implement practically

- Solution: Use multiple channels!
  - If $x_0(t) = x(t)$ and $x_1(t) = x(t - T)$ then both can be sampled at $\frac{1}{2T}$ Hz!
  - Can be generalized

Multichannel Sampling



$$V_i(\omega) = \frac{1}{2T} \sum_{k \in \mathbb{Z}} \widetilde{G}_i \left(\omega + \frac{\pi}{T} k\right) X \left(\omega + \frac{\pi}{T} k\right), \qquad i = 0, 1$$

Since $X(\omega)$ is BL to $[-\pi/T, \pi/T]$, only two spectral components overlap on $[0, \pi/T]$:

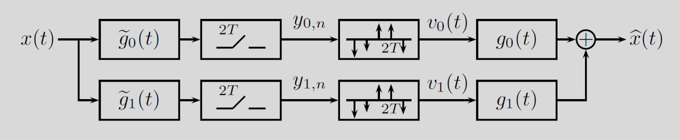
$$\begin{aligned} V_0(\omega) &= \tfrac{1}{2T} \left(\widetilde{G}_0(\omega) X(\omega) + \widetilde{G}_0(\omega - \pi/T) X(\omega - \pi/T)\right), \\ V_1(\omega) &= \tfrac{1}{2T} \left(\widetilde{G}_1(\omega) X(\omega) + \widetilde{G}_1(\omega - \pi/T) X(\omega - \pi/T)\right). \end{aligned}$$

In matrix notation, for $\omega \in [0, \pi/T]$,

$$\begin{bmatrix} V_0(\omega) \\ V_1(\omega) \end{bmatrix} = \tfrac{1}{2T} \begin{bmatrix} \widetilde{G}_0(\omega) & \widetilde{G}_0(\omega - \pi/T) \\ \widetilde{G}_1(\omega) & \widetilde{G}_1(\omega - \pi/T) \end{bmatrix} \begin{bmatrix} X(\omega) \\ X(\omega - \pi/T) \end{bmatrix} = \widetilde{G}(\omega) \begin{bmatrix} X(\omega) \\ X(\omega - \pi/T) \end{bmatrix}$$

As long as *$\widetilde{G}(\omega)$ is nonsingular on the interval $[0, \pi/T]$*, we can recover $X(\omega)$ by choosing $G_0(\omega)$ and $G_1(\omega)$ appropriately.

Multichannel Sampling



Periodic nonuniform sampling

Suppose $x \in \mathrm{BL}[-\pi, \pi]$ and $T = 1$. Choose $\widetilde{G}_0(\omega)$ and $\widetilde{G}_1(\omega)$ to be identity and delay filters:

$$\widetilde{G}_0(\omega) = 1, \qquad \widetilde{G}_1(\omega) = e^{-j\omega\tau}.$$

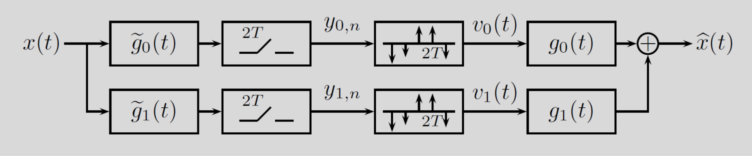
Substituting we get,

$$\begin{bmatrix} V_0(\omega) \\ V_1(\omega) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ e^{-j\omega\tau} & e^{-j(\omega-\pi)\tau} \end{bmatrix} \begin{bmatrix} X(\omega) \\ X(\omega - \pi) \end{bmatrix}.$$

For $\tau \in (0, 2)$ we have $\det(\widetilde{G}(\omega)) = \frac{1}{4} e^{-j\omega\tau}(e^{j\pi\tau} - 1) \neq 0$, $\widetilde{G}(\omega)$ is invertible. Inversion becomes arbitrarily ill-conditioned as τ approaches 0 or 2, as expected.

