Localization and Uncertainty

Mathematical Foundations of Signal Processing

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Outline

- Time and frequency
  - Continuous-time (CT)
  - Discrete Time (DT)
- Uncertainty relations
- Scale and resolution

Goal:

- To understand the intuition behind time-frequency analysis

Why:

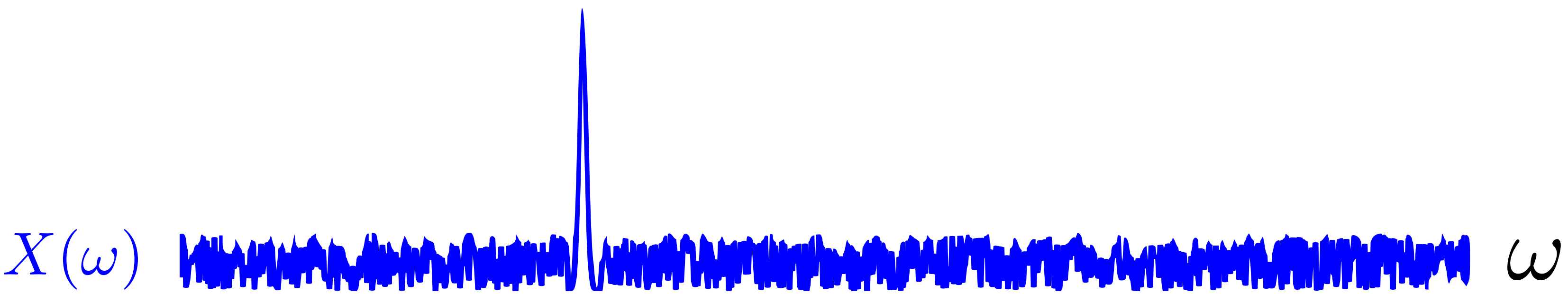
- To understand what we can expect from individual vectors in a basis

Readings:

- Chapter 7

Motivation

- filtering



Motivation

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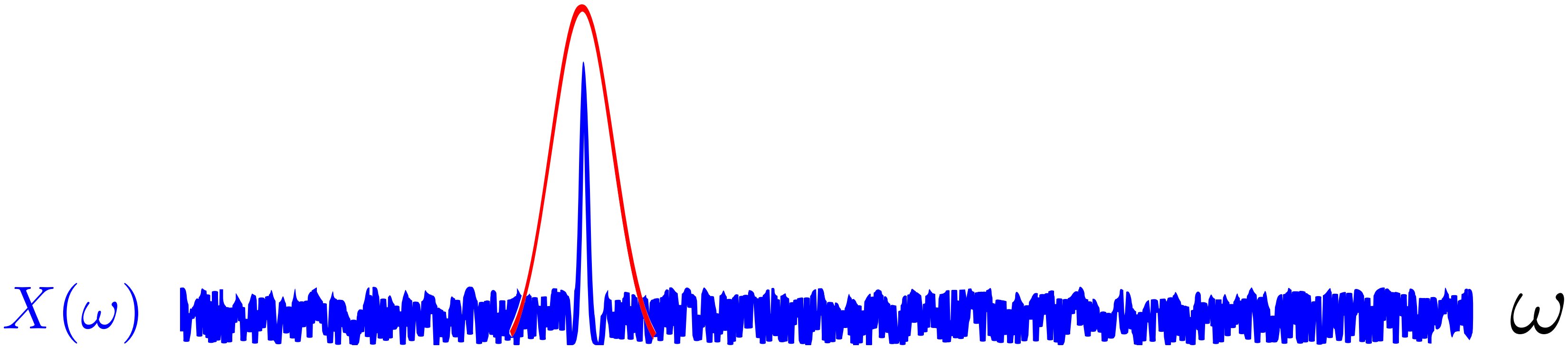

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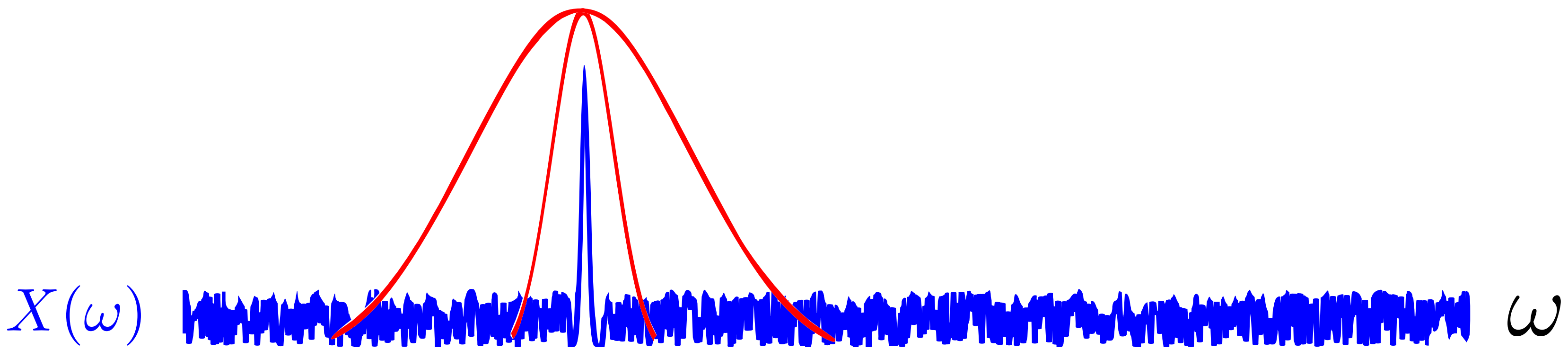

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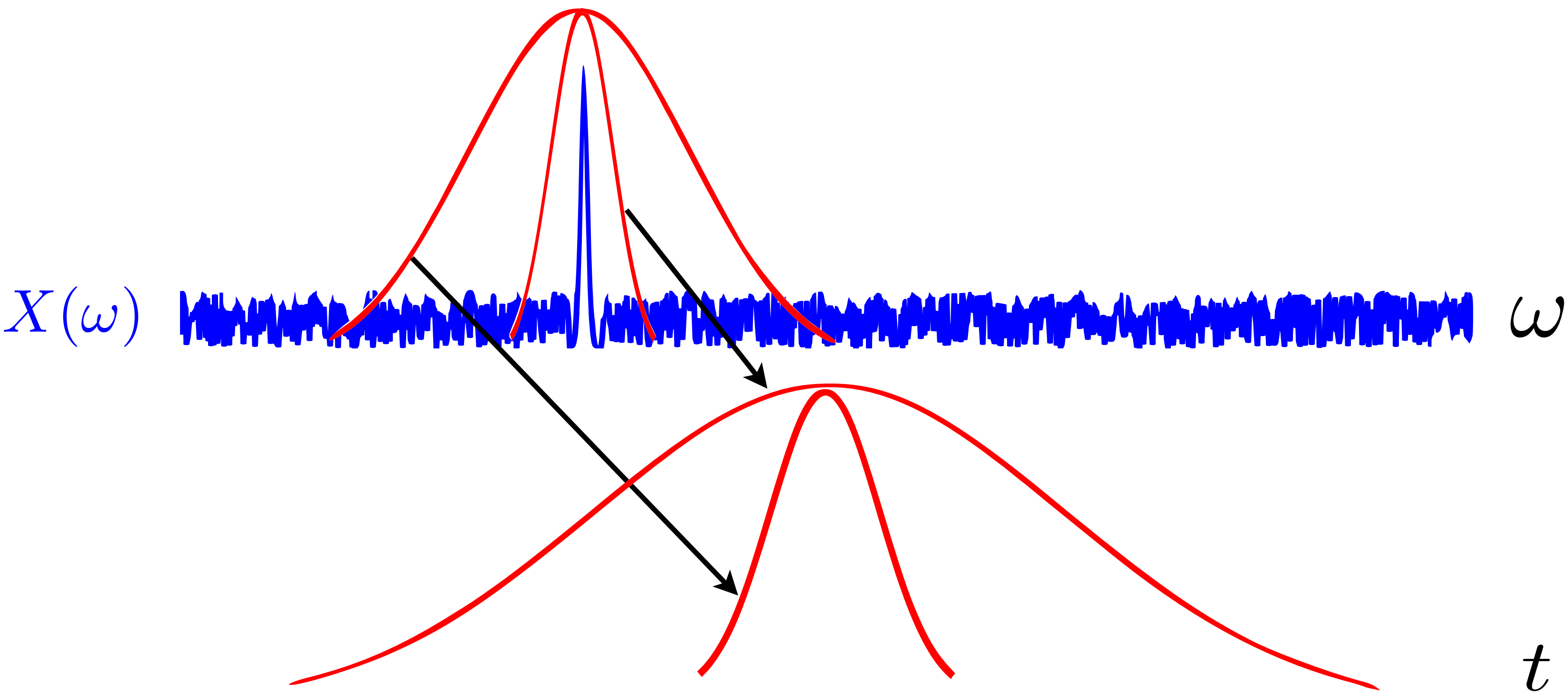
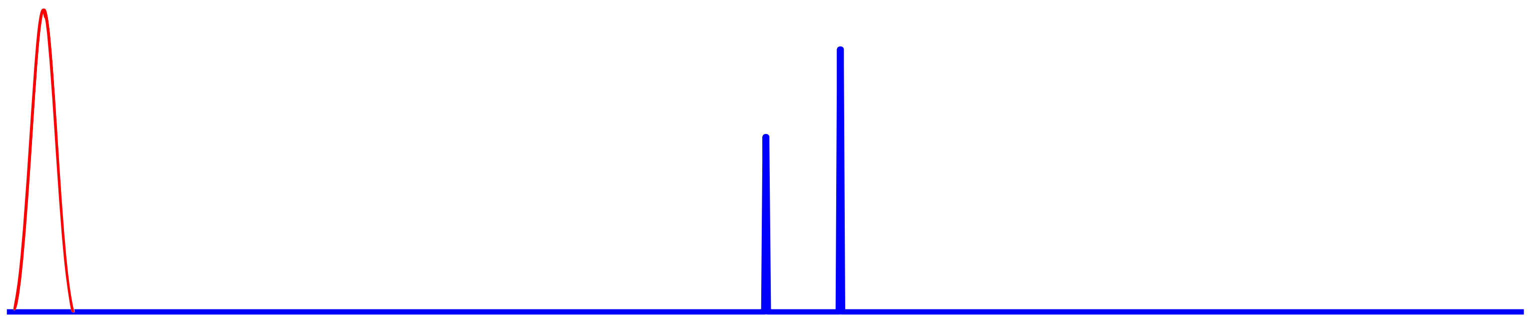


Motivation

- Representation:

$$x(t) = \sum_{n \in \mathbb{Z}} \alpha_n \varphi(t - n t_0)$$



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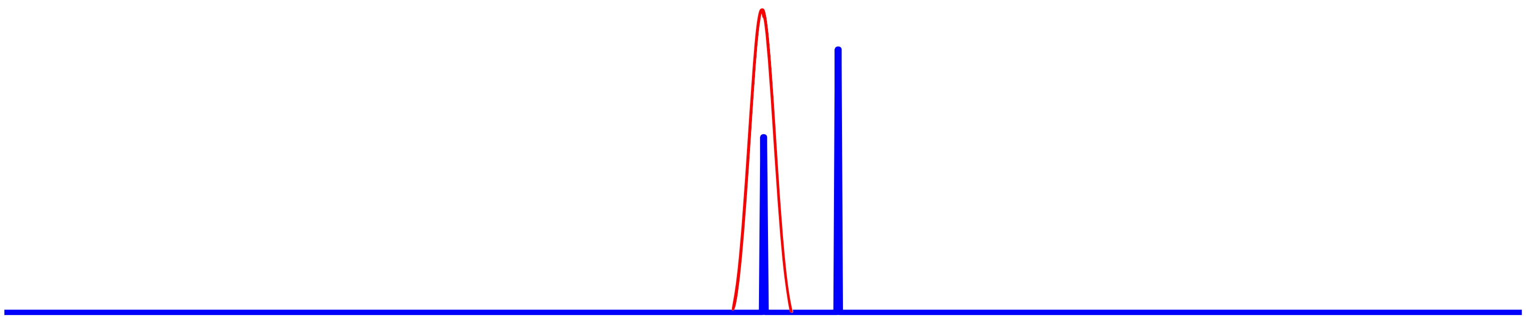

α_n 0000000000000000000000000000000001210231000000000000000000000000000000

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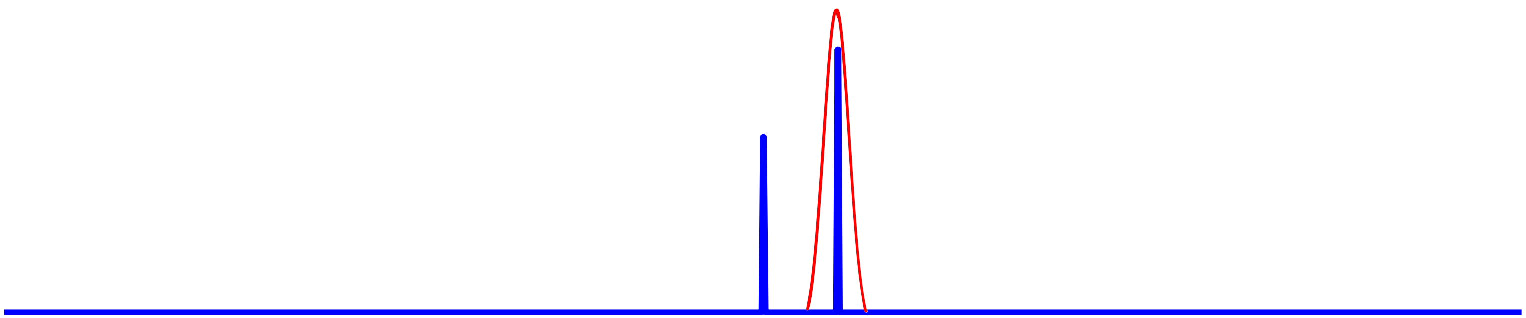

