Mathematical Foundations of Signal Processing
Module 4: Continuous-Time Systems and Signals

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Continuous Time Signals

Modeling Continuous Time Signals

Functions

Any observed signal can be considered as a function (the values of a function)
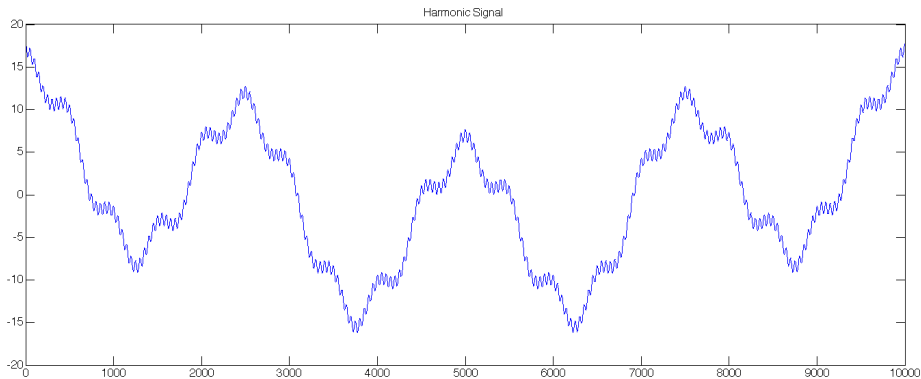
Harmonic Signal

For instance, if we play a tone with a violin, the signal we observe (we hear) can be modeled by a harmonic signal

$$s(t) = \sum_{k=1}^{K} \alpha_k e^{j2\pi f_k t} \,,$$

that is, linear combination of sinusoids with frequencies $f_1, \ldots, f_K$ (see courses Statistical Signal Processing and Signal Processing for Audio & Acoustics)

Functions

Harmonic SIgnal (ct'd)

Harmonic Signal

$$s(t) = 5e^{-i2\pi\nu_0 t} + 10e^{-i2\pi 4\nu_0 t} + 2e^{-i2\pi 20\nu_0 t} + 0.7e^{-i2\pi 200\nu_0 t}\,.$$

Functions: Spaces

- Space of integrable functions $\mathcal{L}^1(\mathbb{R})$ (required to apply integral tools)

$$\mathcal{L}^1(\mathbb{R}) = \left\{ x(t) \text{ s.t. } \int_{\mathbb{R}} |x(t)| dt < \infty \right\} .$$

  Note: $f \in \mathcal{L}^1(\mathbb{R})$ is a sufficient condition to define $\int_{\mathbb{R}} f(t) dt$.

- Finite energy functions $\mathcal{L}^2(\mathbb{R})$ (square integrable functions, a *Hilbert space*)

$$\mathcal{L}^2(\mathbb{R}) = \left\{ x(t) \text{ s.t. } \int_{\mathbb{R}} |x(t)|^2 dt < \infty \right\} .$$

  Note: Being in a Hilbert space is equivalent to observing quantities that make sense from a physical point of view—*signals of finite energy*

In case of periodic functions of period T, we rather consider $\mathcal{L}^1([-T/2, T/2))$ and $\mathcal{L}^2([-T/2, T/2))$ (absolutely and square absolutely integrable over the period)

Functions: Spaces

- Bounded functions $\mathcal{L}^\infty(\mathbb{R})$ (the magnitude of the signal is bounded)

  Recall that $\|s(t)\|_\infty = \operatorname*{ess\,sup}_{t \in \mathbb{R}} |x(t)|$. Notice that *any* measured quantity is bounded by the saturation level of the measuring instrument

- Space of smooth functions C^p (continuously-differentiable functions), with $p = 1, 2, \ldots$

  Functions that are smooth, that is continuously differentiable up to the order p. Such notion is related to the bandwidth of the function (how fast the function varies)

  We should also mention piecewise smooth functions, i.e. functions that are smooth only on intervals

Piecewise smooth function: Linear interpolator

Consider a finite sequences of (real) bounded values $x_0, x_1, \ldots, x_K$ corresponding to the time instants $0, T, \ldots, KT$. For simplicity, we assume $x_0 = x_K = 0$.

The function that interpolates with lines such values is given by

$$x(t) = \begin{cases} x_n + (t - nT)(x_{n+1} - x_n) & nT \le t < (n+1)T, \quad n = 0, \ldots, K-1 \\ 0 & t \notin [0, KT) \end{cases}$$
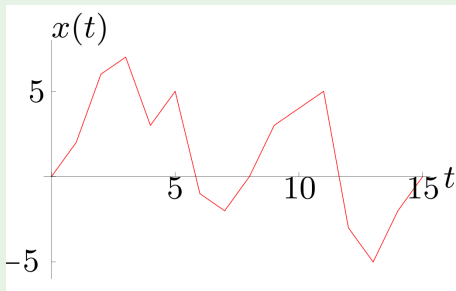
Notice that, by construction, $x(t) \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$.

Special Functions

Dirac delta (generalized function)

The Dirac delta is formally defined as

$$\delta(t) = 0, \forall\, t \neq 0 \quad \text{and} \quad \int_{\mathbb{R}} f(t)\delta(t)dt = f(0), \quad \text{with} \quad \int_{\mathbb{R}} \delta(t)dt = 1.$$

Why it is not a function?

