WKB approximate solutions for standard baroclinic modes

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In this notes I derive approximate WKB solutions to the standard baroclinic modes of physical oceanography. The elementary textbook example with constant buoyancy frequency is recovered as a special case.

1 Pressure modes

The standard baroclinic modes for pressure, here denoted $\mathsf{p}_n(z)$, is defined by the regular Sturm-Liouville eigenproblem

$$\mathsf{L}\mathsf{p}_n = -\kappa_n^2 \mathsf{p}_n \,, \tag{1}$$

with homogeneous Neumann boundary conditions

$$@z = -h,\, 0: \qquad \mathsf{p}_n' = 0\,, \tag{2}$$

and the self-adjoint Linear operator

$$\mathsf{L} \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z}\frac{f_0^2}{N^2}\frac{\mathrm{d}}{\mathrm{d}z}\,. \tag{3}$$

Hence the eigenmodes, p_n, are orthogonal. The real egeinvalues, κ_n, are the deformation wavenumber of the n'th mode. It is convenient to normalize the eigenmodes to have the unit L^2-norm

$$\frac{1}{H}\int_{-h}^{0} \mathsf{p}_n \mathsf{p}_m \mathrm{d}z = \delta_{mn}\,, \tag{4}$$

where δ_{mn} is the Dirac delta. Equation (1) can be rewritten as

$$\left(\tfrac{f_0}{N}\right)^2 \mathsf{p}_n'' + \left[\left(\tfrac{f_0}{N}\right)^2\right]' \mathsf{p}_n' + \kappa_n^2\, \mathsf{p}_n = 0\,. \tag{5}$$

Introducing the following definitions

$$\epsilon \stackrel{\text{def}}{=} \frac{1}{\kappa_n} \qquad \text{and} \qquad S^2(z) \stackrel{\text{def}}{=} \left(\tfrac{N}{f_0}\right)^2\,. \tag{6}$$

we have the renotated equation

$$\epsilon^2\, \mathsf{p}_n'' - \epsilon^2 \left[\log S^2(z)\right]' \mathsf{p}_n' + S^2(z)\mathsf{p}_n = 0\,. \tag{7}$$

In the WKB spirit we assume that $S^2(z)$ is slowly varying i.e., the buoyancy frequency $N^2(z)$ does not vary very fast. (This assumption may be problematic near the base of the mixed-layer.)

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We also assume that ϵ is small; the accuracy of the WKB solution improves with mode number. We now make the exponential approximation (e.g., Bender and Orszag)

$$\mathsf{p}_n^e \stackrel{\text{def}}{=} \mathrm{e}^{Q(z)/\epsilon} . \tag{8}$$

Hence

$$\mathsf{p}_n^{e\,\prime} = \frac{Q'(z)}{\epsilon}\mathsf{p}_n^e , \tag{9}$$

and

$$\mathsf{p}_n^{e\,\prime\prime} = \left[\left(\frac{Q'(z)}{\epsilon}\right)^2 + \frac{Q''(z)}{\epsilon}\right]\mathsf{p}_n^e , \tag{10}$$

Next we expand $Q(z)$ in powers of ϵ

$$Q(z) = Q_0(z) + \epsilon\, Q_1(z) + \epsilon^2\, Q_2(z) + \mathcal{O}(\epsilon^3) . \tag{11}$$

Substituting (11) in (7) we obtain, to lowest order, $\mathcal{O}(\epsilon^0)$,

$$Q_0'^2 + S^2(z) = 0 . \tag{12}$$

Thus

$$Q_0 = \pm\mathrm{i}\int^z S(\xi)\,\mathrm{d}\xi = \pm\mathrm{i}\tfrac{1}{f_0}\int^z N(\xi)\,\mathrm{d}\xi . \tag{13}$$

At next order, $\mathcal{O}(\epsilon)$, we have

$$2\,Q_0'Q_1' + Q_0'' - Q_0'\left[\log S^2(z)\right]' = 0 . \tag{14}$$

Hence

$$Q_1 = \frac{1}{2}\log S(z) - \frac{1}{2}\log \pm\mathrm{i}S(z) + \text{const} \;\; . \tag{15}$$

