

Basis and Dimension

Department of Mathematics

Salt Lake Community College

Definition

For a vector space \mathbb{V} , a **linear combination** of vectors is:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_k$$

where $c_i \in \mathbb{R}$ and $\vec{v}_i \in \mathbb{V}$

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The **span** of a set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ of vectors in a vector space \mathbb{V} is the set of all linear combinations of these vectors. Denoted **span** $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$

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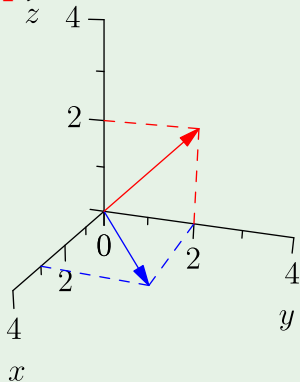
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If the **span** $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \mathbb{V}$ we say the set spans the vector space.

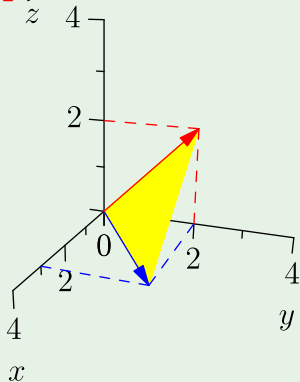
Example 1

Consider $\text{span} \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.



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This spanning set is the plane defined by these two vectors.

Example 2

Let us look closer at this spanning set. Where we give names to the two vectors:

$$\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

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We can then write a general vector in the spanning set as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{u} + b\vec{v}$$

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Example 2

We can write the vector equation as the system:

$$x = 3a \quad \Rightarrow \quad a = \frac{x}{3}$$

$$y = 2a + 2b$$

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Which is equivalent to $2x - 3y + 3z = 0$, the equation of the yellow plane.

Theorem 3

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Example 4

Consider adding $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ to $\text{span} \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

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Consider adding $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Since we can write

$$-1 \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

we see that this doesn't change to the spanning set.

Example 5

Consider adding $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

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To show this, let us try to find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

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Which is equivalent to solving the inconsistent system

$$1 = 3c_1$$

$$1 = 2c_1 + 2c_2$$

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What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

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It turns out the spanning set is \mathbb{R}^3 .

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To show this, we then need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Which has a unique solution for any $x, y, z \in \mathbb{R}$.

Theorem 7

For $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$, a vector $\vec{b} \in \mathbb{R}^n$ is in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ if and only if there is at least one solution to the matrix equation $\mathbf{A}\vec{x} = \vec{b}$. Where \mathbf{A} is formed from the column vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$.

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Note

We can write spanning sets using set builder notation.

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$$\text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} = \left\{ c \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid c \in \mathbb{R} \right\}$$

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$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid c_1, c_2, c_3 \in \mathbb{R} \right\}$$

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$$\begin{aligned} \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} &= \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid c_1, c_2, c_3 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \mid c_1, c_2, c_3 \in \mathbb{R} \right\} \end{aligned}$$

Theorem 10

For $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{V}$, $\text{span} \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \}$ is a subspace of \mathbb{V} .

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Let \vec{u} and \vec{w} be two vectors in the spanning set, which means there are scalars r_i and s_j such that

$$\vec{u} = r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_n \vec{v}_k \quad \text{and} \quad \vec{w} = s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_n \vec{v}_k$$

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So, for any $a, b \in \mathbb{R}$:

$$a\vec{u} + b\vec{w}$$

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So, for any $a, b \in \mathbb{R}$:

$$a\vec{u} + b\vec{w} = a(r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_n \vec{v}_k) + b(s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_n \vec{v}_k)$$

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$$\begin{aligned} a\vec{u} + b\vec{w} &= a(r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_n \vec{v}_k) + b(s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_n \vec{v}_k) \\ &= (ar_1 + bs_1) \vec{v}_1 + (ar_2 + bs_2) \vec{v}_2 + \cdots + (ar_n + bs_n) \vec{v}_k \end{aligned}$$

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So, for any $a, b \in \mathbb{R}$:

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Which means $a\vec{u} + b\vec{w}$ is in the spanning set and we have closure.

