The following assumes that you have installed GLU, for SU(NC), N_d gauge theory. The smearing transformation routines are called by the input file option,

```
MODE = SMEARING
```

1 Wilsonian smearing

Hi, if you are reading this you are probably interested in the wealth of smearing transformations that the library provides. In general, a smearing transformation is (where the prime indicates a replacement of the link $U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)$),

$$U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)' = \exp\left[\frac{\alpha}{2(N_d-1)}\sum_{\nu\neq\mu} Q_{\mu\nu}(x)\right] U_\mu\left(x+a\frac{\hat{\mu}}{2}\right). \quad (1)$$

With $Q_{\mu\nu}(x)$ defined as the log of the surrounding $\mu-\nu$ 1×1 Wilson loops that end with $U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)^\dagger$,

$$\begin{aligned} P_{\mu\nu}(x) &= \left(U_\nu(x+a\hat{\nu}/2)U_\mu(x+a\hat{\nu}+a\hat{\mu}/2)U_\nu(x+a\hat{\mu}+a\hat{\nu}/2)^\dagger U_\mu(x+a\hat{\mu}/2)^\dagger\right), \\ O_{\mu\nu}(x) &= \left(U_\nu(x-a\hat{\nu}/2)^\dagger U_\mu(x-a\hat{\nu}+a\hat{\mu}/2)U_\nu(x+a\hat{\mu}-a\hat{\nu}/2)U_\mu(x+a\hat{\mu}/2)^\dagger\right), \\ Q_{\mu\nu}(x) &= \log(O_{\mu\nu}(x)) + \log(P_{\mu\nu}(x)). \end{aligned} \quad (2)$$

Various (Wilsonian) smearing procedures are based upon approximations of the smaring transformation in Eq.**??**. They are usually called APE, STOUT and LOG.

The smearing routines are called by the following input file options (anything within {} are the available options),

```
SMEARTYPE = {APE,STOUT,LOG}
```

If we wish to perform the smearing for all polarisations of links we specify `ALL` in the input file,

```
DIRECTION = {ALL,SPATIAL}
```

ANd `SPATIAL` if we want to smear only the (N_d-1) spatial directions.

If we wish to perform iterated smearing, we simply specify the maximum number of iterations in the input file,

```
SMITERS = {}
```

Why this is the maximum will be covered in section **??**.

The smearing parameter α should be less than 0.75 for convergence of the routine. For the improved and overimproved smearing (**??**), a different value for convergence should be used. This is set (no matter what dimension $> 2!$ that we use) by,

```
ALPHA1 = {}
```

The other ALPHA's are related to hypercubically blocked smearing (**??**).

2 Projections

For the APE projection, we use the determinant-rescaled nAPE projection. The projection by SU(2) subgroup trace maximisation is also available, with the toggling of

```
--enable-GIVENS_APE
```

in the configure.

The APE projection is of the form,

$$U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)' = P_{SU(N)}\left((1-\alpha)U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)+\frac{\alpha}{2(N_d-1)}\sum_{\nu\neq\mu}N_{\mu\nu}(x)\right). \tag{3}$$

Where the variable $N_{\mu\nu}$ is the open "staple",

$$\begin{aligned} L_{\mu\nu}(x) &= \left(U_\nu(x+a\hat{\nu}/2)U_\mu(x+a\hat{\nu}+a\hat{\mu}/2)U_\nu(x+a\hat{\mu}+a\hat{\nu}/2)^\dagger\right), \\ M_{\mu\nu}(x) &= \left(U_\nu(x-a\hat{\nu}/2)^\dagger U_\mu(x-a\hat{\nu}+a\hat{\mu}/2)U_\nu(x+a\hat{\mu}-a\hat{\nu}/2)\right), \\ N_{\mu\nu}(x) &= L_{\mu\nu}(x)+M_{\mu\nu}(x). \end{aligned} \tag{4}$$

