Dynamical Systems For Engineers

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Introduction

1.1 Introduction

What is a dynamical system? The following answer can be given in common language. It will be made more precise, mathematically, later on.

A dynamical system is a mathematical object that produces through its internal laws a time evolution (or movement) of its variables. It may do that autonomously, or under the influence of an input signal.

In this course, the internal law will be described by ordinary differential equations for continuous-time dynamical systems, and difference equations for discrete-time systems. Other types of dynamical systems are also important, but will not be addressed in this course, in particular

- infinite-dimensional systems, usually described by partial differential equations,
- systems whose time-evolution is not deterministic, but stochastic, in particular systems described by a Markov process.

Dynamical systems can model many "real world" temporal processes, encountered in Physics: Newtonian mechanics (Motion of interacting point masses, pendulum, or coupled pendula), in engineering (Time evolution of voltages and currents in electronic circuits, robot motion), in biology (Time evolution of the membrane potential of a neuron, population dynamics, evolution of epidemics), in economics (Evolution of prices of commodities, stock market evolution). This list can be made arbitrarily long. This shows that it is worthwhile to move from the specific application area to the mathematical abstraction. All concepts and properties established at the abstract level then automatically apply to all the applications.

In this course we will (i) study the relation between the law (equations) that describes a dynamical system and the time evolution it generates, on an abstract level, and (ii) talk from time-to-time about some models, their specific mathematical structure and their relation to the "real world".

However, we will not study the problem of modeling, i.e. how to find the law of the dynamical system from the "real-world", nor the problem of system identification, i.e. how to find the law from given (measured) input-output pairs. This is the topic of the classes in Physics, Engineering, Biology, Economics that lead to the analysis of variables of interest as dynamical systems.

The course is structured according to the mathematical topics. It is divided into a part on Linear Systems, and a part on Nonlinear Systems. The main emphasis is on nonlinear systems, but in order to appreciate fully the wealth of nonlinear dynamics phenomena, a good knowledge of the nature of linear dynamics is necessary.

Supporting material:

- these course notes,
- exercise problems and solutions,
- suggested additional reading, in particular books that give a complementary view on all or part of the subjects treated in the course, and go beyond.

1.2 Dynamical System: Definitions

A dynamical system transforms an input signal u in an output signal y, while having an internal state x. These signals depend on the time $t \in \mathcal{T}$ where $\mathcal{T}$ can be $\mathbb{R}$ (continuous time systems) or $\mathbb{Z}$ (discrete time systems). We will also limit our attention to the signals from some initial time, typically t=0. In this case we would have $\mathcal{T} = \mathbb{R}_+$ or $\mathcal{T} = \mathbb{N}$.

The values of the signals belong to corresponding spaces. Our notation is

$$\begin{array}{rcl} u(t) & \in & \Gamma \\ x(t) & \in & \Omega \\ y(t) & \in & \Theta \end{array}$$

where Γ, Ω and Θ are, respectively the input, state and output spaces.

Definition 1.1 (Solution)**.** *The dynamics of the system is given by an evolution equation. A* solution *of the evolution equation is a function*

$$x : \mathcal{T} \to \Omega$$

It is also called movement. Often it is combined with the observation (or output)

$$y : \mathcal{T} \to \Theta.$$

The set of all pairs composed of the time and the state visited at that time is called the *trajectory* of the solution

$$\{(t, x(t)) \mid t \in \mathcal{T}\} \subseteq \mathcal{T} \times \Omega$$

and the set of all states visited is called the *orbit* of the solution

$$\{x(t) \mid t \in \mathcal{T}\} \subseteq \Omega.$$

1.3 Examples

1.3.1 Example 1: Mass-spring system in Newtonian mechanics

We consider a classical system in mechanics, which is a mass m attached to a wall by a spring of constant k, and to which an external force $f(t)$ is applied to pull the mass away from the wall, as shown in Figure 1.1