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Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

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Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

The column space of \mathbf{B} is a subspace of \mathbb{R}^2 and defined:

$$\mathbf{Col} \mathbf{B} = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 5 \end{bmatrix} \mid c_1, \dots, c_5 \in \mathbb{R} \right\}$$

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A set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ of vectors in a vector space \mathbb{V} is **linearly independent** if no vector of the set can be written as a linear combination of the others. Otherwise it is **linearly dependent**.

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Testing for Linear Independence

To test for linear independence of a set of k vectors $\vec{v}_i \in \mathbb{R}^n$, we consider the system:

$$\begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$$

The column vectors of A are linearly independent if and only if the solution $c_1 = c_2 = \cdots = c_k = 0$ is unique.

Example 12

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ linearly independent?

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To determine if they are, we need to look at the system

$$\mathbf{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Since $|\mathbf{A}| = 5$, we know that \mathbf{A} is invertible and hence a unique solution exists. This means that these vectors are linearly independent.

Example 13

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$ linearly independent?

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We have more columns than rows, which means there will be at least one free variable. Thus, the solution (if one exists) won't be unique, so these vectors are not linearly independent.

Example 14

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ linearly independent?

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To determine if they are, we need to look at the system

$$\mathbf{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 14

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array} \right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

And thus, these vectors are not linearly independent.

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$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

And thus, these vectors are not linearly independent.

Moreover, since

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = 0$$

we can see that any vector can be written as a combination of the others.

Definition

A set of vector functions $\{\vec{v}_1(t), \vec{v}_2(t), \dots, \vec{v}_k\}$ in a vector space \mathbb{V} is **linearly independent** on an interval I if, for *all* $t \in I$, the equation

$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + \cdots + c_k \vec{v}_k(t) = \vec{0} \quad (\text{where } c_i \in \mathbb{R})$$

has the only solution: $c_1 = c_2 = \cdots = c_k = 0$.

If for any value $t_0 \in I$ there is any solution with $c_i \neq 0$, the vector functions $\vec{v}_1(t), \vec{v}_2(t), \dots, \vec{v}_k(t)$ are **linearly dependent**.

Example 15

Are the vectors

$$\vec{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?

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We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0}$$

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$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0}$$

Since this equation has to hold for all t , it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} e^{(0)} \\ 0 \\ 2e^{(0)} \end{bmatrix} + c_2 \begin{bmatrix} e^{-(0)} \\ 3e^{-(0)} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{2(0)} \\ e^{2(0)} \\ e^{2(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Since this equation has to hold for all t , it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Are the following functions linearly independent?

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We can think of each of these as a one-dimensional vector.

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Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

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$$\text{For } t = 0: \quad c_1 + 5c_2 + c_3 = 0$$

$$\text{For } t = 1: \quad ec_1 + \frac{5}{e}c_2 + e^3c_3 = 0$$

$$\text{For } t = -1: \quad \frac{1}{e}c_1 + ec_2 + \frac{1}{e^3}c_3 = 0$$

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$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ e & \frac{5}{e} & e^3 & 0 \\ \frac{1}{e} & e & \frac{1}{e^3} & 0 \end{array} \right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Since we have a unique solution, $c_1 = c_2 = c_3 = 0$, these functions are linearly independent.

Definition

The **Wronskian** of functions f_1, f_2, \dots, f_k on interval I is the determinant:

$$W[f_1, f_2, \dots, f_k](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{vmatrix}$$

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Theorem 17

If $W[f_1, f_2, \dots, f_k](t) \neq 0$ for all $t \in I$, then $\{f_1, f_2, \dots, f_k\}$ is a linearly independent set of functions on I .

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Note

If $\{f_1, f_2, \dots, f_k\}$ are linearly dependent, then $W[f_1, f_2, \dots, f_k](t) = 0$ for all $t \in I$. Thus, to show independence we only need to find a single t that makes the Wronskian nonzero.

Example 18

Use the Wronskian to check that

$$\{t^2 + 1, t^2 - 1, 2t + 5\}$$

are linearly independent on \mathbb{P}_2 .

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$$W(t) = \begin{vmatrix} t^2 + 1 & t^2 - 1 & 2t + 5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

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$$\begin{aligned} W(t) &= \begin{vmatrix} t^2 + 1 & t^2 - 1 & 2t + 5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2 + 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2 - 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t + 5) \begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \end{aligned}$$

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Use the Wronskian to check that

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$$\begin{aligned} W(t) &= \begin{vmatrix} t^2 + 1 & t^2 - 1 & 2t + 5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2 + 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2 - 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t + 5) \begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \\ &= (t^2 + 1)(0 - 4) - (t^2 - 1)(0 - 4) + (2t + 5)(4t - 4t) \\ &= -4t^2 - 4 + 4t^2 - 4 = -8 \end{aligned}$$

Since $W(t) = -8 \neq 0$, this is a set of linearly independent functions.

