Chaos

1. First property of chaos : Irregular, non-periodic trajectories

At first sight, this is the most striking property of chaos. A system with very simple state equations can have very complicated trajectories. The iteration of the logistic map $f(x) = 1 - \mu x^2$ for $\mu = 1.6$ is a case in point (Figure 1).

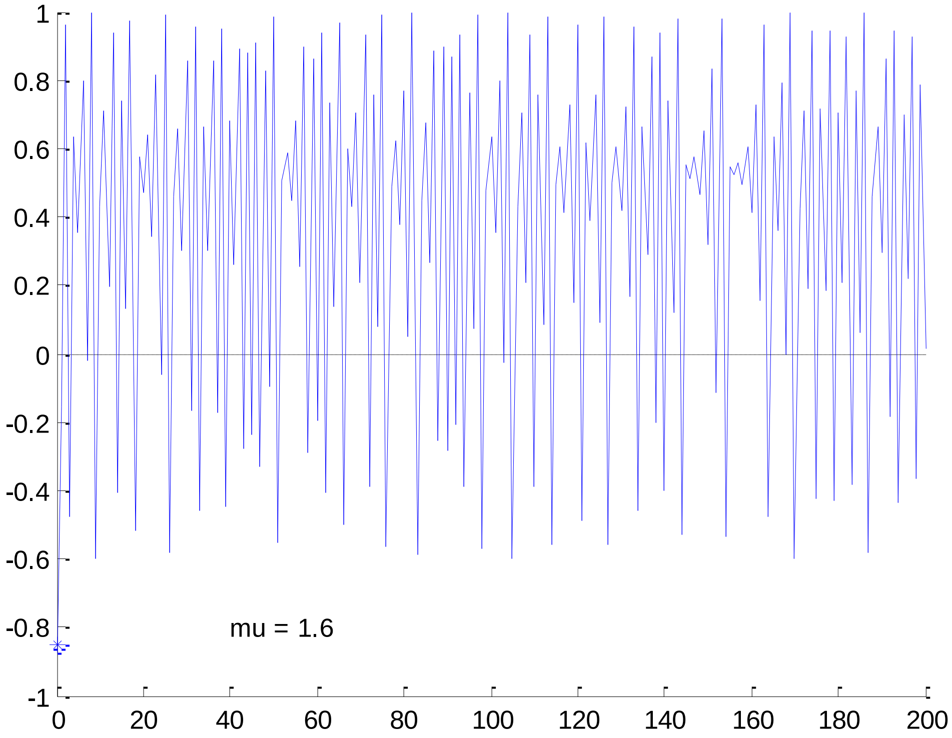


Figure 1 : 200 iterations of the logistic map with $\mu = 1.6$, starting from $x(0) = -0.86$

Not only autonomous discrete time, but also autonomous continuous time systems can have irregular, non-periodic asymptotic behavior. Because of the theorem of Poincaré-Bendixson, that basically says that 2-dimensional continuous-time systems can have as asymptotic behavior only equilibrium constant (equilibrium points) and periodic solutions, the simplest continuous-time chaotic systems have dimension 3. A famous example is the Lorenz system ((Lorenz, 1963) originating from atmospheric physics, and another famous example is Chua's circuit, formulated in terms of electronics. Let us consider the latter. The circuit diagram is drawn in Figure 2

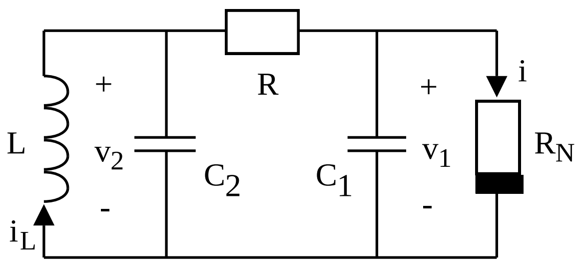


Figure 2 : Chua's circuit

All elements in this circuit are linear, except the nonlinear resistor R_N. Its characteristic is given in Figure 3.

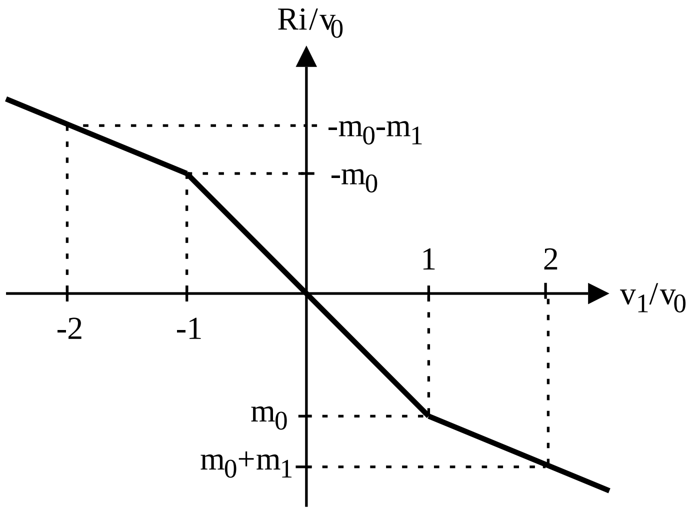


Figure 3 : Characteristic of the nonlinear resistor R_N.

The state equations of this system are

$$\begin{aligned} C_1 \frac{dv_1}{dt} &= \frac{1}{R}(v_2 - v_1) - g(v_1) \\ C_2 \frac{dv_2}{dt} &= -\frac{1}{R}(v_2 - v_1) + i_L \\ L \frac{di_L}{dt} &= -v_2 \end{aligned} \tag{1}$$

It is worthwhile to normalize the currents, voltages and time as follows: $x_1 = v_1/v_0$, $x_2 = v_2/v_0$, $x_3 = Ri_L/v_0$, and $\tau = t/(RC_2)$, where v_0 the voltage where the characteristic of the nonlinear resistor has a breakpoint. The resulting system of equations is then

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha\left(-x_1 - f\left(x_1\right) + x_2\right) \\ \frac{dx_2}{dt} &= x_1 - x_2 + x_3 \\ \frac{dx_3}{dt} &= -\beta x_2 \end{aligned} \tag{2}$$

where $\alpha = C_2 / C_1$, $\beta = R^2 C_2 / L$ and

$$f\left(x\right) = \begin{cases} m_1 x - m_0 + m_1 & \text{for } x < -1 \\ m_0 x & \text{for } \left|x\right| \le 1 \\ m_1 x + m_0 - m_1 & \text{for } x > +1 \end{cases} \tag{3}$$

For the standard choice $\alpha = 9$, $\beta = 100/7$, $m_0 = -8/7$, $m_1 = -5/7$, the trajectories are irregular, non-periodic (Figure 4, Figure 5)