Considering the exponentiation of the Hermitian matrix A to the special Unitary U, $U = e^{iA}, A = -i\log(U)$. The STOUT projection approximates the Logarithm of the link by using the Hermitian projection,

$$A = \frac{1}{2i}(U-U^\dagger) - \frac{1}{NC}\mathrm{Tr}[U-U^\dagger]I_{NC\times NC}) + O(A^3). \tag{5}$$

The LOG projection takes the exact log of the link matrix. For SU(3) we define the LOG by,

$$A = \frac{f_2^* U - f_2 U^\dagger - \Im\left(f_0 f_2^*\right)}{\Im\left(f_1 f_2^*\right)}. \tag{6}$$

And for SU(2) we define it by,

$$A = \frac{U - f_0 I_{2\times 2}}{f_1}. \tag{7}$$

Where the f's are defined by the NC long series,

$$U = f_0 I_{NC\times NC} + f_1 A + f_2 A^2 \ldots f_{NC-1} A^{NC-1}. \tag{8}$$

Where the f's can be obtained from the Eigenvalues of the matrix U or A.

For $N_C > 3$ we use series expansions for the logarithm and exponential.

3 Rectangles

The transformation incorporating rectangle terms is,

$$U_\mu\left(x + a\frac{\hat{\mu}}{2}\right)' = \exp\left\{\frac{\alpha}{2(N_d-1)}\sum_{\nu\neq\mu}\left(c_0 Q_{\mu\nu}(x) + c_1\sum_{i=1}^{6} R^{(i)}_{\mu\nu}\right)\right\} U_\mu\left(x + a\frac{\hat{\mu}}{2}\right). \tag{9}$$

Improved smearing	c_1
Symanzik	$-\frac{1}{12}$
Iwasaki	-0.331
DBW2	-1.4069

Table 1: Table of the parameters $c_0 = 1 - 8c_1$ and c_1 used for different improved smearing techniques.

These are turned on in the configure with,

```
--with-IMPROVED_STAPLE={IWASAKI,DBW2,SYMANZIK}
```

Or, one could specify the terms C_0 and C_1 themselves by the following configure script command

```
--with-IMPROVED_C0={} --with-IMPROVED_C1={}
```

There are six 2×1 rectangular terms $R^{(i)}$ that contribute to the smearing, 3 different rectangles and contributions from $\pm\nu$ I have written the positive ν terms below in Eq.**??**,

$$\begin{aligned}
R^{(1)}_{\mu\nu} = {} & \log\left(U_\nu\left(x+a\frac{\hat{\nu}}{2}\right) U_\nu\left(x+a\frac{3\hat{\nu}}{2}\right) U_\mu\left(x+2a\hat{\nu}+a\frac{\hat{\mu}}{2}\right) U_\nu\left(x+a\hat{\mu}+a\frac{3\hat{\nu}}{2}\right)^\dagger \right. \\
& \left. U_\nu\left(x+a\hat{\mu}+a\frac{\hat{\nu}}{2}\right)^\dagger U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)^\dagger \right). \\
R^{(2)}_{\mu\nu} = {} & \log\left(U_\mu\left(x-a\frac{\hat{\mu}}{2}\right)^\dagger U_\nu\left(x-a\hat{\mu}+a\frac{\hat{\nu}}{2}\right) U_\mu\left(x+a\hat{\nu}-a\frac{\hat{\mu}}{2}\right) U_\mu\left(x+a\hat{\nu}+a\frac{\hat{\mu}}{2}\right) \right. \\
& \left. U_\nu\left(x+a\hat{\mu}+a\frac{\hat{\nu}}{2}\right)^\dagger U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)^\dagger \right). \\
R^{(3)}_{\mu\nu} = {} & \log\left(U_\nu\left(x+a\frac{\hat{\nu}}{2}\right) U_\mu\left(x+a\hat{\nu}+a\frac{\hat{\mu}}{2}\right) U_\mu\left(x+a\hat{\nu}+a\frac{3\hat{\nu}}{2}\right) U_\nu\left(x+2a\hat{\mu}+a\frac{\hat{\nu}}{2}\right)^\dagger \right. \\
& \left. U_\mu\left(x+a\frac{3\hat{\mu}}{2}\right)^\dagger U_\mu\left(x+a\frac{\hat{\nu}}{2}\right)^\dagger \right).
\end{aligned} \tag{10}$$

