Basis and Dimension

Adam Wilson

Salt Lake Community College

Spanning Sets

Definition

For a vector space $\mathbb{V}$, a **linear combination** of vectors is:

$$c_1\vec{\boldsymbol{v_1}} + c_2\vec{\boldsymbol{v_2}} + \cdots + c_n\vec{\boldsymbol{v_k}}$$

where $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{v_i}} \in \mathbb{V}$

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The **span** of a set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of vectors in a vector space $\mathbb{V}$ is the set of all linear combinations of these vectors. Denoted **span** $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$

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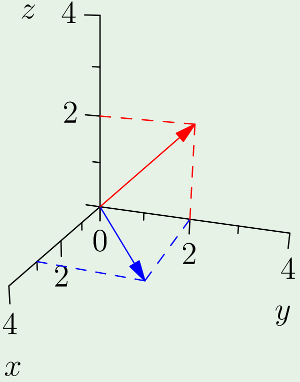
If the $\textbf{span}\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}\} = \mathbb{V}$ we say the set spans the vector space.

Spanning Sets

Example

Consider **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.



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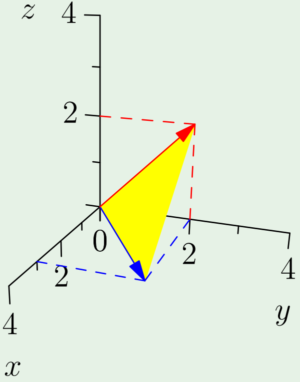

This spanning set is the plane defined by these two vectors.

Spanning Sets

Example

Let up look closer at this spanning set. Where we give names to the two vectors:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

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We can then write a general vector in the spanning set as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{v}}$$

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{v}} = a \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3a \\ 2a + 2b \\ 2b \end{bmatrix}$$

Example

Equation the two vectors gives:

$$\begin{aligned} x &= 3a && \Rightarrow a = \frac{x}{3} \\ y &= 2a + 2b && \\ z &= 2b && \Rightarrow b = \frac{z}{2} \end{aligned}$$

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Which is equivalent to $2x - 3y + 3z = 0$, the equation of the yellow plane.

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Theorem

An additional vector only changes a spanning set if and only if it is not a linear combination of the original vectors in the set.

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Example

Consider adding $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

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Since we can write

$$-1 \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

we see that this doesn't change to the spanning set.

Spanning Sets

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Consider adding $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

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Consider adding $\begin{bmatrix}1\\1\\1\end{bmatrix}$ to **span** $\left\{\begin{bmatrix}3\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\2\end{bmatrix}\right\}$.

This would expand the spanning set.

To show this, let us try to find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix}1\\1\\0\end{bmatrix} = c_1\begin{bmatrix}3\\2\\0\end{bmatrix} + c_2\begin{bmatrix}0\\2\\2\end{bmatrix}$$

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Which is equivalent to solving the inconsistent system

$$\begin{aligned} 1 &= 3c_1 \\ 1 &= 2c_1 + 2c_2 \\ 1 &= 2c_2 \end{aligned}$$

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What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

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To show this, we then need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Which has a unique solution for any $x, y, z \in \mathbb{R}$.

Spanning Sets

We have shown the general idea for:

Theorem

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{R}^n$, a vector $\vec{b} \in \mathbb{R}^n$ is in **span** $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ *if and only if there is at least one solution to the matrix equation $A\vec{x} = \vec{b}$. Where A is formed from the column vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$.*

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We can express spanning sets using set builder notation.

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$$\textbf{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\} = \left\{c\begin{bmatrix}2\\1\end{bmatrix} \;\middle|\; c \in \mathbb{R}\right\}$$

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$$\textbf{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \left\{c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\end{bmatrix}\;\middle|\; c_1,c_2,c_3\in\mathbb{R}\right\}$$

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Theorem

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{V}$, $\textbf{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ *is a subspace of* $\mathbb{V}$.

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The proof comes from the subspace theorem we saw last section.

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Proof

The proof comes from the subspace theorem we saw last section.
Let $\vec{u}$ and $\vec{w}$ be two vectors in the spanning set, which means there are scalars r_i and s_i such that

$$\vec{u} = r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_n\vec{v_k} \quad \text{and } \vec{w} \quad = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_k}$$

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So, for any $a, b \in \mathbb{R}$:

$$a\vec{u} + b\vec{w}$$

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So, for any $a, b \in \mathbb{R}$:

$$a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{w}} = a\big(r_1\vec{\boldsymbol{v_1}} + r_2\vec{\boldsymbol{v_2}} + \cdots + r_n\vec{\boldsymbol{v_k}}\big) + b\big(s_1\vec{\boldsymbol{v_1}} + s_2\vec{\boldsymbol{v_2}} + \cdots + s_n\vec{\boldsymbol{v_k}}\big)$$

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Which means $a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{w}}$ is in the spanning set and we have closure.

