Limited Dependent Variables

Categorical and Limited
Dependent Variables
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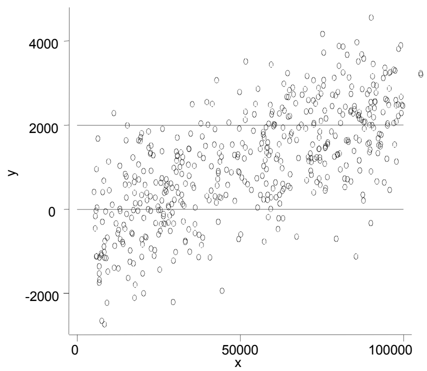
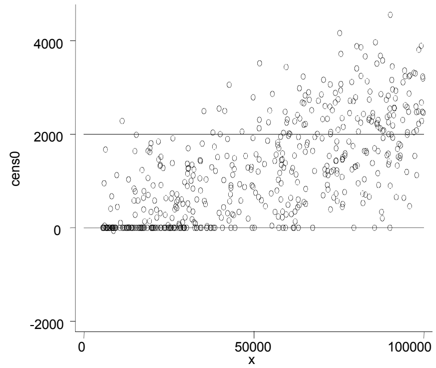
Limited Dependent Variables

- Some continuous variables are affected by limits
  - Values pile up at the limit (censoring)
    - Highest SAT score is 800
    - IRA contributions limits
  - Missing values based on limits (truncation)
    - Minimum wage (people who would earn below the minimum are unemployed)
    - People who want to contribute less than $0 to IRA (in other words take some out)

- These conditions can cause bias in the coefficients, often severe. The bias is towards 0 (no effect).

Censored Data: Misleading Values

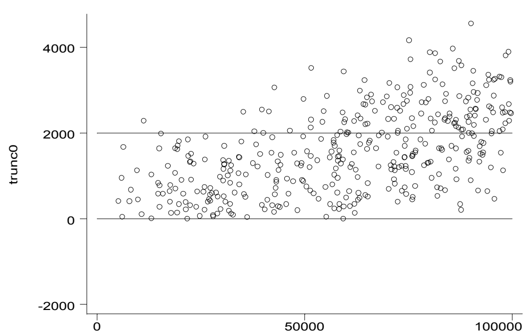




The true values. For example, this could be desired contributions to an IRA account. Negative values mean that you would withdraw money.

Here is the same data, but withdrawals are not allowed, so the recorded value for these people is a contribution of 0. Note how the values "pile up" at zero.

Truncation: Values are Missing (Selected on Value of the DV)



Here we have truncation at zero. People who can't afford to make an IRA contribution of at least $1 are simply not in the dataset. Similarly, people who can't get a job of at least minimum wage are unemployed, and have no observed wage.

The Data (tobit.dta)

Censored means that the data contains "lies," that is a value is recorded but it is not the "true" or desired value. Usually this is because the data are top-coded or bottom-coded or some limit prevents the value from going higher or lower.

Truncated means that the data above or below a limit are missing altogether. In this case we have some information (the x values), but in many cases the truncated individuals are not even in the data set at all.

$$\text{PRF:} \quad Y_i = -500 + 0.03X_i + u_i \quad u_i \sim N\left(0,\sigma^2\right) \quad \sigma = 1000$$

```
. list x y cens0 cens2000 censboth trunc0 trunc2k ira, noobs nod
```

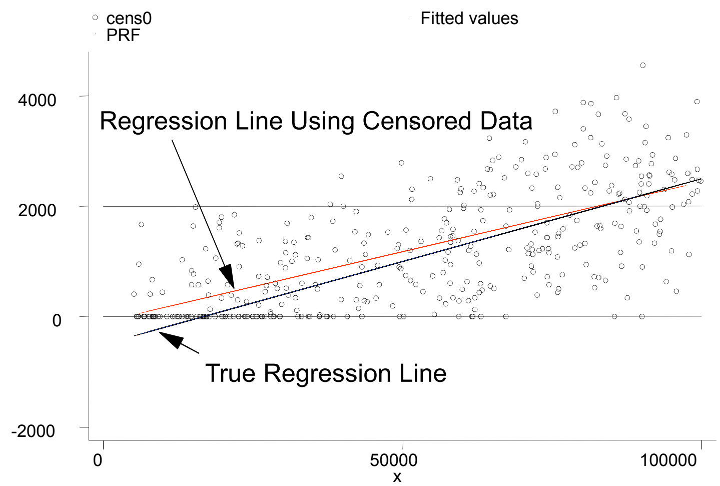
x	y	cens0	cens2000	censboth	trunc0	trunc2k	ira
43186.99	-298.3602	0	-298.3602	0	.	-298.3602	.
56510.91	1562.408	1562.408	1562.408	1562.408	1562.408	1562.408	1562.408
86320.5	2235.013	2235.013	2000	2000	2235.013	.	2000
53107.77	1359.011	1359.011	1359.011	1359.011	1359.011	1359.011	1359.011
57319.36	1698.989	1698.989	1698.989	1698.989	1698.989	1698.989	1698.989
16162.89	-1248.756	0	-1248.756	0	.	-1248.756	.
81442.45	2244.707	2244.707	2000	2000	2244.707	.	2000
34482.09	-1011.442	0	-1011.442	0	.	-1011.442	.
29066.62	510.9071	510.9071	510.9071	510.9071	510.9071	510.9071	510.9071
83317.7	3132.799	3132.799	2000	2000	3132.799	.	2000
27959.77	-319.5778	0	-319.5778	0	.	-319.5778	.
49503.07	-715.4042	0	-715.4042	0	.	-715.4042	.
89647.03	1998.137	1998.137	1998.137	1998.137	1998.137	1998.137	1998.137
5897.91	949.1215	949.1215	949.1215	949.1215	949.1215	949.1215	949.1215

