Nonlinear Models: Logistic Equation

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Nonlinear Differential Equations

Consider the following nonlinear differential equations.

$$y' = y(1-y)$$
$$y' = \cos(y-t)$$
$$y' = \frac{1}{t^2+y^2}$$

What options do we have for solving them?

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Analytical: Sadly, analytical methods cannot always provide formulas for a solutions. Since none of these are linear, the methods we have discussed this chapter won't help us. While the first equation is separable, the other two are not.

Numerical: We could apply a numerical method, though this only gives a single approximate solution. Moreover, the further you move from the initial conditions, the less accurate your numerical solution is likely to be.

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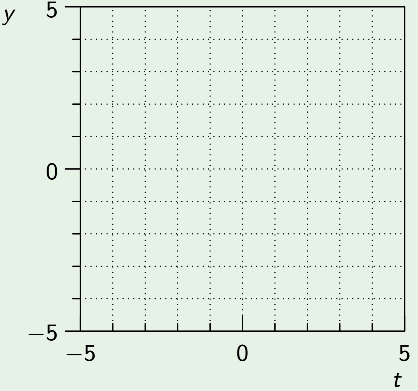

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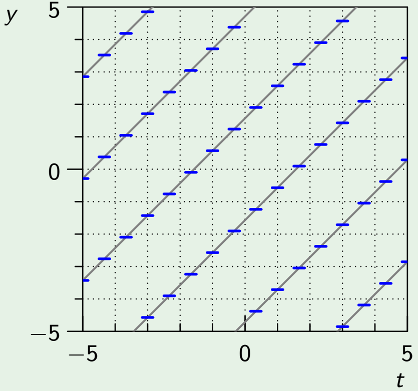

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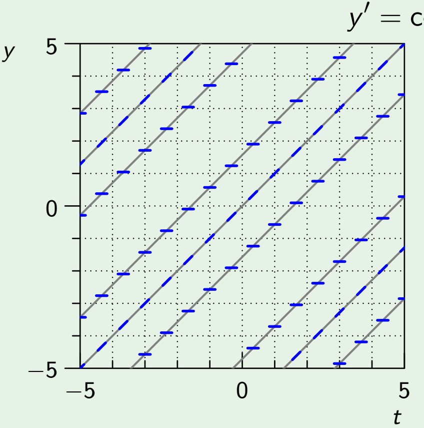

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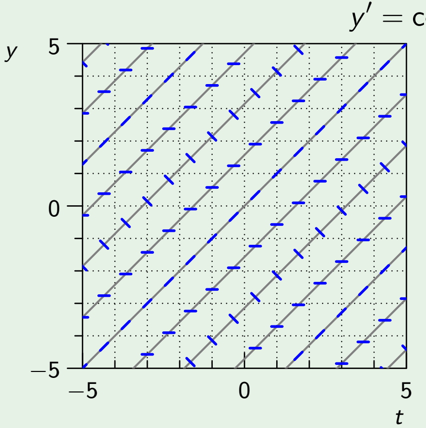

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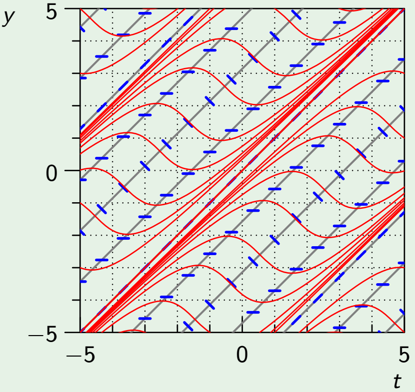

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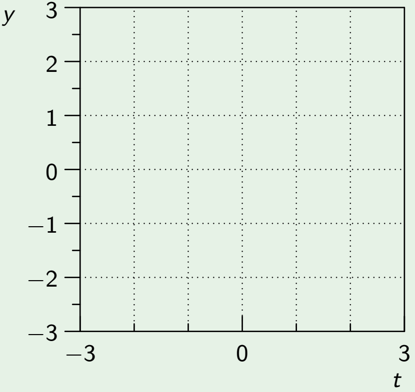

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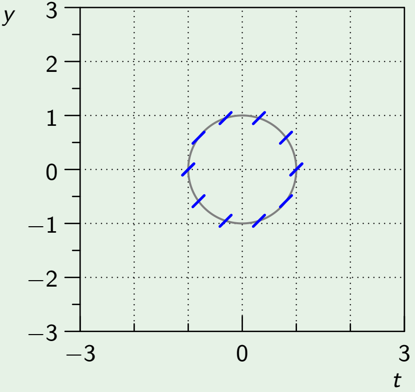

