Opposite_Bias_Addendum

November 14, 2021

1 Opposite Bias Model, Updated

James Yu, 14 November 2021

[1]:
```
from collections import defaultdict
import matplotlib.pyplot as plt
import numpy as np
```

[2]:
```
def M(K, B, R, L, delta):
    """Computes M_{t-1} given B_l \forall l, K_t^l \forall l,
        R_l \forall l, number of strategic agents L, and delta."""
    # handle the generic structure first, with the correct pairings:
    base = [[(B[l_prime].T @ K[l_prime] @ B[l]).item() for l in range(L)] for␣
 ↪l_prime in range(L)]
    # then change the diagonals to construct M_{t-1}:
    for l in range(L): base[l][l] = (B[l].T @ K[l] @ B[l] + R[l]/delta).item()
    return np.array(base, ndmin = 2)

def H(B, K, A, L):
    """Computes H_{t-1} given B_l \forall l, K_t^l \forall l,
        A, and number of strategic agents L."""
    return np.concatenate(tuple(B[l].T @ K[l] @ A for l in range(L)), axis = 0)

def C_l(B, K, k, h, L, c, x, n):
    """Computes C_{t-1}^h (displayed as C_{t-1}^l) given B_l \forall l, K_t^l␣
 ↪\forall l,
        k_t^l \forall l, a specific naive agent h, number of strategic agents␣
 ↪L,
        c_l \forall l, x_l \forall l, and number of naive agents n"""
    return np.concatenate(tuple(B[l].T @ K[l] @ A @ ((x[h] - x[l]) * np.
 ↪ones((n, 1)))
                               + B[l].T @ K[l] @ c[l]
                               + 0.5 * B[l].T @ k[l].T for l in range(L)), axis = 0)

def E(M_, H_):
    """Computes the generic E_{t-1} given M_{t-1} and H_{t-1}."""
    return np.linalg.inv(M_) @ H_
```

```python
def F(M_, C_l_, l):
    """Computes F_{t-1}^l given M_{t-1}, C_{t-1}^l, and specific naive agent l.
 ↪"""
    return (np.linalg.inv(M_) @ C_l_)[l:l+1, :]

def G(A, B, E_, L):
    """Computes the generic G_{t-1} given A, B_l \forall l,
        E_{t-1}, and number of strategic agents L."""
    return A - sum([B[l] @ E_[l:l+1, :] for l in range(L)])

def g_l(B, E_, h, x, F_, L):
    """Computes g_{t-1}^l given B_l \forall l, E_{t-1}^l,
        a particular naive agent h, x_l \forall l, F_{t-1}^l \forall l,
        number of strategic agents L, number of naive agents n, and c_h."""
    return - sum([B[l] @ (E_[l:l+1, :] @ ((x[h] - x[l]) * np.ones((n, 1))) +␣
 ↪F_[l]) for l in range(L)]) + c[h]
```

[3]:
```python
def K_t_minus_1(Q, K, E_, R, G_, L, delta):
    return [Q[l] + E_[l:l+1, :].T @ R[l] @ E_[l:l+1, :]
            + delta * G_.T @ K[l] @ G_ for l in range(L)]

def k_t_minus_1(K, k, G_, g, E_, F_, R, L, delta):
    return [2*delta* g[l].T @ K[l] @ G_ + delta * k[l] @ G_
            + 2 * F_[l].T @ R[l] @ E_[l:l+1, :] for l in range(L)]

def kappa_t_minus_1(K, k, kappa, g_, F_, R, L, delta):
    return [-delta * (g_[l].T @ K[l] @ g_[l] + k[l] @ g_[l] - kappa[l])
            - (F_[l].T @ R[l] @ F_[l]) for l in range(L)]
```

