Solve_Improved

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0.1 Symbolic solutions and some numerical checking

James Yu, 17 October 2021

[1]:
```
from sympy import *
```

0.2 Part 1: a symbolic proof of the simple targeting second agent result with better equations

[2]:
```
K1, K2, K3, a1, a2 = symbols("K1 K2 K3 a1 a2")

K = Matrix(([[K1, K2], [K2, K3]]))
Q = eye(2)
A = Matrix(([[a1, 1 - a1], [a2, a2]]))
B = Matrix(([[0], [1 - 2*a2]]))
K
```

[2]:

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_3 \end{bmatrix}$$

[3]:
```
A
```

[3]:

$$\begin{bmatrix} a_1 & 1 - a_1 \\ a_2 & a_2 \end{bmatrix}$$

[4]:
```
B
```

[4]:

$$\begin{bmatrix} 0 \\ 1 - 2a_2 \end{bmatrix}$$

If we plug these into the equation directly, we obtain:

[5]:
```
K_sol = simplify(A.T *(K - K*B*(B.T * K * B).inv() * B.T * K) * A + Q)
K_sol
```

[5]:

$$\begin{bmatrix} K_1 a_1^2 - \frac{K_2^2 a_1^2}{K_3} + 1 & \frac{a_1\left(-K_1 K_3 a_1 + K_1 K_3 + K_2^2 a_1 - K_2^2\right)}{K_3} \\ \frac{a_1\left(-K_1 K_3 a_1 + K_1 K_3 + K_2^2 a_1 - K_2^2\right)}{K_3} & K_1 a_1^2 - 2K_1 a_1 + K_1 - \frac{K_2^2 a_1^2}{K_3} + \frac{2K_2^2 a_1}{K_3} - \frac{K_2^2}{K_3} + 1 \end{bmatrix}$$

This leads to a system of three equations in three unknowns because of the symmetry, so we need to solve these all at the same time.

$$K_1 = (a_1 - 0)^2(K_1 - \frac{K_2^2}{K_3}) + 1$$

$$K_3 = (a_1 - 1)^2(K_1 - \frac{K_2^2}{K_3}) + 1$$

$$K_2K_3 = (K_1K_3 - K_2^2)a_1(1 - a_1)$$

Notice that $K_1K_3 - K_2^2$ is the determinant of K.

From here, we can use the common terms in the first two expressions to say:

$$\frac{K_1 - 1}{(a_1 - 0)^2} = \frac{K_3 - 1}{(a_1 - 1)^2}$$

[6]:
```
LHS = (K1 - 1)/(a1**2)
RHS = (K3 - 1)/((a1 - 1)**2)
K1_of_K3 = simplify(expand(solve(LHS - RHS, K1)[0]))
K1_of_K3
```

[6]: $$\frac{K_3a_1^2 - 2a_1 + 1}{a_1^2 - 2a_1 + 1}$$

We can also solve for K_1 in the first expression.

[7]:
```
K1_of_K1K2K3 = (a1**2)*(K1 - (K2**2)/K3) + 1
K1_of_K2K3 = solve(K1_of_K1K2K3 - K1, K1)[0]
K1_of_K2K3
```

[7]: $$\frac{K_2^2a_1^2 - K_3}{K_3\left(a_1^2 - 1\right)}$$

and then equate these to solve for K_2^2:

[8]:
```
K2squared_of_K3 = solve(K1_of_K3 - K1_of_K2K3, K2**2)[0]
K2squared_of_K3
```

[8]: $$\frac{K_3\left(K_3a_1 + K_3 - 2\right)}{a_1 - 1}$$

[9]:
```
K2K3_temp = (K1*K3 - K2**2)*a1*(1-a1)
K2K3 = simplify(K2K3_temp.subs(K2**2, K2squared_of_K3))
K2K3
```

[9]: $$K_3a_1\left(-K_1a_1 + K_1 + K_3a_1 + K_3 - 2\right)$$

The above can actually be further simplified to the expression:

$$K_2 = a_1(K_1(1 - a_1) + K_3(1 + a_1) - 2)$$

[10]:
```
K2_of_K3 = K2K3.subs(K1, K1_of_K3) / K3
simplify(K2_of_K3)
```

[10]:
$$\frac{a_1(1 - K_3)}{a_1 - 1}$$

This is a surprisingly nice expression. We now have expressions relating K_3 to K_2 and K_1 to K_3.

This system is:

$$K_2 = \frac{a_1(1 - K_3)}{a_1 - 1}$$

$$K_1 = \frac{K_3 a_1^2 - 2a_1 + 1}{(a_1 - 1)^2}$$

The last step is using the second of the original three expressions to solve for K_3.

