MPO 563 - Convective and Mesoscale Meteorology

HW 4 - Hydrostatic Gravity Waves, the Rossby Radius

100
beautiful!

#1

(i) $0 = u_x + v_y + w_z$ — conservation of mass

(ii) $u_t = -uu_x - vu_y - wu_z + fv - \pi_x + F$ — u-momentum

(iii) $v_t = -uv_x - vv_y - wv_z - fu - \pi_y + G$ — v-momentum ✓

(iv) $w_t = -uw_x - vw_y - ww_z - b - \pi_z$ — w-momentum

(v) $b_t = -ub_x - vb_y - wb_z - N^2 w + q$ — thermodynamic eq.

1) neglecting the advection effects 2) neglecting friction

This results in the desired equations:

(i) $0 = u_x + v_y + w_z$

(ii) $u_t = fv - \pi_x$ ✓

(iii) $v_t = -fu - \pi_y$

(iv) $w_t = b - \pi_z$

(v) $b_t = -N^2 w + q$

Now consider the shallow water equations in (x,t) space:

- wave solutions in x-z plane (vertical modes) ✓ nice!
- hydrostatic limit ✓ Colorized
- set aside the Coriolis term (for now) ✓

This results in:

(i) $0 = u_x + w_z$

(ii) $u_t = -\pi_x$

(iv) $0 = b - \pi_z$

(v) $b_t = -N^2 w + q$

Note: $q = \frac{g}{\theta}\frac{D\theta}{Dt}$ ✓

#2 Look for vertically sinusoidal solutions:

$u(x,z,t) = U(x,t)\cos(mz)$

$\pi(x,z,t) = \Phi(x,t)\cos(mz)$

$w(x,z,t) = W(x,t)\sin(mz)$

$b(x,z,t) = B(x,t)\sin(mz)$

$q(x,z,t) = Q(x,t)\sin(mz)$

Check that $w=0$ @ $z=0$ for all m:

$$w(x,t) = W(x,t)\sin(mz)$$

@ $z=0$, $w(x,t) = W(x,t)\sin(m\cdot 0) = 0$ ← no vertical motion at surface.

(i) $0 = u_x + w_z$

$$0 = U_x(x,t)\cancel{\cos}(mz) + mW(x,t)\cancel{\cos}(mz)$$

$$0 = U_x(x,t) + mW(x,t)$$

$$U_x(x,t) = -mW(x,t) \quad Ⓐ$$

(ii) $u_t = -\pi_x$

$$U_t(x,t)\cdot\cancel{\cos}(mz) = -\Phi_x(x,t)\cancel{\cos}(mz)$$

$$U_t(x,t) = -\Phi_x(x,t) \quad Ⓑ$$

(iv) $0 = b - \pi_z$

$$0 = B(x,t)\cancel{\sin}(mz) + m\Phi(x,t)\cancel{\sin}(mz)$$

$$0 = B(x,t) + m\Phi(x,t)$$

$$B(x,t) = -m\Phi(x,t) \quad Ⓒ$$

(v) $b_t = -N^2 w + q$

$$B_t(x,t)\cancel{\sin}(mz) = -N^2W(x,t)\cancel{\sin}(mz) + Q(x,t)\cancel{\sin}(mz)$$

$$B_t(x,t) = -N^2W(x,t) + Q(x,t) \quad Ⓓ$$

$Ⓒ_t \to Ⓓ$

$$-m\Phi_t(x,t) = -N^2W(x,t) + Q(x,t)$$

$$N^2W(x,t) = m\Phi_t(x,t) + Q(x,t)$$

$$W(x,t) = \frac{m}{N^2}\Phi_t(x,t) + \frac{1}{N^2}Q(x,t) \quad Ⓔ$$

$Ⓔ \to Ⓐ$

$$U_x(x,t) = -\frac{m^2}{N^2}\Phi_t(x,t) - \frac{m}{N^2}Q(x,t)$$

$$\frac{m^2}{N^2}\Phi_t(x,t) = -U_x(x,t) - \frac{m}{N^2}Q(x,t) \quad /\cdot\frac{N^2}{m^2}$$