- A function is defined point-wise;
- If a function is 0 for every $t \neq 0$ (i.e., zero almost everywhere), then its integral is 0, not 1.

Ergo, $\delta(t)$ is a *generalized function*. Within the framework of this course, we can keep treating it as a function with not much risk, but for more complex signal models, a more precise definition is required (see course Mathematical Principles of Signal Processing).

$\delta(t)$ is very handy to model:

- Events localized in time (spikes, impulses)
- Impulse response of a LSI system

Special Functions

Heaviside function

Another important function, particularly used in system theory, for instance to measure the response time of a system. The Heaviside function is defined as

$$u(t) = \begin{cases} 1 & t \geq 0\,; \\ 0 & t < 0\,. \end{cases}$$

Notice that the Dirac delta generalized function is commonly considered to be the derivative of the Heaviside function. Such a result is defined within a specific theory (distribution theory) and it is definitively not trivial

Special Functions

Gaussian

The Gaussian function is widely used:

- It has nice mathematical properties (Gaussian kernel)
- It models important physical phenomena such as diffusion
- It provides a good approximation, via the central limit theorem, of many other phenomena

The Gaussian is a symmetric function with general form given by

$$f(t) = \gamma e^{-\alpha(t-\mu)^2} \text{, where } \mu \text{ defines the symmetry point.}$$

By choosing $\alpha = 1/2\sigma^2$, $\gamma = 1/\sigma\sqrt{2\pi}$ the function is $\mathcal{L}^1$-normalized, i.e.,

$$\|f(t)\|_{\mathcal{L}^1} = 1$$

and it can play the role of a *probability density function*

Deterministic correlation

- *Deterministic autocorrelation*

$$a(\tau) \;=\; \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt \;=\; \langle x(t),\, x(t-\tau)\rangle_t, \qquad \tau \in \mathbb{R}$$

- Properties

$$\begin{aligned} a(\tau) &= a^*(-\tau), \\ a(0) &= \int_{-\infty}^{\infty} |x(t)|^2\, dt \;=\; \|x\|^2 \end{aligned}$$
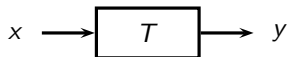
- *Deterministic crosscorrelation*

$$c(\tau) \;=\; \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt \;=\; \langle x(t),\, y(t-\tau)\rangle_t, \qquad \tau \in \mathbb{R}$$

- Properties:

$$c_{x,y}(\tau) \;=\; \left(\int_{-\infty}^{\infty} y(t-\tau)\, x^*(t)\, dt \right)^* \;=\; \left(\int_{-\infty}^{\infty} y(t')\, x^*(t'+\tau)\, dt' \right)^* \;=\; c_{y,x}^*(-\tau)$$

Continuous Time Systems

We are used to see a system as a box that outputs a certain function $y(t)$ when fed by a function $x(t)$

$$x \longrightarrow \boxed{T} \longrightarrow y$$

We can actually elegantly formalize a system by considering it as an *operator* mapping an input function $x \in V$ into an output function $y \in V$

$$y = T(x)$$

The function space V is typically $\mathcal{L}^2(\mathbb{R})$ or $\mathcal{L}^\infty(\mathbb{R})$

- Notice that *a priori* T is time-dependent

Types of Systems

Definition (Linear system)

A continuous-time system T is called *linear* when, for any inputs x and y and any $\alpha, \beta \in \mathbb{C}$,

$$T(\alpha x + \beta y) \;=\; \alpha T(x) + \beta T(y)$$

Definition (Memoryless system)

A continuous-time system T is called *memoryless* when, for any real τ and inputs x and x',

$$1_{\{\tau\}} x \;=\; 1_{\{\tau\}} x' \quad \Rightarrow \quad 1_{\{\tau\}} T(x) \;=\; 1_{\{\tau\}} T(x')$$

Definition (Causal system)

A continuous-time system T is called *causal* when, for any real τ and inputs x and x',

$$1_{(-\infty,\tau]} x \;=\; 1_{(-\infty,\tau]} x' \quad \Rightarrow \quad 1_{(-\infty,\tau]} T(x) \;=\; 1_{(-\infty,\tau]} T(x')$$

Types of Systems

Definition (Shift-invariant system)

A continuous-time system T is called *shift invariant* when, for any real τ and input x,

$$y = T(x) \Rightarrow y' = T(x'), \quad \text{where } x'(t) = x(t - \tau) \text{ and } y'(t) = y(t - \tau)$$

Definition (BIBO stable system)

A continuous-time system T is called *bounded-input bounded-output stable* when a bounded input x produces a bounded output $y = T(x)$:

$$x \in \mathcal{L}^\infty(\mathbb{R}) \quad \Rightarrow \quad y \in \mathcal{L}^\infty(\mathbb{R})$$

Basic Systems

- Shift

$$y(t) = x(t - t_0)$$

- Modulator

$$y(t) = e^{j\omega_0 t} x(t)$$

- Integrator

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

- Averaging operators

$$y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\tau) \, d\tau, \qquad t \in \mathbb{R}$$

