AUDIOVISUAL COMMUNICATIONS LCAV

Mathematical Foundations of Signal Processing

Mathematical Foundations of Signal Processing

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November 3, 2014

Where are we now?

1. Geometrical Tools
   - Hilbert spaces, projections etc.
2. Modeling and Analysis
   - Transforms, DT and CT systems, etc.
3. Measuring and Processing
   - *Sampling and Interpolation*
   - Approximation and Compression
   - Localization and Uncertainty
   - Compressed Sensing
4. Applications

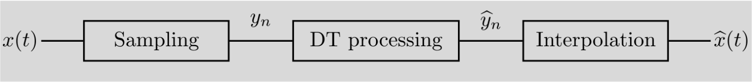
Sampling and Interpolation

1. Why Sampling?
2. Sampling and Interpolation as operators in a Hilbert space
3. Sampling and Interpolation of finite-dim vectors
4. Sampling and Interpolation of sequences in $\ell^2(\mathbb{Z})$
5. Sampling and Interpolation of functions

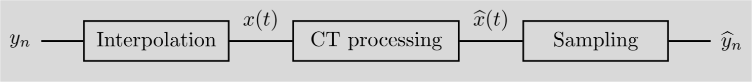
Why Sampling?

Why Sampling?

- World is analog (ch 4). But storing and processing more convenient digitally (ch 3).
- Sampling is the bridge: Given a signal (function) we record its values only at certain instants of time. Trading continuous time description of signal (function) with description based on countable set of values (sequence).
- Convenient but *often lossy*

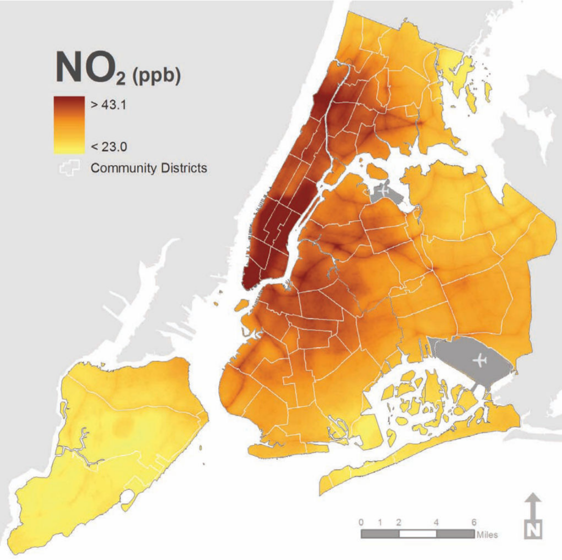


(a) Digital signal processing.

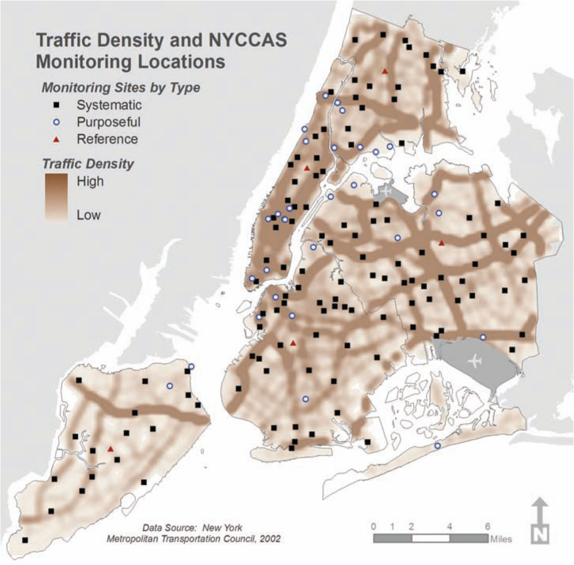


(b) Digital communications.

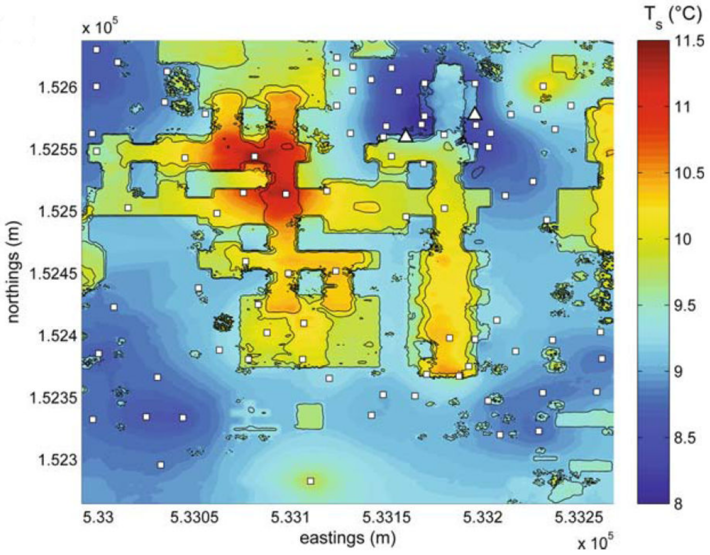
Example: Pollution concentration measurement