OLS Regression on Censored Data

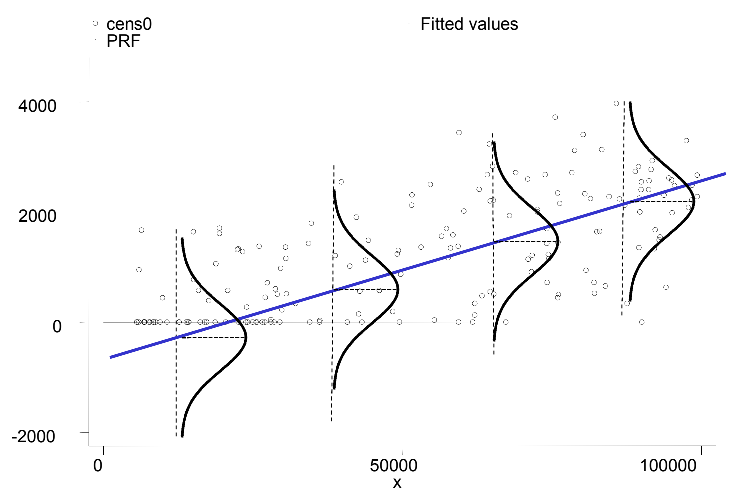
```
. reg y x     The unbiased estimate!
  Source |       SS       df       MS              Number of obs =     500
---------+------------------------------           F(  1,   498) =  430.70
   Model |  408705226     1  408705226             Prob > F      =  0.0000
Residual |  472568086   498  948931.901            R-squared     =  0.4638
---------+------------------------------           Adj R-squared =  0.4627
   Total |  881273312   499  1766078.78            Root MSE      =  974.13
------------------------------------------------------------------------------
       y |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
       x |   .0319565   .0015398     20.753   0.000        .0289312    .0349819
   _cons |  -607.3557   92.22465     -6.586   0.000       -788.5531   -426.1584
------------------------------------------------------------------------------
```

```
. reg cens0 x     OLS on the censored data is biased!!
  Source |       SS       df       MS              Number of obs =     500
---------+------------------------------           F(  1,   498) =  399.45
   Model |  257524250     1  257524250             Prob > F      =  0.0000
Residual |  321062454   498  644703.723            R-squared     =  0.4451
---------+------------------------------           Adj R-squared =  0.4440
   Total |  578586704   499  1159492.39            Root MSE      =  802.93
------------------------------------------------------------------------------
   cens0 |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
       x |   .0253667   .0012692     19.986   0.000         .022873    .0278603
   _cons |  -90.35696    76.0168     -1.189   0.235       -239.7101    58.99619
-----------------------------------------------------------------------------
```

Biased Slope From Censoring at 0

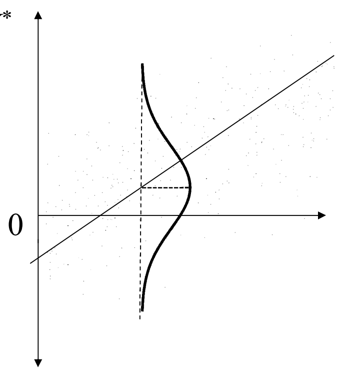


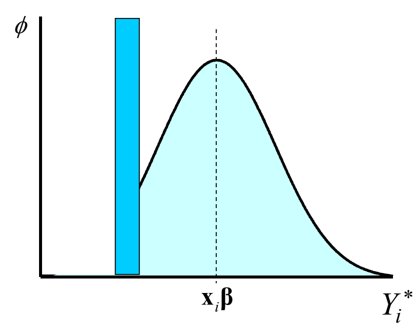
The probability of being at zero depends on the distribution of u_i and X. Is that a problem? If so, why?