Systems: Differential Equation

Linear constant-coefficient differential equations describe LSI systems and are of the form:

$$y(t) \;=\; \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} - \sum_{k=1}^{N} a_k \frac{d^k y(t)}{dt^k}$$

- Solution $y^{(h)}(t)$ to the homogeneous equation obtained by setting the input $x(t)$ to zero
- Particular solution $y^{(p)}(t)$ found typically by assuming the output to be of the same form as the input
- Complete solution formed by superposition of the solution to the homogeneous equation and the particular solution

Linear Shift-Invariant System

Systems that are linear shift invariant (LSI) are the most important ones:

- Several physical systems (real world) can be modeled as such
- They have very nice properties, the salient one being that they can be be described via the impulse response $h(t)$

$$\delta(t) \longrightarrow \boxed{T} \longrightarrow h(t)$$
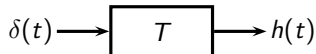
Definition (Impulse response)

A function h is called the *impulse response* of LSI continuous-time system T when input δ produces output h

Remark: *We are used to consider the impulse response as the output when the input is a Dirac impulse. We shall see that the definition of the impulse response of an LSI system finds it natural and most rigorous context in the transform domain*

Linear Shift-Invariant System

Convolution

- We can express an arbitrary input to LSI system T as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

- The output resulting from this input is

$$
\begin{aligned}
y &= Tx = T\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,T\delta(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = h*x
\end{aligned}
$$

Linear Shift-Invariant System

Definition (Convolution)

The *convolution* between functions h and x is defined as

$$(Hx)(t) = (h * x)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau,$$

where H is called the *convolution operator* associated with h.

In signal processing framework, convolution operator is commonly called filtering and the impulse response filter.

Linear Shift-Invariant System

Properties

1) Convolution and BIBO stability

Theorem

An LSI system is BIBO stable if and only if its impulse response $h(t)$ is in $\mathcal{L}^1(\mathbb{R})$ ($h(t)$ is absolutely integrable).

Proof.

a) BIBO stable if $h(t) \in \mathcal{L}^1(\mathbb{R})$: A good exercise, try to do it yourself

b) $h(t) \in \mathcal{L}^1(\mathbb{R})$ if BIBO stable: We proceed by contradiction ... □

Theorem

If the input to a BIBO-stable system is in $\mathcal{L}^p(\mathbb{R})$, the output is in $\mathcal{L}^p(\mathbb{R})$ as well.

Linear Shift-Invariant System

Proof.

b) $h(t) \in \mathcal{L}^1(\mathbb{R})$ if BIBO stable: We proceed by contradiction.
Suppose the BIBO stability, that is,

$$\text{given } |x(t)| \leq k_x, \ \forall t, \ \exists k_y > 0, \text{ such that } |y(t)| \leq k_y, \ \forall t$$

and suppose (by contradiction) that $\int_{\mathbb{R}} |h(t)| dt = \infty$, that is

$$\exists \tilde{t}, \text{ dependent on } k_y/k_x, \text{ such that } \int_{-\tilde{t}}^{\tilde{t}} |h(\tilde{t} - \tau)| d\tau > k_y/k_x$$

Let

$$x(t) = \begin{cases} k_x & \text{when } h(\tilde{t} - t) > 0 \\ 0 & \text{when } h(\tilde{t} - t) = 0 \text{ or } t \notin (-\tilde{t}, \tilde{t}) \\ -k_x & \text{when } h(\tilde{t} - t) < 0 \end{cases}$$

that is, $|x(t)| \leq k_x$
Then

$$y(\tilde{t}) = \int_{\mathbb{R}} h(\tilde{t} - \tau) x(\tau) d\tau = \int_{-\tilde{t}}^{\tilde{t}} h(\tilde{t} - \tau) x(\tau) d\tau = \int_{-\tilde{t}}^{\tilde{t}} |h(\tilde{t} - \tau)| k_x d\tau > k_y,$$

a contradiction. $\qquad\square$

Linear Shift-Invariant System

Linear Convolution with Circularly-Extended Signal

Let a T-periodic signal $x(t) \in \mathcal{L}^1([0, T])$ or $\mathcal{L}^2([0, T])$ be the input of a convolutional system (filter) with impulse response $h(t) \in \mathcal{L}^1(\mathbb{R})$

Call $h_T(t)$ the periodized version of $h(t)$

$$h_T(t) = \sum_{n \in \mathbb{Z}} h(t - nT)$$

Then the output $y(t)$ of the convolutional filter is given

$$(h * x)(t) \ = \ (h_T \circledast x)(t)$$

Definition (Circular convolution)

The *circular convolution* between T-periodic functions h and x is defined as

$$(H x)(t) \ = \ (h \circledast x)(t) \ = \ \int_{-T/2}^{T/2} x(\tau) h(t - \tau) \, d\tau \ = \ \int_{-T/2}^{T/2} x(t - \tau) h(\tau) \, d\tau,$$

where H is called the *circular convolution operator* associated with h

Analyzing Continuous Time Signals

Why do we need Transforms?