Notice that the imaginary part in the log in (15) just contributes an irrelevant constant. Thus

$$Q_1 = \log\sqrt{S(z)} + \text{const} \;\; . \tag{16}$$

In the most common WKB approximation (a.k.a "physical optics") we truncate (11) at $\mathcal{O}(\epsilon)$. The solution to (7), consistent with the bottom boundary condition (2), is

$$\mathsf{p}_n^{po} = A_n\,\sqrt{N(z)}\,\cos\left(\frac{\kappa_n}{f_0}\int_{-h}^z N(\xi)\mathrm{d}\xi\right) , \tag{17}$$

where A_n is a constant. By imposing the boundary condition at $z = 0$ (2), we obtain the eigenvalues κ_n:

$$\kappa_n = \frac{n\pi\, f_0}{\overline{N}\, h} , \qquad n = 0, 1, 2, \ldots , \tag{18}$$

where the mean buoyancy frequency is

$$\overline{N} \stackrel{\text{def}}{=} \frac{1}{h}\int_{-h}^0 N(\xi)\mathrm{d}\xi . \tag{19}$$

The constant A_n is determined by the normalization condition (4). We have

$$A_n^2\int_{-h}^0 N(z)\cos^2\left(\frac{\kappa_n}{f_0}\int_{-h}^z N(\xi)\mathrm{d}\xi\right)\mathrm{d}z = H , \qquad n \ge 1 . \tag{20}$$

The integral in (20) can be evaluated exactly by making the change of variables

$$\eta \stackrel{\text{def}}{=} \frac{\kappa_n}{f_0}\int_{-h}^z N(\xi)\mathrm{d}\xi \qquad \Rightarrow \qquad \mathrm{d}\eta = \frac{\kappa_n}{f_0}N(z)\mathrm{d}z , \tag{21}$$

and using the expression for the eigenvalues (18). We obtain

$$A_n = \left(2/\overline{N}\right)^{1/2}, \qquad n \geq 1\,. \tag{22}$$

Thus the WKB approximate solution to the standard pressure modes is

$$\mathsf{p}_n^{po} = \left[\frac{2\,N(z)}{\overline{N}}\right]^{1/2} \cos\left(\frac{n\pi}{\overline{N}\,h}\int_{-h}^{z} N(\xi)\mathrm{d}\xi\right)\,, \qquad n \geq 1\,. \tag{23}$$

The amplitude of the baroclinic modes at the boundaries is independent of the eigenvalue:

$$\mathsf{p}_n^{po}(z=0) = (-1)^n \left[\frac{2N(0)}{\overline{N}}\right]^{1/2}\,, \tag{24}$$

and

$$\mathsf{p}_n^{po}(z=-h) = \left[\frac{2N(-h)}{\overline{N}}\right]^{1/2}\,. \tag{25}$$

The barotropic mode is not recovered from the WKB solution because $\kappa_0 = 0$. From (1) we have that with $\kappa_0 = 0$, the barotropic mode is constant, independent of the stratification. With the normalization (4) we obtain $\mathsf{p}_0 = 1$.

Constant buoyancy frequency

With $N =$ const. the modes are simple sinusoids. That exact result is recovered as a special case of the WKB solution

$$\mathsf{p}_n^{po} = \sqrt{2}\cos\left[n\pi(1+z/h)\right]\,. \tag{26}$$

2 Density modes

Similarly the baroclinic modes for density, here denoted by r_n, are defined via the eigenproblem

$$\mathsf{r}_n'' = -\kappa_n^2 \left(\tfrac{N}{f_0}\right)^2 \mathsf{r}_n\,, \tag{27}$$

with homogeneous Dirichlet boundary conditions

$$@z = -h, 0: \qquad \mathsf{r}_n = 0\,, \tag{28}$$

and normalization

$$\frac{1}{h}\int_{-h}^{0} \mathsf{r}_n \mathsf{r}_m \mathrm{d}z = \delta_{mn}\,. \tag{29}$$

Alternatively, we can work on the approximation from the beginning. The WKB approximate solution to (27)-(28), consistent with the bottom boundary conditions (28), is

$$\mathsf{r}_n^{po} = \frac{B_n}{\sqrt{N(z)}} \sin\left(\frac{\kappa_n}{f_0}\int_{-h}^{z} N(\xi)\mathrm{d}\xi\right)\,, \tag{30}$$

