

# Classification: basics

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# OVERVIEW

Binary Classification

Multi-class Classification

LDA and QDA

Logistic Regression: binary case

Logistic Regression: multi-class case, sparsity

Others: nonparametric classifier, robust loss, perceptrons

# Binary Classification

# Sensitivity and Specificity

Imagine a scenario where people are tested for a disease:

- ▶ The test outcome: positive (sick) or negative (healthy)
- ▶ The actual status: positive (sick) or negative (healthy)

There are four possible scenarios:

- ▶ True positive (TP): sick people correctly identified as sick
- ▶ False positive (FP) : healthy people incorrectly identified as sick
- ▶ True negative (TN): healthy people correctly identified as healthy
- ▶ False negative (FN): sick people incorrectly identified as healthy

True outcome	Test Outcome		Total
	Positive	Negative	
Positive	True Pos. (TP)	False Neg. (FN)	$P$
Negative	False Pos. (FP)	True Neg. (TN)	$N$
	$P^*$	$N^*$	

True outcome	Test Outcome		Total
	Positive	Negative	
Positive	True Pos. (TP)	False Neg. (FN)	$P$
Negative	False Pos. (FP)	True Neg. (TN)	$N$
	$P^*$	$N^*$	

Accuracy (1-Error):  $(TP + TN)/(TP + TN + FP + FN)$ .

Sensitivity (true positive rate/power/recall): the proportions of positives that are correctly identified

$$\text{Sensitivity} = TP/P = TP/(TP + FN)$$

Specificity (true negative rate): the proportions of negative that are correctly identified

$$\text{Specificity} = TN/N = TN/(FP + TN)$$

- ▶ False positive rate (Type I error):  $= 1 - \text{Specificity}$ .
- ▶ False negative rate (Type II error):  $= 1 - \text{Sensitivity}$
- ▶ False discovery rate (FDR/precision): the proportion of predicted positives that are in fact false positives  $FDR = FP/P^*$

## example 1

	Predicted	
True	email	spam
email	573	40
spam	53	334

spam = presence of disease, email = absence of disease

$$\begin{aligned}\text{specificity} &= 100 \times \frac{573}{573+40} = 93.4\% \\ \text{sensitivity} &= 100 \times \frac{334}{334+53} = 86.3\%\end{aligned}$$

## example 2

Threshold A is chosen to balance sensitivity and specificity without leaning too heavily towards either.

True Positive (TP)	False Negative (FN)
40	5
False Positive (FP)	True Negative (TN)
10	45

With Threshold A, we have: - Sensitivity: 88.9% - Specificity: 81.8%

Threshold B (it has a lower criterion for a positive test result)–adjusted to make the test more sensitive to detecting the disease.

True Positive (TP)	False Negative (FN)
45	0
False Positive (FP)	True Negative (TN)
20	35

With Threshold B, we have: - Sensitivity: 100% - Specificity: 63.6%

# Binary classification: problem

- ▶ input vector  $X \in \mathcal{X} \subset \mathbb{R}^p$
- ▶ output  $Y \in \{0, 1\}$
- ▶ “hard classifier”  $h : \mathcal{X} \longrightarrow \{0, 1\}$

The rule is characterized as

$$h(X) = 1(b(X) > 0)$$

where  $b$  is the boundary function (or discriminant function) that gives the decision boundary  $\{x : b(x) = 0\}$ .

- ▶ If  $b(X)$  is a linear in  $X$ , then the classifier has a linear boundary (in  $X$ -space).
  - ▶ With transformed  $X$  included, the classifier can have a nonlinear boundary (in  $X$ -space).



# Examples of linear boundary

Linear logit model:

Assume that the **logit function** is linear in  $x$ , i.e.,

$$b(x) = \log \frac{\Pr(Y = 1 \mid X = x)}{\Pr(Y = 0 \mid X = x)} = \beta_0 + \beta_1^\top x$$

Thus the classification boundary is given by  $\{x : \beta_0 + \beta_1^\top x = 0\}$

Examples: LDA, Logistic regression (see later)

The classification error rate, of  $h$  is defined as

$$R(h) = E_{X,Y}(1(Y \neq h(X))) = P(Y \neq h(X))$$

The rule  $h$  that minimizes  $R(h)$  is

$$h^*(x) = \begin{cases} 1 & \text{if } m(x) > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

where  $m(x) = P(Y = 1 \mid X = x) = E(Y \mid X = x)$ .

- ▶ This optimal rule is called the **Bayes rule (classifier)** (under equal costs).
- ▶ The risk  $R(h^*)$  is called the **Bayes risk**.
- ▶ The set  $\{x : m(x) - \frac{1}{2} = 0\}$  is called the **Bayes decision boundary**.

Alternatively, the Bayes rule is  $h^*$ , is given by

$$h^*(x) = \begin{cases} 1 & \text{if } P(Y = 1 \mid X = x) > P(Y = 0 \mid X = x) \\ 0 & \text{if } P(Y = 1 \mid X = x) < P(Y = 0 \mid X = x) \end{cases}$$

The classification boundary of the Bayes rule is

$$\begin{aligned} & \{x : P(Y = 1 \mid X = x) = P(Y = 0 \mid X = x)\} \\ &= \{x : P(Y = 1 \mid X = x) - 0.5 = 0\} \end{aligned}$$

From Bayes' theorem

$$p(Y = 1 \mid X = x) = \frac{\pi_1 p_1(x)}{\pi_1 p_1(x) + (1 - \pi_1) p_0(x)}$$

- ▶  $\pi_1 = p(Y = 1), \pi_0 = p(Y = 0)$ : the marginal distribution of  $Y$  (prior class probabilities)
- ▶  $p_j(x) = p(x \mid Y = j)$ : the conditional density of  $X$  given that  $Y = j$ .

$$h^*(x) = \begin{cases} 1 & \text{if } \frac{p_1(x)}{p_0(x)} > \frac{\pi_0}{\pi_1} \\ 0 & \text{otherwise.} \end{cases}$$

This decision-making process balances two types of information:

- ▶ Likelihood ratio (evidence)  $\frac{p_1(x)}{p_0(x)}$ : compares how probable it is that the observed data  $x$  comes from class 1 as opposed to class 0.
- ▶ Prior ratio  $\frac{\pi_0}{\pi_1}$ : our initial belief about the relative frequency of class 0 to class 1 before seeing any data (say  $x$ ).