Transforming a function (here a signal or the impulse response of a system) is motivated by *one or both* of the two following reasons:

- View from a different perspective the characteristics of the function (for example, the energy of the function)

  For instance, the Fourier transform enables to compute the power spectrum and therefore to see the distribution of the energy in the frequency domain

- Move to a space where certain computations and manipulations are simplified

  For instance, a convolution in the time domain becomes a product in the Fourier domain

  Notice that here we need the transform to be *invertible*

Remark: *For a deterministic function, Fourier and Laplace transforms provide a different perspective of its characteristics and a space where certain computations are simplified (the two above motivations are strictly linked). For a stochastic process this is not the case! (see course Mathematical Principles of Signal Processing)*

Fourier transform

Eigenfunctions of the convolution operator

- LSI systems have all unit-modulus complex exponential functions as eigenfunctions

$$v(t) = e^{j\omega t}, \qquad t \in \mathbb{R}$$

$$
\begin{aligned}
(Hv)(t) &= (h * v)(t) = \int_{-\infty}^{\infty} v(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)\,d\tau \\
&= \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau}_{\lambda_\omega} \underbrace{e^{j\omega t}}_{v(t)}
\end{aligned}
$$

- v is an *eigenfunction* of H with corresponding *eigenvalue* λ_ω that we call the *frequency response* of the system $H(\omega)$

$$H\,e^{j\omega t} = h * e^{j\omega t} = H(\omega)\,e^{j\omega t}$$

Fourier Transform

- It is straightforwardly defined for $\mathcal{L}^1(\mathbb{R})$ signals, for $\mathcal{L}^2(\mathbb{R})$ signal one needs to exploit *Hilbert space tools*
- It provides a representation of the signal in the *frequency domain*
- Under *certain conditions*, it is invertible

Fourier Transform

Definition (Fourier transform)

The *Fourier transform* of a function x is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt, \qquad \omega \in \mathbb{R}. \tag{1a}$$

It exists when (1a) is defined and is finite for all $\omega \in \mathbb{R}$; we then call it the *spectrum* of x. The *inverse Fourier transform* of X is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega, \qquad t \in \mathbb{R}. \tag{1b}$$

It exists when (1b) is defined and is finite for all $t \in \mathbb{R}$. When the Fourier transform and inverse Fourier transform exist, we denote the Fourier transform pair as

$$x(t) \quad \stackrel{\mathrm{FT}}{\longleftrightarrow} \quad X(\omega).$$

We discuss two cases:

- $x(t) \in \mathcal{L}^1(\mathbb{R})$
- $x(t) \in \mathcal{L}^2(\mathbb{R})$ – allows the use of all the tools from Hilbert space theory

Fourier Transform

Functions in $\mathcal{L}^1(\mathbb{R})$

- *Existence*:

  Existence of the Fourier transform straightforwardly follows from the existence of the integral for $\mathcal{L}^1(\mathbb{R})$ functions, that is,

$$\int_{\mathbb{R}} |x(t)e^{-j\omega t}|dt = \int_{\mathbb{R}} |x(t)|dt < \infty$$

- *Riemann-Lebesgue Theorem*:

Theorem

Let $x(t) \in \mathcal{L}^1(\mathbb{R})$ and call $X(\omega)$ its Fourier transform. Then

- *$X(\omega)$ is continuous and bounded*

- *$\lim_{\omega \to \pm\infty} |X(\omega)| = 0$*

The Fourier transform is a mapping from $\mathcal{L}^1(\mathbb{R})$ to the space of continuous and bounded functions

Fourier Transform
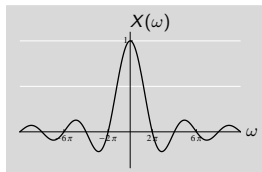
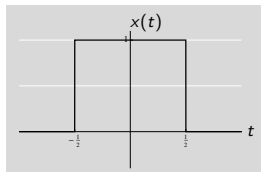### Fourier transform of the box function

- Let x be the box function, which is in $\mathcal{L}^1(\mathbb{R})$

$$x(t) = \begin{cases} 1/\sqrt{t_0}, & \text{for } |t| \leq t_0/2; \\ 0, & \text{otherwise} \end{cases}$$

- Its Fourier transform is given by

$$X(\omega) = \frac{1}{\sqrt{t_0}} \int_{-t_0/2}^{t_0/2} e^{-j\omega t} \, dt = -\frac{1}{j\omega\sqrt{t_0}} \, e^{-j\omega t}\Big|_{-t_0/2}^{t_0/2}$$

$$= \frac{e^{j\omega t_0/2} - e^{-j\omega t_0/2}}{j\omega\sqrt{t_0}} = \sqrt{t_0} \, \text{sinc}\left(\frac{\omega t_0}{2}\right)$$

which is bounded and continuous, but in $\mathcal{L}^2(\mathbb{R})$, and not in $\mathcal{L}^1(\mathbb{R})$

Fourier Transform

Functions in $\mathcal{L}^1(\mathbb{R})$

- *Inverse transform*:
  $x(t) \in \mathcal{L}^1(\mathbb{R})$ does not guarantee that $X(\omega)$ can be inverse-transformed

  In order to be inverse-transformed $X(\omega)$ must be in $\mathcal{L}^1(\mathbb{R})$, and there is no subspace of $x(t) \in \mathcal{L}^1(\mathbb{R})$ that implies this. This is one of the main drawbacks of the Fourier transform of $\mathcal{L}^1(\mathbb{R})$ functions!

