MAT 653: Statistical Simulation

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Metropolis-Hasting algorithm (blockwise)

Target density: $f(x, y)$, which could be unnormalized.

We use two candidate transition densities $q_X(x|x_c, y_c)$ and $q_Y(y|x_c, y_c)$, where x_c, y_c stand for current values.

Algorithm:

1. Update X
   (a) sample $X^* \sim q_X(\cdot|X^{(s)}, Y^{(s)})$
   (b) compute $\gamma_X(X^*|X^{(s)}, Y^{(s)}) = \dfrac{f(X^*, Y^{(s)})}{f(X^{(s)}, Y^{(s)})} \cdot \dfrac{q_X(X^{(s)}|X^*, Y^{(s)})}{q_X(X^*|X^{(s)}, Y^{(s)})}$
   (c) set $X^{(s+1)}$ to X^* with probability $\min(1, \gamma_X)$; set $X^{(s+1)}$ to $X^{(s)}$ otherwise.
2. Update Y
   (a) sample $Y^* \sim q_Y(\cdot|X^{(s+1)}, Y^{(s)})$
   (b) compute $\gamma_Y(Y^*|X^{(s+1)}, Y^{(s)}) = \dfrac{f(X^{(s+1)}, Y^*)}{f(X^{(s+1)}, Y^{(s)})} \cdot \dfrac{q_Y(Y^{(s)}|X^{(s+1)}, Y^*)}{q_Y(Y^*|X^{(s+1)}, Y^{(s)})}$
   (c) set $Y^{(s+1)}$ to Y^* with probability $\min(1, \gamma_Y)$; set $Y^{(s+1)}$ to $Y^{(s)}$ otherwise.

Remark: $p\big((x, y) \to (x^*, y^*)\big) f(x, y) = p\big((x^*, y^*) \to (x, y)\big) f(x^*, y^*)$ does not hold for this algorithm in general.

Proof:

Extra: Multivariate Normal (for R code example in class)

The multivariate normal distribution of a $d-$dimensional random vector $\mathbf{X} = (X_1, \cdots, X_d)^\top \in \mathbb{R}^d$ can be written in the notation $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

When the symmetric covariance matrix $\boldsymbol{\Sigma}$ is positive definite, then the multivariate normal distribution is non-degenerate, and the distribution has density function

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \big(\det(\boldsymbol{\Sigma})\big)^{-\frac{1}{2}} \exp\big(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\big) \\ &\overset{\mathbf{x}}{\propto} \exp(-\frac{1}{2}\mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\mathbf{x} + \mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) \end{aligned}$$

where $\mathbf{x} = (x_1, \cdots, x_d)^\top$.

For comparison, the density function of (univariate) normal distribution $X \sim N(\mu, \sigma^2)$

$$f_X(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\big(-\frac{1}{2\sigma^2}(x-\mu)^2\big) \overset{x}{\propto} \exp\big(-\frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)\big) \overset{x}{\propto} \exp\big(-\frac{1}{2}x^2(\sigma^2)^{-1} + x(\sigma^2)^{-1}\mu\big)$$