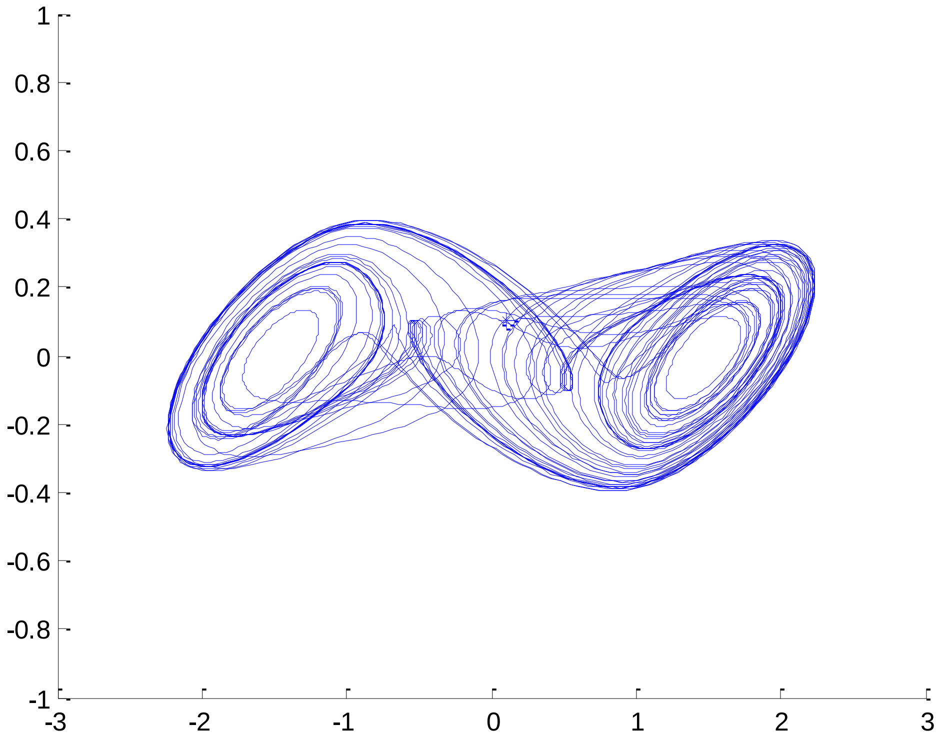


Figure 4 : Projection of the orbit of Chua's circuit with the parameters $\alpha = 9$, $\beta = 100/7$, $m_0 = -8/7$, $m_1 = -5/7$ onto the $x_1 - x_2$ - plane. The time interval represented is [0,200] and the initial state is (0.1, 0.1, 0.1).

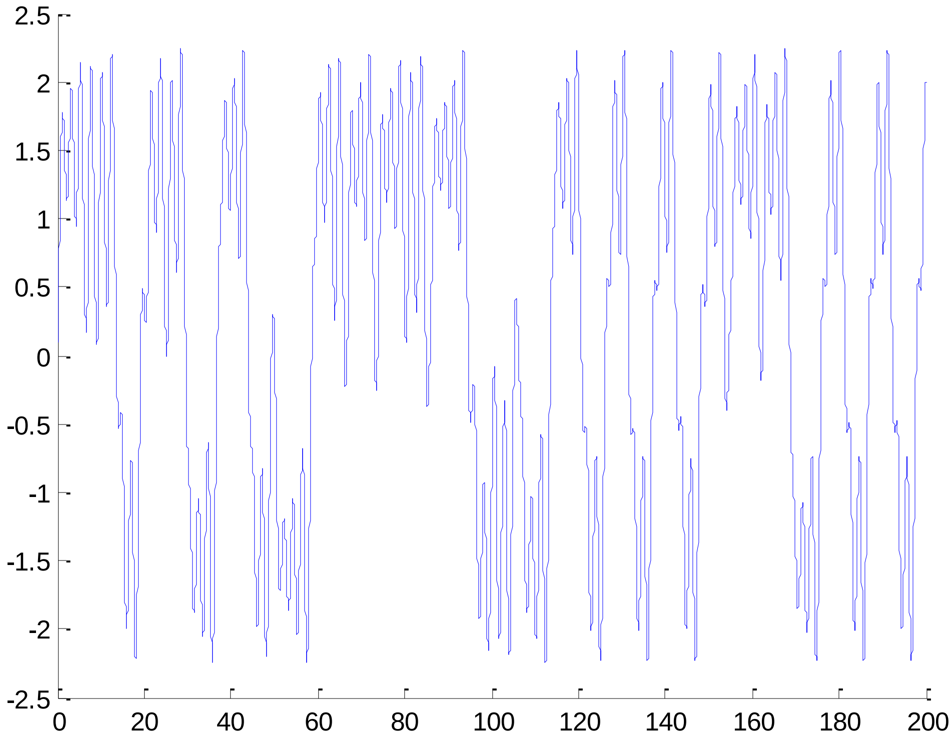


Figure 5 : The state component x_1 as a function of time, for the same solution as in Figure 4

Also in this example, nothing in the state equations indicates the presence of complicated solutions at first sight.

As impressive as this may be, the irregular solutions are not an unmistakable sign of chaos. The signal

$$y(t) = \sin(2t) + 0.7\cos(2\pi t) + 1.3\sin\left(\sqrt{2}\,t\right) \tag{4}$$

is represented in Figure 6. It looks irregular, and it is not periodic, since the frequencies in the sin and cos functions do not have rational ratios. Such signals are called *quasi-periodic*. The signal (4) is the output signal of the linear autonomous continuous-time system (5) in the time interval [0,50], starting from the initial state (0, 1, 0.7, 0, 0, -1.3). This system has its eigenfrequencies on the imaginary axis $\left(\pm 2j,\ \pm\pi j,\ \pm\sqrt{2}\,j\right)$ and therefore it is stable (without being asymptotically stable). Thus, it clearly would not be reasonable to classify this system as being chaotic.

$$
\begin{aligned}
\frac{dx_1}{dt} &= 2x_2 \\
\frac{dx_2}{dt} &= -2x_1 \\
\frac{dx_3}{dt} &= 2\pi x_4 \\
\frac{dx_4}{dt} &= -2\pi x_3 \\
\frac{dx_5}{dt} &= \sqrt{2}x_6 \\
\frac{dx_6}{dt} &= -\sqrt{2}x_5 \\
y(t) &= x_1(t) + x_3(t) + x_5(t)
\end{aligned}
\tag{5}
$$

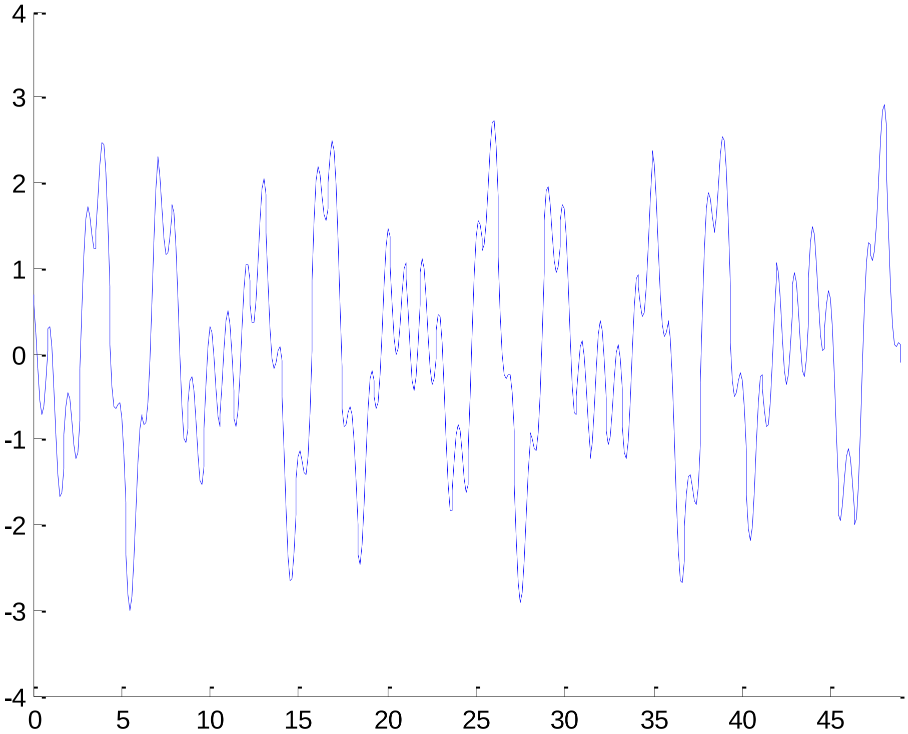


Figure 6 : Quasiperiodic signal given by (4)

2. Second property of chaos: sensitivity to initial conditions

The property that characterizes chaos best is the so-called *sensitivity to initial conditions*, together with the property that the solutions are bounded. In its weakest form, this simply means that all solutions are unstable. Therefore, loosely speaking, any two solutions that start close together will eventually separate from each other. In a stronger form, it is required that this separation is exponentially fast, at least as long as they are close.