# Unequal Losses

For any decision function, there are two possible errors:

- ▶ misclassifying a sample in class 0 to 1 (false positive)
- ▶ misclassifying a sample in class 1 to 0 (false negative)

Each type of error is associated with a cost (the price to pay for the consequence):

- ▶  $L(1, 0)$  is the cost of misclassifying a sample in class 1 to 0
- ▶  $L(0, 1)$  is the cost of misclassifying a sample in class 0 to 1.

We assume  $L(j, j) = 0$  for  $j = 0, 1$ ; but it may not be  $L(0, 1) = L(1, 0)$ .

The loss becomes

$$L(Y, h(X)) = L(1, 0)1(Y = 1, h(X) = 0) + L(0, 1)1(Y = 0, h(X) = 1)$$

For fixed  $x$ , the Bayes rule is given as

$$h^*(x) = \begin{cases} 1 & \text{if } L(1,0)P(Y = 1 \mid X = x) > L(0,1)P(Y = 0 \mid X = x) \\ 0 & \text{if } L(1,0)P(Y = 1 \mid X = x) < L(0,1)P(Y = 0 \mid X = x) \end{cases}$$

Equivalently,

$$h^*(x) = \begin{cases} 1 & \text{if } \frac{P(Y=1|X=x)}{P(Y=0|X=x)} > \frac{L(0,1)}{L(1,0)} \\ 0 & \text{if } \frac{P(Y=1|X=x)}{P(Y=0|X=x)} < \frac{L(0,1)}{L(1,0)} \end{cases}$$

the Bayes rule

$$h^*(x) = 1 \left\{ x : P(Y = 1 \mid X = x) > \frac{L(0,1)}{L(0,1) + L(1,0)} \right\}.$$

In light of the Bayes' theorem,

- ▶  $\pi_1 = p(Y = 1), \pi_0 = p(Y = 0)$ : the marginal distribution of  $Y$  (prior class probabilities)
- ▶  $p_j(x) = p(x \mid Y = j)$ : the conditional density of  $X$  given that  $Y = j$ .

$$h^*(x) = \begin{cases} 1 & \text{if } \frac{p_1(x)}{p_0(x)} > \frac{\pi_0 L(0,1)}{\pi_1 L(1,0)} \\ 0 & \text{if } \frac{p_1(x)}{p_0(x)} < \frac{\pi_0 L(0,1)}{\pi_1 L(1,0)} \end{cases}$$

By changing the weights for  $L(0, 1)$  and  $L(1, 0)$ , we can effectively change the classification threshold.

- ▶  $L(0, 1)$  = the cost of predicting a “non-disease” example to “disease”
- ▶  $L(1, 0)$  = the cost of predicting a “disease” example to “non-disease”

$$h^*(x) = 1 \left\{ x : P(Y = 1 \mid X = x) > \frac{L(0, 1)}{L(0, 1) + L(1, 0)} \right\}.$$

How to increase the sensitivity and decrease the specificity of the rule?

- ▶ Increase  $L(1, 0)$  and decrease  $L(0, 1)$ .

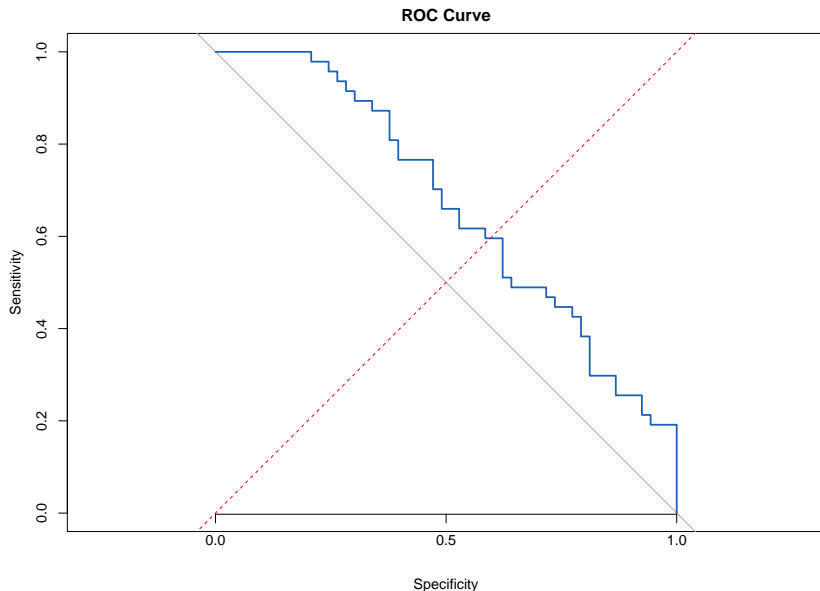
How to increase the specificity and decrease the sensitivity of the rule?

- ▶ Increase  $L(0, 1)$  and decrease  $L(1, 0)$ .



# Receiver Operating Characteristic (ROC) curve

A ROC curve is a plot of sensitivity v.s. specificity:



- ▶ An ideal ROC curve will hug the top right corner.
- ▶ An alternative ROC curve will be a curve plotting the sensitivity (true positive rate or  $1 - \text{Type II error}$ ) against the false positive rate (Type I error).
- ▶ The **area under curve (AUC)** is a commonly used quantitative measure of overall predictive performance.
  - ▶ A value of 0.5 means the predictions were no better than random guessing.

## Bias-variance decomposition for binary response (0-1 loss)

The bias-variance tradeoff behaves differently for 0 – 1/classification loss than it does for squared error loss.

But the prediction error (classification error/0 – 1 loss) is no longer the sum of squared bias and variance.

What matters is that  $E\hat{m}(x_0)$  and  $m(x_0)$  is on the same side of  $1/2$  (thus correct classification).