Example 19

Let us consider the converse:

Does the Wronskian being zero imply dependence?

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In general, the answer is no.

Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

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Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

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So,

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

Definition

The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a **basis** for vector space \mathbb{V} , provided that

- $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a linearly independent set
- $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \mathbb{V}$

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Example 20

The vectors

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for \mathbb{R}^3

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are a basis for \mathbb{R}^3

We saw earlier that these vectors span \mathbb{R}^3 .

Definition

The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a **basis** for vector space \mathbb{V} , provided that

- $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a linearly independent set
- $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \mathbb{V}$

Example 20

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are a basis for \mathbb{R}^3

We saw earlier that these vectors span \mathbb{R}^3 .

It's easy to see that $c_1\vec{i} + c_2\vec{j} + c_3\vec{k} = \vec{0}$ has the unique solution $c_1 = c_2 = c_3 = 0$.

Definition

The **standard basis** for \mathbb{R}^n is $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are the column vectors of the identity matrix I_n .

Example 21

Let us find a basis for the hyperplane in \mathbb{R}^4 that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 2a + 3b - 4c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

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Since $a, b, c \in \mathbb{R}$ were arbitrary, we see these three vectors span the hyperplane.

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Let us find a basis for the hyperplane in \mathbb{R}^4 that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Now, we need to show that the vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

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Let us find a basis for the hyperplane in \mathbb{R}^4 that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The (unique) solution is $c_1 = c_2 = c_3 = 0$, thus these vectors are linearly independent.

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Let us find a basis for the hyperplane in \mathbb{R}^4 that is the solution to

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So, we see that the hyperplane has a basis of three vectors.

It looks like this hyperplane is a three-dimensional subspace of a four-dimensional space.

Example 22

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For \mathbb{R}^2 , one is the standard basis

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For \mathbb{R}^2 , one is the standard basis

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but, another basis is given by

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Theorem 23

The number of vectors in a basis is always the same for a particular vector space

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Proof

The proof is in Appendix LT of your textbook, on page 602.

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Definition

The **dimension** of a vector space \mathbb{V} is the number of vectors in any basis of \mathbb{V} .

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The **dimension** of a vector space \mathbb{V} is the number of vectors in any basis of \mathbb{V} .

Definition

If a vector space is so large that cannot be spanned by a finite set of vectors, it is called **infinite-dimensional**.

Example 24

The solution to the system

$$x_1 + 2x_2 - x_3 + x_4 = 0$$

$$x_1 + 3x_2 + x_3 + 2x_4 = 0$$

is a subspace of \mathbb{R}^4 . (The intersection of two 3D hyperplanes.)

What is its dimension?

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Writing this system in RREF gives

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The two free variables tell us that the solution to this system will be a two-parameter family.

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To find a basis, let $x_3 = a$ and $x_4 = b$, be arbitrary real numbers.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5a + b \\ -2a - b \\ a \\ b \end{bmatrix}$$

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$$\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

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Which means the dimension is 2.

Properties of the Column Space of a Matrix

- ① The pivot columns of a matrix \mathbf{A} form a basis for $\text{Col } \mathbf{A}$.
 - A pivot column is a column of \mathbf{A} that corresponds to a column in the RREF of \mathbf{A} with a leading 1.

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Example 25

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 & 8 \end{bmatrix}$$

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The pivot columns are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which means $\text{rank}(\mathbf{A}) = 2$ and thus the dimension of the column space is 2.

Invertible Matrix Characterization

Let \mathbf{A} be a $n \times n$ matrix. The following statements are equivalent:

- \mathbf{A} is invertible.
- The column vectors of \mathbf{A} are linearly independent.
- Every column of \mathbf{A} is a pivot column.
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Which means **dim** $\mathbb{P}_2 = 3$.

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Note

There are many infinite-dimensional spaces.

We have seen \mathbb{P} , $\mathcal{C}(I)$, and $\mathcal{C}^n(I)$.