Separable Equations

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$\frac{dy}{dt} = 3t^2 \cdot (1 + y)$$

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$\frac{dy}{dt} = 3t^2 \cdot (1 + y)$$

$$\frac{dy}{1+y} = \quad 3t^2 dt$$

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$\frac{dy}{dt} = 3t^2 \cdot (1 + y)$$

$$\int \frac{dy}{1 + y} = \int 3t^2 dt$$

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$.
Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$\frac{dy}{dt} = 3t^2 \cdot (1 + y)$$

$$\int \frac{dy}{1 + y} = \int 3t^2 dt$$

$$\ln|1 + y| = t^3 + c$$

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 \cdot (1 + y) \\ \int \frac{dy}{1+y} &= \int 3t^2 dt \\ \ln|1+y| &= t^3 + c \\ |1+y| &= e^c e^{t^3} \end{aligned}$$

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$
\begin{aligned}
\frac{dy}{dt} &= 3t^2 \cdot (1 + y) \\
\int \frac{dy}{1+y} &= \int 3t^2 dt \\
\ln|1+y| &= t^3 + c \\
|1+y| &= e^c e^{t^3} \\
y &= -1 \pm e^c e^{t^3}
\end{aligned}
$$

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 \cdot (1+y) \\ \int \frac{dy}{1+y} &= \int 3t^2 dt \\ \ln|1+y| &= t^3 + c \\ |1+y| &= e^c e^{t^3} \\ y &= -1 \pm e^c e^{t^3} \\ y &= -1 + ke^{t^3}, \quad k \in \mathbb{R} \end{aligned}$$

Separable Differential Equations

A **separable** differential equation is one that can be written $y' = f(t)g(y)$. Constant solutions $y = c$ can be found by solving $g(y) = 0$.

Example 1

Let use consider

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 \cdot (1+y) \\ \int \frac{dy}{1+y} &= \int 3t^2 dt \\ \ln|1+y| &= t^3 + c \\ |1+y| &= e^c e^{t^3} \\ y &= -1 \pm e^c e^{t^3} \\ y &= -1 + ke^{t^3}, \quad k \in \mathbb{R} \end{aligned}$$

The method we just used is called **Separation of Variables**.

Why does it work?

Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

Why does it work?

Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

Suppose also that $y = y(t)$ is a function defined **implicitly** by

$$G(y) = F(t) + c$$

on some appropriate t-interval for some real constant c.

Why does it work?

Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

Suppose also that $y = y(t)$ is a function defined **implicitly** by

$$G(y) = F(t) + c \quad \Rightarrow \quad G(y(t)) = F(t) + c$$

on some appropriate t-interval for some real constant c.

Why does it work?

Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

Suppose also that $y = y(t)$ is a function defined **implicitly** by

$$G(y) = F(t) + c \quad \Rightarrow \quad G(y(t)) = F(t) + c$$

on some appropriate t-interval for some real constant c.

Then, if y is differentiable, we can differentiate both sides with respect to t.

$$G'(y(t))y'(t) = F'(t)$$

Why does it work?

Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

Suppose also that $y = y(t)$ is a function defined **implicitly** by

$$G(y) = F(t) + c \quad \Rightarrow \quad G(y(t)) = F(t) + c$$

on some appropriate t-interval for some real constant c.

Then, if y is differentiable, we can differentiate both sides with respect to t.

$$G'(y(t))y'(t) = F'(t)$$
$$\frac{y'(t)}{g(y(t))} = f(t)$$

Why does it work?

Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

Suppose also that $y = y(t)$ is a function defined **implicitly** by

$$G(y) = F(t) + c \quad \Rightarrow \quad G(y(t)) = F(t) + c$$

on some appropriate t-interval for some real constant c.

Then, if y is differentiable, we can differentiate both sides with respect to t.

$$\begin{aligned} G'(y(t))y'(t) &= F'(t) \\ \frac{y'(t)}{g(y(t))} &= f(t) \\ y'(t) &= f(t)g(y(t)) \end{aligned}$$

Why does it work?

Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

Suppose also that $y = y(t)$ is a function defined **implicitly** by

$$G(y) = F(t) + c \quad \Rightarrow \quad G(y(t)) = F(t) + c$$

on some appropriate t-interval for some real constant c.

Then, if y is differentiable, we can differentiate both sides with respect to t.

$$\begin{aligned} G'(y(t))y'(t) &= F'(t) \\ \frac{y'(t)}{g(y(t))} &= f(t) \\ y'(t) &= f(t)g(y(t)) \end{aligned}$$

This has shown that y is a solution to $y' = f(t)g(y)$ and explains why the previous example works.

Method of Separation of Variables

Step 1: Set $g(y) = 0$ and solve to find any equilibria.

