Dynamical Systems
for Engineers: Exercise Set 5, Solutions

Exercise 1

The transfer function of the LC series circuit is

$$H(s) = \frac{1}{1 + LCs^2}.$$

It has two poles in $s = \pm j/\sqrt{LC}$, and no zero-pole cancellation.

1. As the poles of the system have a non-negative real part (which is zero here), this system is not B.I.B.O. stable. There will be bounded inputs that will produce an unbounded output. For instance, if we submit the system to the input $u(t) = \cos(\omega_0 t)$ with $\omega_0 = 1/\sqrt{LC}$, we would find that

$$U(s) = \frac{s}{\omega_0^2 + s^2}$$

and therefore that

$$Y(s) = H(s)U(s) = \frac{s\omega_0^2}{(\omega_0^2 + s^2)^2}$$

whose inverse Laplace transform is

$$y(t) = \omega_0 t \sin(\omega_0 t)/2$$

showing that the output diverges to infinity.

2. As there is no zero-pole cancellation, this system is observable. (Another method would be to write the matrices A and C of the ABCD representation of the system, and to verify that M_o is of rank 2 and is thus invertible).

3. As there is no zero-pole cancellation, this system is controllable. (Another method would be to write the matrices A and B of the ABCD representation of the system, and to verify that M_c is of rank 2 and is thus invertible).

Exercise 2

Consider a linear system in the usual ABCD representation, with

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

1. The natural frequencies of the system are -2, -2 and -3 and therefore the poles of the system have all a negative real part. The system is thus B.I.B.O. stable.

2. One computes that

$$M_o = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \\ 4 & 4 & 0 \end{bmatrix}$$

which is of rank $1 < 3$. Hence M_o is not invertible, and the system is not observable.

3. One computes that

$$M_c = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

which is of rank $1 < 3$. Hence M_c is not invertible, and the system is not controllable.

Exercise 3

Consider a continuous-time linear system whose input-output transfer function is

$$H(s) = \frac{s+\alpha}{s^3 + 7s^2 + 14s + 8}$$

where $\alpha \in \mathbb{R}$. Note that $s^3 + 7s^2 + 14s + 8 = (s+1)(s+2)(s+4)$.

1. Since the poles of the system can only be -1, -2 and -4, the system is always B.I.B.O. stable, for any value of α.

2. If $\alpha = 1$ or $\alpha = 2$ or $\alpha = 4$, there is a zero-pole cancellation in $H(s)$ and the system is unobservable and/or uncontrollable. If $\alpha \neq 1, 2, 4$ the system is observable and controllable.

Exercise 4

With

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad C = \begin{bmatrix} c_1 & c_2 \end{bmatrix},$$

we compute that

$$M_o = \begin{bmatrix} c_1 & c_2 \\ c_1 & c_1 + c_2 \end{bmatrix}$$

and

$$M_c = \begin{bmatrix} b_1 & b_1 + b_2 \\ b_2 & b_2 \end{bmatrix}$$

We find that M_o is invertible if and only if $c_1 \neq 0$, and that M_c is invertible if and only if $b_2 \neq 0$. Therefore the condition for the system to be observable and controllable is that $c_1, b_2 \neq 0$.