Multi-vs-Single-updated

September 24, 2021

1 Comparison of Single-Agent-Dual-Message model with Dual-Agent-Single-Message model (asymmetric channels)

James Yu, revised 24 September 2021 with fixed typos, revised 23 September 2021 with different B_i values

[1]:
```
from collections import defaultdict
import matplotlib.pyplot as plt
import numpy as np
```

I moved the code for this notebook to the bottom for readability - the order in which the blocks were run is indicated by the number on the left of each block.

1.1 Comparison between the 1-strategic 2-channel case and the 2-strategic 1-channel case (the baseline multiple strategic agent model)

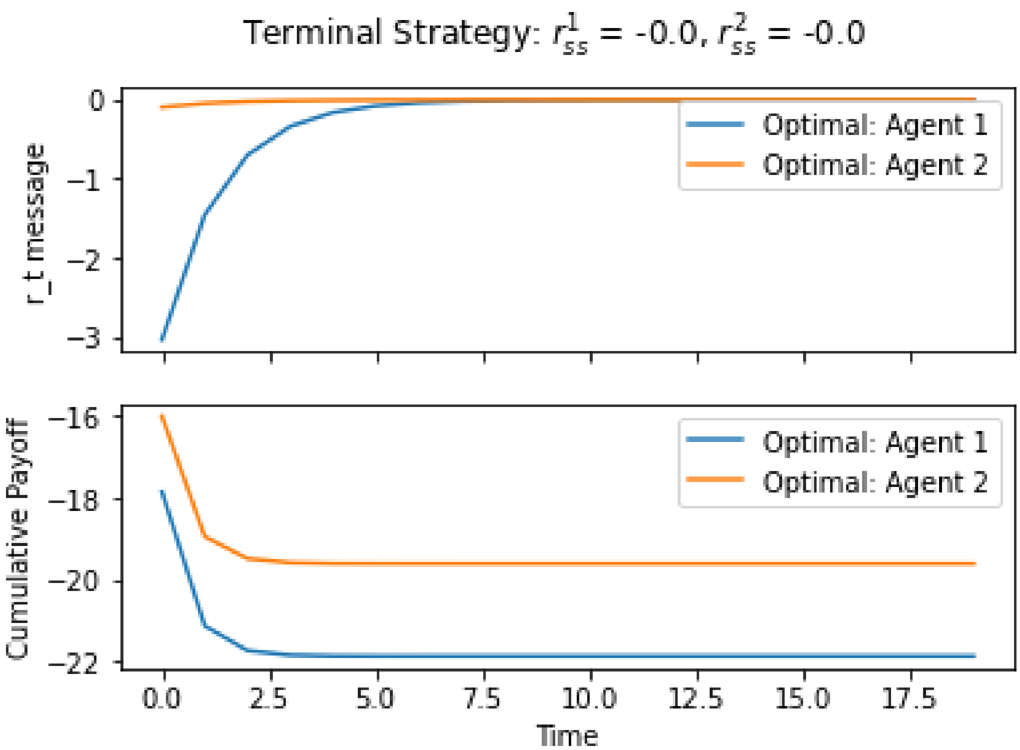
[8]:
```
A = np.array([
  [0.7],
], ndmin = 2)

B_1 = np.array([
  0.29, # split the channel of the strategic agent from the previous model␣
↪amongst the two strategic agents here
], ndmin = 2).T

B_2 = np.array([
  0.01, # split the channel of the strategic agent from the previous model␣
↪amongst the two strategic agents here
], ndmin = 2).T

B = [B_1, B_2]

X_0_1 = np.array([
  4,
], ndmin = 2).T
X_0 = [X_0_1, X_0_1]
```

```
delta = 0.8
n = 1
m = 1
L = 2 # two agents now
Q = [1 * np.identity(n), 1 * np.identity(n)]
R = [0.2 * np.identity(m), 0.2 * np.identity(m)]

x = [0, 0] # identical agents
r = [0, 0]
c_base = sum([B[l] @ np.array([[r[l]]], ndmin = 2) for l in range(L)])
c = [c_base + (A - np.identity(n)) @ (x[l] * np.ones((n, 1))) for l in range(L)]
```