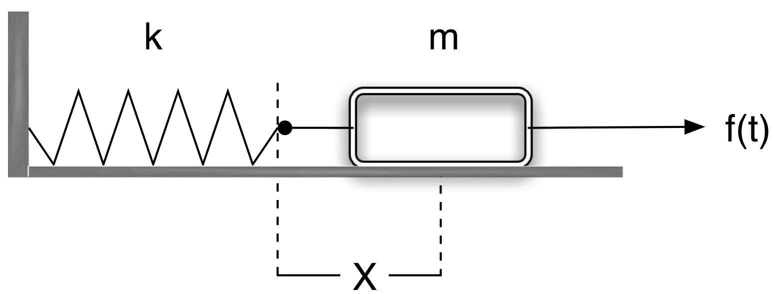

Figure 1.1: Mass-spring system

The input signal is the external force $f(t)$. We are interested in observing the dynamics of x, the position of the mass. Thus, the input space is $\Gamma = \mathbb{R}_+$ and the observation space is also $\Theta = \mathbb{R}_+$. The total force acting on the mass is, assuming frictionless motion, $F = f - kx$. According to Newtonian mechanics, the acceleration of the mass is proportional to the force. This leads to

$$m\frac{d^2x}{dt^2}(t) = -kx(t) + f(t).$$

Instead of writing a second order differential equation, we can write two first order equations by setting $x_1 = x$ and $x_2 = dx_1/dt$, to obtain

$$\frac{dx_1}{dt}(t) = x_2(t) \tag{1.1}$$

$$\frac{dx_2}{dt}(t) = -\frac{k}{m}x_1(t) + \frac{1}{m}f(t). \tag{1.2}$$

The last equation is the output, or observation equation. For reasons that will become clear later, we call the vector

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

the state of the system, and thus $\Omega = \mathbb{R}_+^2$.

1.3.2 Example 2: Commodity market

We want to model the evolution of the price of a farmer's product, say, potatoes. The model implicitly makes the following basic assumptions:

- there are many independent producers of potatoes,
- there are many independent consumers of potatoes,
- the price the consumers pay is what the producers get (intermediaries are not modeled),
- the price p of, say, a ton of potatoes is fixed by the farmer's cooperative once a year, before the farmers plant the potatoes.

Therefore time t for this model is discrete, the time unit being one year, and we are interested in the function $p(t)$.

The behavior of the farmers is modeled by the supply curve $S(p)$. Given the price p, each farmer decides whether or not to plant potatoes, and on what surface, depending on his/her own costs of production and on the prices of alternative products. Disregarding the influence of the weather, diseases, etc., this determines the quantity of potatoes the farmer supplies to the market after harvesting. Collectively, the farmers supply the quantity $S(p)$ (tons of potatoes). Clearly, when the price is low, few potatoes will be produced. In contrast, when the price is high, the incentive for the farmers to produce potatoes is also high. Therefore, $S(p)$ is an increasing function of p.

The behavior of the consumers is modeled by the demand curve $D(p)$. The individual consumer buys a quantity of potatoes that depends on price and many other factors, such as personal taste, etc. Collectively, the consumers buy the quantity $D(p)$ (tons of potatoes). Clearly, when the price is low the consumers will buy lots of potatoes, whereas when the price is high they will rather buy other products, such as pasta. Thus, $D(p)$ is a decreasing function of p.

Both curves are shown in Figure 1.2.

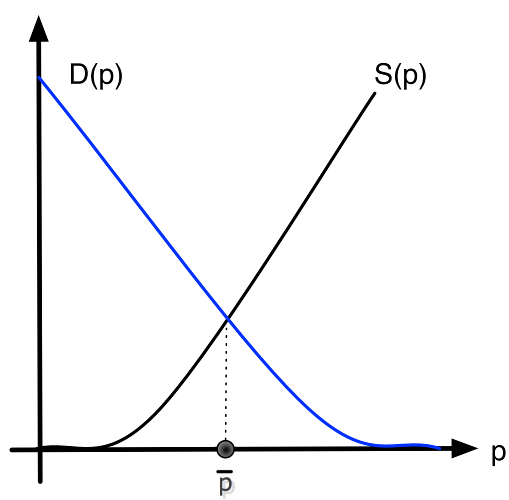

Figure 1.2: The demand curve $D(p)$ and supply curve $S(p)$ as a function of price p.