For  $(X, Y) \in \mathbb{R}^p \times \{0, 1\}$ , consider the regression function,

$$m(x) = \mathbb{E}(Y \mid X = x) = P(Y = 1 \mid X = x)$$

The Bayes classifier is given by

$$h^*(x) = \begin{cases} 0 & \text{if } m(x) \leq 1/2 \\ 1 & \text{if } m(x) > 1/2 \end{cases}$$

The plug-in classifier is given by

$$\hat{h}(x) = \begin{cases} 0 & \text{if } \hat{m}(x) \leq 1/2 \\ 1 & \text{if } \hat{m}(x) > 1/2 \end{cases}$$

$$\begin{aligned} \text{Err}(x_0) &= P(Y_0 \neq \hat{h}(X_0) \mid X_0 = x_0) \\ &= \text{Err}_B(x_0) + |2m(x_0) - 1| P(\hat{h}(X_0) \neq h^*(X_0) \mid X_0 = x_0) \end{aligned}$$

where  $\text{Err}_B(x_0) = P(Y_0 \neq h^*(X_0) \mid X_0 = x_0)$ , the irreducible Bayes error at  $x_0$ .

Using the approximation  $\hat{m}(x_0) \sim N(E\hat{m}(x_0), \text{Var}(\hat{m}(x_0)))$ , it can be shown that

$$\Pr\left(\hat{h}(X^*) \neq h^*(X^*) \mid X^* = x_0\right) \approx \Phi\left(\frac{\text{sign}\left(\frac{1}{2} - m(x_0)\right)(E\hat{m}(x_0) - \frac{1}{2})}{\sqrt{\text{Var}(\hat{m}(x_0))}}\right)$$

The term  $\text{sign}\left(\frac{1}{2} - m(x_0)\right)(E\hat{m}(x_0) - \frac{1}{2})$  is a kind of **boundary-bias term**, as it depends on the true  $m(x_0)$  only through which side of the boundary  $1/2$  that it lies. The bias and variance

combine in a multiplicative rather than additive fashion.

# Multi-class Classification

# Multi-class Classification

- ▶ Class label  $Y \in \{1, \dots, K\}$ ,  $K \geq 3$ .
- ▶ The classifier  $h : \mathbb{R}^d \longrightarrow \{1, \dots, K\}$ .

The loss function  $L(Y, h(X)) = \sum_{k=1}^L \sum_{l=1}^K C(l, k) 1(Y = l, h(X) = k)$  where  $C(l, k) = \text{cost of classifying a sample in class } l \text{ to class } k$ .

The classification risk, or error rate, of  $h$  is defined as

$$R(h) = E_{X,Y}(L(Y, h(X)))$$

Using the 0-1 loss,  $C(k, k) = 0$  for any  $k = 1, \dots, K$ , but equal to 1 otherwise, the rule  $h$  that minimizes  $R(h)$  is

$$h^*(x) = \arg \max_{k=1, \dots, K} P(Y = k \mid x)$$

i.e., assign  $x$  to the most probable class using  $P(Y \mid x)$ .

We generally need to estimate multiple **discriminant functions**

$\delta_k(x), k = 1, \dots, K$

- ▶ Each  $\delta_k(x)$  is associated with class  $k$ .
- ▶  $\delta_k(x)$  represents the evidence strength of a sample  $(x, y)$  belonging to class  $k$ .

The decision rule constructed using  $\delta_k$  's is

$$\hat{h}(x) = k^*, \quad \text{where} \quad k^* = \arg \max_{k=1, \dots, K} \delta_k(x)$$

The decision boundary of the classification rule  $\hat{h}$  between class  $k$  and class  $l$  is defined as

$$\{x : \delta_k(x) = \delta_l(x)\}$$

Note:  $\delta_k(x)$  is related but need not be exact  $P(Y = k | x)$ .



# LDA and QDA

# Gaussian discriminant analysis: Binary classification

If  $X \mid Y = 0 \sim N(\mu_0, \Sigma_0)$  and  $X \mid Y = 1 \sim N(\mu_1, \Sigma_1)$ ,

Using the Bayes' Theorem,

$$h^*(x) = \begin{cases} 1 & \text{if } r_1^2 < r_0^2 + 2 \log \left( \frac{\pi_1}{1-\pi_1} \right) + \log \left( \frac{|\Sigma_0|}{|\Sigma_1|} \right) \\ 0 & \text{otherwise} \end{cases}$$

- $r_i = \sqrt{(x - \mu_i)^\top \Sigma_i^{-1} (x - \mu_i)}$  for  $i = 0, 1$  is the **Mahalanobis distance** between  $x$  and  $\mu_i$ .

Note: LDA is a special case where  $\Sigma_1 = \Sigma_0$ .

# Quadratic discriminant analysis (QDA)

$$\log \frac{\Pr(Y = 1 \mid X = x)}{\Pr(Y = 0 \mid X = x)} = \log \frac{\pi_1 \phi(x; \mu_1, \Sigma_1)}{\pi_0 \phi(x; \mu_0, \Sigma_0)} = \delta_1(x) - \delta_0(x).$$

The Bayes rule is

$$h^*(x) = \operatorname{argmax}_{k \in \{0,1\}} \delta_k(x)$$

where

$$\delta_k(x) = -\frac{1}{2} \log |\Sigma_k| - \frac{1}{2} (x - \mu_k)^\top \Sigma_k^{-1} (x - \mu_k) + \log \pi_k$$

This is called the **Gaussian discriminant function**

- ▶ The decision boundary:  $\{x \in \mathcal{X} : \delta_1(x) = \delta_0(x)\}$ 
  - ▶ **quadratic discriminant analysis (QDA)**: boundary is quadratic

To estimate  $\pi_0, \pi_1, \mu_0, \mu_1, \Sigma_0, \Sigma_1$ :

$$\hat{\pi}_0 = \frac{1}{n} \sum_{i=1}^n (1 - Y_i), \quad \hat{\pi}_1 = \frac{1}{n} \sum_{i=1}^n Y_i$$

$$\hat{\mu}_0 = \frac{1}{n_0} \sum_{i:Y_i=0} X_i, \quad \hat{\mu}_1 = \frac{1}{n_1} \sum_{i:Y_i=1} X_i$$

$$\hat{\Sigma}_0 = \frac{1}{n_0 - 1} \sum_{i:Y_i=0} (X_i - \hat{\mu}_0) (X_i - \hat{\mu}_0)^\top$$

$$\hat{\Sigma}_1 = \frac{1}{n_1 - 1} \sum_{i:Y_i=1} (X_i - \hat{\mu}_1) (X_i - \hat{\mu}_1)^\top$$

# Linear discriminant analysis (LDA)

**LDA** assumes both classes are from Gaussian and they have the same covariance matrix

$$\Sigma_k = \Sigma, \quad k = 0, 1$$

Note that

$$\log \Pr(Y = k \mid X = x) = -\frac{1}{2} (x - \boldsymbol{\mu}_k)^\top \Sigma^{-1} (x - \boldsymbol{\mu}_k) + \log \pi_k + \text{const.}$$

If prior probabilities are same, the LDA classifies  $x$  to the class with centroid closest to  $x$ , using the squared Mahalanobis distance, based on the common covariance matrix.