We have proved a *non-uniform sampling theorem*!
Note: $\tau = 1$ leads to usual sampling of $x(t)$ with even and odd samples in separate channels as we saw earlier

Multichannel Sampling



Sampling function and derivative

Again suppose $x \in \mathrm{BL}[-\pi, \pi]$ and $T = 1$. Choose $\widetilde{G}_0(\omega)$ and $\widetilde{G}_1(\omega)$ to be identity and derivative filters:

$$\widetilde{G}_0(\omega) \;=\; 1, \qquad \widetilde{G}_1(\omega) \;=\; j\omega.$$

In this case

$$\begin{bmatrix} V_0(\omega) \\ V_1(\omega) \end{bmatrix} \;=\; \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j\omega & j(\omega - \pi) \end{bmatrix} \begin{bmatrix} X(\omega) \\ X(\omega - \pi) \end{bmatrix}.$$

And $\det(\widetilde{G}(\omega)) = -\frac{1}{4} j\pi$, is a nonzero constant, making the system invertible

Hence *a bandlimited function can be reconstructed from twice undersampled versions of the function and its derivative!!*

Multichannel Sampling

Theorem (Multichannel sampling (a.k.a. Papoulis' generalized sampling))

Let x belong to $\mathrm{BL}[-\omega_0/2, \omega_0/2]$, and let T be a sampling period with $T < 2\pi/\omega_0$. Consider an N-channel system with filters $\widetilde{g}_i$, $i = 0, 1, \ldots, N-1$, followed by uniform sampling with period NT. A necessary and sufficient condition for recovery of x is that the matrix

$$\widetilde{G}(\omega) = \begin{bmatrix} \widetilde{G}_0(\omega) & \widetilde{G}_0(\omega + \frac{2\pi}{NT}) & \cdots & \widetilde{G}_0(\omega + \frac{2\pi(N-1)}{NT}) \\ \widetilde{G}_1(\omega) & \widetilde{G}_1(\omega + \frac{2\pi}{NT}) & \cdots & \widetilde{G}_1(\omega + \frac{2\pi(N-1)}{NT}) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{G}_{N-1}(\omega) & \widetilde{G}_{N-1}(\omega + \frac{2\pi}{NT}) & \cdots & \widetilde{G}_{N-1}(\omega + \frac{2\pi(N-1)}{NT}) \end{bmatrix}$$

be nonsingular for $\omega \in [0, \frac{2\pi}{NT}]$.

Sampling and Interpolating Bandlimited Stochastic Processes

Theorem (Sampling for continuous-time stochastic processes)

Let x *be a WSS continuous-time stochastic process with autocorrelation function* $a_{\mathrm{x}} \in \mathrm{BL}[-\omega_0/2, \omega_0/2]$*. For any* $T \leq 2\pi/\omega_0$*,*

$$\mathrm{x}(t) \;=\; \sum_{k \in \mathbb{Z}} \mathrm{x}(nT)\, \mathrm{sinc}\left(\tfrac{\pi}{T}(t - nT)\right) \qquad \text{for all } t \in \mathbb{R},$$

in the mean-square sense, meaning

$$\lim_{N \to \infty} \mathrm{E}\left[\, \left| \mathrm{x}(t) - \textstyle\sum_{k=-N}^{N} \mathrm{x}(nT)\, \mathrm{sinc}\left(\tfrac{\pi}{T}(t - nT)\right)^2 \right| \,\right] \;=\; 0 \qquad \text{for all } t \in \mathbb{R}.$$

Convergence in mean-square implies convergence in probability.

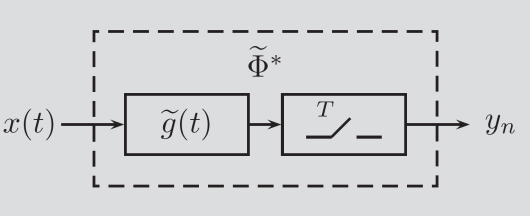
Recap

- Sampling and interpolation as linear transforms between Hilbert spaces
- Sampling and interpolation of functions using *orthonormal* sampling functions

$$\langle \varphi_n, \varphi_k \rangle = \delta_{n-k} \qquad \Leftrightarrow \qquad \langle g(t-nT), g(t-kT) \rangle_t = \delta_{n-k}$$

