Forced Oscillations

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

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and

$$x = c_1 \cos(t) + c_2 \sin(t) - \frac{1}{4}\cos(3t)$$

General Solution

We can now look at the general solution for the undamped system

$$m\ddot{x} + kx = F_0 \cos(\omega_f t)$$

Where ω_f is the **forcing frequency** and F_0 is the **forcing amplitute**.

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This leaves two separate cases for x_p:

1. The frequencies ω_f and ω_0 are different.
2. The frequencies ω_f and ω_0 are the same.

Unmatched Frequencies ($\omega_f \neq \omega_0$)

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So, where c_1 and c_2 are determined by initial conditions, we have

$$x(t) = c_1\cos\left(\omega_0 t\right) + c_2\sin\left(\omega_0 t\right) + \frac{F_0}{m\left(\omega_0^2 - \omega_f^2\right)}\cos\left(\omega_f t\right)$$

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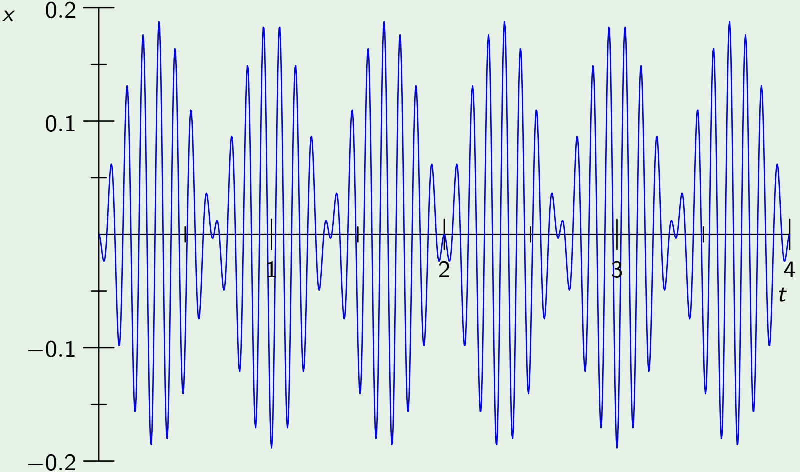
$$\begin{aligned} x(t) &= c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m\left(\omega_0^2 - \omega_f^2\right)}\cos(\omega_f t) \\ &= C\cos(\omega_0 t - \delta) + \frac{F_0}{m\left(\omega_0^2 - \omega_f^2\right)}\cos(\omega_f t) \end{aligned}$$

where $C = \sqrt{c_1^2 + c_2^2}$ and $\tan(\delta) = \dfrac{c_2}{c_1}$.

Example 2

An example of $\omega_f \neq \omega_0$ being different, but close, is



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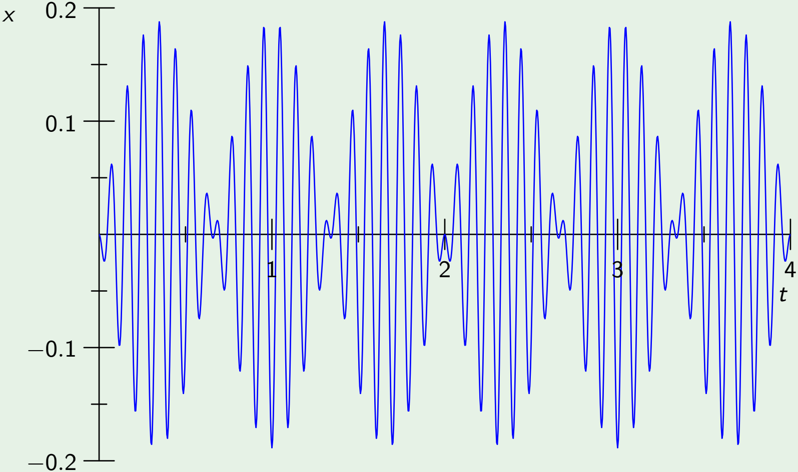

The regular periodic patterns are called **beats**.