  If $X(\omega) \in \mathcal{L}^1(\mathbb{R})$ then

  $$x(t) = \int_{\mathbb{R}} X(\omega) e^{j\omega t} d\omega$$

  (the proof makes use of Gaussian kernel and standard Calculus arguments. Nevertheless, it is a bit tedious)

Fourier Transform

Functions in $\mathcal{L}^2(\mathbb{R})$

If $x \in \mathcal{L}^2(\mathbb{R})$, the Fourier transform integral may or may not be defined for every $\omega \in \mathbb{R}$. The extension of the Fourier transform and its inverse from $\mathcal{L}^1(\mathbb{R})$ to $\mathcal{L}^2(\mathbb{R})$ is technically nontrivial

- *Existence*:
  The Fourier transform gives for any x in $\mathcal{L}^2(\mathbb{R})$ a Fourier transform X itself in $\mathcal{L}^2(\mathbb{R})$ (the Fourier transform in $\mathcal{L}^2(\mathbb{R})$ is a mapping from a Hilbert space to a Hilbert space)

- *Inverse transform*:
  The inverse Fourier transform can be defined similarly for any X in $\mathcal{L}^2(\mathbb{R})$, and it indeed provides an inversion, so x can be recovered from X

  (Going to the Fourier domain for performing computations really makes sense. Knowing that the Fourier transform can be extended rigorously to all of $\mathcal{L}^2(\mathbb{R})$, we can put the technicalities aside)
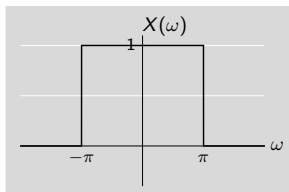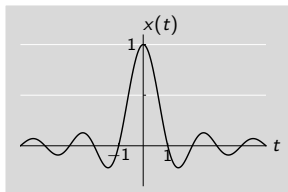
Fourier Transform

Functions in $\mathcal{L}^2(\mathbb{R})$

Fourier transform of the sinc function

$$x(t) = \sqrt{\frac{\omega_0}{2\pi}} \, \text{sinc}\left(\frac{\omega_0 t}{2}\right).$$

- $x(t)$ is in $\mathcal{L}^2(\mathbb{R})$, but not in $\mathcal{L}^1(\mathbb{R})$, and we cannot find the Fourier transform from the defining integral
- We find the Fourier transform by recognizing that the sinc function and the box function are a Fourier transform pair, and verify that

$$X(\omega) = \left\{ \begin{array}{ll} \sqrt{2\pi/\omega_0}, & \text{for } |\omega| \leq \omega_0/2; \\ 0, & \text{otherwise} \end{array} \right.$$

Fourier Transform

Properties

- *Linearity*

$$\alpha x(t) + \beta y(t) \quad \overset{\text{FT}}{\longleftrightarrow} \quad \alpha X(\omega) + \beta Y(\omega)$$

- *Shift in time*

$$x(t - t_0) \quad \overset{\text{FT}}{\longleftrightarrow} \quad e^{-j\omega t_0} X(\omega)$$

- *Shift in frequency*

$$e^{j\omega_0 t} x(t) \quad \overset{\text{FT}}{\longleftrightarrow} \quad X(\omega - \omega_0)$$

- *Differentiation in time*

$$\frac{d^n x(t)}{dt^n} \quad \overset{\text{FT}}{\longleftrightarrow} \quad (j\omega)^n X(\omega)$$

- *Integration*

$$\int_{-\infty}^{t} x(\tau)\, d\tau \quad \overset{\text{FT}}{\longleftrightarrow} \quad \frac{1}{j\omega} X(\omega)$$

- *Convolution in time*

$$(h * x)(t) \quad \overset{\text{FT}}{\longleftrightarrow} \quad H(\omega)X(\omega)$$

- *Convolution in frequency*

$$h(t)\, x(t) \quad \overset{\text{FT}}{\longleftrightarrow} \quad \frac{1}{2\pi}(H * X)(\omega)$$

Fourier Transform

- *Deterministic autocorrelation*

$$a(t) = \int_{-\infty}^{\infty} x(\tau)\, x^*(\tau - t)\, d\tau \quad \overset{\text{FT}}{\longleftrightarrow} \quad A(\omega) = |X(\omega)|^2$$

- *Parseval's equality*: unitary transform up to a scale factor

$$\|x\|^2 = \frac{1}{2\pi}\|X\|^2$$

and

$$\langle x, y \rangle = \frac{1}{2\pi}\langle X, Y \rangle$$

- *Adjoint* The adjoint of the Fourier transform, $F^* : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$, is determined uniquely by

$$\langle Fx, y \rangle = \langle x, F^*y \rangle \qquad \text{for every } x, y \in \mathcal{L}^2(\mathbb{R})$$

Since Parseval's equality shows that $F/\sqrt{2\pi}$ is a unitary operator, we have

$$\left(\frac{1}{\sqrt{2\pi}}F\right)^* = \left(\frac{1}{\sqrt{2\pi}}F\right)^{-1} = \sqrt{2\pi}\, F^{-1}$$