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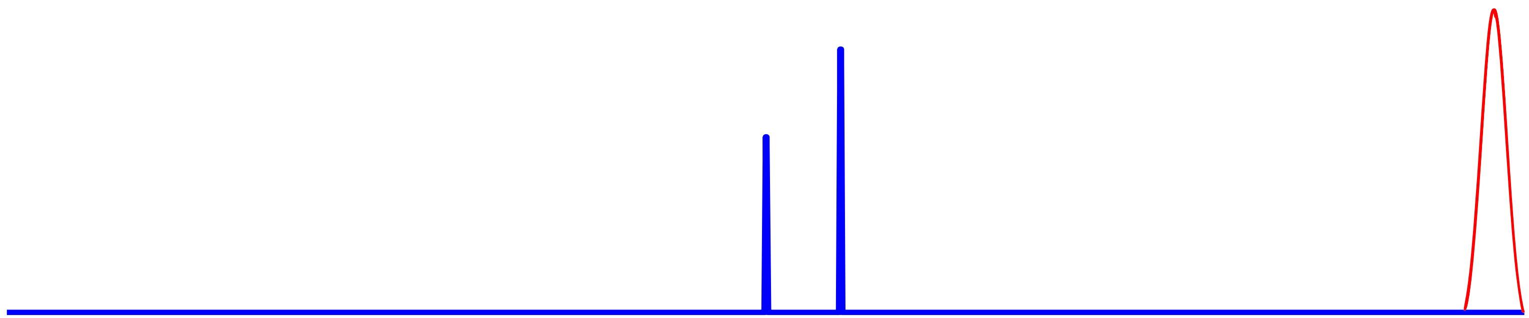

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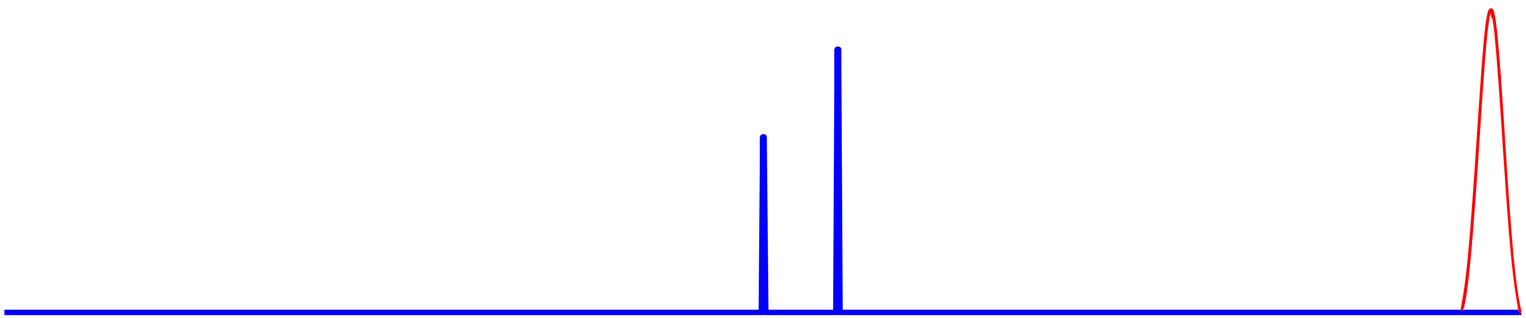

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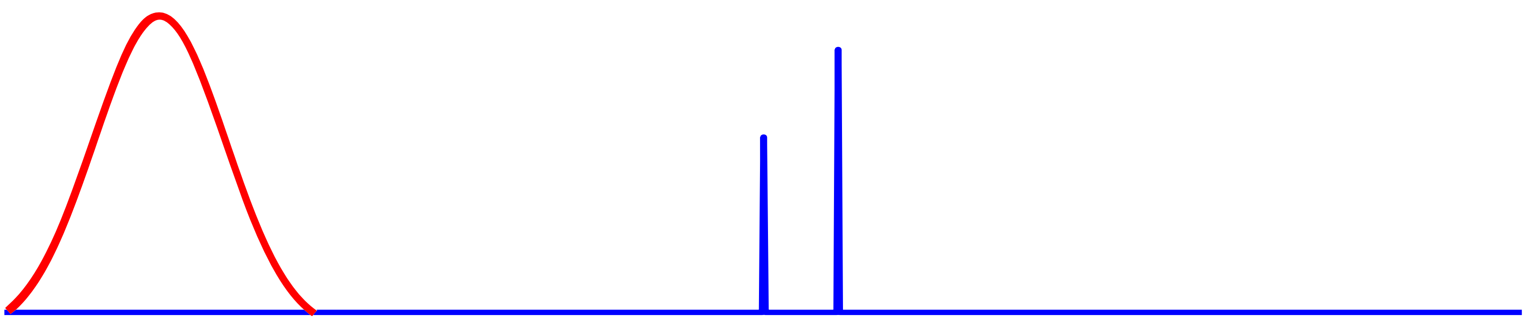

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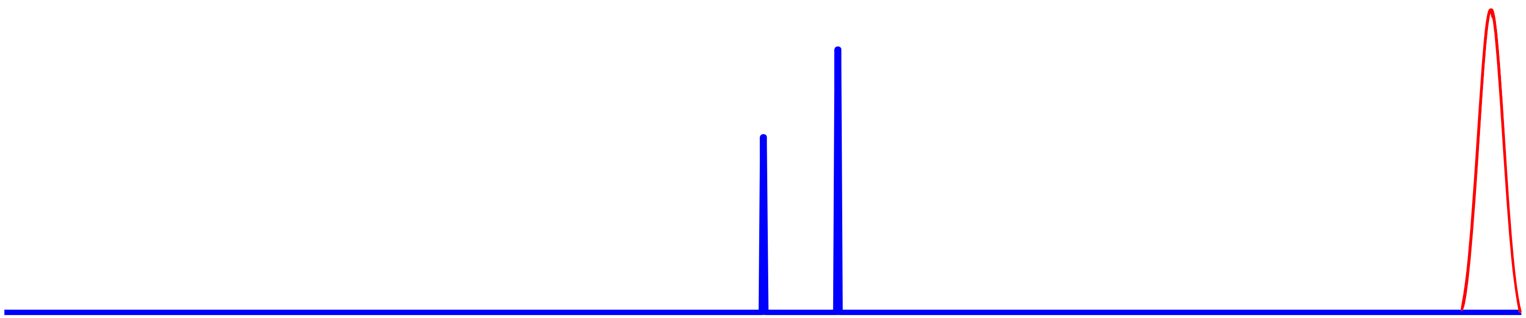

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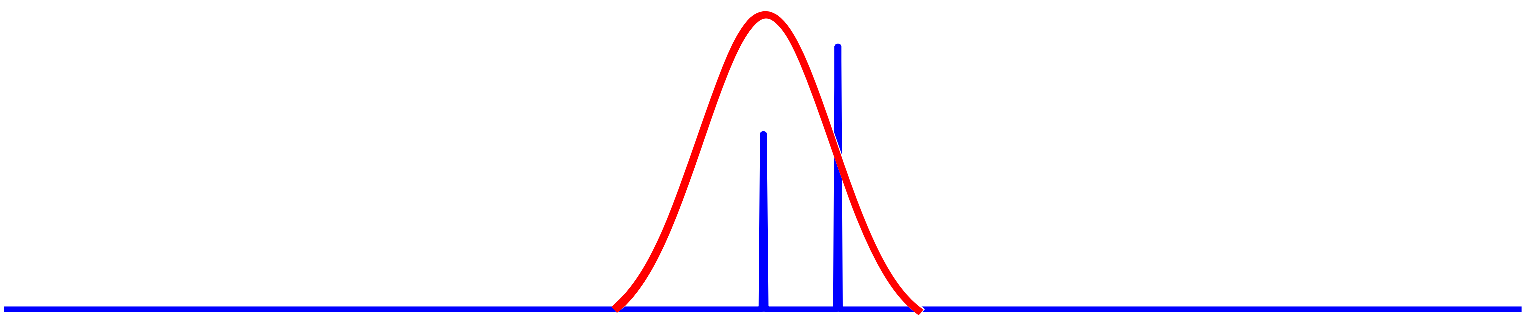

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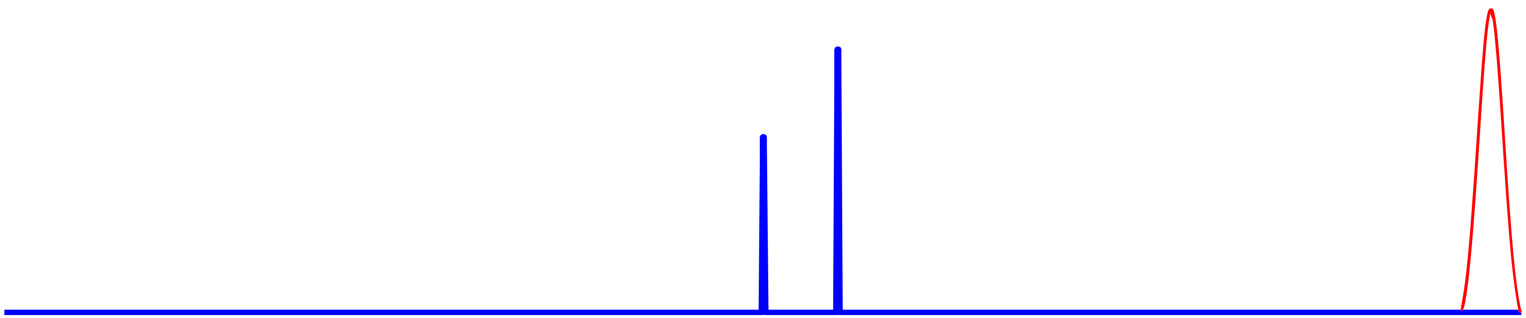

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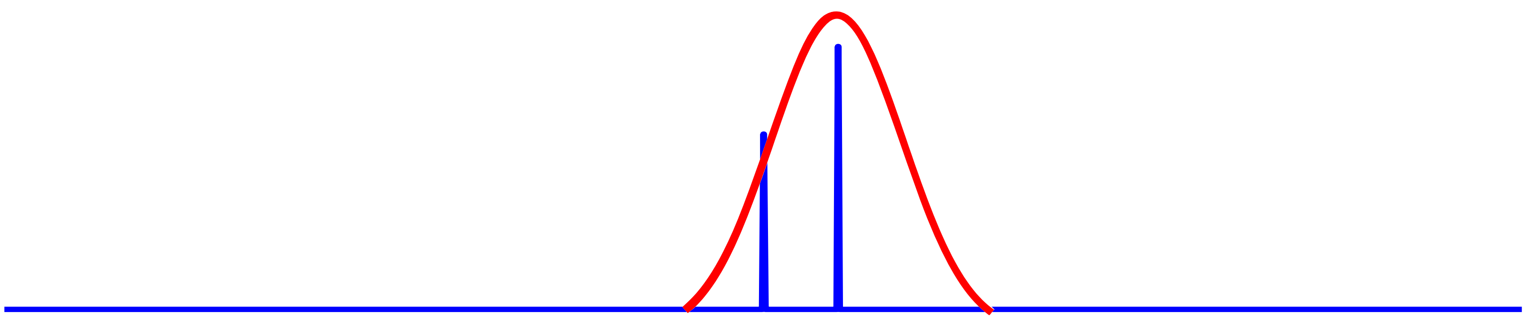

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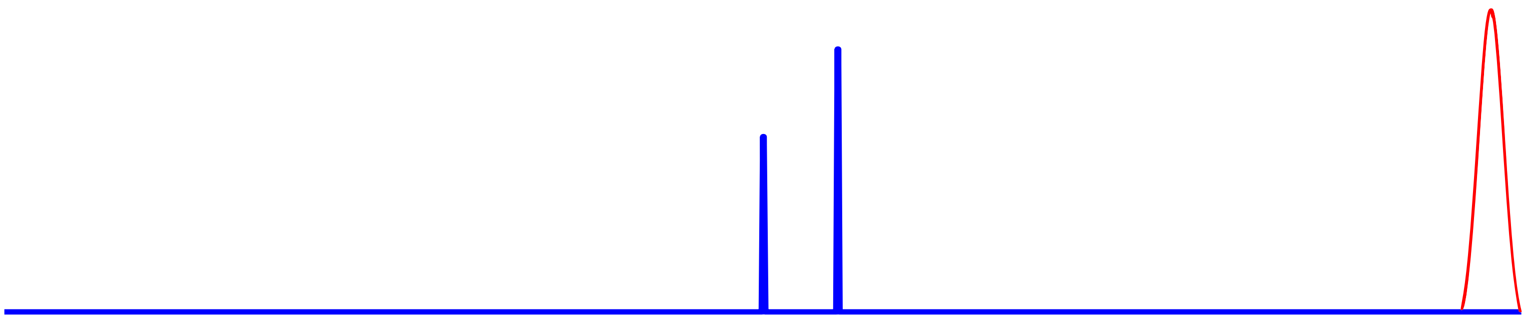

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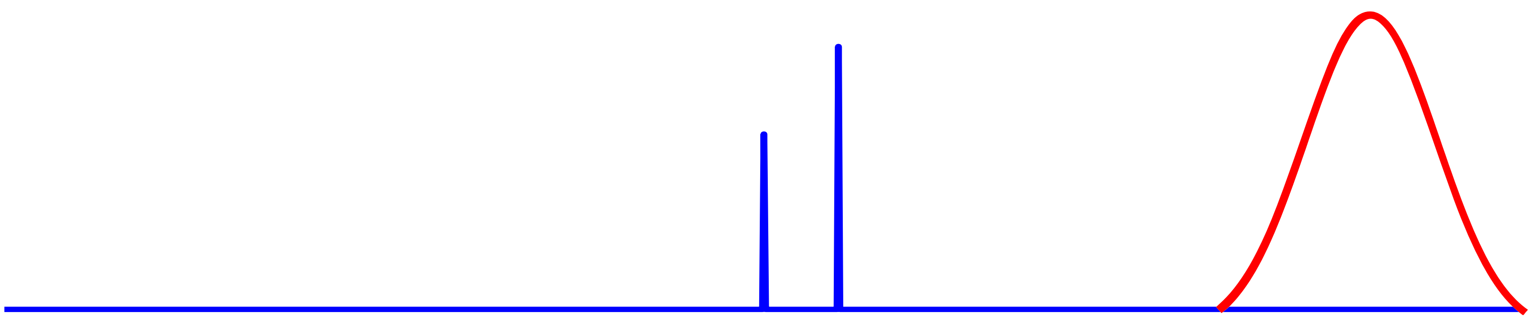
sparse