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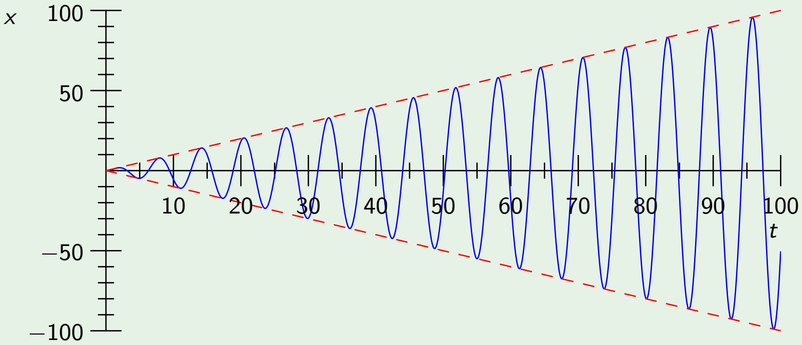

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If the system is initially at rest ($x(0) = 0$ and $\dot{x}(0) = 0$) then the solution is

$$x(t) = -\frac{F_0}{m\left(\omega_0^2 - \omega_f^2\right)} \cos\left(\omega_0 t\right) + \frac{F_0}{m\left(\omega_0^2 - \omega_f^2\right)} \cos\left(\omega_f t\right)$$

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So, we can simplify using the trigonometric identity

$$\cos\left(u\right) - \cos\left(v\right) = -2\sin\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

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The two curves

$$\pm\frac{2F_0}{m\left(\omega_0^2-\omega_f^2\right)}\sin\left(\frac{\omega_0-\omega_f}{2}t\right)$$

form an envelope of the more rapid oscillation and is called the **sinusoidal amplitude**.

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If the system starts from rest ($x(0) = 0$ and $\dot{x}(0) = 0$), the solution can be written as

$$x(t) = \underbrace{\frac{2F_0}{m\left(\omega_0^2 - \omega_f^2\right)} \sin\left(\frac{\omega_0 - \omega_f}{2} t\right)}_{\text{sinusoidal amplitude}} \underbrace{\sin\left(\frac{\omega_0 + \omega_f}{2} t\right)}_{\substack{\text{rapid oscillation} \\ \text{within beats}}}$$

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Thus, envelope curves are

$$y = \pm 1 \cdot \sin(\pi t)$$

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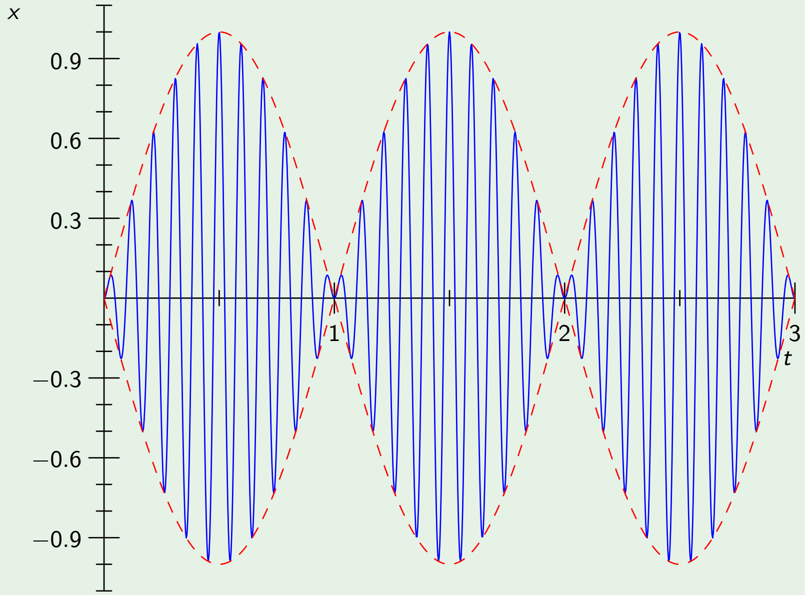

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We can use the method of undetermined coefficients:

$$\begin{array}{rcrcr} x(t) & = & A\cos(3t) & + & B\sin(3t) \\ \dot{x}(t) & = & -3A\sin(3t) & + & 3B\cos(3t) \\ \ddot{x}(t) & = & -9A\cos(3t) & - & 9B\sin(3t) \end{array}$$

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Substituting into the DE gives

$$\begin{aligned}&(-9A\cos(3t) - 9B\sin(3t))\\ &\qquad +4(-3A\sin(3t) + 3B\cos(3t))\\ &\qquad\qquad +5(A\cos(3t) + B\sin(3t)) = 10\cos(3t)\end{aligned}$$