Need a Probability Statement to Use Maximum Likelihood

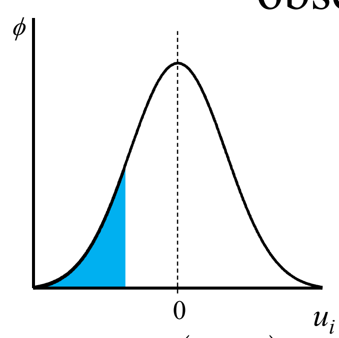
$$Y_i^* = \mathbf{x}_i\boldsymbol{\beta} + u_i \qquad u_i \sim N\left(0, \sigma^2\right) \qquad Y_i = \begin{cases} \boxed{Y_i^* \text{ if } Y_i^* \geq 0} \\ \boxed{0 \text{ if } Y_i^* < 0} \end{cases}$$





We are going to need a two-piece likelihood function!

Given X, what is the probability we observe a zero?



$$u_i \sim N\left(0, \sigma^2\right)$$

$$\begin{aligned} \Pr\left(Y_i = 0 \mid X_i\right) &= \Pr\left(Y_i^* < 0 \mid X_i\right) \\ &= \Pr\left(\mathbf{x}_i\boldsymbol{\beta} + u_i < 0\right) \\ &= \Pr\left(u_i < -\mathbf{x}_i\boldsymbol{\beta}\right) \end{aligned}$$

So this is the probability of censoring given X, as a function of the betas and sigma. Can we convert this problem to a standard normal?

Getting to the Standard Normal

$$\operatorname{var}(cX) = c^2 \operatorname{var}(X)$$

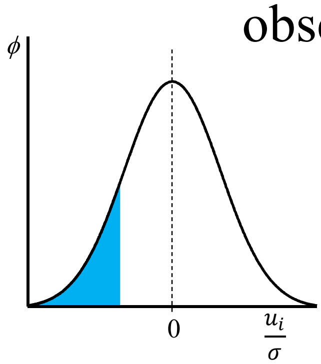
$$\operatorname{var}\left(\frac{u_i}{\sigma}\right) = \operatorname{var}\left(\frac{1}{\sigma}u_i\right) = \frac{1}{\sigma^2}\operatorname{var}(u_i) = \frac{\sigma^2}{\sigma^2} = 1$$

So if $u_i \sim N(0,\sigma^2)$, then $\frac{u_i}{\sigma} \sim N(0,1)$, which is standard normal.

$$\Pr(Y_i = 0 \mid X_i) = \Pr(u_i < -\mathbf{x}_i\boldsymbol{\beta})$$

$$= \Pr\left(\frac{u_i}{\sigma} < \frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right) = \Phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)$$

Given X, what is the probability we observe a zero?



$$\frac{u_i}{\sigma} \sim N(0,1)$$

$$\Pr(Y_i = 0 \mid X_i) = \Pr(Y_i^* < 0 \mid X_i)$$

$$= \Pr(\mathbf{x}_i\boldsymbol{\beta} + u_i < 0)$$

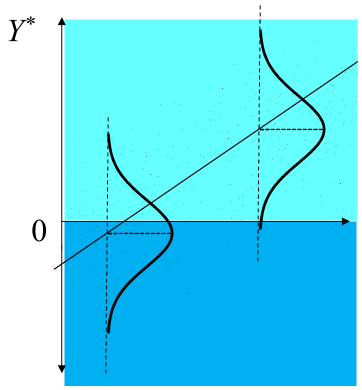
$$= \Pr(u_i < -\mathbf{x}_i\boldsymbol{\beta})$$

$$= \Pr\left(\frac{u_i}{\sigma} < \frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)$$

$$= \Phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)$$

So this is the probability of censoring given X, as a function of the betas and sigma, making use of the machinery of the standard normal.

What about the continuous portion of the distribution of Y?



For $Y_i > 0, P_i = ???$

The probability of any *exact* value in a continuous distribution is zero, but some points are more likely than others. We need to use the density function, which gives the relative likelihood of any value.

Normal Density Functions

$$w_i \sim N\left(\mu, \sigma^2\right) \quad \rightarrow \quad f(w) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w_i - \mu}{\sigma}\right)^2}$$

$$z_i \sim N(0,1) \quad \rightarrow \quad f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad \equiv \phi(z)$$

The disturbance term is a hybrid.

$$u_i \sim N\left(0, \sigma^2\right) \quad \rightarrow \quad f(u_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u_i}{\sigma}\right)^2}$$

The Continuous Part

$$f(u_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u_i}{\sigma}\right)^2} = \frac{1}{\sigma}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u_i}{\sigma}\right)^2}\right]$$

$$= \frac{1}{\sigma}\phi\left(\frac{u_i}{\sigma}\right)$$

$$= \frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)$$

So, for $Y_i > 0$, $\quad P_i = \frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)$