If, by chance, the price was fixed at the value where the supply and demand curves intersect, the consumers buy exactly the quantity of potatoes the producers have brought to the market, and both consumers and producers are satisfied and continue with the same price the next year. If, however, the price was lower and the demand exceeded the supply, the producers find that they can ask for a higher price the next year. The opposite case is that the supply exceeded the demand, in which case the farmers remain with a quantity of unsold potatoes, Consequently, they will lower the price the next year. This mechanism of fixing the price the next year based on the observations of the current year can be modeled by

$$p(t+1) = p(t) + k(D(p(t)) - S(p(t))),$$

where k is a suitable constant that transforms excess demand into price increase and excess supply into price decrease. This constant expresses the experience of the farmer's cooperative in adjusting prices based on market observations. In fact, they observe the difference between the demand and the supply and act accordingly. Hence, the observation signal is simply

$$y = D(p) - S(p).$$

1.4 Important Classes of Dynamical Systems

Evolution equations may take various forms. The following two classes of evolution equations are of particular importance. This course will almost exclusively deal with them.

1.4.1 Continuous time

When $\mathcal{T} = \mathbb{R}_+$, the dynamical system is called an analog or continuous time dynamical system, and is described by

$$\frac{dx}{dt}(t) = F(x(t), u(t)) \tag{1.3}$$

$$y(t) = G(x(t), u(t)) \tag{1.4}$$

where

$$\begin{aligned} u(t) &\in \Gamma = \mathbb{R}^m \\ x(t) &\in \Omega = \mathbb{R}^n \\ y(t) &\in \Theta = \mathbb{R}^p \end{aligned}$$

and where F and G are arbitrary functions

$$\begin{aligned} F : \mathbb{R}^{n+m} &\to \mathbb{R}^n \\ G : \mathbb{R}^{n+m} &\to \mathbb{R}^p. \end{aligned}$$

The first equation (1.3) is called the state equation, or, more precisely, the system of state equations, and the second (1.4) the output, or observation equation, or the system of output or observation equations.

Written explicitly with the different vector components, the system of state equations becomes

$$\begin{aligned} \frac{dx_1}{dt}(t) &= F_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\ \frac{dx_2}{dt}(t) &= F_2(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\ &\vdots \\ \frac{dx_n}{dt}(t) &= F_n(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \end{aligned}$$

and the system of output equations equations becomes

$$\begin{aligned} y_1(t) &= G_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\ y_2(t) &= G_2(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\ &\vdots \\ y_p(t) &= G_p(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)). \end{aligned}$$

Example: Mass-spring system in Newtonian mechanics

In the mass-spring example the functions F and G are, with $u(t) = f(t)$,

$$F_1(x_1, x_2, u) = x_2 \tag{1.5}$$

$$F_2(x_1, x_2, u) = -\frac{k}{m}\, x_1 + \frac{1}{m}\, u \tag{1.6}$$

$$G(x_1, x_2, u) = x_1.$$

1.4.2 Discrete time

When $\mathcal{T} = \mathbb{N}$, the dynamical system is called a discrete time dynamical system, and is described by

$$x(t+1) = F(x(t), u(t)) \tag{1.7}$$

$$y(t) = G(x(t), u(t)) \tag{1.8}$$

where, again,

$$\begin{aligned} u(t) &\in \Gamma = \mathbb{R}^m \\ x(t) &\in \Omega = \mathbb{R}^n \\ y(t) &\in \Theta = \mathbb{R}^p \end{aligned}$$

and where F and G are arbitrary functions

$$\begin{aligned} F : \mathbb{R}^{n+m} &\to \mathbb{R}^n \\ G : \mathbb{R}^{n+m} &\to \mathbb{R}^p. \end{aligned}$$

We can explicit the system of state equations as

$$\begin{aligned} x_1(t+1) &= F_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\ &\vdots \\ x_n(t+1) &= F_n(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \end{aligned}$$

and the output equations are given by the same system as in the previous subsection.

Example: Commodity Price Model

In the commodity price model, functions F and G are, with $x = p$,

$$F(x) = x + k(D(x) - S(x)) \tag{1.9}$$

$$G(x) = D(x) - S(x).$$

1.5 Solutions of the State Equations

Unicity of the Solution

Without defining an initial condition, there are in general infinitely many solutions to the previous dynamical systems. These solutions are usually distinguished by the state $x(t_0)$ at some time t_0 that is usually taken to be 0. The following theorem, which we do not prove here, shows that the solutions are unique under fairly broad conditions.

