Turbulent gravitational convection from maintained and instantaneous sources

By B. R. Morton*, Sir Geoffrey Taylor, F.R.S., and J. S. Turner
The Cavendish Laboratory, University of Cambridge

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[Plate 1]

Theories of convection from maintained and instantaneous sources of buoyancy are developed, using methods which are applicable to stratified body fluids with any variation of density with height; detailed solutions have been presented for the case of a stably stratified fluid with a linear density gradient. The three main assumptions involved are (i) that the profiles of vertical velocity and buoyancy are similar at all heights, (ii) that the rate of entrainment of fluid at any height is proportional to a characteristic velocity at that height, and (iii) that the fluids are incompressible and do not change volume on mixing, and that local variations in density throughout the motion are small compared to some reference density. The governing equations are derived in non-dimensional form from the conditions of conservation of volume, momentum and buoyancy, and a numerical solution is obtained for the case of the maintained source. This leads to a prediction of the final height to which a plume of light fluid will rise in a stably stratified fluid. Estimates of the constant governing the rate of entrainment are made by comparing the theory with some previous results in uniform fluids, and with the results of new experiments carried out in a stratified salt solution.

For the case of an instantaneous source of buoyancy there is an exact solution; the entrainment constant is again estimated from laboratory results for a stratified fluid.

Finally, the analysis is applied to the (compressible) atmosphere, by making the customary substitution of potential temperature for temperature. Predictions are made of the height to which smoke plumes from typical sources of heat should rise in a still, stably stratified atmosphere under various conditions.

Introduction

The convection currents which rise from heated bodies have been discussed previously, but in most cases attention has been directed towards finding the distribution of fluid velocity and temperature near such bodies. The first consideration of what happens to these upward currents at a distance above their source seems to have been by Schmidt (1941). He studied the behaviour of convective plumes of air above steady point and line sources of heat in a uniform, incompressible atmosphere.

Schmidt observed that plumes of hot air rising from small sources tend to be confined within conical regions when the flow is turbulent (just as in the case of forced jets). Using this fact he discussed the dynamics of such cases by supposing that the distribution of temperature and velocity can be found by balancing the horizontal turbulent transfer of heat and momentum against the vertical transfer by convection, allowance being made for the effect of buoyancy. Some assumptions have to be made to connect the horizontal turbulent transfer and the mean vertical flow before the analysis can be carried out. Schmidt assumed that there is geometrical and mechanical similarity of the processes in horizontal sections of the plume, and used mixture length theories of turbulence to find the complete forms of

* Now at University College, London.

the velocity and temperature profiles for both point and line sources of heat in an atmosphere at uniform temperature. His calculated results for the point source were verified by experiments using small electrically heated grids in air.

More recently, Yih (1951) and Rouse, Yih & Humphreys (1952) have given the results of measurements made in plumes above a single gas burner and above a line of gas burners in air. These results, like those of Schmidt, have been for plumes rising through a body fluid at uniform temperature. With the exception of some comments by Batchelor (1954) on convective plumes in unstably stratified body fluids, little attention has been given to cases in which there is a density gradient in the ambient fluid or atmosphere.

When the temperature of the air into which the thermal current is rising is not uniform, the equations governing the motion cannot always be satisfied by a similarity solution of the kind given by Schmidt or Batchelor. In particular, if the ambient fluid is stably stratified the whole motion takes place in a vertical region of limited extent, and it is useless to seek similarity solutions dependent on simple powers of x. Theoretically the method described above could be extended, but because of its dependence on mixture length theories and of the arbitrary assumptions that have to be made, the necessary labour would not be justified and a simpler treatment is to be desired. For this reason a simpler transfer assumption was made by Taylor (1945) in the hope that it would represent the broad outlines of the mechanics of a rising plume of buoyant air without the necessity for understanding in detail how the turbulent eddies mix the heated and the ambient air.

The present treatment is based on this assumption, which relates the inflow into the edge of a convective plume to some characteristic velocity in the plume. By this means it is possible to use the equations representing conservation of volume, momentum and heat to study vertical convection currents in stably stratified, incompressible body fluids. The method can be applied to any variation of temperature with height, but the particular case chosen is that of a linear gradient. Experiments are described corresponding to this case, in which a lighter fluid is discharged into a heavier, stably stratified fluid with which it can mix. A similar discussion is given for the case in which a known quantity of heat is released suddenly from a source, and as a result a 'cloud' of heated fluid rises through a stably stratified body fluid.

An immediate application can be made to the atmosphere, although this can no longer be regarded as an incompressible fluid. The compressibility of air is allowed for by using the potential temperature in place of the absolute temperature throughout the above treatment. With this modification the theoretical and experimental results are combined to give an estimate of the heights to which plumes and clouds may be expected to rise from characteristic heat sources and under conditions common in the atmosphere.

Gravitational convection

Although the most obvious application of the results will be to the convection of heat, the essential feature of gravitational convection is the existence of a source of buoyancy. Thus, essentially the same pattern of convection can be produced, for example, by a source of heat in a gas, by a source of light liquid in a heavier liquid with which it is freely miscible, or by a source of heavy fluid (or of salt) in a lighter miscible fluid in which case the flow is directed downwards. In these and other similar cases the motion of convection will have the same general form, provided that local changes in density are small in comparison with some reference density in the fluid, and that the source does not produce an appreciable amount of momentum at its origin. It is also assumed in the cases in which the source delivers a lighter (or heavier) fluid, that when this mixes with the ambient fluid there is no change in volume due to mixing.

In problems of convection of heat it has been usual to specify the thermal field by means of the excess of temperature over that of the surroundings. However, for a unified treatment of all the kinds of convection that can take place in a gravitational field, this excess temperature must be replaced by the equivalent deficiency of density under that of the surroundings. But the sole relevant effect of a deficiency of density is to create a buoyancy force. That is, if ρ_0 is the (possibly variable) density in the ambient fluid and ρ is that in the convecting fluid, $(\rho_0-\rho)$ enters the equations governing the motion essentially with g, which does not appear separately. Thus the problem is best specified by the variables, buoyancy force $(\rho_0-\rho)g$ and velocity. The acceleration due to gravity g may normally be taken as constant throughout the regions considered.

The strength of a source of heat in a region of fluid is measured by the total heat output to the neighbouring fluid in unit time. Similarly, the strength of a source of buoyancy is the total rate of release of buoyancy to the nearby fluid, but care is needed since there is no absolute reference of measurement (cf. temperature). In a uniform ambient fluid the source strength can be taken simply as the discharge of weight deficiency in unit time. However, in a stably stratified fluid the strength of the source must be defined relative to some chosen density, and this standard density for the particular case will be taken as that of the ambient fluid level with the source.

The following treatment is given with the buoyancy as one of the dependent variables defining the field of convective motion. To transfer into terms of temperature, use may be made of the relation

$$\frac{\rho_0-\rho}{\rho_1} = \beta(T-T_0), \tag{1}$$

where ρ_1 is the reference density, ρ_0 and ρ are the densities in the ambient fluid and in the plume respectively, with the absolute temperatures (T) similarly defined, and β is the coefficient of cubical expansion (for gases $\beta \sim 1/T_1$).

The main assumptions

The three main assumptions are:

(i) That the rate of entrainment at the edge of the plume or cloud is proportional to some characteristic velocity at that height,

(ii) That the profiles of mean vertical velocity and mean buoyancy force in horizontal sections are of similar form at all heights,

(iii) That the largest local variations of density in the field of motion are small in comparison with some chosen reference of density, this reference being taken as the density of the ambient fluid at the level of the source.

These assumptions are consistent with previous work; they are further discussed below.

When a stream of fluid is in contact with another stream the eddies which cause transfer of matter between them are characterized by velocities proportional to the relative velocity of the two streams. This can be seen from dimensional considerations, since if the mutual entrainment is turbulent the only quantity determining the motion is the relative velocity of the two streams. It is also shown by the experiments of Kuethe (1935) on the mixing at the edge of a jet. The rate at which the bounding edge of a heated plume expands and absorbs the surrounding air into the plume may similarly be assumed as proportional to the vertical velocity at that level. In the following treatment the rate at which fluid is entrained into plumes is taken as proportional to the vertical velocity on the axis of the plume.

To apply this assumption it is necessary to take some particular form for the profiles of fluid velocity and buoyancy force in horizontal sections of the plume, and to characterize each section by a horizontal length scale, b, which is proportional to the breadth or radius of the plume, for the motion at that height.

