MPO 663 – Spring 2019 was Due March 9, 2017

Hydrostatic gravity waves and the Rossby radius

The point of this problem set is to see how gravity waves exist, as sine & cosine solutions to the equations governing a stratified fluid with a static stability measured by N^2. We will initially neglect the Coriolis force, to focus on high-frequency waves for which the Earth doesn't rotate importantly during an oscillation. Keep your eyes peeled for why gravity wave speed has a strong dependence on vertical wavelength: *vertically thick warm or cool anomalies travel faster in the horizontal*.

Then, reintroducing the Coriolis force to discuss longer-term consequences of a heating event (like a convective event with net rainfall and thus net latent heating), you will consider a balanced vortex flow. This is again a solution to the same simple equation set, this time characterized by exponential decay away from the vortex center. It should not surprise you the exponentials and sine-cosine functions are solutions to second-order differential equations! Indeed, both are special cases of the complex exponential function.

You'll see that the *Rossby radius* is related to the distance a gravity wave can travel in an 'inertial' time 1/f, the time for the ground to rotate half a circle under the fluid.

We use the **simplest possible** equation framework: *hydrostatic, Boussinesq (constant density) uniformly stratified (constant N^2)* atmosphere, linearized for small perturbations about a resting state (that is, with advection terms neglected entirely).

Subscripts denote derivatives here: for example, the pressure gradient force can be written as $\pi_x \equiv p'_x / \rho$ (π is sometimes called the Exner function).

1. **Write down** our full equation set, from the confidence of your own knowledge. **Simplify** them down to this form:

$u_t = fv - \pi_x$	u momentum equation
$v_t = -fu - \pi_y$	v momentum equation
$w_t = b - \pi_z$	w momentum equation (=0 for hydrostatic)
$u_x + v_y + w_z = 0$	continuity equation
$b_t = -N^2 w + q$	thermodynamic equation w/ heating ($q = g/\theta\, D\theta/Dt$)

Let's study 2D wave solutions in the x-z plane only, in the hydrostatic limit. In that case, we can separate the vertical and horizontal structure equations, and study *shallow water equations* in (x,t) space that represent different *vertical modes* (sines and cosines in z). So, neglect y variations, and set aside the Coriolis term, which can be added back in later. **Show** that this simplifies the equation set to:

$u_t = -\pi_x$	u momentum equation
$0 = b - \pi_z$	w momentum equation (hydrostatic)
$u_x + w_z = 0$	continuity equation
$b_t = -N^2 w + q$	thermodynamic equation, with heating q

2. Let's look for vertically sinusoidal solutions of the form $u = U(x,t)\cos(mz)$, $\pi = \Phi(x,t)\cos(mz)$, $w = W(x,t)\sin(mz)$, $b = B(x,t)\sin(mz)$, $q = Q(x,t)\sin(mz)$.

Confirm (write from the confidence of your own knowledge) that these satisfy the atmospherically relevant boundary condition w=0 at z=0, for all *half-integer* vertical wavenumbers m =1/2, 1, 3/2,…

Plug these forms into the 4 equations above, then **substitute** the hydrostatic equation into the thermo equation and substitute the result into the continuity equation to **show:**

$U_t = -\Phi_x$	u momentum equation
$\Phi_t = -N^2/m^2\,(U_x + \delta_d)$	height equation w/ source term δ_d

Here the effect of heating can be expressed as the *diabatic divergence* $\delta_d \equiv \partial/\partial z\,(q/N^2) = mQ\cos(mz)/N^2$ in our case. **What are its units?** Why do I call it a *divergence*?

Rewrite these to make clearer that they are mathematically the same as the linear *shallow water equations*, which govern surface waves on a shallow layer of water (or any inviscid fluid), after they are created by a *mass source* δ_d. In the shallow water interpretation, the water surface height anomalies h have amplitude $h = \Phi/g$, and they travel on a shallow (compared to wavelength) layer with a depth $H = N^2/(gm^2)$. In other words, *there is a shallow water set for each vertical mode m*, and these can be multiplied by sin(mz) or cos(mz) as appropriate for each field and summed up to give the total solution for arbitrary vertical structures of forcing. **What are approximate values of H for typical tropospheric values of N and m? Confirm the units.** You will need to calculate what N is in the troposphere, from its definition $N^2 = (g/\theta)\,\theta_z$. What is the *period* of a BV oscillation, $2\pi/N$, the shortest period gravity wave in the atmosphere (with purely vertical parcel oscillations)?

