Basics of Markov Chains (continuous state space)

Instructor: Dr. Wei Li Scribe: Liming Zhao

Nov 11th, 2021

Markov Chain with continuous state space

We consider Markov chains in the continuous state space, that is, S = uncountable set.

Transition probability from each possible state x to each possible set of states A:

$$P(x_0, A) := P(X^{(n+1)} \in A | X^{(n)} = x_0)$$
$$\text{CDF} : F(v|x_0) := P(X^{(n+1)} \leq v | X^{(n)} = x_0)$$
$$\text{Transition density function} : p(v|x_0) := \frac{\partial F(v|x_0)}{\partial v} = P(X^{(n+1)} = v | X^{(n)} = x_0)$$

Transition density

Define one-step transition density:

$f^{[1]}(x_n, x_{n+1}) := p(x_{n+1}|x_n) = P(X^{(n+1)} = x_{n+1}|X^{(n)} = x_n)$

Two-step transition density:

$f^{[2]}(x_n, x_{n+2}) := P(X^{(n+2)} = x_{n+2}|X^{(n)} = x_n)$

$$\begin{aligned}
f^{[2]}(x_n, x_{n+2}) &= \int P(X^{(n+2)} = x_{n+2}|X^{(n+1)} = x_{n+1}, X^{(n)} = x_n) \cdot P(X^{(n+1)} = x_{n+1}|X^n = x_n) dx_{n+1} \\
&= \int P(X^{(n+2)} = x_{n+2}|X^{(n+1)} = x_{n+1}) \cdot P(X^{(n+1)} = x_{n+1}|X^{(n)} = x_n) dx_{n+1} \\
&= f^{[1]}(x_{n+1}, x_{n+2}) \cdot f^{[1]}(x_n, x_{n+1}) dx_{n+1}
\end{aligned}$$

Three-step transition density:

$f^{[3]}(x_n, x_{n+3}) := P(X^{(n+3)} = x_{n+3}|X^{(n)} = x_n)$

$$\begin{aligned}
f^{[3]}(x_n, x_{n+3}) &= \int P(X^{(n+3)} = x_{n+3}|X^{(n+2)} = x_{n+2}) \cdot P(X^{(n+2)} = x_{n+2}|X^{(n)} = x_n) dx_{n+2} \\
&= \int f^{[1]}(x_{n+2}, x_{n+3}) \cdot f^{[2]}(x_n, x_{n+2}) dx_{n+2} \\
&= \int \int f^{[1]}(x_{n+2}, x_{n+3}) \cdot f^{[1]}(x_{n+1}, x_{n+2}) \cdot f^{[1]}(x_n, x_{n+1}) dx_{n+1} dx_{n+2}
\end{aligned}$$

Arguing as above, we can obtain the m-step transition density:

$f^{[m]}(x_n, x_{n+m}) = \int \cdots \int \prod_{k=n+1}^{n+m} f^{[1]}(x_{k-1}, x_k) dx_{n+1} \cdots dx_{n+m-1}$

The corrresponding m-step transition probability can be written as $P(X^{(n+m)} \in A | X^{(n)} = x_n) = \int_A f^{[m]}(x_n, x_{n+m}) dx_{n+m}$

Some important properties

If the M.C. possesses a limiting transition density independent of the initial state, that is

$$\lim_{n\to\infty} f^{[n]}(x, v) = g(v)$$

then $g(v)$ is called "steady-state density" (long-term probability density) of M.C. and it is a solution to steady-state equation:

$$g(v) = \int_{-\infty}^{+\infty} g(w) f(w, v) dw \qquad (*)$$

$$(\text{discrete space}: \pi_j = \sum_{i=1}^{k} \pi_i p_{i,j} \qquad \forall j = 1, 2, \cdots, k)$$

where $g(w)$ is the start distribution and $f(w, v)$ is the transition density function. Note that an equivalent expression to $(*)$ is

$$\int_A g(v) dv = \int_{-\infty}^{+\infty} g(w) P(w, A) dw, \text{ for all set } A$$

Let $g(x)$ be the steady-state density and $f(x, v)$ be the density function of one-step transition, "**detailed balance condition**" is given by

$$g(x) f(x, v) = g(v) f(x, v) \qquad \forall x, v \qquad (**)$$

As in the discrete case, it can be shown that any Markov chain satisfying the "detailed balance condition" $(**)$ will have g as the steady state density, i.e. $(*)$ holds.

Proof:

Ergodic Theorem

If $(X^{(1)}, X^{(2)}, \cdots)$ is an ergodic M.C. whose steady-state density is given by g, then for $n \to \infty$,

$$P(X^{(n)} \in A) \to P(X^{(\infty)} \in A) = \int_A g(x) dx.$$

In addition,

$$\frac{1}{T} \sum_{n=1}^{T} h(X^{(n)}) \to E[h(X^{(\infty)})] = \int h(x) g(x) dx, \text{ for } T \to \infty.$$