2.3 Problems: properties of the Boussinesq set

The Boussinesq set for a stratified rotating fluid may be written:

$$\begin{aligned}
u_t &= -(uu+\pi)_x - (uv)_y - (uw)_z + fv + F_x \\
v_t &= -(vu)_x - (vv+\pi)_y - (vw)_z - fu + F_y \\
w_t &= -(wu)_x - (wv)_y - (ww+\pi)_z + b \\
b_t &= -(bu)_x - (bv)_y - (bw)_z + N^2 w + Q_b
\end{aligned}$$

Plus continuity, which can be expressed as giving us the pressure:

$$\pi = \boldsymbol{\nabla}^{-2}[-\boldsymbol{\nabla}\cdot[(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}] + f\zeta - u\beta + div(\boldsymbol{F}) + b_z]$$

2.3.1 *Energetics of a whole horizontally unbounded fluid*

Multiply the 4 prognostic equations by u,v,w,b respectively to generate prognostic equations for KE = (uu+vv+ww)/2 and PE = bb/($2N^2$). Integrate over the whole fluid, with no lateral boundary fluxes (unbounded, like an atmosphere) and with w=0 at the top and bottom. Show that

$$\begin{aligned}
[KE]_t &= [wb] - \boldsymbol{F}.\boldsymbol{V} \\
[PE]_t &= [Jb] - [wb] \\
[PE+KE]_t &= [Jb] - \boldsymbol{F}.\boldsymbol{V}
\end{aligned}$$

where $J = Q_b/N^2$

Interpret the terms: what causes fluid motion? How is a steady state achieved in a long-lasting moving atmosphere?

How do internal shear instabilities (KH, barotropic) fit into this framework?

2.3.2 *Vorticity equation for the nondivergent motions*

Since the curl of a gradient vanishes in 3D vector calculus, the PGF can be eliminated to form a vorticity equation. Derive at whatever mathematical detail length you like, but with the clear line of reasoning, one of the horizontal components of vorticity. You may like to consult book derivations.

Emphasize especially: What is the role of the *horizontal gradient of buoyancy*?

2.3.3 *Stokes' theorem and vorticity patch reasoning*

Write Stokes' theorem, also known as the Circulation Theorem, for a circle of radius r centered on an isolated small circular patch of vertical vorticity. In this way, show that the tangential component of flow "induced by" an element of vorticity decays with distance as 1/r.

Interpret the result: how can vorticity at one point "induce" flow out to infinite radius? Think about what was specified in this problem.

2.3.4