# Vector Spaces

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## Definition

A **vector space** $\mathbb{V}$ is a nonempty collection of objects called **vectors** for which the following operations

- Vector addition, denoted $\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}$
- Scalar multiplication, denoted $c\vec{\boldsymbol{x}}$

satisfy the nine properties on the following slide. (For all $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \in \mathbb{V}$ and all $c, d \in \mathbb{R}$)

## Closure

1. $c\vec{\boldsymbol{x}} + d\vec{\boldsymbol{y}} \in \mathbb{V}$

## Addition

2. There exists a **zero vector** $\vec{\mathbf{0}} \in \mathbb{V}$ such that $\vec{\boldsymbol{x}} + \vec{\mathbf{0}} = \vec{\boldsymbol{x}}$
3. For all $\vec{\boldsymbol{x}} \in \mathbb{V}$ there exists $-\vec{\boldsymbol{x}} \in \mathbb{V}$ such that $\vec{\boldsymbol{x}} + (-\vec{\boldsymbol{x}}) = \vec{\mathbf{0}}$
4. $(\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}) + \vec{\boldsymbol{z}} = \vec{\boldsymbol{x}} + (\vec{\boldsymbol{y}} + \vec{\boldsymbol{z}})$
5. $\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} = \vec{\boldsymbol{y}} + \vec{\boldsymbol{x}}$

## Scalar Multiplication

6. $1\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}$
7. $c(\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}) = c\vec{\boldsymbol{x}} + c\vec{\boldsymbol{y}}$
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All vectors $< x_1, x_2, \ldots, x_n >$ in $\mathbb{R}^n$ satisfy these properties.
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## Example 2

Thinking back, we can see that the properties for addition and scalar multiplication of matrices we saw in section 3.1 satisfy all nine requirements to be a vector space.
Which means, for any $m, n \in \mathbb{R}$, $\mathbb{M}_{mn}$ is a vector space.

## Definition

A **function space** is a vector space where the “vectors” are functions defined on an interval $I$. The addition and scalar multiplication operations are defined in the usual way:

- $(f+g)(t) = f(t) + g(t)$, for all $t \in I$
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## Note

Solutions to *linear homogeneous* DEs form a vector space.

## Example 3

The set of all solutions of the first order linear homogeneous DE

$$y' + p(t)y = 0$$

(where $p$ and $y$ are defined on some interval $I$) is a vector space.

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For solutions $u(t)$ and $v(t)$, as well as scalars $a$ and $b$, we need to verify that $a \cdot u(t) + b \cdot v(t)$ is a solution.

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The set of all solutions of the second order linear homogeneous DE

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&= a \cdot 0 + b \cdot 0 \\
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## Example 6

Consider the collection of all polynomials of degree $\leq 3$. A vector in this space is given by

$$P(t) = a_3x^3 + a_2x^2 + a_1x + a_0$$

where $a_3, a_2, a_1, a_0 \in \mathbb{R}$.
This collection is a vector space, verifiable using basic algebra.

## Prominent Vector Spaces

- $\mathbb{R}^2$, the space of all real ordered pairs.
- $\mathbb{R}^3$, the space of all real ordered triples.
- $\mathbb{R}^n$, the space of all real ordered $n$-tuples.
- $\mathbb{C}^n$, the space of all complex $n$-tuples.
- $\mathbb{P}$, the space of all polynomials.
- $\mathbb{P}_n$, the space of all polynomials of degree $\leq n$
- $\mathbb{M}_{mn}$, the space of all $m \times n$ matrices.
- $\mathcal{C}(I)$, the space of all continuous functions defined on the interval $I$.
- $\mathcal{C}^n(I)$, the space of all functions, defined on the interval $I$, having $n$ continuous derivatives.
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## Prominent Vector Spaces

- $\mathbb{R}^2$, the space of all real ordered pairs.
- $\mathbb{R}^3$, the space of all real ordered triples.
- $\mathbb{R}^n$, the space of all real ordered $n$-tuples.
- $\mathbb{C}^n$, the space of all complex $n$-tuples.
- $\mathbb{P}$, the space of all polynomials.
- $\mathbb{P}_n$, the space of all polynomials of degree $\leq n$
- $\mathbb{M}_{mn}$, the space of all $m \times n$ matrices.
- $\mathcal{C}(I)$, the space of all continuous functions defined on the interval $I$.
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## Theorem 7

*A nonempty subset,* $\mathbb{W}$, *of a vector space* $\mathbb{V}$ *is a* **subspace** *of* $\mathbb{V}$ *if*

- $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} \in \mathbb{W}$ *for all* $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{W}$
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The definition of a subspace guarantees closure, everything else is inherited from the parent vector space.

For example, given $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{W}$, consider $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$.