Step 2: Now, assume that $g(y) \neq 0$. Rewrite the equation in separated form:

$$\frac{dy}{g(y)} = f(t)dt$$

Step 3: Integrate, if possible, each side:

$$\int \frac{dy}{g(y)} = \int f(t)dt$$

(obtaining the implicit one-parameter family of solutions.)

Step 4: If possible, solve for y in terms of t, getting the explicit solution $y = y(t)$

Step 5: If you have an IVP, use the initial condition to evaluate c.

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y}$

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y} \quad \Rightarrow \quad y\ dy = -t\ dt$

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y} \quad \Rightarrow \quad y\ dy = -t\ dt$
2. $\dfrac{dy}{dt} = t^2 y$

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y} \quad \Rightarrow \quad y\ dy = -t\ dt$
2. $\dfrac{dy}{dt} = t^2 y \quad \Rightarrow \quad \dfrac{dy}{y} = t^2 dt$

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y} \quad \Rightarrow \quad y\ dy = -t\ dt$
2. $\dfrac{dy}{dt} = t^2 y \quad \Rightarrow \quad \dfrac{dy}{y} = t^2 dt$
3. $\dfrac{dy}{dt} = y + 1$

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y} \quad \Rightarrow \quad y\ dy = -t\ dt$
2. $\dfrac{dy}{dt} = t^2 y \quad \Rightarrow \quad \dfrac{dy}{y} = t^2 dt$
3. $\dfrac{dy}{dt} = y + 1 \quad \Rightarrow \quad \dfrac{dy}{y+1} = dt$

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y} \quad \Rightarrow \quad y\ dy = -t\ dt$
2. $\dfrac{dy}{dt} = t^2 y \quad \Rightarrow \quad \dfrac{dy}{y} = t^2 dt$
3. $\dfrac{dy}{dt} = y + 1 \quad \Rightarrow \quad \dfrac{dy}{y+1} = dt$
4. $\dfrac{dy}{dt} = t + y$

Example 2

Are the following differentiable equations are separable?

1. $\dfrac{dy}{dt} = -\dfrac{t}{y} \quad \Rightarrow \quad y\ dy = -t\ dt$
2. $\dfrac{dy}{dt} = t^2 y \quad \Rightarrow \quad \dfrac{dy}{y} = t^2 dt$
3. $\dfrac{dy}{dt} = y + 1 \quad \Rightarrow \quad \dfrac{dy}{y+1} = dt$
4. $\dfrac{dy}{dt} = t + y \quad \Rightarrow$ (not separable)

Example 3

What are the solutions for the following separable differential equation?

$$y' = \frac{t^2}{1 - y^2}, \quad y \neq \pm 1$$

Example 3

What are the solutions for the following separable differential equation?

$$y' = \frac{t^2}{1 - y^2}, \quad y \neq \pm 1$$

First we need to rewrite the equation in separated form.

$$\left(1 - y^2\right) dy = t^2 \; dt$$

Example 3

What are the solutions for the following separable differential equation?

$$y' = \frac{t^2}{1 - y^2}, \quad y \neq \pm 1$$

First we need to rewrite the equation in separated form.

$$\left(1 - y^2\right) dy = t^2 \ dt$$

Next, we can integrate both sides to get:

$$y - \frac{y^3}{3} = \frac{t^3}{3} + c$$

Example 3

What are the solutions for the following separable differential equation?

$$y' = \frac{t^2}{1-y^2}, \quad y \neq \pm 1$$

First we need to rewrite the equation in separated form.

$$\left(1-y^2\right) dy = t^2\ dt$$

Next, we can integrate both sides to get:

$$\begin{aligned} y - \frac{y^3}{3} &= \frac{t^3}{3} + c \\ -t^3 + 3y - y^3 &= k \qquad\qquad \text{where } k = 3c \end{aligned}$$

Example 3

What are the solutions for the following separable differential equation?

$$y' = \frac{t^2}{1 - y^2}, \quad y \neq \pm 1$$

First we need to rewrite the equation in separated form.

$$\left(1 - y^2\right) dy = t^2 \ dt$$

Next, we can integrate both sides to get:

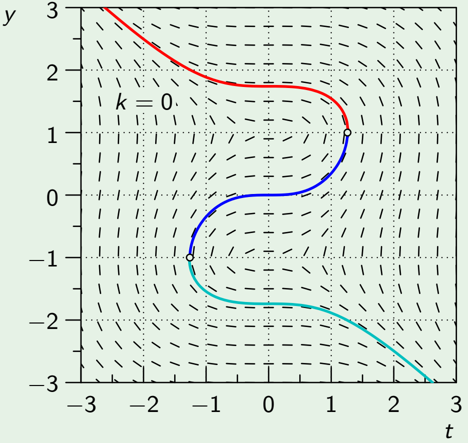
$$\begin{aligned} y - \frac{y^3}{3} &= \frac{t^3}{3} + c \\ -t^3 + 3y - y^3 &= k \qquad \text{where } k = 3c \end{aligned}$$

Note that because of the restrictions, each solution curve in the direction field will be a piecewise combination of several functions. A particular solution of an IVP for this DE would only be *one* of these.