Putting the Pieces Together

$$\mathcal{L} = \prod_{i=1}^{n}\left[P_i\right]$$

$$= \prod_{Y_i=0}\left[\Phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right]\prod_{Y_i>0}\left[\frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right]$$

Censored Uncensored

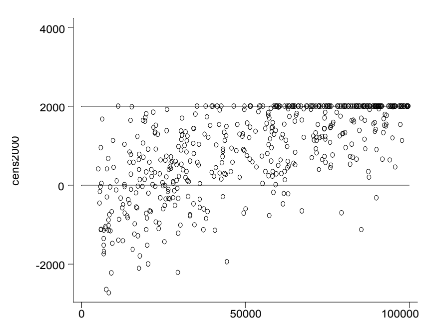
$$\ln\mathcal{L} = \sum_{Y_i=0}\ln\left[\Phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right] + \sum_{Y_i>0}\ln\left[\frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right]$$

Now apply maximum likelihood to estimate $\boldsymbol{\beta}$ and σ.

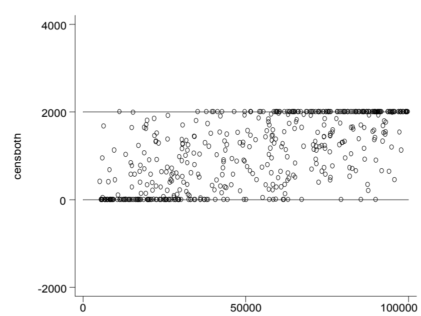
Tobit Analysis

```
tobit   depvar [indepvars] [weight] [if exp] [in range], ll[(#)] ul[(#)]
                                [ level(#) offset(varname) maximize_options ]

. tobit cens0 x, ll(0)

Tobit estimates                                   Number of obs   =        500
                                                  LR chi2(1)      =     305.98
                                                  Prob > chi2     =     0.0000
Log likelihood =  -3348.696                       Pseudo R2       =     0.0437

------------------------------------------------------------------------------
   cens0 |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
       x |   .0313236   .0016149     19.397   0.000       .0281508    .0344964
   _cons |   -553.965   100.5348     -5.510   0.000      -751.4887   -356.4412
---------+--------------------------------------------------------------------
     _se |    946.287   34.75281              (Ancillary parameter)
------------------------------------------------------------------------------

Obs. summary:       105 left-censored observations at cens0<=0
                    395 uncensored observations
```

Censoring, continued



The maximum allowable contribution is 2000. So for people who desire to contribute more, we observe 2000.



Censoring from above and below. Note how data piles up at the limits.

Does OLS give biased results for top censoring?

Bias from Top Censoring

```
. reg cens2000 x

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =  391.21
       Model |  258948923     1  258948923             Prob > F      =  0.0000
    Residual |  329634450   498  661916.566            R-squared     =  0.4400
-------------+------------------------------           Adj R-squared =  0.4388
       Total |  588583373   499   1179525.8            Root MSE      =  813.58

------------------------------------------------------------------------------
    cens2000 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0254367    .001286    19.78   0.000       .02291    .0279635
       _cons |  -449.9395   77.02489    -5.84   0.000    -601.2733   -298.6057
------------------------------------------------------------------------------
```

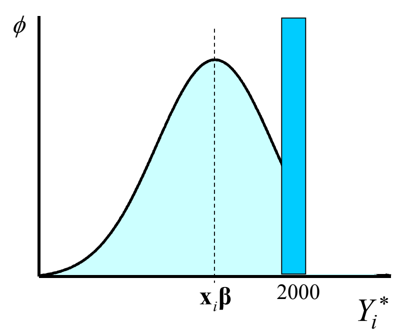
Likelihood for Top Censoring

$$\Pr\left(Y_i = 2000 \mid X_i\right) = \Pr\left(\mathbf{x_i\beta} + u_i > 2000\right) = \Pr\left(u_i > 2000 - \mathbf{x_i\beta}\right)$$

$$= \Pr\left(\frac{u_i}{\sigma} > \frac{2000 - \mathbf{x_i\beta}}{\sigma}\right)$$

$$= 1 - \Pr\left(\frac{u_i}{\sigma} < \frac{2000 - \mathbf{x_i\beta}}{\sigma}\right)$$

$$= 1 - \Phi\left(\frac{2000 - \mathbf{x_i\beta}}{\sigma}\right)$$



The Fix using Tobit

```
. tobit cens2000 x, ul(2000)

Tobit regression                                  Number of obs   =        500
                                                  LR chi2(1)      =     301.28
                                                  Prob > chi2     =     0.0000
Log likelihood = -3145.3849                       Pseudo R2       =     0.0457

------------------------------------------------------------------------------
    cens2000 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0332636   .0017547    18.96   0.000     .0298162    .0367111
       _cons |  -645.1107   97.89438    -6.59   0.000    -837.4467   -452.7748
-------------+----------------------------------------------------------------
      /sigma |   1006.317   38.64256                      930.3951    1082.239
------------------------------------------------------------------------------
  Obs. summary:          0  left-censored observations
                       366     uncensored observations
                       134 right-censored observations at cens2000>=2000
```