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Example

Consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

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Consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

The column space of $\boldsymbol{B}$ is a subspace of $\mathbb{R}^2$ and defined:

$$\text{Col } \boldsymbol{B} = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 5 \end{bmatrix} \;\middle|\; c_1, \ldots, c_5 \in \mathbb{R} \right\}$$

Linear Independence

Definition

A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of vectors in a vector space $\mathbb{V}$ is **linearly independent** if no vector of the set can be written as a linear combination of the others. Otherwise it is **linearly dependent**.

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Testing for Linear Independence

To test for linear independence of a set of k vectors $\vec{v_i} \in \mathbb{R}^n$, we consider the system:

$$\begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_k} \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$$

The column vectors of A are linearly independent if and only if the solution $c_1 = c_2 = \cdots = c_k = 0$ is unique.

Linear Independence

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Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ linearly independent?

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To determine if they are, we need to look at the system

$$\boldsymbol{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Since $|\boldsymbol{A}| = 5$, we know that $\boldsymbol{A}$ is invertible and hence a unique solution exists. This means that these vectors are linearly independent.

Linear Independence

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Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\3\\2\end{bmatrix}$, and $\begin{bmatrix}5\\-1\\0\end{bmatrix}$ linearly independent?

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$$\boldsymbol{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 1 & 3 & -1 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 1 & 3 & -1 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have more columns than rows, which means there will be at least one free variable. Thus, the solution (if one exists) won't be unique, so these vectors are not linearly independent.

Linear Independence

Example

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ linearly independent?

Linear Independence

Example

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & -1 & -2\\1 & 0 & 1\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Linear Independence

Example

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

Linear Independence

Example

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Linear Independence

Example

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And thus, these vectors are not linearly independent.

Linear Independence

Example

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And thus, these vectors are not linearly independent. Moreover, since

$$\begin{bmatrix}1\\1\\1\end{bmatrix} + 3\begin{bmatrix}-1\\0\\1\end{bmatrix} - \begin{bmatrix}-2\\1\\4\end{bmatrix} = 0$$

we can see that any one can be written as a combination of the others.

Linear Independence of Functions

Definition

A set of vector functions $\{\vec{\boldsymbol{v}}_{\boldsymbol{1}}(t), \vec{\boldsymbol{v}}_{\boldsymbol{2}}(t), \ldots, \vec{\boldsymbol{v}}_{\boldsymbol{k}}\}$ in a vector space $\mathbb{V}$ is **linearly independent** on an interval I if, for *all* $t \in I$, the equation

$$c_1 \vec{\boldsymbol{v}}_{\boldsymbol{1}}(t) + c_2 \vec{\boldsymbol{v}}_{\boldsymbol{2}}(t) + \cdots + c_k \vec{\boldsymbol{v}}_{\boldsymbol{k}}(t) = \vec{\boldsymbol{0}} \quad (\text{where } c_i \in \mathbb{R})$$

has the only solution: $c_1 = c_2 = \cdots = c_k = 0$.

If for any value $t_0 \in I$ there is any solution with $c_i \neq 0$, the vector functions $\vec{\boldsymbol{v}}_{\boldsymbol{1}}(t), \vec{\boldsymbol{v}}_{\boldsymbol{2}}(t), \ldots, \vec{\boldsymbol{v}}_{\boldsymbol{k}}(t)$ are **linearly dependent**.

Linear Independence of Functions

Example

Are the vectors

$$\vec{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?

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linearly independent on $(-\infty, \infty)$?

We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1\vec{v}_1(t) + c_2\vec{v}_2(t) + c_3\vec{v}_3(t) = \vec{0}$$

Linear Independence of Functions

Example

Are the vectors

$$\vec{\boldsymbol{v_1}}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{\boldsymbol{v_2}}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{\boldsymbol{v_3}}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

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We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{\boldsymbol{v_1}}(t) + c_2 \vec{\boldsymbol{v_2}}(t) + c_3 \vec{\boldsymbol{v_3}}(t) = \vec{\boldsymbol{0}}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} e^{(0)} \\ 0 \\ 2e^{(0)} \end{bmatrix} + c_2 \begin{bmatrix} e^{-(0)} \\ 3e^{-(0)} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{2(0)} \\ e^{2(0)} \\ e^{2(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example

Are the vectors

$$\vec{\boldsymbol{v}}_{\mathbf{1}}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{\boldsymbol{v}}_{\mathbf{2}}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{\boldsymbol{v}}_{\mathbf{3}}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

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$$c_1 \vec{\boldsymbol{v}}_{\mathbf{1}}(t) + c_2 \vec{\boldsymbol{v}}_{\mathbf{2}}(t) + c_3 \vec{\boldsymbol{v}}_{\mathbf{3}}(t) = \vec{\mathbf{0}}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Independence of Functions

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Are the vectors

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$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Independence of Functions

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$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right]$$

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Since this equation has to hold for all t, it has to hold for $t = 0$:

Since the unique solution is $c_1 = c_2 = c_3 = 0$, these vectors are linearly independent.