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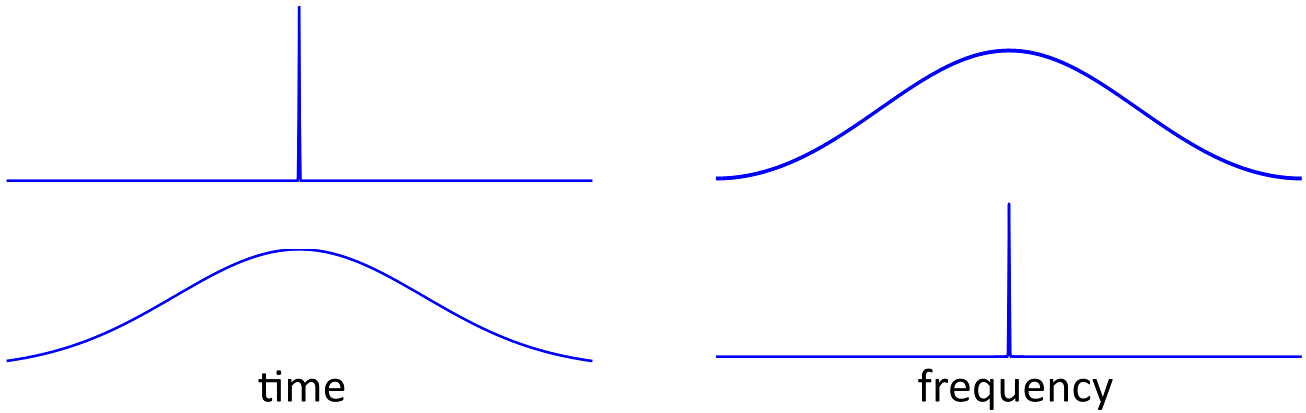
not sparse

What?

- Most of the times we want something “localized” (= “compact”= “sharp”) in time and frequency
  – in frequency -> more selective
  – in time -> better resolution and sparsity of coefficients
- Is it possible to be compact both in time and in frequency?!
  – Theme: localization both in **time** and **frequency** is limited!



$$\delta(t) \quad \overset{\text{FT}}{\longleftrightarrow} \quad 1,$$
$$1 \quad \overset{\text{FT}}{\longleftrightarrow} \quad 2\pi\delta(\omega).$$

We want to tell you about…

- The **spread** of a signal in time and frequency
- The trade off: **uncertainty principle**
  - Concentrated in one domain ⟶ spread in the other domain

$$x(t) \longrightarrow X(\omega)$$
$$X(t) \longrightarrow 2\pi x(-\omega)$$

  - The product of the spreads has a lower bound
- **Scaling** trades between the time and frequency spreads
- **Resolution** is related to
  - Bandwidth of the signal
  - Number of degrees of freedom per unit of measure

How do we define localization?

- Probability density function (PDF)

$$Z \sim f_Z(z)$$

$$\Pr(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_2} f_Z(z)\,dz \qquad \int f_Z(z)\,dz = 1$$

How do we define localization?

- Probability density function (PDF)

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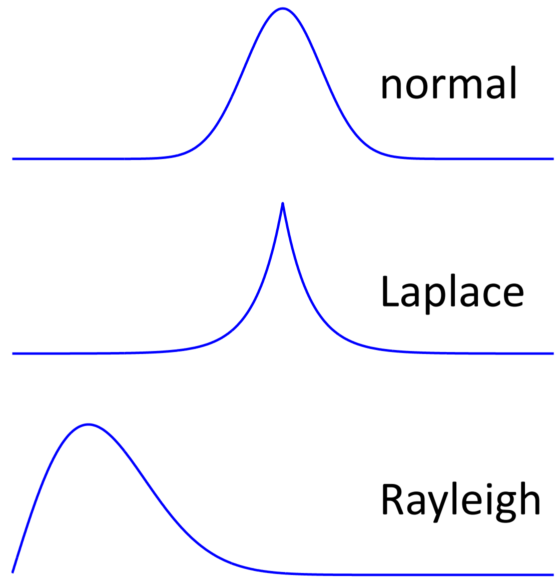

$$f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

$$f_Z(z) = \frac{1}{2b} e^{-\frac{|z-\mu|}{b}}$$

$$f_Z(z) = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}}$$

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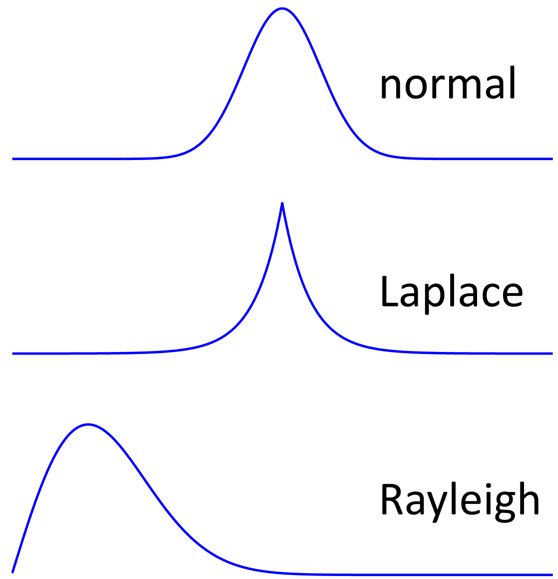

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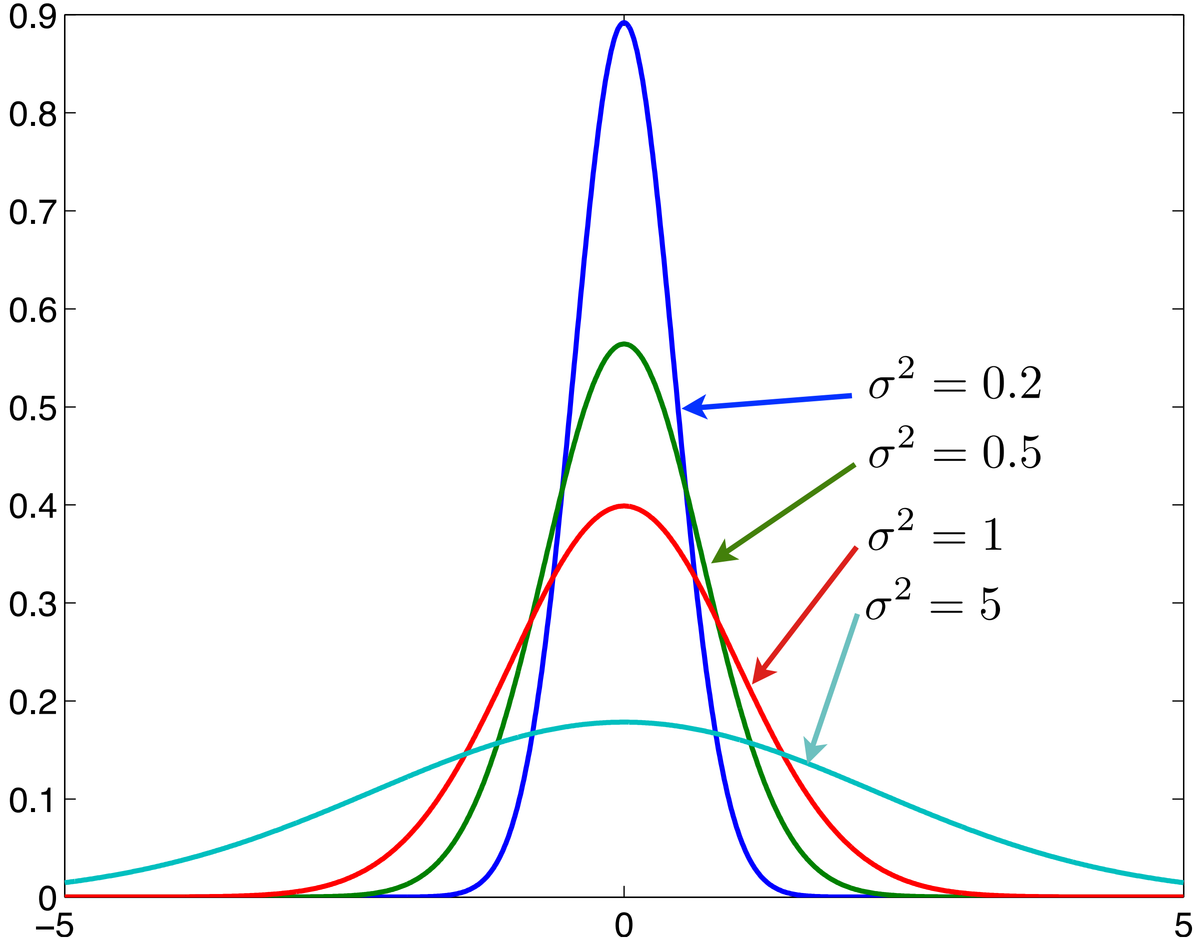
$$f_Z(z) = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}}$$

center: $\mu_Z = \mathbb{E}[Z] = \int z f_Z(z)\, dz$

variance: $\sigma_Z^2 = \mathbb{E}[(Z-\mu_Z)^2] = \int (z-\mu_Z)^2 f_Z(z)\, dz$

Variance versus Spread



- larger variance = more spread

Time localization

- Time **center** and **spread**

$$\int_{-\infty}^{\infty} \frac{|x(t)|^2}{\|x\|^2} \, dt = 1$$

$$\boxed{\begin{aligned} \mu_t &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} t \, \frac{|x(t)|^2}{\|x\|^2} \, dt \\ \Delta_t^2 &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (t - \mu_t)^2 \, \frac{|x(t)|^2}{\|x\|^2} \, dt \end{aligned}}$$

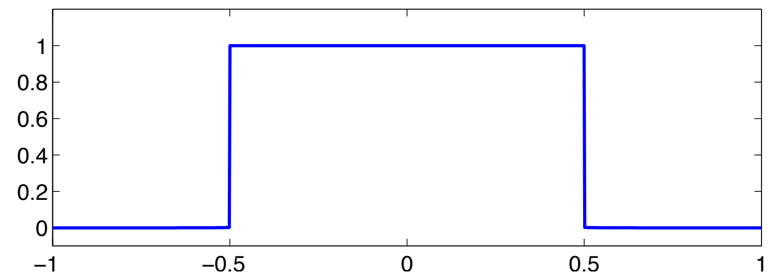
- This is like the variance of t, where $t \sim \dfrac{|x(t)|^2}{\|x\|^2}$

Some examples

i. **The box function**

$$b(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

has $\mu_t = 0$ and $\Delta_t^2 = \dfrac{1}{12}$

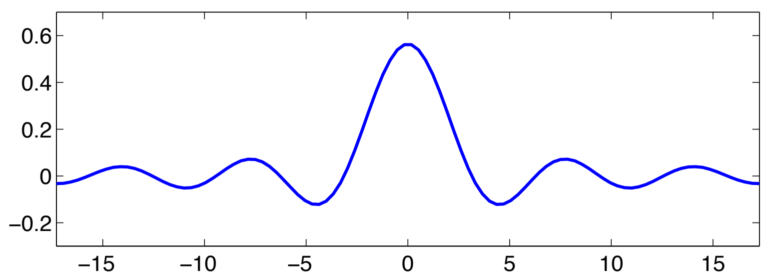


ii. **The sinc function**

$$x(t) = \frac{1}{\sqrt{\pi}} \frac{\sin t}{t}$$

$|x(t)|^2$ decays only as $\dfrac{1}{|t|^2}$

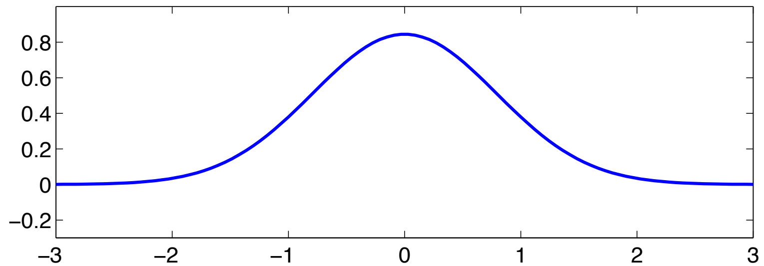
so Δ_t^2 is infinite



iii. **Gaussian function**

$$g(t) = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha t^2}$$

has $\mu_t = 0$ and $\Delta_t^2 = \dfrac{1}{4\alpha}$



Frequency Localization

- Frequency center and spread

$$\|X(\omega)\|^2 = 2\pi\|x(t)\|^2$$

$$\int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{2\pi\|x\|^2}\,dt = 1$$

$$\boxed{\begin{aligned} \mu_{\mathrm{f}} &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \omega\,\frac{|X(\omega)|^2}{2\pi\|x\|^2}\,dt \\ \Delta_{\mathrm{f}}^2 &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (\omega-\mu_{\mathrm{f}})^2\,\frac{|X(\omega)|^2}{2\pi\|x\|^2}\,dt \end{aligned}}$$

- This is like the variance of ω where $\omega \sim \dfrac{|X(\omega)|^2}{2\pi\|x\|^2}$

The Heisenberg Uncertainty Principle

Given a function $x \in \mathcal{L}^2(\mathbb{R})$, the product of its squared time and frequency spreads is lower bounded as

$$\Delta_t^2 \, \Delta_\mathrm{f}^2 \geq \frac{1}{4}$$

- The lower bound is attained by **Gaussian** functions:

$$x(t) \;=\; \gamma e^{-\alpha t^2}, \quad \alpha > 0$$

- Proof: on the board!