OLS Regression on Doubly Censored Data

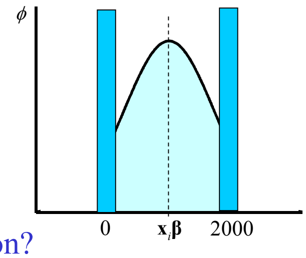
```
. reg censboth x

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =  403.79
       Model |  142157758     1  142157758             Prob > F      =  0.0000
    Residual |  175324537   498  352057.303            R-squared     =  0.4478
-------------+------------------------------           Adj R-squared =  0.4467
       Total |  317482295   499  636237.064            Root MSE      =  593.34

------------------------------------------------------------------------------
    censboth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0188469   .0009379    20.09   0.000     .0170041    .0206896
       _cons |   67.05927    56.1741     1.19   0.233    -43.30818    177.4267
------------------------------------------------------------------------------
```

Regression slope is biased toward zero. Intercept is also biased. Standard error of the estimate (sigma) biased toward zero.

Double Censoring

Censoring above at 2000 & below at 0.

$$Y_i = \begin{cases} 0 \text{ if } Y_i^* \leq 0 \\ Y_i^* \text{ if } 0 < Y_i^* < 2000 \\ 2000 \text{ if } Y_i^* \geq 2000 \end{cases}$$



How many pieces in the likelihood function?

"Fixing" Upper and Lower Censoring with Tobit

```
. tobit censboth x, ll(0) ul(2000)

Tobit estimates                                   Number of obs   =        500
                                                  LR chi2(1)      =     294.89
                                                  Prob > chi2     =     0.0000
Log likelihood = -2345.4808                       Pseudo R2       =     0.0591

------------------------------------------------------------------------------
censboth |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
       x |   .0328139   .0019722     16.638   0.000       .0289391    .0366888
   _cons |  -614.9721   112.6902     -5.457   0.000      -836.3778   -393.5664
---------+--------------------------------------------------------------------
     _se |   985.9899   48.16598             (Ancillary parameter)
------------------------------------------------------------------------------

Obs. summary:       105  left-censored observations at censboth<=0
                    261     uncensored observations
                    134 right-censored observations at censboth>=2000
```

Interpreting the results

What is the desired contribution for a person with $50,000 in income?

$\hat{Y}_i^* = -615 + 0.0328X_i \quad \hat{\sigma} = 986$

For $X_i = 50,000$: $\hat{Y}_i^* = ?$

$$\begin{aligned}\hat{Y}_i^* &= \mathbf{x}_i\hat{\boldsymbol{\beta}} \\ &= -615 + 0.0328(50,000) \\ &= \$1,025\end{aligned}$$

But this is not the only possible outcome! Given the disturbance term, any given individual my desire to contribute more or less than the predicted amount.

Probability of a Censored Outcome

$\widehat{\Pr}(Y_i = 0 \mid X_i = 50000) = ?$

$$\begin{aligned}\Phi\left(\frac{-\mathbf{x}_i\hat{\boldsymbol{\beta}}}{\hat{\sigma}}\right) &= \Phi\left(\frac{-1025}{986}\right) \\ &= \Phi(-1.04) \\ &= 0.15\end{aligned}$$

$\widehat{\Pr}(Y_i = 2000 \mid X_i = 50000) = ?$

$$\begin{aligned}1-\Phi\left(\frac{2000-\mathbf{x}_i\hat{\boldsymbol{\beta}}}{\hat{\sigma}}\right) &= 1-\Phi\left(\frac{2000-1025}{986}\right) \\ &= 1-\Phi(0.99) \\ &= 0.16\end{aligned}$$

Marginal Effect of X

Effect on desired contribution:

$$Y_i^* = \mathbf{x}_i\boldsymbol{\beta} + u_i \qquad \frac{\partial Y_i^*}{\partial X_k} = ? \quad \beta_k \quad \text{...just like OLS!}$$

Effect on probability of being censored:

$$P_0 = \Pr(Y_i = 0 \mid X_k) = \Phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right) \qquad \frac{\partial P_0}{\partial X_k} = \phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\left(\frac{-\beta_k}{\sigma}\right)$$

$$P_{2000} = \Pr(Y_i = 2000 \mid X_k) = 1 - \Phi\left(\frac{2000 - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)$$

$$\frac{\partial P_{2000}}{\partial X_k} = -\phi\left(\frac{2000 - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\left(\frac{-\beta_k}{\sigma}\right) = \phi\left(\frac{2000 - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\left(\frac{\beta_k}{\sigma}\right)$$