The eigenvalues κ_n are the same as before (18). (This should be no surprise because it follow from the definition of p_n and r_n. Nonetheless, the verification is a good sanity check.) To find B_n we use the normalization (29)

$$B_n^2 \int_{-h}^{0} \frac{1}{N(z)} \sin\left(\frac{\kappa_n}{f_0}\int_{-h}^{z} N(\xi)\mathrm{d}\xi\right) \mathrm{d}z = h\,, \qquad n \geq 1\,. \tag{31}$$

We use a similar trick as above i.e., we change variables with

$$\eta \stackrel{\text{def}}{=} \frac{\kappa_n}{N^2(z) f_0} \int_{-h}^{z} N(\xi) \mathrm{d}\xi \qquad \Rightarrow \qquad \mathrm{d}\eta = \frac{\kappa_n}{N(z) f_0} \mathrm{d}z \,, \tag{32}$$

where, in the WKB spirit, we used the fact that $N(z)$ is slowly varying when differentiating the relation above. We obtain

$$B_n = \left(2N^2(0)/\overline{N}\right)^{1/2} . \tag{33}$$

Thus the WKB approximate solution to the density modes is

$$\mathsf{r}_n^{po} = \left(\frac{2N^2(0)}{\overline{N} N(z)}\right)^{1/2} \sin\left(\frac{n\pi}{\overline{N}\, h} \int_{-h}^{z} N(\xi) \mathrm{d}\xi\right) , \qquad n \geq 1 \,. \tag{34}$$

Finally, note that the modes are simply related

$$\frac{\mathrm{d}\mathsf{r}_n^{po}}{\mathrm{d}z} = N(0) \underbrace{\frac{n\pi}{\overline{N}\, h}}_{=\kappa_n/f_0} \mathsf{p}_n^{po} \,. \tag{35}$$

Constant buoyancy frequency

Again we recover the $N = $ const. special case from (34):

$$\mathsf{r}_n^{po} = \sqrt{2} \sin\left[n\pi(1 + z/h)\right] \,. \tag{36}$$

3 Probing WKB

How accurate is the WKB approximate solution for the standard modes? To answer this questions, we first recall that the fundamental assumption in WKB is the slowly varying nature of $N^2(z)$. Second, the introduction perturbation series (11) hinges on the smallness of the parameter ϵ. Hence we expect the accuracy to increase with mode number. Specifically, the physical optics approximation truncates the series at first order, so the error decreases as n^{-2}.

To verify the predictions above for the accuracy of the WKB solution, we consider an example with exponential stratification: $N^2(z) = N_0^2 \mathrm{e}^{-\alpha z}$, where N_0 is the stratification frequency at the surface and α^{-1} is the e-folding depth (see figure 1). In this simple example, the standard modes can be calculated analytically in terms of Bessel functions (see LaCasce JPO 2012). Figure 3 shows the stratification profiles for two different e-folding scales: $\alpha = 2$ and $\alpha = 5$. With $\alpha = 5$, the buoyancy frequency is strongly surface intensified as opposed to the slowly varying $\alpha = 2$ profile. Indeed, the relative error of the WKB eigenvalue (18) as compared to the exact eigenvalue is more than 10 times smaller for $\alpha = 2$ than for $\alpha = 5$ (see figure 3) — the relative error in the first deformation wavenumber for $\alpha = 2$ is spetacularly less than 1%! As expected, the error decreases with modes number n as n^{-2}. The eigenmodes are also well approximated by the WKB approximation (see figure 3 for the 4 gravest baroclinic modes with $\alpha = 5$). While there are some differences in amplitude, particularly at depth, the overall structure and zero-crossing depths are well captured by the WKB approximation.

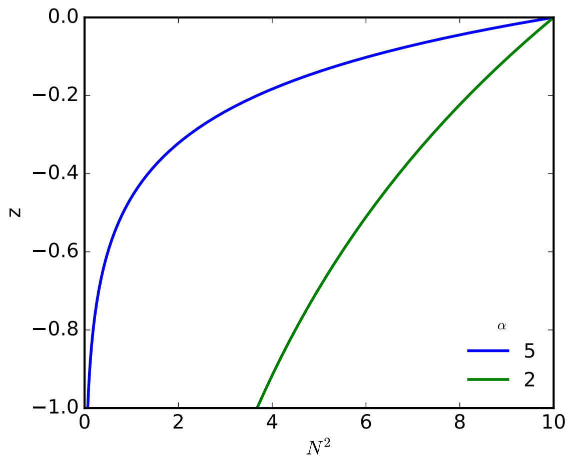

Figure 1: The exponential stratification $N^2(z) = N_0^2 \mathrm{e}^{-\alpha z}$ with $\alpha = 5$ and $\alpha = 2$.