$$\Phi_t(x,t) = -\frac{N^2}{m^2}U_x(x,t) - \frac{1}{m}Q(x,t)$$

$$\Phi_t(x,t) = -\frac{N^2}{m^2}\left[U_x(x,t) + \frac{m}{N^2}Q(x,t)\right]$$

$$\Phi_t(x,t) = -\frac{N^2}{m^2}\left[U_x(x,t) + \delta_d\right] \quad Ⓕ$$

$$\boxed{\begin{aligned} &\delta_d\cos(mz) = \frac{\partial}{\partial z}\left[\frac{q}{N^2}\right] \\ &\delta_d\cancel{\cos}(mz) = \frac{m}{N^2}Q(x,t)\cancel{\cos}(mz) \\ &\delta_d = \frac{m}{N^2}Q(x,t) \\ &\text{Units: }\left[\frac{1}{\cancel{m}}\cdot s^2\cdot\frac{\cancel{m}\cdot\cancel{K}}{\cancel{s^2}\cdot\cancel{K}\cdot s}\right] = \left[\frac{1}{s}\right] \end{aligned}}$$

$\boxed{\#3}$ If we look at the intermediate equation set from #1:

(i) $0 = u_x + v_y + w_z$

(ii) $u_t = fv - \pi_x + F$

(iii) $v_t = -fu - \pi_y + G$

(iv) $\cancel{w_t} = b - \pi_z$

(v) $b_t = -N^2 w + q$

• wave solutions in (x,y,z,t) space

• hydrostatic limit

and assuming waveform solutions for new variables in (x,y,z,t) space:

$u(x,y,z,t) = U(x,y,t)\cos(mz)$

$v(x,y,z,t) = V(x,y,t)\cos(mz)$

$w(x,y,z,t) = W(x,y,t)\sin(mz)$

$b(x,y,z,t) = B(x,y,t)\sin(mz)$

$q(x,y,z,t) = Q(x,y,t)\sin(mz)$

$F(x,y,z,t) = \mathcal{F}(x,y,t)\cos(mz)$

$G(x,y,z,t) = \mathcal{G}(x,y,t)\cos(mz)$

$\pi(x,y,z,t) = \Phi(x,y,t)\cos(mz)$

(i) $0 = u_x + v_y + w_z$

$0 = U_x \cancel{\cos(mz)} + V_y \cancel{\cos(mz)} + mW\cancel{\cos(mz)}$

$\underline{mW(x,y,t) = -U_x - V_y}$ (A) ✓

(ii) $u_t = fv - \pi_x + F$

$U_t \cancel{\cos(mz)} = fV\cancel{\cos(mz)} - m\Phi_x \cancel{\cos(mz)} + \mathcal{F}\cancel{\cos(mz)}$

$\underline{U_t = fV - m\Phi_x + \mathcal{F}}$ (B) ✓

(iii) $v_t = -fu - \pi_y + G$

$V_t \cancel{\cos(mz)} = -fU\cancel{\cos(mz)} - \Phi_y \cancel{\cos(mz)} + \mathcal{G}\cancel{\cos(mz)}$

$\underline{V_t = -fU - \Phi_y + \mathcal{G}}$ (C) ✓

(iv) $0 = b - \pi_z$

$0 = B\cancel{\sin(mz)} + m\Phi\cancel{\sin(mz)}$

$\underline{B = -m\Phi}$ (D) ✓

(v) $b_t = -N^2 w + q$

$B_t \cancel{\sin(mz)} = -N^2 W \cancel{\sin(mz)} + Q\cancel{\sin(mz)}$

$\underline{B_t = -N^2 W + Q}$ (E) ✓

No vertical differentiation of these terms, so they flow right through! ✓

This yields the desired set of equations:

$U_t(x,t) = -\Phi_x(x,t)$ (B)

$\Phi_t(x,t) = -\frac{N^2}{m^2}\left[U_x(x,t) + \delta_d\right]$ (F)

The units of δ_d are s^{-1}. ✓

In tropical convection, diabatic divergence δ_d is nearly equal to actual horizontal divergence (Mapes & Houze, 1995).