Linear Independence of Functions

Example

Are the following functions linearly independent?

$$\vec{v_1}(t) = e^t, \quad \vec{v_2}(t) = 5e^{-t}, \quad \vec{v_3}(t) = e^{3t}$$

Linear Independence of Functions

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Are the following functions linearly independent?

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We can think of each of these as one-dimensional vectors.

Linear Independence of Functions

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We can think of each of these as one-dimensional vectors.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad \text{(for all } t \in \mathbb{R}\text{)}$$

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For $t =$ 0: $\quad c_1 \cdot 5e^{(0)} + \quad c_2 \cdot e^{-(0)} + \quad c_3 \cdot e^{3(0)} = 0$

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$$\begin{array}{llll} \text{For } t = \ 0: & c_1 \cdot 5e^{(0)} + & c_2 \cdot e^{-(0)} + & c_3 \cdot e^{3(0)} = 0 \\ \text{For } t = \ 1: & c_1 \cdot 5e^{(1)} + & c_2 \cdot e^{-(1)} + & c_3 \cdot e^{3(1)} = 0 \end{array}$$

Linear Independence of Functions

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$$\begin{array}{lrrr}
\text{For } t = \ \ 0: & c_1 \cdot 5e^{(0)} + & c_2 \cdot e^{-(0)} + & c_3 \cdot e^{3(0)} = 0 \\
\text{For } t = \ \ 1: & c_1 \cdot 5e^{(1)} + & c_2 \cdot e^{-(1)} + & c_3 \cdot e^{3(1)} = 0 \\
\text{For } t = -1: & c_1 \cdot 5e^{(-1)} + & c_2 \cdot e^{-(-1)} + & c_3 \cdot e^{3(-1)} = 0
\end{array}$$

Linear Independence of Functions

Example

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$$\begin{array}{lrcrcrl}
\text{For } t = 0: & c_1 & + & 5c_2 & + & c_3 & = 0 \\
\text{For } t = 1: & ec_1 & + & \frac{5}{e}c_2 & + & e^3 c_3 & = 0 \\
\text{For } t = -1: & \frac{1}{e}c_1 & + & ec_2 & + & \frac{1}{e^3}c_3 & = 0
\end{array}$$

Linear Independence of Functions

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$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ e & \frac{5}{e} & e^3 & 0 \\ \frac{1}{e} & e & \frac{1}{e^3} & 0 \end{array}\right]$$

Linear Independence of Functions

Example

Are the following functions linearly independent?

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We can think of each of these as one-dimensional vectors.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since we have the unique solution $c_1 = c_2 = c_3 = 0$, these functions are linearly independent.

The Wronskian and Linear Independence

Definition

The **Wronskian** of functions $f_1, f_2, \ldots, f_k$ on interval I is the determinant:

$$W[f_1, f_2, \ldots, f_k](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{vmatrix}$$

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Theorem

If $W[f_1, f_2, \ldots, f_k](t) \neq 0$ for all $t \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is a linearly independent set of functions on I.

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If $W[f_1, f_2, \ldots, f_k](t) \neq 0$ for all $t \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is a linearly independent set of functions on I.

If $\{f_1, f_2, \ldots, f_k\}$ are linearly dependent, then $W[f_1, f_2, \ldots, f_k](t) = 0$ for all $t \in I$. Thus, to show independence we only need to find a single t that makes the Wronskian nonzero.

The Wronskian and Linear Independence

Example

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

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$$W(t) = \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

The Wronskian and Linear Independence

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$$\begin{aligned} W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \end{aligned}$$

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$$\begin{aligned}
W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\
&= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \\
&= (t^2+1)(0-4) - (t^2-1)(0-4) + (2t+5)(4t-4t) \\
&= -4t^2 - 4 + 4t^2 - 4 = -8
\end{aligned}$$

Since $W(t) = -8 \neq 0$, this is a set of linearly independent functions.

The Wronskian and Linear Independence

Example

Let us consider the converse:

Does the Wronskian being zero imply dependence?

The Wronskian and Linear Independence

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Does the Wronskian being zero imply dependence?

In general, the answer is no. Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

The Wronskian and Linear Independence

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Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

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In general, the answer is no. Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

So,

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

Basis of a Vector Space

Definition

The set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a **basis** for vector space $\mathbb{V}$, provided that

- $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is linearly independent
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Example

The vectors

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for $\mathbb{R}^3$

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We saw earlier that these vectors span $\mathbb{R}^3$.