Beron: Two Limit Tobit

The amount people are willing to pay is cs_i. Assume people pay only between \$1 and the amount due, $csdue_i$. If they desire to pay less than zero, they are observed at \$0. If the would be willing to pay more, they still don't pay more than $csdue_i$.

$$cspaid_i = \begin{cases} csdue_i & \text{if } \mathrm{cs}_i^* \geq csdue_i \\ cs_i^* & \text{if } 0 > \mathrm{cs}_i^* > csdue_i \\ 0 & \text{if } \mathrm{cs}_i^* \leq 0 \end{cases}$$

Note the subscript "i" – *the upper limit varies* depending on the child support order in a particular case. Therefore:

$$\mathcal{L} = \prod_{Y_i=0}\left[\Phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right] \prod_{\substack{0<\\cspaid_i\\<csdue_i}}\left[\frac{1}{\sigma}\phi\left(\frac{cspaid_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right] \prod_{\substack{cspaid_i=\\csdue_i}}\left[1 - \Phi\left(\frac{csdue_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right]$$

Beron's Notation (page 656)

Beron defines $\frac{\mathbf{x}_i\boldsymbol{\beta}}{\sigma} = z$, so....

$$\mathcal{L} = \prod_{Y_i=0}\left[\Phi(-z)\right] \prod_{\substack{0< \\ cspaid_i \\ <csdue_i}} \left[\frac{1}{\sigma}\phi\left(\frac{cspaid_i}{\sigma} - z\right)\right] \prod_{\substack{cspaid_i= \\ csdue_i}} \left[1-\Phi\left(\frac{csdue_i}{\sigma} - z\right)\right]$$

Can be estimate using "interval regression" -- `intreg`

`intreg depvar1 depvar2 [indepvars] [if] [in] [weight] [, options]`

```
        depvar1 and depvar2 should have the following form:

                Type of data                  depvar1  depvar2
                ----------------------------------------------
                 point data          a = [a,a]    a        a
                 interval data           [a,b]    a        b
                 left-censored data   (-inf,b]    .        b
                 right-censored data   [a,inf)    a        .
                ----------------------------------------------
```

Intreg:

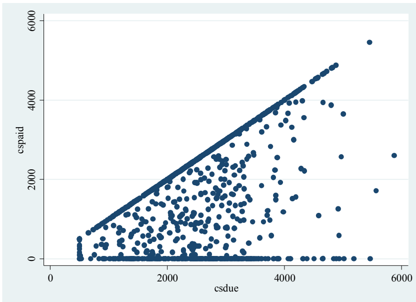
You need to create two dependent variables to specify they range which includes the true value: {Lower < DV < Upper}

	lower	upper
A single value, e.g. $1,317, in continuous data:	1317	1317
A value in a range, say $1-$500:	1	500
A left censored value, -∞ up to $0:	.	0
A right censored value, $2000 to + ∞:	2000	.

```
gen lower = cspaid if (cspaid>0) & (cspaid<=csdue)
gen upper = cspaid if (cspaid>=0) & (cspaid<csdue)
```

Beron Pseudo data

```
clear
set obs 1000
set seed 1

gen csdue = max(500, 2500 + 1000*invnorm(uniform()))
gen randnum = uniform()
gen cspaid = csdue
replace cspaid = 0 if randnum<0.25
replace cspaid = csdue*uniform() if randnum>0.75

scatter cspaid csdue
gen lower = cspaid if (cspaid>0) & (cspaid<=csdue)
gen upper = cspaid if (cspaid>=0) & (cspaid<csdue)
list csdue cspaid lower upper in 1/20
```

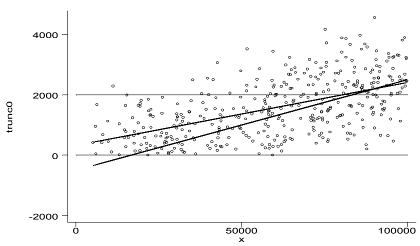


csdue	cspaid	lower	upper
1391.749	1391.749	1391.749	.
1403.395	1403.395	1403.395	.
2377.407	2166.737	2166.737	2166.737
500	251.4147	251.4147	251.4147
1733.313	0	.	0
2294.699	0	.	0
1824.811	0	.	0

```
intreg lower upper x1 x2 …
```

Regression on Data **Truncated** from Below

```
. reg trunc0 x

  Source |       SS       df       MS              Number of obs =     395
---------+------------------------------           F(  1,   393) =  173.45
   Model |  113701840     1   113701840            Prob > F      =  0.0000
Residual |  257630493   393  655548.329            R-squared     =  0.3062
---------+------------------------------           Adj R-squared =  0.3044
   Total |  371332333   394  942467.851            Root MSE      =  809.66

------------------------------------------------------------------------------
  trunc0 |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
       x |   .0209047   .0015873     13.170   0.000       .0177841    .0240254
   _cons |   319.3285   104.0796      3.068   0.002       114.7061    523.9509
------------------------------------------------------------------------------
```