Thus,

$$F^* = 2\pi\, F^{-1}$$

Fourier Transform

Transform pairs

- *Dirac delta*

$$1 \quad \overset{\mathrm{FT}}{\longleftrightarrow} \quad 2\pi\,\delta(\omega)$$

- *Box Function*

$$x(t) = \left\{ \begin{array}{ll} 1/\sqrt{t_0}, & \text{for } |t| \le t_0/2; \\ 0, & \text{otherwise,} \end{array} \right. \quad \overset{\mathrm{FT}}{\longleftrightarrow} \quad X(\omega) = \sqrt{t_0}\,\operatorname{sinc}(\omega t_0/2)$$

- *Sinc function*

$$x(t) = \sqrt{\frac{\omega_0}{2\pi}}\,\operatorname{sinc}(\omega_0 t/2) \quad \overset{\mathrm{FT}}{\longleftrightarrow} \quad X(\omega) = \left\{ \begin{array}{ll} \sqrt{2\pi/\omega_0}, & \text{for } |\omega| \le \omega_0/2; \\ 0, & \text{otherwise.} \end{array} \right.$$

- *Heaviside function*

$$u(t) = \left\{ \begin{array}{ll} 1, & \text{for } t \ge 0; \\ 0, & \text{otherwise.} \end{array} \right. \quad \overset{\mathrm{FT}}{\longleftrightarrow} \quad U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

- *Gaussian function*

$$g(t) = \gamma\,e^{-\alpha t^2} \quad \overset{\mathrm{FT}}{\longleftrightarrow} \quad G(\omega) = \gamma\,\sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/(4\alpha)}$$

Fourier Transform

- If

$$|X(\omega)| \leq \frac{\gamma}{1 + |\omega|^{1+\varepsilon}}, \qquad \text{for all } \omega \in \mathbb{R},$$

  for some positive constants γ and ε, then $|X(\omega)|$ is integrable, or, $X \in \mathcal{L}^1(\mathbb{R})$. Therefore, the inverse Fourier transform x is bounded and continuous, or, $x \in C^0$

- If

$$|X(\omega)| \leq \frac{\gamma}{1 + |\omega|^{p+1+\varepsilon}} \qquad \text{for all } \omega \in \mathbb{R},$$

  for some nonnegative integer p and positive constants γ and ε, then $x \in C^p$

- The Fourier transform of any x in C^p is bounded by

$$|X(\omega)| \leq \frac{\gamma}{1 + |\omega|^{p+1}}, \qquad \text{for all } \omega \in \mathbb{R},$$

  for some positive constant γ

- Fourier transform of any continuous function is bounded by

$$|X(\omega)| \leq \frac{\gamma}{1 + |\omega|}, \qquad \text{for all } \omega \in \mathbb{R},$$

  for some positive constant γ

Fourier Series

- The Fourier series arises from identifying the eigenfunctions of the circular convolution operator, which have the following form

$$v(t) = e^{j(2\pi/T)kt}, \qquad t \in \mathbb{R}$$

$$
\begin{aligned}
(H\,v)(t) &= (h \circledast v)(t) = \int_{-T/2}^{T/2} v(t-\tau)h(\tau)\,d\tau \\
&= \int_{-T/2}^{T/2} e^{j(2\pi/T)k(t-\tau)}h(\tau)\,d\tau \\
&= \underbrace{\int_{-T/2}^{T/2} h(\tau)e^{-j(2\pi/T)k\tau}\,d\tau}_{\lambda_k} \underbrace{e^{j(2\pi/T)kt}}_{v(t)}
\end{aligned}
$$

- v is an *eigenfunction* of H with corresponding *eigenvalue* λ_k that we call the *frequency response* H_k

$$H\,e^{j(2\pi/T)kt} = h \circledast e^{j(2\pi/T)kt} = H_k\,e^{j(2\pi/T)kt}$$

Fourier Series

Projecting onto the subspaces generated by each of the eigenfunctions

Definition (Fourier series)

The *Fourier series coefficients* of a periodic function x with period T are

$$X_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j(2\pi/T)kt}\,dt, \qquad k \in \mathbb{Z}. \tag{4a}$$

They exist when (4a) is defined and finite for all $k \in \mathbb{Z}$; we then call them the *spectrum* of x. The *Fourier series* reconstruction from X is

$$x(t) = \sum_{k\in\mathbb{Z}} X_k\, e^{j(2\pi/T)kt}, \qquad t \in [-T/2, T/2).$$

When the Fourier series coefficients exist and the reconstruction converges, we denote the Fourier series pair as

$$x(t) \quad \overset{\text{FS}}{\longleftrightarrow} \quad X_k.$$

Fourier Series

Spectrum in $\ell^1(\mathbb{Z})$

- Let the T-periodic function x have spectrum $X \in \ell^1(\mathbb{Z})$. Then Fourier series converges absolutely for every t since

$$\sum_{k \in \mathbb{Z}} \left| X_k e^{j(2\pi/T)kt} \right| \;=\; \sum_{k \in \mathbb{Z}} |X_k| \left| e^{j(2\pi/T)kt} \right| \;=\; \sum_{k \in \mathbb{Z}} |X_k| \;=\; \|X\|_1 \;<\; \infty.$$

  Moreover, the Fourier series converges to a continuous function $\widehat{x}(t)$

- When x is continuous and $X \in \ell^1(\mathbb{Z})$,

$$x(t) \;=\; \widehat{x}(t), \qquad \text{for all } t \in \mathbb{R}$$

Fourier series

Fourier Series as an Orthonormal Basis Expansion

Theorem (Orthonormal basis from Fourier series)

The set $\{\varphi_k\}_{k \in \mathbb{Z}}$ with

$$\varphi_k(t) \; = \; \frac{1}{\sqrt{T}} \, e^{j(2\pi/T)kt}, \qquad t \in [-T/2, T/2),$$

forms an orthonormal basis for $\mathcal{L}^2([-T/2, T/2))$.

