

# Growth and Decay Phenomena

Department of Mathematics

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(Slides by Adam Wilson)

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### Note

We last saw this equation in section 1.1 where Thomas Malthus used it to estimate global population growth.

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Note that  $e^C > 0$  for all  $C \in \mathbb{R}$ . Thus, if we replace  $e^C$  with an arbitrary constant  $A$ , which could be negative, we can dispose of the absolute value bars.

$$y(t) = Ae^{kt}, \quad A \in \mathbb{R}$$

## Growth and Decay

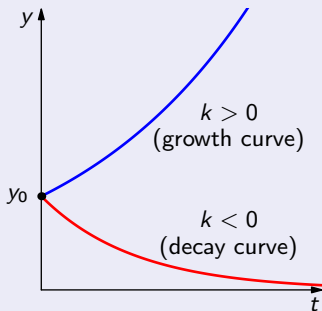
For each  $k$ , the solution of the IVP

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is given by

$$y(t) = y_0 e^{kt}$$

Solution curves are **growth curves** for  $k > 0$  and **decay curves** for  $k < 0$ .



## Example 1

**Radiocarbon Dating.** Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity  $Q$  of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

*What is the value of the decay constant  $k$ ?*

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Let us say that the initial amount of Carbon-14 is  $Q(0) = Q_0$ . So, after 5600 years, the half-life, there will be  $\frac{1}{2}Q_0$  remaining. That is,

$$Q(5600) = Q_0 e^{5600k} = \frac{1}{2}Q_0$$

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Since we know  $k$ , we can plug it into the general solution:

$$Q(t) = Q_0 e^{-t \ln(2)/5600} \approx Q_0 e^{-0.00012378t}$$

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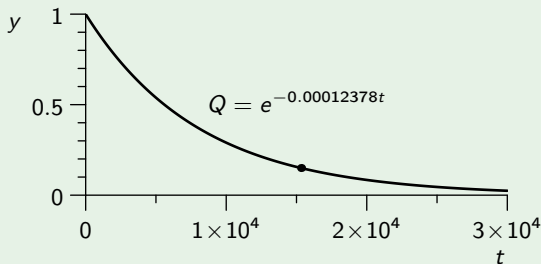
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## Compounded Interest

If an initial amount of  $A_0$  dollars is deposited at an **annual interest rate** of  $r$ , **compounded**  $n$  times per year, the **future value** of the deposit at time  $t$  is given by

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We call this **continuously compounded** interest.

## Continuously Compounded Interest

If an initial amount of  $A_0$  dollars is deposited at an annual interest rate of  $r$ , compounded continuously, the future value  $A(t)$  of the deposit at time  $t$  satisfies the initial-value problem

$$\frac{dA}{dt} = rA, \quad A(0) = A_0$$

and is therefore given by

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## Example 2

Suppose you came across a time machine and decided to travel to London on 27 July 1694 and deposit £20 (about \$25.78 USD) during the Bank of England's grand opening. Let us suppose you got the founder's day savings account with an annual interest rate of 8% compounded continuously. How much money would be in the account on the bank's tricentennial in 1994?

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So we would have the queenly sum of £529,782,442,597 pounds (which is about \$682,836,581,952 USD).

## The Savings Problem

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We now have the case where we have an annual interest rate of  $r$  and a annual deposit amount of  $a$  dollars. So, if the account starts with  $A_0$  dollars, then it satisfies the initial-value problem:

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It is left as an exercise to the student that we can use the methods from earlier in the chapter solve this nonhomogenous linear IVP, obtaining:

$$A(t) = A_0 e^{rt} + \frac{a}{r}(e^{rt} - 1)$$

Here the first term represents the accumulation due to the starting deposit, and the second term the accumulation due to the subsequent deposits and the interest that they earn.

### Example 3

Ravi has just entered college at age 18 and has decided to improve his health and save money by quitting smoking. He figures he can save \$30 per week in this way. If he deposits the amount in an account paying 10% annual interest compounded continuously how much will he have in the account when he retires at age 65?

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After  $65 - 18 = 47$  years, Ravi will have

$$A(47) = 15600(e^{0.1(47)} - 1) \approx \$1,699,575.89$$