  - Sampling and interpolation are *adjoints* of each other
  - Consistency: $\Phi^* \Phi = I$
  - Ideally matched: $\Phi \Phi^* = P$ is *orthogonal projection* onto $S = \mathcal{R}(\Phi) = \overline{\mathrm{span}}(\{\varphi_k\}_{k \in \mathbb{Z}})$
- Sampling without prefilter; CT convolution via discrete processing; Multichannel sampling
- Coming up: Sampling with non-orthogonal functions, Sampling and Interpolation in $\ell^2(\mathbb{Z})$, Other topics

Sampling



$$y_k \;=\; \int_{-\infty}^{\infty} x(\tau)\, \widetilde{g}(kT - \tau)\, d\tau \;=\; \langle x,\, \widetilde{\varphi}_k \rangle$$

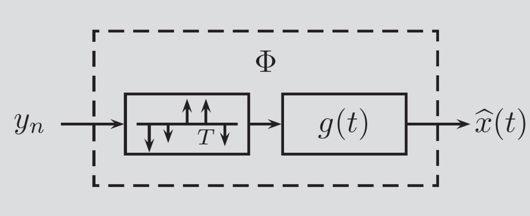
where $\widetilde{\varphi}_k(t) \;=\; \widetilde{g}^*(kT - t)$.

Suppose shifts of $\tilde{g}$ are not orthogonal

Sampling operator $\widetilde{\Phi}^*$ and $\widetilde{S} = \mathcal{N}(\widetilde{\Phi}^*)^\perp = \overline{\operatorname{span}}(\{\widetilde{\varphi}_k\}_{k\in\mathbb{Z}})$

Sampling and Interpolating with non-orthogonal functions

Interpolation



Interpolation

$$\widehat{x}(t) = \sum_{k\in\mathbb{Z}} y_k g(t - kT) = \left(\sum_{k\in\mathbb{Z}} y_k \varphi_k\right)(t)$$

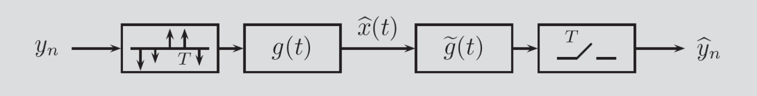
Operator notation

$$\widehat{x} = \Phi y$$

As before let $S = \mathcal{R}(\Phi)$

Sampling and Interpolating with non-orthogonal functions

Interpolation followed by sampling



For *consistency* we need

$$\widetilde{\Phi}^* \Phi \;=\; I \qquad \Leftrightarrow \qquad \langle \varphi_k, \widetilde{\varphi}_n \rangle \;=\; \delta_{k-n}$$

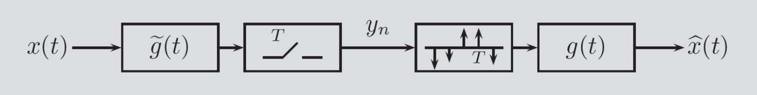
or equivalently,

$$\langle g(t-kT), \widetilde{g}^*(nT-t) \rangle_t \;=\; \delta_{k-n},$$

i.e., the vectors are *biorthogonal*.

We require shifts of g to be orthogonal to time reversed and conjugated shifts of $\tilde{g}$

Sampling followed by interpolation



Operator

$$P = \Phi\widetilde{\Phi}^*$$

forms a projection whenever consistency condition holds, i.e., Φ is a right inverse of $\widetilde{\Phi}^*$

Forms an orthogonal projection when Φ is the "pseudoinverse" of $\widetilde{\Phi}^*$. In this case, $S = \widetilde{S}$, i.e., $\mathcal{R}(\Phi) = \mathcal{N}(\widetilde{\Phi}^*)^\perp$ and we say sampling and interpolation operators are *ideally matched*

- Note: Identifying pseudoinverses of an operator on an infinite-dimensional space is non-trivial. In practice we only verify the $S = \widetilde{S}$ condition, in which case $\{\varphi_n : n \in \mathbb{Z}\}$ and $\{\widetilde{\varphi}_n : n \in \mathbb{Z}\}$ form a biorthogonal pair of bases for S.