Up to scaling by the constant $\sqrt{T}$, the Fourier series is an orthonormal basis expansion. The orthonormality leads to geometrically-intuitive properties using Hilbert space tools

Fourier series

Theorem (Fourier series on $\mathcal{L}^2([-T/2, T/2])$)

Let $x \in \mathcal{L}^2([-T/2, T/2])$ have Fourier series coefficients $X = Fx$. Then the following hold:

- *$\mathcal{L}^2$ inversion: Let $\widehat{x}$ be the Fourier series reconstruction from X. Then*

$$\|x - \widehat{x}\| = 0$$

  using the $\mathcal{L}^2([-T/2, T/2])$ norm.

- *Norm conservation: The linear operator $\sqrt{T}F : \mathcal{L}^2([-T/2, T/2]) \rightarrow \ell^2(\mathbb{Z})$ is unitary. This yields the Parseval's equalities*

$$\|x\|^2 = \int_{-T/2}^{T/2} |x(t)|^2 \, dt = T \sum_{k \in \mathbb{Z}} |X_k|^2,$$

$$\langle x, y \rangle = \int_{-T/2}^{T/2} x(t) y^*(t) \, dt = T \sum_{k \in \mathbb{Z}} X_k Y_k^*,$$

  where T-periodic function y has Fourier series coefficients Y.

- *Least-squares approximation: The function*

$$\widehat{x}_N(t) = \sum_{k=-N}^{N} X_k \, e^{j(2\pi/T)kt}$$

  is the least-squares approximation of x on the subspace spanned by $\{\varphi_k(t)\}_{k=-N}^{N}$.

Fourier series

- *Linearity*

$$\alpha x(t) + \beta y(t) \quad \xleftrightarrow{\text{FS}} \quad \alpha X_k + \beta Y_k$$

- *Shift in time*

$$x(t - t_0) \quad \xleftrightarrow{\text{FS}} \quad e^{-j(2\pi/T)kt_0} X_k$$

- *Shift in frequency*

$$e^{j(2\pi/T)k_0 t} x(t) \quad \xleftrightarrow{\text{FS}} \quad X_{k-k_0}$$

- *Differentiation*

$$\frac{d^n x(t)}{dt^n} \quad \xleftrightarrow{\text{FS}} \quad \left(j\frac{2\pi}{T}k \right)^n X_k$$

- *Integration*

$$\int_{-T/2}^{t} x(\tau)\, d\tau \quad \xleftrightarrow{\text{FS}} \quad \frac{T}{j2\pi k} X_k, \quad \text{for } k \neq 0$$

- *Circular convolution in time*

$$(h \circledast x)(t) \quad \xleftrightarrow{\text{FS}} \quad T\, H_k\, X_k$$

- *Circular convolution in frequency*

$$h(t)\, x(t) \quad \xleftrightarrow{\text{FS}} \quad (H * X)_k$$

- *Circular deterministic autocorrelation*

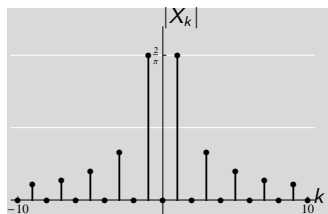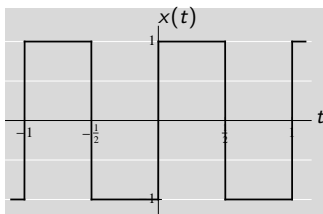$$a(t) = \int_{-T/2}^{T/2} x(\tau) x^*(\tau - t)\, d\tau \quad \xleftrightarrow{\text{FS}} \quad A_k = T\, |X_k|^2$$

Fourier series

- *Square wave*

$$x(t) = \begin{cases} 1, & \text{for } -1/2 \le t < 0; \\ -1, & \text{for } 0 \le t < 1/2; \end{cases}$$

$$X_k = \begin{cases} 2j/(\pi k), & \text{for } k \text{ odd}; \\ 0, & \text{for } k \text{ even} \end{cases}$$

$$x(t) = \sum_{\ell \in \mathbb{Z}} \frac{2j}{\pi(2\ell+1)} e^{j2\pi(2\ell+1)t} = \sum_{\ell=0}^{\infty} \frac{4}{\pi(2\ell+1)} \sin(2\pi(2\ell+1)t).$$
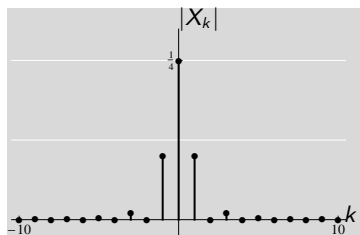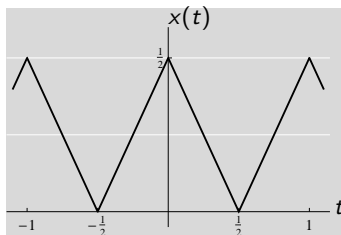
Fourier series