Example: Traffic density measurement

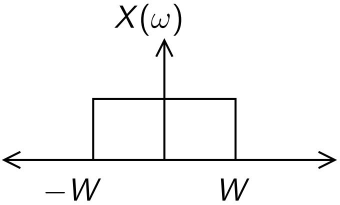


Example: Temperature distribution on campus

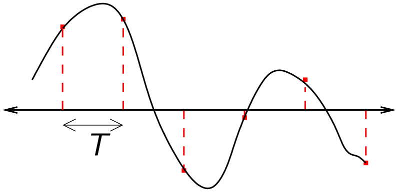


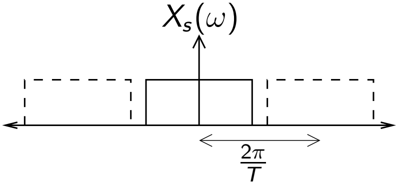
Classical sampling

- Given 1-D bandlimited signal



- Perfect recovery via uniform sampling provided $T \leq \frac{\pi}{W}$

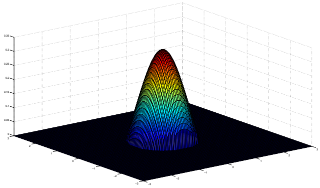




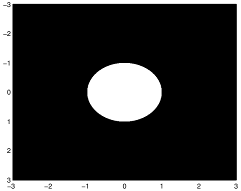
Classical sampling in higher dimensions

- Given: spatially bandlimited field $f : \mathbb{R}^d \mapsto \mathbb{C}$

$$\mathcal{F}(\omega) := \int f(r) e^{-j\langle \omega, r \rangle} dr = 0 \text{ for } \omega \notin \Omega$$



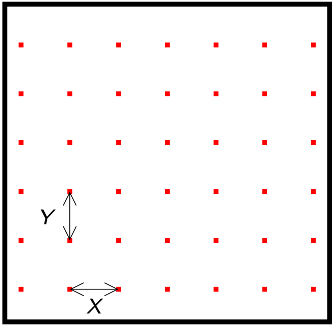
Spectrum $|\mathcal{F}(\omega_x, \omega_y)|$



Support of spectrum Ω

Classical sampling in higher dimensions

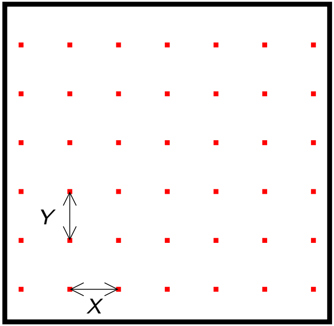
- Sampling on a lattice



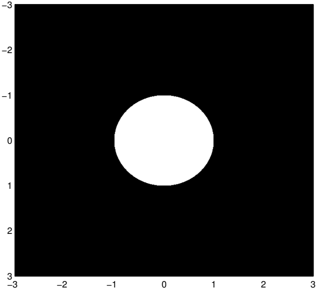
Sampling lattice

Classical sampling in higher dimensions

- Sampling on a lattice



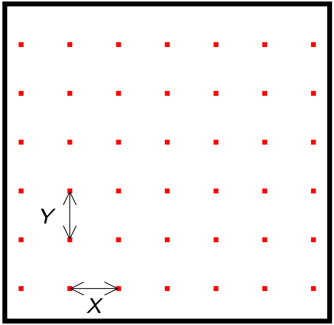
Sampling lattice



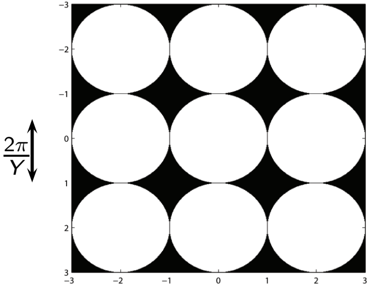
Original spectrum

Classical sampling in higher dimensions

- Sampling on a lattice



Sampling lattice

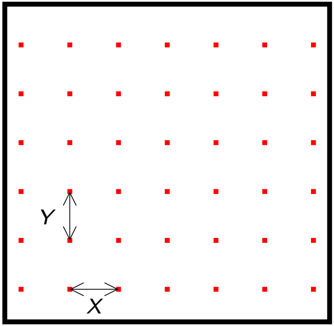


No aliasing in sampled spectrum for

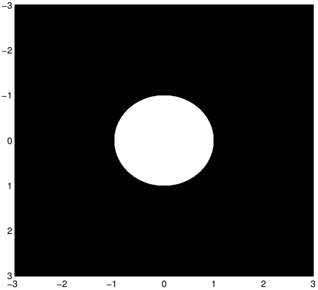
$$X = Y \leq \frac{\pi}{R}$$

Classical sampling in higher dimensions

- Sampling on a lattice



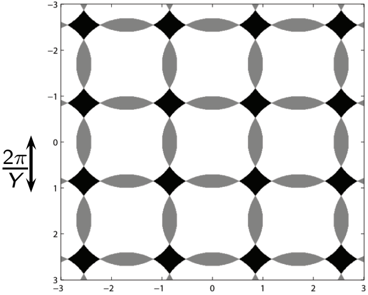
Sampling lattice



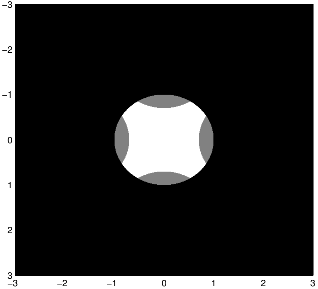
Perfect recovery of original spectrum

Classical sampling in higher dimensions

- Sampling on a lattice



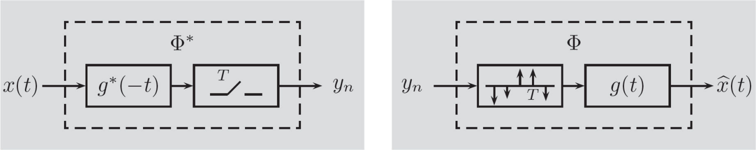
Aliased sampled spectrum



Perfect recovery impossible

- Lattice should be fine enough $\equiv$ Nyquist criterion in $\mathbb{R}^d$

Sampling and Interpolation as Operators



(a) Sampling. (b) Interpolation.

