SympyDemo

August 21, 2021

[98]:
```
from sympy import *

Q = eye(2)
A = Matrix(([[symbols("a11"), symbols("a12")], [symbols("a21"), 1 -␣
↪symbols("a21")]]))
B = Matrix(([[1 - symbols("a11") - symbols("a12")], [0]]))
```

First, a symbolic model of the first three rounds of iterating the Riccati equation. Here $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 1 - a_{21} \end{bmatrix}$ and $B = \begin{bmatrix} 1 - a_{11} - a_{12} \\ 0 \end{bmatrix}$ such that the sum of the rows of A and B are all 1. Also, let $\delta = 1$ and $R = 0$ for now, and let $Q = I$ as in the note.

[104]:
```
K1 = simplify(A.T *(Q - Q*B*(B.T * Q * B).inv() * B.T * Q.T) * A + Q)
K1
```

[104]:
$$\begin{bmatrix} a_{21}^2 + 1 & a_{21}(1 - a_{21}) \\ a_{21}(1 - a_{21}) & a_{21}^2 - 2a_{21} + 2 \end{bmatrix}$$

[107]:
```
K2 = A.T *(K1 - K1*B*(B.T * K1 * B).inv() * B.T * K1.T) * A + Q
simplify(K2)
```

[107]:
$$\begin{bmatrix} \frac{2a_{21}^4 - 2a_{21}^3 + 3a_{21}^2 + 1}{a_{21}^2 + 1} & \frac{2a_{21}(-a_{21}^3 + 2a_{21}^2 - 2a_{21} + 1)}{a_{21}^2 + 1} \\ \frac{2a_{21}(-a_{21}^3 + 2a_{21}^2 - 2a_{21} + 1)}{a_{21}^2 + 1} & \frac{2a_{21}^4 - 6a_{21}^3 + 9a_{21}^2 - 6a_{21} + 3}{a_{21}^2 + 1} \end{bmatrix}$$

[109]:
```
K3 = A.T *(K2 - K2*B*(B.T * K2 * B).inv() * B.T * K2.T) * A + Q
simplify(K3)
```

[109]:
$$\begin{bmatrix} \frac{4a_{21}^6 - 8a_{21}^5 + 13a_{21}^4 - 8a_{21}^3 + 6a_{21}^2 + 1}{2a_{21}^4 - 2a_{21}^3 + 3a_{21}^2 + 1} & \frac{a_{21}(-4a_{21}^5 + 12a_{21}^4 - 19a_{21}^3 + 17a_{21}^2 - 9a_{21} + 3)}{2a_{21}^4 - 2a_{21}^3 + 3a_{21}^2 + 1} \\ \frac{a_{21}(-4a_{21}^5 + 12a_{21}^4 - 19a_{21}^3 + 17a_{21}^2 - 9a_{21} + 3)}{2a_{21}^4 - 2a_{21}^3 + 3a_{21}^2 + 1} & \frac{4a_{21}^6 - 16a_{21}^5 + 33a_{21}^4 - 38a_{21}^3 + 29a_{21}^2 - 12a_{21} + 4}{2a_{21}^4 - 2a_{21}^3 + 3a_{21}^2 + 1} \end{bmatrix}$$

Now, if instead I explicitly set B to be $\begin{bmatrix} b \\ 0 \end{bmatrix}$, the result is:

[110]:
```
Q = eye(2)
A = Matrix(([[symbols("a11"), symbols("a12")], [symbols("a21"), 1 -␣
↪symbols("a21")]]))
B = Matrix(([[symbols("b")], [0]]))
```

[111]:
```
K1 = simplify(A.T *(Q - Q*B*(B.T * Q * B).inv() * B.T * Q.T) * A + Q)
K1
```

[111]:

$$\begin{bmatrix} a_{21}^2+1 & a_{21}(1-a_{21}) \\ a_{21}(1-a_{21}) & (a_{21}-1)^2+1 \end{bmatrix}$$

Clearly, no change to K_{T-1} because B cancels out (due to no R cost). The next period:

[112]:
```
K2 = A.T *(K1 - K1*B*(B.T * K1 * B).inv() * B.T * K1.T) * A + Q
simplify(K2)
```

