Solving DEs and IVPs with Laplace Transforms

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$$= e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt$$

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This process can be repeated to determine the Laplace transform for any derivative of f.

Derivative Theorem for Laplace Transforms

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$, $f^{(n)}$ is piecewise continuous on $[0, \infty)$, and $f, f', \ldots, f^{(n)}$ are of exponential order α, then for $s > a$, and $n = 1, 2, \ldots$

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Strategy to Solve DEs with Laplace Transforms

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2. Solve the algebraic problem for $Y(s)$.
3. Manipulating $Y(s)$ algebraically if necessary, use the inverse Laplace transform to transform $Y(s)$ into the IVP solution $y(t)$.

Example

Consider

$$y'' - 2y' - 3y = 0 \quad \text{where} \quad y(0) = 2,\ y'(0) = -10$$

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(s^2+1)Y(s) &= \frac{s^2+2}{s^2+1} \\
Y(s) &= \frac{s^2+2}{(s^2+1)(s^2+4)}
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$$y'' + 4y = \sin(t) \quad \text{where} \quad y(0) = 0,\ y'(0) = 1$$

Applying the derivative theorem and solving gives

$$\begin{aligned}
s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} &= \mathcal{L}\{\sin(t)\} \\
s^2Y(s) - 1 + 4Y(s) &= \frac{1}{s^2+1} \\
(s^2+1)Y(s) &= \frac{s^2+2}{s^2+1} \\
Y(s) &= \frac{s^2+2}{(s^2+1)(s^2+4)} \\
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Thus, the solution is

$$y(t) = \frac{1}{3}\sin(t) + \frac{1}{3}\sin(2t)$$

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Translation Property for Multiplication by e^{at}

If the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad \text{for } s > a + \alpha$$

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Multiplication by t^n Rule for the Laplace Transform

If $f(t)$ is a piecewise continuos function on $[0, \infty)$ and is of exponential order α, then for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s) \quad \text{where} \quad n \in \mathbb{N}^+$$

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This process can be repeated for an arbitrary n.

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