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Which gives the system

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Thus,

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Next, we need to plug in the initial conditions

$$\begin{aligned} x(0) = 0 \quad &\Rightarrow \quad c_1 - \frac{1}{4} = 0\\ \dot{x}(0) = 0 \quad &\Rightarrow \quad c_2 + \frac{7}{4} = 0\end{aligned}$$

Example 5

The general solution is

$$x = e^{-2t}\left(c_1 \cos(t) + c_2 \sin(t)\right) - \frac{1}{4}\cos(3t) + \frac{3}{4}\sin(3t)$$

To solve the IVP, we need to calculate

$$\begin{aligned} \dot{x} = &-2e^{-2t}\left(c_1 \cos(t) + c_2 \sin(t)\right) \\ &+ e^{-2t}\left(-c_1 \sin(t) + c_2 \cos(t)\right) \\ &+ \frac{1}{4}\sin(3t) - \frac{3}{4}\cos(3t) \end{aligned}$$

Next, we need to plug in the initial conditions

$$\begin{aligned} x(0) = 0 \quad &\Rightarrow \quad c_1 - \frac{1}{4} = 0 \quad &\Rightarrow \quad c_1 = \frac{1}{4} \\ \dot{x}(0) = 0 \quad &\Rightarrow \quad c_2 + \frac{7}{4} = 0 \quad &\Rightarrow \quad c_2 = -\frac{7}{4} \end{aligned}$$

Example 5

The solution to the IVP is

$$x = e^{-2t}\left(\frac{1}{4}\cos(t) - \frac{7}{4}\sin(t)\right) - \frac{1}{4}\cos(3t) + \frac{3}{4}\sin(3t)$$

Example 5

The solution to the IVP is

$$x = \underbrace{e^{-2t}\left(\frac{1}{4}\cos(t) - \frac{7}{4}\sin(t)\right)}_{\text{Transient}} \underbrace{-\frac{1}{4}\cos(3t) + \frac{3}{4}\sin(3t)}_{\text{Steady-State}}$$

We call x_h **transient**, because for $b > 0$ the solution tends towards zero.

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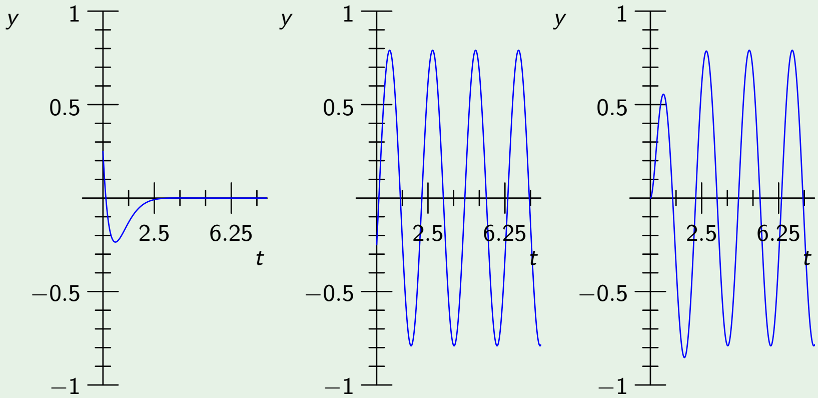
We call x_h **transient**, because for $b > 0$ the solution tends towards zero.

The particular solution x_p may either be constant or a periodic **steady-state** solution.

Example 5



(a) Transient Solution (b) Steady-State (c) IVP Solution

Particular Solution x_p of a Damped Mass-Spring System

The damped mass-spring system

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_f t)$$

has particular solution

$$x_p = A\cos(\omega_f t) + B\sin(\omega_f t)$$

with

$$A = \frac{m\left(\omega_0^2 - \omega_f^2\right)F_0}{m^2\left(\omega_0^2 - \omega_f^2\right)^2 + (b\omega_f)^2} \quad \text{and} \quad B = \frac{b\omega_f F_0}{m^2\left(\omega_0^2 - \omega_f^2\right)^2 + (b\omega_f)^2}$$

with natural circular frequency $\omega_0 = \sqrt{\frac{k}{m}}$.

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with natural circular frequency $\omega_0 = \sqrt{\frac{k}{m}}$.

Note

You will verify this in the homework.