- Two questions arise:
  1) *How much information* about the signal is contained in the samples?
  2) To what extent can we *recover* the signal from the sequence of samples?

- Classical sampling theorem: If x is in $\mathrm{BL}[-\frac{\pi}{T}, \frac{\pi}{T}]$ and $g(t) = \operatorname{sinc}\left(\frac{\pi t}{T}\right)$ then $\hat{x}(t) = x(t)$

Classical sampling theorem

Theorem (Sampling theorem)

If function x is in $\mathrm{BL}[-\pi/T, \pi/T]$,

$$x(t) \;=\; \sum_{n\in\mathbb{Z}} x(nT)\,\mathrm{sinc}\left(\tfrac{\pi}{T}(t-nT)\right), \qquad t \in \mathbb{R}.$$

- We will see:
  1) Why is this true?
  2) What are the properties of Φ that make this true?
  3) What happens when x is not in $\mathrm{BL}[-\frac{\pi}{T}, \frac{\pi}{T}]$?
  4) Can we use different filters in place of g^* and g?
  5) What properties do $\hat{x}$ have in such a case?
- All answers provided via *Hilbert space interpretation*

Sampling and Interpolation as operators in a Hilbert space

If you think about it...

- Classical sampling is a *linear transform* from *Hilbert space* $\mathcal{L}^2(\mathbb{R})$ to *Hilbert space* $\ell^2(\mathbb{Z})$ that admits a more compact representation
  - *Potentially lossy*: Only bandlimited signals can be recovered from the samples
- Classical interpolation is a *linear transform* from *Hilbert space* $\ell^2(\mathbb{Z})$ to *Hilbert space* $\mathcal{L}^2(\mathbb{R})$
  - Embeds information within the bandlimited subspace of $\mathcal{L}^2(\mathbb{R})$

Other kinds of Sampling and Interpolation

- Typical definition of sampling and interpolation:

$$\text{discrete-time signal } (\ell^2(\mathbb{Z})) \quad \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} \quad \text{continuous-time signal } (\mathcal{L}^2(\mathbb{R}))$$

- It could also be

$$\text{low-rate sequence } (\ell^2(\mathbb{Z})) \quad \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} \quad \text{high-rate sequence } (\ell^2(\mathbb{Z}))$$

- Can be extended to

$$\text{shorter finite-length vector } \mathbb{C}^N \quad \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} \quad \text{longer finite-length vector } \mathbb{C}^M$$

- All the above can be interpreted as *linear operators between two Hilbert spaces*

Sampling and Interpolation Operators

We shall discuss sampling and interpolation in the following cases:

- Finite dimensional vectors
- Sequences in $\ell^2(\mathbb{Z})$
- Functions in $\mathcal{L}^2(\mathbb{R})$

Sampling and Interpolation of finite-dim vectors

Sampling and Interpolating Finite dimensional vectors

- Sampling and interpolation are linear operators between finite dimensional subspaces, *for example,* , $\mathbb{R}^N$ and $\mathbb{R}^M$ (or $\mathbb{C}^N$ and $\mathbb{C}^M$) with $N < M$.
  - Represented by *matrix multiplication*

- Sampling will take M values and produce $N < M$ values
  - Sampling matrix is *fat* - i.e., has more columns than rows

- Interpolation will take N values and produce $M > N$ values
  - Interpolation matrix is *tall* - i.e., has more rows than columns

Sampling and interpolation with orthonormal vectors

Sampling

$$y = \begin{bmatrix} \text{———} & \varphi_0^* & \text{———} \\ \text{———} & \varphi_1^* & \text{———} \\ & \vdots & \\ \text{———} & \varphi_{N-1}^* & \text{———} \end{bmatrix}_{N\times M} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}_{M\times 1} = \Phi^* x \,.$$

where φ_n^* is the n-th row of Φ^*.

Equivalently, $y_n = \langle x, \varphi_n \rangle$.

Here Φ^* is a $N \times M$ (*fat*) matrix, or equivalently an operator:

$$\Phi^* : \mathbb{C}^M \mapsto \mathbb{C}^N$$

We assume φ_n, $n = 0, \ldots, N-1$ to be *orthonormal*

$$\langle \varphi_n, \varphi_k \rangle = \delta_{n-k} \quad \Leftrightarrow \quad \Phi^*\Phi = I \,, \text{ where } N < M \,.$$

Sampling and interpolation with orthonormal vectors

Sampling

$$y = \begin{bmatrix} \text{———} & \varphi_0^* & \text{———} \\ \text{———} & \varphi_1^* & \text{———} \\ & \vdots & \\ \text{———} & \varphi_{N-1}^* & \text{———} \end{bmatrix}_{N\times M} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}_{M\times 1} = \Phi^* x\,.$$

Since $N < M$ sampling is a lossy operation

Sampling operator Φ^* has max rank N and $M - N$ dimensional null space $\mathcal{N}(\Phi^*)$ with orthogonal complement $S = \mathcal{N}(\Phi^*)^\perp = \text{span}(\{\varphi_n\}_{n=0,\ldots,N-1})$.

