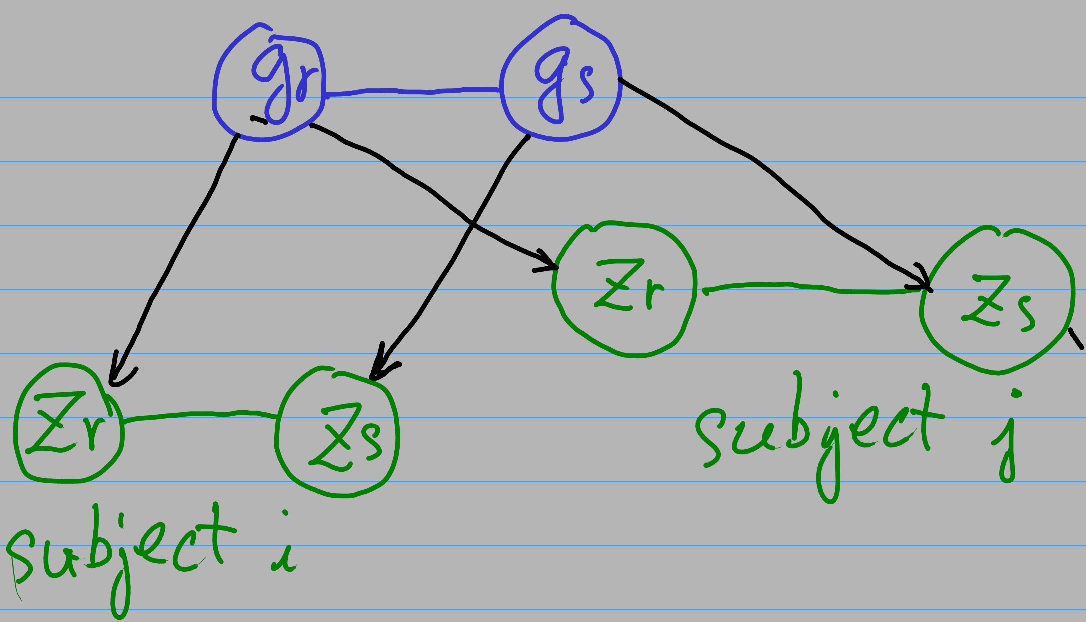

G: group level label map

g_s: label at site s. g_s is discrete.

$p(G)$ is Gibbs and MRFs

$$p(G) = \frac{1}{Z} \exp\{-U(G)\}$$

$$U(G) = \sum_{(r\ s)} \beta [g_r \neq g_s]$$ Potts Model

$$p(G) = \frac{1}{Z} \exp\{-\sum_{(r,s)} \beta [g_r \neq g_s]\}$$

$$p(z|G) = \frac{1}{C} \exp\{-U(z|G)\}$$

$$= \frac{1}{C} \exp\{-(\alpha \sum_{s} [g_r \neq g_s] + \beta \sum_{(r.s)} [z_r \neq z_s])\}$$

$$p(z_s | g_s) = \frac{1}{C_s} \exp\{-\alpha [g_r \neq g_s] + \beta \sum_{r \in N_s} [z_r \neq z_s]\}$$

• How to sample G, group label map.

$$p(G|Z,X) = p(G|Z^j) \quad j = 1 \dots J$$

$$p(g_s|Z^j) \propto p(g_s) \cdot \prod_j p(z_s^j|g_s)$$

$$\log p(g_s|Z^j) = \log p(g_s) + \sum_j \log p(z_s^j|g_s)$$

$$\underbrace{-\log p(g_s|Z^j)}_{U(g_s|z_s^j)} = \underbrace{-\log p(g_s)}_{U(g_s)} \underbrace{- \sum_j \log p(z_s^j|g_s)}_{U(z_s^j|g_s)}$$

$$U(g_s|z_s^j) = U(g_s|g_{N_s}) + \sum_j U(z_s^j|g_s)$$

$$= \beta_g \sum_{r\in N_s} [g_r \neq g_s] + \sum_j \left(\alpha [g_s \neq z_s^j] + \beta_z \sum_{r\in N_s} [z_r \neq z_s] \right)$$

$$= \beta_g \sum_{r\in N_s} [g_r \neq g_s] + \sum_j \alpha [g_s \neq z_s^j] + \text{const}$$