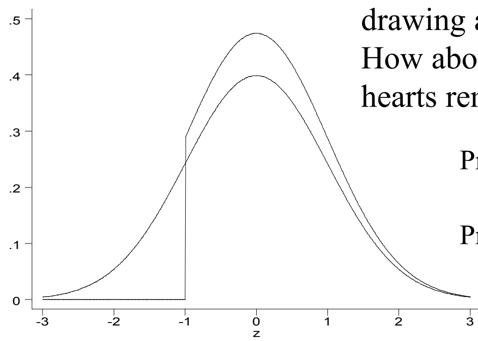


Notice: 1) bias towards zero, 2) only 395 observations (Y is missing for the rest).

Truncated Normal Density Function

Normal curve after truncating at z = -1 and renormalizing

$$f(z \mid z > -1) = \frac{\phi(z)}{1-\Phi(-1)} = \frac{\phi(z)}{\Phi(1)} = \frac{\phi(z)}{0.8413}$$

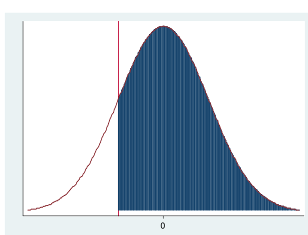


Example: what is the probability of drawing a spade from a deck of cards? How about a deck of cards with all the hearts removed?

$$\Pr(spade) = \frac{1}{4} = 0.25$$

$$\Pr(spade \mid no\ hearts) = \frac{0.25}{0.75} = \frac{1}{3}$$

Truncation at 0 Likelihood Function



$$P_i = \frac{\frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)}{1-\Phi\left(\frac{-\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)} = \frac{\frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)}{\Phi\left(\frac{\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)}$$

$$\mathcal{L} = \prod_{i=1}^{n}\left[P_i\right] = \prod_{i=1}^{n}\left[\frac{\frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)}{\Phi\left(\frac{\mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)}\right]$$

Only one piece. All the observed points are in the continuous part of the distribution.

Estimating a "Truncit"

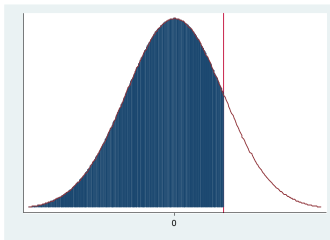
```
------------------------------------------------------------------------------
help for truncreg                                   (manual:  [R] truncreg)
------------------------------------------------------------------------------
       truncreg depvar [varlist] [weight] [if exp] [in range]
                       [, ll(varname | #) ul(varname | #)
                       noconstant offset(varname) marginal at(matname)
                       robust cluster(varname) score(newvarlist)
                       level(#) constraints(numlist) noskip nolog
                       maximize_options ]

. truncreg trunc0 x, ll(0)

Truncated regression
Limit:    lower =          0                       Number of obs =     395
          upper =       +inf                       Wald chi2(1)  =  126.25
Log likelihood = -3169.2288                        Prob > chi2   =  0.0000
------------------------------------------------------------------------------
      trunc0 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
eq1          |
           x |    .029533   .0026284    11.24   0.000     .0243814    .0346845
       _cons |  -422.5624   199.0235    -2.12   0.034    -812.6412   -32.48358
-------------+----------------------------------------------------------------
sigma        |
       _cons |   939.7842   47.10231    19.95   0.000     847.4654    1032.103
------------------------------------------------------------------------------
```

OLS on Upper Truncation

```
. reg trunc2k x

      Source |       SS       df       MS              Number of obs =     366
-------------+------------------------------           F(  1,   364) =  147.83
       Model |  105189846     1   105189846            Prob > F      =  0.0000
    Residual |  259003007   364  711546.723            R-squared     =  0.2888
-------------+------------------------------           Adj R-squared =  0.2869
       Total |  364192853   365  997788.639            Root MSE      =  843.53

------------------------------------------------------------------------------
     trunc2k |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0206337    .001697    12.16   0.000     .0172964    .0239709
       _cons |  -412.3078   86.14517    -4.79   0.000    -581.7125   -242.9031
------------------------------------------------------------------------------
```

The bias is even worse than with censoring.