Alternatively,

$$h^*(x) = \begin{cases} 1 & \text{if } \delta_1(x) > \delta_0(x) \\ 0 & \text{otherwise} \end{cases}$$

where the Gaussian discriminant function can be simplified

$$\delta_k(x) = x^\top \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^\top \Sigma^{-1} \mu_k + \log \pi_k.$$

- ▶ The decision boundary:  $\{x \in \mathcal{X} : \delta_1(x) = \delta_0(x)\}$ 
  - ▶ **linear discriminant analysis** (LDA): boundary is linear

Pooled estimate of the  $\Sigma$ :

$$\hat{\Sigma} = \frac{(n_0 - 1) \hat{\Sigma}_0 + (n_1 - 1) \hat{\Sigma}_1}{n_0 + n_1 - 2}$$

# Multi-class classification (trivial extension)

QDA assume that  $X | Y = k \sim N(\mu_k, \Sigma_k)$ .

$$h^*(x) = \operatorname{argmax}_k \delta_k(x)$$

where

$$\delta_k(x) = -\frac{1}{2} \log |\Sigma_k| - \frac{1}{2} (x - \mu_k)^\top \Sigma_k^{-1} (x - \mu_k) + \log \pi_k.$$

If all Gaussians assumed to have equal variance  $\Sigma$ ,

$$\delta_k(x) = x^\top \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^\top \Sigma^{-1} \mu_k + \log \pi_k.$$

The corresponding estimates are given by

$$\begin{aligned} \hat{\pi}_k &= \frac{1}{n} \sum_{i=1}^n 1(y_i = k), \quad \hat{\mu}_k = \frac{1}{n_k} \sum_{i: Y_i = k} X_i \\ \hat{\Sigma}_k &= \frac{1}{n_k - 1} \sum_{i: Y_i = k} (X_i - \hat{\mu}_k)(X_i - \hat{\mu}_k)^\top \\ \hat{\Sigma} &= \frac{\sum_{k=0}^{K-1} (n_k - 1) \hat{\Sigma}_k}{n - K}. \end{aligned}$$

## Logistic Regression: binary case



# Binary case

The logistic regression assumes that

$$p_1(x; \beta_0, \beta_1) := P(Y = 1 \mid X = x) = \frac{\exp(\beta_0 + x^\top \beta_1)}{1 + \exp(\beta_0 + x^\top \beta_1)}$$

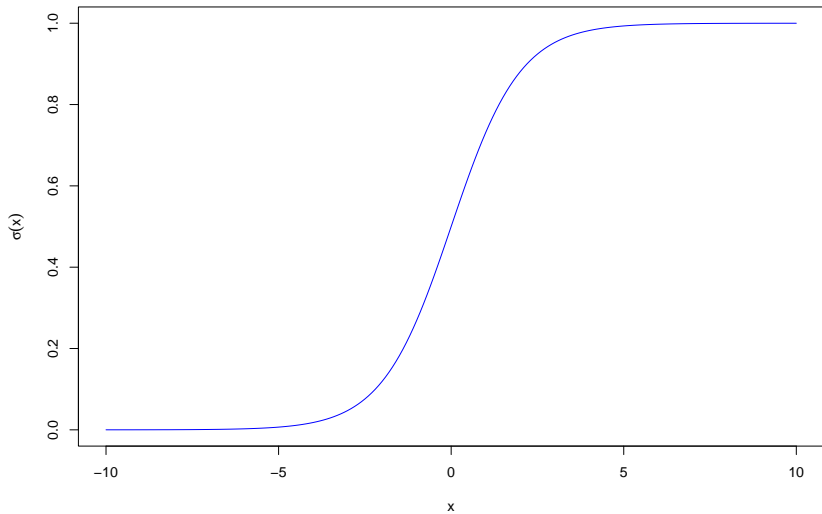
The model can be written as

$$\text{logit}(x) := \text{logit}(\Pr(Y = 1 \mid X = x)) = \log \frac{\Pr(Y = 1 \mid X = x)}{\Pr(Y = 0 \mid X = x)} = \beta_0 + \beta_1^\top x$$

- ▶ **logit function:**  $\text{logit}(a) = \log(a/(1-a)) : (0, 1) \mapsto \mathbb{R}$ 
  - ▶  $\beta_0 + \beta_1^\top x$ : **logits, net input or pre-activation value**
- ▶ The inverse of logit function (**logistic function** or **sigmoid function**):

$$\sigma(a) = \exp(a)/(1 + \exp(a)) : \mathbb{R} \mapsto (0, 1)$$

### Sigmoid Function



# MLE for logistic models

Notations: assuming  $x_i$  contains the constant term 1 (thus a  $p + 1$  vector).

$$\boldsymbol{\beta} := \{\beta_0, \boldsymbol{\beta}_1^\top\}^\top$$

$$\mathbf{y} := [y_1, \dots, y_n]^\top$$

$$\mathbf{p} := \mathbf{p}(\boldsymbol{\beta}) = [p(x_1; \boldsymbol{\beta}), \dots, p(x_n; \boldsymbol{\beta})]^\top$$

$$\mathbf{W} := \mathbf{W}(\boldsymbol{\beta}) = \text{diag}\{p(x_i; \boldsymbol{\beta})(1 - p(x_i; \boldsymbol{\beta}))\} : n \times n$$