Example 3

What are the solutions for the following separable differential equation?

$$y' = \frac{t^2}{1-y^2}, \quad y \neq \pm 1$$



Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2 + 1}{1 + 2y}, \quad y(0) = 1$$

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

This separable equation has no equilibrium solutions. Moreover, $y \neq -\frac{1}{2}$.

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

This separable equation has no equilibrium solutions. Moreover, $y \neq -\frac{1}{2}$.

To find any other solutions we need to separate the DE.

$$(1+2y)dy = (3t^2+1)dt$$

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

This separable equation has no equilibrium solutions. Moreover, $y \neq -\frac{1}{2}$.

To find any other solutions we need to separate the DE.

$$(1+2y)dy = (3t^2+1)dt$$
$$\int (1+2y)dy = \int (3t^2+1)dt$$

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

This separable equation has no equilibrium solutions. Moreover, $y \neq -\frac{1}{2}$.

To find any other solutions we need to separate the DE.

$$(1+2y)dy = (3t^2+1)dt$$
$$\int (1+2y)dy = \int (3t^2+1)dt$$
$$y + y^2 = t^3 + t + c$$

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

To satisfy the initial conditions we must have $y = 1$ when $t = 0$.

$$1 + 1^2 = 0^3 + 0 + c$$

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

To satisfy the initial conditions we must have $y = 1$ when $t = 0$.

$$1 + 1^2 = 0^3 + 0 + c \quad \to \quad c = 2$$

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

To satisfy the initial conditions we must have $y = 1$ when $t = 0$.

$$1 + 1^2 = 0^3 + 0 + c \quad \to \quad c = 2$$

Which give the solution:

$$y + y^2 = t^3 + t + 2$$

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$

To satisfy the initial conditions we must have $y = 1$ when $t = 0$.

$$1 + 1^2 = 0^3 + 0 + c \quad \to \quad c = 2$$

Which give the solution:

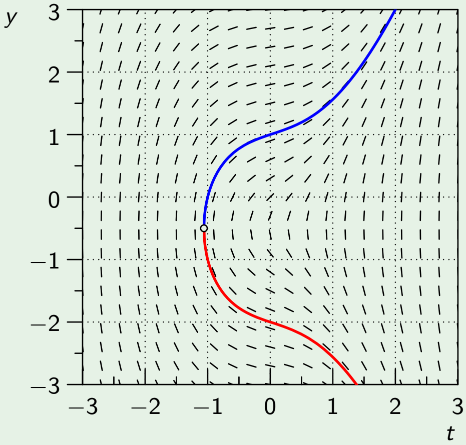
$$\begin{aligned} y + y^2 &= t^3 + t + 2 \\ y &= \frac{1}{2}\left(-1 \pm \sqrt{4t^3 + 4t + 9}\right) \end{aligned}$$

Again, the solution curve is made up of multiple parts.

Example 4

Solve the following IVP.

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1$$



Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

There are no equilibrium solutions and $y \neq 0$.

We can separate the DE.

$$y \; dy = -t \; dt$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

There are no equilibrium solutions and $y \neq 0$.

We can separate the DE.

$$y \ dy = -t \ dt$$

$$\int y \ dy = \int -t \ dt$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

There are no equilibrium solutions and $y \neq 0$.

We can separate the DE.

$$\begin{aligned} y\ dy &= -t\ dt \\ \int y\ dy &= \int -t\ dt \\ \frac{y^2}{2} &= -\frac{t^2}{2} + c \end{aligned}$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

There are no equilibrium solutions and $y \neq 0$.

We can separate the DE.

$$y\ dy = -t\ dt$$

$$\int y\ dy = \int -t\ dt$$

$$\frac{y^2}{2} = -\frac{t^2}{2} + c$$

We can then use the initial condition to find c.

$$\frac{1}{2} = -\frac{0^2}{2} + c \quad \Rightarrow \quad c = \frac{1}{2}$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

We have two explicit solutions:

$$\frac{y^2}{2} = -\frac{t^2}{2} + \frac{1}{2}$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

We have two explicit solutions:

$$\frac{y^2}{2} = -\frac{t^2}{2} + \frac{1}{2}$$
$$y^2 = -t^2 + 1$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$

We have two explicit solutions:

$$\begin{aligned} \frac{y^2}{2} &= -\frac{t^2}{2} + \frac{1}{2} \\ y^2 &= -t^2 + 1 \\ y &= \pm\sqrt{1 - t^2} \end{aligned}$$

Example 5

Solve the IVP.

$$y' = -\frac{t}{y}, \quad y(0) = 1$$