Transform pairs

- *Triangle wave*

$$y(t) = \tfrac{1}{2} - |t|, \qquad \text{for } t \in [-1/2, 1/2)$$

$$Y_0 = 0$$

$$Y_k = \begin{cases} 1/(\pi k)^2, & \text{for } k \text{ odd}; \\ 0, & \text{for } k \text{ nonzero and even.} \end{cases}$$

Fourier series

Transform pairs

- *Dirac comb*

$$s_T(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT)$$

$$
\begin{aligned}
S_{T,k} &= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n \in \mathbb{Z}} \delta(t - nT) \, e^{-j(2\pi/T)kt} \, dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \, e^{-j(2\pi/T)kt} \, dt \\
&= \frac{1}{T}, \qquad k \in \mathbb{Z}
\end{aligned}
$$

$$\frac{1}{T} \sum_{k \in \mathbb{Z}} e^{j(2\pi/T)kt} = \sum_{n \in \mathbb{Z}} \delta(t - nT)$$

Fourier series

Relation of the Fourier series coefficients to the Fourier transform

- For a function that is identically zero outside of $[-T/2, T/2)$, the Fourier series integral is the same as the Fourier transform integral for frequency $\omega = (2\pi/T)k$

- An arbitrary T-periodic function x can be restricted to $[-T/2, T/2)$ by defining $\widetilde{x} = 1_{[-T/2, T/2)} x$

- It is then easy to verify that

$$X_k = \frac{1}{T} \, \widetilde{X}\left(\frac{2\pi}{T}k\right), \qquad k \in \mathbb{Z},$$

where $\widetilde{X}$ is the Fourier transform of $\widetilde{x}$

Fourier series

Fourier transform of periodic functions

Let x be a T-periodic function, and let $\widetilde{x}$ be the restriction of x to $[-T/2, T/2)$:

$$\widetilde{x}(t) \;=\; 1_{[-T/2, T/2)}\, x(t), \qquad t \in \mathbb{R}$$

Using the Dirac comb, we can write

$$x(t) \;=\; (s_T * \widetilde{x})(t), \qquad t \in \mathbb{R}$$

Assuming we may apply the convolution theorem, we have

$$X(\omega) \;=\; S_T(\omega)\, \widetilde{X}(\omega), \qquad \omega \in \mathbb{R}$$

We can write the Fourier transform of the Dirac comb as

$$S_T(\omega) \;=\; \sum_{n \in \mathbb{Z}} e^{-j\omega n T}$$

The Dirac delta function S_T can be expressed as a scaled Dirac comb with spacing $2\pi/T$:

$$S_T(\omega) \;=\; \sum_{n \in \mathbb{Z}} e^{-j\omega n T} \;=\; \tfrac{2\pi}{T} \sum_{k \in \mathbb{Z}} \delta\left(\omega - \tfrac{2\pi}{T}k\right) \;=\; \tfrac{2\pi}{T} s_{2\pi/T}(\omega).$$

Finally, we may write the Fourier transform of $X(\omega)$ using the Fourier series coefficients of x:

$$X(\omega) \;=\; 2\pi \sum_{k \in \mathbb{Z}} X_k\, \delta\left(\omega - \tfrac{2\pi}{T}k\right)$$

Fourier series

Poisson sum formula

Theorem (Poisson sum formula)

Let s_T be the Dirac comb and let x be a function with decay sufficient for the periodization

$$(s_T * x)(t) \; = \; \sum_{n \in \mathbb{Z}} x(t - nT)$$

to converge absolutely for all t. Then

$$\sum_{n \in \mathbb{Z}} x(t - nT) \; = \; \frac{1}{T} \sum_{k \in \mathbb{Z}} X \left(\tfrac{2\pi}{T} k \right) e^{j(2\pi/T)kt},$$

where X is the Fourier transform of x. Specializing to $T = 1$ and $t = 0$,

$$\sum_{n \in \mathbb{Z}} x(n) \; = \; \sum_{k \in \mathbb{Z}} X(2\pi k).$$

Fourier series

Regularity and spectral decay

Note

- A discontinuous function such as the square wave has Fourier series coefficients decaying only as $O(1/k)$
- A continuous function such as the triangle wave leads to a decay of $O(1/k^2)$

Generally

-
$$|X_k| \leq \frac{\gamma}{1 + |k|^{q+1+\varepsilon}} \qquad \text{for all } k \in \mathbb{Z}$$

  for some nonnegative integer q and positive constants γ and ε, implies x has q continuous derivatives

- If the periodic function x has q continuous derivatives, then its Fourier series coefficients satisfy

$$|X_k| \leq \frac{\gamma}{1 + |k|^{q+1}} \qquad \text{for all } k \in \mathbb{Z}$$