Unstable linear systems have also sensitivity to initial conditions, but almost all their solutions are unbounded. In general, nonlinear systems with bounded solutions may have simultaneously stable and unstable solutions, the unstable ones being the exception. As an example, the 2-dimensional continuous-time system with the flow represented in Figure 7 has two asymptotically stable and one unstable equilibrium points. All other solutions are asymptotically stable, except those that are starting on the stable manifold of the unstable equilibrium point (a saddle point). They converge to the saddle point, but an arbitrarily small perturbation off the stable manifold will cause such a solution to converge to either of the stable equilibrium points and thus become asymptotically stable

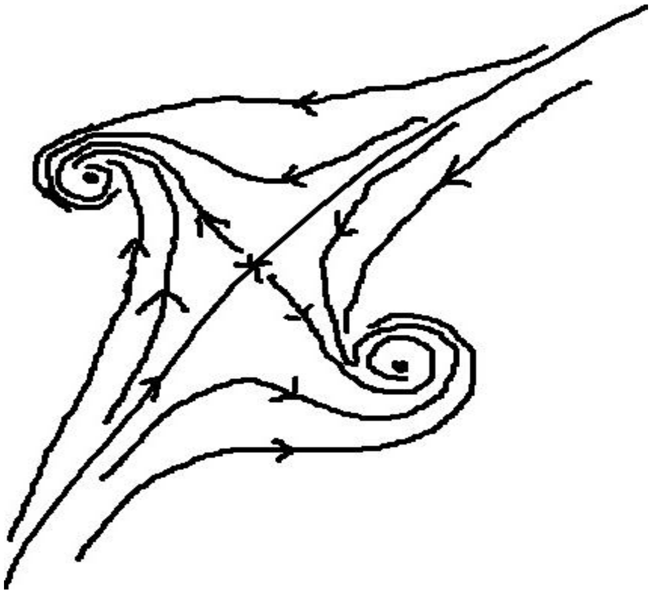
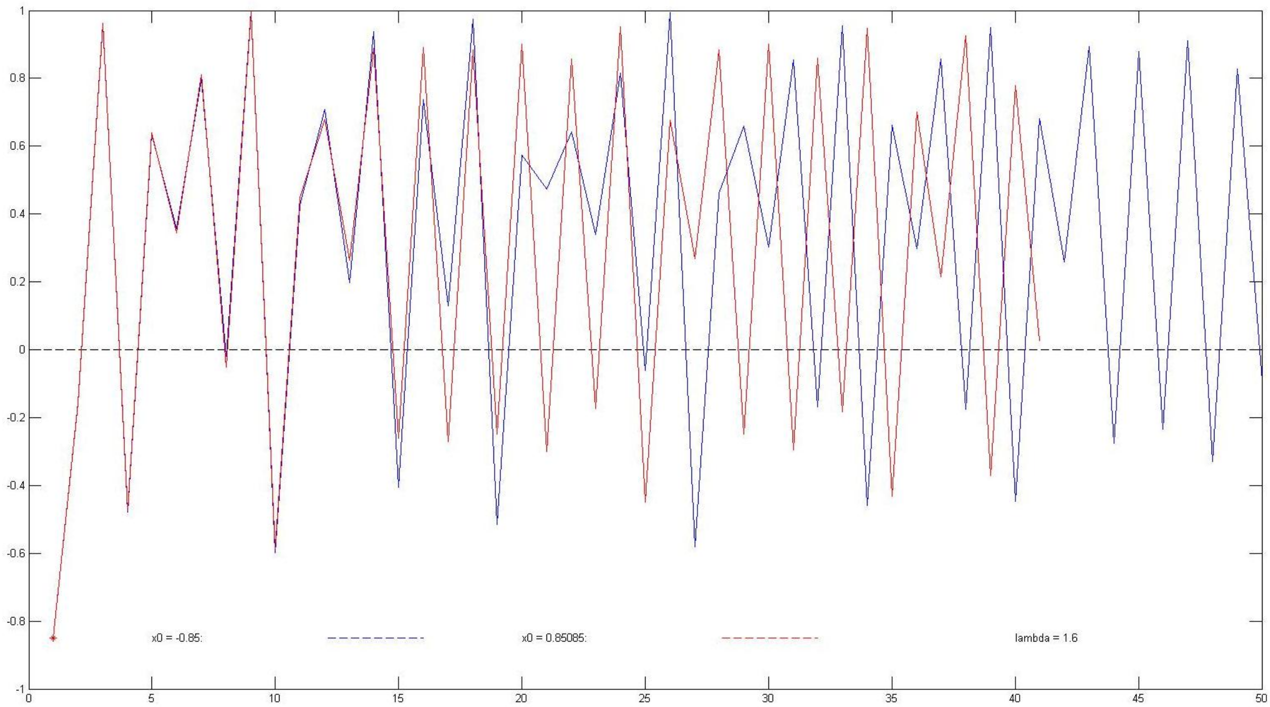


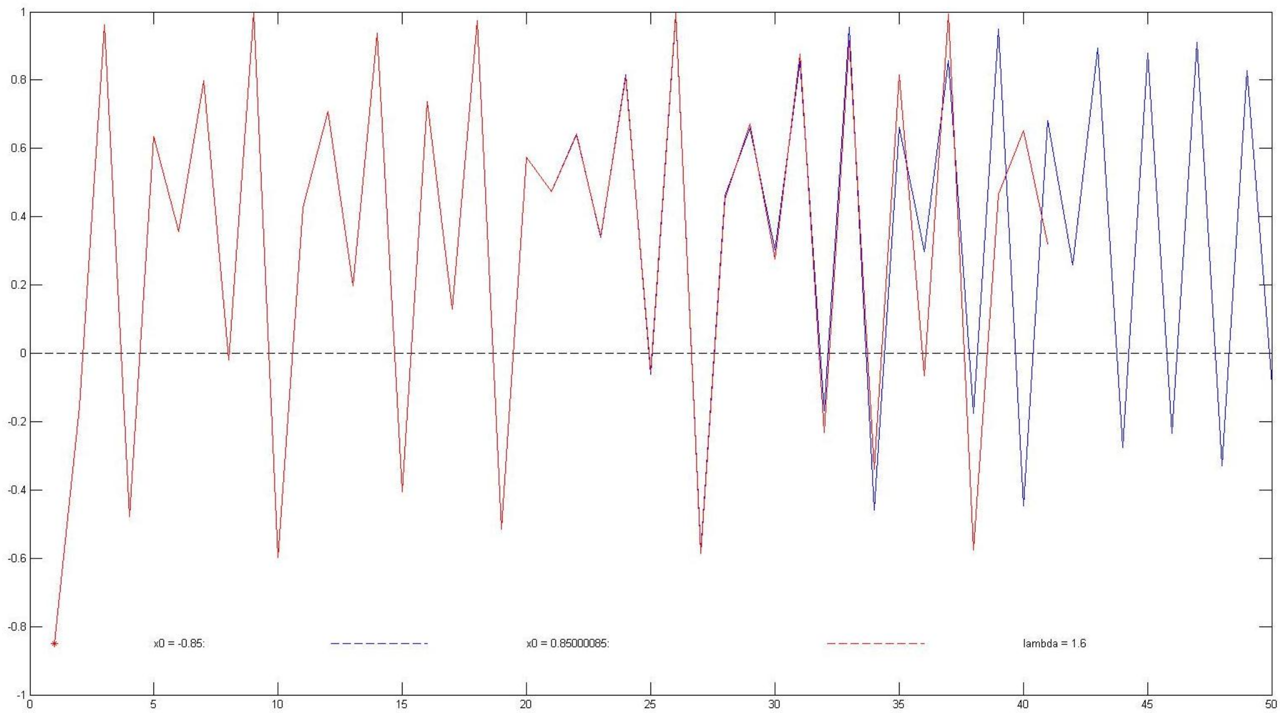
Figure 7 : Flow of a 2-dimensional continuous time system with two asymptotically stable equilibrium points (focus) and one unstable equilibrium point (saddle). Almost all solutions are asymptotically stable, only those on the stable manifold of the saddle are unstable.