For the lower region of the flow of a plume in a stably stratified ambient fluid the assumptions concerning entrainment and similarity may be expected to be satisfactory. However, in the upper region, where the flow in the plume is diverted horizontally, the nature of the turbulent flow must be modified considerably owing to the change in the shape of the plume. Probably neither of these assumptions will then represent the mean motion correctly, but as the velocities are small in this upper part of the plume there is relatively little entrainment there and the error introduced into the ultimate height should be small.

That variations in density must be small relative to some reference density has been shown to be important by Batchelor (1954), with particular reference to clouds or bubbles of very light fluid released in a heavier ambient fluid. This restriction simplifies the mathematical treatment considerably, and on the physical side it confines the treatment to cases in which turbulent entrainment takes place throughout the ascent of the lighter fluid.

Maintained sources

Steady point and line sources may be treated in the same way, in either case only the mean motion being considered explicitly. Attention will be concentrated on the case of the steady point source.

Take cylindrical polar co-ordinates (x, r), with the x-axis vertical and the source at the origin. Let $u = u(x, r)$ be the vertical velocity, and $\rho = \rho(x, r)$ and $\rho_0 = \rho_0(x)$ be the fluid density inside and outside the plume respectively, where the plume is regarded as having a horizontal length scale $b = b(x)$ which will be defined in each case. Put $\rho_0(0) \equiv \rho_1$ and choose ρ_1 as the reference of density for the system, the body fluid being taken as incompressible.

The conservation equations used below represent;

(i) conservation of volume,

(ii) conservation of momentum,

(iii) conservation of density deficiency (the equivalent of heat).

The maintained plume in a uniform ambient fluid

As a simple illustration of the method it is sufficient to assume that the velocity and buoyancy force are constant across the plume, and zero outside it (i.e. a 'top hat' profile). The conservation relations may then be written:

$$\left.\begin{aligned}
&\text{(i)} \quad \frac{\mathrm{d}}{\mathrm{d}x}(\pi b^2 u) = 2\pi b\alpha u && \text{(volume)},\\
&\text{(ii)} \quad \frac{\mathrm{d}}{\mathrm{d}x}(\pi b^2 u^2 \rho) = \pi b^2 g(\rho_0-\rho) && \text{(momentum)},\\
&\text{(iii)} \quad \frac{\mathrm{d}}{\mathrm{d}x}[\pi b^2 u(\rho_1-\rho)] = 2\pi b\alpha u(\rho_1-\rho_0) && \text{(density deficiency)},
\end{aligned}\right\} \quad (2)$$

where α is the proportionality constant relating the inflow velocity at the edge of the plume to the vertical velocity within the plume, and b is the actual radius to the plume edge. Equation (2, iii), in which the density deficiency is measured relative to fluid at the level of the source, can be transformed with the use of (2, i),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\pi b^2 u(\rho_1-\rho)] &= (\rho_1-\rho_0)\frac{\mathrm{d}}{\mathrm{d}x}(\pi b^2 u)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}[\pi b^2 u(\rho_1-\rho_0)] - \pi b^2 u\frac{\mathrm{d}}{\mathrm{d}x}(\rho_1-\rho_0),
\end{aligned}$$

and hence into the form

$$\frac{\mathrm{d}}{\mathrm{d}x}[\pi b^2 u(\rho_0-\rho)] = \pi b^2 u\frac{\mathrm{d}\rho_0}{\mathrm{d}x}.$$

Since the density variations considered are small with respect to ρ_1 the density ρ which occurs in the left-hand side of equation (2, ii) may be taken as ρ_1 without appreciable loss of accuracy. The conservation relations may then be written:

$$\left.\begin{aligned}
&\text{(i)} \quad \frac{\mathrm{d}}{\mathrm{d}x}(b^2 u) = 2\alpha b u,\\
&\text{(ii)} \quad \frac{\mathrm{d}}{\mathrm{d}x}(b^2 u^2) = b^2 g\frac{\rho_0-\rho}{\rho_1},\\
&\text{(iii)} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(b^2 u g\frac{\rho_0-\rho}{\rho_1}\right) = b^2 u\frac{g}{\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x},
\end{aligned}\right\} \quad (3)$$

An exact solution can be found for the case of an ambient fluid of uniform density ($\rho_0 \equiv \rho_1$). For boundary conditions it may be assumed that at the source the radius and the flux of momentum in the plume are zero, and that weight deficiency (or, equivalently, heat) is released at a known and constant rate. Then the term on the right of equation (3, iii) vanishes and the equation can be integrated immediately to give

$$b^2ug\frac{\rho_1-\rho}{\rho_1} = Q = \text{constant},$$

showing that the vertical flux of buoyancy remains constant at all heights. The density difference may then be eliminated from the second equation and the solution to the remaining two equations determined in the form

$$bu = (\tfrac{9}{10}\alpha Q)^{\frac{1}{3}}\, x^{\frac{2}{3}}, \quad b^2u = \frac{6\alpha}{5}(\tfrac{9}{10}\alpha Q)^{\frac{1}{3}}\, x^{\frac{5}{3}}.$$

Thus $$b = \frac{6\alpha}{5}x, \quad u = \frac{5}{6\alpha}(\tfrac{9}{10}\alpha Q)^{\frac{1}{3}}\, x^{-\frac{1}{3}}, \quad g\frac{\rho_1-\rho}{\rho_1} = \frac{5Q}{6\alpha}(\tfrac{9}{10}\alpha Q)^{-\frac{1}{3}}\, x^{-\frac{5}{3}}. \tag{4}$$

This solution is of the same form as that of Schmidt (1941), and could be compared with his results to determine a value for α. At a later stage this will be done with a profile that resembles Schmidt's solution more closely than does the 'top hat' The solution calculated in this approximate way gives only a little more information than can be found by dimensional analysis, but it illustrates the fact that the entrainment assumption is consistent with Schmidt's mixture length assumptions.

The point source in a stratified fluid

In a stably stratified body fluid the plume density increases steadily owing to entrainment, while that of the body fluid decreases steadily with height. At some level the density difference vanishes, and above this the forces on the plume fluid act so as to reduce its momentum and ultimately to bring it to rest. On the assumption that the rate of entrainment at the plume surface is proportional to the vertical velocity, the amount of fluid entrained in this upper region will not be large. Thus the fluid in the topmost part of the plume will fall back some distance as it spreads sideways, although the solution cannot be expected to show this.

As a particular example consider the case of a constant density gradient in the ambient fluid, and for more direct comparison of the theoretical results with previous experiments assume that the velocity and the buoyancy profiles are normal distribution curves centred about the axis of symmetry. These profiles may be taken as

$$\left.\begin{aligned} u(x,r) &= u(x)\,\mathrm{e}^{-r^2/b^2}, \\ g\frac{\rho_0-\rho}{\rho_1}(x,r) &= g\frac{\rho_0-\rho}{\rho_1}(x)\,\mathrm{e}^{-r^2/b^2}. \end{aligned}\right\} \tag{5}$$

The horizontal scale b for the plume at any particular height is then the distance from the axis of symmetry to points at which the velocity and buoyancy amplitudes are $1/e$ of those on the axis. Such a length is characteristic of the normal distribution profile, and is a convenient scale to use for purposes of calculation with the maintained plume. It does not, of course, correspond with the radius that would be observed in experiments on plumes, but is proportional to it. An 'effective' radius can be defined for purposes of observation as the distance from the vertical axis of points at which the velocity or density deficiency have fallen to, say, 1 % of the axial values. Integration over a section of the plume is taken to mean integration over the entire horizontal plane, and this is approximately equivalent to integration within the effective radius.