Proof for Absents

- STEP 1

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} t\, x(t)\, x'(t)\, dt\right|^2 &\overset{(a)}{\leq} \int_{-\infty}^{\infty} |t\, x(t)|^2\, dt \int_{-\infty}^{\infty} |x'(t)|^2\, dt \\
&\overset{(b)}{=} \underbrace{\int_{-\infty}^{\infty} |t\, x(t)|^2\, dt}_{=\Delta_{\mathrm{t}}^2} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |j\omega\, X(\omega)|^2\, d\omega}_{=\Delta_{\mathrm{f}}^2} = \Delta_{\mathrm{t}}^2\, \Delta_{\mathrm{f}}^2,
\end{aligned}
$$

(a) Cauchy Schwarz, (b) Parseval

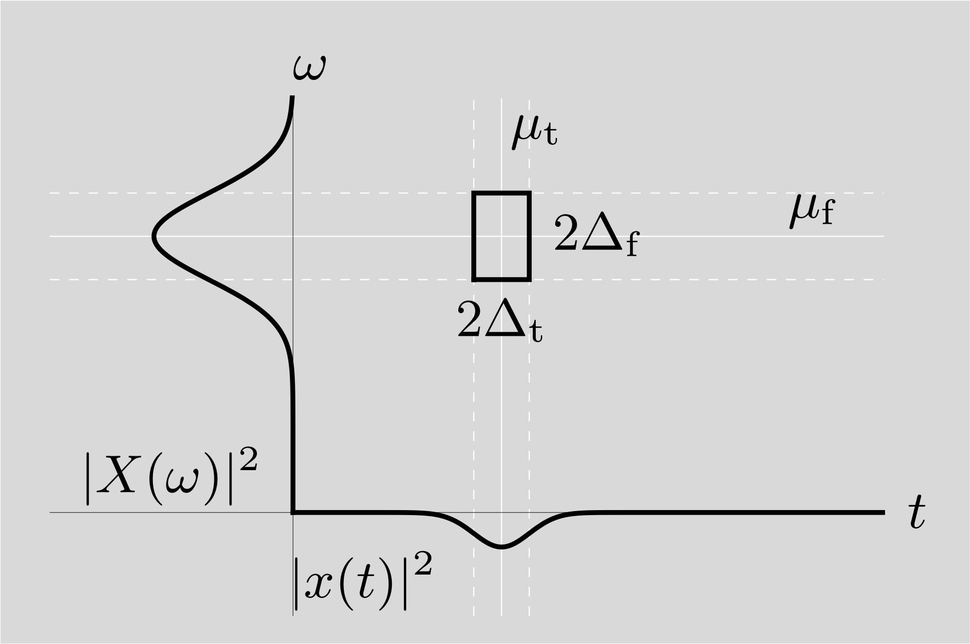
- STEP 2

$$
\begin{aligned}
\int_{-\infty}^{\infty} t\, x(t)\, x'(t)\, dt &\overset{(a)}{=} \frac{1}{2}\int_{-\infty}^{\infty} t\, \frac{dx^2(t)}{dt}\, dt \overset{(b)}{=} \frac{1}{2}\, t\, x^2(t)\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty} x^2(t)\, dt \\
&\overset{(c)}{=} -\frac{1}{2}\int_{-\infty}^{\infty} x^2(t)\, dt \overset{(d)}{=} -\frac{1}{2},
\end{aligned}
$$

(a) $(x^2(t))' = 2x'(t)x(t)$ (b) integration by parts (c) $\lim_{t\to\infty} tx^2(t) = 0$

Heisenberg Boxes

- Time-frequency plane and H-boxes
  - Time/frequency centered at (μ_t, μ_f)
  - Extent of rectangular box $(2\Delta_t, 2\Delta_\mathrm{f})$



Heisenberg: The box area is at least 2!

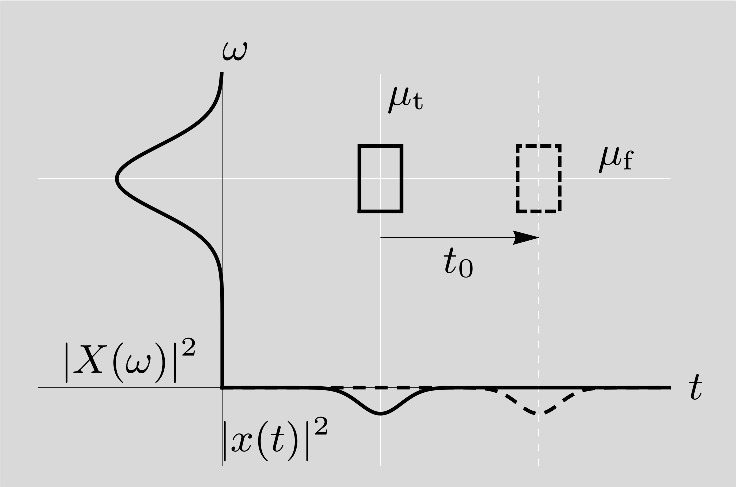
Time Shift, Frequency Shift

- Shift in time

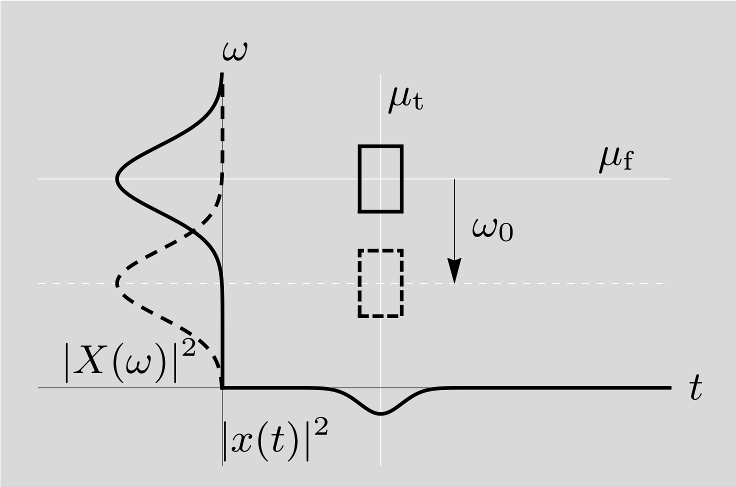
$$y(t) \;=\; x(t - t_0) \quad \overset{\text{FT}}{\longleftrightarrow} \quad Y(\omega) \;=\; e^{-j\omega t_0}\, X(\omega).$$

- Shift in frequency

$$y(t) \;=\; e^{j\omega_0 t}\, x(t) \quad \overset{\text{FT}}{\longleftrightarrow} \quad Y(\omega) \;=\; X(\omega - \omega_0).$$



shift in time



shift in frequency

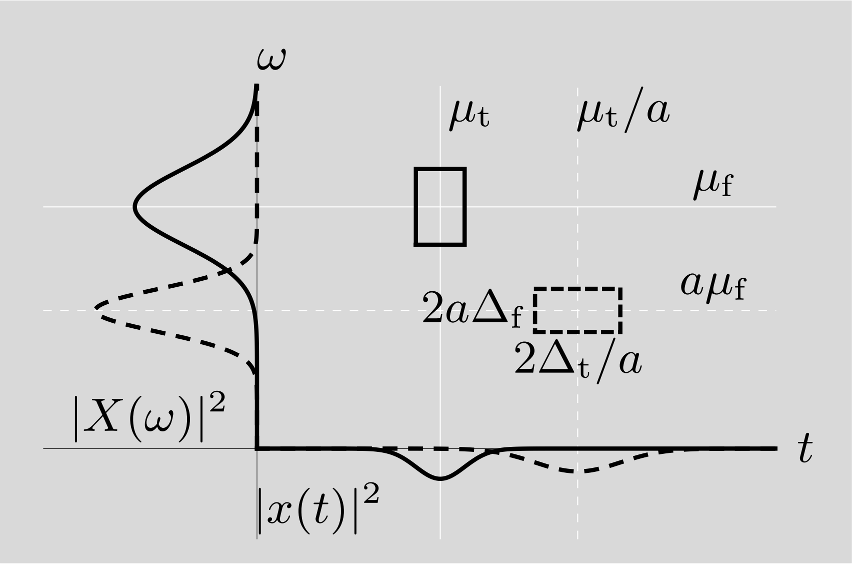
- Any volunteers?

Energy Conserving Scaling

- scaling

$$y(t) = \sqrt{a}\, x(at) \quad \stackrel{\text{FT}}{\longleftrightarrow} \quad Y(\omega) = \frac{1}{\sqrt{a}} X\left(\frac{\omega}{a}\right).$$

$$\|y\|^2 = \|x\|^2$$



- shifting or scaling does not change the area of the box!
  - narrowing a signal in one domain must widen it in the other!

Shift and Scaling: Summary

Function	Time center	Time spread	Fourier transform	Frequency center	Frequency spread
$x(t)$	μ_{t}	Δ_{t}	$X(\omega)$	μ_{f}	Δ_{f}
$x(t-t_0)$	$\mu_{\mathrm{t}}+t_0$	Δ_{t}	$e^{-j\omega t_0}X(\omega)$	μ_{f}	Δ_{f}
$e^{j\omega_0 t}x(t)$	μ_{t}	Δ_{t}	$X(\omega-\omega_0)$	$\mu_{\mathrm{f}}+\omega_0$	Δ_{f}
$\sqrt{a}x(at)$	μ_{t}/a	Δ_{t}/a	$X(\omega/a)/\sqrt{a}$	$a\mu_{\mathrm{f}}$	$a\Delta_{\mathrm{f}}$

Discrete-time sequences

- Similar definitions to continuous-time
- Sequences with 2π-periodic DTFT

$$\sum_{n\in\mathbb{Z}} \frac{|x_n|^2}{\|x\|^2} = 1 \qquad \int_{-\pi}^{\pi} \frac{|X(e^{j\omega})|^2}{2\pi\|x\|^2}\, dt = 1$$

- centers and spreads

$$\mu_n \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} n \frac{|x_n|^2}{\|x\|^2} \qquad \mu_{\text{f}} \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} \omega \frac{|X(e^{j\omega})|^2}{2\pi\|x\|^2}\, dt$$

$$\Delta_n^2 \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} (n-\mu_n)^2 \frac{|x_n|^2}{\|x\|^2} \qquad \Delta_{\text{f}}^2 \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} (\omega-\mu_{\text{f}})^2 \frac{|X(e^{j\omega})|^2}{2\pi\|x\|^2}\, dt$$

Heisenberg Uncertainty Principle

Given a sequence $x \in \ell^2(\mathbb{Z})$, with $X(e^{j\pi}) = 0$, the product of its squared time and frequency spreads is lower bounded as

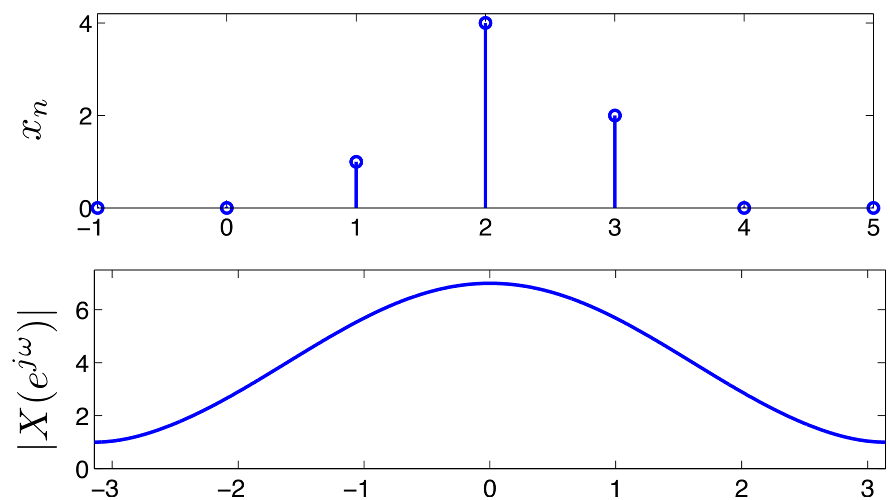
$$\Delta_n^2 \, \Delta_{\mathrm{f}}^2 > \frac{1}{4}$$

- lower bound cannot be achieved
- Necessary condition

$$x_n = 1 + 4\,\delta_{n-1} + 2\,\delta_{n-2}$$

$$X(e^{j\pi}) \neq 0$$

$$\Delta_n^2 \, \Delta_{\mathrm{f}}^2 = 0.239 < \frac{1}{4}$$



Time Shift, Frequency Shift

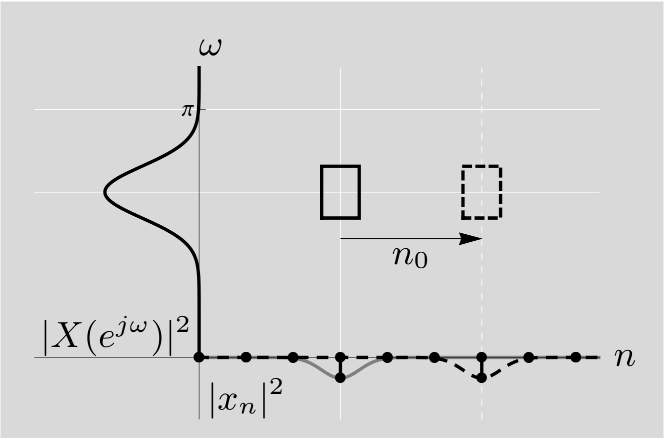
- Shift in time

$$y_n \;=\; x_{n-n_0} \qquad \stackrel{\text{DTFT}}{\longleftrightarrow} \qquad Y(e^{j\omega}) \;=\; e^{-j\omega n_0}\, X(e^{j\omega}).$$