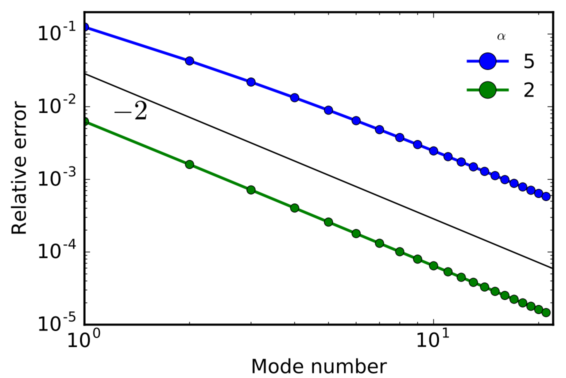

Figure 2: The relative error in deformation wavenumber square (the eigenvalue) for exponential stratification $N^2(z) = N_0^2 \mathrm{e}^{-\alpha z}$ with $\alpha = 5$ and $\alpha = 2$. The magnitude of the error decreases with α — the buoyancy frequency is more slowly varying with $\alpha = 2$. The error decreases with mode number as n^{-2} (-2 slope in log×log space).

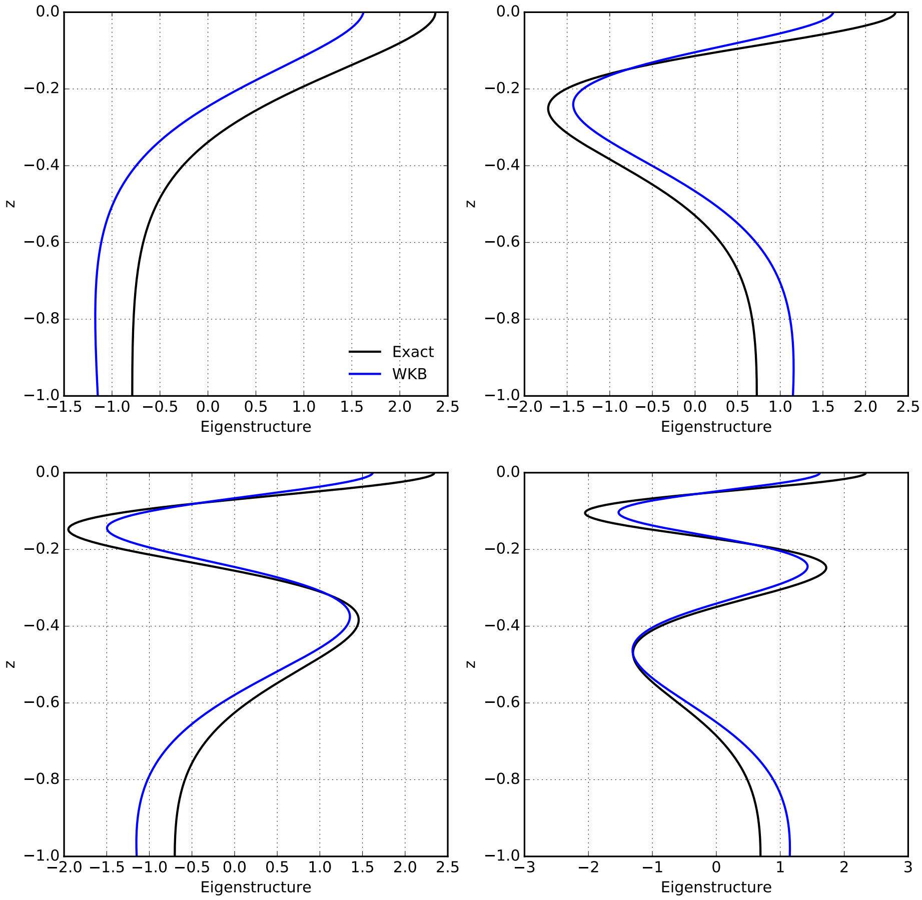

Figure 3: The pressure modes for exponential stratification $N^2(z) = N_0^2 \mathrm{e}^{-\alpha z}$ with $\alpha = 5$ calculated exactly (black) and using WKB (blue).