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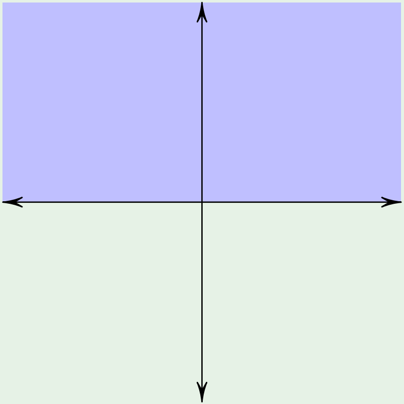
## Note

A vector space is a subspace of itself.

## Example 8

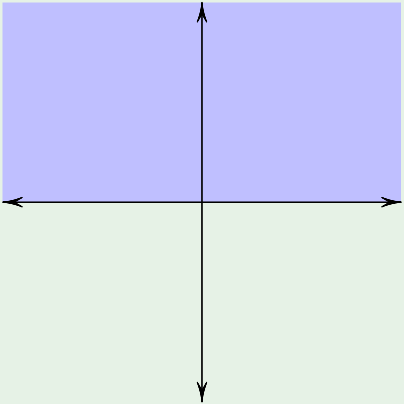
Is the upper half plane a subspace of $\mathbb{R}^2$?



## Example 8

Is the upper half plane a subspace of $\mathbb{R}^2$?

No, points in the upper half plane are not closed under scalar multiplication.



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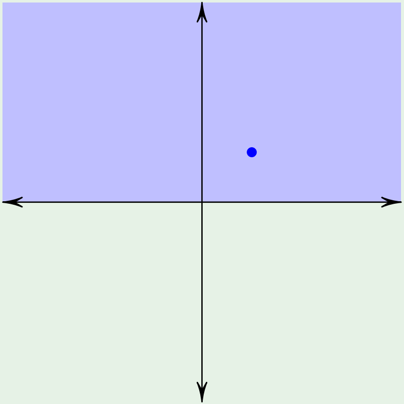

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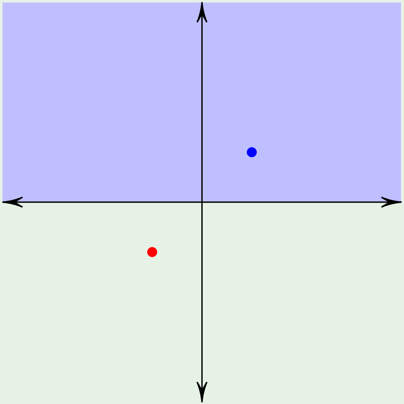
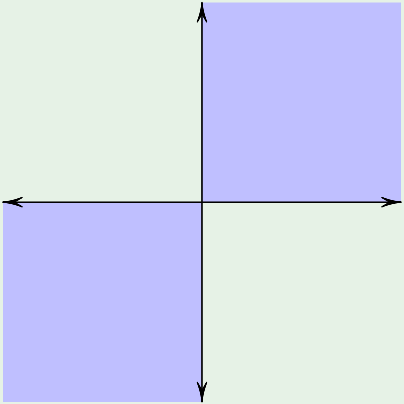
Consider $(1, 1)$.

Multiplying by the scalar -1 gives $(-1 \cdot 1, -1 \cdot 1) = (-1, -1)$, a point in Q3.



## Example 9

Is the set containing Q1 and Q3 a subspace of $\mathbb{R}^2$?



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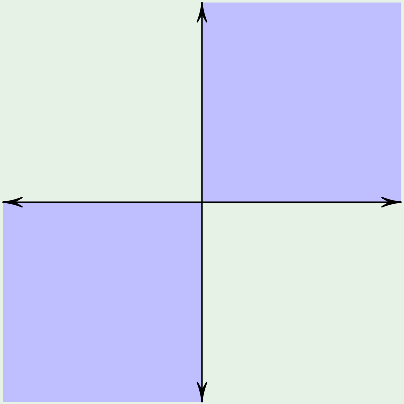
Is the set containing Q1 and Q3 a subspace of $\mathbb{R}^2$?

No, points in the set containing Q1 and Q3 are not closed under addition.



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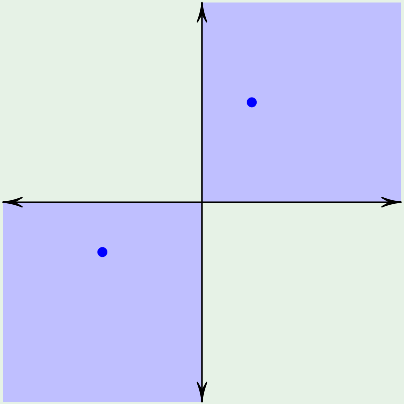

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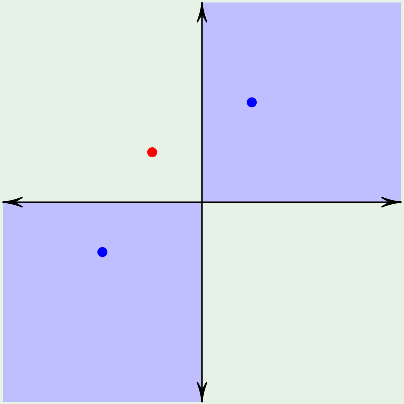
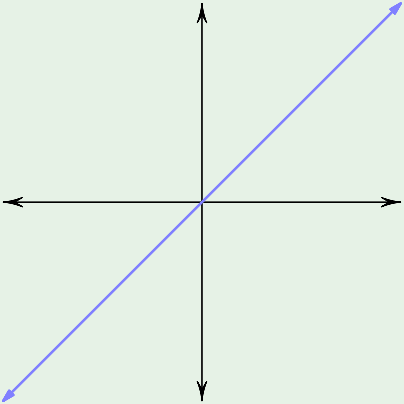
Adding these points gives $(1 + (-2), 2 + (-1)) = (-1, 1)$, a point in Q2.