It is not difficult to imagine that when all solutions are repelling each other and when they nevertheless stay in a bounded region of state space (or at least those that start in a bounded region) they must have a very disordered behavior (the first property of chaos).

The sensitivity to initial conditions is illustrated in Figure 8 for the iterations of the logistic map for $\mu = 1.6$ and in Figure 9 for Chua's circuit. In Figure 8, evidence is given for the exponentially fast separation of closeby trajectories. While the attractor of the logistic map systemin the case of chaotic behavior covers part of the interval [-1,+1], in the case of Chua's circuit it is a complicated lower dimensional geometric object (dimension < 3) that can be seen in a 2-dimensional projection in Figure 9 a) to a certain approximation. This attractor governs the asymptotic behavior of the solutions. Even if this asymptotic behavior is not unique, in fact there are infinitely many solutions in the attractor, the large time behavior of the solutions is nevertheless similar. This can be seen in Figure 9.

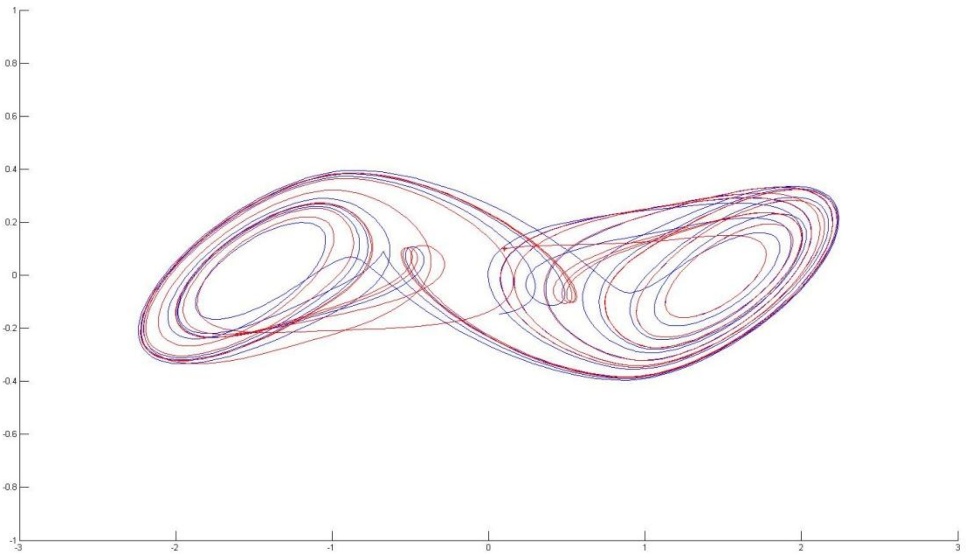


a) Trajectories starting at -0.85 (blue) and -0.00085 (red) at time t=1. Within the resolution of this figure, they are indistinguishable until about time t=11. After time t = 30, the two trajectories become approximately uncorrelated.

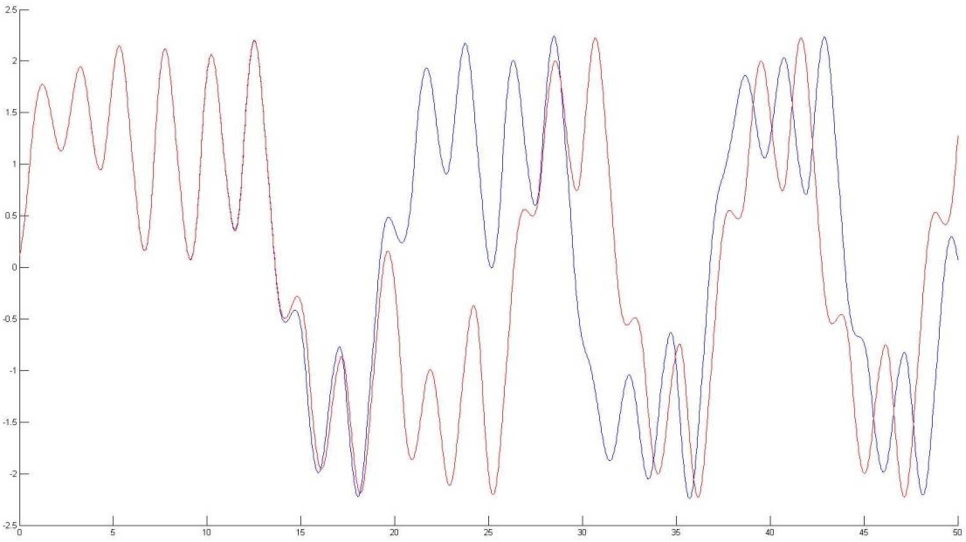


b) Trajectories starting at -0.85 (blue) and -0.00000085 (red) at time t=1. Within the resolution of this figure, they are indistinguishable until about t=30. If the two trajectories were continued in time, they would become less and less correlated.

Figure 8 : Two trajectories of the logistic map system starting close drift exponentially fast apart. In figure a) the initial separation is 10^{-3} and it takes about 10 iterations until the separation becomes visible, whereas in figure b) the initial separation is 10^{-6} and it takes about 30 iterations until the two trajectories can be distinguished in the picture.



a) Projection onto the x_1-x_2-plane of two orbits of Chua's circuit with standard parameters, starting from the initial state (0.1, 0.1, 0.1) (blue) and (0.1001, 0.1, 0.1) (red). For a short time, the two orbits cannot be distinguished in the figure, but then they become different. However, the general nature of both orbits is the same.



b) Same two solutions as in a), but here the $x_1(t)$ is represented.

Figure 9 : Two solutions of Chua's circuit starting at two initial states whose first component is different by 10^{-3}. They separate rapidly, but their general aspect is similar.

Since sensitivity to initial conditions is such an important property, is it possible to quantify it? Yes, it is possible to compute the exponential speed of separation of two nearby solutions, as long as they remain close and the linear approximation of the time evolution of the differences is valid. This naturally recalls the use of the variational equations that were successful in deciding the stability or instability of periodic solutions. In fact, exactly the same formalism can be applied to arbitrary solutions, not only periodic ones. Let us first discuss only 1-dimensional discrete-time systems, such as e.g. the iterations of the logistic map.

3. Lyapunov exponent of autonomous 1-dimensional discrete-time systems

Consider the discrete time system

$$x(t+1) \quad = \quad f\left(x(t)\right), \quad \text{where } f:\mathbb{R}\to\mathbb{R} \tag{6}$$

and *f* is continuously differentiable. Let $x(t)$ be an arbitrary solution of (6) and $\tilde{x}(t)$ a second solution such that $\Delta x(0)=\tilde{x}(0)-x(0)$ is small. Then, the time evolution of the increment $\Delta x(t)=\tilde{x}(t)-x(t)$ is, up to first order approximation, given by the solution of the variational equation

$$M(t+1) \quad = \quad \frac{df}{dx}\left(x(t)\right)\cdot M(t) \quad \text{with } M(0)=1 \tag{7}$$

and by setting

$$\Delta x(t) \quad \approx \quad M(t)\cdot\Delta x(0) \tag{8}$$

The solution of the variational equation (7) along the solution $x(t)$ of the dynamical system (6) is given by

$$M(t) \quad = \quad \frac{df}{dx}\left(x(t-1)\right)\cdot\frac{df}{dx}\left(x(t-2)\right)\cdot\ \cdots\ \cdot\frac{df}{dx}\left(x(0)\right) \tag{9}$$

And therefore the relative growth or shrinking of the increment is given by

$$\left|\frac{\Delta x(t)}{\Delta x(0)}\right| \quad \approx \quad \left|\frac{df}{dx}\left(x(t-1)\right)\right|\cdot\left|\frac{df}{dx}\left(x(t-2)\right)\right|\cdot\ \cdots\ \cdot\left|\frac{df}{dx}\left(x(0)\right)\right| \tag{10}$$

We are here interested in the exponential speed or convergence of the two solutions and therefore we set

$$\left|\frac{\Delta x(t)}{\Delta x(0)}\right| \approx e^{\alpha(t)\cdot t} \tag{11}$$

and thus

$$\alpha(t) = \frac{1}{t}\sum_{\tau=0}^{t-1}\ln\left|\frac{df}{dt}\left(x(\tau)\right)\right| \tag{12}$$

This function of time, representing the average exponential speed of growth or contraction in the time interval [0,t] can be calculated along any solution $x(t)$. Accordingly, we define

Definition 1 :

The *Lyapunov exponent* of a solution $x(t)$ of the autonomous 1-dimensional discrete time system is given by

$$\alpha = \lim_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}\ln\left|\frac{df}{dt}\left(x(\tau)\right)\right| \tag{13}$$

if the limit exists.

Quite naturally the following questions arise:

- Does the limit of $\alpha(t)$ as $t\to\infty$ exist?
- If so, to what extent does it depend on the solution $x(t)$

The answer, at least probabilistically, is given by the application of Birkhoff's ergodic theorem.

Theorem 1 :

Consider the autonomous 1-dimensional discrete-time system (6). Let μ be an invariant measure under the function f. Then, for μ-almost all solutions the Lyapunov exponent exists. If, in addition, μ is ergodic with respect to f, then for μ-almost all solutions the Lyapunov exponent is the same and its value is given by

$$\alpha = \int_{-\infty}^{+\infty}\ln\left|\frac{df}{dx}(x)\right|d\mu(x) \tag{14}$$

Proof:

Application of Birkhoffs ergodic theorem for $\Omega = \mathbb{R}$ and $F = \ln\left|\frac{df}{dx}\right|$.

Remarks:

a) Clearly, a positive Lyapunov exponent of a solution $x(t)$ implies that the solutions is unstable and a negative one that the solution is asymptotically stable. However, a solution with a 0 Lyapunov exponent can be unstable, stable or even asymptotically stable. In fact, the Lyapunov exponent distinguishes only exponential speeds of expansion or contraction, but not anything slower.

b) When the solution $x(t)$ converges to a fixed point $\bar{x}$ then its Lyapunov exponent equals

$$\alpha = \ln\left|\frac{df}{dx}(\bar{x})\right| \tag{15}$$

This implies that all solutions starting in the basin of attraction of an asymptotically stable fixed point have the same (non-positive) Lyapunov exponent.

c) Similarly, when the solution $x(t)$ converges to the T-periodic solution $\bar{x}_1, \cdots, \bar{x}_T$ then its Lyapunov exponent is

$$\alpha = \frac{1}{T}\sum_{i=1}^{T} \ln\left|\frac{df}{dx}(\bar{x}_i)\right| \tag{16}$$

Again, if the periodic solution is asymptotically stable, then all solutions starting in the basin of attraction of this solution, as well as a cyclic permutation of it, have the same (non-positive) Lyapunov exponent.

d) Remarks b) and c) show that the notion of Lyapunov exponent is not really interesting for fixed points and periodic solutions, since it coincides with the notion of eigenvalue of the Jacobian or a product of Jacobians (here just numbers). However, it is a good tool to distinguish between chaotic and non-chaotic behavior.

e) If the ergodic invariant measure is given by a density $\rho(x)$, then for Lebesgue-almost all solutions the Lyapunov exponent is

$$\alpha = \int_{-\infty}^{+\infty} \ln\left|\frac{df}{dx}(x)\right| \rho(x)\,dx \tag{17}$$

This means that if we choose an initial condition at random on the real line (or in an interval, if the system and the invariant measure is restricted to an interval), we will obtain, by calculating the right hand side of (12) for sufficiently large t, a good approximation of the Lyapunov exponent. Since determining an invariant measure most of the time cannot be done explicitly, this is in fact the way Lyapunov exponents are computed.

In Figure 10, $\alpha(t)$ is represented for a few solutions of the logistic map for μ = 0.9, where there is an asymptotically stable period-2 solution (to be precise, there are two of them). The theoretical (negative) value for the solutions converging to the period-2 solution according to (16) is indicated by a horizontal black line. Two other solutions nicely converge to this value. The only exception is the solution starting exactly at the unstable fixed point (or at its other pre-image). It remains constant at a positive value, as it should be. How is this compatible with Theorem 1? The fact is that the two exceptional initial points, i.e. the unstable fixed point and its pre-image) have Lebesgue measure 0 and thus for Lebesgue-almost all initial point in the interval [-1,+1] we get the same value for the Lyapunov exponent, namely the value corresponding to the period-2 solution.

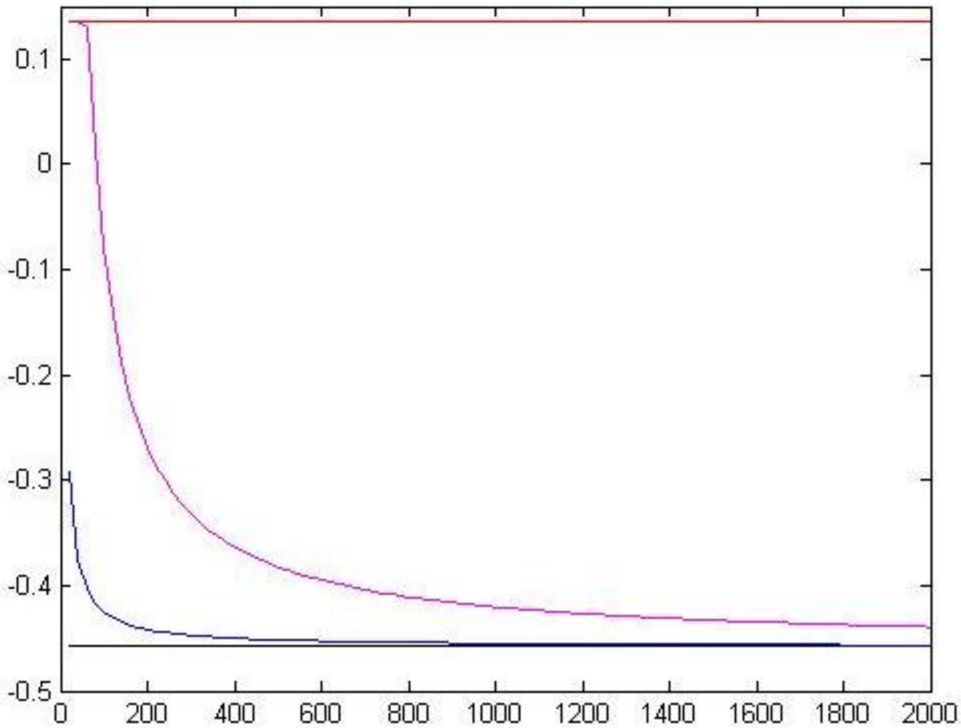


Figure 10 : Calculation of the Lyapunov exponent for the logistic map system with μ = 0.9. Horizontal black line: Lyapunov exponent calculated according to (16) for the asymptotically stable period-2 solution. Blue: $\alpha(t)$ according to (12) for the solution starting at $x(0)$ = 0.37. Magenta: $\alpha(t)$ according to (12) for the solution starting at $x(0)$ = fixed point – 0.0001. Since the initial point is so close to the fixed point that has a different Lyapunov exponent, it takes longer to converge to the Lyapunov exponent of the period-2 solution. Horizontal red line: Lyapunov exponent if the fixed point calculated according to (15).