[9]:
```
max_distances, historical_K, historical_k, historical_kappa = solve(Q, [np.
↪zeros((1, n)), np.zeros((1, n))], [0, 0], A, B, delta, n, m, L, Q, R, x, c,␣
↪tol = 1000)
```

First, we take the baseline two-agent model. Agent 1 has $b_1 = 0.29$ and Agent 2 has $b_2 = 0.01$:

[10]:
```
xs2, rs2, payoffs2 = optimal(X_0, historical_K, historical_k, historical_kappa)
do_plot(rs2, r, payoffs2, num_agents = 2, set_cap = 20)
```



Observe that the agent with the smaller b_i has greater payoff and smaller messages.

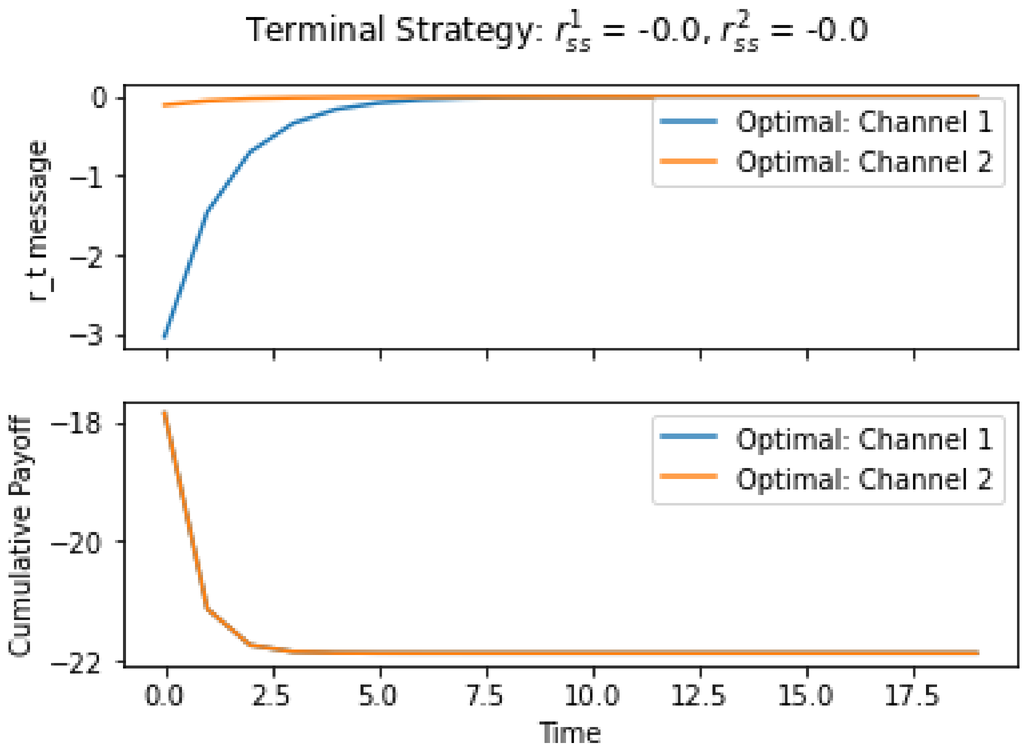
[11]:
```
for i in range(len(rs2[1])):
    if rs2[0][i].item() > rs2[1][i].item():
        print(rs2[0][i], rs2[1][i])
print("done")
```

```
done
```

The above demonstrates that there is no crossover, since one is always bounded by the other.