Theorem 1.1 (Unique Solution)**.** *For each initial state $x(0) \in \mathbb{R}^n$, a discrete-time dynamical system admits exactly one solution $x : \mathbb{N} \to \mathbb{R}^n$. If $F(x(t), u(t))$ is continuous and locally Lipschitz with respect to x, then for each initial state $x(0) \in \mathbb{R}^n$, a continuous-time dynamical systems admits exactly one solution $x : \mathbb{R}_+ \to \mathbb{R}^n$.*

For discrete time systems, this property is obvious, but for continuous time systems, one has to impose a few constraints on F. This is usually formulated as a precise theorem (Picard-Lindelöf Theorem), see e.g. Theorem 3.1 in J.K.Hale, "Ordinary Differential Equations", Pure and Applied Mathematics, vol. XXI, Wiley-Interscience, New York, 1969. $F(x, u)$ is locally Lipschitz with respect to x if and only if for any closed bounded set $X \in \Omega \subseteq \mathbb{R}^n$ there is $k > 0$ such that $\|F(x, u) - F(x', u)\| \leq k \|x - x'\|$ for all $(x, u), (x', u) \in X \times \Gamma$. If F has continuous first partial derivatives with respect to x, it is automatically locally Lipschitz in x. We shall only remark here that the constraints are not very strong and that we may assume henceforth the property to hold, unless otherwise stated.

This property only mentions the existence and uniqueness of the solution for positive times, starting at an initial state at time 0, but nothing is said about negative times. For discrete time systems one has to impose the condition that the map $F(., u) : \Omega \to \Omega$ is invertible for any value of the input signal u. If this is the case, there is a unique solution defined for all positive and negative times with the given initial state $x(0)$. This property does not automatically hold for all systems of interest. In particular, the logistic map $F(x) = 1 - \lambda x^2$, parametrized by $\lambda > 0$, is not invertible.

In the case of continuous-time systems, the above-mentioned theorem actually guarantees the existence and uniqueness of the solution, for a given state at time 0, for an open time interval (t_1, t_2) where $t_1 < 0 < t_2$. Of course, we would like to have $t_1 = -\infty$ and $t_2 = +\infty$. In most models of physical systems, it is easy to prove that $t_2 = +\infty$, because of dissipativity or energy conservation. On the other hand, a finite negative t_1 is not uncommon, but it is not of any special concern, except that certain notions that require the existence of the solution for all negative times cannot be applied.

From a purely mathematical point of view, it is not unusual that in continuous time systems a solution only exists up to a finite positive time t_2 (or to a finite negative time t_1, if the solution is continued beyond 0 to negative times). What happens is that $\|x(t)\| \to \infty$ as $t \to t_2$. This phenomenon is called *finite escape time*.

Example: Mass-spring system in Newtonian mechanics

In the mass-spring example, supposing that the force is constant, i.e. $u(t) = f(t) = f$, we first determine the constant solution:

$$
\begin{aligned}
0 &= \overline{x}_2 && (1.10)\\
0 &= -\frac{k}{m}\,\overline{x}_1 + \frac{1}{m}\,f, && (1.11)
\end{aligned}
$$

which is written in vector form as

$$
\overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = \begin{bmatrix} f/k \\ 0 \end{bmatrix}.
$$

The state $\overline{x}$ where the constant solution sits, is called the *equilibrium* point of the system. We now introduce the increments with respect to the equilibrium point: $\Delta x(t) = x(t) - \overline{x}$. Substracting (1.10) and (1.11) respectively from (1.1) and (1.2), we get

$$\begin{aligned} \frac{d\Delta x_1}{dt}(t) &= \Delta x_2(t) \\ \frac{d\Delta x_2}{dt}(t) &= -\frac{k}{m}\Delta x_1(t), \end{aligned}$$

whose general solution, parametrized by the initial state, is

$$\begin{aligned} \Delta x_1(t) &= \Delta x_1(0)\cos(\omega t) + \frac{\Delta x_2(0)}{\omega}\sin(\omega t) \\ \Delta x_2(t) &= -\Delta x_1(0)\omega\sin(\omega t) + \Delta x_2(0)\cos(\omega t), \end{aligned}$$

where $\omega = \sqrt{k/m}$. Translated back into the original states, we find

$$x_1(t) = \left(x_1(0) - \frac{f}{k}\right)\cos(\omega t) + \frac{x_2(0)}{\omega}\sin(\omega t) + \frac{f}{k} \tag{1.12}$$

$$x_2(t) = -\left(x_1(0) - \frac{f}{k}\right)\omega\sin(\omega t) + x_2(0)\cos(\omega t). \tag{1.13}$$

We see explicitly that for each initial state there is exactly one solution and this solution exists on the whole real time axis.