Upper Truncation at 2000 Likelihood Function

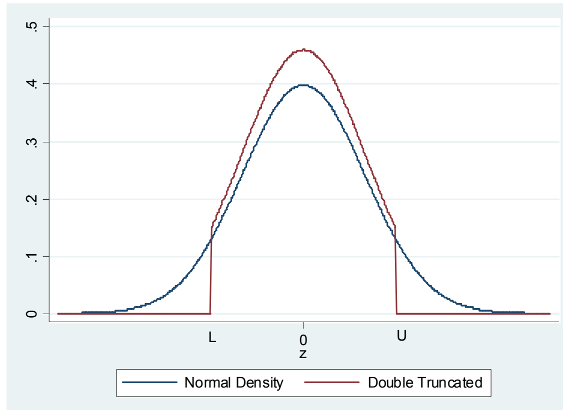


Fix using truncreg

```
. truncreg trunc2k x, ul(2000)
(note: 0 obs. truncated)


Truncated regression
Limit:    lower =       -inf                      Number of obs =     366
          upper =       2000                      Wald chi2(1)  =   98.32
Log likelihood = -2939.6585                       Prob > chi2   =  0.0000

------------------------------------------------------------------------------
     trunc2k |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0330026   .0033283     9.92   0.000     .0264793    .0395258
       _cons |  -607.5885   121.5944    -5.00   0.000    -845.9091   -369.2678
-------------+----------------------------------------------------------------
      /sigma |   1027.542   58.65219    17.52   0.000     912.5859    1142.498
------------------------------------------------------------------------------
```

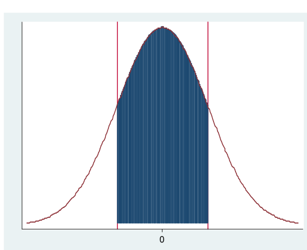
Double Truncation

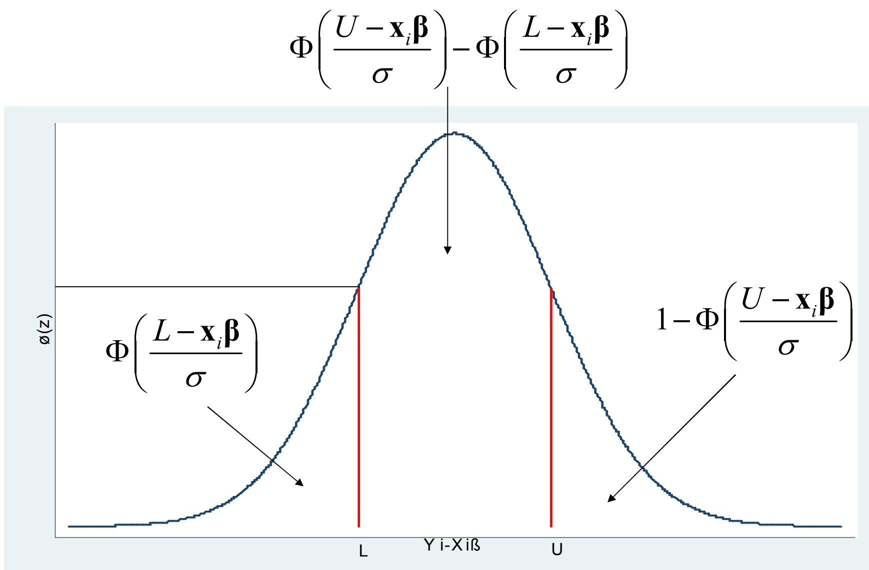
- Truncation from above (2000) and below (0).
- Observations missing beyond limits.
- What is the likelihood function?



Variable	Obs	Mean	Std. Dev.	Min	Max
truncboth	261	1007.618	556.0427	1.898094	1998.137

Likelihood for Double Truncation





Results

```
. reg truncboth x
      Source |       SS       df       MS              Number of obs =     261
-------------+------------------------------           F(  1,   259) =   32.93
       Model |     9067352     1     9067352           Prob > F      =  0.0000
    Residual |  71320352.6   259  275368.157           R-squared     =  0.1128
-------------+------------------------------           Adj R-squared =  0.1094
       Total |  80387704.6   260  309183.479           Root MSE      =  524.76

------------------------------------------------------------------------------
   truncboth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0075317   .0013125     5.74   0.000     .0049471    .0101162
       _cons |    620.947   74.80434     8.30   0.000     473.6449    768.2492

. truncreg truncboth x, ll(0) ul(2000)
Limit:     lower =          0                           Number of obs =    261
           upper =       2000                           Wald chi2(1)  =   5.31
Log likelihood = -1967.4219                             Prob > chi2   = 0.0212
------------------------------------------------------------------------------
   truncboth |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0270894   .0117534     2.30   0.021     .0040531    .0501257
       _cons |  -361.9088   604.5535    -0.60   0.549    -1546.812    822.9943
-------------+----------------------------------------------------------------
      /sigma |    1003.06   228.4255     4.39   0.000     555.3546    1450.766
```

IRA Data

What is the likelihood for the IRA data analysis (truncated at 0, censored at 2000)? Write it down…