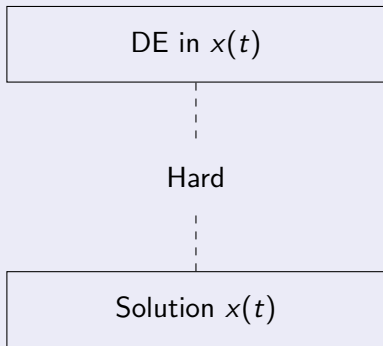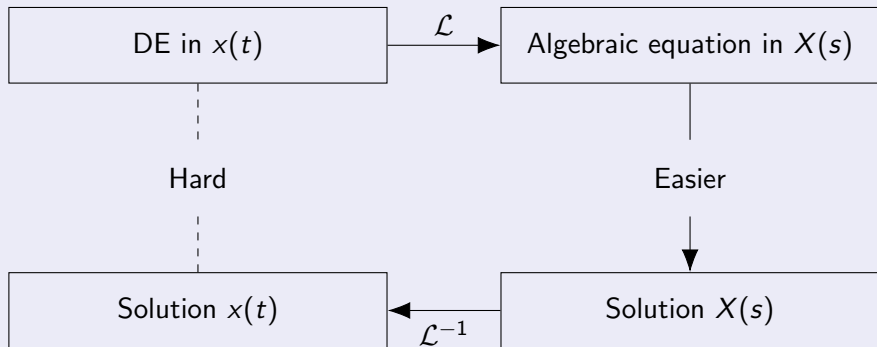# Laplace Transforms

Adam Wilson

Salt Lake Community College

In this chapter we shall study the Laplace transformation, which will allow us to reframe some hard problems into equivalent easier problems.

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DE in $x(t)$

Hard

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┌─────────────────────┐      𝓛       ┌──────────────────────────┐
│    DE in x(t)       │ ───────────▶ │ Algebraic equation in X(s)│
└─────────────────────┘              └──────────────────────────┘
           ┊                                      │
         Hard                                  Easier
           ┊                                      │
           ▼                                      ▼
┌─────────────────────┐      𝓛⁻¹     ┌──────────────────────────┐
│   Solution x(t)     │ ◀─────────── │     Solution X(s)        │
└─────────────────────┘              └──────────────────────────┘
```

Laplace Transform

The **Laplace Transform** $\mathcal{L}\{f(t)\}$ of a suitable function $f(t)$ defined on $[0, \infty)$ is the function $F(s)$ given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t) dt$$

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Linearity of the Laplace Transform

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then by the properties of integrals

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(t) \quad \text{for } a, b \in \mathbb{C}$$

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Existence Theorem for Laplace Transform

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

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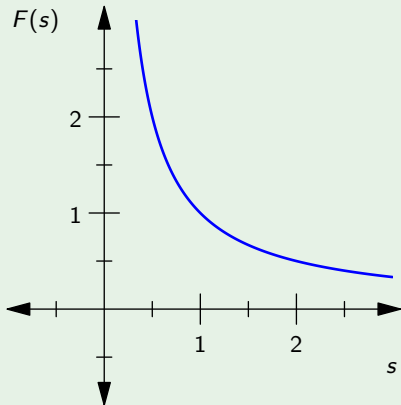Let us consider the constant function $f(t) = 1$.

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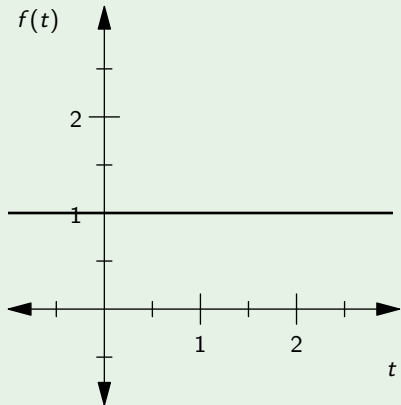Let us consider the constant function $f(t) = 1$.

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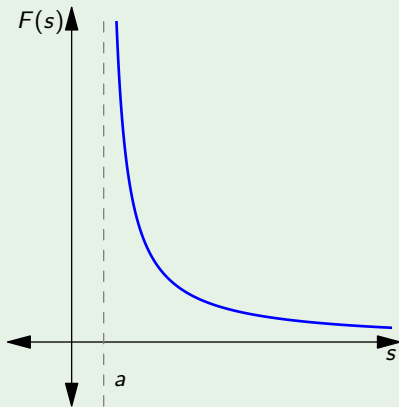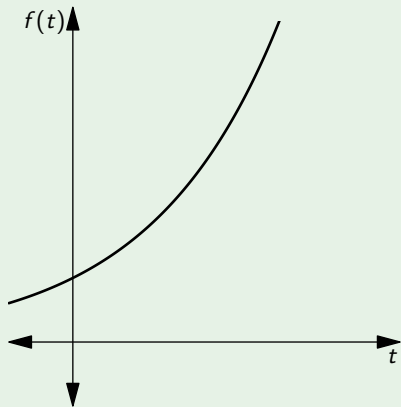
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$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{5 \cdot 1 - 3e^{-2t}\} \\
&= 5\mathcal{L}\{1\} - 3\mathcal{L}\{e^{-2t}\} \\
&= 5\left(\frac{1}{s}\right) - 3\left(\frac{1}{s+2}\right) \\
&= \frac{2s+10}{s(s+2)}
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\end{aligned}$$

So, if we equate the real and imaginary parts, we get:

$$\mathcal{L}\{\cos{(kt)}\} = \frac{s}{s^2 + k^2} \quad \text{and} \quad \mathcal{L}\{\sin{(kt)}\} = \frac{k}{s^2 + k^2}$$

Inverse Laplace Transform

A function $f(t)$ whose transform if $F(s)$ is called the **inverse Laplace transform** of F, and we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

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Thus,

$$f(t) = 4e^{-t} - 2e^{3t}$$

Some Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\}$	
1	$\dfrac{1}{s}$	$s > 0$
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0,\ n \in \mathbb{N}^+$
e^{at}	$\dfrac{1}{s - a}$	$s > a$
$t^n e^{at}$	$\dfrac{n!}{(s - a)^{n+1}}$	$s > a,\ n \in \mathbb{N}^+$
$\sin(bt)$	$\dfrac{b}{s^2 + b^2}$	$s > 0$
$\cos(bt)$	$\dfrac{s}{s^2 + b^2}$	$s > 0$

Some More Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\}$	
$e^{at}\sin{(bt)}$	$\dfrac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos{(bt)}$	$\dfrac{s-a}{(s-a)^2+b^2}$	$s>a$
$\sinh{(bt)}$	$\dfrac{b}{s^2-b^2}$	$s>|b|$
$\cosh{(bt)}$	$\dfrac{s}{s^2-b^2}$	$s>|b|$

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$$F(s) = 2\underbrace{\left(\frac{1}{s-1}\right)}_{\mathcal{L}\{e^t\}} - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{\sqrt{5}}{s^2 + \left(\sqrt{5}\right)^2}\right)}_{\mathcal{L}\{\sin\left(\sqrt{5}t\right)\}}$$

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Thus, $f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^t - \frac{1}{\sqrt{5}}\sin\left(\sqrt{5}t\right)$.

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$$F(s) = \frac{s+1}{s^2 + 4s + 13}$$

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$$= \mathcal{L}\{e^{-2t}\cos(3t)\} - \frac{1}{3}\mathcal{L}\{s^{-2t}\sin(3t)\}$$

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\end{aligned}
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Thus, by linearity we have $f(t) = e^{-2t}\cos(3t) - \frac{1}{3}s^{-2t}\sin(3t)$.