Dynamical Systems
for Engineers: Exercise Set 11, Solutions

Exercise 1

The first-order continuous-time system (from S. Strogatz, Nonlinear Dynamics and Chaos, Addison-Wesley, 1994, p.46)

$$\dot{x} = r - x - e^{-x}$$

undergoes a bifurcation as r is varied.

The equilibrium points satisfy $F(x) = r - x - e^{-x} = 0$. But now we run into a difficulty as we cannot find the fixed points explicitly as a function of r. Instead we adopt a geometric approach. Two functions $r - x$ and e^{-r} have much more familiar graphs than their difference $r - x - e^{-x}$. So we plot $r - x$ and e^{-x} on the same picture (Figure 1(left)). At the intersection between the curves $r - x$ and e^{-x}, we have $r - x = e^{-x}$ and so $F(x) = 0$. The intersection gives therefore all the equilibrium point(s) of the system. This picture also allows us to read off the direction of flow on the x-axis: the flow goes to the right when the line $r - x$ lies above the curve e^{-x}, because $\dot{x} = r - x - e^{-x} > 0$. Hence, the equilibrium point on the right is stable, and the one on the left is unstable.

Now imagine we start decreasing the parameter r. The line $r - x$ slides down and the equilibrium points approach each other. At some critical value $r = r_c$, the line $r - x$ becomes tangent to the curve e^{-x} and the equilibrium points coalesce in a fold, or saddle-node bifurcation (Figure 1(center)). For r below this critical value, the line $r - x$ lies below the curve e^{-x} and there are no fixed points (Figure 1(right)).

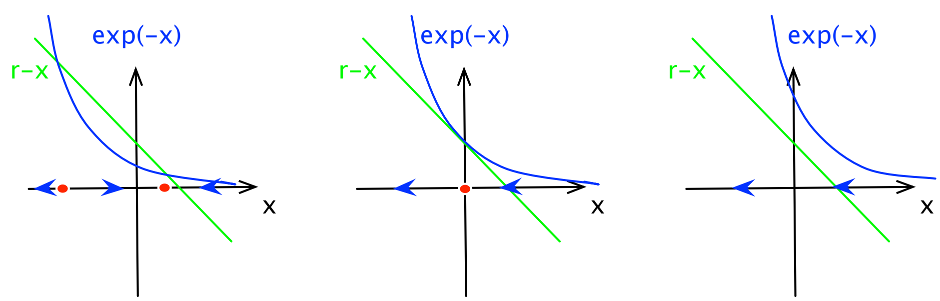

Figure 1: Intersection of the cruves $r - x$ and e^{-x} for different values of r.

We see that for $r > r_0$, there are two equilibrium points, one asymptotically stable, the other unstable, and for $r < r_0$ there is none. One can therefore suspect that there is a fold bifurcation at $(\overline{x}_0, r_0)$. To find the bifurcation point r_0, we impose the condition that the graphs of $r - x$ and e^{-x} intersect tangentially, i.e. the point $(\overline{x}_0, r_0)$ such that

$$\begin{aligned} F(\overline{x}_0, r_0)) &= 0 \\ \frac{\partial F}{\partial x}(\overline{x}_0, r_0)) &= 0 \end{aligned}$$

i.e.

$$\begin{aligned} r_0 - \overline{x}_0 - e^{-\overline{x}_0} &= 0 \\ -1 + e^{-\overline{x}_0} &= 0 \end{aligned}$$

which yields that $(\overline{x}_0, r_0) = (0, 1)$. Since in addition,

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2}(\overline{x}_0, r_0)) &= -e^{-\overline{x}_0} = -1 \neq 0 \\ \frac{\partial F}{\partial r}(\overline{x}_0, r_0)) &= 1 \neq 0 \end{aligned}$$

we have proven that the system undergoes a fold bifurcation at $(0, 1)$.

Exercise 2

To verify the existence of Flip bifurcation, one applies the corresponding theorem. With $F(x, \mu) = -\mu \text{Arctan}(x)$, we find that $x = 0$ is the only fixed point for all $\mu \in \mathbb{R}$, and since

$$\frac{\partial F}{\partial x}(x, \mu) = -\frac{\mu}{1 + x^2}$$

we have that for $(\overline{x}_0, \mu_0) = (0, 1)$,

$$\begin{aligned}
F(\overline{x}_0, \mu_0) &= \overline{x}_0 \\
\frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= -1 \\
\left[\frac{\partial^2 F}{\partial \mu \partial x} + \frac{1}{2}\left(\frac{\partial F}{\partial \mu}\right)\left(\frac{\partial^2 F}{\partial x^2}\right)\right](\overline{x}_0, \mu_0) = \alpha &= -1 \neq 0 \\
\frac{1}{6}\frac{\partial^3 F}{\partial x^3}(\overline{x}_0, \mu_0) + \left(\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0)\right)^2 = \beta &= \frac{1}{3} \neq 0.
\end{aligned}$$

showing the system undergoes a flip bifurcation at $(0, 1)$. One can check that $\overline{x} = 0$ is stable for $\mu < 1$ and unstable for $\mu > 1$ (and one could compute that in addition,there is an asymptotically stable 2-cycle), hence the flip bifurcation is supercritical.

Exercise 3

The origin is an equilibrium of

$$\begin{aligned}
\dot{x_1} &= \mu x_1 - x_2 + x_1 x_2^2 \\
\dot{x_2} &= x_1 + \mu x_2 + x_2^3
\end{aligned}$$

for all $\mu = 0$. The Jacobian at the origin is

$$\frac{\partial F}{\partial x}((0, 0), \mu) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

hence its eigenvalues are $\mu \pm j$. For $\mu = \mu_0 = 0$, both eigenvalues are thus imaginary. Moreover, for all μ,

$$\frac{d\Re(\lambda(\mu))}{d\mu} = 1, \tag{1}$$

and we can conclude that system undergoes an Andronov-Hopf bifurcation at $((0, 0), 0)$ since the last condition is assumed to be valid in this course.

To decide about the sub/supercritical nature of the bifurcation, let

$$\begin{aligned}
r &= \left(x_1^2 + x_2^2\right)^{1/2} &(2) \\
\varphi &= \arctan\left(\frac{x_2}{x_1}\right). &(3)
\end{aligned}$$

Then the system becomes

$$\begin{aligned}
\dot{r} &= \mu r + r^2 x_2^2 &(4) \\
\dot{\varphi} &= 1. &(5)
\end{aligned}$$

(4) means that $\dot{r} \geq \mu r$ for the entire system. Therefore, for $\mu > 0$, the radius r of the solution grows exponentially fast and tends to infinity if $r(0) > 0$. Therefore the system definitely does not

contain any closed orbits for $\mu > 0$ and has only an unstable equilibrium at the origin. As a result, we know that the bifurcation is not supercritical: we cannot have a stable limit cycle smoothly emerging from a stable equilibrium.

Now the bifurcation could also be degenerate, but this is not the case here (this can be checked by the fourth criterion that we did not evaluate). If this was the case the origin would have to be a center for $\mu = 0$, but for $\mu = 0$ we have $\dot{r} = r^2 x_2^2 > 0$, so no closed orbits can exist either.

By elimination we have found that the bifurcation must be subcritical: an unstable limit cycle surrounds the stable equilibrium for $\mu < 0$ and for $\mu \to 0$, it encloses the equilibrium and makes the system globally unstable. You can also observe why subcritical bifurcations are sometimes called catastrophic bifurcations: they can make a stable equilibrium globally unstable – all of a sudden all solutions "explode" to infinity upon perturbation of the bifurcation parameter.