The log (conditional) likelihood function is

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^n \{y_i \log p(x_i; \boldsymbol{\beta}) + (1 - y_i) \log [1 - p(x_i, \boldsymbol{\beta})]\}$$

The loss function, as a negative loglikelihood function, is called **binomial deviance (loss)**:

$$L(Y, p_Y(X)) = -\{1(Y = 0) \log(\Pr(Y = 0 \mid X; \boldsymbol{\theta})) + 1(Y = 1) \log(\Pr(Y = 1 \mid X; \boldsymbol{\theta}))\}$$

$$\begin{aligned}
-\ell(\boldsymbol{\beta}) &= -\sum_{i=1}^n \{y_i \log p(x_i; \boldsymbol{\beta}) + (1 - y_i) \log [1 - p(x_i, \boldsymbol{\beta})]\} \\
&= -\sum_{i=1}^n \left\{ y_i \boldsymbol{\beta}^\top x_i - \log \left[ 1 + \exp \left( \boldsymbol{\beta}^\top x_i \right) \right] \right\}.
\end{aligned}$$

The score and Hessian are given by

$$\begin{aligned}
\frac{\partial(-\ell(\boldsymbol{\beta}))}{\partial \boldsymbol{\beta}} &= -\sum_{i=1}^n x_i [y_i - p(x_i; \boldsymbol{\beta})] = -\mathbf{X}^\top (\mathbf{y} - \mathbf{p}) \\
\frac{\partial^2(-\ell(\boldsymbol{\beta}))}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^\top} &= \sum_{i=1}^n x_i x_i^\top p(x_i; \boldsymbol{\beta}) [1 - p(x_i; \boldsymbol{\beta})] = \mathbf{X}^\top \mathbf{W} \mathbf{X} \quad (p.s.d)
\end{aligned}$$

Gradient descent step: In the k-th sep,

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} + \alpha \times \mathbf{X}^\top (\mathbf{y} - \mathbf{p}^{(k)})$$

Newton-Raphson step: In the k-th sep,

$$\begin{aligned}\boldsymbol{\beta}^{(k+1)} &= \boldsymbol{\beta}^{(k)} + \left( \mathbf{X}^\top \mathbf{W}^{(k)} \mathbf{X} \right)^{-1} \mathbf{X}^\top (\mathbf{y} - \mathbf{p}^{(k)}) \\ &= \left( \mathbf{X}^\top \mathbf{W}^{(k)} \mathbf{X} \right)^{-1} \mathbf{X}^\top \mathbf{W}^{(k)} \left( \mathbf{X} \boldsymbol{\beta}^{(k)} + \mathbf{W}^{(k)-1} (\mathbf{y} - \mathbf{p}^{(k)}) \right) \\ &= \left( \mathbf{X}^\top \mathbf{W}^{(k)} \mathbf{X} \right)^{-1} \mathbf{X}^\top \mathbf{W}^{(k)} \mathbf{z}^{(k)}\end{aligned}$$

where we defined the adjusted response

$$\mathbf{z}^{(k)} = \mathbf{X} \boldsymbol{\beta}^{(k)} + \mathbf{W}^{(k)-1} (\mathbf{y} - \mathbf{p}^{(k)})$$

The Newton-Raphson's approach is equivalent to **Iteratively Reweighted Least Squares Algorithm**:

$$\boldsymbol{\beta}^{(k+1)} = \arg \min_{\boldsymbol{\beta}} (\mathbf{z}^{(k)} - \mathbf{X} \boldsymbol{\beta})^\top \mathbf{W}^{(k)} (\mathbf{z}^{(k)} - \mathbf{X} \boldsymbol{\beta}).$$

## Interpretation of $\beta_j$

$$e^{\beta_j} = \underbrace{\frac{P(Y = 1 | \dots, X_j = x + 1, \dots) / P(Y = 0 | \dots, X_j = x + 1, \dots)}{P(Y = 1 | \dots, X_j = x, \dots) / P(Y = 0 | \dots, X_j = x, \dots)}}_{\text{odds ratio}}$$

When an increase of  $X_j$  by one unit from  $x$  to  $x + 1$ , while keeping all other predictors fixed, it multiplies the odds by  $e^{\beta_j}$  (relative change from the odds when  $X_j = x$ );

e.g.

- ▶ If  $X_j = 0$  or 1, then for the group with  $X_j = 1$ , the odds of the event are  $e^{\beta_j}$  times that of the group with  $X_j = 0$ , with other values of  $X_{-j}$  fixed.
  - ▶ When  $\beta_j > 0$ , the group with  $X_j = 1$  has  $100(e^{\beta_j} - 1)\%$  more odds than the group with  $X_j = 0$ , with other values of  $X_{-j}$  fixed.
  - ▶ When  $\beta_j < 0$ , then it is a decrease in the odds by  $100(1 - e^{\beta_j})\%$ .

For intercept,  $e^{\beta_0} \div (1 + e^{\beta_0})$  is the probability of the event for the base group when all  $X$ 's = 0.

## Inferences: logistic regression

Assuming correct model specification, by central limit theorem, the MLE estimator

$$\hat{\beta} \rightarrow N\left(\beta^*, (\mathbf{X}'\mathbf{W}(\beta^*)\mathbf{X})^{-1}\right).$$

Here,  $\beta^*$  is the truth. The estimator for the variance of  $\hat{\beta}$  is given by

$$\hat{\text{var}}(\hat{\beta}) = \left(\mathbf{X}'\mathbf{W}(\hat{\beta})\mathbf{X}\right)^{-1}.$$

So as  $n \rightarrow \infty$ ,

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim N(0, 1),$$

where

$$se(\hat{\beta}_j) = \left[ \left( \mathbf{X}'\mathbf{W}(\hat{\beta})\mathbf{X} \right)^{-1} \right]_{j,j}^{1/2}$$

## Alternative formulation: logistic regression

Suppose that

$$Y_i = 1\{\beta_0 + X_i^\top \beta_1 + \varepsilon_i\}$$

where  $\varepsilon_i$  is independent of  $X_i$ , following a logistic distribution (mean 0 and standard deviation  $\pi/\sqrt{3}$ ), aka, the c.d.f. of  $\varepsilon_i$  given by

$$F(\epsilon) = \frac{\exp(\epsilon)}{1 + \exp(\epsilon)}.$$

Then

$$P(Y_i = 1|X_i = x) = \frac{\exp(\beta_0 + x^\top \beta_1)}{1 + \exp(\beta_0 + x^\top \beta_1)}.$$

If important predictors are omitted from the model (model misspecified), the usual interpretation linking the coefficients to (true) log-odds will break down.