Sampling and Interpolating with non-orthogonal functions

Theorem (Recovery for functions, nonorthogonal)

Suppose sampling prefilter $\widetilde{g}$ *and interpolation postfilter* g *satisfy consistency condition. Then,*

$$\widehat{x}(t) = \sum_{k\in\mathbb{Z}} y_k \, g(t - kT), \qquad t \in \mathbb{R},$$

where

$$y_k = \int_{-\infty}^{\infty} x(\tau)\, \widetilde{g}(kT - \tau)\, d\tau, \qquad k \in \mathbb{Z},$$

satisfies $\widehat{x} = P\,x$*, with* $P = \Phi\widetilde{\Phi}^*$*. Furthermore:*

1. P *is a projection operator with range* $S = \mathcal{R}(\Phi)$*, and* $x - \widehat{x} \perp \widetilde{S} = \mathcal{N}(\widetilde{\Phi}^*)^{\perp}$*. In particular,* $\widehat{x} = x$ *when* $x \in S$*.*
2. *If* Φ *is the "pseudoinverse" of* $\widetilde{\Phi}^*$*,* P *is an orthogonal projection operator and* $S = \widetilde{S}$*.*

Sampling and Interpolating with non-orthogonal functions

Consistent sampling and interpolation filters

Suppose $T = 1$ and the postfilter is

$$g(t) \; = \; \begin{cases} 1 - |t|, & \text{for } |t| < 1; \\ 0, & \text{otherwise.} \end{cases}$$

$S = \overline{\text{span}}(\{g(t-k)\}_{k \in \mathbb{Z}})$, is a shift-invariant subspace with respect to integer shifts. Qn: *What is S?*
Several choices for $\widetilde{g}$ satisfy consistency condition:

$$\langle g(t-kT), \widetilde{g}^*(nT-t) \rangle_t \; = \; \delta_{k-n}$$

Suppose we choose $\widetilde{g}$ of form

$$\widetilde{g}(t) \; = \; \begin{cases} a(b - |t|), & \text{for } |t| < 1/2; \\ 0, & \text{otherwise.} \end{cases}$$

Sampling and Interpolating with non-orthogonal functions

Consistent sampling and interpolation filters

We need

$$
\begin{aligned}
1 &= \langle g(t), \widetilde{g}^*(-t)\rangle_t = \int_{-1/2}^{1/2} (1-|t|)\, a(b-|t|)\, dt = \frac{1}{12} a(9b-2), \\
0 &= \langle g(t), \widetilde{g}^*(1-t)\rangle_t = \int_{1/2}^{1} (1-t)\, a(b-(1-t))\, dt = \frac{1}{24} a(3b-1).
\end{aligned}
$$

Other constraints are met automatically because $\widetilde{g}$ and g have finite supports.
Gives solution $a = 12$, $b = 1/3$, or

$$
\widetilde{g}(t) = \begin{cases} 4 - 12|t|, & \text{for } |t| < 1/2; \\ 0, & \text{otherwise.} \end{cases}
$$

Note: *Not ideally matched* because $\widetilde{S} = \overline{\text{span}}(\{\widetilde{g}(t-k)\}_{k\in\mathbb{Z}}) \neq S$ (e.g., $\widetilde{g}$ is not continuous while all functions in S are continuous)

Sampling and Interpolating with non-orthogonal functions

Ideally matched sampling and interpolation filters

To ensure $S = \widetilde{S}$ we just need to choose $\widetilde{g}$ such that $\widetilde{\varphi}_0$ is in S, since S and $\widetilde{S}$ are shift-invariant spaces with shift T. Let

$$\widetilde{g}(t) \;=\; \sum_{\ell\in\mathbb{Z}} \alpha_\ell\, g^*(-t-\ell T)$$

Consistency condition becomes

$$\begin{aligned}
\delta_k \quad &= \quad \langle g(t-kT),\, \widetilde{g}^*(-t)\rangle_t \\
&= \quad \sum_{\ell\in\mathbb{Z}} \alpha_\ell \,\langle g(t-kT),\, g(t-\ell T)\rangle_t \\
&= \quad \sum_{\ell\in\mathbb{Z}} \alpha_\ell\, a_{\ell-k}
\end{aligned}$$