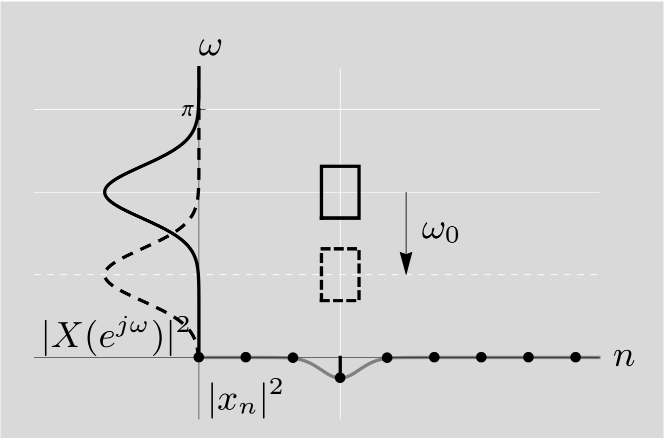
- Shift in frequency

$$y_n \;=\; e^{j\omega_0 n}\, x_n \qquad \stackrel{\text{DTFT}}{\longleftrightarrow} \qquad Y(e^{j\omega}) \;=\; X(e^{j(\omega-\omega_0)}).$$

$$\begin{cases} X(e^{j\omega}) = 0, & \omega \in (\pi-\omega_0, \pi] \quad \text{if } \omega_0 \in (0,\pi] \\ X(e^{j\omega}) = 0, & \omega \in (-\pi, -\pi-\omega_0] \quad \text{if } \omega_0 \in (-\pi, 0] \end{cases}$$



shift in time



shift in frequency

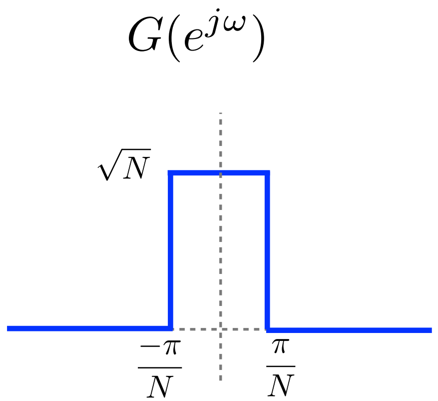
Upsampling Sequences

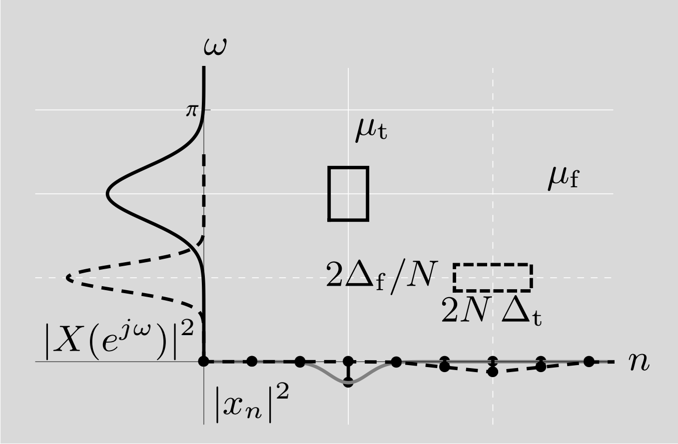
- Upsampling by $N \in \mathbb{Z}^+$ followed by ideal lowpass filtering

$$y_n = g_n *_n \begin{cases} x_{n/N}, & \text{for } n/N \in \mathbb{Z}; \\ 0, & \text{otherwise,} \end{cases} \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad Y(e^{j\omega}) = \sqrt{N}\, X(e^{jN\omega}).$$

$$\mu_t(y) = N\mu_t(x), \qquad \Delta_t(y) = N\Delta_t(x).$$

$$\mu_f(y) = \frac{1}{N}\mu_f(x), \qquad \Delta_f(y) = \frac{1}{N}\Delta_f(x).$$





Upsampling + filtering

Downsampling Sequences

- Downsampling by $N \in \mathbb{Z}^+$ preceded by ideal lowpass filtering

$$y_n = (g * x)_{Nn} \quad \stackrel{\text{DTFT}}{\longleftrightarrow} \quad Y(e^{j\omega}) = \frac{1}{\sqrt{N}} X(e^{j\omega/N}), \quad \omega \in [-\pi, \pi);$$

- provided that $x \in \mathrm{BL}[-\pi/N, \pi/N]$ (filtering does not affect the signal)

$$\mu_{\mathrm{t}}(y) = \frac{1}{N}\mu_{\mathrm{t}}(x), \qquad \Delta_{\mathrm{t}}(y) = \frac{1}{N}\Delta_{\mathrm{t}}(x),$$

$$\mu_{\mathrm{f}}(y) = N\mu_{\mathrm{f}}(x), \qquad \Delta_{\mathrm{f}}(y) = N\Delta_{\mathrm{f}}(x),$$

Shift and Scaling for Sequences: Summary

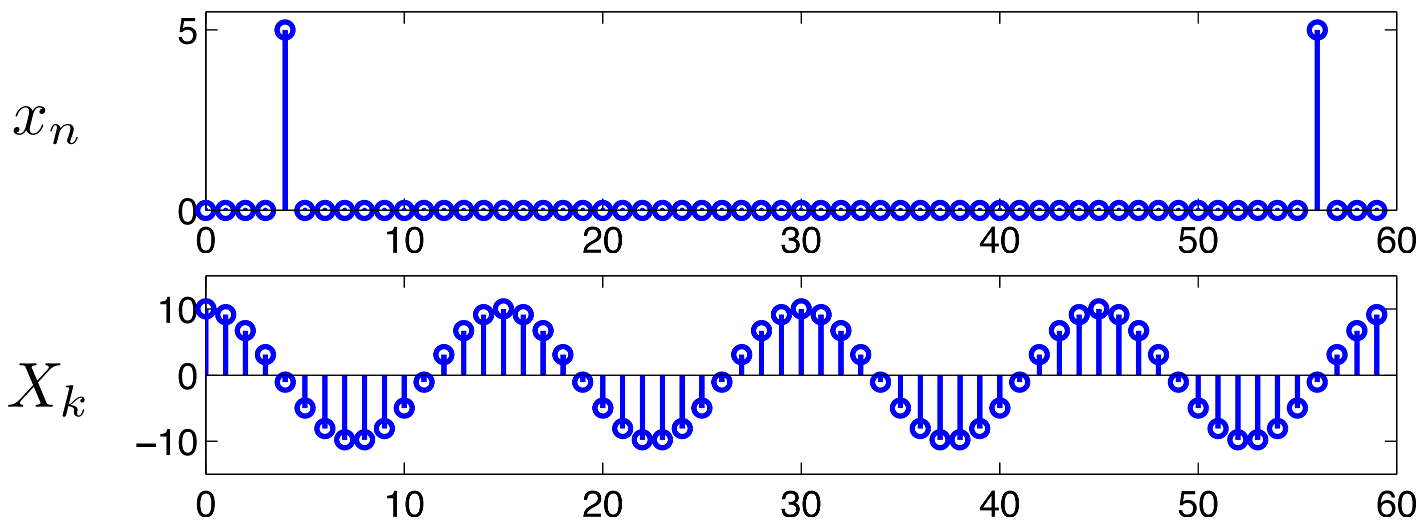
Sequence	Time center	Time spread	DTFT	Frequency center	Frequency spread
x_n	μ_t	Δ_t	$X(e^{j\omega})$	μ_f	Δ_f
x_{n-n_0}	$\mu_\mathrm{t} + n_0$	Δ_t	$e^{-j\omega n_0} X(e^{j\omega})$	μ_f	Δ_f
$e^{j\omega_0 n} x_n$	μ_t	Δ_t	$X(e^{j(\omega-\omega_0)})$	$\mu_\mathrm{f} + \omega_0$	Δ_f
upsampled & postfiltered	$N\mu_\mathrm{t}$	$N\Delta_\mathrm{t}$	$\sqrt{N} X(e^{jN\omega})$	μ_f/N	Δ_f/N
prefiltered & downsampled	μ_t/N	Δ_t/N	$\frac{1}{\sqrt{N}} X(e^{j\omega/N})$	$N\mu_\mathrm{f}$	$N\Delta_\mathrm{f}$

Uncertainty Principle for finite-length sequences

Let $x_n \in \mathbb{C}^N$ with DFT X_k . Let N_n and N_k denote the number of nonzero components of x and X , respectively. Then

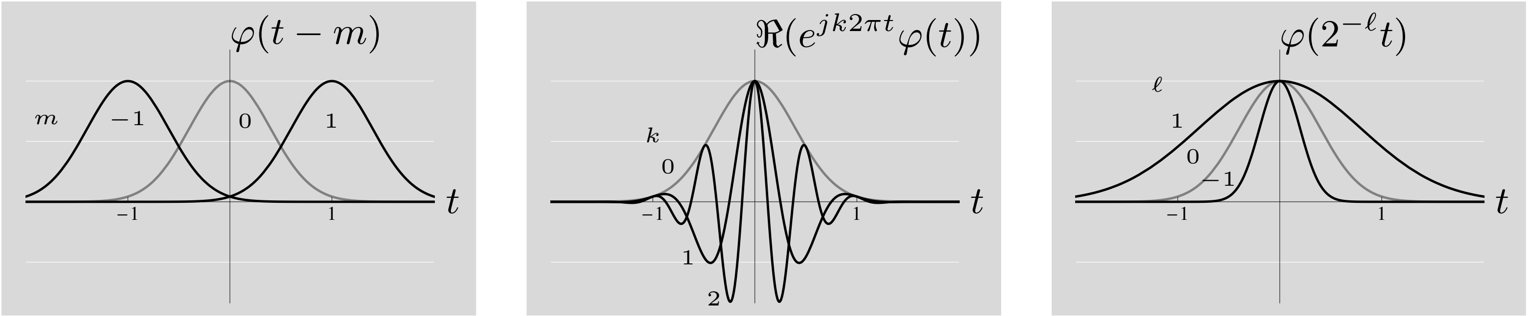
$$N_n \, N_k \geq N$$

- sequences cannot be sparse in both domains!



Localization for set of functions

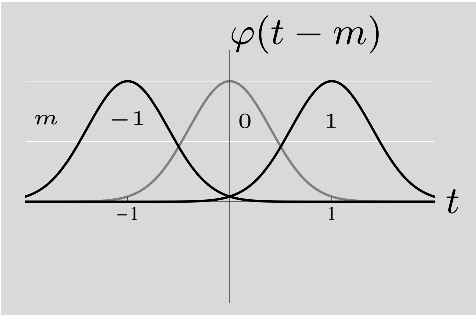
- Prototype function: $\varphi(t): \quad (\mu_t, \Delta_t, \mu_f, \Delta_f)$
- Shifts in time: $\Phi_1 = \{\varphi(t - mt_0)\}_{m\in\mathbb{Z}}$
- Shifts in frequency: $\Phi_2 = \{e^{jk\omega_0 t}\varphi(t)\}_{k\in\mathbb{Z}}$
- Scale: $\Phi_3 = \{\varphi(a^{-\ell}t)\}_{\ell\in\mathbb{Z}}$

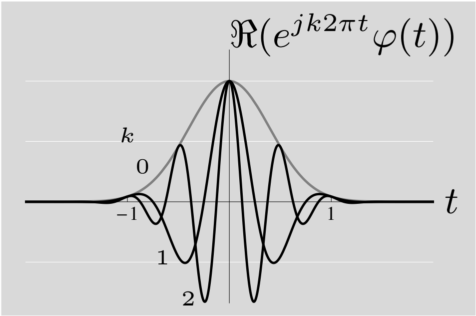


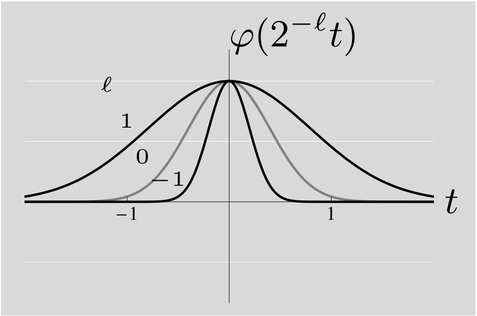
shifts in time   shifts in frequency   scales in time/freq.

Localization for set of functions

- Prototype function: $\varphi(t): \quad (\mu_t, \Delta_t, \mu_f, \Delta_f)$
- Shifts in time: $\Phi_1 = \{\varphi(t - mt_0)\}_{m\in\mathbb{Z}}$
- Shifts in frequency: $\Phi_2 = \{e^{jk\omega_0 t}\varphi(t)\}_{k\in\mathbb{Z}}$
- Scale: $\Phi_3 = \{\varphi(a^{-\ell}t)\}_{\ell\in\mathbb{Z}}$



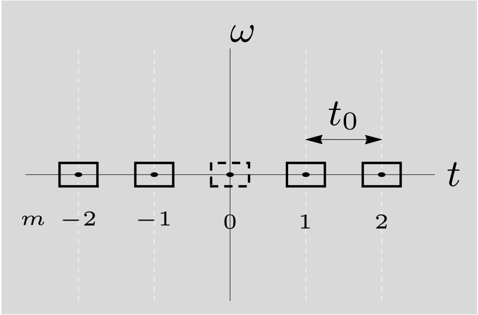


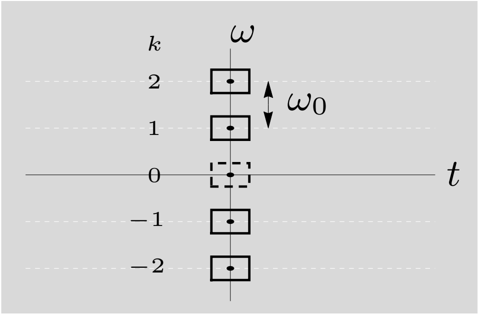


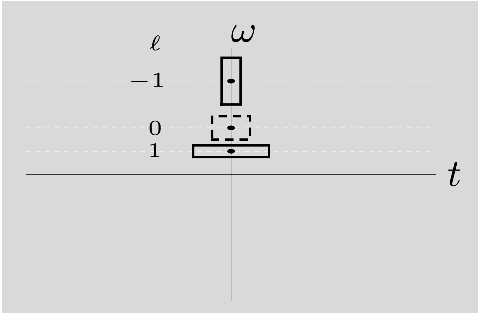
shifts in time

shifts in frequency

scales in time/freq.