[11]:
```
K3_of_K1K2K3 = ((a1 - 1)**2)*(K1 - (K2**2)/K3) + 1
K3_tosolve = simplify(K3_of_K1K2K3.subs(K1, K1_of_K3).subs(K2, K2_of_K3))
K3_tosolve
```

[11]:
$$2a_1^2 - 2a_1 + 2 - \frac{a_1^2}{K_3}$$

[12]:
```
possibly_K3 = solve(K3_tosolve - K3, K3)[0]
possibly_K3
```

[12]:
$$a_1^2 - a_1 + (1 - a_1)\sqrt{a_1^2 + 1} + 1$$

[13]:
```
possibly_not_K3 = solve(K3_tosolve - K3, K3)[1]
possibly_not_K3
```

[13]:
$$a_1^2 - a_1 + (a_1 - 1)\sqrt{a_1^2 + 1} + 1$$

K_2 if K_3 is positive:

[14]:
```
possibly_K2 = simplify(K2_of_K3.subs(K3, possibly_K3))
possibly_K2
```

[14]:
$$a_1\left(-a_1 + \sqrt{a_1^2 + 1}\right)$$

K_1 if K_3 is positive:

[15]:
```
possibly_K1 = simplify(K1_of_K3.subs(K3, possibly_K3))
possibly_K1
```

[15]:
$$\frac{a_1^3 - a_1^2\sqrt{a_1^2 + 1} + a_1 - 1}{a_1 - 1}$$

K_2 if K_3 is negative:

[16]:
```
possibly_not_K2 = simplify(K2_of_K3.subs(K3, possibly_not_K3))
possibly_not_K2
```

[16]:

$$-a_1\left(a_1+\sqrt{a_1^2+1}\right)$$

K_1 if K_3 is negative:

[17]:
```
possibly_not_K1 = simplify(K1_of_K3.subs(K3, possibly_not_K3))
possibly_not_K1
```

[17]:

$$\frac{a_1^3+a_1^2\sqrt{a_1^2+1}+a_1-1}{a_1-1}$$

The next question to answer is whether K has a positive or negative K_2. Recall that we start with $K_T = Q = cI$ meaning the diagonal is positive and the off-diagonal is zero.

What follows is a proof by induction using $K_T = Q$ as the base case:

[18]:
```
K_TM1 = simplify(A.T *(Q - Q*B*(B.T * Q * B).inv() * B.T * Q) * A + Q)
K_TM1
```

[18]:

$$\begin{bmatrix} a_1^2+1 & a_1(1-a_1) \\ a_1(1-a_1) & a_1^2-2a_1+2 \end{bmatrix}$$

K_1 must be positive and K_2 must be non-negative, and K_3 simplifies to $(a_1-1)^2+1$ which is also positive.

Also, $K_2 < K_1$ and $K_2 < K_3$ holds since we know $K_2 < 1$, $K_1 \geq 1$, $K_3 \geq 1$ all hold. So $det(K) > 0$.

Now, the induction step. Suppose K_t has a positive diagonal and non-negative off-diagonal such that $det(K_t) > 0$.

$$K_{1t-1} = (a_1-0)^2(K_{1t} - \frac{K_{2t}^2}{K_{3t}}) + 1$$

We want to determine if $K_1 > K_2^2/K_3$. Since we know $det(K_t) > 0$, it follows that $K_1K_3 > K_2^2$. Thus, since $K_3 > 0$, we know $K_1 > K_2^2/K_3$, so $K_{1t-1} > 0$.

$$K_{3t-1} = (a_1-1)^2(K_{1t} - \frac{K_{2t}^2}{K_{3t}}) + 1$$

The same conclusion results in $K_{3t-1} > 0$.

$$K_{2t-1} = \frac{(K_{1t}K_{3t} - K_{2t}^2)a_1(1-a_1)}{K_{3t}}$$

Finally, the same conclusion, combined with $K_3 > 0$, implies $K_2 \geq 0$.

Now, we wish to check if $det(K_{t-1}) > 0$. This requires checking $K_1K_3 > K_2^2$ in K_t.

[19]:
```
K1tm1 = a1**2 * (K1 - K2**2 / K3) + 1
K1tm1
```

[19]:

$$a_1^2\left(K_1 - \frac{K_2^2}{K_3}\right) + 1$$

[20]:
```
K3tm1 = (a1-1)**2 * (K1 - K2**2 / K3) + 1
K3tm1
```

[20]:

$$\left(K_1 - \frac{K_2^2}{K_3}\right)(a_1 - 1)^2 + 1$$

[21]:
```
EXPR1 = simplify(K1tm1 * K3tm1)
EXPR1
```

[21]:

$$\frac{\left(K_3 + a_1^2\left(K_1K_3 - K_2^2\right)\right)\left(K_3 + (a_1 - 1)^2\left(K_1K_3 - K_2^2\right)\right)}{K_3^2}$$

[22]:
```
K2tm1 = (K1 * K3 - K2**2)*a1*(1-a1)/K3
K2tm1
```

[22]:

$$\frac{a_1(1 - a_1)\left(K_1K_3 - K_2^2\right)}{K_3}$$

[23]:
```
EXPR2 = K2tm1 * K2tm1
EXPR2
```

[23]:

$$\frac{a_1^2(1 - a_1)^2\left(K_1K_3 - K_2^2\right)^2}{K_3^2}$$

Since $K_3 > 0$ we just need to compare the numerators.