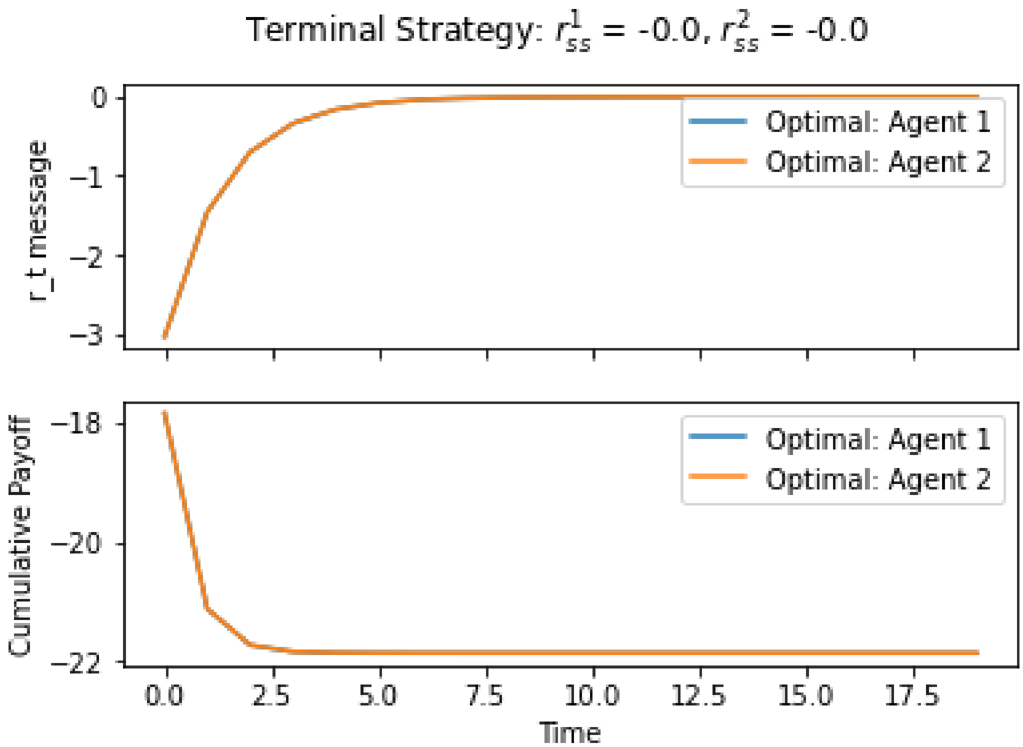
Next, we can look at the two-message one-agent case on its own:

[12]:
```
A = np.array([
  [0.7],
], ndmin = 2)

B = np.array([
  [0.29, 0.01] # here the agent now has two channels through which to send
], ndmin = 2)

delta = 0.8
Q = 1 * np.identity(1)
R = 0.2 * np.identity(2)
x = np.array([
  [4],
], ndmin = 2)

r_ts, x_ts, payoffs, Ks = optimal_single(1, delta, A, B, R, Q, x)

do_plot({0:[i[0] for i in r_ts], 1:[i[1] for i in r_ts]}, [0, 0], {0:payoffs, 1:
 ↪payoffs}, num_agents = 2, set_cap = 20, flag = True)
```