## Example 10

Is the line $y = x$ a subspace of $\mathbb{R}^2$?



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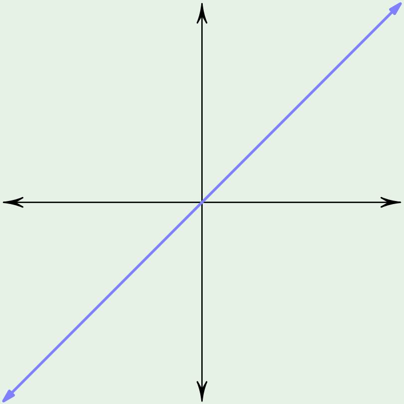
Is the line $y = x$ a subspace of $\mathbb{R}^2$?

Yes. Given $(s, s)$ and $(t, t)$, two points on the line, then

$a \cdot (s, s) + b \cdot (t, t) =$



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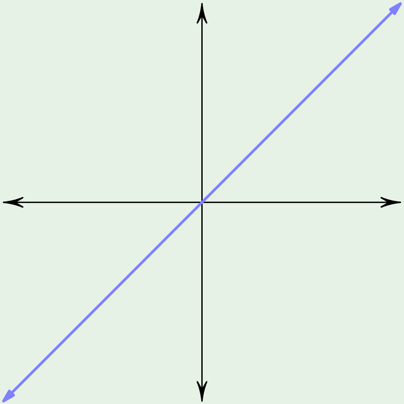

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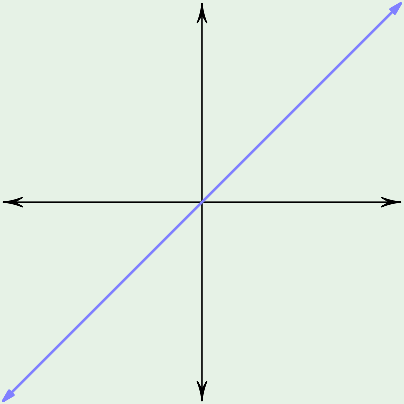

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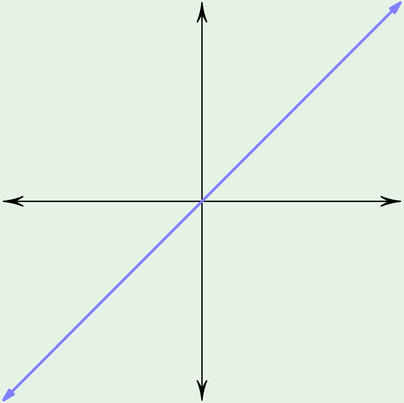
which is a point on the line.



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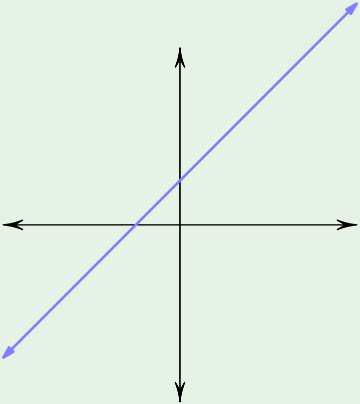

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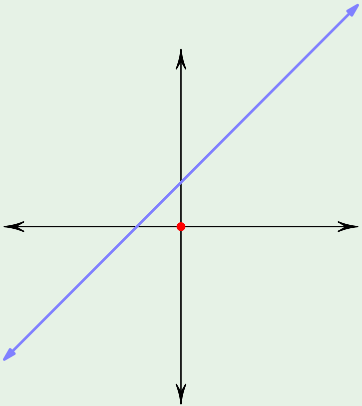
Is the line $y = x + 1$ a subspace of $\mathbb{R}^2$?

No, the zero vector, $(0, 0)$ is not on the line.



## Corollary 12

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We call a subspace of $\mathbb{V}$ **trivial** if it is the subspace containing just the zero vector, or $\mathbb{V}$ itself. All other subspaces are called **nontrivial**.

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*The set of solutions of the linear system $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ is a subspace of $\mathbb{R}^m$, where $A$ is a $m \times n$ matrix and $\vec{\boldsymbol{x}} \in \mathbb{R}^m$, is a subspace of $\mathbb{R}^m$.*

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## Proof

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Since solutions to $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ are vectors in $\mathbb{R}^m$, the remaining properties are inherited from $\mathbb{R}^m$