[24]:
```
DETERMINANT_SIGNED = expand(fraction(EXPR1)[0]) - expand(fraction(EXPR2)[0])
DETERMINANT_SIGNED
```

[24]:

$$2K_1K_3^2a_1^2 - 2K_1K_3^2a_1 + K_1K_3^2 - 2K_2^2K_3a_1^2 + 2K_2^2K_3a_1 - K_2^2K_3 + K_3^2$$

The above is EXPR1, which is $K_1 * K_3$, minus EXPR2, which is K_2^2, so we have $K_1K_3 - K_2^2$. If this is positive, then the determinant is positive.

[25]:
```
simplify(DETERMINANT_SIGNED / K3)
```

[25]:

$$2K_1K_3a_1^2 - 2K_1K_3a_1 + K_1K_3 - 2K_2^2a_1^2 + 2K_2^2a_1 - K_2^2 + K_3$$

Rearranging this expression gives

$$K_1K_3a_1^2 + K_1K_3(a_1^2 - 2a_1 + 1) - (2K_2^2a_1^2 - 2K_2a_1 + K_2^2) + K_3$$

which is

$$K_1K_3a_1^2 + K_1K_3(a_1-1)^2 - K_2^2a_1^2 - K_2^2(a_1-1)^2 + K_3$$

which simplifies to

$$a_1^2(K_1K_3 - K_2^2) + (a_1-1)^2(K_1K_3 - K_2^2) + K_3$$

which is positive, since $K_1K_3 - K_2^2 > 0$ is given. Thus, $det(K_{t-1}) > 0$, and the proof is complete.

It follows that $K_1 > 0, K_2 \geq 0, K_3 > 0$ holds for all t.

Thus, we take the $K_3 > 0$ case.

Note that $K_2 = 0$ only happens when $a_1 = 1$ or $a_1 = 0$, in which case the system is not fully connected. Otherwise, $K_2 > 0$ holds by the above process.

We can also double-check that the expression is correct:

[26]:
```
K_steady = Matrix(([[possibly_K1, possibly_K2], [possibly_K2, possibly_K3]]))
K_steady
```

[26]:

$$\begin{bmatrix} \frac{a_1^3 - a_1^2\sqrt{a_1^2+1} + a_1 - 1}{a_1 - 1} & a_1\left(-a_1 + \sqrt{a_1^2+1}\right) \\ a_1\left(-a_1 + \sqrt{a_1^2+1}\right) & a_1^2 - a_1 + (1-a_1)\sqrt{a_1^2+1} + 1 \end{bmatrix}$$

[27]:
```
K_steady.subs(a1, 0.5)
```

[27]:

$$\begin{bmatrix} 1.30901699437495 & 0.309016994374947 \\ 0.309016994374947 & 1.30901699437495 \end{bmatrix}$$

[28]:
```
K_check = simplify(A.T *(K_steady - K_steady*B*(B.T * K_steady * B).inv() * B.T␣
↪* K_steady) * A + Q)
K_check
```

[28]:

$$\begin{bmatrix} \frac{2a_1^5-2a_1^4\sqrt{a_1^2+1}-2a_1^4+2a_1^3\sqrt{a_1^2+1}+3a_1^3-2a_1^2\sqrt{a_1^2+1}-3a_1^2+2a_1\sqrt{a_1^2+1}+2a_1-\sqrt{a_1^2+1}-1}{a_1^3-a_1^2\sqrt{a_1^2+1}-2a_1^2+2a_1\sqrt{a_1^2+1}+2a_1-\sqrt{a_1^2+1}-1} & \frac{a_1(-2a_1^3+2a_1^2\sqrt{a_1^2+1}+2a_1^2-2a_1\sqrt{a_1^2+1}-2a_1}{a_1^2-a_1\sqrt{a_1^2+1}-a_1+\sqrt{a_1^2+1}+1} \\ \frac{a_1\left(-2a_1^3+2a_1^2\sqrt{a_1^2+1}+2a_1^2-2a_1\sqrt{a_1^2+1}-2a_1+\sqrt{a_1^2+1}+1\right)}{a_1^2-a_1\sqrt{a_1^2+1}-a_1+\sqrt{a_1^2+1}+1} & \frac{2a_1^4-2a_1^3\sqrt{a_1^2+1}-4a_1^3+4a_1^2\sqrt{a_1^2+1}+5a_1^2-4a_1\sqrt{a_1^2+}}{a_1^2-a_1\sqrt{a_1^2+1}-a_1+\sqrt{a_1^2+1}+1} \end{bmatrix}$$

[29]:
```
K_check.subs(a1, 0.5)
```

[29]:

$$\begin{bmatrix} 1.30901699437495 & 0.309016994374947 \\ 0.309016994374947 & 1.30901699437495 \end{bmatrix}$$

[30]:
```
import numpy as np
for res in np.linspace(0, 0.99999999, 500):
    print(np.allclose(np.array(K_steady.subs(a1, res)).astype(np.float64), np.
↪array(K_check.subs(a1, res)).astype(np.float64)), end = " ")
```

```
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True True True True
```

This says that, for 500 evenly-spaced values of a_1 from 0 to about 1 (if its exactly 1, the system doesn't work because the stubborn agent is fixed), with the distance between values being 1/500, the solutions are always identical. This on its own is not a proof that this really is the steady-state, but it means there were no significant typos in the derivation of the steady-state matrix, so barring any other small typos, the answer should be correct.