The entrainment constant α is selected in this case so that the rate of entrainment of volume at any particular height is $2\pi b\alpha u(x)$, where b is the length scale and $u(x)$ the axial vertical velocity at that height. If the equations representing conservation of volume, momentum and density deficiency are integrated over the plume section they take the form,

$$\left.\begin{aligned}&\text{(i)}\ \frac{\mathrm{d}}{\mathrm{d}x}(\pi b^2 u) = 2\pi b\alpha u,\\ &\text{(ii)}\ \frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}\pi b^2 u^2 \rho) = \pi b^2 g(\rho_0-\rho),\\ &\text{(iii)}\ \frac{\mathrm{d}}{\mathrm{d}x}[\tfrac{1}{2}\pi b^2 u(\rho_0-\rho)] = \pi b^2 u\frac{\mathrm{d}\rho_0}{\mathrm{d}x}.\end{aligned}\right\} \tag{6}$$

As above, equations (6) may be rearranged and the density retained only to the necessary order of accuracy in each term to give the reduced set,

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(b^2u) &= 2\alpha bu,\\ \frac{\mathrm{d}}{\mathrm{d}x}(b^2u^2) &= 2b^2 g\frac{\rho_0-\rho}{\rho_1},\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(b^2ug\frac{\rho_0-\rho}{\rho_1}\right) &= 2b^2u\frac{g}{\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}.\end{aligned}\right\} \tag{7}$$

Select new variables; $V = bu$, $W = b^2u$, $F^* = b^2ug\dfrac{\rho_0-\rho}{\rho_1}$ and put $G = -\dfrac{g}{\rho_1}\dfrac{\mathrm{d}\rho_0}{\mathrm{d}x}$. Equations (7) can then be written,

$$\frac{\mathrm{d}W}{\mathrm{d}x} = 2\alpha V, \quad \frac{\mathrm{d}V^4}{\mathrm{d}x} = 4F^*W, \quad \frac{\mathrm{d}F^*}{\mathrm{d}x} = -2WG. \tag{8}$$

Three boundary conditions are required to determine a solution, and these can be imposed at the origin if it is assumed that the width of the plume (as defined above) and the momentum associated with the plume are both zero at the source. The third condition concerns the rate of release of total buoyancy (or weight deficiency) from the source; from above

$$\begin{aligned}F^* &= \frac{2}{\pi\rho_1}\ \text{(flux of buoyancy relative to the ambient fluid at the same level past a given horizontal plane)}\\ &= \frac{2}{\pi}F,\end{aligned}$$

and

$$F_0^* = \frac{2}{\pi}F_0 = \frac{2}{\pi\rho_1}\ \text{(flux of buoyancy per second from the source)}.$$

With F, F_0 defined by these relations, the boundary conditions at $x = 0$ are,

$$V = 0, \quad W = 0, \quad F = F_0. \tag{9}$$

Reduction of the equations to non-dimensional form

Two governing parameters are provided by the physical conditions of the problem, F_0 and G, and these must determine the scale of the motion. The following transformations are chosen so as to reduce equations (8), together with the boundary conditions (9), to their simplest non-dimensional form, and can be derived from dimensional arguments:

$$\left.\begin{aligned} x &= 2^{-\frac{7}{8}}\pi^{-\frac{1}{4}}\alpha^{-\frac{1}{2}}F_0^{\frac{1}{4}}G^{-\frac{3}{8}}x_1, \\ V &= 2^{\frac{3}{4}}\pi^{-\frac{1}{2}}F_0^{\frac{1}{2}}G^{-\frac{1}{4}}v, \\ W &= 2^{\frac{7}{8}}\pi^{-\frac{3}{4}}\alpha^{\frac{1}{2}}F_0^{\frac{3}{4}}G^{-\frac{5}{8}}w, \\ F^* &= 2\pi^{-1}F_0 f. \end{aligned}\right\} \tag{10}$$

Table 1. The tabulated numerical solutions to the non-dimensional equations for the steady plume in a stably stratified ambient fluid

x_1	v	w	f
0	0	0	1·0000
0·1	0·1310	0·0079	0·9997
0·2	0·2080	0·0250	0·9981
0·3	0·2724	0·0490	0·9945
0·4	0·3297	0·0792	0·9881
0·5	0·3821	0·1148	0·9785
0·6	0·4307	0·1555	0·9650
0·7	0·4761	0·2009	0·9472
0·8	0·5189	0·2506	0·9247
0·9	0·5592	0·3046	0·8970
1·0	0·5971	0·3624	0·8636
1·1	0·6327	0·4239	0·8244
1·2	0·6660	0 4888	0·7787
1·3	0·6971	0·5570	0·7265
1·4	0·7257	0·6282	0·6672
1·5	0·7519	0·7021	0·6008
1·6	0·7753	0·7785	0·5267
1·7	0·7958	0·8570	0·4450
1·8	0·8130	0·9375	0·3553
1·9	0·8266	1·0195	0·2574
2·0	0·8360	1·1027	0·1513
2·1	0·8406	1·1866	+0·0369
2·2	0·8393	1·2706	−0·0860
2·3	0·8308	1·3542	−0·2172
2·4	0·8128	1·4365	−0·3568
2·5	0·7814	1·5163	−0·5045
2·6	0·7289	1·5920	−0·6599
2·7	0·6341	1·6607	−0·8226
2·8	0·3035	1·7049	−0·9915

On substituting these relations (10) into equations (8), together with the boundary conditions (9), the system is in its simplest and most general form,

$$\frac{dw}{dx_1} = v, \quad \frac{dv^4}{dx_1} = fw, \quad \frac{df}{dx_1} = -w, \tag{11}$$

subject to the boundary conditions,

$$v = 0, \quad w = 0, \quad f = 1 \quad \text{on } x_1 = 0. \tag{12}$$

These equations are in suitable form for numerical integration. The solution can be started from $x_1 = 0$ with the starting series:

$$\left.\begin{aligned} f &= 1 - 0{\cdot}1368x_1^{8/3} + 0{\cdot}0005x_1^{16/3} - \dots, \\ v^4 &= 0{\cdot}1368x_1^{8/3} - 0{\cdot}0098x_1^{16/3} + 0{\cdot}0001x_1^{21/3} - \dots, \\ w &= 0{\cdot}3649x_1^{5/3} - 0{\cdot}0025x_1^{13/3} + \dots. \end{aligned}\right\} \tag{13}$$

The first terms of these series solutions give the solution for a uniform body fluid as found by Schmidt. The series solutions (13) can be used with sufficient accuracy for about a third of the total range. After this the solution is most readily continued by

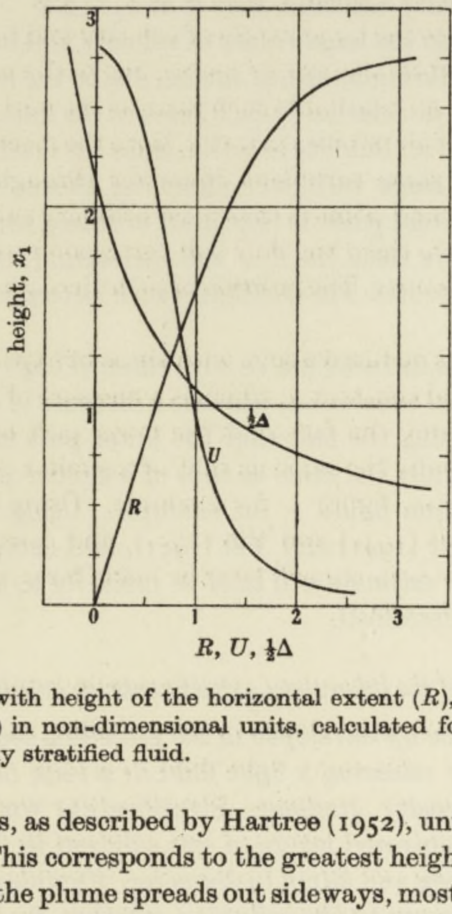

FIGURE 1. The variation with height of the horizontal extent (R), the vertical velocity (U), and the buoyancy (Δ) in non-dimensional units, calculated for the turbulent plume in a uniformly and stably stratified fluid.

finite difference methods, as described by Hartree (1952), until u ($= V^2/W$) vanishes very close to $x_1 = 2{\cdot}8$. This corresponds to the greatest height reached by the plume fluid. As this fluid from the plume spreads out sideways, most of it will fall back some distance, but certainly not as far as the height corresponding with $x_1 = 2{\cdot}125$ at which the density difference first vanishes. In table 1 the solutions for v, w and f are tabulated. If $R = w/v$, $U = v^2/w$, $\Delta = f/w$ then, on rearranging equations (10),

$$\left.\begin{aligned} x &= 0{\cdot}410\alpha^{-\frac{1}{2}}F_0^{\frac{1}{4}}G^{-\frac{3}{8}}x_1, \\ b &= 0{\cdot}819\alpha^{\frac{1}{2}}F_0^{\frac{1}{4}}G^{-\frac{3}{8}}R, \\ u &= 1{\cdot}158\alpha^{-\frac{1}{2}}F_0^{\frac{1}{4}}G^{\frac{1}{8}}U, \\ g\frac{\rho_0-\rho}{\rho_1} &= 0{\cdot}819\alpha^{-\frac{1}{2}}F_0^{\frac{1}{4}}G^{\frac{5}{8}}\Delta. \end{aligned}\right\} \tag{14}$$

The solution curves for R, U, Δ, corresponding with the numerical solutions, tabulated in table 1, are given in figure 1.