It's easy to see that $c_1\vec{i} + c_2\vec{j} + c_3\vec{k} = \vec{0}$ has the unique solution $c_1 = c_2 = c_3 = 0$.

Basis of a Vector Space

Definition

The **standard basis** for $\mathbb{R}^n$ is $\{\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}, \ldots, \vec{\boldsymbol{e_n}}\}$ where

$$\vec{\boldsymbol{e_1}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{\boldsymbol{e_2}} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \cdots, \ \vec{\boldsymbol{e_n}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are the column vectors of the identity matrix $\boldsymbol{I}_n$.

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Example

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 2a + 3b - 4c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

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Since $a, b, c \in \mathbb{R}$ were arbitrary, we see these three vectors span the hyperplane.

Basis of a Vector Space

Example

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Now, we need to show that these vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

Basis of a Vector Space

Example

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\left[\begin{array}{ccr|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The (unique) solution is $c_1 = c_2 = c_3 = 0$, thus these vectors are linearly independent.

Basis of a Vector Space

Example

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

So, we see that the hyperplane has a basis of three vectors.

It looks like this hyperplane is a three-dimensional subspace of a four-dimensional space.

Dimension of a Vector Space

Example

It is possible for a vector space to have multiple bases.

Dimension of a Vector Space

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For $\mathbb{R}^2$, one is the standard basis

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It is possible for a vector space to have multiple bases.
For $\mathbb{R}^2$, one is the standard basis

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but, another basis is given by

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Dimension of a Vector Space

Theorem

The number of vectors in a basis is always the same for a particular vector space

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Proof

The proof is in Appendix LT of your textbook, on page 602.

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Definition

The **dimension** of a vector space $\mathbb{V}$ is the number of vectors in any basis of $\mathbb{V}$.

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The number of vectors in a basis is always the same for a particular vector space

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Definition

The **dimension** of a vector space $\mathbb{V}$ is the number of vectors in any basis of $\mathbb{V}$.

Definition

If a vector space is so large that cannot be spanned by a finite set of vectors, it is called **infinite-dimensional**.

Dimension of a Vector Space

Example

The solution to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two three-dimensional hyperplanes.) What is its dimension?

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Writing this system in RREF gives

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The two free variables tell us that the solution to this system will be a two-parameter family.

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To find a basis, let $x_3 = a$ and $x_4 = b$, arbitrary real numbers.

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The two vectors

$$\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

span the subspace.

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Which means the dimension is 2.

Dimension of the Column Space of a Matrix

Properties of the Column Space of a Matrix

- The pivot columns of a matrix $\boldsymbol{A}$ form a basis for Col $\boldsymbol{A}$.
    - A pivot column is a column of $\boldsymbol{A}$ that corresponds to a column in the RREF of $\boldsymbol{A}$ with a leading 1.

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Example

Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 & 8 \end{bmatrix}$$

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The pivot columns are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which means $\textbf{rank}(\boldsymbol{A}) = 2$ and thus the dimension of the column space is 2.

Invertible Matrices and Dimension

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent:

- $\boldsymbol{A}$ is invertible.
- The column vectors of $\boldsymbol{A}$ are linearly independent.
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Example

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$$\text{For } x = -1: \qquad c_3(-1)^2 + \quad c_2(-1) + \quad c_1 = \quad 0$$

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$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

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So, since we have the unique solution $c_1 = c_2 = c_3 = 0$, these functions are linearly independent and thus form a basis of $\mathbb{P}_2$.

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So, since we have the unique solution $c_1 = c_2 = c_3 = 0$, these functions are linearly independent and thus form a basis of $\mathbb{P}_2$.
Which means $\mathbf{dim}\,\mathbb{P}_2 = 3$.

Invertible Matrices and Dimension

Example

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Suppose we have a possible finite basis of $\mathbb{P}$, where the highest degree polynomial in this basis is k.

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But, there would be no way, using the basis to generate:

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What is the dimension of $\mathbb{P}$, the space of all polynomials?

Suppose we have a possible finite basis of $\mathbb{P}$, where the highest degree polynomial in this basis is k.

But, there would be no way, using the basis to generate:

$$t^{k+1}$$

Thus, $\mathbb{P}$ is infinite-dimensional. ($\mathbf{dim}(\mathbb{P}) = \infty$).

Invertible Matrices and Dimension

Example

What is the dimension of $\mathbb{P}$, the space of all polynomials?

Suppose we have a possible finite basis of $\mathbb{P}$, where the highest degree polynomial in this basis is k.

But, there would be no way, using the basis to generate:

$$t^{k+1}$$

Thus, $\mathbb{P}$ is infinite-dimensional. ($\mathbf{dim}(\mathbb{P}) = \infty$).

There are many infinite-dimensional spaces.
We have seen $\mathbb{P}$, $\mathcal{C}(I)$, and $\mathcal{C}^n(I)$.