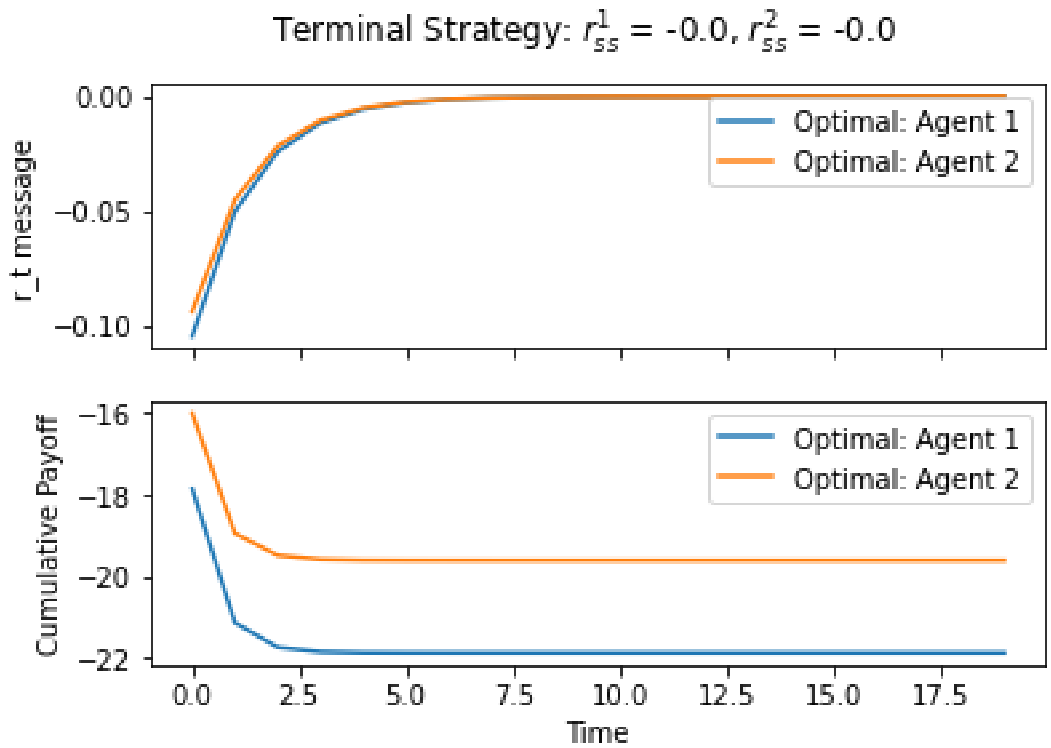
Payoff is worse, due to the one agent having to take the cost of both messages. The r_t sequences are close, but as demonstrated in some of the following superimposed plots, they are not the same:

[13]:
```
do_plot({0:[i[0] for i in r_ts], 1:rs2[0]}, [0, 0], {0:payoffs, 1:payoffs2[0]},
↪num_agents = 2, set_cap = 20)
```



Agent 1 is the first channel of the two-message case, and Agent 2 is the first agent of the two-agent case. Their strategies are almost exactly the same (further down I demonstrate there is a slight margin).

[14]:
```
do_plot({0:[i[1] for i in r_ts], 1:rs2[1]}, [0, 0], {0:payoffs, 1:payoffs2[1]},␣
↪num_agents = 2, set_cap = 20)
```



Here I take the second channel and the second agent, respectively. Now the difference is more clear.

The differences in r_0^l for channel 1 and channel 2 are:

[15]:
```
print(r_ts[0][0] - rs2[0][0], r_ts[0][1] - rs2[1][0])
```

```
[[-8.55325669e-05]] [[-0.01084864]]
```

The channels of smaller influence exhibit greater difference between the two models. And over both models, we have the messages of the two-agent case being closer to zero.

1.2 Primary Code:

[2]:
```
def M(K, B, R, L, delta):
    """Computes M_{t-1} given B_l \forall l, K_t^l \forall l,
        R_l \forall l, number of strategic agents L, and delta."""
    # handle the generic structure first, with the correct pairings:
    base = [[(B[l_prime].T @ K[l_prime] @ B[l]).item() for l in range(L)] for␣
 ↪l_prime in range(L)]
    # then change the diagonals to construct M_{t-1}:
    for l in range(L): base[l][l] = (B[l].T @ K[l] @ B[l] + R[l]/delta).item()
    return np.array(base, ndmin = 2)

def H(B, K, A, L):
```

```python
    """Computes H_{t-1} given B_l \forall l, K_t^l \forall l,
        A, and number of strategic agents L."""
    return np.concatenate(tuple(B[l].T @ K[l] @ A for l in range(L)), axis = 0)

def C_l(B, K, k, h, L, c, x, n):
    """Computes C_{t-1}^h (displayed as C_{t-1}^l) given B_l \forall l, K_t^l \forall l,
        k_t^l \forall l, a specific naive agent h, number of strategic agents L,
        c_l \forall l, x_l \forall l, and number of naive agents n"""
    return np.concatenate(tuple(B[l].T @ K[l] @ A @ ((x[h] - x[l]) * np.ones((n, 1)))
                                + B[l].T @ K[l] @ c[l]
                                + 0.5 * B[l].T @ k[l].T for l in range(L)), axis = 0)

def E(M_, H_):
    """Computes the generic E_{t-1} given M_{t-1} and H_{t-1}."""
    return np.linalg.inv(M_) @ H_

def F(M_, C_l_, l):
    """Computes F_{t-1}^l given M_{t-1}, C_{t-1}^l, and specific naive agent l."""
    return (np.linalg.inv(M_) @ C_l_)[l:l+1, :]

def G(A, B, E_, L):
    """Computes the generic G_{t-1} given A, B_l \forall l,
        E_{t-1}, and number of strategic agents L."""
    return A - sum([B[l] @ E_[l:l+1, :] for l in range(L)])

def g_l(B, E_, h, x, F_, L):
    """Computes g_{t-1}^l given B_l \forall l, E_{t-1}^l,
        a particular naive agent h, x_l \forall l, F_{t-1}^l \forall l,
        number of strategic agents L, number of naive agents n, and c_h."""
    return - sum([B[l] @ (E_[l:l+1, :] @ ((x[h] - x[l]) * np.ones((n, 1))) + F_[l]) for l in range(L)]) + c[h]
```

