Linear Systems of Differential Equations

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Linear First-Order DE System

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If $\vec{\boldsymbol{f}}(t) = \vec{\boldsymbol{0}}$, the system is **homogeneous**

$$\vec{\boldsymbol{x}}'(t) = \boldsymbol{A}(t)\vec{\boldsymbol{x}}(t)$$

Example

Consider the homogeneous linear first-order system

$$\begin{aligned} x' &= 3x - 2y \\ y' &= x \\ z' &= -x + y + 3z \end{aligned}$$

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Initial-Value Problem for a Linear DE System

For a linear DE system, an **initial-value problem** is the combination of a linear DE system and an initial value vector.

$$\vec{\boldsymbol{x}'}(t) = \boldsymbol{A}(t)\vec{\boldsymbol{x}}(t) + \vec{\boldsymbol{f}}(t), \qquad \vec{\boldsymbol{x}}(t_0) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

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Existence and Uniqueness Theorem for Linear DE Systems

Given an $n \times n$ matrix function $\boldsymbol{A}(t)$ and a $n \times 1$ vector function $\vec{\boldsymbol{f}}(t)$, both continuous on an open interval I containing t_0, and a constant n-vector $\vec{\boldsymbol{x}_0}$, there exists a unique vector function $\vec{\boldsymbol{x}}(t)$ such that

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}(t)\vec{\boldsymbol{x}} + \vec{\boldsymbol{f}}(t) \quad \text{and} \quad \vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{x}_0}$$

The Superposition Principle for Homogeneous Linear DE Systems

Let $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ be solution vectors for the homogenous equation

$$\vec{x'} = \boldsymbol{A}(t)\vec{x} \quad \text{on } I$$

Then, any linear combination of these solution vectors is also a solution vector for the system.

That is,

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + \cdots + c_n\vec{x_n}$$

is also a solution on I for any $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Solution Space Theorem for Homogeneous Linear DE Systems

If

$$\vec{\boldsymbol{x}'} = \boldsymbol{A}(t)\vec{\boldsymbol{x}}$$

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Solution Theorem for Homogenous Linear DE Systems

For n linearly independent solutions $\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \ldots, \vec{\boldsymbol{x_n}}$ of

$$\vec{\boldsymbol{x'}} = \boldsymbol{A}(t)\vec{\boldsymbol{x}}$$

the general solution is

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For the system in the last example we have three solutions

$$\vec{x_1} = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x_2} = \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x_3} = \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}$$

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To show that $\{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$ are linearly independent on $(-\infty, \infty)$ choose a point, say $t_0 = 0$.

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Calculate $\vec{x_1}(t_0)$, $\vec{x_2}(t_0)$, and $\vec{x_3}(t_0)$. Then construct the column space matrix:

$$\boldsymbol{C} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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So, the general solution is $\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$.

We have a few ways to express solutions:

$$\vec{\boldsymbol{x}} = c_1 \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}$$

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Fundamental Matrix

For a basis of n linearly independent solutions of $\vec{\boldsymbol{x'}} = \boldsymbol{A}\vec{\boldsymbol{x}}$, the matrix $\boldsymbol{X}(t)$ whoose *columns* are the vector solutions $\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \ldots, \vec{\boldsymbol{x_n}}$ is called the **fundamental matrix** for the system.

$$\vec{\boldsymbol{x}} = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{\boldsymbol{x_1}} & \vec{\boldsymbol{x_2}} & \cdots & \vec{\boldsymbol{x_n}} \\ | & | & & | \end{bmatrix}}_{\boldsymbol{X}(t)} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad c_1, c_2, c_3 \in \mathbb{R}$$

Graphical Views

- The t-x and t-y graphs showing the individual solution functions $x(t)$ and $y(t)$ are called **component graphs**, **solution graphs**, or **time series**.
- The x-y graph is the **phase plane**. The **trajectories** in the phase plane are the parametric curves described by $x(t)$ and $y(t)$.

Trajectories on a phase plane create a **phase portrait**.