Example: Commodity Price Model

If in the population dynamics model we take a linear supply curve $S(p) = \alpha p$ and an affine (linear + constant) demand curve $D(p) = -\gamma p + \delta$, we obtain the state equation

$$p(t+1) = (1 - k(\alpha+\gamma))p(t) + k\delta. \tag{1.14}$$

Again, we first look for a constant solution $\overline{p}$. We easily find that

$$\overline{p} = \frac{\delta}{\alpha+\gamma}.$$

Note that this is the price where the demand and the supply curve intersect. Defining the increment with respect to this solution $\Delta p(t) = p(t) - \overline{p}$ and substracting $\overline{p}$ from (1.14), we get

$$\Delta p(t+1) = (1 - k(\alpha+\gamma))\Delta p(t)$$

whose general solution reads

$$\Delta p(t) = (1 - k(\alpha+\gamma))^t \Delta p(0).$$

Getting rid of the increments gives

$$p(t) = (1 - k(\alpha+\gamma))^t \left(p(0) - \frac{\delta}{\alpha+\gamma}\right) + \frac{\delta}{\alpha+\gamma}.$$

We can see that for each price at $t = 0$ there is exactly one solution which exists for all integer times.

1.5.1 Notion of State

The property that the solutions of the state equations are characterized by the state at time 0 can actually be taken as the basic property of the state. If we know the state at present time (time 0) then we can predict the solution in the future (positive times), given the input signal. Therefore, it is not necessary to know in detail what happened to the system in the past (negative times), i.e. it is not necessary to know the past input signal nor the solution the system has followed in the past. The state at present can be considered as the condensed information about the past, the minimum of information necessary to predict the future (given the input signal for present and future).

Example: Mass-spring system in Newtonian mechanics

If, in the mass-spring system, we take the evolution equation

$$\frac{d^2x}{dt^2}(t) = -\frac{k}{m}\,x(t) + \frac{f(t)}{m},$$

we are tempted to consider x as the state. However, $x(0)$ does not by itself determine the state. We have to add to it a second component, e.g. $dx/dt(0)$. Indeed, as can be seen from (1.12) and (1.13), there is an infinity of solutions with the same $x(0)$ and the same input signal.

1.5.2 Notion of Flow

It is often useful to consider not only a single solution, but all solutions simultaneously. We can imagine putting at time 0 an infinitesimally small particle at each possible state and then let the particles move according to the state equation. This generates a “flow” of particles. Mathematically, the flow is a function

$$\Phi : T \times \Omega \to \Omega$$

defined by

$$\Phi(t, x(0)) \to x(t).$$

A flow of a 2-dimensional dynamical system can be represented on a phase-plot, by drawing the trajectories of a certain number of solutions, as shown on Figure 1.3 for the mass-spring system.

1.6 Asymptotic Behavior

One distinguishes between the properties of the solutions for small time t, which strongly depend on the initial state $x(0)$ (and, of course, on the input signal u), the transient behavior and the behavior for large time t, which may or may not depend on the initial state (but also depend on the input signal), the asymptotic behavior, also called steady state behavior. For the sake of mathematical rigor, the asymptotic behavior is defined to be the behavior for $t \to +\infty$. Of course, it may be practically achieved at a finite time, maybe even at a time that is not exceedingly large.

Note that the asymptotic behavior is not necessarily constant, it may be also oscillatory or even more complicated. In the case of systems with an input signal, the asymptotic behavior depends on the input signal, and almost any kind of asymptotic time-dependence can be obtained with a suitable time-dependent input. Nevertheless, the following general notion is applicable even in this context.

Definition 1.2. *A dynamical system has unique asymptotic behavior if for any two solutions $x(t)$ and $\tilde{x}(t)$,*

$$\|x(t) - \tilde{x}(t)\| \to 0$$

as $t \to \infty$.

This definition avoids specifying the nature of the asymptotic behavior. It states that all solutions after some time approximately have the same behavior. Since different solutions correspond to different initial states, after some time the solution “forgets” his initial state. For this reason the term system with fading memory is also used.