From the theoretical treatment two heights are found for maintained plumes, that at which the buoyancy first vanishes, and that at which the (vertical) velocity first vanishes. The bulk of the fluid in the plume will rise almost to the greater of these heights, and then in spreading horizontally it will fall back again, and the mean height of the sideways flow will lie between the two levels given by the solution. However, no allowance has been made for entrainment from above into the top of the plume, and the result of this will be that some marked fluid will remain near the highest point. The top of the plume should thus be approximately at the level at which the vertical velocity vanishes, that is at $x_1 = 2{\cdot}8$.

It should be noted that the large values of velocity and buoyancy near the source predicted by the above formulae are, of course, due to the mathematical singularity at the origin. They bear no relation to such cases as the initial cloud from the atomic bomb or to the release of air bubbles in water, since the mechanism of entrainment is assumed to be of the same turbulent character throughout the motion in the theoretical model. A second point is that most ordinary sources cover a finite area, and sufficiently far above them the flow will correspond to that from some virtual source below the actual source. The position of such virtual sources can be calculated, as will be shown below.

To connect the results deduced above with those of experiment some value must be given to the numerical constant α, which is a measure of the rate of entrainment. This can be done by using the fact that the lower part of the plume in a stably stratified fluid is essentially the same as that of a similar plume in a uniform body fluid, as can be seen from figure 1, for example. Using the results of previous experiments by Schmidt (1941) and Yih (1951), and comparing them with those of the present paper, an estimate will later be made for α, after the present experimental work has been described.

Description of the laboratory experiments on maintained plumes

A simple test of the theory developed in the preceding sections was carried out on the laboratory scale by releasing a light fluid in a tank of heavier fluid in which there was a stable density gradient. Stratification was produced by adding successively more concentrated layers of salt solution to the bottom of the tank, which was about 1 m deep and 30 cm in diameter. Provided that care was taken to give no vertical momentum to the inflowing solution, no large-scale disturbances were set up, although there was sufficient mixing between adjacent layers to give some smoothing out of the density profile. A tank prepared in this way can remain with the density gradient substantially unchanged for some days if necessary.

The maximum overall density difference used was 15 %, and by varying the total depth and the concentration of layers density gradients differing by a factor of 80 were produced. These were checked by putting into the tank a number of small glass bubbles of known density and measuring the heights at which they floated.

In all cases the liquid released was methylated spirits coloured with methyl violet so that its motion could be followed. In order to study the plumes outlet nozzles of

FIGURE 2

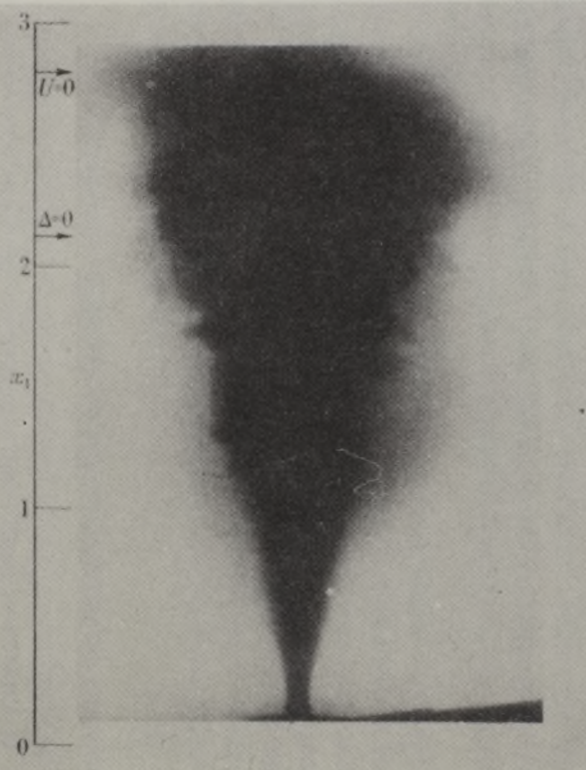

FIGURE 3

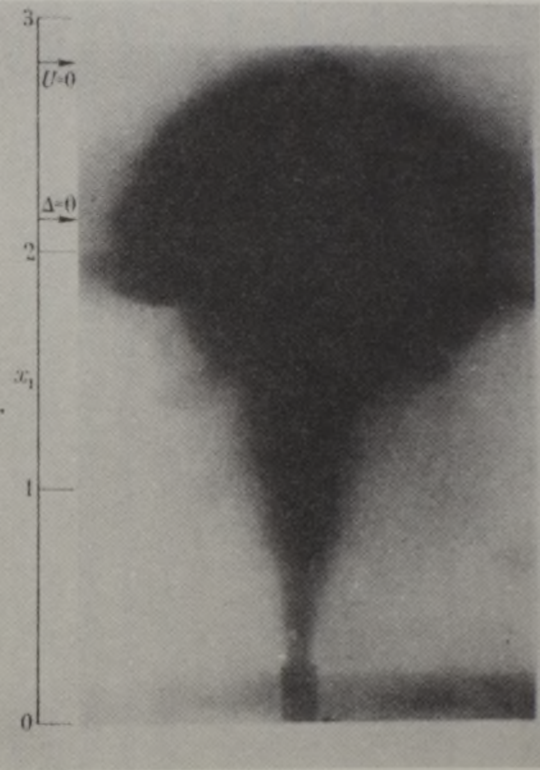

FIGURE 4

FIGURE 2. The maintained plume in a uniform ambient fluid.

FIGURE 3. The maintained plume in a stably stratified fluid; a time exposure during an early stage of the release of dyed fluid so that the edges of the plume are not obscured by the top layer.

FIGURE 4. The maintained plume in a stably stratified fluid; a time exposure during a later stage of the release when a layer of dyed fluid is growing at the top.

various sizes were used, and these were connected to a constant head of pressure, while a standard volume of the plume liquid was released. The time taken was recorded to give a measure of the rate of supply of total buoyancy at the source; this time must be long enough for the release to be regarded as a continuous plume rather than a cloud, but short enough for the amount of fluid released not to fill the tank and obscure the results completely. The final volume of dyed fluid was of course much larger than that released, owing to the entrainment. Since it was prevented from spreading out into a thin layer by the sides of the tank, the dyed fluid eventually formed quite a thick horizontal layer or band in the tank. The heights of the top and bottom of these bands were measured; it seemed probable that the top corresponds to the maximum height reached by the material in the plume, that is to the value of x_1 for which $U = 0$.

Three photographs are reproduced (figures 2 to 4, plate 1) to show the appearance of the plumes during a typical experiment. Figure 2 is an exposure of 1/25 s showing the irregular turbulent edges at any particular instant. It was actually taken in a uniform tank, but the lower parts of a plume in a stratified fluid look exactly the same as this. Figure 3 is an exposure of 5 s in a stratified tank, and this time exposure has effectively averaged out the random variations in the position of the plume edge. A scale of x_1 has been drawn in so that the various levels referred to in the theory can be identified, and for this purpose the maximum height has been assumed to lie at $x_1 = 2{\cdot}8$. Figure 4 is another time exposure of a plume in a stratified fluid, but taken at a later stage so that the spreading out and descending of the plume fluid is clearly visible in the upper region.

Experimental results for the maintained plumes

It has been predicted that the characteristic heights in a plume are proportional to $F_0^{\frac{1}{4}}G^{-\frac{3}{8}}$, where F_0 and G are both directly measurable quantities which do not depend on the detailed profiles that are assumed. The constant of entrainment does depend on the assumptions however, so that a convenient comparison of theory and experiment can be obtained by plotting $F_0^{\frac{1}{4}}G^{-\frac{3}{8}}$ against measured heights.