0.2.1 Part 2: what about targeting the stubborn agent? Why is it positive?

[31]:
```
K1, K2, K3, a11, a12 = symbols("K1 K2 K3 a11 a12")

K = Matrix(([[K1, K2], [K2, K3]]))
Q = eye(2)
A = Matrix(([[a11, a12], [0.5, 0.5]]))
B = Matrix(([[1 - a11 - a12], [0]]))
K
```

[31]: $\begin{bmatrix} K_1 & K_2 \\ K_2 & K_3 \end{bmatrix}$

[32]:
```
A
```

[32]: $\begin{bmatrix} a_{11} & a_{12} \\ 0.5 & 0.5 \end{bmatrix}$

[33]:
```
B
```

[33]: $\begin{bmatrix} -a_{11} - a_{12} + 1 \\ 0 \end{bmatrix}$

[34]:
```
K_sol = simplify(A.T *(K - (K*B*(B.T * K * B).inv() * B.T * K)) * A + Q)
K_sol
```

[34]: $\begin{bmatrix} 0.25K_3 + 1.0 - \frac{0.25K_2^2}{K_1} & 0.25K_3 - \frac{0.25K_2^2}{K_1} \\ 0.25K_3 - \frac{0.25K_2^2}{K_1} & 0.25K_3 + 1.0 - \frac{0.25K_2^2}{K_1} \end{bmatrix}$

So clearly here $K_1 = K_3 = K_2 + 1$.

[35]:
```
K2_of_K2 = 0.25*(K2+1) - (0.25 * K2**2)/(K2+1)
K2_of_K2
```

[35]: $$-\frac{0.25K_2^2}{K_2 + 1} + 0.25K_2 + 0.25$$

[36]:
```
solve(K2_of_K2 - K2, K2)
```

[36]:
```
[-0.809016994374947, 0.309016994374947]
```

These are the two solutions we saw in a previous notebook and now we wish to know why K_2 should be the solution on the right. Recall that $K_T = Q$.

Then:

[37]:
```
simplify(A.T *(Q - (Q*B*(B.T * Q * B).inv() * B.T * Q)) * A + Q)
```

[37]: $\begin{bmatrix} 1.25 & 0.25 \\ 0.25 & 1.25 \end{bmatrix}$

This is clearly a positive matrix with a positive determinant. If we then use a proof by induction here, assume we start with a fully positive K as above such that the determinant is positive.

[38]:
```
K_sol[1, 0]
```

[38]: $$0.25K_3 - \frac{0.25K_2^2}{K_1}$$

This is the expression for K_2 at $t-1$ as a function of K at t. This simplifies to $0.25(K_3 - K_2^2/K_1)$ and since we know $K_1 > 0$, we check the sign of $K_1K_3 - K_2^2$, which is the determinant, which is positive, so $K_2 > 0$. Since $K_1 = K_3 = K_2 + 1$, it follows that $K_1 > 0$ and $K_3 > 0$ too.

Finally, $K_1K_3 - K_2^2 = (K_2+1)^2 - K_2^2$ and since $K_2 > 1$, this is always going to be positive, so the determinant is always positive (this holds without needing to use induction). Thus, for all t, $det(K) > 0$ and all entries of K are positive. So we take the positive K_2.

0.2.2 Part 3: robustness through adding a cost term R

[39]:
```python
R = Matrix(([[symbols("R")]])) # this is a scalar written as a 1x1 matrix so␣
↪Python is happy with it
K_sol = simplify(A.T *(K - (K*B*(B.T * K * B + R).inv() * B.T * K)) * A + Q)
K_sol[0, 0]
```

[39]:
$$\frac{0.25K_1K_3a_{11}^2 + 0.5K_1K_3a_{11}a_{12} - 0.5K_1K_3a_{11} + 0.25K_1K_3a_{12}^2 - 0.5K_1K_3a_{12} + 0.25K_1K_3 + 1.0K_1Ra_{11}^2 + 1.0K_1a_1^2}{1.0K_1a_{11}^2 +}$$

The cost term increases the symbolic complexity severely.

[40]:
```python
K_sol[1, 1]
```

[40]:
$$\frac{0.25K_1K_3a_{11}^2 + 0.5K_1K_3a_{11}a_{12} - 0.5K_1K_3a_{11} + 0.25K_1K_3a_{12}^2 - 0.5K_1K_3a_{12} + 0.25K_1K_3 + 1.0K_1Ra_{12}^2 + 1.0K_1a_1^2}{1.0K_1a_{11}^2 +}$$

These (K_1 and K_3, respectively), are also no longer the same value.