For the case of autonomous systems, more elaborate notions are used that we will introduce hereafter.

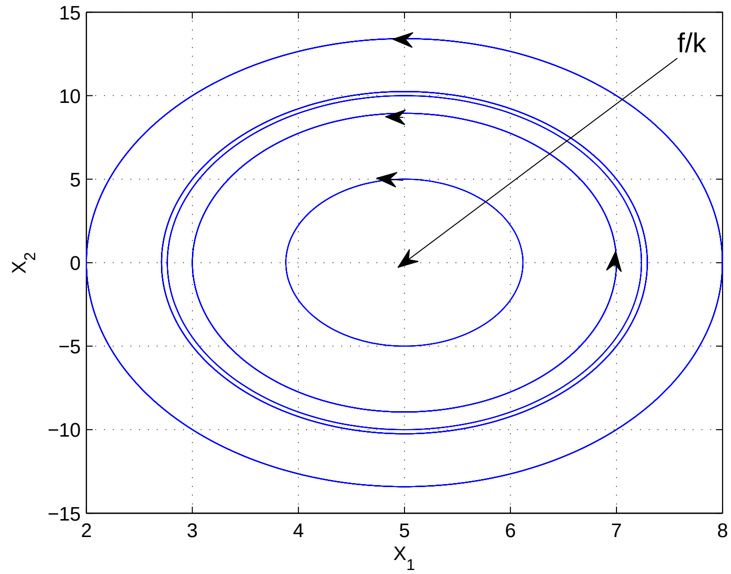

Figure 1.3: Phase plot of the mass-spring system (here $m = 0.1$, $f = 10$ and $k = 2$).

Example: Mass-spring system in Newtonian mechanics

This is a somewhat peculiar case in the sense that all solutions are periodic (see (1.12) and (1.12)), i.e. there is a time $T > 0$ such that for all t, $x(t + T) = x(t)$. Thus, they are identical to their asymptotic behavior, since right from the beginning they move around their closed trajectories (see Figure 1.3). Thus, there is no transient behavior; the asymptotic behavior depends on the initial state and is therefore not unique.

Example: Commodity Price Model

In this discrete time system, supposing positive constants, three qualitatively different asymptotic behaviors are possible, depending on whether (i) $-1 < 1 - k(\alpha + \gamma) < 1$, (ii) $1 - k(\alpha + \gamma) = -1$ or (iii) $1 - k(\alpha + \gamma) < -1$.

(i) In the first case, the solution $p(t)$ converges to the fixed point $\overline{p}$, either monotonically if $0 < 1 - k(\alpha + \gamma) < 1$, or with alternating signs if $-1 < 1 - k(\alpha + \gamma) < 0$. The asymptotic behavior is constant and it does not depend on the initial price $p(0)$. The transient behavior is the exponentially fast convergence to the fixed point.

(ii) In the second case, the solution oscillates between two values symmetrically positioned around the fixed point. One of the values is the initial price $p(0)$. This is qualitatively similar to the behavior of the mass-spring system.

(iii) In the last case, the solutions diverge to infinity, except when the initial point is exactly the fixed point ($p(0) = \overline{p}$).

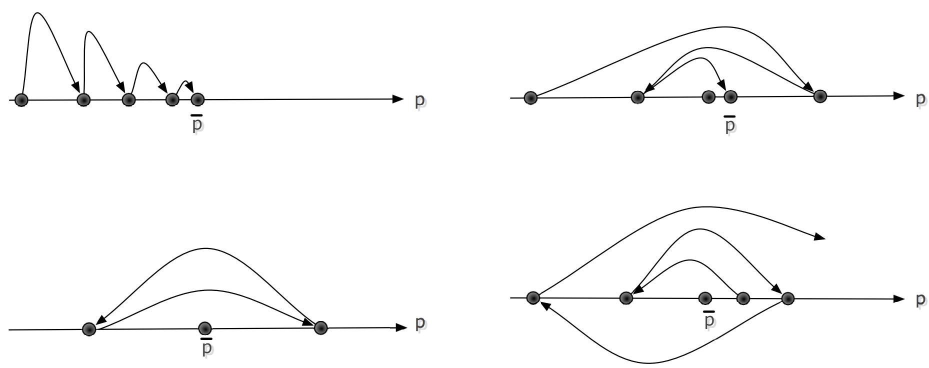

Figure 1.4: Asymptotic and transient behaviors for the commodity price model: (i) Exponential convergence to the fixed point $\overline{p}$ (top left: monotone; top right: alternating signs); (ii) periodic solution (bottom left) and (iii) divergence to infinity (bottom right).