One can incorporate a term ε that interpolates between the full rectangle term and the standard, Wilsonian smearing transformation. This is called "over-improved" smearing, and is set by the configure command,

```
--with-OVERIMPROVED_EPSILON={}
```

And some number.

The coefficients for the over-improved smearing are.

Improved smearing	c_0	c_1
Symanzik	$1+\frac{2}{3}(1-\varepsilon)$	-$\frac{1}{12}(1-\varepsilon)$
Iwasaki	$1+2.648(1-\varepsilon)$	$-0.331(1-\varepsilon)$
DBW2	$1+11.2536(1-\varepsilon)$	$-1.4069(1-\varepsilon)$

Table 2: Table of the over improved smearing parameters used in this study.

4 Hypercubic blocking

The 4 dimensional general, hypercubic transformation is,

$$\begin{aligned}
V_{\mu,\nu\rho}(x) &= \exp\left\{ \frac{\alpha_3}{2(N_d-3)} \sum_{\sigma\neq\mu\nu\rho} Q_{\mu\sigma}(U(x)) \right\} U_\mu\left(x+a\frac{\hat{\mu}}{2}\right), \\
W_{\mu,\nu}(x) &= \exp\left\{ \frac{\alpha_2}{2(N_d-2)} \sum_{\rho\neq\mu\nu} Q_{\mu\rho}(V(x)) \right\} U_\mu\left(x+a\frac{\hat{\mu}}{2}\right), \\
U_\mu\left(x+a\frac{\hat{\mu}}{2}\right)' &= \exp\left\{ \frac{\alpha_1}{2(N_d-1)} \sum_{\nu\neq\mu} Q_{\mu\nu}(W(x)) \right\} U_\mu\left(x+a\frac{\hat{\mu}}{2}\right).
\end{aligned} \tag{11}$$

We use precomputed forms for the top two levels if possible. Otherwise these are (slowly) computed in-step. For the N_d-generic hypercubic smearing routine one notes that the blocking is N_D-1 recursions down (and hence N_d-1 coefficients α_i) from the link level. This is the form we use. The usual projections APE, STOUT and LOG are available for these routines and have the special monikers HYP, HEX and HYL respectively. The rectangle terms are not included.

The Hypercubically-blocked smearing routines are called by the following input file commands,

```
SMTYPE = {HYP,HEX,HYL}
```

The input file options for N_d larger than 4 dimensional smearing is,

```
ALPHA1 = {}
ALPHA2 = {}
......
ALPHA{ND-1} = {}
```

Where ALPHA1 is one level of the blocking down, ALPHA2 is two and so on.

5 Topological measurements

If we configure the code with the argument,

```
--with-TOP_VALUE={}
```

some number. This will tell the code to begin topological measurements from `TOP_VALUE`'s numerical value. The code uses the naive gauge field strength tensor definition for the topological charge,

$$Q_{\text{top}}^{\text{Latt}} = \frac{1}{32\pi^2} \sum_x \varepsilon_{\mu\nu\rho\sigma} \text{Tr}\left[F_{\mu\nu}^{\text{Latt}}(x) F_{\rho\sigma}^{\text{Latt}}(x) \right]. \tag{12}$$

If the topological charge is *close enough* to an integer, the routine breaks and quotes an integer for the topological charge.

I would suggest also using the highly improved field strength tensor the configure command should be called,

```
--enable-CLOVER_IMPROVE
```

and if a value for k_5 is to be used (apart from 0), the command in the configure,

```
--with-CLOVER_K5={}
```

should be used.