[41]:
```python
K_sol[1, 0]
```

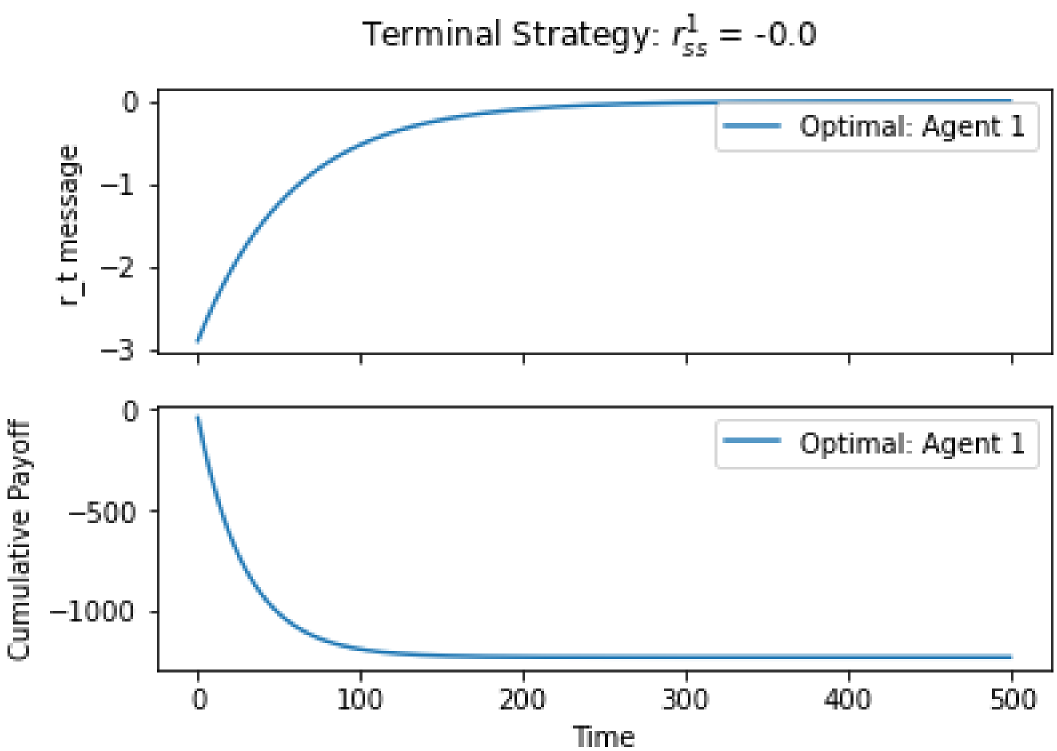
[41]:
$$\frac{0.25K_1K_3a_{11}^2 + 0.5K_1K_3a_{11}a_{12} - 0.5K_1K_3a_{11} + 0.25K_1K_3a_{12}^2 - 0.5K_1K_3a_{12} + 0.25K_1K_3 + 1.0K_1Ra_{11}a_{12} - 0.25K}{1.0K_1a_{11}^2 + 2.0K_1a_{11}a_{12} - 2.0K_1a_{11} + 1.0}$$

For now, I explore this numerically instead:

[42]:
```python
from collections import defaultdict
import matplotlib.pyplot as plt

def do_plot(rs, r, payoffs, num_agents = 1, set_cap = np.inf, flag = False,␣
↪legend = True):
    fig, sub = plt.subplots(2, sharex=True)
    if legend:
        fig.suptitle(f"Terminal Strategy: {', '.join(['$r_{ss}^' + str(l+1) +␣
↪'$ = ' + str(round(rs[l][:min(len(rs[l]), set_cap)][-1].item() + r[l], 2))␣
↪for l in range(num_agents)])}")

    for l in range(num_agents):
        sub[0].plot(range(min(len(rs[l]), set_cap)), [a.item() + r[l] for a in␣
↪rs[l][:min(len(rs[l]), set_cap)]], label = f"Optimal: {['Agent',␣
↪'Channel'][flag]} {l+1}")
    sub[0].set(ylabel = "r_t message")

    for l in range(num_agents):
```

```
        sub[1].plot(range(min(len(payoffs[l]), set_cap)), payoffs[l][:
↪min(len(payoffs[l]), set_cap)], label = f"Optimal: {['Agent',␣
↪'Channel'][flag]} {l+1}")
    sub[1].set(xlabel = "Time", ylabel =  "Cumulative Payoff")
    if legend:
        sub[0].legend()
        sub[1].legend()
    plt.show()

A = np.array([
  [0.99 * 0.99, 0.01 * 0.99],
  [0.5, 0.5],
], ndmin = 2)

B = np.array([
  [0.01],
  [0]
], ndmin = 2)

delta = 1
Q = 1 * np.identity(2)
x = np.array([
  [4],
  [4]
], ndmin = 2)

K = np.zeros((2, 2))
K_t = [Q]
K = Q

R = 1

while True:
    K_new = delta * (A.T @ (K - (K @ B @ np.linalg.inv(B.T @ K @ B + R) @ B.T @␣
↪K)) @ A) + Q
    K_t.insert(0, K_new)
    current_difference = np.max(np.abs(K - K_new))
    K = K_new
    if current_difference < 10**(-14):
        break

def L_single(K_ent):
    return -1 * np.linalg.inv(B.T @ K_ent @ B + R) @ B.T @ K_ent @ A

x_t = x
r_ts = []
```

```
payoff = 0
payoffs = []
x_ts = [x]
i = 0
while True:
    r_t = L_single(K_t[0]) @ x_t
    r_ts.append(r_t)
    payoff += (-1 * delta**i * (x_t.T @ Q @ x_t)).item() + (-1 * delta**i *␣
 ↪R*r_t*r_t).item()
    payoffs.append(payoff)
    x_t_new = A @ x_t + B @ r_t
    x_ts.append(x_t_new)
    if np.max((x_t_new - x_t)**2) == 0:
        break
    x_t = x_t_new
    i += 1


do_plot({0:r_ts}, [0], {0:payoffs}, num_agents = 1, set_cap = 500)
```