Remember, m corresponds to the vertical wavenumber ($m = 2\pi/$ vertical wavelength) in the stratified fluid. Each half-integer vertical wavenumber (obeying the boundary conditions) has a different value of H, called its *equivalent depth*, and thus a different wave speed.

3. **Confirm** by revisiting your very simple derivation in 2. that the v equation, Coriolis terms, and u and v momentum source terms F and G could have been carried along in the same way, to give the *forced* shallow water set for each vertical mode (value of m):

$U_t = fV - \Phi_x + F$	u momentum equation
$V_t = -fU - \Phi_y + G$	v momentum equation
$\Phi_t = -N^2/m^2\,(U_x + V_y + \delta_d)$	how divergence changes geopotential Φ

4. Now instead of wave solutions, let's think about a *steady, geostrophically balanced vortex* solution to this equation set. Physically, we solve for the structure of an inviscid (F=G=0), steady, balanced, 2-dimensional *line vortex* (that is, a shear line), created by a some *past* heating along x = 0, on an f-plane with $f = 10^{-4}\ s^{-1}$. The solution is shown in Matlab figures at the end of this homework.

Suppose that the past heating or δ_d which created the balanced vortex was a narrow line of heating along x=0, which only excited a single vertical wavenumber m. **Confirm** that this balanced flow should have no y or t derivatives (i.e., the flow is steady state), with $\delta_d = 0$ at the current time. **Write** the steady state version of the 3-equation set from part 3. **What is the familiar name (physical meaning) of that steady state equation set**?

Does it give us the structure of the vortex**?** Why not? Does it contain the information that we are specifically seeking a vortex *created by a past heating along x = 0***?** If not, how can we put that information into the solution? The answer is...

5. To define our balanced vortex structure, you need to **express mathematically** the statement that *PV is constant and equal to its pre-heating value everywhere except along the line x=0 where the heating occurred*. **Solve** this equation for $\Phi(x)$ and $V(x)$.

Now interpret the horizontal length scale for the vortex's exponentially decaying structure in x: this length is called the *Rossby deformation radius*. What is its dependence on m? (**Extra credit**: solve in radial coordinates for a circular vortex if you can. Not so easy, Bessel functions.)

6. Let's think about and visualize (by computer) the balanced vortex created by *2-vertical-mode* heating at the origin. That is, suppose the heating profile (which occurred at x=0 in the distant past) was $q = Q\,[\sin(m_1 z) - S \sin(m_2 z)]$, where $m_1 = \pi/14\text{km}$ and $m_2 = 2\pi/14\text{km}$.

Plots are shown for the heating profiles in the domain $z \in [0,14\text{km}]$ for S = -0.5, 0, and 0.5. [S may be viewed as a "stratiform fraction" when positive, or "shallow convection fraction" when negative, based on typical moist convective heating profiles.]

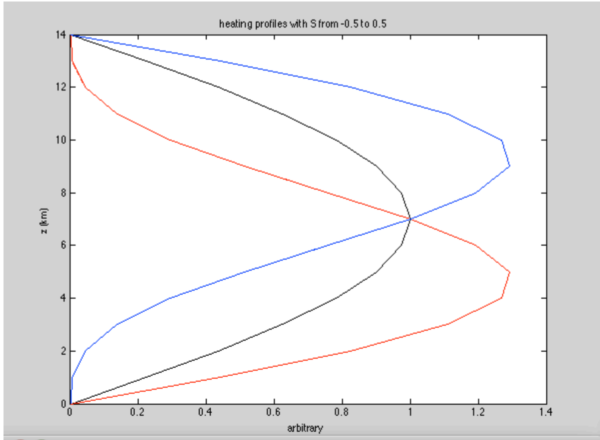
Verify that the total heat added to the fluid (the vertical integral of the heating profile curve) is the same in all cases.