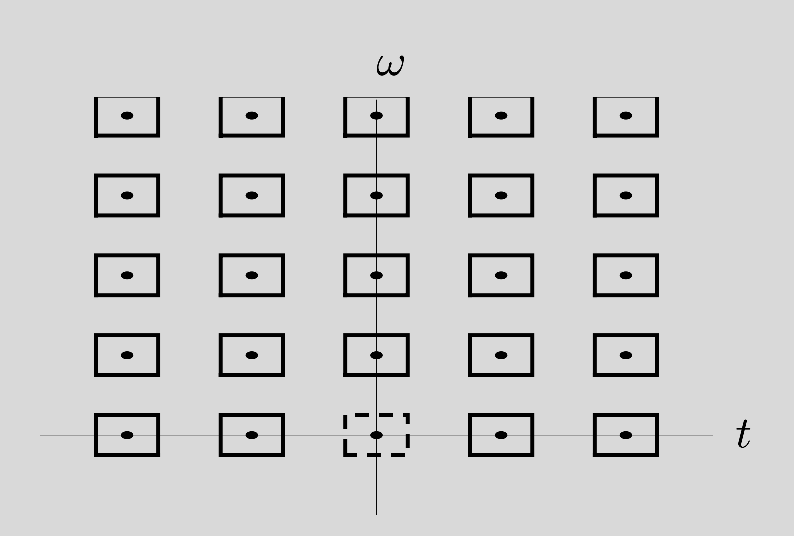




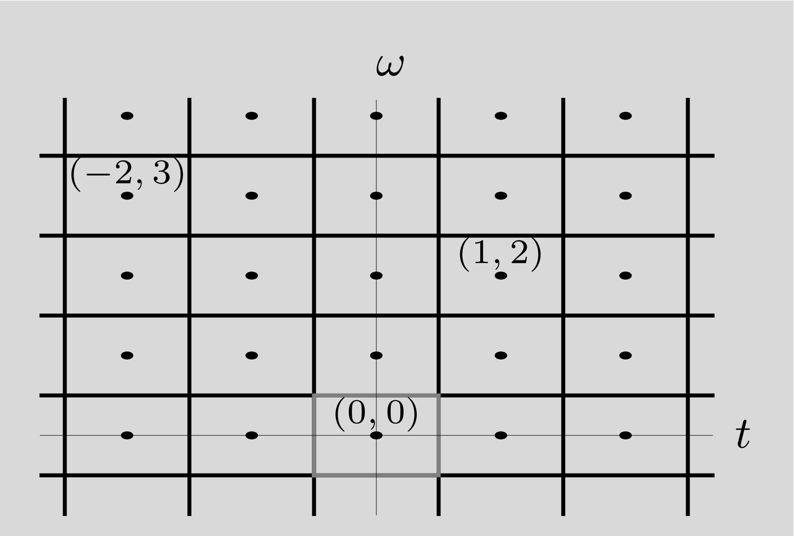
Time-Frequency Tilings

- Time shift and modulation

$$\varphi_{k,m}(t) = e^{jk\omega_0 t}\varphi(t - mt_0), \qquad k, m \in \mathbb{Z}.$$



actual plot



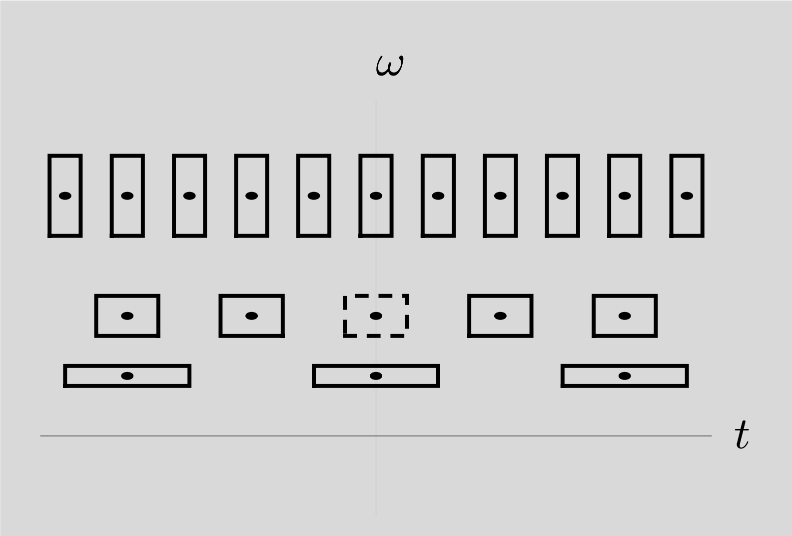
idealized tiling

assumption here: $\mu_t = 0$, $\mu_f = 0$

Time-Frequency Tilings

- Time shift and scaling

$$\varphi_{\ell,m}(t) \;=\; \varphi(a^{-\ell}t - mt_0), \qquad \ell, m \in \mathbb{Z}.$$



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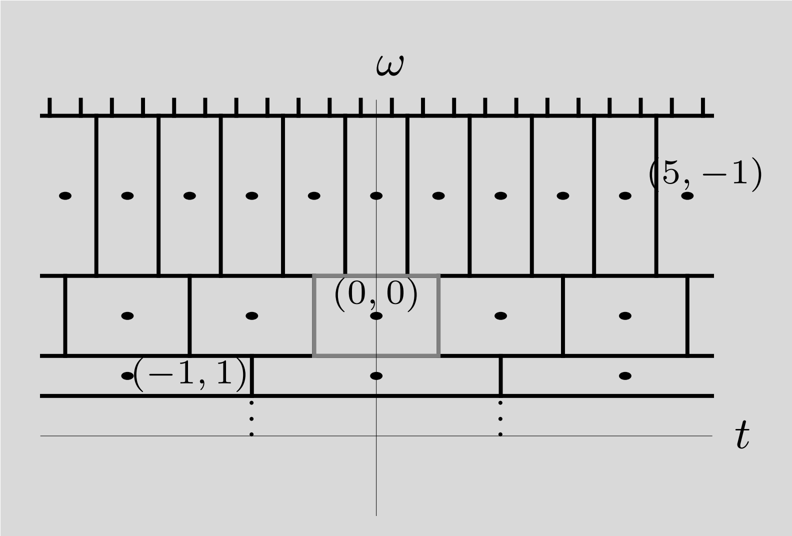


idealized tiling

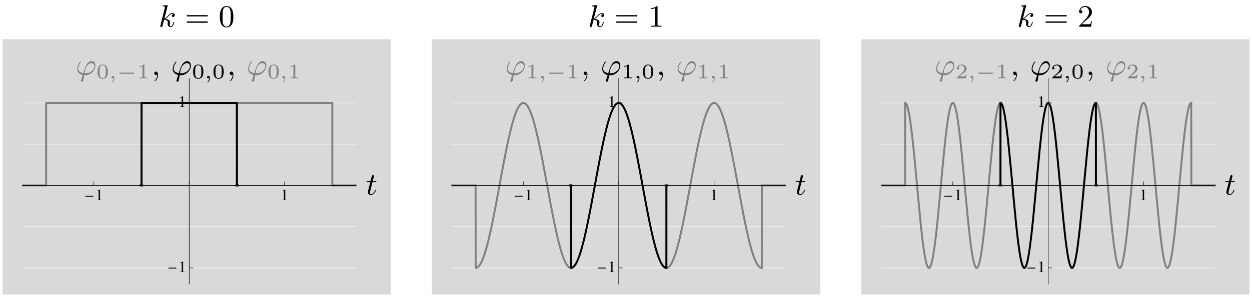
assumption here: $\mu_t = 0$

Idealized vs. Actual Tilings

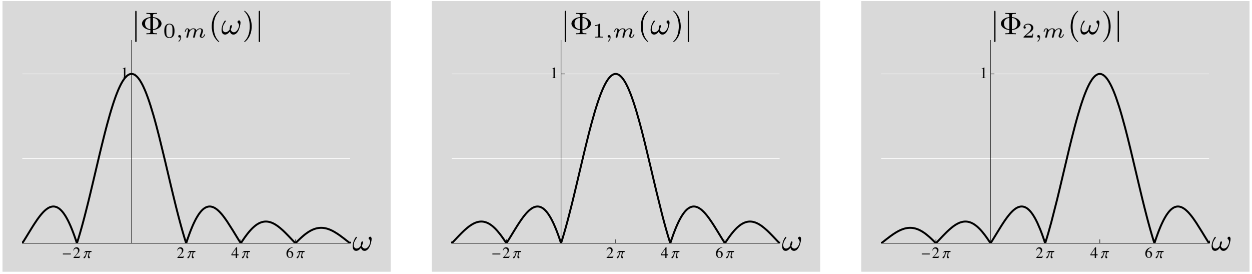
- Example (local Fourier basis)

$$\varphi(t) = \begin{cases} 1 & \text{for } |t| \le 1/2; \\ 0 & \text{otherwise} \end{cases} \qquad t_0 = 1, \quad \omega_0 = 2\pi$$

$$\varphi_{k,m}(t) = e^{jk2\pi t}\varphi(t-m)$$



Basis functions (real parts only).



Magnitudes of the Fourier transform.

Idealized vs. Actual Tilings

- Example (local Fourier basis)

$$\varphi(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2; \\ 0 & \text{otherwise} \end{cases} \qquad t_0 = 1, \quad \omega_0 = 2\pi$$

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- For the box function

$$\Delta_t = \frac{1}{2\sqrt{3}} \qquad \Delta_{\mathrm{f}} = \infty$$

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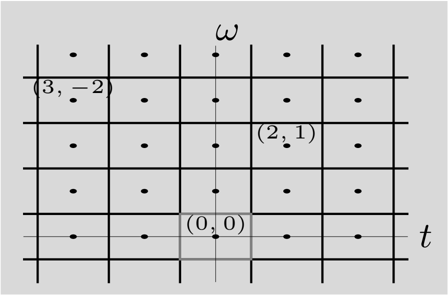
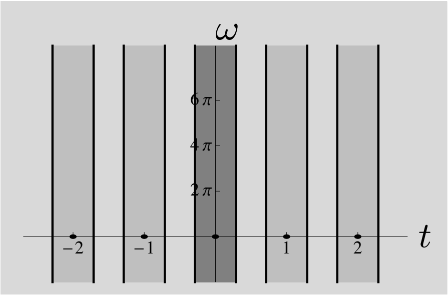
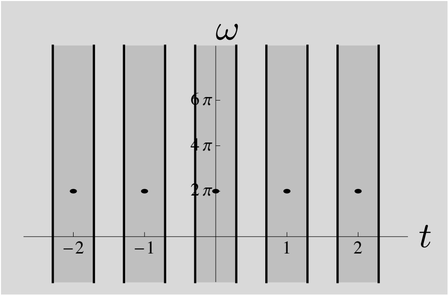


Idealized tiling



Actual tiling for k = 0



Actual tiling for k = 1

Resolution

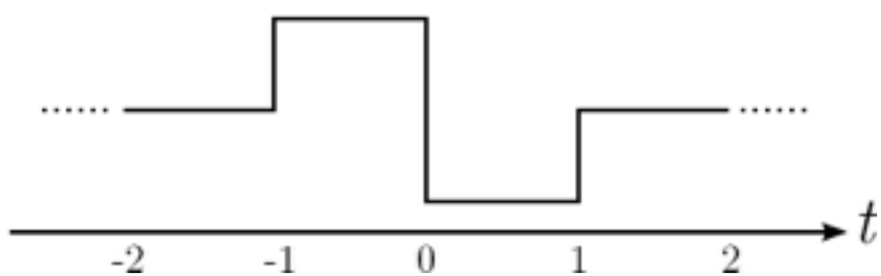
- Sharp and blurry images
  - Even with the same number of samples
  - Intuitively related to the bandwidth

Resolution

- Intuitively related to bandwidth
  - Sharp and blurry images
- More universal : Number of degrees of freedom (DoF) per unit time (space)
  - A signal in $\mathrm{BL}[-\omega_0/2, \omega_0/2]$ has $\omega_0/2\pi$ DoF per unit time
    - It can be specified with this many samples (Nyquist)

Bandlimited $\Rightarrow$ Small DoF

- Functions can have infinite bandwidth and finite resolution



Piece-wise constant

Wrap Up

- Filtering and representation
- Localization in time and frequency
  - Measuring the spreads
  - Continuous/continuous
  - Discrete/continuous
  - Discrete/discrete
- Uncertainty Principle
  - No free lunch as usual
- Time-frequency tilings
- Resolution
  - Degrees of freedom

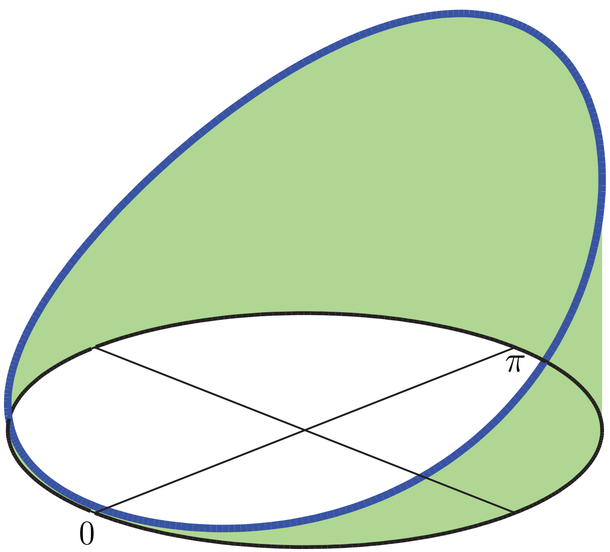
Bad rumors about DTFT variance!