I think it's a divergence, because it is a vertical divergence of diabatic heating, q, modulated by the buoyancy frequency (N^2), a measure of stability. ✓

Could define $w_d = \left(\frac{q}{N^2}\right)$, as a vertical (cross-isentropic) velocity ~~or~~ flux, and δ_d is the (vertical) divergence of that flux.

To rewrite these as shallow water equations, use:

- δ_d – mass source
- $h = \frac{\Phi}{g}$ – surface height anomalies ($\Phi = gh$)
- $H = \frac{N^2}{gm^2}$ – mean depth of layer

(B) $U_t(x,t) = -\Phi_x(x,t)$

$U_t(x,t) = -gh_x(x,t)$ (G) $\quad \longleftrightarrow \quad u_t = gh_x$

(F) $\Phi_t(x,t) = -\frac{N^2}{m^2}\left[U_x(x,t) + \delta_d\right]$

$gh_t(x,t) = -\frac{N^2}{m^2}\left[U_x(x,t) + \delta_d\right]$

$h_t(x,t) = -H\left[U_x(x,t) + \delta_d\right]$ (H) $\quad \longleftrightarrow \quad h_t = -H\left[u_x + \delta_d\right]$

simplified SWE ✓

For atmospheric values of buoyancy frequency $\frac{2\pi}{N} = 10\text{ min} = 600\text{s}$, $N = 0.01\ s^{-1}$ and a vertical wavenumber such that $\frac{2\pi}{m} = 10000\text{m}$, $m = 6.28\cdot10^{-4}\text{m}^{-1}$, this would yield an equiv. depth $H = \frac{N^2}{gm^2} = \frac{(0.01)^2}{9.81\cdot(6.28\cdot10^{-4})^2} = 25.8\text{ m}$. For a higher wavenumber, eg. $\frac{2\pi}{m} = 5000\text{m}$, $m = 1.26\cdot10^{-3}\text{m}^{-1}$, the equivalent depth would be $H = \frac{N^2}{gm^2} = \frac{(0.01)^2}{9.81\cdot(1.26\cdot10^{-3})^2} = 6.42\text{ m}$. ✓

Since the shallow water wave speed is directly proportional to fluid depth H, we can confirm that vertically extensive waves (lower wavenumber) travel faster. ✓

$\frac{\partial}{\partial t}$ D $\longrightarrow$ E

$$-m\Phi_t = -N^2 W + Q$$

$$N^2 W = m\Phi_t + Q$$

$$\underline{W = \frac{m}{N^2}\Phi_t + \frac{1}{N^2}Q} \quad \text{(F)}$$

F $\longrightarrow$ A

$$\frac{m^2}{N^2}\Phi_t + \frac{m}{N^2}Q = -U_x - V_y$$

$$\frac{m^2}{N^2}\Phi_t = -U_x - V_y - \frac{m}{N^2}Q \quad / \cdot \frac{N^2}{m^2}$$

$$\Phi_t = -\frac{N^2}{m^2}\left[U_x + V_y + \frac{m}{N^2}Q\right]$$

$$\underline{\Phi_t = -\frac{N^2}{m^2}\left[U_x + V_y + \delta_d\right]} \quad \text{(G)}$$

This yields the desired equation set:

$$U_t = fV - \Phi_x + F \quad \text{(B)}$$

$$V_t = -fU - \Phi_y + G \quad \text{(C)}$$

$$\Phi_t = -\frac{N^2}{m^2}\left[U_x + V_y + \delta_d\right] \quad \text{(G)}$$

This shows they could have been carried along the entire time. ✓

$\boxed{\#4}$ Now we solve the equations for an inviscid $(F = G = 0)$, steady $(X_t = 0)$, balanced, 2-dimensional line vortex, created by some past heating along $x=0$, on an f-plane with $f = 10^{-4}s^{-1}$.