[4]:
```python
def solve(K_t, k_t, kappa_t, A, B, delta, n, m, L, Q, R, x, c, tol = 300):
    historical_K = [K_t]
    historical_k = [k_t]
    historical_kappa = [kappa_t]
    max_distances = defaultdict(list)
    counter = 0
    while True:
        M_ = M(K_t, B, R, L, delta)
        H_ = H(B, K_t, A, L)
        E_ = E(M_, H_)
        G_ = G(A, B, E_, L)
        K_new = K_t_minus_1(Q, K_t, E_, R, G_, L, delta)
        F_ = [F(M_, C_l(B, K_t, k_t, l, L, c, x, n), l) for l in range(L)]
        g = [g_l(B, E_, h, x, F_, L) for h in range(L)]
        k_new = k_t_minus_1(K_t, k_t, G_, g, E_, F_, R, L, delta)
        kappa_new = kappa_t_minus_1(K_t, k_t, kappa_t, g, F_, R, L, delta)
        cd_K = [np.max(np.abs(K_t[l] - K_new[l])) for l in range(L)]
```

```python
        cd_k = [np.max(np.abs(k_t[l] - k_new[l])) for l in range(L)]
        cd_kappa = [np.max(np.abs(kappa_t[l] - kappa_new[l])) for l in range(L)]
        K_t = K_new
        k_t = k_new
        kappa_t = kappa_new
        historical_K.insert(0, K_t)
        historical_k.insert(0, k_t)
        historical_kappa.insert(0, kappa_t)
        for l in range(L):
            max_distances[(l+1, "K")].append(cd_K[l])
            max_distances[(l+1, "k")].append(cd_k[l])
            max_distances[(l+1, "kappa")].append(cd_kappa[l])
        counter += 1
        if sum(cd_K + cd_k + cd_kappa) == 0 or counter > tol:
            return max_distances, historical_K, historical_k, historical_kappa
```

[5]:
```python
def optimal(X_init, historical_K, historical_k, historical_kappa, infinite =␣
 ↪True):
    X_t = [a.copy() for a in X_init]
    xs = defaultdict(list)
    for l in range(L):
        xs[l].append(X_t[l])

    rs = defaultdict(list)
    payoffs = defaultdict(list)
    payoff = defaultdict(lambda: 0)
    i = 0
    while [i < len(historical_K), True][infinite]:
        K_t = historical_K[[i, 0][infinite]]
        k_t = historical_k[[i, 0][infinite]]
        M_ = M(K_t, B, R, L, delta)
        H_ = H(B, K_t, A, L)
        E_ = E(M_, H_)
        G_ = G(A, B, E_, L)
        F_ = [F(M_, C_l(B, K_t, k_t, l, L, c, x, n), l) for l in range(L)]
        g = [g_l(B, E_, h, x, F_, L) for h in range(L)]
        for l in range(L):
            Y_new = -1 * E_[l:l+1, :] @ X_t[l] - F(M_, C_l(B, K_t, k_t, l, L,␣
↪c, x, n), l)
            rs[l].append(Y_new)
            payoff[l] += (-1 * delta**i * (X_t[l].T @ Q[l] @ X_t[l])).item() +␣
↪(-1 * delta**i * (Y_new.T @ R[l] @ Y_new)).item()
            payoffs[l].append(payoff[l])
            X_new = G_ @ X_t[l] + g[l]
            xs[l].append(X_new)
            if l == L - 1 and infinite == True and np.max(X_t[l] - X_new) == 0:
                return xs, rs, payoffs
```

```
            X_t[l] = X_new
        i += 1

    return xs, rs, payoffs
```