In Figure 11, $\alpha(t)$ is represented for two solutions of the logistic map system for μ = 1.6. They appear to converge to the same positive asymptotic value. One could choose many other initial points at random to obtain the same positive asymptotic value. This is an indication of chaos. Note that the two functions are not very smooth. This is also typical of chaos. It comes from the fact that within the chaotic attractor there are infinitely many unstable periodic solutions, each one with a somewhat different Lyapunov exponent. Even though the set of initial points that lead to periodic solutions has Lebesgue measure 0, these solutions are everywhere present and perturb the calculation of the Lyapunov exponent of the chaotic solutions.

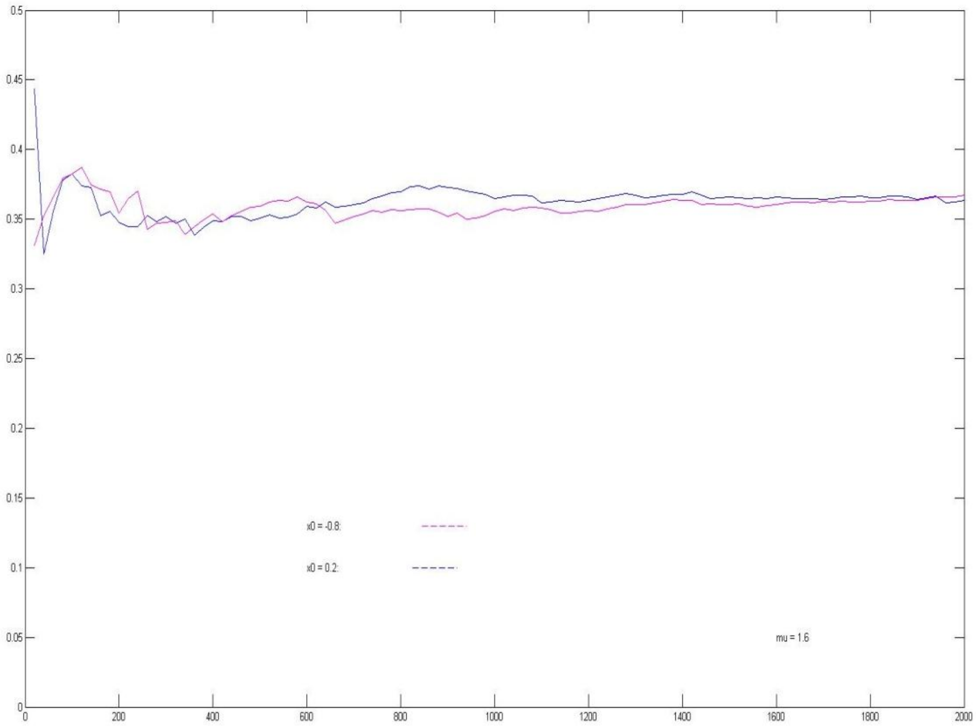


Figure 11 : Calculation of the Lyapunov exponent for the logistic map system for μ = 1.6. For two different solutions, one starting at $x(0)$ = -0.8 and the other at $x(0)$ = 0.2, $\alpha(t)$ calculated according to (12) is represented. Both functions appear to converge to the same positive value, an indication of chaos.

4. Lyapunov exponents of higher dimensional autonomous discrete-time and continuous-time systems

How can we generalize the notion of Lyapunov exponent to higher dimensional discrete-time and to continuous-time systems? The basic ideas and constructs are the same. We again consider two close solutions $\boldsymbol{x}$(t) and $\tilde{\boldsymbol{x}}(t)$ of the dynamical system $\boldsymbol{x}(t+1)=\boldsymbol{F}(\boldsymbol{x}(t))$ resp. $d\boldsymbol{x}/dt=\boldsymbol{F}(\boldsymbol{x}(t))$, where $\boldsymbol{F}:\mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, and we define the increments $\Delta\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$. Up to first order approximation, the time evolution of the increments is given by the variational equations

$$\boldsymbol{M}(t+1) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}(t)\right)\cdot \boldsymbol{M}(t) \quad \text{resp.} \quad \frac{d\boldsymbol{M}}{dt} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}(t)\right)\cdot \boldsymbol{M}(t) \quad \text{and} \quad \boldsymbol{M}(0) = I \tag{18}$$

and

$$\Delta\boldsymbol{x}(t) \approx \boldsymbol{M}(t)\Delta\boldsymbol{x}(0) \tag{19}$$

Therefore, the expansion or contraction of the initial increment vector is approximated by

$$\frac{\left\|\Delta\boldsymbol{x}(t)\right\|^2}{\left\|\Delta\boldsymbol{x}(0)\right\|^2} = \frac{\Delta\boldsymbol{x}(t)^T\,\Delta\boldsymbol{x}(t)}{\left\|\Delta\boldsymbol{x}(0)\right\|^2} \approx \frac{\Delta\boldsymbol{x}(0)^T\,\boldsymbol{M}(t)^T\,\boldsymbol{M}(t)\Delta\boldsymbol{x}(0)}{\left\|\Delta\boldsymbol{x}(0)\right\|^2} \tag{20}$$

Contrary to the 1-dimensionl case, this expansion/contraction ratio depends on $\Delta\boldsymbol{x}(0)$. Therefore, we have to consider not only a number, but the whole matrix $\boldsymbol{M}(t)^T\,\boldsymbol{M}(t)$. This matrix describes different exponential expansion/contraction in different directions in state space. To express this more clearly, we set

$$\boldsymbol{M}(t)^T\,\boldsymbol{M}(t) = e^{2\boldsymbol{\Lambda}(t)\cdot t} \quad \Leftrightarrow \quad \boldsymbol{\Lambda}(t) = \frac{1}{2t}\ln\left(\boldsymbol{M}(t)^T\,\boldsymbol{M}(t)\right) \tag{21}$$

This is possible, because $\boldsymbol{M}(t)^T\,\boldsymbol{M}(t)$ is a positive definite matrix. The factor 2 in (21) comes from the squares in (20). Again, we can ask whether $\boldsymbol{\Lambda}$(t) converges as $t \to \infty$ and if so, whether the limit depends on the solution $\boldsymbol{x}$(t).

Definition 2 :

The *Lyapunov exponents* of a solution $\boldsymbol{x}$(t) of a dynamical system are the eigenvalues of the matrix

$$\boldsymbol{\Lambda} = \lim_{t\to\infty}\frac{1}{2t}\ln\left(\boldsymbol{M}(t)^T\,\boldsymbol{M}(t)\right) \tag{22}$$

if this limit exists, where $\boldsymbol{M}$(t) is the solution of the variational equations (18).

The following generalization of Theorem 1 follows from Oseledecs ergodic theorem, a generalization of Birkhoffs ergodic theorem. The only part of Theorem 1 that does not generalize is (14).

Theorem 2 :

Let μ be an invariant measure with respect a dynamical system. The the Lyapunov exponents exist for μ-almost all solutions. If the measure is ergodic, then μ-almost all solutions have the same Lyapunov exponents.