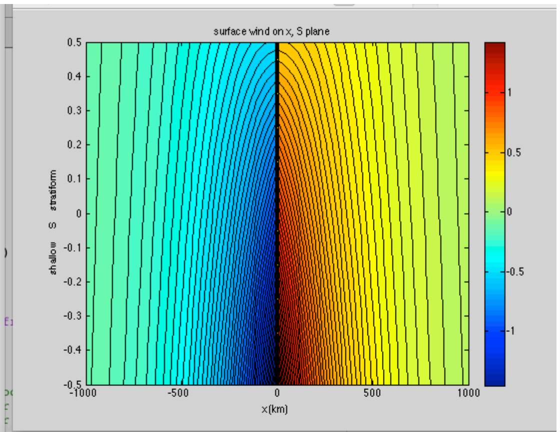
Because the equation set is linear, you can simply evaluate the vortex formulas for v(z,x) for the two values of m separately, and then construct the S-weighted sum of these two arrays. Also, because the equations are linear, the absolute magnitude of the total solution is arbitrary. With $N = 2\pi/600\text{s}$ (a typical tropospheric stratification), you can give distance an actual physical scale (based on the Rossby radius).

I. **What are the Rossby radii** for the two vertical modes?

II. **Consider** the domain-integrated KE versus S for a range of values of S in [-0.5,0.5]. Why is the curve nonlinear?

III. Surface flux is approximately proportional to *surface* wind speed. **Examine** the domain-averaged surface wind speed versus S for values of S in [-0.5,0.5]. **Discuss** the implications for tropical cyclogenesis, in terms of convective heating profiles and their dependence on low-level rain evaporation and downdrafts (which change S, and thus make the net heating felt by the dynamics more top-heavy in shape).


heating profiles with S from -0.5 to 0.5
z (km)
arbitrary


surface wind on x, S plane
shallow S stratiform
x(km)

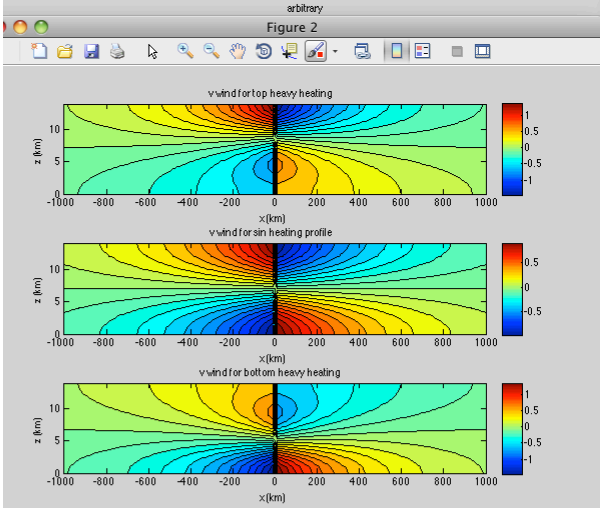


Figure 2
v wind for top heavy heating
v wind for sin heating profile
v wind for bottom heavy heating
z (km)
x(km)