When a vector $x \in \mathbb{R}^M$ is sampled *information about the component of x in S is preserved* and is captured by $\Phi^* x$, while the *component in the null space $\mathcal{N}(\Phi^*)$ is lost*. I.e., $y = \Phi^* x_S$.

Sampling and interpolation with orthonormal vectors

Interpolation

$$\widehat{x} = \begin{bmatrix} | & | & & | \\ \varphi_0 & \varphi_1 & \cdots & \varphi_{N-1} \\ | & | & & | \end{bmatrix}_{M \times N} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}_{N \times 1} = \Phi y = \sum_{n=0}^{N-1} y_k \varphi_k \,,$$

where φ_n is the n-th column of Φ.

Since $N < M$, Φ is a *tall matrix*.

As was true for Φ^*, Φ has maximum rank N and the interpolation operator has an N dimensional range $S = \text{span}(\{\varphi_n\}_{n=0,\ldots,N-1})$. This subspace is the same as the orthogonal complement of the null space of the sampling operator,

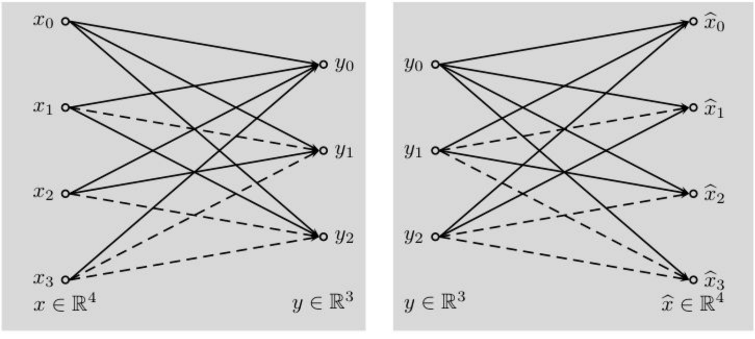
$$\mathcal{R}(\Phi) = S = \mathcal{N}(\Phi^*)^\perp.$$

Sampling and Interpolation Operators

Finite dimensional vectors

Sampling and Interpolation in $\mathbb{R}^4$

Let us define sampling of $x \in \mathbb{R}^4$ to obtain three samples $y \in \mathbb{R}^3$, where solid lines have weight $1/2$, while dashed lines have weight $-1/2$; for example, $y_1 = (x_0 - x_1 + x_2 - x_3)/2$.



(a) Sampling. (b) Interpolation.

An example

Sampling and Interpolation in $\mathbb{R}^4$

Consider sampling matrix with orthonormal rows

$$\Phi^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}_{3\times 4},$$

with

$$\mathcal{N}(\Phi^*) = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}, \text{ and } S = \left\{ \alpha_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

For instance

$$\Phi^* \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \Phi^* \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right) = \Phi^* \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \Phi^* \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

An example

Sampling and Interpolation in $\mathbb{R}^4$

Now the interpolator operator can be written as

$$\Phi = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}_{4\times 3} ,$$

The range of Φ is given by

$$S \;=\; \left\{ \alpha_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \;\middle|\; \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C} \right\}.$$

Can verify

$$\mathcal{R}(\Phi) = S = \mathcal{N}(\Phi^*)^{\perp}.$$

Interpolation followed by sampling

- Since sampling vectors are orthonormal we have

$$\Phi^*\Phi = I \,.$$

- This means that

$$\Phi^*\Phi y = y \,, \text{ for all } y \in \mathbb{C}^N$$

  i.e., any vector y in the smaller space can be recovered perfectly by interpolating followed by sampling

- In this case, we say sampling and interpolation operators are *consistent*

Sampling followed by interpolation

- Sampling followed by interpolation

$$\Phi\Phi^* = P$$

Here P is an *orthogonal projection operator* because P is

idempotent:

$$P^2 = \Phi\Phi^*\Phi\Phi^* = \Phi(\Phi^*\Phi)\Phi^* = \Phi\Phi^* = P$$

and *self-adjoint*

$$P^* = (\Phi\Phi^*)^* = \Phi\Phi^* = P$$

- Therefore, $\hat{x} = Px$ is the *best least square approximation* of x in $S = (\mathcal{N}(\Phi^*))^\perp = \mathcal{R}(\Phi)$.

$$\hat{x} \; = \; \arg\min_{x_S \in S} \|x - x_S\|, \qquad x - \hat{x} \; \perp \; S.$$

In particular, $\hat{x} = x$ *when* $x \in S$

Aside: In general Hilbert spaces

- Same ideas extend to general Hilbert spaces, e.g., sequences ($\ell^2(\mathbb{Z})$) or functions ($\mathcal{L}^2(\mathbb{R})$)

- Sampling using *orthonormal vectors* in finite-dimensional vector spaces is analogous to classical sampling in $\mathcal{L}^2(\mathbb{R})$ with *sinc-filter* for filtering and reconstructing
    - There subspace $S = \mathrm{BL}[\frac{-\pi}{T}, \frac{\pi}{T}]$
    - To be discussed later

Sampling and interpolation with non-orthonormal vectors

Sampling

$$y = \begin{bmatrix} \text{———} & \widetilde{\varphi}_0^* & \text{———} \\ \text{———} & \widetilde{\varphi}_1^* & \text{———} \\ & \vdots & \\ \text{———} & \widetilde{\varphi}_{N-1}^* & \text{———} \end{bmatrix}_{N \times M} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}_{M \times 1} = \widetilde{\Phi}^* x \,.$$

where $\widetilde{\varphi}_n^*$ is the n-th row of $\widetilde{\Phi}^*$.