1.7 Autonomous systems

A dynamical system is *autonomous*, if it has no input signal.

1.7.1 Invariant set

Consider an autonomous system in $\mathbb{R}^n$ with solutions existing on the whole time axis $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = \mathbb{Z}$. An invariant set of the system is defined as follows.

Definition 1.3 (Invariant Set)**.** *(i) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is forward invariant if for any solution x such that some time $t \in \mathcal{T}$ $x(t) \in \mathcal{S}$, it follows that $x(t') \in \mathcal{S}$ for all $t' \geq t$.*

(ii) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is backward invariant if for any solution x such that some time $t \in \mathcal{T}$ $x(t) \in \mathcal{S}$, it follows that $x(t') \in \mathcal{S}$ for all $t' \leq t$.

(iii) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is invariant if it is both forward and backward invariant.

If the solutions of a dynamical system are only defined for $\mathcal{T} = \mathbb{R}^+$ or $\mathcal{T} = \mathbb{N}$, and if they cannot be extended to the whole $\mathbb{R}$ or $\mathbb{Z}$, then part (i) of the definition for a forward invariant set applies without problem but (ii) and (iii) are not a priori applicable as such. However, it may still be that the solutions with $x(0) \in \mathcal{S}$ exist also for negative times and are unique. In this case, the notion of backward invariant and invariant set for $\mathcal{S}$ makes still sense.

Examples

- The whole space $\Omega = \mathbb{R}^n$ and the empty set $\emptyset$ are always invariant.
- Any orbit is forward invariant.
- If a solution is defined for all positive and negative times, and if there is no other solution with the same state at time 0, its orbit is invariant.
- A fixed point of a discrete time system and an equilibrium point of a continuous time system are always invariant.
- The orbit of a periodic solution is always invariant.

1.7.2 Limit sets

The following notion concerns the asymptotic behavior of an individual solution of an autonomous system.

Definition 1.4 (α- and ω-limit sets)**.** *Consider a solution $x(t)$ of an autonomous dynamical system in $\mathbb{R}^n$, defined for $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = \mathbb{Z}$ and unique for a given $x(0)$.*

(i) The ω-limit set of the solution is the set of points $\xi \in \mathbb{R}^n$ such that there exists a sequence of times $t_1 < t_2 < \ldots < t_i < \ldots$ with $t_i \to \infty$ when $i \to \infty$, such that

$$\lim_{i\to\infty} x(t_i) = \xi. \tag{1.15}$$

(ii) The α-limit set of the solution is the set of points $\xi \in \mathbb{R}^n$ such that there exists a sequence of times $t_1 > t_2 > \ldots > t_i > \ldots$ with $t_i \to -\infty$ when $i \to \infty$, such that (1.15) holds.

Note that if the solution is only defined for positive times, the definition of ω-limit set is still applicable.

Theorem 1.2. *Let $x(t)$ be a solution of the discrete-time system $x(t+1) = F(x(t))$, where F is continuous, invertible and its inverse is also continuous. Then*

(i) If the solution is bounded for $t \to +\infty$, its ω-limit set is compact (bounded and closed), non-empty and invariant.

(ii) If the solution is bounded for $t \to -\infty$, its α-limit set is compact, non-empty and invariant.

Proof:

It follows from basic calculus that the ω- and α-limit sets set are compact if the solution is bounded for positive, resp. negative times.

Let $\mathcal{S}_\omega$ be the ω-limit set. To prove the forward-invariance when the solution is bounded for $t \to +\infty$, we have to show that $\xi \in \mathcal{S}_\omega$ implies that $F(\xi) \in \mathcal{S}_\omega$. Now $\xi \in \mathcal{S}_\omega$ means that there is a sequence of times $t_1 < t_2 < \ldots < t_i < \ldots$ with $t_i \to \infty$ when $i \to \infty$, such that $x(t_i) \to \xi$. Therefore there is a sequence of times $t'_1 < t'_2 < \ldots < t'_i < \ldots$ with $t'_i \to \infty$ when $i \to \infty$,given by $t'_1 = t_1 + 1, \ldots, t'_i = t_i + 1, \ldots$, such that $x(t'_i) \to F(\xi)$. Therefore $F(\xi) \in \mathcal{S}_\omega$.