[112]:

$$\begin{bmatrix} \frac{2a_{21}^4-2a_{21}^3+3a_{21}^2+1}{a_{21}^2+1} & \frac{2a_{21}\left(-a_{21}^3+2a_{21}^2-2a_{21}+1\right)}{a_{21}^2+1} \\ \frac{2a_{21}\left(-a_{21}^3+2a_{21}^2-2a_{21}+1\right)}{a_{21}^2+1} & \frac{2a_{21}^4-6a_{21}^3+9a_{21}^2-6a_{21}+3}{a_{21}^2+1} \end{bmatrix}$$

[113]:
```
B.T * K1 * B
```

[113]:

$$\left[b^2\left(a_{21}^2+1\right)\right]$$

This is, from my perspective, surprising, because it means B does not factor into K_t at all here. For example, K_{T-3} is:

[115]:
```
K3 = A.T *(K2 - K2*B*(B.T * K2 * B).inv() * B.T * K2.T) * A + Q
simplify(K3)
```

[115]:

$$\begin{bmatrix} \frac{4a_{21}^6-8a_{21}^5+13a_{21}^4-8a_{21}^3+6a_{21}^2+1}{2a_{21}^4-2a_{21}^3+3a_{21}^2+1} & \frac{a_{21}\left(-4a_{21}^5+12a_{21}^4-19a_{21}^3+17a_{21}^2-9a_{21}+3\right)}{2a_{21}^4-2a_{21}^3+3a_{21}^2+1} \\ \frac{a_{21}\left(-4a_{21}^5+12a_{21}^4-19a_{21}^3+17a_{21}^2-9a_{21}+3\right)}{2a_{21}^4-2a_{21}^3+3a_{21}^2+1} & \frac{4a_{21}^6-16a_{21}^5+33a_{21}^4-38a_{21}^3+29a_{21}^2-12a_{21}+4}{2a_{21}^4-2a_{21}^3+3a_{21}^2+1} \end{bmatrix}$$

Note that K_{T-2} and K_{T-3} with the generic B are identical to the version where we explicitly defined B. This means that, in this 2 by 2 case, B simply isn't a factor.

Suppose we expanded to 3 by 3. Then:

[117]:
```
Q = eye(3)
A =␣
↪Matrix(([[symbols("a11"),symbols("a12"),symbols("a13")],[symbols("a21"),symbols("a22"),1-sy
B = Matrix(([[symbols("b")], [0], [0]]))
```

[118]:
```
K1 = simplify(A.T *(Q - Q*B*(B.T * Q * B).inv() * B.T * Q.T) * A + Q)
K1
```

[118]:

$$\begin{bmatrix} a_{21}^2+a_{31}^2+1 & a_{21}a_{22}+a_{31}a_{32} & -a_{21}(a_{21}+a_{22}-1)-a \\ a_{21}a_{22}+a_{31}a_{32} & a_{22}^2+a_{32}^2+1 & -a_{22}(a_{21}+a_{22}-1)-a \\ -a_{21}(a_{21}+a_{22}-1)-a_{31}(a_{31}+a_{32}-1) & -a_{22}(a_{21}+a_{22}-1)-a_{32}(a_{31}+a_{32}-1) & (a_{21}+a_{22}-1)^2+(a_{31} \end{bmatrix}$$

This is of course more complicated, but still no B. K_{T-2} is:

[119]:
```
K2 = A.T *(K1 - K1*B*(B.T * K1 * B).inv() * B.T * K1.T) * A + Q
simplify(K2)
```

[119]:

$$\begin{bmatrix} \frac{a_{21}^2+a_{21}\left(a_{21}\left(-(a_{21}a_{22}+a_{31}a_{32})^2+\left(a_{21}^2+a_{31}^2+1\right)\left(a_{22}^2+a_{32}^2+1\right)\right)-a_{31}\left(-(a_{21}a_{22}+a_{31}a_{32})(a_{21}(a_{21}+a_{22}-1)+a_{31}(a_{31}+a_{32}-1))+(a_{22}(a_{21}+\right.\right.}{} \\ \frac{a_{21}\left(a_{22}\left(-(a_{21}a_{22}+a_{31}a_{32})^2+\left(a_{21}^2+a_{31}^2+1\right)\left(a_{22}^2+a_{32}^2+1\right)\right)-a_{32}\left(-(a_{21}a_{22}+a_{31}a_{32})(a_{21}(a_{21}+a_{22}-1)+a_{31}(a_{31}+a_{32}-1))+(a_{22}(a_{21}\right.\right.}{} \\ \frac{a_{21}\left(\left(-(a_{21}a_{22}+a_{31}a_{32})(a_{21}(a_{21}+a_{22}-1)+a_{31}(a_{31}+a_{32}-1))+(a_{22}(a_{21}+a_{22}-1)+a_{32}(a_{31}+a_{32}-1))\left(a_{21}^2+a_{31}^2+1\right)\right)(a_{31}+a_{32}-1)-\left(-(a_{21}a_{22}+a_{31}a_{32})^2+\right.\right.}{} \end{bmatrix}$$