where a_m denotes autocorrelation sequence

$$a_m \;=\; \langle g(t),\, g(t-m)\rangle, \qquad m\in\mathbb{Z}.$$

Sampling and Interpolating with non-orthogonal functions

Ideally matched sampling and interpolation filters

To solve for α rewrite as convolution:

$$\delta_k = (\alpha_\cdot * a_{-\cdot})_k$$

In z-transform domain, we get

$$\alpha(z)A(z^{-1}) \;=\; 1.$$

Substituting $A(z) \;=\; (z + 4 + z^{-1})/6$ we get

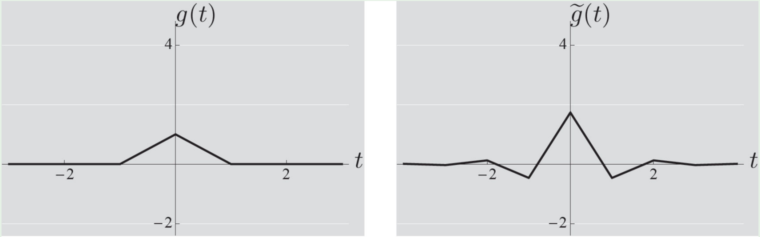
$$\begin{aligned}
\alpha(z) \quad &= \quad \frac{1}{A(z^{-1})} \;=\; \frac{6}{z^{-1} + 4 + z} \;=\; \frac{6c}{(1 + cz^{-1})(1 + cz)} \\
&= \quad \frac{6c}{1 - c^2}\left(\frac{1}{1 + cz^{-1}} - \frac{cz}{1 + cz}\right),
\end{aligned}$$

where $c = 2 - \sqrt{3}$. Inverting we get

$$\alpha_k \;=\; \frac{6c}{1 - c^2}(-c)^{|k|}, \qquad k \in \mathbb{Z},$$

Sampling and Interpolating with non-orthogonal functions

Ideally matched sampling and interpolation filters



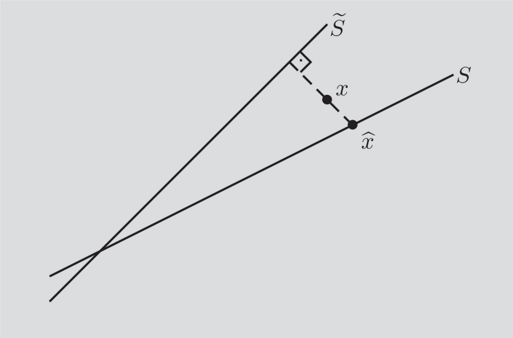
(a) Interpolation postfilter g. (b) Sampling prefilter $\widetilde{g}$.

Ideally matched sampling prefilter lies in shift invariant space generated by conjugated and time-reversed version of interpolation postfilter

In this case $P = \Phi\widetilde{\Phi}^*$ implements orthogonal projection onto S, the space of piecewise-linear and continuous functions that are smooth everywhere except at the integers.
Will be generalized to *splines* later.

Sampling and interpolation with non-orthonormal vectors

Subspaces defined in sampling and interpolation



$\widetilde{S}$ represents what can be measured; it is the orthogonal complement of the null space of the sampling operator $\widetilde{\Phi}^*$. S represents what can be reproduced; it is the range of the interpolation operator Φ. When sampling and interpolation are *consistent*, $\Phi\widetilde{\Phi}^*$ is a projection and $x - \widehat{x}$ is orthogonal to $\widetilde{S}$. When furthermore $S = \widetilde{S}$, the projection becomes an orthogonal projection and the sampling and interpolation are *ideally matched*.

Other examples of sampling

- *Non-uniform sampling* of bandlimited signals
  - Perfect reconstruction conditions and algorithms (e.g., POCS)

- Sampling non-bandlimited signals
  - Sparse discrete signals: *Compressed sensing*

    $$y = \Phi^* x + w$$

    where x is *sparse*. Key difference: *non-convex constraint*
  - Sparse continuous signals: *Finite rate of innovation*
  - Stochastic spatial fields: *Kriging*, an interpolation technique that uses specific assumptions on the correlation structure of the stochastic process

- Sampling for mobile sensing
  - Designing sensor trajectories that minimize sensor movement
  - Spatial anti-aliasing via time-domain filtering