As before assume $\widetilde{\Phi}^*$ has full rank N and $M - N$ dimensional null space $\mathcal{N}(\Phi^*)$ with orthogonal complement $\widetilde{S} = \mathcal{N}(\widetilde{\Phi}^*)^\perp = \operatorname{span}(\{\widetilde{\varphi}_n\}_{n=0,\ldots,N-1})$.

When a vector $x \in \mathbb{R}^M$ is sampled *information about the component of x in $\widetilde{S}$ is preserved* and is captured by $\widetilde{\Phi}^* x$, while the *component in the null space $\mathcal{N}(\widetilde{\Phi}^*)$ is lost*.

Sampling with non-orthonormal vectors: Example

Sampling and Interpolation in $\mathbb{R}^4$

Consider sampling matrix with non-orthonormal rows

$$\widetilde{\Phi}^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4},$$

with

$$\mathcal{N}(\widetilde{\Phi}^*) = \left\{ \beta \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}, \text{ and } \widetilde{S} = \left\{ \beta_0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Sampling and interpolation with non-orthonormal vectors

Interpolation

Interpolation represented by $M \times N$ matrix Φ, but is not the adjoint of $\widetilde{\Phi}^*$

Interpolation output lies in

$$S \;=\; \mathcal{R}(\Phi) \;=\; \left\{ \sum_{k=0}^{N-1} \alpha_k \varphi_k \;\middle|\; \alpha \in \mathbb{C}^N \right\}.$$

A possible choice of Φ is the *pseudoinverse of $\widetilde{\Phi}^*$*:

$$\Phi \;=\; \widetilde{\Phi}(\widetilde{\Phi}^* \widetilde{\Phi})^{-1}$$

In such a case $S = \widetilde{S}$

Sampling and interpolation with non-orthonormal vectors

Interpolation followed by sampling

Interpolation followed by sampling is defined by $\widetilde{\Phi}^* \Phi$

We say sampling and interpolation operators are *consistent* when Φ is a right inverse of $\widetilde{\Phi}^*$:

$$\widetilde{\Phi}^* \Phi \;=\; I \qquad \Leftrightarrow \qquad \langle \varphi_n, \widetilde{\varphi}_k \rangle \;=\; \delta_{n-k}.$$

In this case, the vectors are biorthogonal

They form a *biorthogonal pair* of bases for S when $S = \widetilde{S}$, e.g., when Φ is the pseudoinverse of $\widetilde{\Phi}^*$. In this case they are called *ideally matched*.

Sampling and interpolation with non-orthonormal vectors

Interpolation followed by sampling

Consider sampling operator

$$\widetilde{\Phi}^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4}.$$

Two possible consistent interpolators (i.e., right-inverses):

$$\Phi_1 = \frac{1}{2}\begin{bmatrix} 3 & -2 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \\ 1 & -2 & 3 \end{bmatrix}$$

Pseudoinverse. Ideally matched

$$\Phi_2 = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

Not pseudoinverse. Not ideally matched

Sampling and interpolation with non-orthonormal vectors

Sampling followed by interpolation

Sampling followed by interpolation is defined by $P = \Phi\widetilde{\Phi}^*$

When sampling and interpolation operators are *consistent* P is a *projection operator*

$$P^2 \;=\; (\Phi\widetilde{\Phi}^*)\,(\Phi\widetilde{\Phi}^*) \;=\; \Phi\,(\widetilde{\Phi}^*\Phi)\,\widetilde{\Phi}^* \;=\; \Phi\, I\, \widetilde{\Phi}^* \;=\; \Phi\widetilde{\Phi}^* \;=\; P$$

If Φ is the *pseudoinverse* of $\widetilde{\Phi}^*$, then P is self-adjoint and hence is an *orthogonal projection operator*

$$\begin{aligned} P^* \quad &= \quad (\Phi\widetilde{\Phi}^*)^* \;=\; (\widetilde{\Phi}(\widetilde{\Phi}^*\widetilde{\Phi})^{-1}\widetilde{\Phi}^*)^* \;=\; \widetilde{\Phi}((\widetilde{\Phi}^*\widetilde{\Phi})^{-1})^*\widetilde{\Phi}^* \\ &= \quad \widetilde{\Phi}(\widetilde{\Phi}^*\widetilde{\Phi})^{-1}\widetilde{\Phi}^* \;=\; \Phi\widetilde{\Phi}^* \;=\; P \end{aligned}$$

In this case $S = \widetilde{S}$ and sampling and interpolation operators are called *ideally matched*.

