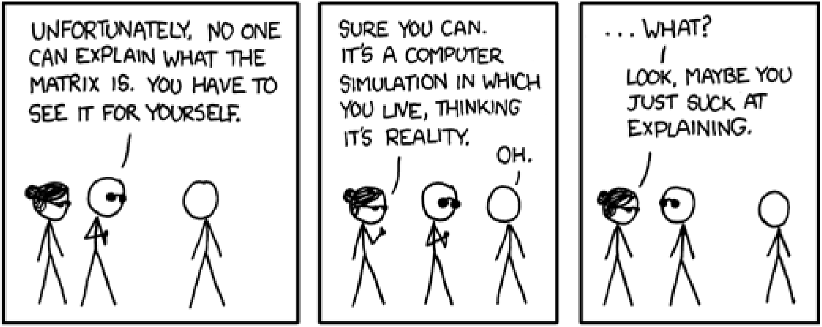
Matrices: Sum and Products

Department of Mathematics

Salt Lake Community College



Matrix

A **matrix** is a rectangular array of **elements** or **entries** (numbers or functions) arranged in **rows** (horizontal) and **columns** (vertical).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

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Equal Matrices

Two matrices of the same order are **equal** if their corresponding entries are equal. If matrices $A = [a_{ij}]$ and $B = [a_{ij}]$ are both $m \times n$, then

$$A = B \Leftrightarrow a_{ij} = b_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n$$

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- The $n \times n$ **identity matrix**, denoted $\mathbf{I}_n$ is:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix Addition

Two matrices of the same order are added (or subtracted) by adding (or subtracting) corresponding entries and recording the results in a matrix of the same size. Using matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$.

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$
$$A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$$

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Multiplication by a Scalar

To find the product of a matrix and a scalar (a complex number), multiply each entry of the matrix by that number. This is called **multiplication by a scalar**. Using matrix notation, if $A = [a_{ij}]$, then

$$c \cdot A = [c \cdot a_{ij}] = [a_{ij} \cdot c] = A \cdot c$$

Properties of Matrix Addition and Scalar Multiplication

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Vectors (are just tiny matrices)

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Vector addition and Scalar Multiplication

Let

$$\vec{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

be vectors in $\mathbb{R}^n$ and c be any scalar. Then, we have:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c \cdot x_1 \\ \vdots \\ c \cdot x_n \end{bmatrix}$$

Properties of Vector Addition and Multiplication

For vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^n$ and scalars c and k.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutativity)
- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (Associativity)
- $c(k\vec{v}) = (ck)\vec{v}$ (Associativity)
- $\vec{u} + \vec{0} = \vec{u}$ (Zero Element)
- $\vec{u} + (-\vec{u}) = \vec{0}$ (Inverse Element)
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (Distributivity)
- $(c + k)\vec{u} = c\vec{u} + k\vec{u}$ (Distributivity)

Dot Product (also called the Scalar Product)

The **dot product** of a row vector $\vec{\boldsymbol{x}}$ and a column vector $\vec{\boldsymbol{y}}$ of equal length n is the result of adding the products of the corresponding entries as follows:

$$
\begin{aligned}
\vec{\boldsymbol{x}} \bullet \vec{\boldsymbol{y}} &= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \bullet \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
&= x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n \\
&= \sum_{k=1}^{n} x_k \cdot y_k
\end{aligned}
$$

Orthogonality

Two vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ in $\mathbb{R}^n$ are called **orthogonal** when:

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Magnitude

For any vector $\vec{v}$ in $\mathbb{R}^n$ the **length**, or **magnitude**, of $\vec{v}$ is a nonnegative scalar, denoted by $\|\vec{v}\|$ and defined to be

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

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Unit Vectors

Vectors of unit length are called **unit vectors**.

Matrix Product

The **matrix product** of a $m \times r$ matrix A and a $r \times n$ matrix B is denoted

$$C = A \cdot B = AB$$

where the ijth entry of C is the dot product of the ith row vector of A and the jth column vector of B:

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{2j} & \cdots & a_{ir} \end{bmatrix} \bullet \begin{bmatrix} b_{1j} \\ \vdots \\ b_{rj} \end{bmatrix} = \sum_{k=1}^{r} a_{ik} b_{kj}$$

The matrix C has order $m \times n$.

Example 1

Perform AB where

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -4 \\ -1 & 0 \end{bmatrix}$$

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Properties of Identity Matrices

For a $m \times n$ matrix A:

- $A \cdot I_n = A$ and $I_m \cdot A = A$
- $A \cdot \mathbf{0}_n = \mathbf{0}_{mn}$ and $\mathbf{0}_m \cdot A = \mathbf{0}_{mn}$

Transpose

For a matrix $A = [a_{ij}]$ the **transpose** of the $m \times n$ matrix A is defined as the $n \times m$ matrix:

$$A^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

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Matrices with Function Entries

Matrices can have functions as entries, not just real numbers.

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m1}(t) & \cdots & a_{mn}(t) \end{bmatrix}$$

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Example 4

$$A(t) = \begin{bmatrix} t^2 & \sin(2t) & 5t-1 \\ t^3 & \frac{1}{3t} & \ln(t+1) \end{bmatrix}$$

Where,

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Derivative of a Matrix

For a differentiable matrix A, the derivative of A is defined as:

$$A'(t) = \frac{dA}{dt} = \begin{bmatrix} a'_{11}(t) & a'_{12}(t) & \cdots & a'_{1n}(t) \\ a'_{21}(t) & a'_{22}(t) & \cdots & a'_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a'_{m1}(t) & a'_{m1}(t) & \cdots & a'_{mn}(t) \end{bmatrix}$$

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Matrix Differentiation Rules

For differentiable matrices $A(t)$ and $B(t)$ and scalar constant c.

- $(A(t) + B(t))' = A'(t) + B'(t)$
- $(cA(t))\prime = cA'(t)$
- $(A(t) \cdot B(t))' = A(t) \cdot B'(t) + A'(t) \cdot B(t)$

Example 5

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Example 6

$$A(t) = \begin{bmatrix} e^t & t^2 \\ \sin t & 2t \end{bmatrix} \qquad A'(t) = \begin{bmatrix} & \\ & \end{bmatrix}$$

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$$g(t) = \begin{bmatrix} \ln t \\ -t^3 \\ \cos 2t \end{bmatrix} \qquad g'(t) = \begin{bmatrix} \dfrac{1}{t} \\ -\,3t^2 \\ -\,2\sin 2t \end{bmatrix}$$

Example 6

$$A(t) = \begin{bmatrix} e^t & t^2 \\ \sin t & 2t \end{bmatrix} \qquad A'(t) = \begin{bmatrix} e^t & \\ & \end{bmatrix}$$

Example 5

$$g(t) = \begin{bmatrix} \ln t \\ -t^3 \\ \cos 2t \end{bmatrix} \qquad g'(t) = \begin{bmatrix} \dfrac{1}{t} \\ -3t^2 \\ -2\sin 2t \end{bmatrix}$$

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Example 6

$$A(t) = \begin{bmatrix} e^t & t^2 \\ \sin t & 2t \end{bmatrix} \qquad A'(t) = \begin{bmatrix} e^t & 2t \\ \cos t & 2 \end{bmatrix}$$