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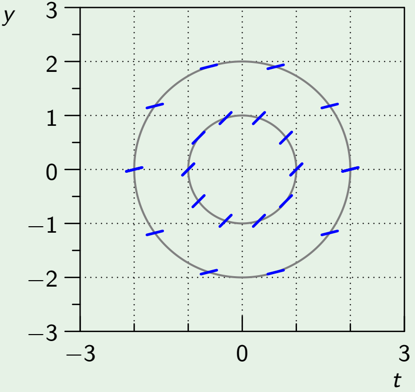

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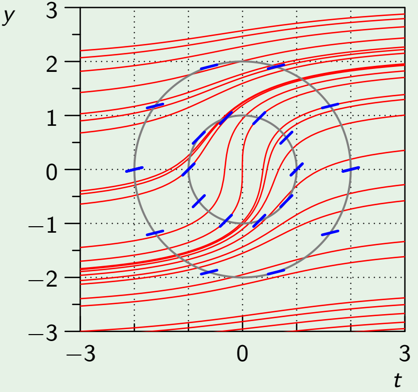

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- When $y' = 1$: $t^2 + y^2 = 1^2$.
- When $y' = 2$: $t^2 + y^2 = 2^2$.
- When $t^2 + y^2 \to \infty$, Slope $\to 0$.
- When $t^2 + y^2 \to 0$, Slope $\to$ vertical.

Autonomous Differential Equation

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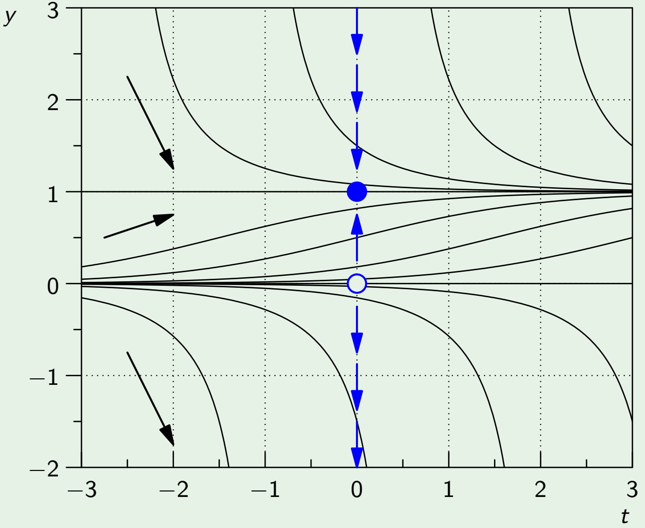
Phase Line

Thus, for a given y value, all solutions are horizontal translations. Which means we can encapsulate information about all solutions with a vertical line, called a **phase line**.

Example 3

$$y' = y(1-y)$$



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Semistable If the phase-line one of the arrows above or below the equilibrium point towards the equilibrium and the other points away. (Also called a **node**.)

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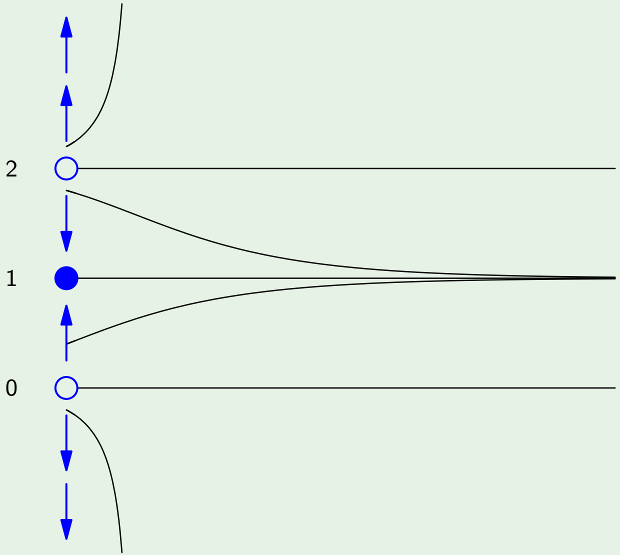
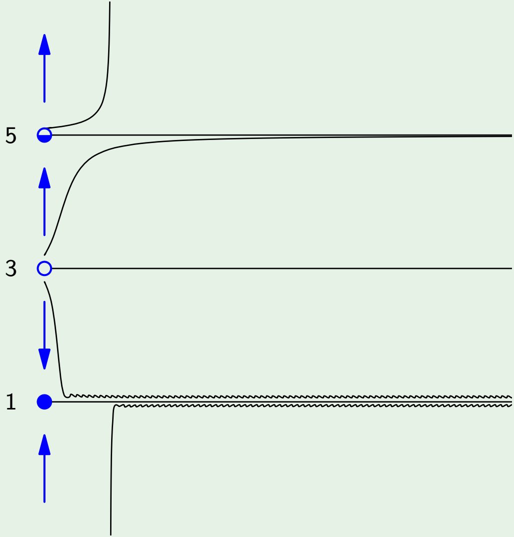
$$y' = y(1-y)(2-y)$$