## Logistic Regression: multi-class case, sparsity

# Logistic Regression: multi-class case, sparsity

Suppose there are  $K$  groups. Let the  $K$ -th group be the base group. One may model  $Pr(Y = k|x; \beta_0, \beta)$  as

$$Pr(Y = k|x; \beta_0, \beta) = \frac{\exp(x^\top \beta_k + \beta_{k0})}{\sum_{k'=1}^K \exp(x^\top \beta_{k'} + \beta_{k'0})}$$

- ▶  $\beta_0 := (\beta_{10}, \dots, \beta_{K0})$  the vector of  $K$  intercepts.
- ▶  $\beta := (\beta_1, \dots, \beta_K)^\top$ , a  $K$  by  $p$  matrix.
- ▶  $\eta := \beta x + \beta_0$  the vector of  $K$  logits

To ensure identification of the parameters:

- ▶ treat  $K$  as the base group
- ▶ set  $\beta_{K0} = 0, \beta_K = 0$  to avoid overparametrization.

The multi-class logistic regression (or multinomial logistic regression) models  $K - 1$  logits:

- ▶ treat  $K$  as the base group
- ▶ set  $\beta_{K0} = 0, \beta_K = 0$  to avoid overparametrization.

$$\log \frac{\Pr(Y = 1 \mid X = x)}{\Pr(Y = K \mid X = x)} = \beta_{10} + \beta_1^\top x$$

$$\log \frac{\Pr(Y = 2 \mid X = x)}{\Pr(Y = K \mid X = x)} = \beta_{20} + \beta_2^\top x$$

$$\log \frac{\Pr(Y = K - 1 \mid X = x)}{\Pr(Y = K \mid X = x)} = \beta_{(K-1)0} + \beta_{K-1}^\top x$$

**interpretation of coefficients:**

$$e^{\beta_{kj}} = \frac{P(Y = k \mid \dots, X_j = x + 1, \dots) / P(Y = K \mid \dots, X_j = x + 1, \dots)}{P(Y = k \mid \dots, X_j = x, \dots) / P(Y = K \mid \dots, X_j = x, \dots)}$$

Equivalently,

$$p_k(x) \equiv \Pr(Y = k \mid x) = \frac{\exp(\beta_{k0} + \beta_k^\top x)}{1 + \sum_{l=1}^{K-1} \exp(\beta_{l0} + \beta_l^\top x)}, \quad k = 1, \dots, K-1$$

$$p_K(x) \equiv \Pr(Y = K \mid x) = \frac{1}{1 + \sum_{l=1}^{K-1} \exp(\beta_{l0} + \beta_l^\top x)}$$

Clearly  $\sum_{k=1}^K p_k(x) = 1$ . The parameter vector

$$\boldsymbol{\theta} = \left\{ \beta_{10}, \beta_1^\top, \dots, \beta_{(K-1)0}, \beta_{K-1}^\top \right\}^\top$$

Let  $p_{k,i} := \Pr(Y_i = k | X = x_i, \boldsymbol{\theta})$  and  $p_{y_i}(x_i; \boldsymbol{\theta}) = \Pr(Y_i = y_i | X = x_i, \boldsymbol{\theta})$ .

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= \sum_{i=1}^n \log p_{y_i}(x_i; \boldsymbol{\theta}) = \log \left( \prod_{i=1}^n \prod_{k=1}^K p_{k,i}^{1(y_i=k)} \right) \\ &= \sum_{i=1}^n \sum_{k=1}^K 1(y_i = k) \log(p_{k,i})\end{aligned}$$

Since  $\beta_{K0} := 0$  and  $\beta_K := 0$ , the log-likelihood:

$$\sum_{i=1}^n \left\{ \beta_{y_i 0} + \beta_{y_i}^\top x_i - \log \left[ 1 + \sum_{k=1}^{K-1} \exp(\beta_{k0} + \beta_k^\top x_i) \right] \right\}$$

The loss function, as a negative loglikelihood function, is called **multinomial deviance** (loss):

$$L(Y, p_Y(X)) = -\log p_Y(X; \theta) = -\sum_{k=1}^K 1(Y = k) \log(\Pr(Y = k | X; \theta)).$$

## Side note

If prediction is the only concern, we may fit a overparametrized model. This is the approach taken in the perceptron-based model.

The **S** is the **softmax function**  $\mathbb{R}^K \mapsto (0, 1)^K$ , defined as

$$\mathbf{S}(\boldsymbol{\eta})_k = \frac{e^{\eta_k}}{\sum_{k'}^K e^{\eta_{k'}}}, \quad k = 1, \dots, K; \quad \boldsymbol{\eta} = (\eta_1, \dots, \eta_K)^\top$$

$$Pr(Y = k|x; \beta_0, \beta) := Pr(Y = k|\boldsymbol{\eta}) = \mathbf{S}(\boldsymbol{\eta})_k$$

**interpretation of coefficients:**

$$e^{\beta_{kj} - \beta_{k'j}} = \frac{P(Y = k \mid \dots, X_j = x + 1, \dots) / P(Y = k' \mid \dots, X_j = x + 1, \dots)}{P(Y = k \mid \dots, X_j = x, \dots) / P(Y = k' \mid \dots, X_j = x, \dots)}$$

# Compare logistic regression with LDA

Logistic regression:

- ▶ Maximizing the conditional likelihood, the multinomial likelihood with probabilities  $\Pr(Y = k \mid \mathbf{X})$
- ▶ The marginal density  $\Pr(X)$  is ignored (fully nonparametric)
  - ▶ **discriminative approach**: only modelling  $\Pr(Y = k \mid x)$

LDA:

- ▶ Maximizing the full log-likelihood based on the joint density

$$\Pr(X, Y = k) = \phi(X; \boldsymbol{\mu}_k, \Sigma) \pi_k$$

- ▶ Marginal density does play a role  $\Pr(\mathbf{X}) = \sum_k \pi_k \phi(X; \boldsymbol{\mu}_k, \Sigma)$ 
  - ▶ **generative approach**: modelling  $\Pr(x \mid Y = k)$  (usually joint modelling of  $\Pr(X, Y = k)$ )

# Regularized logistic regression

The idea is to minimize the penalized negative likelihood function (binary response):

$$\min_{\beta_0, \beta_1} \sum_{i=1}^n \left( -y_i (\beta_0 + x_i^\top \beta_1) + \log \left( 1 + e^{\beta_0 + x_i^\top \beta_1} \right) \right) + \lambda J(\beta_1)$$

The update is equivalent to solving the weighted LS till convergence (Iteratively Reweighted Least Squares Algorithm):

$$(\beta_0^{(k+1)}, \beta_1^{(k+1)}) = \arg \min_{\beta} \left\{ (\mathbf{z}^{(k)} - \mathbf{X}\beta)^\top \mathbf{W}^{(k)} (\mathbf{z}^{(k)} - \mathbf{X}\beta) + \lambda J(\beta_1) \right\}.$$

For the Lasso penalty, it can be solved using coordinate descent.



# Sparse logistic regression

**Algorithm** (Coordinate descent for sparse logistic regression):

Let  $\hat{\beta}^{(0)} = (\hat{\beta}_0^{(0)}, \hat{\beta}_1^{(0)}, \dots, \hat{\beta}_p^{(0)})^T$

For  $k = 0, 1, 2, \dots$ ,

- ▶ Compute  $p_1(x_i; \hat{\beta}^{(k)}), z_i^{(k)}, w_{ii}^{(k)}, i = 1, \dots, n$ .
- ▶ Let  $\beta_0 = \sum_i [w_{ii}^{(k)}(z_i^{(k)} - \sum_{l=1}^p \hat{\beta}_l^{(k)} x_{il})] / \sum_i w_{ii}^{(k)}, \quad \beta_l = \hat{\beta}_l^{(k)}, l = 1, \dots, p$ .
  - ▶ for  $j = 1, \dots, p$  do
    - ▶ compute  $r_{ij} = z_i^{(k)} - \beta_0 - \sum_{l \neq j} \beta_l x_{il}, i = 1, \dots, n$
    - ▶ compute  $u_j^{(k)} = \sum_{i=1}^n w_{ii}^{(k)} r_{ij} x_{ij}$ ,
    - ▶ compute  $v_j^{(k)} = \sum_{i=1}^n w_{ii}^{(k)} x_{ij}^2$ ,
    - ▶ compute  $\beta_j = \text{soft}(u_j^{(k)} / v_j^{(k)}, \lambda / v_j^{(k)})$
  - ▶  $\hat{\beta}_0^{(k+1)} = \beta_0, \hat{\beta}_j^{(k+1)} = \beta_j, j = 1, \dots, p$

until convergence

Others: nonparametric classifier, robust loss,  
perceptrons

# KNN

For any given  $X = x_0$ , we find the  $K$  closest neighbors to  $X = x_0$  in the training data, and examine their corresponding  $Y$ :

$$P(Y = j \mid X = x_0) = \frac{1}{K} \sum_{i \in N_K(x_0)} 1(y_i = j)$$

Estimate the conditional probability for group  $j$  by the proportion out of the  $k$  neighbors that are in group  $j$ .

The smaller that  $K$  is the more flexible the method will be.

Note: more on nonparametric method (e.g., nonparametric logistic regression) in future lessons.

## Others: Alternative loss functions

For binary classification,

$$\min_{f \in \mathcal{F}_{0/1}} \mathbb{E}_{X,Y}(1(Y \neq f(X)))$$

where  $\mathcal{F}_{0/1}$  consists of function that maps to  $\{0, 1\}$ .

The ERM solution is

$$\hat{f}_n = \arg \min_{f \in \mathcal{F}_{0/1}} \frac{1}{n} \sum_{i=1}^n 1(y_i \neq f(x_i))$$

► the choice of  $\mathcal{F}_{0/1}$  **hard-classifier**:

► perceptron:  $\mathcal{F}_{0/1} = \{1(\beta_0 + \beta^\top x) : \beta_0, \beta\}$

Equivalent formulation, using  $y_i \in \{-1, 1\}$ , and  $\mathcal{F}_{\pm 1}$  consists of function that maps to  $\{-1, 1\}$ :

$$\hat{f}_n = \arg \min_{f \in \mathcal{F}_{\pm 1}} \frac{1}{n} \sum_{i=1}^n 1(-y_i f(x_i) > 0)$$

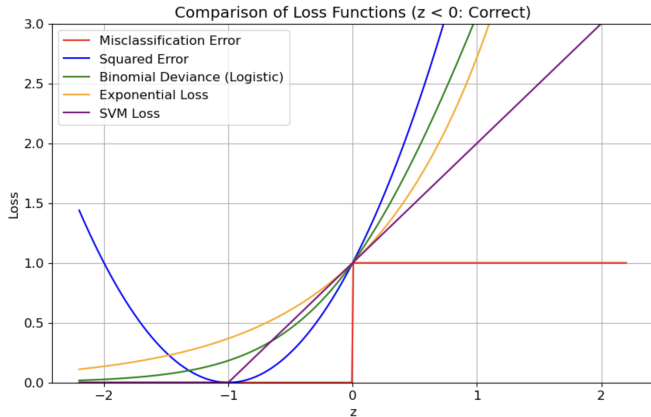
Using some smooth and convex surrogate function  $\psi(z)$  for  $1(z > 0)$  and relaxing the class of functions  $\mathcal{F}$ , solve

$$\hat{f}_n = \arg \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n L(y_i, f(x_i)) = \frac{1}{n} \sum_{i=1}^n \psi(-y_i f(x_i))$$

