The great equation-set divide: Boussinesq for theory and understanding, Primitive Equations for realism

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1 Strategy: seeking equation sets we can work with

The Navier-Stokes set of PDE's for fluid flow are highly general, but not tractable. (There is a million-dollar prize for finding a non-trivial solution!) They are excessively general: sound waves are among their solutions, and we don't care about those for meteorology. The hydrostatic Primitive Equations are good for synoptic meteorology, but the thermodynamic equation is still a bit tedious. Meanwhile, nonhydrostatic pressure perturbations p' acting in the z direction as well as the x and y directions are crucially important for supercell thunderstorms, which a curious meteorology student will wants to understand. So it is time to make a split: For global modeling, study and use the Primitive Equations. For convective meteorology, and for any kind of theory (like the baroclinic instability chapter in Lackmann), it's nicer to use the Boussinesq equation set.

2 The Primitive Equations

In defining the Primitive Equations, we conveniently eliminated density by folding in the ideal gas law, so that we have just 5 equations in 5 unknowns. For your NWP module, you should know these by heart. Please write them below by hand, in the form relevant to modeling. For a model, we set up a 3D grid of fixed points (x,y,p) and then integrate the differential equations with respect to time. That is, we take equations like

$$\frac{\partial u}{\partial t} = -\vec{V} \cdot \nabla u + fv + F_x$$

and express them for integration, like stepping forward from time t to $t + \Delta t$:

$$u_{t+\Delta t} = u_t + [-\vec{V} \cdot \nabla u + fv + F_x]\Delta t$$

Your exercise, therefore, is to write all 5 of the Primitive Equations in a form friendly for integration. Remember, these are equations for 5 unknowns (u,v,ω,T,Φ) with coordinates (x,y,p,t) for convenience of writing. Of course, on the globe, you would use latitide and longitude instead of x and y.

Zonal momentum:

$\partial u/\partial t =$

Meridional momentum:

$\partial v/\partial t =$

Vertical momentum (hydrostatic force balance):

$\partial \Phi/\partial p =$

Mass continuity:

$\partial \omega/\partial p =$

Thermodynamic equation:

$\partial T/\partial t =$

3 The other approach: simplify thermodynamics some more

To simplify the equations for studying our changes and motions, it helps to define a motionless background or *reference state*, with a specified background mass field (density ρ_0) that is held up by a background or reference pressure $p_0(z)$ in hydrostatic balance. That force balance is Gravity = vertical PGF, or

$$g = -1/\rho_0 \partial p_0/\partial z$$

Then, to study the PGF in motions like weather, we only need to consider small perturbations p' as deviations from that background pressure p_0. The PGF for weather motions is then

$$PGF' = -1/\rho_0 \nabla p'$$

From that equation, you can actually calculate p' everywhere.

There are two common choices for ρ_0: *Anelastic* where we allow $\rho(z)$ to vary with height, and *Boussinesq* where we assume $\rho_0 = constant$. Both of these *incompressible* equation sets forbid sound waves by setting $\partial\rho/\partial t = 0$. Essentially this sets the speed of sound to infinity. The pressure field p' is defined by the job it has to do, instantly, everywhere: it has to keep density constant, even when other forces try to push mass together or apart (Coriolis, buoyancy, or momentum advection which is sometimes called "inertial force" if the momentum or inertia of the winds would have parcels crashing together or flying apart). That is,

$$\nabla \cdot PGF' = -(\nabla \cdot Forces)$$

Ignoring density variations with height (Boussinesq) is not literally accurate for deep tropospheric weather, so we should probably prefer the Anelastic set. But why lug around ρ_0 in the equations if we are not computing quantitative solutions or writing a numerical model?

4 Simplest: the Boussinesq equation set

The idea of the Boussinesq equations is that density deviations from a reference state ρ_o are neglected, EXCEPT when multiplied by gravity.

With constant density, PGF$_x$ =-1/ρ $\partial p/\partial x = -\partial/\partial x(\pi)$ where $\pi = p/\rho_o$.

But there is a special vertical force called "buoyancy", which is actually the hydrostatic background pressure gradient $\partial p_0/\partial z$ acting on a density that is not quite the background ρ_0 that it is in hydrostatic balance with. If the air is a little less dense (warmer, for instance), then the same pressure gradient will be a larger force *per unit mass*, and this excess upward force is what we call buoyacy. Read the Wikipedia page about Archimedes for instance, to understand the buoyancy force better. Its strength is proportional to g of course, and it depends on the density perturbation, and is positive (upward) for lighter air (negative ρ'), so its form must be:

$$b = -g\rho'/\rho_o$$

The same *concept* of buoyancy carries over to real air, where the ideal gas allows us to write it as:

$b(z) = gT'/T_o = g\theta'/\theta_o$ for dry air at a constant altitude or pressure level, or
$b(z) = gT_v'/T_{v_o}$ using the virtual temperature correction for air + water mixtures

4.1 The full nonlinear nonhydrostatic Boussinesq equations, flux form

Mass continuity:

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

Using mass continuity, <u>*show*</u> that 3D transports of any quantity ψ can be expressed as either advection or flux convergence, interchangeably. That is, show:

$$\nabla \cdot (\psi \vec{V}) = \vec{V} \cdot \nabla \psi$$

Using that, let's write the 3 momentum equations with the usual approximations (neglect of vertical coriolis terms, etc.) in flux form, using the subscript notation for partial derivatives in Cartesian (xyz) coordinates:

$$u_t = -\nabla \cdot (u\vec{V}) - \pi_x + fv - F_x \tag{2}$$

$$v_t = -\nabla \cdot (v\vec{V}) - \pi_y - fu - F_y \tag{3}$$

$$w_t = -\nabla \cdot (w\vec{V}) - \pi_z + b - F_z \tag{4}$$

There is also a thermodynamic equation for b. In addition to diabatic heating Q, the effect of static stability (rising air becomes negatively buoyant, sinking air becomes positively buoyant) can be expressed with the Brunt-Vaisala frequency $N^2 = g/\theta_v(\partial\theta_v/\partial z)$

$$b_t = -\nabla \cdot (b\vec{V}) - N^2 w + Q \tag{5}$$

The last term is the diabatic buoyancy source, $Q = Q_T \times g/T_o$ for heating rate Q_T.

That's it! Five equations in five unknowns.

The pressure field is completely determined by its role in enforcing mass continuity, everywhere and at all times, in the face of all other forces. To get the diagnostic π equation, take the 3D divergence of the momentum equation and use (1).

$$\nabla^2 \pi = b_z - \nabla \cdot (\vec{V} \cdot \nabla \vec{V}) \tag{6}$$

4.2 SUMMARY: The full Boussinesq set

Let's make one more simplification of notation: replace $\frac{\partial \psi}{\partial t}$ with the subscript ψ_t, and likewise for x,y,z coordinates. With that, the full set can be written by gathering the above (1)-(6):

$u_t = -\nabla \cdot (u\vec{V}) - \pi_x + fv + F_x$
$v_t = -\nabla \cdot (v\vec{V}) - \pi_y - fu + F_y$
$w_t = -\nabla \cdot (w\vec{V}) - \pi_z + b + F_z$
$b_t = -\nabla \cdot (b\vec{V}) - N^2 w + Q$
$\nabla^2 \pi = b_z - \nabla \cdot (\vec{V} \cdot \nabla \vec{V})$
5 equations, 5 unknowns. The continuity equation is redundant but may be kept:
$\nabla \cdot \vec{V} = 0$

4.3 Simplifications:

The Boussinesq equations can be simplified: for instance, here are the set for frictionless, linear (small amplitude), hydrostatic gravity waves in a resting atmopshere with no background winds:

$u_t = -\pi_x + fv$

$v_t = -\pi_y - fu$

$0 = -\pi_z + b$ Note: this is now the equation for π directly.

$b_t = -N^2 w + Q$

$u_x + v_y + w_z = 0$

4.4 Exercises:

1. Write out the full Boussinesq set of section 4.2, with the transport terms in advective form rather than flux form. I'll get you started:

$u_t = -uu_x - vu_y - wu_z - \pi_x + fv + F_x$

2. Write out the u equation in exercise 1. in normal Leibnitz notation for derivatives, using $\frac{\partial}{\partial t}$ rather than the subscript $_t$

3. Using the subscript notation, eliminate π from your u and v equations in 1. to form a vorticity equation. Don't forget to retain the term f_y. Circle the terms that are retained in our most meteorological form, with only stretching on the RHS: $d\zeta_a/dt = -\zeta_a(\nabla \cdot \vec{V}_h)$

4. In Lackmann Chapter 8, we are going to study the evolution of small-amplitude weather fluctuations in the presence of a sheared westerly jet stream which is in a state of thermal wind balance. That strong background jet stream can be specified as U(y,z), with no dependence on x or t. Its state of thermal wind balance requires a strong pressure field structure $\Pi(y, z)$ which is hydrostatically balanced with a buoyancy structure (meridional temperature gradient) B(y,z).

Let's denote the small amplitude of the weather fluctuations by ' symbols, like u', v', w', b'. These are the weather fluctuations we want to study, so they are functions of every coordinate, like u'(x,y,z,t).

In order to proceed, write our equation set governing u', v', w', b'. To do this, **substitute** u = U(y,z) + u'(x,y,z,t), v = v'(x,y,z,t), w = w'(x,y,z,t), b = B(y,z) + b'(x,y,z,t), and $\pi = \Pi(y, z) + \pi'$(x,y,z,t) into the equations from exercise 1. Obviously, this just involves writing the ' for v and w, but for u and b the substitution will make the equation longer. Now because the product of two small numbers is extremely small, cross out all the "non-linear" terms that involve the product of two primes, like $v'\partial w'/\partial y$ and $w'\partial u'/\partial z$ and all the rest. But of course you keep the terms with just one prime ' in them, like $v'\partial B/\partial y$ and $U\partial w'/\partial x$. In this way, **write** the 5 equations in 5 unknowns that we will need to solve: