Hierarchical Graphical Lasso

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This notes is for solving the graphical Lasso problem in a hierarchical fashion. Define the population mean of the inverse covariance matrix of a multi-variate Gaussian as Ω. Give Ω, the subject's precision matrix is

$$\Phi_j = \Omega + \Theta_j.$$

The objective function is defined as

$$f(\Omega, \Phi_j) = \sum_j (\log\det\Phi_j - \mathrm{tr}(S_j\Phi_j) - \rho\,||\Theta_j||) + \lambda\,||\Omega||$$

It should be easy to prove this is a hierarchical Bayesian model.

$$\frac{\partial f(\Omega, \Phi_j)}{\partial \Phi_j} = \Phi^{-1} - S_j - \rho\Gamma_j = 0$$

Define $W = \Phi^{-1}$ we have

$$\frac{\partial f(\Omega, \Phi_j)}{\partial \Phi_j} = W - S_j - \rho\Gamma_j = 0$$

This is same with equation (2.6) of Friedman et al. [2008], and can be solved by a iterative Lasso method. For Ω, rewrite f as

$$\begin{aligned} f(\Omega) &= \sum_j (\log\det(\Omega + \Theta_j) - \mathrm{tr}(S_j(\Omega + \Theta_j)) - \rho\,||\Theta_j||) + \lambda\,||\Omega|| \\ \frac{\partial f(\Omega, \Phi_j)}{\partial \Omega} &= \sum_j ((\Omega + \Theta_j)^{-1} - S_j) - \lambda\Gamma \\ &= \sum_j (\Omega + \Theta_j)^{-1} - \sum_j S_j - \lambda\Gamma = 0. \end{aligned}$$

Define $Q = \sum_j (\Omega + \Theta_j)^{-1}$, we have $Q - \sum_j S_j - \lambda\Gamma = 0$, which again is same with equation (2.6) of Friedman et al. [2008]. so we can use Friedman et al. [2008] to solve Q, and then solve Ω from $Q = \sum_j (\Omega + \Theta_j)^{-1}$.

A few results are available for matrix differentiation

$$
\begin{aligned}
||X||_F^2 &= \mathrm{tr}(X^\top X) \\
\frac{\partial\, ||X||_F^2}{\partial X} &= 2X \\
\frac{\partial X^{-1}}{\partial X} &= -(X^{-\top} \otimes X^\top) \\
\frac{\partial X^\top X}{\partial X} &= (\mathbf{I}_{n^2} + \mathbf{T}_{n,n})(\mathbf{I}_n \otimes X^\top),
\end{aligned}
$$

where $\mathbf{T}$ is a vector permutation matrix that transform $vec\mathbf{X}$ to $vec\mathbf{X}^\top$.

References

J. Friedman, T. Hastie, and R. Tibshirani. Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441, 2008.