path cylinders $C_i$ with bases $\bar{E}_i^{n-1}$, $\bar{E}_{i+1}^{n-1}$, $\bar{E}_2^{n-1}=\bar{E}_0^{n-1}$, which are transversal to $\gamma$ at $p_i$, $p_{i+1}$, respectively. Relative to orbits in $C_i$, let $T_i: \bar{E}_i^{n-1} \to \bar{E}_{i+1}^{n-1}$, $i=0, 1$, be the maps taking $p'$ in $\bar{E}_i^{n-1}$ into the point $q'$ of the intersection of the orbit through $p'$ with $\bar{E}_{i+1}^{n-1}$. By choosing the diameter of $E_0^{n-1}$ sufficiently small, one can be sure there is a closed $(n-1)$-cell $\Gamma$ in $E_0^{n-1}$ such that the composite map $T=T_2 \circ T_1$ is a homeomorphism of $\Gamma \subset E_0^{n-1}$ into a closed $(n-1)$-cell in $E_0^{n-1}$. The union of the paths $p'Tp'$ is the desired path ring. This proves the lemma.

It may be that a solution of an autonomous equation is not defined for all $t$ in $R$ as the example $\dot{x}=x^2$ shows. In the applications, one is usually only interested in studying the behavior of the solutions in some bounded set $G$ and it is very awkward to have to continually speak of the domain of definition of a solution. We can avoid this situation by replacing the original differential equation by another one for which all solutions are defined on $(-\infty, \infty)$ and the paths defined by the solutions of the two coincide inside $G$. When the paths of two autonomous differential equations coincide on a set $G$, we say the differential equations are *equivalent on* $G$.

LEMMA 7.6. If $f$ in (7.1) is defined on $R^n$ and $G \subset R^n$ is open and bounded, there exists a function $g\colon R^n \to R^n$ such that $\dot{x}=g(x)$ is equivalent to (7.1) on $G$ and the solutions of this latter equation are defined on $(-\infty, \infty)$.

PROOF. If $f=(f_1, \ldots, f_n)$, we may suppose without loss of generality that $G \subset \{x\colon |f_j(x)| \leqq 1, j=1, 2, \ldots, n\}$. Define $g=(g_1, \ldots, g_n)$ by $g_j=f_j\phi_j$, where each $\phi_j$ is defined by

$$\phi_j(x)=\begin{cases} 1 & \text{if } |f_j(x)| \leqq 1, \\ \dfrac{1}{f_j(x)} & \text{if } f_j(x)>1, \\ -\dfrac{1}{f_j(x)} & \text{if } f_j(x)<-1. \end{cases}$$

Corollary 6.3 implies that $g$ satisfies the conditions of the lemma since $|g(x)|$ is bounded in $R^n$.

## I.8. Autonomous Systems—Limit Sets, Invariant Sets

In this section we consider system (7.1) and suppose $f$ satisfies enough conditions on $R^n$ to ensure that the solution $\phi(t, p)$, $\phi(0, p)=p$, is defined for all $t$ in $R$ and all $p$ in $R^n$ and satisfies the conditions (i)–(iii) listed at the beginning of Section I.7.

The orbit $\gamma(p)$ of (7.1) through $p$ is defined by $\gamma(p) = \{x: x = \phi(t, p), -\infty < t < \infty\}$. If $q$ belongs to $\gamma(p)$, then $\gamma(q) = \gamma(p)$ as remarked earlier. The positive semiorbit through $p$ is $\gamma^+(p) = \{x: x = \phi(t, p), t \geqq 0\}$ and the negative semiorbit through $p$ is $\gamma^-(p) = \{x: x = \phi(t, p), t \leqq 0\}$. If we do not wish to distinguish a particular point on an orbit, we will write $\gamma, \gamma^+, \gamma^-$ for the orbit, positive semiorbit, negative semiorbit, respectively.

The positive or $\omega$-limit set of an orbit $\gamma$ of (7.1) is the set of points in $R^n$ which are approached along $\gamma$ with increasing time. More precisely, a point $q$ belongs to the *$\omega$-limit set* or *positive limit set* $\omega(\gamma)$ of an orbit $\gamma$ if there exists a sequence of real numbers $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that $\phi(t_k, p) \to q$ as $k \to \infty$. Similarly, a point $q$ belongs to the *$\alpha$-limit set* or *negative limit set* $\alpha(\gamma)$ if there is a sequence of real numbers $\{t_k\}$, $t_k \to -\infty$ as $k \to \infty$ such that $\phi(t_k, p) \to q$ as $k \to \infty$.

It is easy to prove that equivalent definitions of the $\omega$-limit set and $\alpha$-limit set are

$$\omega(\gamma) = \bigcap_{p \in \gamma} \overline{\gamma^+(p)} = \bigcap_{\tau \in (-\infty, \infty)} \overline{\bigcup_{t \geqq \tau} \phi(t, p)}$$

$$\alpha(\gamma) = \bigcap_{p \in \gamma} \overline{\gamma^-(p)} = \bigcap_{\tau \in (-\infty, \infty)} \overline{\bigcup_{t \leqq \tau} \phi(t, p)}$$

where the bar denotes closure.

A set $M$ in $R^n$ is called an *invariant set* of (7.1) if, for any $p$ in $M$, the solution $\phi(t, p)$ of (7.1) through $p$ belongs to $M$ for $t$ in $(-\infty, \infty)$. Any orbit of (7.1) is obviously an invariant set of (7.1). A set $M$ is called *positively (negatively) invariant* if for each $p$ in $M$, $\phi(t, p)$ belongs to $M$ for $t \geqq 0$ $(t \leqq 0)$.

THEOREM 8.1. The $\alpha$- and $\omega$-limit sets of an orbit $\gamma$ are closed and invariant. Furthermore, if $\gamma^+(\gamma^-)$ is bounded, then the $\omega$-($\alpha$-) limit set is nonempty compact and connected, $\operatorname{dist}(\phi(t, p), \omega(\gamma(p))) \to 0$ as $t \to \infty$ and $\operatorname{dist}(\phi(t, p), \alpha(\gamma(p))) \to 0$ as $t \to -\infty$.

PROOF. The closure is obvious from the definition. We now prove the positive limit sets are invariant. If $q$ is in $\omega(\gamma)$, there is a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\phi(t_n, p) \to q$ as $n \to \infty$. Consequently, for any fixed $t$ in $(-\infty, \infty)$, $\phi(t + t_n, p) = \phi(t, \phi(t_n, p)) \to \phi(t, q)$ as $n \to \infty$ from the continuity of $\phi$. This shows that the orbit through $q$ belongs to $\omega(\gamma)$ or $\omega(\gamma)$ is invariant. A similar proof shows that $\alpha(\gamma)$ is invariant.

If $\gamma^+$ $(\gamma^-)$ is bounded, then the $\omega$- ($\alpha$-) limit set is obviously nonempty and bounded. The closure therefore implies compactness. It is easy to see that $\operatorname{dist}(\phi(t, p), \omega(\gamma(p))) \to 0$ as $t \to \infty$, $\operatorname{dist}(\phi(t, p), \alpha(\gamma(p))) \to 0$ as $t \to -\infty$. This latter property clearly implies that $\omega(\gamma)$ and $\alpha(\gamma)$ are connected and the theorem is proved.