(i) $0 = fV - \Phi_x$

(ii) $0 = -fU - \Phi_y$ } geostrophic balance

(iii) $0 = -\frac{N^2}{m^2}\left[U_x + \delta_d\right]$

$\rightarrow$ $fV = \Phi_x$ (i)

$\rightarrow$ $fU = -\Phi_y$ (ii) ✓

This equation set does not give information about the vortex structure because information about Φ is lost. ✓

$\boxed{\#5}$ Let's construct a PV equation from the 2D shallow water set with $F = G = Q = 0$.

A $U_t = fV - \Phi_x$

B $V_t = -fU - \Phi_y$

C $h_t = -H\left[U_x + V_y\right]$

$\frac{\partial}{\partial y}$ Ⓐ : $\underline{U_{ty} = fV_y + Vf_y - \Phi_{xy}}$ Ⓓ

$\frac{\partial}{\partial x}$ Ⓑ : $\underline{V_{tx} = -fU_x - \Phi_{xy}}$ Ⓔ

Ⓔ − Ⓓ $V_{tx} - U_{ty} = -fU_x - fV_y - V\cancel{f_y} - \cancel{\Phi_{xy}} + \cancel{\Phi_{xy}}$ Let $V_x - U_y = \zeta$

$\underline{\zeta_t = -f(U_x + V_y)}$ Ⓕ f-plane

Ⓒ → Ⓕ $\zeta_t = -f\left[\frac{h_t}{H}\right] \rightarrow \underline{\underline{\frac{\zeta_t}{f} = \frac{h_t}{H}}}$ PV equation

$\boxed{\#5}$ If we integrate the PV equation in time,

$\frac{\zeta}{f} = \frac{h}{H}$ $\zeta = V_x - U_y$ If we are still considering a single shear line, $U_y = 0 \rightarrow \zeta = V_x$

$\frac{V_x}{f} = \frac{h}{H} \rightarrow V_x = \frac{hf}{H}$

Since the vortex is geostrophically balanced, $V = \frac{gh_x}{f} \rightarrow V_x = \frac{g}{f}h_{xx}$

$V_x = \frac{g}{f}h_{xx} = \frac{hf}{H}$ f, H, g are constants

$$h_{xx} - \frac{f^2}{gH}h = 0$$

g, H, f^2 are all non-negative, so the solution to this system is either exponential growth or decay:

$$h(x) = c_1 e^{\frac{f}{\sqrt{gH}}x} + c_2 e^{-\frac{f}{\sqrt{gH}}x}$$

Since we want the structure to be bounded, $h(x \rightarrow \infty) = 0$ choose the decaying part of the solution (i.e., $c_1 = 0,\ c_2 = 1$)

$$\underline{h(x) = e^{-\frac{f}{\sqrt{gH}}x}}$$

at that value $h(x) = \frac{1}{e}\cdot h(0)$

The length scale for the structure of the vortex is $\frac{\sqrt{gH}}{f}$. The gravity wave speed is $\sqrt{gH}$. Recall $H = \frac{N^2}{gm^2}$, so the Rossby deformation radius $L_R = \sqrt{\frac{gH}{f^2}} = \sqrt{\frac{\cancel{g}N^2}{f^2\cancel{g}m^2}} = \frac{N}{fm}$. Thus the Rossby ~~number~~ radius is inversely proportional to vertical wavenumber, modulated by latitude (f) and stability (N).

$\boxed{\#7}$ $q = Q\sin\left(\frac{\pi z}{14}\right) - S\sin\left(\frac{\pi z}{7}\right)$

$$I = \int_0^{14\,km} q\,dz = \int_0^{14} Q\sin\left(\frac{\pi z}{14}\right)dz - \int_0^{14} S\sin\left(\frac{\pi z}{7}\right)dz \qquad Q,S \neq Q,S(z)$$

$$= Q\int_0^{14}\sin\left(\frac{\pi z}{14}\right)dz - S\int_0^{14}\sin\left(\frac{\pi z}{7}\right)dz$$

$$= -\frac{14Q}{\pi}\left[\cos\left(\frac{\pi z}{14}\right)\right]_0^{14} + \frac{7S}{\pi}\left[\cos\left(\frac{\pi z}{7}\right)\right]_0^{14}$$

$$= -\frac{14Q}{\pi}\left[\cos(\pi) - 1\right] + \frac{7S}{\pi}\left[\cos(2\pi) - 1\right]$$

$$= -\frac{14Q}{\pi}\left[-1-1\right] + \frac{7S}{\pi}\left[1-1\right]$$

$$= \frac{28Q}{\pi}$$

This is independent of S, so the total heat added to the fluid is the same for all cases of S. ✓