Since the nozzles used were far from being point sources, it might be expected that the relevant heights should be measured from a virtual source somewhere below the nozzle opening. To start with, this difficulty has been avoided by using the same nozzle (2·8 cm in diameter) for each experiment. Figure 5 shows the actual heights above the nozzle of the top and bottom of the final bands plotted against $F_0^{\frac{1}{4}}G^{-\frac{3}{8}}$ for a series of experiments in which both F_0 and G were varied over a wide range. A straight line has been fitted to the points by the method of least squares; the slope of this line is 3·79, and its intercept on the x-axis is $-5{\cdot}19$ cm. The correlation coefficient between the height and $F_0^{\frac{1}{4}}G^{-\frac{3}{8}}$ is $r = 0{\cdot}98$.

Also shown in figure 5 are the lines corresponding to predictions of the maximum height reached by the plume made by comparing the present theory with the experiments of Schmidt (1941) and of Rouse *et al.* (1952). Since these earlier experiments were carried out in air at a constant ambient temperature, a comparison of this kind is only valid over the lower region of the plumes. The details of this comparison will be explained in the next section and the notation given.

A final point that may be made in recommendation of the type of measurement chosen in the above experiments is that an averaging process is carried out during the motion and before any particular measurement is taken. Previous experiments on convective plumes have taken averages over time for the velocity and temperature at given points in the field. For instance, Schmidt watched the oscillations of the pointer of a galvanometer and chose a mean value by eye. In the present experiments where the relevant measurement is the height of the plume or cloud top, the particular quantity chosen for observation has already been subjected to an averaging process and can then be measured directly.

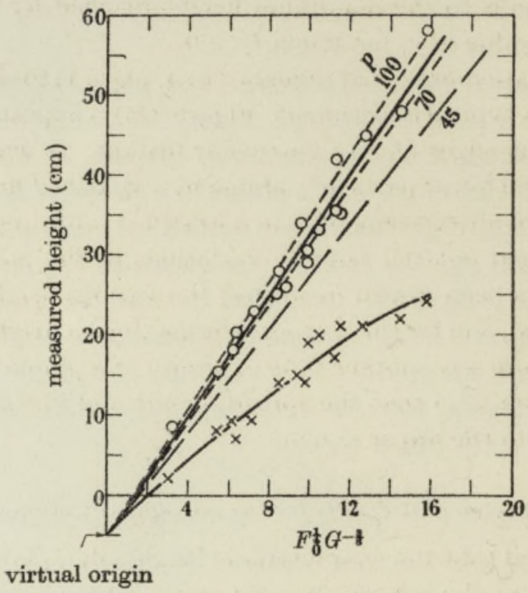

FIGURE 5. Experimentally determined heights of maintained plumes compared with the height predictions made by comparing previous experimental data on unstratified fluids with the present theory. ○, top of the layer; ×, bottom of the layer. Nozzle diameter 2·8 cm. The continuous straight line is the line of best fit with the experimental points for the plume top.

A comparison with previous experiments

Schmidt (1941) measured temperatures and velocities above an electrically heated wire grid in a uniform atmosphere of air. Measurements were made for a single source strength, traverses of the plume being taken at five distinct heights above the source. As a hot wire was used for the velocity measurements the component velocities were not separated. However, a profile of the plume was given for temperature, and it is found that these experimental points are fitted quite well by the normal distribution profile $\exp(-45r^2/x^2)$.

Rouse *et al.* (1952) gave measurements of temperature and vertical velocity above point and line sources of gas burners in a uniform atmosphere of air. In these experiments the measurements were taken at a variety of heights and for a variety of source strengths, the results being combined in non-dimensional form to give a vertical velocity profile and a buoyancy profile for each of the two types of source.

For the point source they chose the profile $\exp(-96r^2/x^2)$ for the vertical velocity, and the profile $\exp(-71r^2/x^2)$ for the buoyancy, as giving the best fit with the quoted experimental results. It seems doubtful that these numerical constants can be determined so critically from the experimental points shown, and values of 100 and 70 will be used for the comparison to be made.

In the theoretical treatment given above for a stably stratified ambient fluid, normal distribution curves have been taken for each of the profiles of vertical velocity and of buoyancy, although it should be noted that identical profiles are used. From the fact that the lower part of the plume in a stably stratified fluid is essentially the same as that from the same source in a uniform fluid, as can be seen for example in figure 1, it is possible to compare the theoretical plume derived above with the experimental results of Schmidt and Rouse *et al.*, as well as with the experiments described above. The most direct way of doing this, since the profiles are of similar form, is to compare the plume angles defined by the horizontal length scale b (defined above) with those of experiment. If the various experimental

TABLE 2. THE VARIATION WITH p OF b/x, α, AND THE SLOPE $x/(F^{\frac{1}{4}}G^{-\frac{3}{8}})$ FOR VARIOUS VALUES OF x_1

p ...	45	70	85	100
b/x	0·149	0·120	0·108	0·100
α	0·125	0·100	0·090	0·083
$x/(F^{\frac{1}{4}}G^{-\frac{3}{8}}x_1)$	1·160	1·296	1·367	1·423
$x/(F^{\frac{1}{4}}G^{-\frac{3}{8}})$ for $x_1 = 2{\cdot}125$	2·464	2·755	2·904	3·024
$x/(F^{\frac{1}{4}}G^{-\frac{3}{8}})$ for $x_1 = 2{\cdot}8$	3·247	3·630	3·827	3·985

profiles are written in the form $\exp(-pr^2/x^2)$ and are compared with the assumed theoretical profile $\exp(-r^2/b^2)$ it is seen that $b/x = p^{-\frac{1}{2}}$. Moreover, from equations (4) representing the plume in a uniform fluid, or from equations (14) if only the lower region of the motion is considered,

$$\frac{b}{x} = 2\alpha\frac{R}{x_1} = \frac{6}{5}\alpha, \tag{15}$$

where the initial slope $R/x_1 = \frac{3}{5}$ can be measured from figure 1 or taken from the numerical calculations.

Table 2 gives the variation with p of the quantities b/x, α, the slope $x/(F^{\frac{1}{4}}G^{-\frac{3}{8}}x_1)$, and the slopes $x/(F^{\frac{1}{4}}G^{-\frac{3}{8}})$ for values of $x_1 = 2{\cdot}125$ and $2{\cdot}8$. The value of p in best agreement with the present experimental results is $p = 80$, for which $\alpha = 0{\cdot}093$.

The results derived above have been developed with the use of the horizontal length scale b, which is the distance from the axis to points at which the profile amplitude is $1/e$ of its axial value. In order to consider the question of virtual sources in a simple way it is convenient to use the 'effective' plume radius, which is the radius to points at which the profile amplitude has fallen to say c % of its axial value, where c is small.

Determination of the virtual source

As the experimental source had a finite diameter it is necessary to determine the position of a virtual source for each experimental nozzle before comparison is made with the theoretical results. This position can be determined by analysis of the experimental results, or by using the theory in the following way. If near the nozzle the effective radius is taken as an actual radius defining a bounding surface for the plume, then this surface may be considered as passing through the edge of the nozzle and defining a source point below. The displacement of the virtual source will depend on the choice of the effective radius (i.e. of c). By taking $c = 1\,\%$ the radius of the nozzle is $2{\cdot}146b$, and from equation (15) the displacement of the virtual source along the x axis is $-5{\cdot}8$ cm for the nozzle of diameter $2{\cdot}8$ cm, where α has been taken as $0{\cdot}093$. Alternatively, for $c = \frac{1}{10}\,\%$ the nozzle radius must be given by $2{\cdot}626b$ and the corresponding displacement is $-4{\cdot}7$ cm for this particular nozzle. These are both in reasonable agreement with the measured intercept from figure 5 of $-5{\cdot}19$ cm.

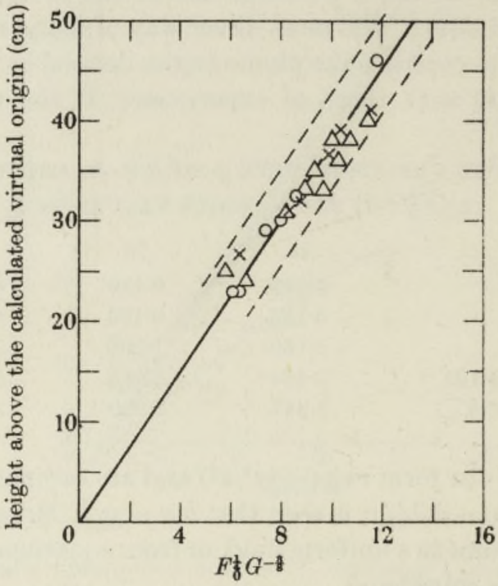

FIGURE 6. Comparison of the plume heights for a particular density gradient with nozzles of various sizes (diameters: ○, 9 mm; Δ, 19 mm; $\times$, 28 mm). The line of best fit for the results in figure 5 is also drawn, together with the limits of twice the standard error for those results.