[3]:

```python
def K_t_minus_1(Q, K, E_, R, G_, L, delta):
    return [Q[l] + E_[l:l+1, :].T @ R[l] @ E_[l:l+1, :]
            + delta * G_.T @ K[l] @ G_ for l in range(L)]

def k_t_minus_1(K, k, G_, g, E_, F_, R, L, delta):
    return [2*delta* g[l].T @ K[l] @ G_ + delta * k[l] @ G_
            + 2 * F_[l].T @ R[l] @ E_[l:l+1, :] for l in range(L)]

def kappa_t_minus_1(K, k, kappa, g_, F_, R, L, delta):
    return [-delta * (g_[l].T @ K[l] @ g_[l] + k[l] @ g_[l] - kappa[l])
            - (F_[l].T @ R[l] @ F_[l]) for l in range(L)]
```

[4]:
```python
def solve(K_t, k_t, kappa_t, A, B, delta, n, m, L, Q, R, x, c, tol = 300):
    historical_K = [K_t]
    historical_k = [k_t]
    historical_kappa = [kappa_t]
    max_distances = defaultdict(list)
    counter = 0
    while True:
        M_ = M(K_t, B, R, L, delta)
        H_ = H(B, K_t, A, L)
        E_ = E(M_, H_)
        G_ = G(A, B, E_, L)
        K_new = K_t_minus_1(Q, K_t, E_, R, G_, L, delta)
        F_ = [F(M_, C_l(B, K_t, k_t, l, L, c, x, n), l) for l in range(L)]
        g = [g_l(B, E_, h, x, F_, L) for h in range(L)]
        k_new = k_t_minus_1(K_t, k_t, G_, g, E_, F_, R, L, delta)
        kappa_new = kappa_t_minus_1(K_t, k_t, kappa_t, g, F_, R, L, delta)
        cd_K = [np.max(np.abs(K_t[l] - K_new[l])) for l in range(L)]
        cd_k = [np.max(np.abs(k_t[l] - k_new[l])) for l in range(L)]
        cd_kappa = [np.max(np.abs(kappa_t[l] - kappa_new[l])) for l in range(L)]
        K_t = K_new
        k_t = k_new
        kappa_t = kappa_new
        historical_K.insert(0, K_t)
        historical_k.insert(0, k_t)
        historical_kappa.insert(0, kappa_t)
        for l in range(L):
            max_distances[(l+1, "K")].append(cd_K[l])
            max_distances[(l+1, "k")].append(cd_k[l])
            max_distances[(l+1, "kappa")].append(cd_kappa[l])
        counter += 1
        if sum(cd_K + cd_k + cd_kappa) == 0 or counter > tol:
            return max_distances, historical_K, historical_k, historical_kappa
```