No sign of B here either. Note:

[120]:
```
B.T * K1 * B
```

[120]:

$$\left[b^2\left(a_{21}^2+a_{31}^2+1\right)\right]$$

Once again, this is a single number with b^2, so that term cancels out with $B'B$.

In particular, BXB' will always cancel out with $B'B$ given that B only has one entry in it. If for example we go to 10 dimensions:

[128]:
```
B = Matrix(([[symbols("b")], [0], [0], [0], [0], [0], [0], [0], [0], [0]]))
A = Matrix(([[symbols("a_{" + str(i) + str(j) + "}") for i, j in zip(range(10),␣
↪range(10))] for i in range(10)]))
A
```

[128]:

$$\begin{bmatrix} a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \\ a_{00} & a_{11} & a_{22} & a_{33} & a_{44} & a_{55} & a_{66} & a_{77} & a_{88} & a_{99} \end{bmatrix}$$

[129]:
```
B.T * A * B
```

[129]: $\left[a_{00}b^2\right]$

Same principle. We will always return a scalar containing b^2 multiplied by the corresponding diagonal of the central matrix, so b^2 will always cancel out with $B'B = b^2$.

0.0.1 Conclusions so far:

- with $\delta = 1$ and $R = 0$ and a single listening naive agent out of all naive agents, K_t is fully dependent on A (which makes sense, because B is mathematically connected to A through subtraction of one of the rows).

0.1 Next check: what if more than one naive agent is listening

[130]:
```
Q = eye(2)
A = Matrix(([[symbols("a11"), symbols("a12")], [symbols("a21"),␣
↪symbols("a22")]]))
B = Matrix(([[symbols("b1")], [symbols("b2")]]))
```

[131]:
```
K1 = simplify(A.T *(Q - Q*B*(B.T * Q * B).inv() * B.T * Q.T) * A + Q)
K1
```

[131]:

$$\begin{bmatrix} \frac{-a_{11}b_2(-a_{11}b_2+a_{21}b_1)-a_{21}b_1(a_{11}b_2-a_{21}b_1)+b_1^2+b_2^2}{b_1^2+b_2^2} & \frac{a_{12}b_2(a_{11}b_2-a_{21}b_1)-a_{22}b_1(a_{11}b_2-a_{21}b_1)}{b_1^2+b_2^2} \\ \frac{a_{11}b_2(a_{12}b_2-a_{22}b_1)-a_{21}b_1(a_{12}b_2-a_{22}b_1)}{b_1^2+b_2^2} & \frac{-a_{12}b_2(-a_{12}b_2+a_{22}b_1)-a_{22}b_1(a_{12}b_2-a_{22}b_1)+b_1^2+b_2^2}{b_1^2+b_2^2} \end{bmatrix}$$

Here we see B explicitly show up in the first term, meaning it shows up later. But again, recall that B is mathematically connected to A, so we can instead do:

[132]:
```
A = Matrix(([[symbols("a11"), symbols("a12")], [symbols("a21"),␣
↪symbols("a22")]]))
B = Matrix(([[1 - symbols("a11") - symbols("a12")], [1 - symbols("a21") -␣
↪symbols("a22")]]))
```

```
K1 = simplify(A.T *(Q - Q*B*(B.T * Q * B).inv() * B.T * Q.T) * A + Q)
K1
```