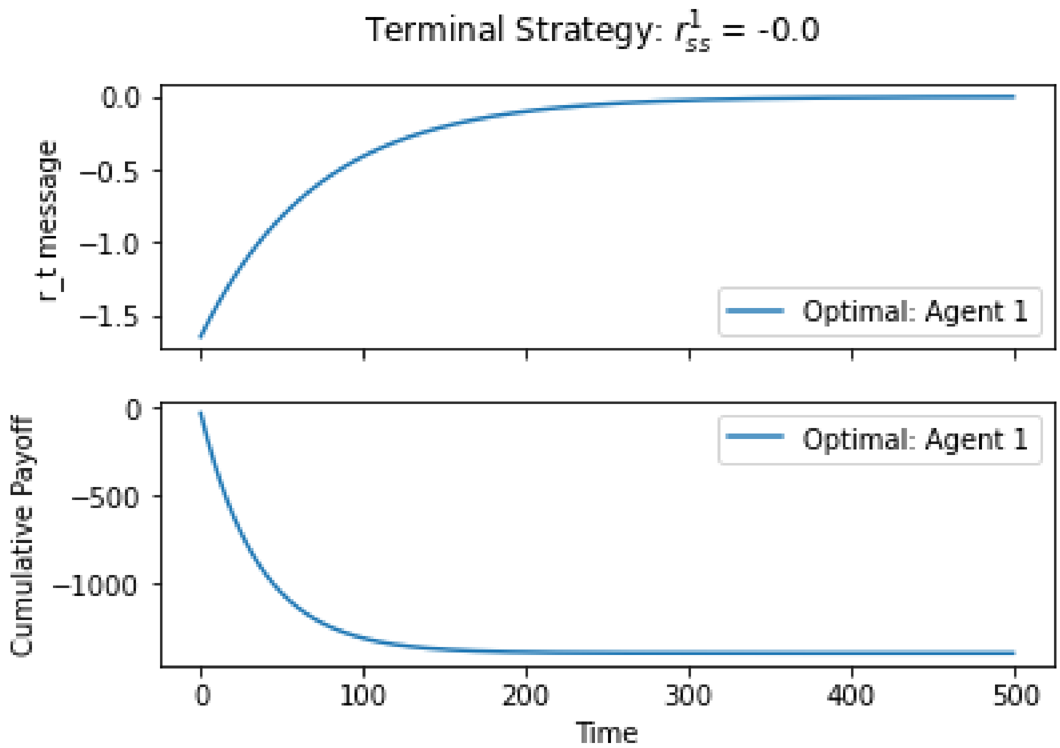
So here $R = 1$ targeting the stubborn agent. K in the steady-state is:

[43]:
```
K_t[0]
```

[43]: array([[71.79273815, 2.03297403],
       [2.03297403, 1.36915552]])

This does not have $K_1 = K_3$, but is still fully positive with positive determinant.

[44]:
```
K = np.zeros((2, 2))
K_t = [Q]
K = Q
R = 2

while True:
    K_new = delta * (A.T @ (K - (K @ B @ np.linalg.inv(B.T @ K @ B + R) @ B.T @␣
↪K)) @ A) + Q
    K_t.insert(0, K_new)
    current_difference = np.max(np.abs(K - K_new))
    K = K_new
    if current_difference < 10**(-14):
        break

x_t = x
r_ts = []

payoff = 0
payoffs = []
x_ts = [x]
i = 0
while True:
    r_t = L_single(K_t[0]) @ x_t
    r_ts.append(r_t)
    payoff += (-1 * delta**i * (x_t.T @ Q @ x_t)).item() + (-1 * delta**i *␣
↪R*r_t*r_t).item()
    payoffs.append(payoff)
    x_t_new = A @ x_t + B @ r_t
    x_ts.append(x_t_new)
    if np.max((x_t_new - x_t)**2) == 0:
        break
    x_t = x_t_new
    i += 1


do_plot({0:r_ts}, [0], {0:payoffs}, num_agents = 1, set_cap = 500)
```