[5]:
```python
def optimal(X_init, historical_K, historical_k, historical_kappa, infinite =␣
 ↪True):
    X_t = [a.copy() for a in X_init]
    xs = defaultdict(list)
    for l in range(L):
        xs[l].append(X_t[l])

    rs = defaultdict(list)
    payoffs = defaultdict(list)
    payoff = defaultdict(lambda: 0)
    i = 0
    while [i < len(historical_K), True][infinite]:
        K_t = historical_K[[i, 0][infinite]]
        k_t = historical_k[[i, 0][infinite]]
```

```python
        M_ = M(K_t, B, R, L, delta)
        H_ = H(B, K_t, A, L)
        E_ = E(M_, H_)
        G_ = G(A, B, E_, L)
        F_ = [F(M_, C_l(B, K_t, k_t, l, L, c, x, n), l) for l in range(L)]
        g = [g_l(B, E_, h, x, F_, L) for h in range(L)]
        for l in range(L):
            Y_new = -1 * E_[l:l+1, :] @ X_t[l] - F(M_, C_l(B, K_t, k_t, l, L,␣
↪c, x, n), l)
            rs[l].append(Y_new)
            payoff[l] += (-1 * delta**i * (X_t[l].T @ Q[l] @ X_t[l])).item() +␣
↪(-1 * delta**i * (Y_new.T @ R[l] @ Y_new)).item()
            payoffs[l].append(payoff[l])
            X_new = G_ @ X_t[l] + g[l]
            xs[l].append(X_new)
            if infinite == True and np.max(X_t[l] - X_new) == 0 and l == L - 1:
                return xs, rs, payoffs
            X_t[l] = X_new
        i += 1

    return xs, rs, payoffs
```

[6]:
```python
def do_plot(rs, r, payoffs, num_agents = 1, set_cap = np.inf, flag = False,␣
↪legend = True):
    fig, sub = plt.subplots(2, sharex=True)
    if legend:
        fig.suptitle(f"Terminal Strategy: {', '.join(['$r_{ss}^' + str(l+1) +␣
↪'$ = ' + str(round(rs[l][:min(len(rs[l]), set_cap)][-1].item() + r[l], 2))␣
↪for l in range(num_agents)])}")

    for l in range(num_agents):
        sub[0].plot(range(min(len(rs[l]), set_cap)), [a.item() + r[l] for a in␣
↪rs[l][:min(len(rs[l]), set_cap)]], label = f"Optimal: {['Agent',␣
↪'Channel'][flag]} {l+1}")
    sub[0].set(ylabel = "r_t message")

    for l in range(num_agents):
        sub[1].plot(range(min(len(payoffs[l]), set_cap)), payoffs[l][:
↪min(len(payoffs[l]), set_cap)], label = f"Optimal: {['Agent',␣
↪'Channel'][flag]} {l+1}")
    sub[1].set(xlabel = "Time", ylabel =  "Cumulative Payoff")
    if legend:
        sub[0].legend()
        sub[1].legend()
    plt.show()
```

[7]:
```python
def optimal_single(num_agents, delta, A, B, R, Q, x):
    K = np.zeros((num_agents, num_agents))
    K_t = [Q]
    K = Q
    while True:
        K_new = delta * (A.T @ (K - (K @ B @ np.linalg.inv((B.T @ K @ B) + R/
↪delta) @ B.T @ K)) @ A) + Q
        K_t.insert(0, K_new)
        current_difference = np.max(np.abs(K - K_new))
        K = K_new
        if current_difference == 0:
            break

    def L_single(K_ent):
        return -1 * np.linalg.inv((B.T @ K_ent @ B) + R/delta) @ B.T @ K_ent @ A

    x_t = x
    r_ts = []

    payoff = 0
    payoffs = []
    x_ts = [x]
    i = 0
    while True:
        r_t = L_single(K_t[0]) @ x_t
        r_ts.append(r_t)
        payoff += (-1 * delta**i * (x_t.T @ Q @ x_t)).item() + (-1 * delta**i *␣
↪(r_t.T @ R @ r_t)).item()
        payoffs.append(payoff)
        x_t_new = A @ x_t + B @ r_t
        x_ts.append(x_t_new)
        if np.max(x_t_new - x_t) == 0:
            break
        x_t = x_t_new
        i += 1

    return r_ts, x_ts, payoffs, K_t
```