- zero-at-pi condition!

$$x_n = 1 + 4\,\delta_{n-1} + 2\,\delta_{n-2} \qquad X(e^{j\pi}) \neq 0 \qquad \eta_\ell = 0.239 < \frac{1}{4}$$

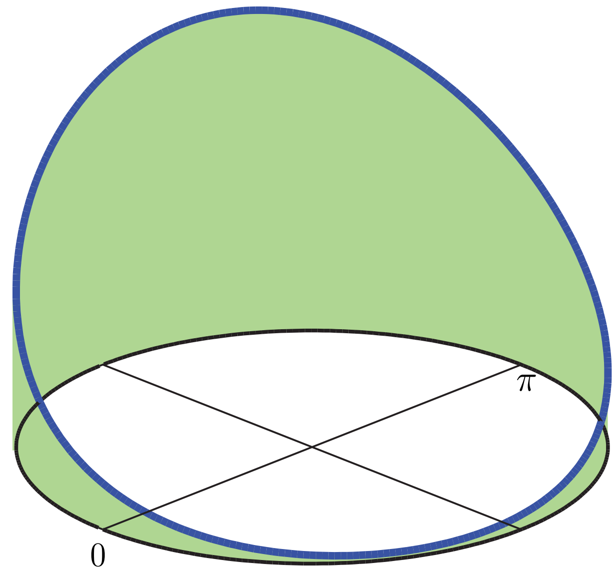
- if I am periodic, then I want something periodic!

$\mathrm{DTFT}[x]$



$\Delta_{\omega_\ell} = 1.3$

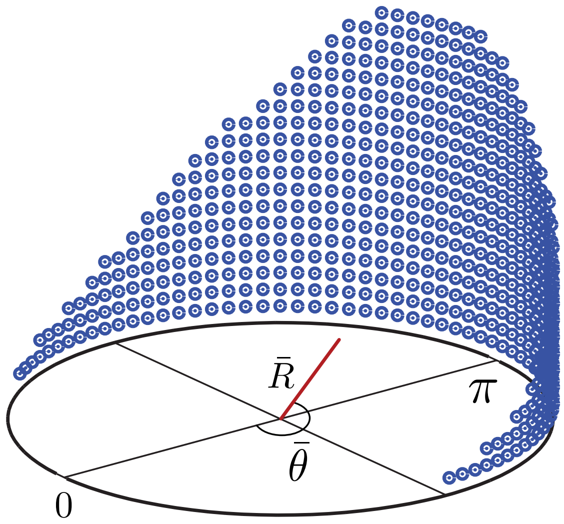
$\mathrm{DTFT}[e^{j\omega_0 n}x]$



$\Delta_{\omega_\ell} = 1.1$

Suspect no.2: circular statistics

- distributions on a circle!



$$\boldsymbol{x}_n = e^{j\theta_n}$$

$$\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_n = \bar{R}e^{j\bar{\theta}}$$

$$\bar{\theta} \neq \frac{1}{n}\sum_{i=1}^{n} \theta_n$$

- mean and variance:

Suspect no.2: circular statistics

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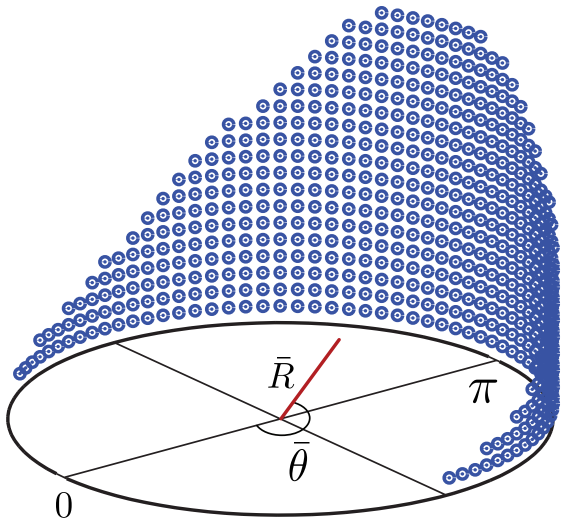

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$$\mathbb{E}[\theta] \approx \arg[\bar{\boldsymbol{x}}]$$

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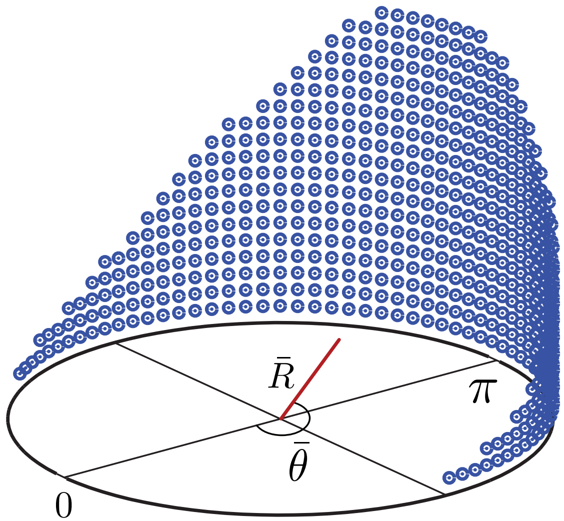

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$$\bar{\theta} \neq \frac{1}{n}\sum_{i=1}^{n}\theta_n$$

- mean and variance:

$$\mathbb{E}[\theta] \approx \arg[\bar{\boldsymbol{x}}]$$

$$\begin{aligned} \mathrm{var}[\theta] &\approx 2(1-\bar{R}) \\ \text{or } &\approx 1-\bar{R}^2 \\ \text{or } &\approx \frac{1}{\bar{R}^2}-1 \end{aligned}$$

Suspect no.2: circular statistics

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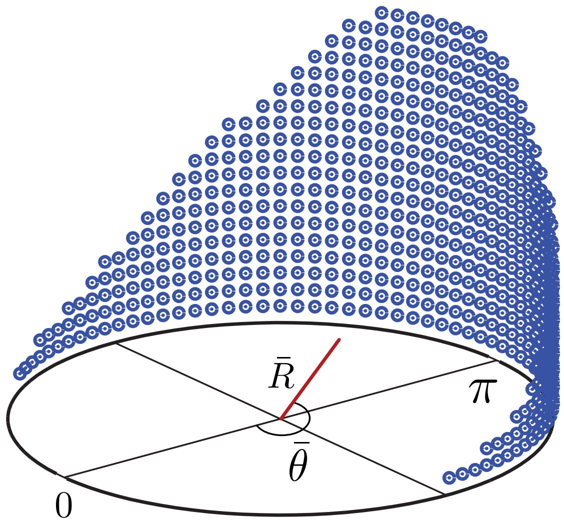

$$\boldsymbol{x}_n = e^{j\theta_n}$$

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$$\mathbb{E}[\theta] \approx \arg[\bar{\boldsymbol{x}}]$$

$$\mathrm{var}[\theta] \approx 2(1-\bar{R})$$

$$\text{or} \approx 1-\bar{R}^2$$

$$\boxed{\text{or} \approx \frac{1}{\bar{R}^2}-1}$$

Circular statistics for the DTFT

- First trigonometric moment

$$\tau(x) = \int_{-\pi}^{\pi} e^{j\omega} \frac{|X(e^{j\omega})|^2}{2\pi\|x\|^2} d\omega = \frac{1}{\|x\|^2} \sum_{i\in\mathbb{Z}} x_i x_{i+1}^*$$

- Periodic mean and variance

$$\mu_{\omega_p} = \text{arg}[\tau(x)]$$

$$\Delta_{\omega_p}^2 = \frac{1}{|\tau(x)|^2} - 1 = \left| \frac{\|x\|^2}{\sum_{n\in\mathbb{Z}} x_n \, x_{n+1}^*} \right|^2 - 1$$

- These definitions are periodic!

$$\mu_{\omega_p}(x_n e^{-j\omega_0 n}) = \mu_{\omega_p}(x_n) - \omega_0$$

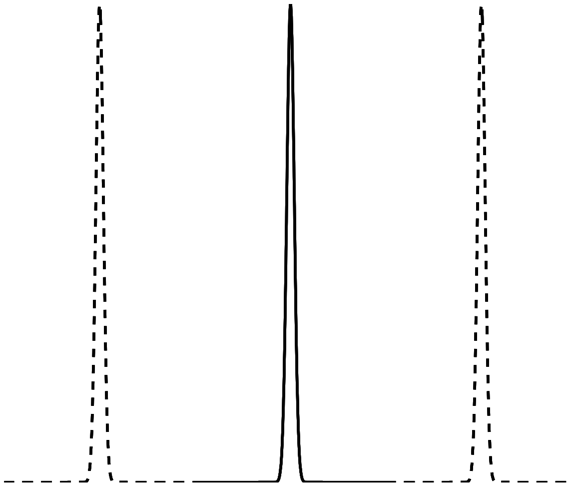
$$\Delta_{\omega_p}(x_n e^{-j\omega_0 n}) = \Delta_{\omega_p}(x_n)$$

Mr. Heisenberg meets suspect no.2

- Heisenberg uncertainty principle

$$\text{For } x_n \in \ell^2(\mathbb{Z}), \text{ with } \|x_n\|_0 > 1,$$
$$\eta_p = \Delta_n \, \Delta_{\omega_p} > \frac{1}{4}$$

- [Prestin99]: no sequence can achieve the lower bound!
- Asymptotically wrapped Gaussians can achieve it!
  - But then they are not really periodic anymore ;)



maximally compact sequences

- Find the most compact sequence in time for a given frequency spread

$$\begin{aligned} \Delta^2_{n,\text{opt}} &= \underset{x_n}{\text{minimize}} \quad \Delta^2_n \\ &\ \text{subject to} \quad \Delta^2_{\omega_p} = \text{fixed} \end{aligned}$$

- Call the solution **maximally compact**

find maximally compact sequences	=	find zero-mean, real, positive, unit norm maximally compact sequences

Find them

- A whole season in one slide!

$$\begin{aligned}\Delta^2_{n,\mathrm{opt}} &= \underset{x_n}{\text{minimize}} \quad \sum_{n\in\mathbb{Z}} n^2 x_n^2 \\ &\quad \text{subject to} \quad \sum_{n\in\mathbb{Z}} x_n x_{n+1} = \frac{1}{\sqrt{1+\sigma^2}}, \\ &\qquad\qquad\qquad \sum_{n\in\mathbb{Z}} x_n^2 = 1.\end{aligned}$$

Find them

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$$\Delta^2_{n,\mathrm{opt}} = \underset{x_n}{\text{minimize}} \quad \sum_{n\in\mathbb{Z}} n^2 x_n^2$$
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$$\longrightarrow \quad \underset{\boldsymbol{x}}{\text{minimize}} \quad \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$$
$$\text{subject to} \quad \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} = \alpha\,, \quad \boldsymbol{x}^T\boldsymbol{x} = 1\,,$$

$$\boldsymbol{A} = \begin{bmatrix} \ddots & & & & & \\ & 2^2 & & & 0 & \\ & & 1^2 & & & \\ & & & 0 & & \\ & & & & 1^2 & \\ & 0 & & & & 2^2 \\ & & & & & & \ddots \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \ddots & & & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & & & & \ddots \end{bmatrix}$$

Find them

- A whole season in one slide!

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & \operatorname{tr}(\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^T) \\
\text{subject to} \quad & \operatorname{tr}(\boldsymbol{B}\boldsymbol{x}\boldsymbol{x}^T) = \alpha \\
& \operatorname{tr}(\boldsymbol{x}\boldsymbol{x}^T) = 1\,.
\end{aligned}
\qquad \longleftarrow \qquad
\begin{aligned}
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$$

$$
\boldsymbol{A} = \begin{bmatrix}
\ddots & & & & & & \\
& 2^2 & & & & 0 & \\
& & 1^2 & & & & \\
& & & 0 & & & \\
& & & & 1^2 & & \\
& 0 & & & & 2^2 & \\
& & & & & & \ddots
\end{bmatrix}, \quad
\boldsymbol{B} = \begin{bmatrix}
\ddots & & & & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
& \frac{1}{2} & 0 & \frac{1}{2} & \\
& & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & & & & \ddots
\end{bmatrix}
$$

Find them

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\begin{array}{ll}
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& \operatorname{tr}(\boldsymbol{X}) = 1 \\
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$$

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\boldsymbol{A} = \begin{bmatrix}
\ddots & & & & & & \\
& 2^2 & & & & 0 & \\
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\end{bmatrix},
\quad
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\ddots & & & & 0 \\
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$$

Semidefinite program

- relaxation

$$\begin{aligned} \underset{\boldsymbol{X}}{\text{minimize}} \quad & \mathrm{tr}(\boldsymbol{A}\boldsymbol{X}) \\ \text{subject to} \quad & \mathrm{tr}(\boldsymbol{B}\boldsymbol{X}) = \alpha \\ & \mathrm{tr}(\boldsymbol{X}) = 1 \\ & \boldsymbol{X} \succeq 0, \ \mathrm{rank}(\boldsymbol{X}) = 1\,. \end{aligned}$$

- tight relaxation [shapiro82] (m: number of constraints)

$$\mathrm{rank}(\boldsymbol{X}^{\mathrm{opt}}) \leq \lfloor (\sqrt{8m+1} - 1)/2 \rfloor \leq 1$$

- here $m = 2$, thus

$$\mathrm{rank}(\boldsymbol{X}^{\mathrm{opt}}) \leq 1$$

- we can drop the rank constraint!
- now just use cvx!