To find the effect of the output conditions on the heights finally measured, a second series of experiments was carried out with a fixed value of G but with varying F_0 and with three different diameters of nozzle. The heights of the tops of the resulting bands in the tank from experiments with these three nozzles are plotted in figure 6 relative to their respective virtual sources (which are calculated by taking the radius of the nozzle as equivalent to $2{\cdot}146b$). The regression line of figure 5 is included, together with the limits of plus and minus twice the standard error of the estimate for the more extensive set of results plotted there. All the experimental points for the three nozzles lie within these limits, as they do for all reasonable assumptions concerning the choice of c in finding the virtual source. The conclusion is that, within the experimental error, the systematic differences due to the output conditions may be taken into account just by measuring from the virtual source in each case.

Instantaneous sources

If a given amount of weight deficiency (e.g. a given quantity of heat or volume of miscible fluid which is lighter than the body fluid) is released instantaneously from a point or line source in a stably stratified fluid it will rise as a convective cloud. After rising for some distance it will eventually settle back into a layer at some ultimate height, after first overshooting that level. During its ascent the shape of the cloud will change constantly and may vary considerably from that of a sphere (or cylinder). In spite of this, provided that the ratio of the surface area to the volume varies as the ratio of the square to the cube of some length scale, it is possible to make an estimate of the ultimate height. This length scale, which will vary with height, can be thought of as proportional to a 'radius' for the cloud, which will be assumed to retain its mean form throughout the motion. That is, contours of equal buoyancy will be assumed to retain the same average shape and their scale will be given by the 'radius'. In the upper region of its motion the cloud will be considerably flattened, but if it is still assumed that the rate of entrainment is proportional in this case to the mean vertical velocity, the cloud will be growing slowly at this stage and its ultimate height should be little affected.

Again, the case of an instantaneous point source in a stably and uniformly stratified fluid is taken as a characteristic example.

The point source in a uniformly stratified fluid

The same assumptions are made concerning the rate of entrainment and the limitation of the total density variations in order that a similarity solution may again be sought. In this case any assumptions as to the nature of the profiles are likely to be too artificial, as it is much more difficult to picture the detail of the motion and as there are no prior experiments from which information can be obtained.

Consider variation with respect to time with the motion beginning at $t = 0$. Assume that the profiles of velocity and buoyancy through the cloud have the same form, and that the mean velocity over the cloud is $ku(t)$. Then if b is the 'mean radius' to the edge of the cloud in this case, the rate of entrainment at the cloud surface can be taken as αku (which may be regarded as a definition of α for this case). From the appropriate equations of conservation

$$\left.\begin{aligned}
&\text{(i)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{4}{3}\pi b^3\right) = 4\pi b^2\alpha ku,\\
&\text{(ii)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{4}{3}\pi b^3\rho ku\right) = \tfrac{4}{3}\pi b^3 kg(\rho_0-\rho),\\
&\text{(iii)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\int(\rho_1-\rho)\,\mathrm{d}\,(\text{volume})\right] = 4\pi b^2\alpha ku(\rho_1-\rho_0),
\end{aligned}\right\} \tag{16}$$

where α is the constant of entrainment, k is defined above, and the other quantities have been defined before. The integration in (iii) is taken over the whole volume of the cloud. The third of equations (16) can be simplified as in the earlier equations to

$$\text{(iii)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{4}{3}\pi b^3 kg\frac{\rho_0-\rho}{\rho_1}\right) = -\tfrac{4}{3}\pi b^3 Gu,$$

where $G = -\dfrac{g}{\rho_1}\dfrac{\mathrm{d}\rho_0}{\mathrm{d}x}$. A fourth equation is given by

$$\text{(iv)}\quad \frac{\mathrm{d}x}{\mathrm{d}t} = u.$$

A convenient change of variables is made by taking

$$M = b^3u, \quad V = b^3 \quad \text{and} \quad F^* = b^3g\frac{\rho_0-\rho}{\rho_1},$$

where $F^* = \dfrac{3}{4\pi k}F = \dfrac{3}{4\pi k\rho_1}$ (buoyancy force acting on cloud at time t)

and $F_0^* = \dfrac{3}{4\pi k}F_0$

$= \dfrac{3}{4\pi k\rho_1}$ (total buoyancy released from the source at $t = 0$).

If the cloud has zero radius and no momentum at its release, the boundary conditions are given at $t = 0$ by,

$$x = 0, \quad M = 0, \quad V = 0, \quad F = F_0. \tag{17}$$

Non-dimensional form of the equations

The scale of the motion is governed by the constants defining the system, F_0 and G. The transformation

$$F = F_0 f, \quad M = \frac{3}{4\pi}k^{-\frac{1}{4}}F_0G^{-\frac{1}{2}}m, \quad V = \left(\frac{3}{\pi}\right)^{\frac{1}{4}}(\alpha k)^{\frac{3}{4}}F_0^{\frac{3}{4}}G^{-\frac{3}{4}}v,$$

$$x = \frac{1}{4}\left(\frac{3}{\pi}\right)^{\frac{1}{4}}(\alpha k)^{-\frac{1}{4}}F_0^{\frac{1}{4}}G^{-\frac{3}{4}}X, \quad t = k^{\frac{1}{2}}G^{-\frac{1}{2}}t_1,$$

is chosen to reduce the equations (16) and the boundary conditions (17) to the simple non-dimensional set

$$\frac{\mathrm{d}v^{\frac{4}{3}}}{\mathrm{d}t_1} = m, \quad \frac{\mathrm{d}m}{\mathrm{d}t_1} = f, \quad \frac{\mathrm{d}f}{\mathrm{d}t_1} = -m, \quad \frac{\mathrm{d}X}{\mathrm{d}t_1} = \frac{m}{v}, \tag{18}$$

where certain second-order terms have been neglected. The boundary conditions at $t_1 = 0$ are,

$$X = 0, \quad m = 0, \quad v = 0, \quad f = 1. \tag{19}$$

Actually, the entrainment velocity should be taken as proportional to $|u|$ so that when the cloud has reached its maximum height and falls back, fluid is still entrained into the cloud and not out of it. For the downwards motion the first of equations (18) must be replaced by

$$\frac{\mathrm{d}v^{\frac{4}{3}}}{\mathrm{d}t_1} = -m,$$

and new, appropriate boundary conditions must be used to start the downwards motion.

For the case of a constant, negative density gradient in the ambient fluid there is the exact solution,

$$0 \leqslant t_1 \leqslant \pi, \quad \left.\begin{aligned} R &= v^{\frac{1}{3}} = (1-\cos t_1)^{\frac{1}{4}}, \\ U &= \frac{m}{v} = \frac{\sin t_1}{(1-\cos t_1)^{\frac{3}{4}}}, \\ \Delta &= \frac{f}{v} = \frac{\cos t_1}{(1-\cos t_1)^{\frac{3}{4}}}, \\ X &= 4(1-\cos t_1)^{\frac{1}{4}}. \end{aligned}\right\} \tag{20}$$

$$\pi \leqslant t_1 \leqslant 2\pi, \quad \left.\begin{aligned} R &= (3+\cos t_1)^{\frac{1}{4}}, \\ U &= \frac{\sin t_1}{(3+\cos t_1)^{\frac{3}{4}}}, \\ \Delta &= \frac{\cos t_1}{(3+\cos t_1)^{\frac{3}{4}}}, \\ X &= 8\cdot 2^{\frac{1}{4}} - 4(3+\cos t_1)^{\frac{1}{4}}. \end{aligned}\right\} \tag{21}$$

The theoretical solution can be continued, reversing the sign of m every time t_1 is increased by π, until eventually X reaches the ultimate value of 4·2, slightly greater than the height at which the buoyancy forces first vanish, as might be expected. Curves for the variation of R, U, Δ, X with time t_1 for the first three quarter-periods of the theoretical motion are given in figure 7. In figure 8 are given the curves for R, U, Δ with height X for the rising cloud up to its maximum height.