[132]:

$$\begin{bmatrix} \frac{a_{11}^2 a_{22}^2 - 2a_{11}^2 a_{22} + 2a_{11}^2 - 2a_{11}a_{12}a_{21}a_{22} + 2a_{11}a_{12}a_{21} + 2a_{11}a_{12} + 2a_{11}a_{21}a_{22} - 2a_{11}a_{21} - 2a_{11} + a_{12}^2 a_{21}^2 + a_{12}^2 - 2a_{12}a_{21}^2 - 2a_{12} + 2a_{21}^2 + 2a_{21}a_{22} - 2a_{21} + a_{22}^2 - 2a_{22} + 2}{a_{11}^2 + 2a_{11}a_{12} - 2a_{11} + a_{12}^2 - 2a_{12} + a_{21}^2 + 2a_{21}a_{22} - 2a_{21} + a_{22}^2 - 2a_{22} + 2} \\ \frac{-a_{11}^2 a_{22}^2 + a_{11}^2 a_{22} + 2a_{11}a_{12}a_{21}a_{22} - a_{11}a_{12}a_{21} - a_{11}a_{12}a_{22} + a_{11}a_{12} - a_{11}a_{21}a_{22} + a_{11}a_{22}^2 - a_{11}a_{22} - a_{12}^2 a_{21}^2 + a_{12}^2 a_{21} + a_{12}a_{21}^2 - a_{12}a_{21}a_{22} - a_{12}a_{21} + a_{21}a_{22}}{a_{11}^2 + 2a_{11}a_{12} - 2a_{11} + a_{12}^2 - 2a_{12} + a_{21}^2 + 2a_{21}a_{22} - 2a_{21} + a_{22}^2 - 2a_{22} + 2} \end{bmatrix}$$

This is of course more complicated, but it demonstrates that B does not need to be analyzed on its own.

0.1.1 Next check: add δ

[133]:

```
Q = eye(2)
A = Matrix(([[symbols("a11"), symbols("a12")], [symbols("a21"), 1 -␣
↪symbols("a21")]]))
B = Matrix(([[1 - symbols("a11") - symbols("a12")], [0]]))

delta = symbols("\delta")

K1 = simplify(delta * A.T *(Q - Q*B*(B.T * Q * B).inv() * B.T * Q.T) * A + Q)
K1
```

[133]:

$$\begin{bmatrix} \delta a_{21}^2 + 1 & \delta a_{21}(1 - a_{21}) \\ \delta a_{21}(1 - a_{21}) & \delta a_{21}^2 - 2\delta a_{21} + \delta + 1 \end{bmatrix}$$

[136]:

```
K2 = delta * A.T *(K1 - K1*B*(B.T * K1 * B).inv() * B.T * K1.T) * A + Q
simplify(K2)
```

[136]:

$$\begin{bmatrix} \frac{2\delta^2 a_{21}^4 - 2\delta^2 a_{21}^3 + \delta^2 a_{21}^2 + 2\delta a_{21}^2 + 1}{\delta a_{21}^2 + 1} & \frac{\delta a_{21}\left(-2\delta a_{21}^3 + 4\delta a_{21}^2 - 3\delta a_{21} + \delta - a_{21} + 1\right)}{\delta a_{21}^2 + 1} \\ \frac{\delta a_{21}\left(-2\delta a_{21}^3 + 4\delta a_{21}^2 - 3\delta a_{21} + \delta - a_{21} + 1\right)}{\delta a_{21}^2 + 1} & \frac{2\delta^2 a_{21}^4 - 6\delta^2 a_{21}^3 + 7\delta^2 a_{21}^2 - 4\delta^2 a_{21} + \delta^2 + 2\delta a_{21}^2 - 2\delta a_{21} + \delta + 1}{\delta a_{21}^2 + 1} \end{bmatrix}$$

[137]:

```
A
```

[137]:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 1 - a_{21} \end{bmatrix}$$

This then allows us to see, for example, how δ relates to K_{T-2}, although this is tedious.

[138]:

```
K3 = delta * A.T *(K2 - K2*B*(B.T * K2 * B).inv() * B.T * K2.T) * A + Q
simplify(K3)
```

[138]:

$$\begin{bmatrix} \frac{4\delta^3 a_{21}^6 - 8\delta^3 a_{21}^5 + 8\delta^3 a_{21}^4 - 4\delta^3 a_{21}^3 + \delta^3 a_{21}^2 + 5\delta^2 a_{21}^4 - 4\delta^2 a_{21}^3 + 2\delta^2 a_{21}^2 + 3\delta a_{21}^2 + 1}{2\delta^2 a_{21}^4 - 2\delta^2 a_{21}^3 + \delta^2 a_{21}^2 + 2\delta a_{21}^2 + 1} & \frac{\delta a_{21} (-4\delta^2 a_{21}^5 + 12\delta^2 a_{21}^4 - 16\delta^2 a_{21}^3 + 12\delta^2 a_{21}^2 - 5}{2\delta^2 a_{21}^4 - 2\delta^2 a_{21}^3 +} \\ \frac{\delta a_{21}\left(-4\delta^2 a_{21}^5 + 12\delta^2 a_{21}^4 - 16\delta^2 a_{21}^3 + 12\delta^2 a_{21}^2 - 5\delta^2 a_{21} + \delta^2 - 3\delta a_{21}^3 + 5\delta a_{21}^2 - 3\delta a_{21} + \delta - a_{21} + 1\right)}{2\delta^2 a_{21}^4 - 2\delta^2 a_{21}^3 + \delta^2 a_{21}^2 + 2\delta a_{21}^2 + 1} & \frac{4\delta^3 a_{21}^6 - 16\delta^3 a_{21}^5 + 28\delta^3 a_{21}^4 - 28\delta^3 a_{21}^3 + 17\delta^3 a_{21}^2 - 6\delta^3 a_{21} + \delta^3 +}{2\delta^2 a_{21}^4 - 2\delta^2 a_{21}^3 +} \end{bmatrix}$$

0.1.2 Next check: add R

[144]:

```
Q = eye(2)
A = Matrix(([[symbols("a11"), symbols("a12")], [symbols("a21"), 1 -␣
↪symbols("a21")]]))
B = Matrix(([[1 - symbols("a11") - symbols("a12")], [0]]))
```

```python
delta = symbols("\delta")
R = Matrix(([["R"]]))

K1 = simplify(delta * A.T *(Q - Q*B*(B.T * Q * B + (R/delta)).inv() * B.T * Q.
↪T) * A + Q)
K1
```

[144]:

$$\begin{bmatrix} -\delta a_{11}^2\left(\frac{\delta(a_{11}+a_{12}-1)^2}{R+\delta a_{11}^2+2\delta a_{11}a_{12}-2\delta a_{11}+\delta a_{12}^2-2\delta a_{12}+\delta}-1\right)+\delta a_{21}^2+1 & -\delta a_{11}a_{12}\left(\frac{\delta(a_{11}+a_{12}-1)^2}{R+\delta a_{11}^2+2\delta a_{11}a_{12}-2\delta a_{11}+\delta a_{12}^2-2\delta a_{12}+\delta}-\right. \\ -\delta a_{11}a_{12}\left(\frac{\delta(a_{11}+a_{12}-1)^2}{R+\delta a_{11}^2+2\delta a_{11}a_{12}-2\delta a_{11}+\delta a_{12}^2-2\delta a_{12}+\delta}-1\right)-\delta a_{21}\left(a_{21}-1\right) & -\delta a_{12}^2\left(\frac{\delta(a_{11}+a_{12}-1)^2}{R+\delta a_{11}^2+2\delta a_{11}a_{12}-2\delta a_{11}+\delta a_{12}^2-2\delta a_{12}+\delta}-1\right) \end{bmatrix}$$

0.1.3 Next check: what about the steady state?

[152]:

```python
K = Matrix(([[symbols("K11"), symbols("K12")], [symbols("K12"),␣
↪symbols("K22")]]))
Q = eye(2)
A = Matrix(([[symbols("a11"), symbols("a12")], [symbols("a21"), 1 -␣
↪symbols("a21")]]))
B = Matrix(([[1 - symbols("a11") - symbols("a12")], [0]]))
```

[153]:

```python
simplify(A.T *(K - K*B*(B.T * K * B).inv() * B.T * K.T) * A + Q)
```

[153]:

$$\begin{bmatrix} K_{22}a_{21}^2+1-\frac{K_{12}^2a_{21}^2}{K_{11}} & \frac{a_{21}\left(-K_{11}K_{22}a_{21}+K_{11}K_{22}+K_{12}^2a_{21}-K_{12}^2\right)}{K_{11}} \\ \frac{a_{21}\left(-K_{11}K_{22}a_{21}+K_{11}K_{22}+K_{12}^2a_{21}-K_{12}^2\right)}{K_{11}} & K_{22}a_{21}^2-2K_{22}a_{21}+K_{22}+1-\frac{K_{12}^2a_{21}^2}{K_{11}}+\frac{2K_{12}^2a_{21}}{K_{11}}-\frac{K_{12}^2}{K_{11}} \end{bmatrix}$$

Note that solving for $K =$ the above matrix is the steady-state solution. Observe the above matrix is symmetric, as K is.

[154]:

```python
K
```

[154]:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}$$

So this gives a system of four equations.