Dynamical Systems for Engineers: Exercise Set 8, Solutions

Exercise 1

The system we consider here is an oscillator that is used in front ends of some mobile phones. A model for this oscillator is given by the following three differential equations

$$\begin{aligned}\dot{x_1} &= 20\left(-n(x_2)+x_3\right)\\ \dot{x_2} &= 20x_3\\ \dot{x_3} &= \frac{1}{40}\left(x_1+x_2\right)-x_3\end{aligned}$$

where $n(x_2) = e^{-x_2} - 1$.

1. The only equilibrium point is the origin.

2. The stability analysis follows from the eigenvalues of the Jacobian matrix evaluated at the origin (linearization of the system around the equilibrium point). The Jacobian matrix of the system is

$$J(x_1,x_2,x_3) = \begin{bmatrix} 0 & 20e^{-x_2} & 20 \\ 0 & 0 & 20 \\ 1/40 & 1/40 & -1 \end{bmatrix}.$$

At the origin, the characteristic polynomial is

$$\det(\lambda I_3 - J(0,0,0)) = \lambda^3 + \lambda^2 - \lambda - 10.$$

Because it is a third degree polynomial, it has at least one real root. After some trial and error of different we find that it is $\lambda = 2$, hence one of the eigenvalues of the Jacobian has a real poistive part, and the origin is unstable.

We can factorize the characteristic polynomial as

$$(\lambda - 2)(\lambda^2 + 3\lambda + 5),$$

and we can find the other two eigenvalues, which are a complex conjugated pair with negative real part. So the origin is a saddle-focus equilibrium point.

Exercise 2

In the context of immunology, we would like to analyze an infection model. Consider two types of cells: the target cells T that might become infected, and the cells I infected by some virus.

The following events affect the cell populations sizes:

- target cells are produced at a constant rate σ cells per time unit;
- a fixed fraction δ_T of all target cells dies every unit of time;
- the infected cells have the ability to infect target cells when they encounter them; assume that target cells become infected at rate β per day per infected cell, in other words, a proportion β of all infected cells will infect the target cells in a time unit;
- a fixed fraction δ_I of all infected cells dies every unit of time.

1. The differential equations describing the dynamics of the populations of target (T) and infected (I) cells are

$$\begin{aligned}\frac{dT}{dt}(t) &= \sigma - \delta_T T(t) - \beta T I(t)\\ \frac{dI}{dt}(t) &= \beta T I(t) - \delta_I I(t).\end{aligned}$$

2. Setting the left hand side to zero in the above differential equations and solving, we find that the system has up to two equilibria, which are

$$(\overline{T}_h, \overline{I}_h) = \left(\frac{\sigma}{\delta_T}, 0\right) \tag{1}$$

$$(\overline{T}_i, \overline{I}_i) = \left(\frac{\delta_I}{\beta}, \frac{\sigma}{\delta_I} - \frac{\delta_T}{\beta}\right). \tag{2}$$

We see that in the first equilibrium point (1), there are no infected cells, so we call it the "healthy" steady state. The second equilibrium point (2) represents a chronic infection, where both target and infected cells coexist. Needless to say, the first equilibrium (1) is preferred. Note that the healthy equilibrium (1) always exists, but that the "infected" equilibrium (2) exists if and only if

$$\sigma\beta > \delta_T \delta_I. \tag{3}$$

If $\sigma\beta \leq \delta_T\delta_I$, then the system has only the first equilibrium (1).

3. In order to study the stability of each equilibrium point, we can linearize the system around them and make a stability analysis of the linearized system. The Jacobian of the system is:

$$J(T, I) = \begin{bmatrix} -\delta_T - \beta I & -\beta T \\ \beta I & \beta T - \delta_I \end{bmatrix}.$$

The eigenvalues of this matrix evaluated at the first (healthy) equilibrium point $(\overline{T}_h, \overline{I}_h)$ are

$$\begin{aligned}\lambda_1^h &= -\delta_T\\ \lambda_2^h &= \frac{\beta\sigma}{\delta_T} - \delta_I\end{aligned}$$

We see that $\lambda_1^h < 0$ and that $\lambda_2^h < 0$ if and only if $\sigma\beta < \delta_T\delta_I$. Therefore the healthy equilibrium (1) is asymptotically stable if $\sigma\beta < \delta_T\delta_I$ and unstable if $\sigma\beta > \delta_T\delta_I$, i.e. if (3) is verified. If $\sigma\beta = \delta_T\delta_I$, the linearization does not allow us to conclude that the equilibrium point is stable or not.

The eigenvalues of this matrix evaluated at the second (infected) equilibrium point $(\overline{T}_i, \overline{I}_i)$ are

$$\lambda_{1,2}^i = -\frac{\sigma\beta}{2\delta_I} \pm \frac{1}{2}\sqrt{\left(\frac{\sigma\beta}{\delta_I}\right)^2 - 4(\sigma\beta - \delta_T\delta_I)}.$$

If the eigenvalues are complex, they both have the same negative real part, and the equilibrium $(\overline{T}_i, \overline{I}_i)$ is asymptotically stable. If the eigenvalues are real, then (3) yields that the largest of the two is negative, because

$$\begin{aligned}\lambda_1 &= -\frac{\sigma\beta}{2\delta_I} + \frac{1}{2}\sqrt{\left(\frac{\sigma\beta}{\delta_I}\right)^2 - 4(\sigma\beta - \delta_T\delta_I)}\\ &< -\frac{\sigma\beta}{2\delta_I} + \frac{\sigma\beta}{2\delta_I} = 0\end{aligned}$$

Therefore, for any set of values of the parameters such that the equilibrium point $(\overline{T}_i, \overline{I}_i)$ exists (i.e. such that (3) holds), it is asymptotically stable.