Sampling and interpolation with non-orthonormal vectors

Sampling followed by interpolation

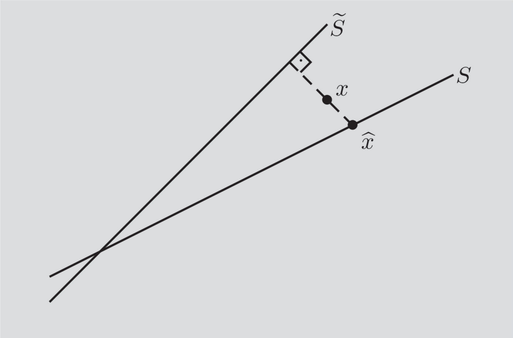
Theorem (Recovery for vectors, nonorthogonal)

Let sampling operator $\widetilde{\Phi}^* : \mathbb{C}^M \to \mathbb{C}^N$ *and interpolation operator* $\Phi : \mathbb{C}^N \to \mathbb{C}^M$ *satisfy consistency condition* $\widetilde{\Phi}^* \Phi = I$*. Then, with* $S = \mathcal{R}(\Phi)$, $\widetilde{S} = \mathcal{N}(\widetilde{\Phi}^*)^\perp$, $P = \Phi \widetilde{\Phi}^*$, *and* $\hat{x} = P\,x$*:*

1. P *is a projection operator with range* S, *and* $x - \hat{x} \perp \widetilde{S}$. *In particular,* $\hat{x} = x$ *when* $x \in S$.
2. *If* Φ *is the pseudoinverse of* $\widetilde{\Phi}^*$, *then* $S = \widetilde{S}$ *and* P *is an orthogonal projection operator onto* S. *Hence* Px *gives best approximation of* x *in* S.

Sampling and interpolation with non-orthonormal vectors

Subspaces defined in sampling and interpolation



$\widetilde{S}$ represents what can be measured; it is the orthogonal complement of the null space of the sampling operator $\widetilde{\Phi}^*$. S represents what can be reproduced; it is the range of the interpolation operator Φ. When sampling and interpolation are *consistent*, $\Phi\widetilde{\Phi}^*$ is a projection and $x - \widehat{x}$ is orthogonal to $\widetilde{S}$. When furthermore $S = \widetilde{S}$, the projection becomes an orthogonal projection and the sampling and interpolation are *ideally matched*.

Recap

- Sampling and interpolation as *linear operators* between Hilbert spaces
  - Simplest example: Finite dimensional vector spaces

- Sampling matrix Φ^* is *fat* and interpolation matrix Φ is *tall*

- Case 1: Orthogonal sampling vectors (columns of Φ). Then:

$$\Phi^*\Phi = I \quad \text{and} \quad \Phi\Phi^* \text{ is an orthogonal projection operator}$$

- Case 2: Non-orthogonal sampling vectors
  - Sampling $\widetilde{\Phi}^*$ and interpolation Φ are *consistent* when

$$\widetilde{\Phi}^*\Phi \ = \ I$$

  - If Φ is pseudoinverse of $\widetilde{\Phi}^*$ then sampling and interpolation operators are *ideally matched* and $\Phi\widetilde{\Phi}^*$ forms an orthogonal projection

- Read: Chapter 5, sections 5.1-5.2

Sampling and Interpolation of sequences in $\ell^2(\mathbb{Z})$

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

A different Hilbert Space: Sequences in ℓ^2

We will study downsampling and upsampling of sequences in $\ell^2(\mathbb{Z})$ using Hilbert space framework

- *Shift invariant subspaces of ℓ^2*
  A subspace $S \in \ell^2$ is a shift-invariant subspace with respect to shift $L \in \mathbb{Z}^+$ when $x_n \in S$ implies $x_{n-kL} \in S$ for every integer k.
- *Subspace of bandlimited sequences*
  A sequence $x_n \in \ell^2(\mathbb{Z})$ is said to have bandwidth $\omega_0 \in (0, 2\pi]$ if the discrete time Fourier transform $X(e^{j\omega})$ satisfies

  $$X(e^{j\omega}) = 0 \text{ for all } |\omega| > \frac{\omega_0}{2} .$$

  We define then the subspace of ω_0 bandlimited sequences as

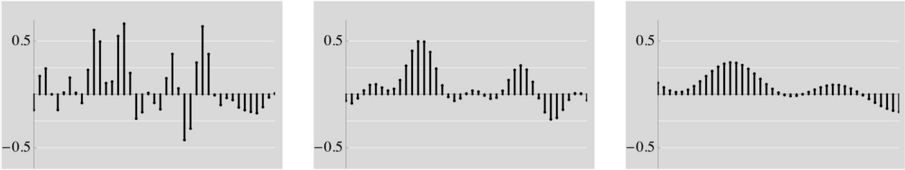
  $$BL[-\omega_0/2, \omega_0/2] = \{x_n \mid x_n \text{ has bandwidth at most } \omega_0\} .$$

  Remark: *Subspace of bandlimited sequences is shift invariant (prove it!)*

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sequences in ℓ^2

Bandlimited sequences



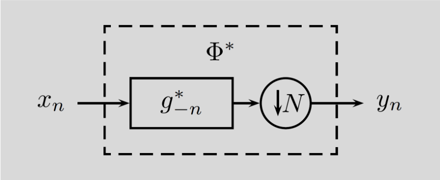
(a) $\mathrm{BL}[-\pi/2, \pi/2]$. (b) $\mathrm{BL}[-\pi/4, \pi/4]$. (c) $\mathrm{BL}[-\pi/8, \pi/8]$.