From above, the solution is given by

$$\left.\begin{aligned} u &= \frac{M}{V} = \frac{1}{4}\left(\frac{3}{\pi}\right)^{\frac{1}{4}} \alpha^{-\frac{3}{4}} k^{-\frac{5}{4}} F_0^{\frac{1}{4}} G^{\frac{1}{4}} U, \\ b &= V^{\frac{1}{3}} = \left(\frac{3}{\pi}\right)^{\frac{1}{4}} (\alpha k)^{\frac{1}{4}} F_0^{\frac{1}{4}} G^{-\frac{1}{4}} R, \\ x &= \frac{1}{4}\left(\frac{3}{\pi}\right)^{\frac{1}{4}} (\alpha k)^{-\frac{3}{4}} F_0^{\frac{1}{4}} G^{-}\, X, \\ g\frac{\rho_0-\rho}{\rho_1} &= \frac{F^*}{V} = \frac{1}{4}\left(\frac{3}{\pi}\right)^{\frac{1}{4}} \alpha^{-\frac{3}{4}} k^{-\frac{7}{4}} F_0^{\frac{1}{4}} G^{\frac{3}{4}} \Delta. \end{aligned}\right\} \tag{22}$$

To determine these quantities completely both α and k must be known. However, if the ultimate height of the cloud is the only feature of the motion that is required it is sufficient to find the product αk. Particular assumptions about profiles of the cloud would seem to have so little significance that the present experimental results will instead be used to find a value for αk without more detailed study of the motion itself.

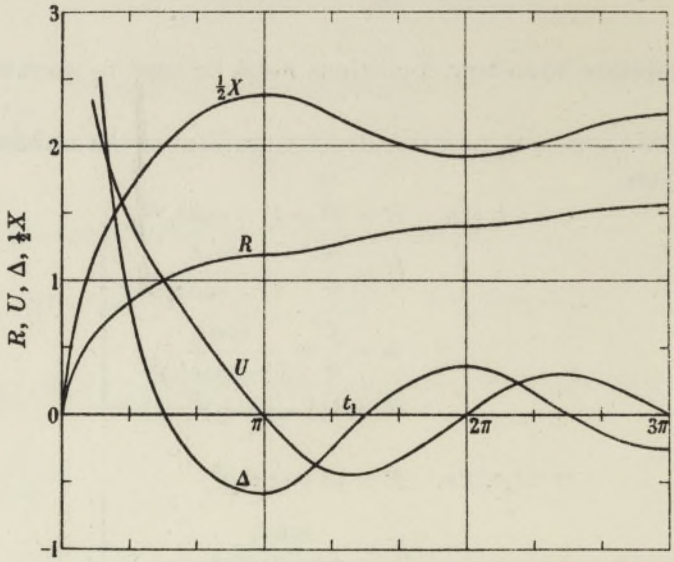

FIGURE 7. The solution curves for the turbulent cloud in a stably stratified fluid showing the oscillatory nature of the theoretical solution.

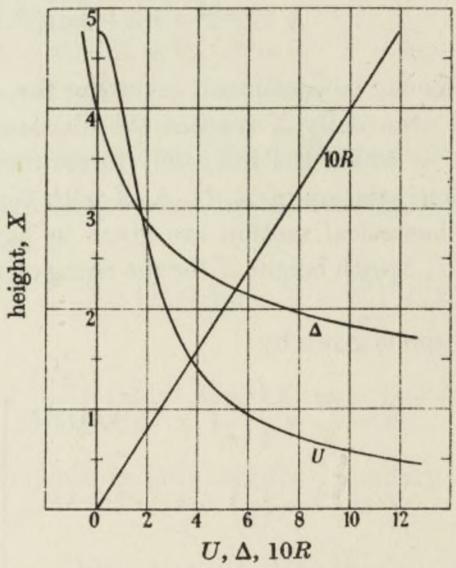

FIGURE 8. The solution curves for the turbulent cloud in a uniformly and stably stratified ambient fluid up to the height at which the vertical velocity first vanishes.

Experiments on clouds

Experiments were carried out in the stably stratified salt solution in which discrete clouds were released suddenly by removing the cover from the top of a small reservoir, which contained a known volume of light fluid.

The heights of the tops of the clouds above the top of the reservoir are plotted against $F_0^{\frac{1}{4}}G^{-\frac{1}{2}}$ in figure 9. The regression line is

$$H = 2{\cdot}66 F_0^{\frac{1}{4}}G^{-\frac{1}{2}} - 4{\cdot}51, \tag{23}$$

where H is the ultimate height of the top of the cloud, and the correlation coefficient between H and $F_0^{\frac{1}{4}}G^{-\frac{1}{2}}$ is $r = 0{\cdot}98$. Again there is a virtual source, but in this case it is difficult to give any meaning to it since the reservoir was filled to different depths for the various values of F_0.

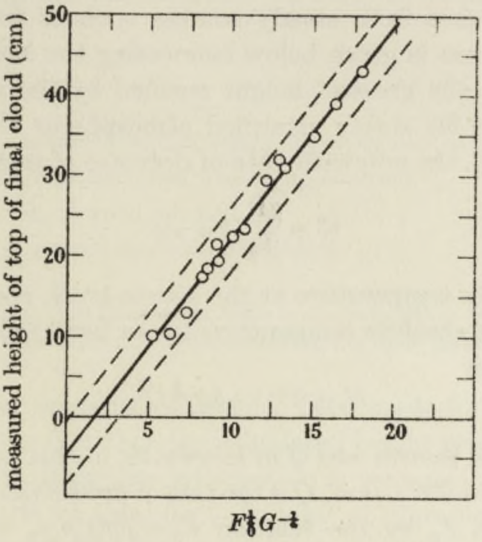

FIGURE 9. Measurements of the ultimate heights of discrete clouds. The line of best fit is drawn in, and also limits of twice the standard error of the estimate.

The slope of 2·66 from the graph leads to a value of $\alpha k = 0{\cdot}285$, corresponding to $X = 4{\cdot}2$ for the ultimate height of the cloud in equilibrium. In this case there is reason to believe that this ultimate height is the one that is measured experimentally, since experimental clouds are seen to oscillate about their final height. However, the limits to the values for αk imposed by fitting to the heights at which Δ and U first vanish are respectively,

$$\alpha k = 0{\cdot}27 \quad \text{and} \quad \alpha k = 0{\cdot}34.$$

AN APPLICATION TO ATMOSPHERIC CONVECTION

The treatment given above applies directly to incompressible fluids. In atmospheric convection the vertical extent of the motion will be such that it is impossible to treat the air as an incompressible fluid. In such cases it has been customary to appeal to the intuitive idea that real motions in the earth's atmosphere, described by velocities, potential temperatures and potential densities, are formally equivalent to motions in a similar but incompressible atmosphere, described by velocities, absolute temperatures and densities. If this idea is used, then by substituting potential temperatures and potential densities for absolute temperatures and densities in the equations derived above, the analysis can be applied to the atmosphere. In particular, for thermal convection in a still atmosphere which has a constant positive gradient of the potential temperature in the vertical direction it is possible to predict the height to which smoke will rise from various sources. It must be stressed that these calculations apply to a still atmosphere. If the medium is itself moving horizontally the plume will be carried along in the direction of the wind, and will have suffered more entrainment by the time a given height is reached. Thus the estimated heights will be too great under windy conditions, but the magnitude of this effect will not be considered at present.

For maintained plumes from steady sources such as fires, smoke stacks, or burning forests a relation is given below connecting the height of the top of the plume (assumed to be the greatest height reached by the plume fluid) with the strength of the source for stably stratified atmospheres of known temperature gradient. In terms of Γ, the adiabatic rate of decrease of temperature with height,

$$G = \frac{g\Gamma}{T_1}(1+n), \tag{24}$$

where T_1 is the absolute temperature at the source level, and n is the ratio of the vertical gradient of the absolute temperature to the lapse rate Γ. The height of the top of the plume is then

$$H = 31(1+n)^{-\frac{3}{8}} Q^{\frac{1}{4}}, \tag{25}$$

where H is measured in metres and Q in kilowatts; in this calculation T_1 has been taken as 288° K and Γ as 9·8° C/km. Q is the rate of production of heat at the source, and is connected with F_0 by the relation $F_0 = \beta g Q/\rho_1 c_p$ where $\beta = 1/T_1$ is the coefficient of expansion and c_p is the specific heat at constant pressure for air. The value of the constant in equation (14) has been taken from the experimental results as $\alpha = 0{\cdot}093$. As an example of equation (25) for the I.C.A.N. standard atmosphere (having $T_1 = 288°$ K and a rate of decrease in temperature with height of 6·5° C/km),

$$H_{n=-0{\cdot}66} = 46 Q^{\frac{1}{4}}. \tag{26}$$

A particular application of these results might be made to a burning forest. In a conflagration in which 1000 tons of forest wood is burnt each hour, assuming that burning wood gives off 4000 cal/g, heat is produced at the rate of 5×10^6 kW, approximately. According to equation (26) if the temperature in the atmosphere decreases vertically at 6·5° C/km and the ground temperature of the atmosphere is 15° C then the plume of smoke will rise to about 2200 m, or approximately 7000 ft. In this way it would be possible to estimate the rate at which a forest fire is burning by observing the height to which the smoke rises. Another interesting application is to determine the total rate of heat output from a volcano to the air by measuring the height of the fume or ash cloud above the crater. For example, under the above conditions a cloud reaching 5000 m would imply an output of about $1{\cdot}4 \times 10^8$ kW.

In table 3 other typical examples are given for the case of an I.C.A.N. standard atmosphere. As x depends on $G^{-\frac{3}{8}}$, small variations in the temperature gradient in the undisturbed atmosphere have comparatively little influence on the height of the plume. For example, for temperature gradients that differ from the adiabatic lapse rate by $+20°$ C and $+\frac{1}{2}°$ C/km the heights of the plume top in metres are $22Q^{\frac{1}{4}}$ and $94Q^{\frac{1}{4}}$ respectively. However, on autumm evenings strong inversions may be produced near the ground because of radiation. Under such conditions the temperature may actually rise by as much as 5° C in the first 40 m. Under this inversion a bonfire burning the equivalent of about a hundred-weight of wood per hour would send up a plume to a height of about 150 ft.

TABLE 3. THE HEIGHT IN METRES TO WHICH SMOKE WILL RISE FROM VARIOUS SOURCES IN A WIND-FREE I.C.A.N. STANDARD ATMOSPHERE (LAPSE RATE 6·5° C/KM)

source	assumed rate of consumption of fuel	Q, rate of release of heat (kW)	H, approx. height (m)
household chimney	4 lb./h of coal, half wasted up chimney	8	80
bonfire	2 cwt. of wood per hour	450	200
large power station chimney	heat wasted equivalent to $\frac{1}{4}$MkW	5×10^5	1200
burning town	250 to 500 houses per hour containing 10 to 20 tons of combustible material, i.e. 5000 tons/h	$2{\cdot}5 \times 10^7$	3200

In none of the above cases has allowance been made for the finite area of the source and hence for the fact that the calculated height refers to a virtual source. For any particular case, if further information is provided about the source itself a correction can be made for this effect. To take an example, if the hot part of the autumn bonfire mentioned above is assumed to be 1 m in diameter the virtual source is approximately 2 m below the fire.

For instantaneous point sources, such as explosions or the rapid burning of highly inflammable material, the ultimate height of the cloud can again be estimated. For the I.C.A.N. standard atmosphere the relation is

$$H = 1{\cdot}87Q^{\frac{1}{4}}, \tag{27}$$

where H is in metres and Q in joules.

As an example, consider the cloud of smoke from the explosion of 100 lb. of T.N.T. The heat released is roughly 4×10^7 cal, or 17×10^7 J, so that the height reached in such a standard atmosphere in the absence of winds is about 200 m.

It has already been pointed out that conditions in the cloud formed at the explosion of an atomic bomb do not fall within the scope of this treatment. If the photographs of this initial cloud are studied, it is seen that there is strongly turbulent interior motion confined within a surface which is relatively smooth during the early stages of the ascent. During this time the high internal temperature of the cloud will fall by radiation and conduction across the intersurface rather than by turbulent entrainment. This explains why the ultimate height reached by the cloud is greater than that predicted by the above theory. Indeed, in the early stages of its motion the atomic bomb cloud behaves more as a bubble of air in water, in which case there is no entrainment at all, and this analogy has already been discussed by Taylor (1950).

Note added, August 1955

Since the above treatment was submitted for publication Priestley & Ball (1955) have given a slightly different solution to the problem of the convective plume. The most striking difference between the two solutions is that according to equation (16) of their paper all buoyant plumes are straight-sided, no matter what the stratification of the ambient fluid may be. This is clearly not true for plumes in a stably stratified fluid, because these reach a maximum height below which the plume fluid spreads out horizontally. Their solution satisfies the requirements of continuity in the upper region of the flow only because the temperature goes to minus infinity and the density to infinity at the height where the velocity vanishes. The treatment given in the present paper is more realistic in this respect, although it is still only approximate since the dynamical effects of lateral flow near the level where the plume spreads out are neglected.

In addition, Priestley & Ball criticize previous treatments (and presumably that above) because of the unreal prediction of a vertical velocity which is greater than zero at the source and decreases steadily upwards, and they state that 'previous solutions are not the most general ones, even in neutral conditions'. This is obviously true, for any values of b, u and ρ may be chosen at a given height, and the equations (2) can be used to find the appropriate solution. Our choice of Schmidt's similarity solution to represent the conditions near the source was dictated by the fact that Schmidt's laboratory experiments showed that they do in fact represent what is observed over a concentrated source in a uniform atmosphere. The consistency of the results given in figure 6 shows that when the discharge momentum is small, the plume from a real source of finite area quickly adopts the form characteristic of turbulent buoyant plumes, such as would rise from a suitably placed virtual source according to the theoretical treatment given above. This result is analogous to that of Kuethe for forced jets. The solution proposed by Priestley & Ball may perhaps be regarded as providing a more realistic description of upward flow from an extended hot area on the ground but it should be noted that it involves an infinite temperature instead of an infinite velocity at the ground level.

The experimental results of Railston (1954) quoted by Priestley & Ball were not used in the present treatment as they were not considered to be relevant to the case of the free turbulent plume. Since Railston discards several of his experimental results on the ground that they correspond with measurements in 'completely turbulent' parts of the flow the inference must be that the bulk of his measurements were on plumes which had not yet attained a steady state. In addition, his experiments cannot be regarded as characteristic of a free plume since he placed his source at the bottom of a shield which would prevent free horizontal inflow into the jet—a condition which is essential both to the theoretical treatment and to any experiments intended to simulate occurrences in the atmosphere.

Note by Sir Geoffrey Taylor

The work described in this paper was undertaken by the two junior authors at the suggestion of Dr G. K. Batchelor. When they had nearly completed their work it was found that most of the theoretical part of it is almost identical with a treatment which I had written, but had not published, some years ago and had described at the New Zealand meeting of the Pacific Science Association in 1949. A recent paper by Dr Batchelor (1954) revived my interest in the subject and I was preparing a much delayed note on it when the identity of my treatment with that of my co-authors was discovered. For this reason my name appears with theirs, though I contributed very little to the paper. We have greatly benefited by discussions with Dr Batchelor, who first pointed out the exact solutions of the equations for the cloud.

References

Batchelor, G. K. 1954 *Quart. J. R. Met. Soc.* **80**, 339.
Goldstein, S. (ed.) 1938 *Modern developments in fluid dynamics.* Oxford: Clarendon Press.
Hartree, D. R. 1952 *Numerical analysis.* Oxford: Clarendon Press.
Kuethe, A. M. 1935 *J. Appl. Mech.* **2**, 87.
Priestley, C. H. B. & Ball, F. K. 1955 *Quart. J.R. Met. Soc.* **81**, 144.
Railston, W. 1954 *Proc. Phys. Soc.* B, **67**, 42.
Rouse, H., Yih, C. S. & Humphreys, H. W. 1952 *Tellus*, 4, 201.
Schmidt, W. 1941 *Z. angew. Math. Mech.* **21**, 265, 351.
Taylor, Sir Geoffrey 1945 *Dynamics of a mass of hot gas rising in air.* U.S. Atomic Energy Commission MDDC 919. LADC 276.
Taylor, Sir Geoffrey 1950 *Proc. Roy. Soc.* A, **201**, 175.
Yih, C. S. 1951 *Proc. 1st Nat. Congr. Appl. Mech.* p. 941.