$\left(\sum_{r\in N_s} [z_r \neq z_s] \text{ term is not a fun of } g_s \right)$

$$= \beta_g \sum_{r\in N_s} [g_r \neq g_s] + \alpha \sum_j [g_s \neq z_s^j] + \text{const}$$

$$= \beta_g \sum_{r\in N_s} [g_r \neq g_s] + \alpha \left(J - \#(g_s = z_s) \right) + \text{const}$$

$$= \beta_g \sum_{r\in N_s} [g_r \neq g_s] - \alpha \cdot \#(g_s = z_s) + \text{const}$$

$$= \beta_g \sum_{r\in N_s} [g_r \neq g_s] - \alpha \cdot h(g_s) + \text{const}$$

($h(g_s)$ is the histogram of z_s over all J subs.
So $h(g_s)$ is the number of z_s that equal to g_s)

Comments: why after sampling G and Z, both label map became smooth, and thus the estimated β_z and β_g are much larger than the true value? Is that because the way I generated the label map is incorrect? Instead of generating G first, then generate Z given G, I should generate them symmutaneously. i.e. iteratively sample G and Z.

Sample $p(z|G,X)$

$$p(z|G,X) \propto p(z,G,X) \propto p(z|G) \cdot p(X|z,G)$$

$$\propto p(z|G) \cdot p(X|z)$$

$$\log p(z|G,X) = \log p(z|G) + \log p(X|z)$$

$$-\log p(z|G,X) = -\log p(z|G) - \log p(X|z)$$

$$U(z|G,X) = U(z|G) + U(X|z)$$

$$U(z_s|g_s, x_s) = U(z_s|g_s) + U(x_s|z_s)$$

$$U(z_s|g_s) = \alpha\,[z_s \neq g_s] + \beta \sum_{r \in N_s} [z_s \neq z_r]$$

$$U(x_s|z_s) = \log \sigma + \frac{(x-\mu)^2}{2\sigma^2}$$

$$U(z_s|g_s, x_s) = \alpha\,[z_s \neq g_s] + \beta \sum_{r \in N_s} [z_s \neq z_r] + \log \sigma + \frac{(x-\mu)^2}{2\sigma^2}$$

(μ, σ is fun of g_s)

G: group label map
Z: subject label map. Z^j for jth subject
X: observed data.

obj fun $= \log P(G, Z, X)$

parameter set $\theta = \{\mu_k, \sigma_k, k = 1 \ldots K\}$

$$\hat{\theta} = \arg\max_{\theta} \log p(G, Z, X; \theta)$$
$$= \arg\min_{\theta} -\log p(G, Z, X; \theta)$$

Since we do not know (G, Z).

$$\hat{\theta} = \arg\min_{\theta} - E_{p(G, Z \mid X)}[\log p(G, Z, X; \theta)]$$
$$= Q_1 + Q_2 + Q_3$$

$$Q_1 = -E[\log p(G)] \approx -\frac{1}{M} \sum_m \log P(G^m)$$
$$= -\frac{1}{M} \sum_m \sum_s \log P(g_s^m)$$

pseudo likelihood Approximation

Because $p(g_s) = \frac{1}{Z_s} \exp\{-\sum_{r \in N_s} [g_r \neq g_s]\}$

$$Z_s = \sum_{k=1}^{K} \exp\{-\sum_{r \in N_s} [k \neq g_r]\}$$

$$\log p(g_s) = -\sum_{r \in N_s} [g_r \neq g_s] - \log Z_s$$

$$Q_1(\beta g) = -\frac{1}{M}\sum_{m,s}\left(-\beta\sum_{r\in N_s}[g_r \neq g_s] - \log Z_s\right)$$

$$= -\frac{1}{M}\sum_{m,s}\left(-\beta\sum_{r\in N_s}[g_r \neq g_s] - \log\sum_k e^{-\beta\sum_{r\in N_s}[k\neq g_r]}\right)$$

$$= \frac{1}{M}\sum_{m,s}\left(\beta\sum_{r\in N_s}[g_r \neq g_s] + \log\sum_k e^{-\beta\sum_{r\in N_s}[k\neq g_r]}\right)$$