Example 5

$$y' = (y-1)(y-3)(y-5)^2;$$



Population Models

Consider the unrestricted growth equation:

$$\frac{dy}{dt} = ky, \quad k > 0$$

which assumes that the rate of growth of a population is always proportional to it's size. This equation predicts exponential growth that cannot continue indefinitely.

For long-range predictions we need to consider how the population interacts with it's environment. That is, as a population will level off as it reaches a limited food supply, increased disease, crowding, etc.

To build a model that includes these factors we need to replace the constant growth rate k with a variable growth rate $k(y)$ that depends on the population size:

$$\frac{dy}{dt} = k(y) \cdot y, \quad k > 0$$

Logistic Equation

A population may be modeled using

$$\frac{dy}{dt} = r\left(1 - \frac{y}{L}\right)y$$

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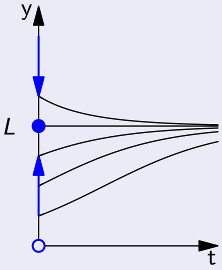

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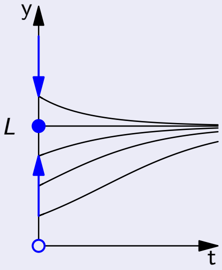

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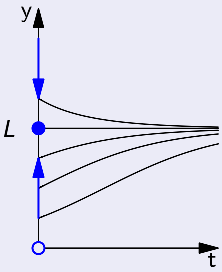

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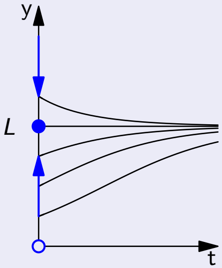

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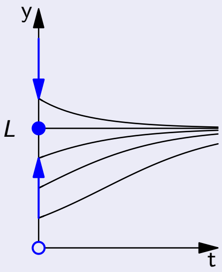

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- There is an inflection point between 0 and L.

Analytic Solution of the Logistic Equation

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So, we are really solving

$$\left(\frac{1}{y} + \frac{\frac{1}{L}}{1-\frac{1}{L}}\right)dy = r\ dt$$

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Where C will be determined by the initial condition $y(0) = y_0$.

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Note: If $y_0 > L$, we will arrive at the same solution.

Initial-Value Problem for the Logistic Equation

The solution for $t \geq 0$ of the logistic IVP

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where $r > 0$ is the **initial growth rate** and $L > 0$ is the **carrying capacity**.

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To model the U.S. population using the logistic equation, we will let $t = 0$ represent the year 1990 and $t = 1$ the year 2000.

So, $t = 1.5$ would be the year 1950 and $t = 1.3$ would be the year 2030.

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Given that we need to find both r, L, and y_0 we will need three data points:

$$y(0) = y_0 = 76.1, \quad y(0.5) = 151.1, \quad y(1) = 271.3$$

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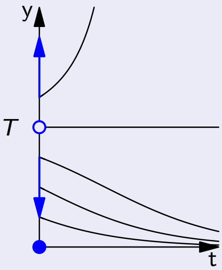
- The projected population in 2030 is $y(1.3) \approx 360.7$ million.
- The backward projected population in 1790 is $y(-1.1) \approx 14.3$ million. (The actual population was 4 million. Why the discrepancy?)

Threshold Equation

For some species there is a critical population size, such that if the population ever falls below this the species will go extinct. This level T, called the **threshold level** behaves like a carrying capacity, except solutions need to tend away from T.



The **threshold equation** is the logistic equation with a negative sign:

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{L}\right) y$$

Initial-Value Problem for the Threshold Equation

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$$\frac{dy}{dt} = -r\left(1 - \frac{y}{L}\right)y, \quad y(0) = y_0$$

is given by

$$y(t) = \frac{T}{1 + \left(\frac{T}{y_0} - 1\right)e^{rt}}$$

where $r > 0$ is the **initial growth rate** and $T > 0$ the **threshold level**.