$\boxed{\text{I.}}$ $f = 10^{-4}\,s^{-1}$ $\quad N = 0.01\,s^{-1} = 10^{-2}\,s^{-1}$

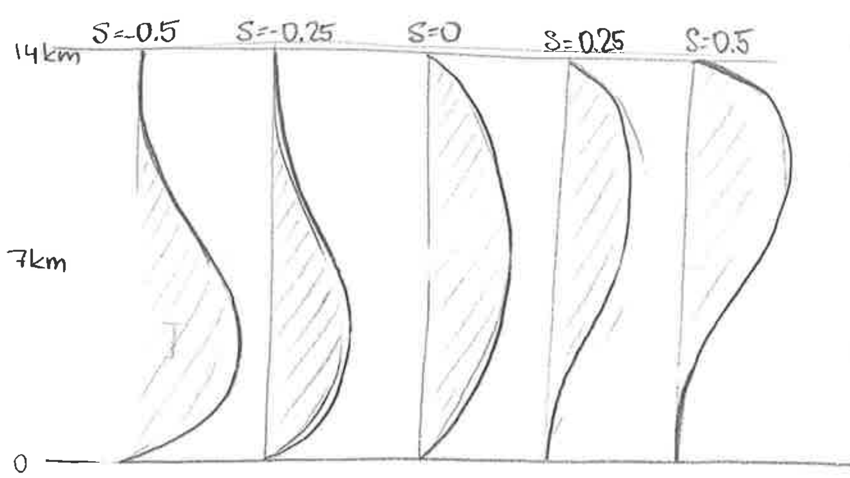
$$m_1 = \frac{\pi}{14000\,m} \rightarrow L_{R_1} = \frac{N}{fm_1} = \frac{10^{-2}\,\cancel{s^{-1}}\,14000\,m}{10^{-4}\,\cancel{s^{-1}}\,\pi} = \frac{14\cdot10^3\,m\cdot10^{-2}\cdot10^4}{\pi} = \frac{14\cdot10^5\,m}{\pi}$$

$$= \underline{445.6\ km}$$ ✓

$$m_2 = \frac{\pi}{7000\,m} \rightarrow L_{R_2} = \frac{N}{fm_2} = \frac{10^{-2}\,\cancel{s^{-1}}\cdot7000\,m}{10^{-4}\,\cancel{s^{-1}}\,\pi} = \frac{7\cdot10^3\,m\cdot10^{-2}\cdot10^4}{\pi} = \frac{7\cdot10^5\,m}{\pi}$$

$$= \underline{222.8\ km}$$ ✓

$\boxed{\text{II.}}$ The curve is non-linear (looks more like quadratic in S). ✓ I think this is due to the following:



$\delta_d \propto \frac{\partial q}{\partial z}$ (vertical gradient in the heating profile). Looking at figures, that increases the tangential wind speed, But $KE \propto V^2$, so regardless of the sign of V, the KE will increase,

$$KE = KE_1 + S^2 KE_2$$ ✓

follows from $U = U_1 + SU_2$, since $\{U_1, U_2\} = 0$ (orthogonal vertical structures).

$\boxed{\text{II.}}$ Low-level rain evaporation and downdrafts change S (increase S) and make the heating more top-heavy in shape. That, according to the surface wind KE v. S, will decrease the KE of the surface winds, thus decreasing surface fluxes, and reducing the strength of surface cyclonic circulation. In this case, the reduced surface fluxes would either hinder the development of a tropical cyclone, or prevent it completely. through lack of sensible and latent heat input from the surface.

lovely treatment!