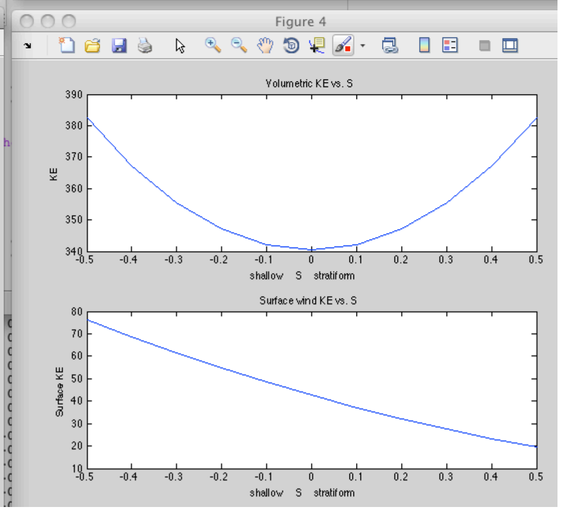
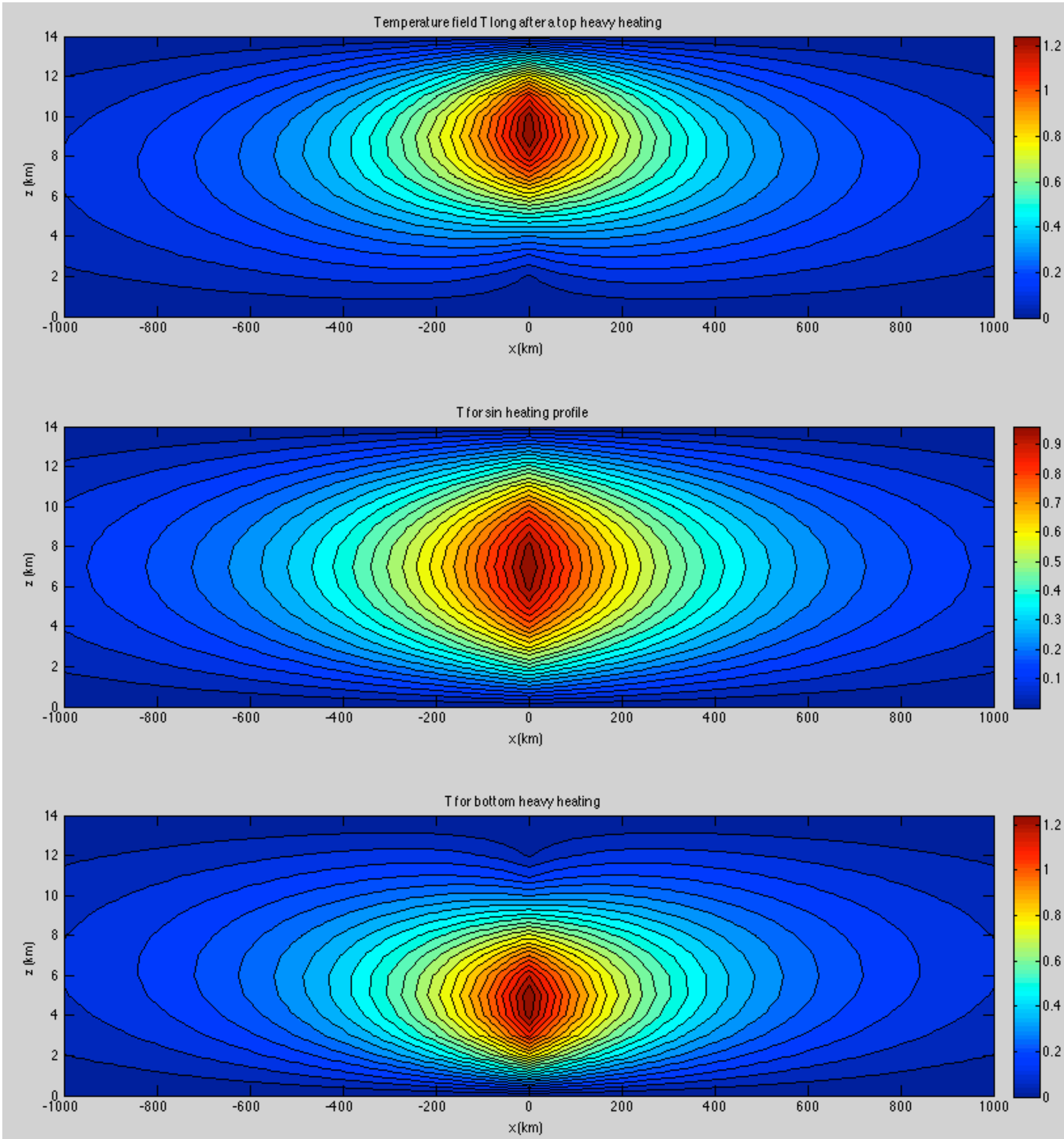


Figure 4
Volumetric KE vs. S
KE
shallow S stratiform
Surface wind KE vs. S
Surface KE
shallow S stratiform


Temperature field T long after a top heavy heating
z (km)
x(km)
-1000
-800
-600
-400
-200
0
200
400
600
800
1000
0
2
4
6
8
10
12
14
0
0.2
0.4
0.6
0.8
1
1.2
T for sin heating profile
z (km)
x(km)
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
T for bottom heavy heating
z (km)
x(km)