Semidefinite program

- relaxation

$$\begin{array}{ll} \underset{\boldsymbol{X}}{\text{minimize}} & \text{tr}(\boldsymbol{A}\boldsymbol{X}) \\ \text{subject to} & \text{tr}(\boldsymbol{B}\boldsymbol{X}) = \alpha \\ & \text{tr}(\boldsymbol{X}) = 1 \\ & \boldsymbol{X} \succeq 0 \end{array}$$

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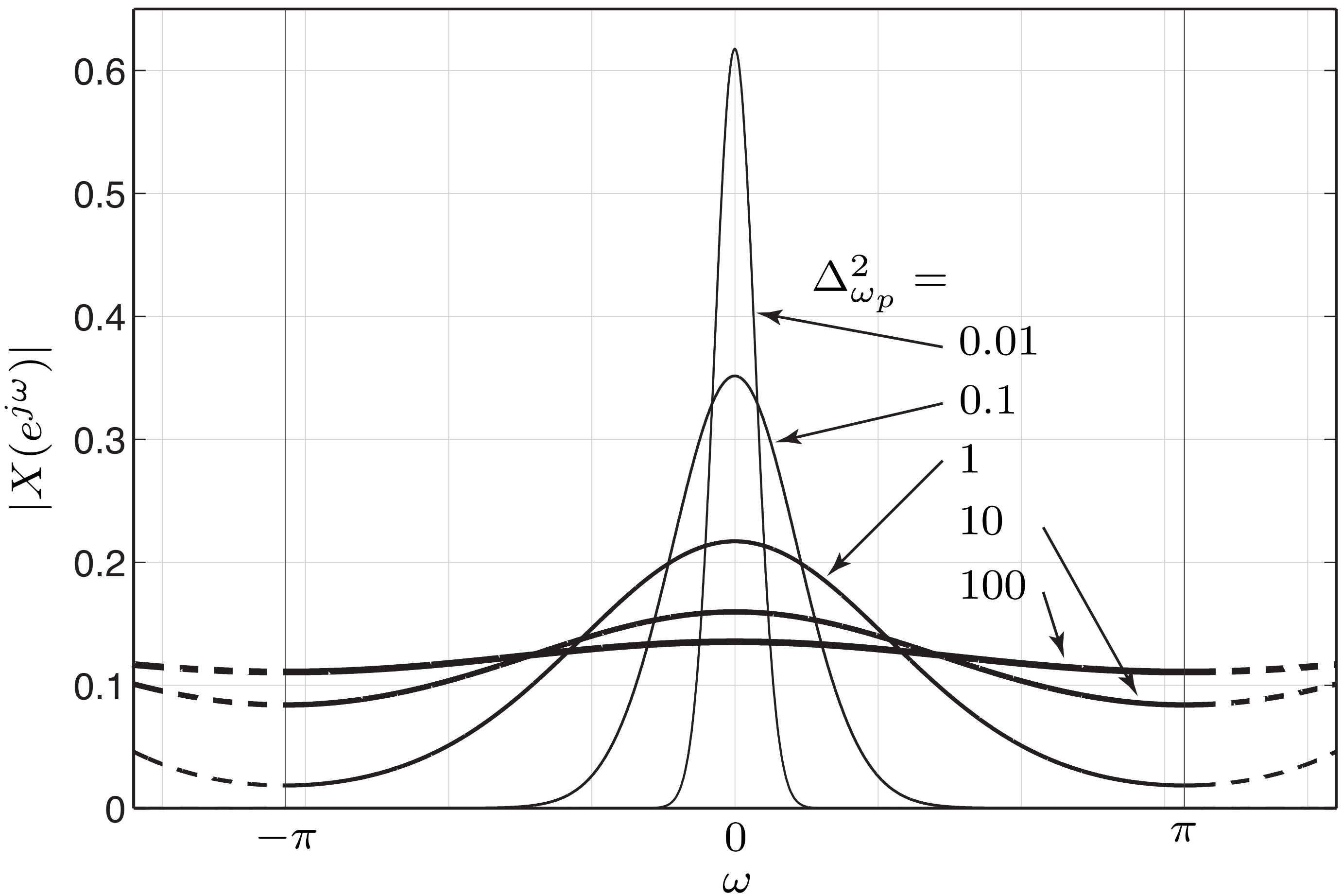
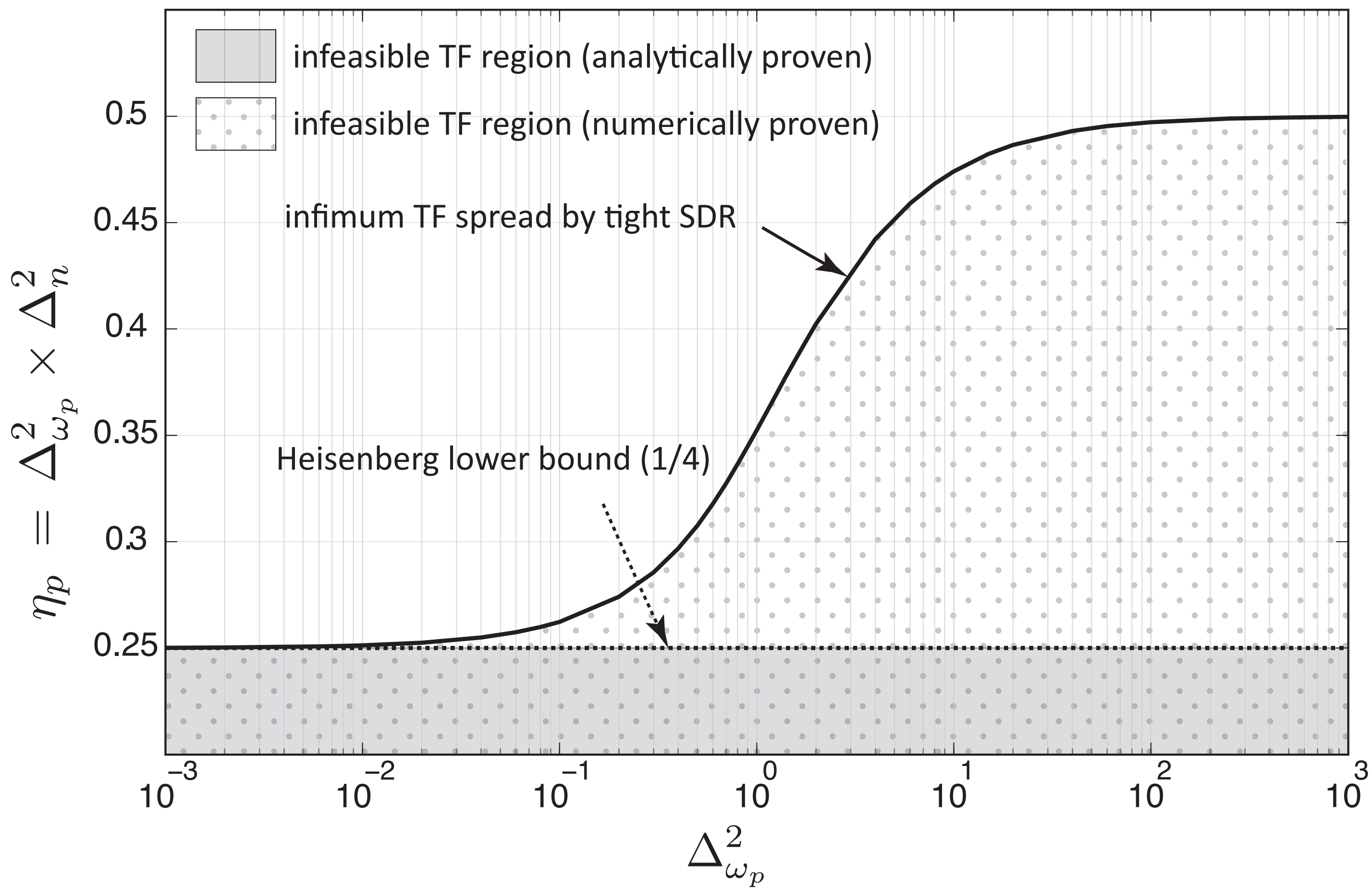
Experiments



Mr. Heisenberg is no good here!



Solve the dual instead

- Primal

$$\begin{aligned} \underset{\boldsymbol{X}}{\text{minimize}} \quad & \operatorname{tr}(\boldsymbol{A}\boldsymbol{X}) \\ \text{subject to} \quad & \operatorname{tr}(\boldsymbol{B}\boldsymbol{X}) = \alpha \\ & \operatorname{tr}(\boldsymbol{X}) = 1 \\ & \boldsymbol{X} \succeq 0 \end{aligned}$$

- Dual

$$\begin{aligned} \underset{\lambda_1, \lambda_2}{\text{maximize}} \quad & \alpha\, \lambda_1 + \lambda_2 \\ \text{subject to} \quad & \boldsymbol{A} - \lambda_1 \boldsymbol{B} - \lambda_2 \boldsymbol{I} \succeq 0 \end{aligned}$$

for the primal and dual, strong duality holds!

Analytic lower bound

- A sufficient condition

$$\text{If } \lambda_2 \leq 1 - \sqrt{1+\lambda_1^2}\,, \text{ then} \qquad \boldsymbol{A} - \lambda_1\,\boldsymbol{B} - \lambda_2\,\boldsymbol{I} \succ 0$$

- Lower bound

$$\begin{aligned} \underset{\lambda_1,\lambda_2}{\text{maximize}} \quad & \alpha\,\lambda_1 + \lambda_2 \\ \text{subject to} \quad & \lambda_2 \leq 1 - \sqrt{1+\lambda_1^2} \end{aligned}$$

- et voilà!

If x_n is maximally compact for a given $\Delta^2_{\omega_p} = \sigma^2$, then

$$\eta_p = \Delta n^2\,\Delta^2_{\omega_p} \geq \sigma^2 \left(1 - \sqrt{\frac{\sigma^2}{1+\sigma^2}}\right)$$

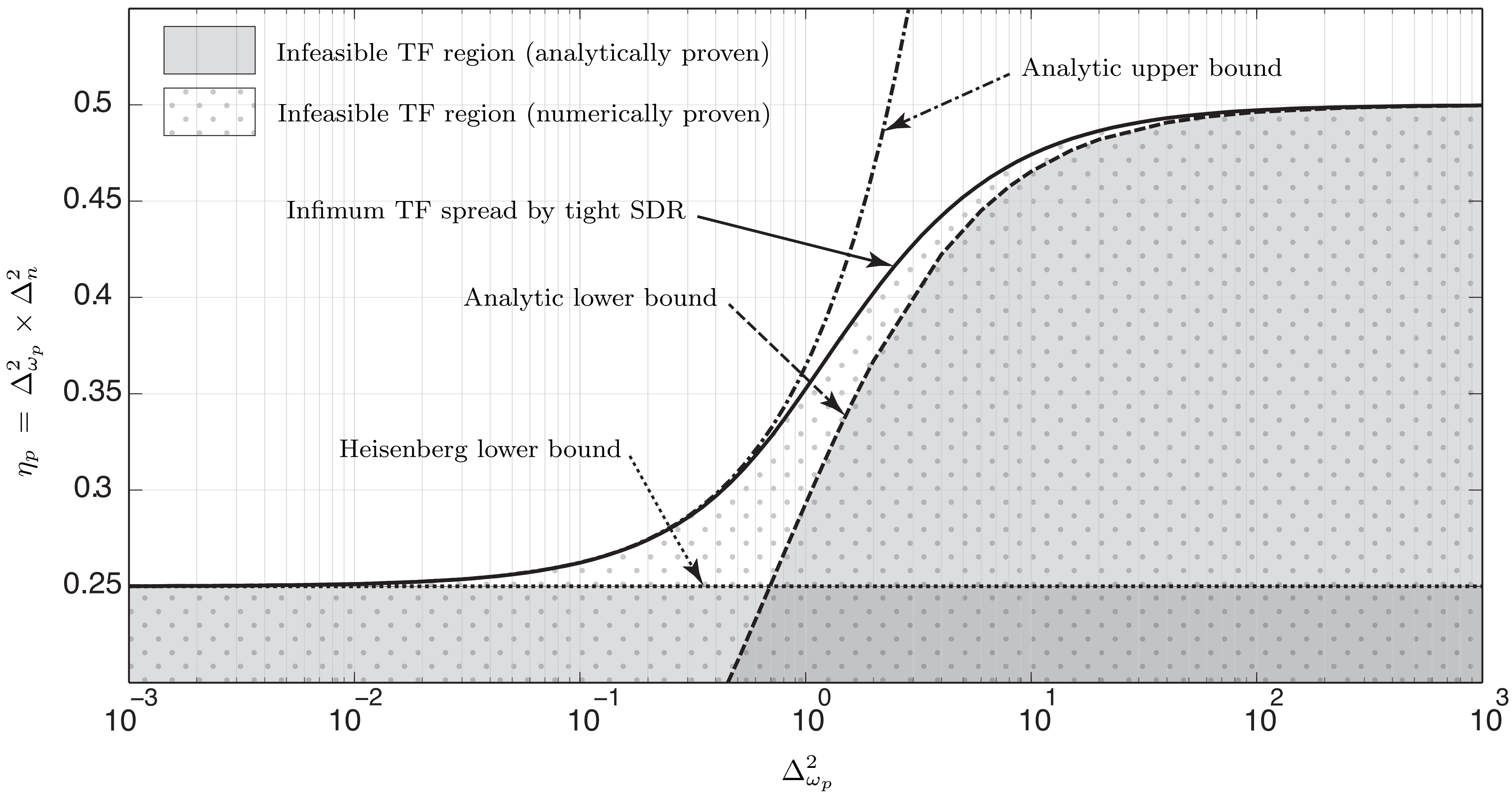
Analytic Upper Bound

- we can also find an analytic upper bound

if x_n is maximally compact for a given $\Delta_{\omega_p} = \sigma^2$, then for small values of σ, we have

$$\eta_p = \Delta_n^2 \ \Delta_{\omega_p}^2 \leq \frac{\sigma^2}{8} \left(\frac{\sqrt{1+\sigma^2}}{\sqrt{1+\sigma^2}-1} - \frac{1}{2} \right) .$$

Together we can …



A Benchmark for compactness