- ▶ Choice of  $L(y, f)$  :
  - ▶ Squared error:  $(y - f)^2 = (1 - yf)^2$
  - ▶ Binomial deviance (logistic):  $\log(1 + \exp(-2yf))$
  - ▶ Exponential loss:  $\exp(-yf)$
  - ▶ SVM loss:  $(1 - yf)_+$
- ▶ Choice of  $\mathcal{F}$ : **soft-classifier**  $f \in [-1, 1]$  or  $\mathbb{R}$ 
  - ▶ decision:  $\hat{Y}_i = \text{sign}(\hat{f}_n(X_i))$

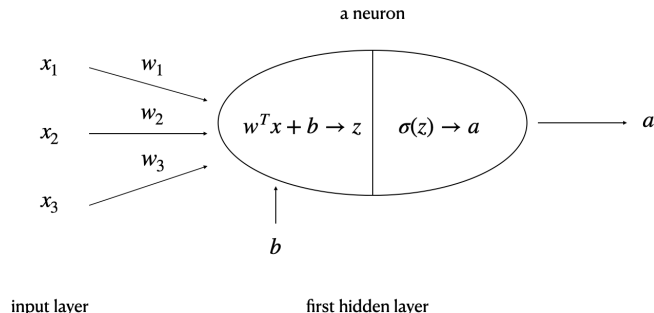
Let  $z = -yf > 0$  indicate misclassification.

- ▶ Misclassification error:  $1(z > 0)$
- ▶ Squared error:  $(1 + z)^2$
- ▶ Binomial deviance (scaled):  $\log(1 + \exp(2z))/\log(2)$
- ▶ Exponential loss:  $\exp(z)$
- ▶ SVM loss:  $(1 + z)_+$



More in future lessons...

# Perceptron

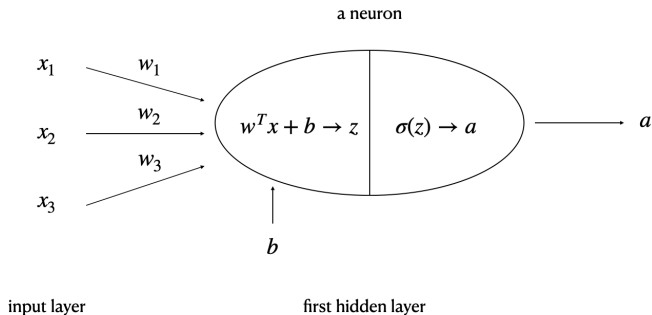


$\mathbf{x} = (x_1, x_2, x_3)$ : three input features

$\sigma(\cdot)$ : activation function– three simplest cases

- i. identity  $\sigma(v) = v$
- ii. indicator (Heaviside step)  $\sigma(v) = 1(v > 0)$
- iii. sigmoid  $\sigma(v) = 1/(1 + e^{-v})$

The output  $\hat{a}$  will be measured against the actual outcome value under some loss function.



Formally,

$$\sigma \left( \sum_{j=1}^p x_j w_j + b \right) = \sigma \left( \mathbf{x}^\top \mathbf{w} + b \right), \quad \beta := (\mathbf{w}, b)$$

Note: OLS in regression is a special case with  $\sigma(\cdot)$  being identity function and the squared error loss (see Adaline).



# Rosenblatt's Perceptron

Rosenblatt's Perceptron:  $\sigma(v) = 1(v > 0)$

Let  $\beta$  include the  $b$ ,  $\mathbf{x}$  includes the intercept,  $\alpha > 0$  small constant (learning rate). Start with random weight, then iteratively update the weight

- ▶ For  $k = 1, 2, \dots, K$ :
  - ▶ for each  $i = 1, \dots, n$ :
    - ▶ compute  $\hat{y}_i^{(k)} = 1(\mathbf{x}_i^\top \beta^{(k)} > 0)$  (Heaviside)
    - ▶ compute  $e_i^{(k)} = y_i - \hat{y}_i^{(k)}$
    - ▶ update  $\beta^{(k+1)} = \beta^{(k)} + \alpha \times e_i^{(k)} \times \mathbf{x}_i$
- ▶ a **training epoch**: *one* loop over the whole training data
- ▶ the whole process repeats for  $K$  times or **epochs**

A training algorithm that updates the parameter after seeing one example is called **on-line training** (stochastic gradient descent).

## Compared with logistic regression

Recall logistic regression, the update at the  $k$ -th iteration

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} + \alpha \times \mathbf{X}^\top (\mathbf{y} - \mathbf{p}^{(k)}) = \boldsymbol{\beta}^{(k)} + \alpha \times \sum_i^n (y_i - p(x_i; \boldsymbol{\beta}^{(k)})) \mathbf{x}_i$$

The **on-line training** process is

- ▶ For  $k = 1, 2, \dots, K$ :
  - ▶ for each  $i = 1, \dots, n$ :
    - ▶ compute  $\hat{y}_i^{(k)} = p(\mathbf{x}_i; \boldsymbol{\beta}^{(k)})$  (sigmoid)
    - ▶ compute  $e_i^{(k)} = y_i - \hat{y}_i^{(k)}$
    - ▶  $\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} + \alpha \times e_i^{(k)} \times \mathbf{x}_i$

Note

- ▶ Perceptron uses hard-classifier, while logistic uses soft-classifier
- ▶ Perceptron algorithm does not converge if the data is not linearly separable
- ▶ SGD (WNLS) for logistic regression converges to global minimum of the binary entropy loss function (even when the data is not linearly separable)

# Adaline (ADaptive LInear NEuron)/OLS

Adaline for classification is a special case when  $\sigma(\cdot)$  is identity function and the loss is squared error loss.

$$\text{MSE} = \frac{1}{n} \sum_{i=1}^n (\hat{a}_i - y_i)^2, \quad \hat{a}_i = \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}$$

It can be implemented using gradient descent for OLS.

$$\blacktriangleright \boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} + \alpha \times 2(y_i - \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}^{(k)}) \times \mathbf{x}_i$$

To classify based on the hard rule

$$\hat{y}_i = 1 \left( \mathbf{x}_i^\top \hat{\boldsymbol{\beta}} > 0.5 \right)$$

A major difference from Rosenblatt's Perceptron is that

- ▶  $\hat{a}_i$  is the continuous output (as opposed to  $\hat{y}_i$ ) that is measured against the true  $y_i$  in the loss calculation
  - ▶ That is, the hard-classification step is not backpropagated.