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sequences in ℓ^2: Sampling

We define as sampling of a sequence $x_n \in \ell^2$ the operation of filtering by g^*_{-n} and downsampling by integer $N > 1$ and we denote it with the operator Φ^*

$$y_n = (\Phi^* x)_n$$



$$\begin{aligned} y_k &= (\Phi^* x)_k = \left(g^*_{-n} *_n x_n\right)\Big|_{n=kN} = \left(\sum_{m\in\mathbb{Z}} x_m\, g^*_{m-n}\right)\Bigg|_{n=kN} \\ &= \sum_{m\in\mathbb{Z}} x_m\, g^*_{m-kN} = \langle x_m, g_{m-kN}\rangle_m = \langle x, \varphi_k\rangle, \end{aligned}$$

where $\varphi_{k,n} = g_{n-kN}, \qquad n \in \mathbb{Z}.$

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sequences in ℓ^2: Sampling

The sampling operator Φ^* is now an infinite matrix with rows equal to φ^* and its shifts by integer multiples of N.

We assume these rows to be *orthonormal*,

$$\langle \varphi_k, \varphi_\ell \rangle \;=\; \delta_{k-\ell} \qquad \Leftrightarrow \qquad \langle g_{n-kN}, g_{n-\ell N} \rangle_n \;=\; \delta_{k-\ell}$$

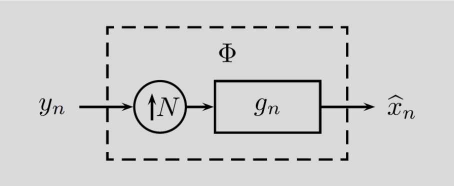
or equivalently, $\Phi^*\Phi = I$.

The sampling operator Φ^* has a nontrivial null space $\mathcal{N}(\Phi^*)$ and $S = \mathcal{N}(\Phi^*)^\perp = \mathrm{span}(\{\varphi_n\}_{n\in\mathbb{Z}})$.

Sequences in ℓ^2: Interpolation

We define as interpolation of a sequence $y_n \in \ell^2$ the operation of upsampling by integer $N > 1$ and filtering by g_n, and we denote it with the operator Φ

$$\widehat{x}_n = (\Phi y)_n$$



$$\widehat{x}_n \;=\; (\Phi y)_n \;=\; \sum_{k \in \mathbb{Z}} y_k \, g_{n-kN} \;=\; \left(\textstyle\sum_{k \in \mathbb{Z}} y_k \varphi_k\right)_n ,$$

The interpolation operator Φ is now an infinite matrix with columns equal to φ and its shifts by integer multiples of N.

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sampling and Interpolation in ℓ^2

Set $N = 2$ and choose

$$g_{-n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cdots & 0 & 1 & \boxed{1} & 0 & 0 & \cdots \end{bmatrix} .$$

Then the sampling reads

$$\begin{bmatrix} \vdots \\ \boxed{y_0} \\ y_1 \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \boxed{1} & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \boxed{x_0} \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \Phi^* x .$$

For every two inputs samples x_{2k} and x_{2k+1}, we get one output sample $y_k = (x_{2k} + x_{2k+1})\sqrt(2)$, and we have

$$\mathcal{N}(\Phi^*) = \left\{ x \in \ell^2(\mathbb{Z}) \mid x_{2k} = -x_{2k+1} \,, k \in \mathbb{Z} \right\} .$$

Sampling and Interpolation in ℓ^2

$$
\begin{aligned}
S &= \mathcal{N}(\Phi^*)^\perp = \{x \in \ell^2(\mathbb{Z}) \mid x_{2k} = x_{2k+1} \text{ for all } k \in \mathbb{Z}\} \\
&= \left\{ \cdots + \alpha_{-1} \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ \boxed{0} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \alpha_0 \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \boxed{1} \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \alpha_1 \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \boxed{0} \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \cdots \;\middle|\; \alpha \in \ell^2(\mathbb{Z}) \right\}.
\end{aligned}
$$

Sampling and Interpolation in ℓ^2

$$\text{We have } g = \frac{1}{\sqrt{2}} \begin{bmatrix} \dots & 0 & 0 & \boxed{1} & 1 & 0 & \dots \end{bmatrix}^T .$$

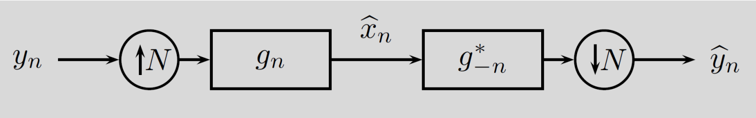
The output of interpolation with $N = 2$ and postfilter g is

$$\begin{bmatrix} \vdots \\ \boxed{\widehat{x}_0} \\ \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} & \vdots & \vdots & \\ \cdots & \boxed{1} & 0 & \cdots \\ \cdots & 1 & 0 & \cdots \\ \cdots & 0 & 1 & \cdots \\ \cdots & 0 & 1 & \cdots \\ & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ \boxed{y_0} \\ y_1 \\ \vdots \end{bmatrix} = \Phi\, y.$$

For every input sample y_k, we get two output samples $x_{2k} = x_{2k+1} = y_k/\sqrt{2}$.