Remarks:

a) For continuous time systems with bounded solutions, there is always a Lyapunov exponent 0. It corresponds to increments in the direction of the solution:

$$\begin{aligned} \Delta\boldsymbol{x}(0) &= \boldsymbol{x}(\tau) - \boldsymbol{x}(0) \approx \frac{d\boldsymbol{x}}{dt}(0)\cdot\tau = \boldsymbol{F}\left(\boldsymbol{x}(0)\right)\cdot\tau \\ \Delta\boldsymbol{x}(t) &= \boldsymbol{x}(t+\tau) - \boldsymbol{x}(t) \approx \frac{d\boldsymbol{x}}{dt}(t)\cdot\tau = \boldsymbol{F}\left(\boldsymbol{x}(t)\right)\cdot\tau \end{aligned} \tag{23}$$

The solution ***x***(t) being bounded ***F***(***x***(t)) is also bounded and therefore, this increment neither expands exponentially nor contracts exponentially.

b) Again, invariant measures cannot be determined explicitly in most dynamical systems. Therefore, one chooses a solution at random and one computes the Lyapunov exponents based on Definition 2. The presence of at least one positive Lyapunov exponent indicates the presence of chaos. In the case of Chua's circuit with the standard parameters, one obtains $\alpha_1 = 0.23,\ \alpha_2 = 0,\ \alpha_3 = -1.75$. Here, the positive value of α_1 indicates chaos.

c) It can be shown that volumes in state space contract/expand with exponential speed $\alpha_1 + \cdots + \alpha_n$. In the case of Chua's circuit with standard parameters this sum is -1.52, which indicates an exponentially fast contraction of volumes. Therefore, the attractor cannot occupy a positive volume in state space. In fact, it is a complicated lower dimensional geometric object. The 2-dimensional projection of a typical trajectory, as represented in Figure 4 gives only a poor glimpse of the "thin" and complicated geometry, full of layers and holes of the attractor in 3 dimensional state space. Such attractors are loosely called *strange attractor*.

d) The computation of Lyapunov exponents is a challenge for numerical analysis. On the one hand, convergence is slow because of the perturbation by unstable periodic solutions (cf. Figure 11) and on the other hand, ***M***(t) in (22) either grows or shrinks exponentially fast and becomes singular. There is a considerable literature on this subject.

5. Third property of chaos: presence of a dense set of (unstable) periodic solutions

The asymptotic behavior of the iterations of the logistic map $f(x) = 1 - \mu x^2$ as a function of the parameter μ is visualized in the bifurcation diagram of Figure 12. From the left, it clearly shows the succession of asymptotically stable 2^n-periodic solutions, followed by chaos and other asymptotically stable periodic solutions in the windows in chaos. After the flip bifurcations from a T-periodic to a 2T-periodic solution, the asymptotically stable T-periodic solution continues to exist, but becomes unstable. Therefore, in the chaotic region, infinitely many unstable periodic solutions are present.

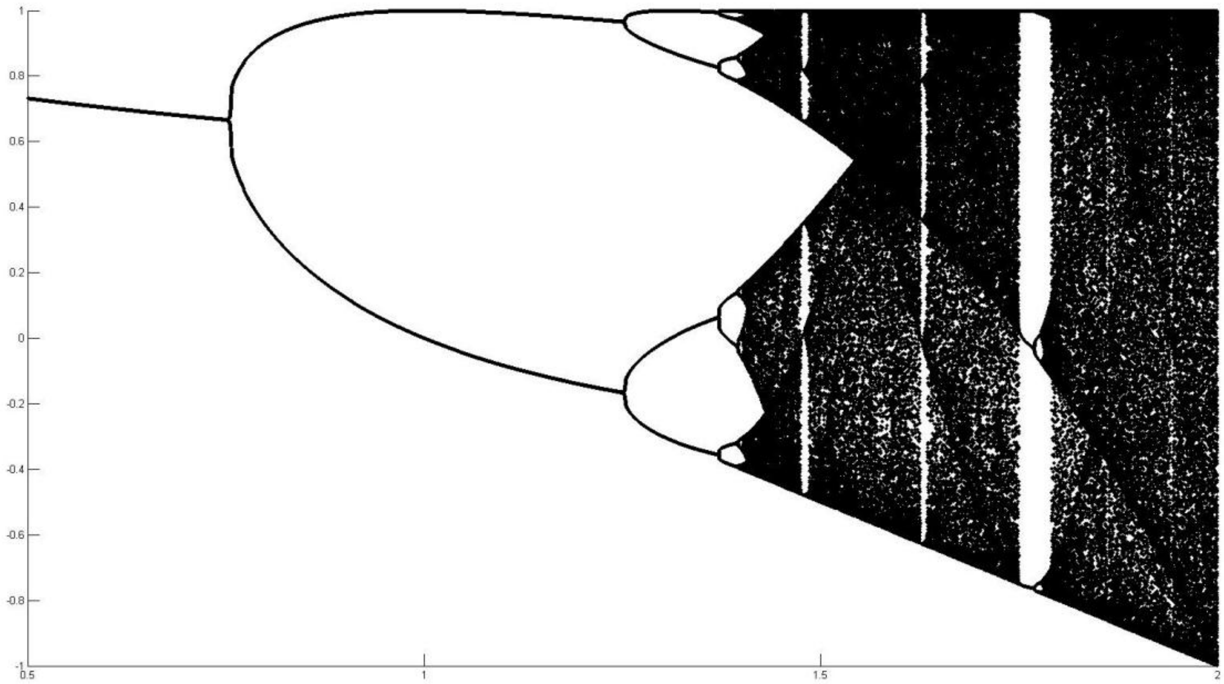


Figure 12 : Bifurcation diagram of the logistic map system. For each value of μ between 0.5 and 2, the solution x(t) is computed from suitable initial states x(0). After a number of iterations such that the transient effects have died out, 200 points are represented vertically. Thus, an asymptotically stable T-periodic solution will be represented by the T points of its orbit. On can clearly see the flip bifurcation at μ = 0.75 where the fixed point becomes unstable and an asymptotically stable 2-periodic solution is born, another flip bifurcation at m = 1.25 where the 2-periodic bifurcation becomes unstable and an asymptotically stable 4-periodic solution is born, etc. After approximately μ = 1.4, chaos appears where all 200points are distinct. In the same parameter region, however, there are also small subintervals where there is no chaos, but again asymptotically stable periodic solutions. They are called *windows in chaos*. The largest window contains a 3-periodic solution that undergoes again a cascade of flip bifurcations that create 6-, 12-, 24-, ...-periodic solutions.

The orbits of the unstable periodic solutions are actually dense in the chaotic attractors. This is illustrated in the following figures. The chaotic trajectory is first approximated by 16-periodic trajectory (Figure 13), then by an 8-periodic trajectory (Figure 14) and then then by a trajectory of period 11 (Figure 15). This can be continued. Thus, the chaotic trajectory navigates between the periodic trajectories, getting close to one, then to another, etc.

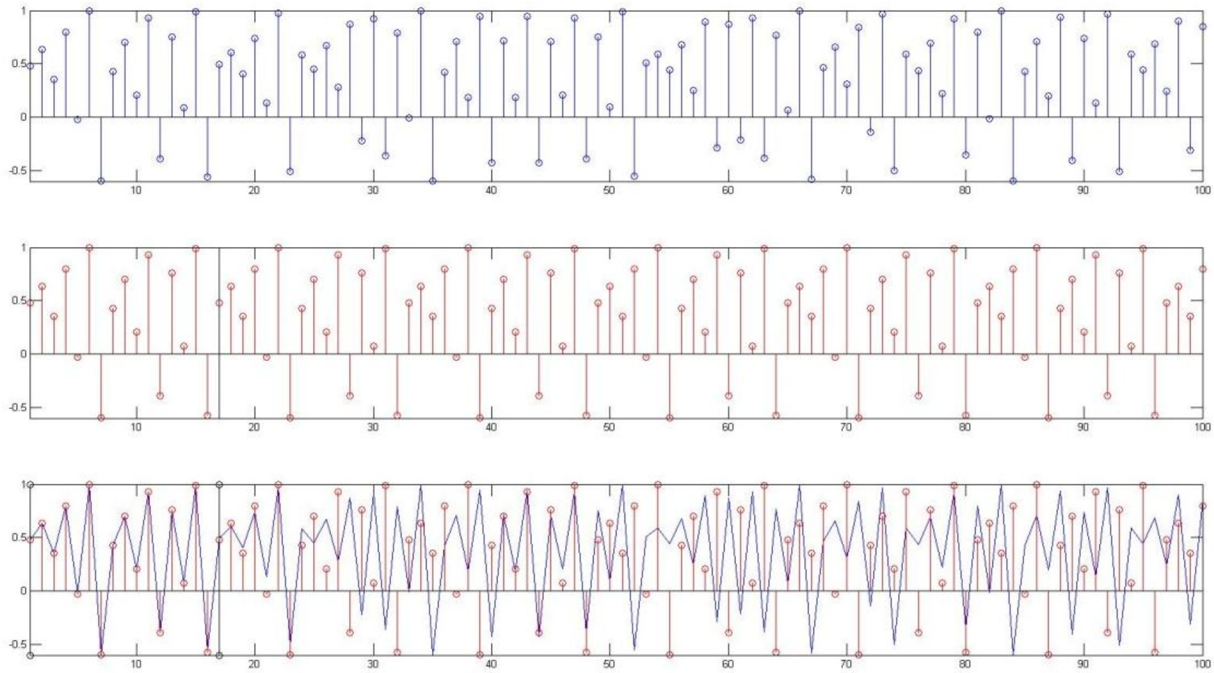


Figure 13 : Approximation of part of a chaotic trajectory by a periodic trajectory. Top: Trajectory (chaotic) of the logistic map system with μ = 1.6 starting at $x(1)$ = 0.4772. Middle: 16-periodic trajectory of the same system. Bottom: Superposition of the chaotic and the periodic trajectory. In the resolution of this figure, both trajectories are almost indistinguishable from time t = 1 to t = 18. After that they separate visibly.