1.1 Control: $x_0 = 0$

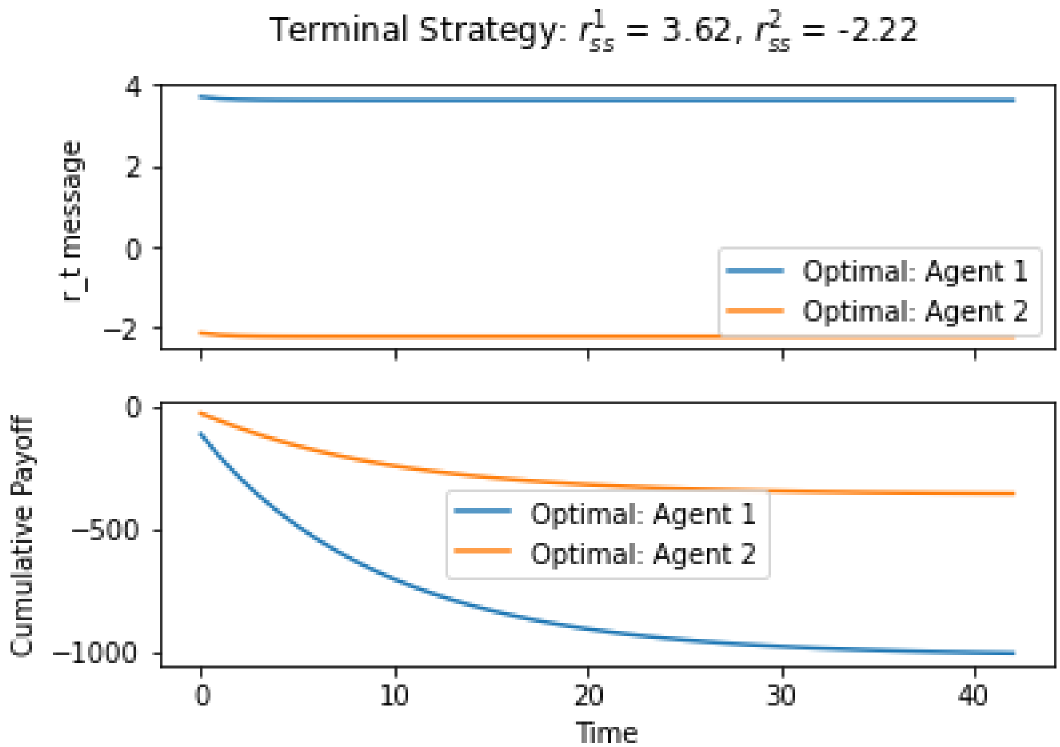
Note that in the multiple agent model, x_0 is translated to $\chi_0 = x_0 - x1_n$ where x is the agenda of the particular strategic agent.

[6]:
```
A = np.array([
  [0.5],
], ndmin = 2)

B_1 = np.array([
  0.25,
], ndmin = 2).T

B_2 = np.array([
  0.25,
], ndmin = 2).T

B = [B_1, B_2]

x0 = 0

X_0_1 = np.array([ # \chi_0^1
  x0 - 10, # agenda here is 10
], ndmin = 2).T

X_0_2 = np.array([ # \chi_0^2
  x0 + 5, # agenda here is -5
], ndmin = 2).T
X_0 = [X_0_1, X_0_2]

delta = 0.9
n = 1
m = 1
L = 2
Q = [1 * np.identity(n), 1 * np.identity(n)]
R = [1 * np.identity(m), 1 * np.identity(m)]

x = [10, -5] # here we have difference in agendas
r = [0, 0] # a message of nonzero weight has cost
c_base = sum([B[l] @ np.array([[r[l]]], ndmin = 2) for l in range(L)])
c = [c_base + (A - np.identity(n)) @ (x[l] * np.ones((n, 1))) for l in range(L)]
```

[7]:
```
max_distances, historical_K, historical_k, historical_kappa = solve(Q, [np.
↪zeros((1, n)), np.zeros((1, n))], [0, 0], A, B, delta, n, m, L, Q, R, x, c,␣
↪tol = 1000)
```

[8]:
```
save_K = historical_K
xs, rs, payoffs = optimal(X_0, historical_K, historical_k, historical_kappa)
save_xs = xs
```

[9]:
```
def do_plot(rs, r, payoffs, num_agents = 1, set_cap = np.inf, flag = False,␣
↪legend = True):
    fig, sub = plt.subplots(2, sharex=True)
    if legend:
        fig.suptitle(f"Terminal Strategy: {', '.join(['$r_{ss}^' + str(l+1) +␣
↪'$ = ' + str(round(rs[l][:min(len(rs[l]), set_cap)][-1].item() + r[l], 2))␣
↪for l in range(num_agents)])}")

    for l in range(num_agents):
        sub[0].plot(range(min(len(rs[l]), set_cap)), [a.item() + r[l] for a in␣
↪rs[l][:min(len(rs[l]), set_cap)]], label = f"Optimal: {['Agent',␣
↪'Channel'][flag]} {l+1}")
    sub[0].set(ylabel = "r_t message")

    for l in range(num_agents):
        sub[1].plot(range(min(len(payoffs[l]), set_cap)), payoffs[l][:
↪min(len(payoffs[l]), set_cap)], label = f"Optimal: {['Agent',␣
↪'Channel'][flag]} {l+1}")
    sub[1].set(xlabel = "Time", ylabel =  "Cumulative Payoff")
    if legend:
        sub[0].legend()
        sub[1].legend()
    plt.show()
do_plot(rs, r, payoffs, num_agents = 2, set_cap = 1000)
```