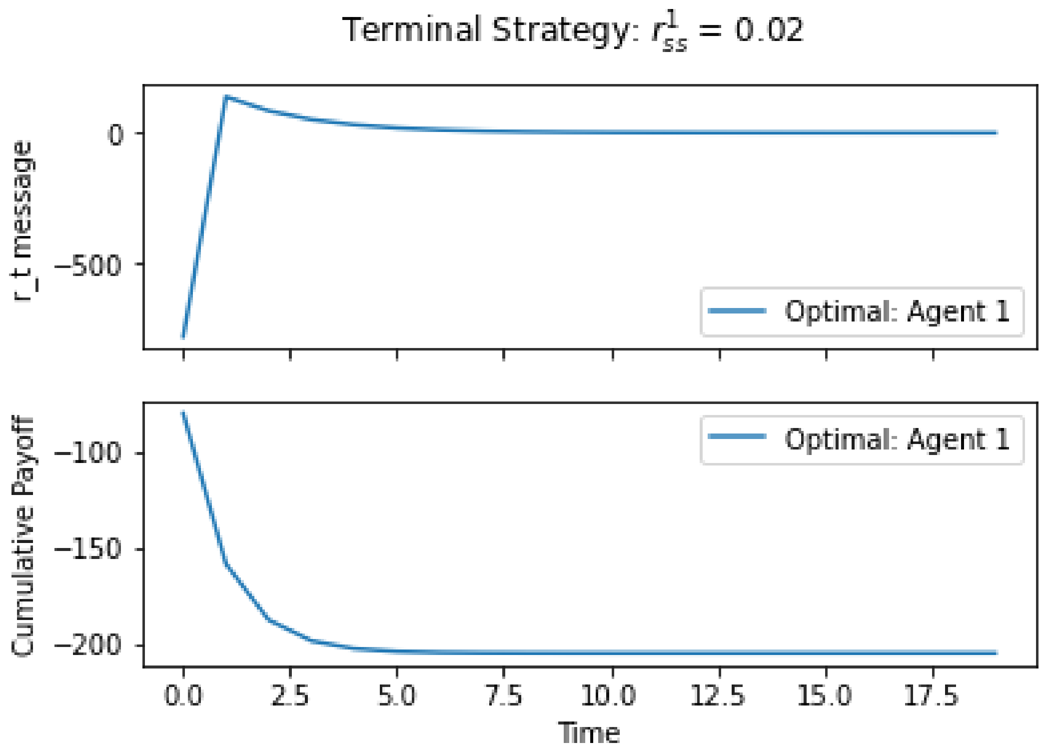


Increasing the cost brings the messages closer to zero and, compared to the $R = 0$ case from one of the previous notebooks, has longer convergence time.

0.2.3 Part 4: robustness through adding duplicate naive agents

[45]:
```
A = np.array([
  [0.99 * 0.99, 0.0025 * 0.99, 0.0025 * 0.99, 0.0025 * 0.99, 0.0025 * 0.99],
  [0.2, 0.2, 0.2, 0.2, 0.2],
  [0.2, 0.2, 0.2, 0.2, 0.2],
  [0.2, 0.2, 0.2, 0.2, 0.2],
  [0.2, 0.2, 0.2, 0.2, 0.2],
], ndmin = 2)

B = np.array([
  [0.01],
  [0],
  [0],
  [0],
  [0]
], ndmin = 2)

delta = 1
Q = 1 * np.identity(5)
```

```
x = np.array([
  [4],
  [4],
  [4],
  [4],
  [4]
], ndmin = 2)

K = np.zeros((5, 5))
K_t = [Q]
K = Q

while True:
    K_new = delta * (A.T @ (K - (K @ B @ np.linalg.inv(B.T @ K @ B) @ B.T @ K))
 ↪@ A) + Q
    K_t.insert(0, K_new)
    current_difference = np.max(np.abs(K - K_new))
    K = K_new
    if current_difference < 10**(-14):
        break

def L_single(K_ent):
    return -1 * np.linalg.inv(B.T @ K_ent @ B) @ B.T @ K_ent @ A

x_t = x
r_ts = []

payoff = 0
payoffs = []
x_ts = [x]
i = 0
while True:
    r_t = L_single(K_t[0]) @ x_t
    r_ts.append(r_t)
    payoff += (-1 * delta**i * (x_t.T @ Q @ x_t)).item()
    payoffs.append(payoff)
    x_t_new = A @ x_t + B @ r_t
    x_ts.append(x_t_new)
    if np.max((x_t_new - x_t)**2) == 0:
        break
    x_t = x_t_new
    i += 1


do_plot({0:r_ts}, [0], {0:payoffs}, num_agents = 1, set_cap = 20)
```



