MAT 653: Statistical Simulation

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Oct 5st, 2021

Random Variable Generation

Reading Parts:

1.Chapter 2 from Introducing Monte Carlo Methods with R, by Christian. P. Robert and George Casella.

2.Section 1.4.1, 1.4.2, 1.4.3, 1.4.4 from An Introduction to Statistical Computing: A Simulation-based Approach, by Jochen Voss.

Definition

A random variable U has a uniform distribution $U \sim U(0,1)$ if $P(U \in (a,b)) = b - a$ for $0 < a < b < 1$.

Definition

A random variable has PDF (probability density function) $f(x)$ if $P(X \in A) = \int_A f(x)dx$.

- **PDF** $f(x)$:

1. $f(x) \geq 0$
2. $\int_{\mathbb{R}} f(x)dx = 1$

The support of X is $\{x : f(x) > 0\}$ denoted by $\mathcal{X}$.

- **CDF**: the CDF of X (on $\mathbb{R}$) is given by $F(a) = P(X \leq a)$, for all $a \in \mathbb{R}$.

1. $\lim\limits_{x \to -\infty} F(a) = 0$
2. $\lim\limits_{x \to \infty} F(a) = 1$
3. $F(a)$ is nondecreasing function.
4. $\lim\limits_{h_n \to 0^+} F(a + h_n) = F(a)$. i.e., F(a) is right continuous.

Fact: Any non-negative function $\tilde{f}$ that is integrable on its support can be used to construct a PDF by normalization.

Examples:

1.$\tilde{f}(x) = e^{-\frac{x^2}{2}}$, $\mathcal{X} = (-\infty, \infty)$

$f(x) = \frac{\tilde{f}(x)}{\int_{-\infty}^{\infty} \tilde{f}(x)dx}$ (normalization) $= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ where $e^{-\frac{x^2}{2}}$ is the kernel of $N(0,1)$.

$f(x) \propto e^{-\frac{x^2}{2}}$.

2.$\tilde{f}(x) = x^{\alpha-1}e^{-\beta x}$, $x > 0, \alpha, \beta > 0$

$f(x) = \frac{\tilde{f(x)}}{\int_0^\infty x^{\alpha-1}e^{-\beta x}dx} = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ $(x > 0) \sim Gamma(\alpha, \beta)$

$f(x) \propto x^{\alpha-1}e^{-\beta x}$

Probability integral transform

(works only for $\mathbb{R}$)

Suppose X has a CDF F, i.e. $F(x) = P(X \leq x)$. If F is invertible and $Y = F^{-1}(U)$. $U \sim U(0,1)$.

Then $Y \sim F$.

Proof:

More generally, define the inverse of F as $F^{-1}(u) = inf\{x \in \mathbb{R} : F(x) \geq u\}$.

Fact1: For every $0 < p < 1$, $F \circ F^{-1}(p) \geq p$. The equality holds iff (if and only if) p in the range of F.

Fact2: For every $0 < p < 1$ and $x_0 \in \mathbb{R}$, $p \leq F(x_0)$ iff $F^{-1}(p) \leq x_0$.

Proof:

Example 2.1 from R.C.:

$X \sim exp(1)$, the CDF of $exp(1)$ is $F(x) = 1 - e^{-x}$.

Then $F^{-1}(u) = -log(1-u)$.

1.Generate $u \sim U(0,1)$.

2.Evaluate $F^{-1}(u)$, gives one sample.