The same holds for the α-limit set.

$\blacksquare$

Theorem 1.3. *Let $x(t)$ be a solution of the continuous-time system $dx/dt(t) = F(x(t))$.*

(i) If the solution is bounded for $t > 0$, its ω-limit set is compact (bounded and closed), non-empty and connected.

(ii) If the solution is bounded for $t < 0$, its α-limit set is compact, non-empty and connected.

The proof is given in J.K.Hale, "Ordinary Differential Equations", Pure and Applied Mathematics, vol. XXI, Wiley-Interscience, New York, 1969.

Example: Mass-spring system in Newtonian mechanics

The orbit of a solution is itself its ω- and α-limit set.

Example: Commodity Price Model

(i) If $-1 < 1 - k(\alpha + \gamma) < 1$, the ω-limit set of every solution $p(t)$ is composed of a single point, which is the fixed point $\overline{p}$, whereas the α-limit set of every non-constant solution is empty.

(ii) When $1 - k(\alpha + \gamma) = -1$, both the ω-limit set and the α-limit set of any solution $p(t)$ consist of the two points $p(0)$ and $p(1)$.

(iii) The last case where $1 - k(\alpha + \gamma) < -1$ is the opposite of the first one (i): the α-limit set of every solution $p(t)$ is composed of a single point, the fixed point $\overline{p}$, whereas the ω-limit set of every non-constant solution is empty.

1.7.3 Attractor

The definition of attractor varies in the literature by some subtle points. We do not want to discuss them here, and adopt the following definition, which is appropriate for most cases of interest.

Definition 1.5 (Attractor)**.** *A non-empty compact (bounded and closed) set of the state space $\mathcal{A} \subseteq \Omega$ is an attractor of the system, if the following conditions hold:*

(i) $\mathcal{A}$ is forward invariant.

(ii) There exists a neighborhood $\mathcal{U}$ of $\mathcal{A}$, which is an open set $\mathcal{U} \supset \mathcal{A}$ such that all solutions starting in $\mathcal{U}$ converge to $\mathcal{A}$ as $t \to +\infty$.

(iii) There is no proper non-empty compact subset of $\mathcal{A}$ that has properties (i) and (ii).

In the above definition, a solution $x(t)$ *converges* to a set $\mathcal{A}$ if the distance between $x(t)$ and $\mathcal{A}$ converges to 0 as $t \to +\infty$. The distance between a point ξ and a set $\mathcal{S}$ is the distance between ξ and the point $\eta \in \mathcal{S}$ that is closest to ξ, or if there is no such point, then more generally, it is defined by

$$\mathrm{dist}(\xi, \mathcal{S}) = \inf_{\eta \in \mathcal{S}} \mathrm{dist}(\xi, \eta).$$

Because of point (ii) in the definition, the attractor $\mathcal{A}$ contains the ω-limit sets of all solutions starting in $\mathcal{U}$.

Typical attractors are single points, unions of several points, closed curves, etc., i.e. “thin” sets, whereas the neighborhood $\mathcal{U}$ of $\mathcal{A}$, being an open set, is “thick”, because every point of an open set is the center of a small sphere that lies entirely in $\mathcal{U}$. Most attractors are simple objects of integer dimensions (points, curves), but there can be many other geometrical sets that have possible non integer dimensional (the so-called “strange” attractors). A system may have a single, several or no attractor. In order to have an attractor, it must have some sort of dissipativity which causes $\mathcal{U}$ to shrink to $\mathcal{A}$.

Example: Mass-spring system in Newtonian mechanics

This (non-dissipative) system has no attractor. It has an infinity of periodic solutions that do not converge to each other.

Example: Commodity Price Model

(i) If $-1 < 1 - k(\alpha + \gamma) < 1$, the attractor is the fixed point $\overline{p}$.

(ii) If $1 - k(\alpha + \gamma) = -1$, the system has no attractor.

(iii) Likewise, if $1 - k(\alpha + \gamma) < -1$, the system has no attractor.