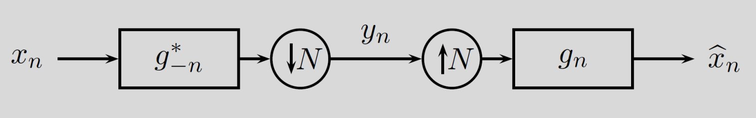
Sampling and Interpolation in $\ell^2(\mathbb{Z})$

- Interpolation followed by sampling



Since $\Phi^*\Phi = I$ we have $\hat{y}_n = y_n$.

- Sampling followed by interpolation



$$\Phi\Phi^* = P.$$

As before P is an orthogonal projection operator. Therefore, Px is the best least square approximation of x in S.

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Theorem (Recovery for sequences, orthogonal)

Assume filter g is such that,

$$\langle \varphi_k, \varphi_\ell \rangle = \delta_{k-\ell} \qquad \Leftrightarrow \qquad \langle g_{n-kN}, g_{n-\ell N} \rangle_n = \delta_{k-\ell}.$$

Then,

$$\widehat{x}_n = \sum_{k \in \mathbb{Z}} y_k \, g_{n-kN}, \qquad n \in \mathbb{Z},$$

where

$$y_k = \sum_{m \in \mathbb{Z}} x_m \, g^*_{m-kN}, \qquad k \in \mathbb{Z},$$

is the best approximation of x in $S = \mathcal{R}(\Phi)$:

$$\widehat{x} = \arg\min_{x_S \in S} \|x - x_S\|, \qquad x - \widehat{x} \perp S.$$

In particular, $\widehat{x} = x$ when $x \in S$.

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sequences in $BL[-\omega_0/2, \omega_0/2] \subset \ell^2(\mathbb{Z})$

$$g_n \;=\; \frac{1}{\sqrt{N}}\,\mathrm{sinc}\left(\frac{\pi n}{N}\right), \quad n \in \mathbb{Z}, \qquad \stackrel{\text{DTFT}}{\longleftrightarrow} \qquad G(e^{j\omega}) \;=\; \begin{cases} \sqrt{N}, & |\omega| \le \pi/N; \\ 0, & \text{otherwise,} \end{cases}$$

Like in continuous time, we have that g is a generator with shift N of $\mathrm{BL}[-\pi/N, \pi/N]$ (Prove it!). Moreover, as before, shifted versions are orthonormal,

$$\begin{aligned} \langle g_{n-kN}, g_{n-\ell N}\rangle_n &= \frac{1}{2\pi}\langle e^{-j\omega kN}G(e^{j\omega}), e^{-j\omega \ell N}G(e^{j\omega})\rangle_\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(k-\ell)N}|G(e^{j\omega})|^2\, d\omega \\ &= \frac{N}{2\pi}\int_{-\pi/N}^{\pi/N} e^{-j\omega(k-\ell)N}\, d\omega \;=\; \delta_{k-\ell}. \end{aligned}$$

Theorem (Projection to bandlimited subspace)

Using sinc filter for g we have

$$\widehat{x}_n \;=\; \frac{1}{\sqrt{N}} \sum_{k\in\mathbb{Z}} y_k \operatorname{sinc}\left(\tfrac{\pi}{N}(n-kN)\right), \qquad n \in \mathbb{Z},$$

where

$$y_k \;=\; \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}} x_n \operatorname{sinc}\left(\tfrac{\pi}{N}(n-kN)\right), \qquad k \in \mathbb{Z},$$

is the best approximation of x *in* $\mathrm{BL}[-\pi/N, \pi/N]$*:*

$$\widehat{x} \;=\; \operatorname*{arg\,min}_{x_{\mathrm{BL}}\in\mathrm{BL}[-\pi/N,\pi/N]} \|x - x_{\mathrm{BL}}\|, \qquad x - \widehat{x} \;\perp\; \mathrm{BL}[-\pi/N, \pi/N].$$

In particular, $\widehat{x} = x$ *when* $x \in \mathrm{BL}[-\pi/N, \pi/N]$*.*

Sampling and Interpolation in $\ell^2(\mathbb{Z})$

Sequences in $BL[-\omega_0/2, \omega_0/2] \subset \ell^2(\mathbb{Z})$

Other results from sampling of functions can be generalized:

- Sampling without prefilter
- Sampling with non-orthogonal functions

Summary

- Sampling and Interpolation as linear operators between Hilbert spaces
  - Intuition from finite dimensional Euclidean spaces ($\mathbb{C}^M \rightleftarrows \mathbb{C}^N$)
  - Generalizes to sampling of functions ($\mathcal{L}^2(\mathbb{R}) \rightleftarrows \ell^2(\mathbb{Z})$) and sequences ($\ell^2(\mathbb{Z}) \rightleftarrows \ell^2(\mathbb{Z})$)
- *Consistency*: Interpolation followed by Sampling is identity
- *Ideally matched*: Sampling followed by Interpolation is orthogonal projection onto $S = \mathcal{R}(\Phi) = \mathcal{N}(\widetilde{\Phi}^*)^\perp$
  - Ideally matched interpolator: Pseudoinverse of Sampling operator
  - For orthonormal vectors, pseudoinverse is the adjoint!
- Reading: Sections 5.1, 5.2, 5.3.1 and parts of 5.3.2 up to Theorem 5.7. Shannon's original paper sections I and II.