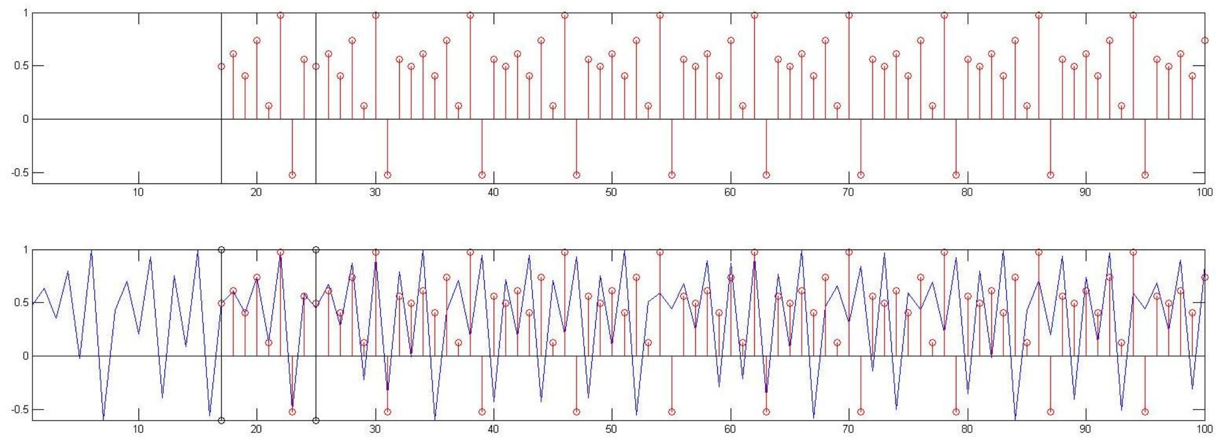


Figure 14 : Same system as in Figure 13. Top: 8-periodic trajectory. Superposition of the chaotic trajectory of Figure 13 and the 8-periodic trajectory. They are indistinguishable in this figure from t = 17 to t = 25.

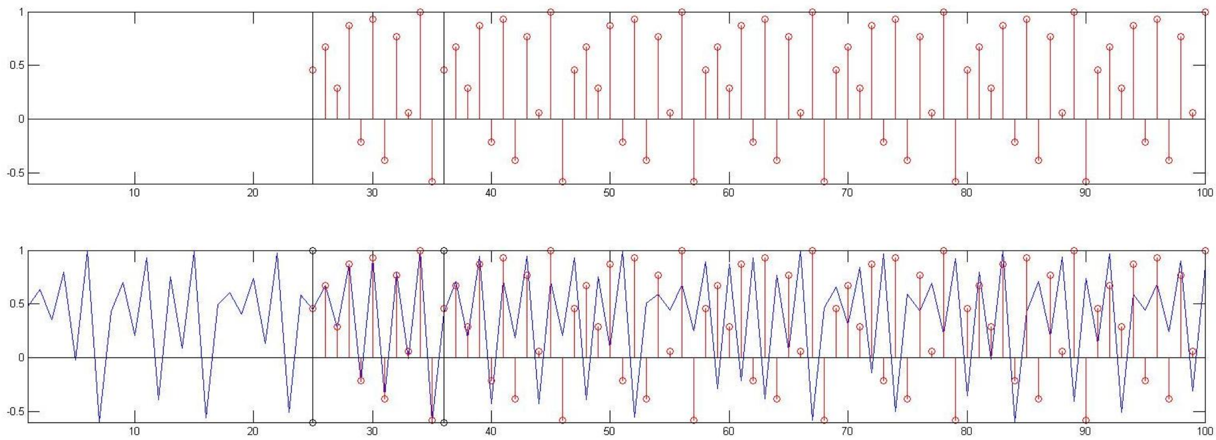


Figure 15 : Same system as in Figure 13. Top: 11-periodic trajectory. Bottom: superposition of the chaotic trajectory and the 11-periodic trajectory. In this figure they are indistinguishable from t = 25 until t = 37

To understand the intertwining of chaotic and periodic trajectories better, it is instructive to consider another 1-dimensional discrete-time system, the iterations of the Bernoulli map on the interval [0,1):

$$f(x) \quad = \quad 2x \operatorname{mod} 1 \quad = \quad \begin{cases} 2x & \text{for } 0 \le x < 0.5 \\ 2x-1 & \text{for } 0.5 \le x \end{cases} \qquad (24)$$

This function has a discontinuity at x = 0.5, but (Lebesgue-) almost all trajectories do not pass through this point. Therefore, the Lyapunov exponent of almost all trajectories exists and since the derivative of f is 2 at all points, except at 0.5, the Lyaponov exponent is for almost all trajectories $\alpha = \ln(2)$. As this is a positive number, the behavior of the system is chaotic.

In the chapter on ergodic theory, it has been shown that this dynamical system is equivalent to the left shift on the set of binary sequences, excluding those that contain only 1's after a certain index. The mapping between the points x in the interval 1 and the binary sequences $(b_1, b_2, \cdots),\ b_i \in \{0,1\}$ is given by the binary fraction expansion

$$x \quad = \quad \sum_{i=1}^{\infty} b_i 2^{-i} \qquad (25)$$

It is not difficult to see that the relation between the trajectory $x(1) = x,\ x(2) = f(x),\ x(3) = f(f(x)),\ \ldots$ starting at x and the binary sequence $(b_1, b_2, b_3, \cdots)$ is

$$b_i \quad = \quad \begin{cases} 0 & \text{if } 0 \le \mathrm{x}(i) < 0.5 \\ 1 & \text{if } 0.5 \le \mathrm{x}(i) < 1 \end{cases} \qquad (26)$$

Hence, a periodic trajectory is generated by an initial state x whose corresponding binary sequence is periodic. On the other hand, since the action of f corresponds to the left shift on the binary sequences, a periodic binary sequence generates a periodic trajectory in [0,1). From this, we can deduce easily the following properties (among others):

- There is an infinity of periodic trajectories in [0,1). Their Lyapunov exponent is also ln(2) and therefore they are all unstable.

- The orbits of the periodic trajectories consist of rational numbers in [0,1).

- The union of the orbits of the periodic trajectories is dense in [0,1). Indeed, take any x in [0,1). Determine its corresponding (generally nonperiodic) binary sequence up to index N. Repeat this binary sequence of length N indefinitely. This generates an N-periodic sequence to which there corresponds an N-periodic trajectory in [0,1). Its initial point is not farther than 2^{-N} from x. As N can be chosen arbitrarily large, there are points of periodic orbits arbitrarily close to x.

Note that without the correspondence between the discrete-time dynamical system on [0,1) and the left shift on binary sequences, it would be difficult to prove these properties. This way of analyzing a dynamical system is called *symbolic analysis*.