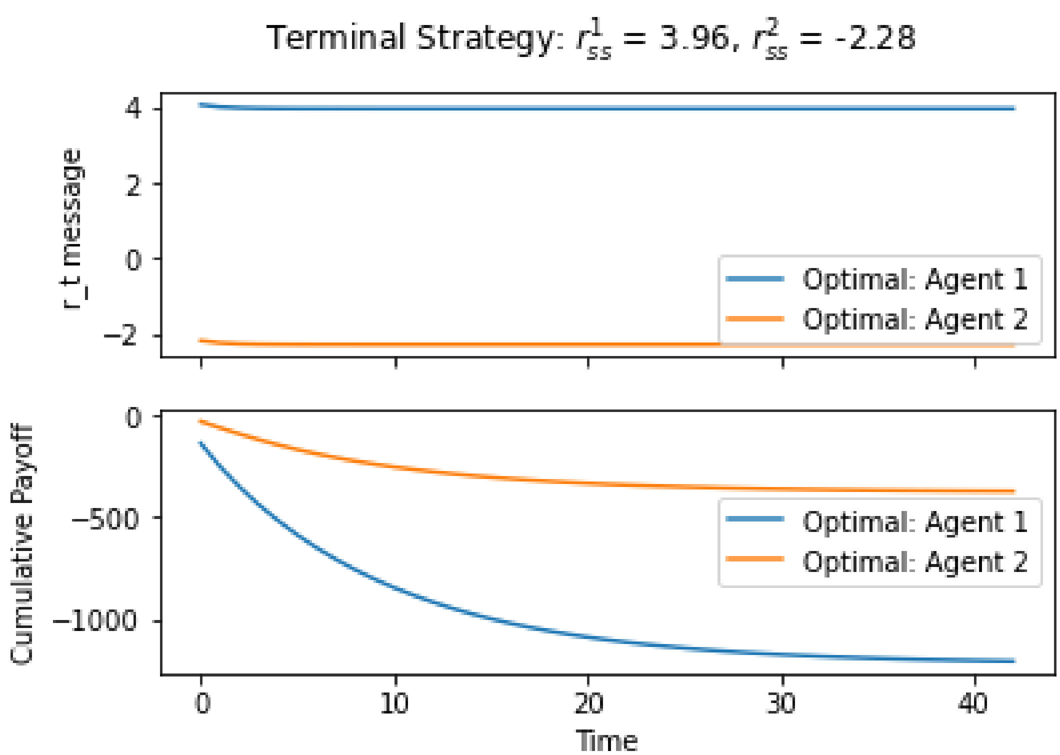


1.2 The important question: aren't r^1 and r^2 inversely related? Why do they move together?

Actually they are inversely related. This is what happens if we shift the agenda of agent 1 from 10 to 11:

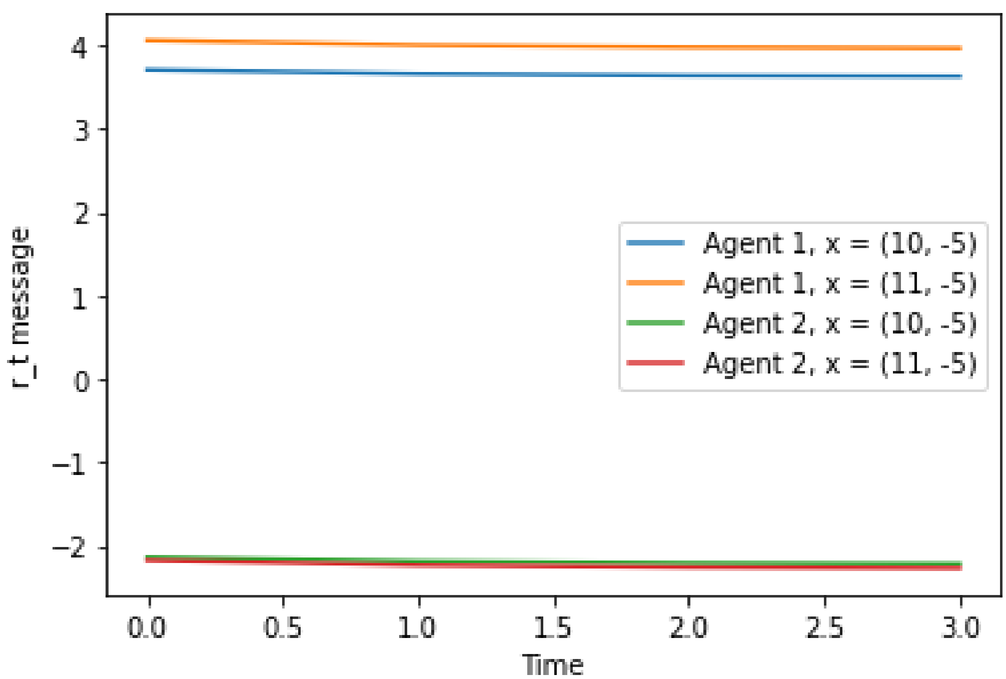
[10]:
```
A = np.array([
    [0.5],
], ndmin = 2)

B_1 = np.array([
    0.25,
], ndmin = 2).T

B_2 = np.array([
    0.25,
], ndmin = 2).T

B = [B_1, B_2]

x0 = 0

X_0_1 = np.array([ # \chi_0^1
```

```
  x0 - 11, # agenda here is 10
], ndmin = 2).T

X_0_2 = np.array([ # \chi_0^2
  x0 + 5, # agenda here is -5
], ndmin = 2).T
X_0 = [X_0_1, X_0_2]

delta = 0.9
n = 1
m = 1
L = 2
Q = [1 * np.identity(n), 1 * np.identity(n)]
R = [1 * np.identity(m), 1 * np.identity(m)]

x = [11, -5] # here we have difference in agendas
r = [0, 0] # a message of nonzero weight has cost
c_base = sum([B[l] @ np.array([[r[l]]], ndmin = 2) for l in range(L)])
c = [c_base + (A - np.identity(n)) @ (x[l] * np.ones((n, 1))) for l in range(L)]
max_distances, historical_K, historical_k, historical_kappa = solve(Q, [np.
 ↪zeros((1, n)), np.zeros((1, n))], [0, 0], A, B, delta, n, m, L, Q, R, x, c,␣
 ↪tol = 1000)
xs2, rs2, payoffs2 = optimal(X_0, historical_K, historical_k, historical_kappa)
do_plot(rs2, r, payoffs2, num_agents = 2, set_cap = 1000)
```



Notice both strategies moved in opposite directions away from zero when one agent's agenda moved away from zero. I can plot them together as follows:

[11]:
```
plt.plot(range(4), [a.item() for a in rs[0][:4]], label = "Agent 1, x = (10,␣
↪-5)")
plt.plot(range(4), [a.item() for a in rs2[0][:4]], label = "Agent 1, x = (11,␣
↪-5)")
plt.plot(range(4), [a.item() for a in rs[1][:4]], label = "Agent 2, x = (10,␣
↪-5)")
plt.plot(range(4), [a.item() for a in rs2[1][:4]], label = "Agent 2, x = (11,␣
↪-5)")
plt.legend()
plt.xlabel("Time")
plt.ylabel("r_t message")
plt.show()
```



[12]:
```
rs[0][:4] # agent 1 with x = (10, -5) case
```

[12]:
```
[array([[3.70923401]]),
 array([[3.6615178]]),
 array([[3.64055569]]),
 array([[3.63134687]])]
```

[13]:
```
rs[1][:4] # agent 2 with x = (10, -5) case
```

[13]:
```
[array([[-2.13683068]]),
 array([[-2.18454689]]),
 array([[-2.205509]]),
 array([[-2.21471782]])]
```

[14]:
```
rs2[0][:4] # agent 1, x = (11, -5)
```

[14]:
```
[array([[4.06134317]]),
 array([[4.00408371]]),
 array([[3.97892918]]),
 array([[3.96787859]])]
```

[15]:
```
rs2[1][:4] # agent 2, x = (11, -5)
```

[15]:
```
[array([[-2.17445917]]),
 array([[-2.23171862]]),
 array([[-2.25687316]]),
 array([[-2.26792374]])]
```