Let $b = -\sum_{r\in N_s}[g_r \neq g_s]$

$$Q_1(\beta g) = \frac{1}{M}\sum_{m,s}\left(-b\beta + \log\sum_k e^{b\beta}\right)$$

$$q_1 = -b\beta + \log\sum_k e^{b\beta}, \quad Q_1(\beta g) = \frac{1}{M}\sum_{m,s} q_1$$

$$\frac{\partial q_1}{\partial \beta} = -b + \frac{\sum_k b\cdot e^{b\beta}}{\sum_k e^{b\beta}}$$

(b is fun of k, so can not take out of $\sum_k$)

$$\frac{\partial q_1}{\partial \beta^2} = \frac{\left(\sum_k b^2\cdot e^{b\beta}\right)\left(\sum_k e^{b\beta}\right) - \left(\sum_k b\cdot e^{b\beta}\right)^2}{\left(\sum_k e^{b\beta}\right)^2}$$

Let $M_0 = \sum_k e^{b\beta}$, $M_1 = \sum_k b\cdot e^{b\beta}$.

$$M_2 = \sum_k b^2\cdot e^{b\beta}$$

$$\frac{\partial^2 q}{\partial \beta^2} = \frac{M_2\cdot M_0 - M_1^2}{M_0^2}$$

$$Q_2(\alpha, \beta_z) = -E\left[\sum_j \log p(z^j | G)\right]$$

$$\approx -\frac{1}{M} \sum_{m,j} \log p(z^{mj} | G)$$

Pseduo-Likelihood Approximation

$$\approx -\frac{1}{M} \sum_{m,j,s} \log p(z_s^{mj} | g_s^m)$$

$$p(z_s | g_s) = \frac{1}{C_s} \exp\left\{-\alpha [z_s \neq g_s] - \beta_z \sum_{r \in N_s} [z_r \neq z_s]\right\}$$

$$C_s = \sum_k \exp\left\{-\alpha [k \neq g_s] - \beta_z \sum_{r \in N_s} [k \neq z_r]\right\}$$

$$\log p(z_s | g_s) = -\alpha [z_s \neq g_s] - \beta_z \sum_{r \in N_s} [z_r \neq z_s] - \log \sum_k \exp\left\{-\alpha [k \neq g_s] - \beta_z \sum_{r \in N_s} [k \neq z_r]\right\}$$

$$Q_2(\alpha, \beta_z) = -\frac{1}{M} \sum_{j,m,s} \left(-\alpha [z_s \neq g_s] - \beta_z \sum_{r \in N_s} [z_r \neq z_s] - \log \sum_k \exp\left\{-\alpha [k \neq g_s] - \beta_z \sum_{r \in N_s} [k \neq z_r]\right\}\right)$$

$$a = -[z_s \neq g_s], \quad b = -\sum_{r \in N_s} [z_r \neq z_s]$$

$$Q_2(\alpha, \beta_z) = -\frac{1}{M} \sum_{j,m,s} \left(a\alpha + b\beta - \log \sum_k e^{a\alpha + b\beta}\right)$$

$$= \frac{1}{M} \sum_{j,m,s} \left(-a\alpha - b\beta + \log \sum_k e^{a\alpha + b\beta}\right)$$

$$q_2 = -a\alpha - b\beta + \log \sum_k e^{a\alpha+b\beta}$$

$$\frac{\partial q_2}{\partial \alpha} = -a + \frac{\sum_k a \cdot e^{a\alpha+b\beta}}{\sum_k e^{a\alpha+b\beta}}, \quad \frac{\partial q_2}{\partial \beta} = -b + \frac{\sum_k b \cdot e^{a\alpha+b\beta}}{\sum_k e^{a\alpha+b\beta}}$$

$$\frac{\partial^2 q_2}{\partial \alpha^2} = \frac{\left(\sum a^2 \cdot e^{a\alpha+b\beta}\right) \cdot \left(\sum_k e^{a\alpha+b\beta}\right) - \left(\sum_k a \cdot e^{a\alpha+b\beta}\right)^2}{\left(\sum_k e^{a\alpha+b\beta}\right)^2}$$

$$\frac{\partial^2 q_2}{\partial \beta^2} = \frac{\left(\sum_k b^2 \cdot e^{a\alpha+b\beta}\right) \cdot \left(\sum_k e^{a\alpha+b\beta}\right) - \left(\sum_k b \cdot e^{a\alpha+b\beta}\right)^2}{\left(\sum_k e^{a\alpha+b\beta}\right)^2}$$