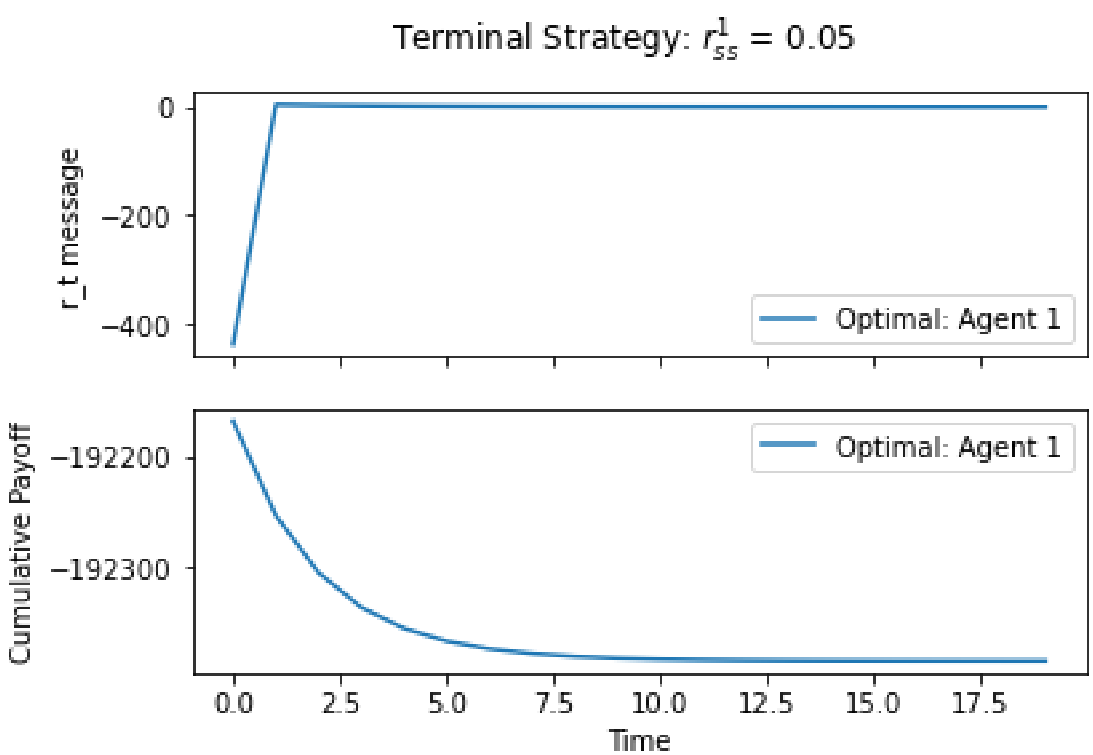
[46]:
```
K_t[0]
```

[46]:
```
array([[1.31231056, 0.31231056, 0.31231056, 0.31231056, 0.31231056],
       [0.31231056, 1.31231056, 0.31231056, 0.31231056, 0.31231056],
       [0.31231056, 0.31231056, 1.31231056, 0.31231056, 0.31231056],
       [0.31231056, 0.31231056, 0.31231056, 1.31231056, 0.31231056],
       [0.31231056, 0.31231056, 0.31231056, 0.31231056, 1.31231056]])
```

Note that this is very very similar to the case with only one easily-influenced agent. This case uses $R = 0$. The big difference is the messages are further away from zero.

[47]:
```
K = np.zeros((5, 5))
K_t = [Q]
K = Q
R = 1

while True:
    K_new = delta * (A.T @ (K - (K @ B @ np.linalg.inv(B.T @ K @ B + R) @ B.T @␣
 ↪K)) @ A) + Q
    K_t.insert(0, K_new)
    current_difference = np.max(np.abs(K - K_new))
    K = K_new
    if current_difference < 10**(-14):
```

```
        break

x_t = x
r_ts = []

payoff = 0
payoffs = []
x_ts = [x]
i = 0
while True:
    r_t = L_single(K_t[0]) @ x_t
    r_ts.append(r_t)
    payoff += (-1 * delta**i * (x_t.T @ Q @ x_t)).item() + (-1 * delta**i *␣
 ↪R*r_t*r_t).item()
    payoffs.append(payoff)
    x_t_new = A @ x_t + B @ r_t
    x_ts.append(x_t_new)
    if np.max((x_t_new - x_t)**2) == 0:
        break
    x_t = x_t_new
    i += 1


do_plot({0:r_ts}, [0], {0:payoffs}, num_agents = 1, set_cap = 20)
```



[48]:
```
r_ts[0].item()**2
```

[48]:
```
192088.66466466917
```

This is with $R = 1$. Payoff is extremely bad because of the extremely large $t = 0$ message being factored into quadratic cost.

[49]:
```
print(K_t[0])
```

```
[[132.55617878   3.50273146   3.50273146   3.50273146   3.50273146]
 [  3.50273146   1.48254381   0.48254381   0.48254381   0.48254381]
 [  3.50273146   0.48254381   1.48254381   0.48254381   0.48254381]
 [  3.50273146   0.48254381   0.48254381   1.48254381   0.48254381]
 [  3.50273146   0.48254381   0.48254381   0.48254381   1.48254381]]
```

There's some extra structural symmetry in the matrix.