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Any version of the these equations produces solutions of the form

$$\begin{aligned} x(t) &= c_1 \cos\left(t\sqrt{0.1}\right) + c_2 \sin\left(t\sqrt{0.1}\right) \\ y(t) &= x'(t) = -c_1\sqrt{0.1}\sin\left(t\sqrt{0.1}\right) + c_2\sqrt{0.1}\cos\left(t\sqrt{0.1}\right) \end{aligned}$$

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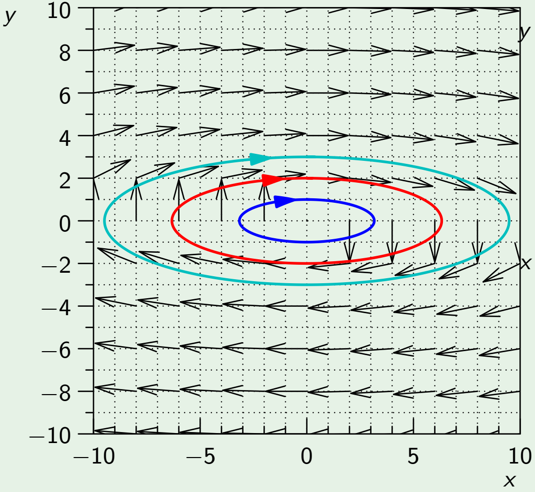

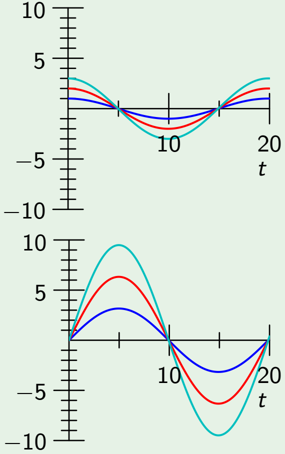

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With solutions of the (approximate) form

$$\begin{aligned} x(t) &\approx e^{-0.025t}\left(c_1 \cos(0.32t) + c_2 \sin(0.32t)\right) \\ y(t) &\approx e^{-0.025t}\left(-0.32c_1 \sin(0.32t) + 0.32c_2 \cos(0.23t)\right) \\ &\quad - 0.025e^{-0.025t}\left(c_1 \cos(0.32t) + c_2 \sin(0.32t)\right) \end{aligned}$$

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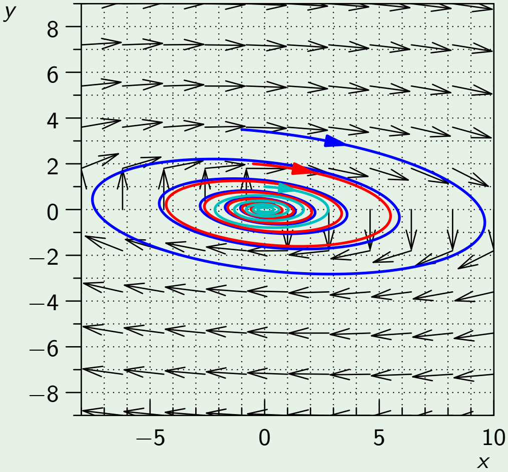

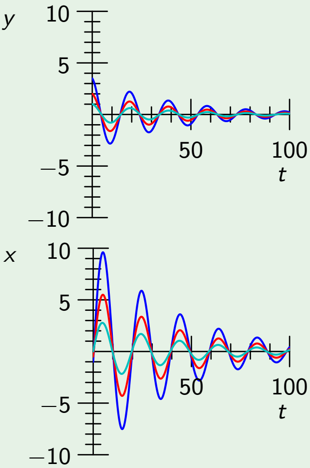

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$$x'' + 0.1x = 0.5\cos(t) \qquad x(0) = 1, \quad x'(0) = 0$$

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This DE represents a periodically forces harmonic oscillator and has system form:

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It is not easy to find an analytic solution to this DE, but we can draw the solutions using numerical calculations.

We can use Euler's method, which we have seen before.

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$$x_{n+1} = x_n + h \cdot x'(t_n) = x_n + h \cdot y_n$$
$$y_{n+1} = y_n + h \cdot y'(t_n) = y_n + h \cdot (-0.1x + 0.5\cos(t_n))$$

With step size $h = 0.1$, $x(0) = 1$, and $y(0) = 0$.

t_n	x_n	y_n	x'	y'
0.0	1.0000	0.0000	0.0000	0.4000
0.1	1.0000	0.0400	0.0400	0.3975
0.2	1.0040	0.0798	0.0798	0.3896
0.3	1.0120	0.1187	0.1187	0.3765
0.4	1.0238	0.1564	0.1564	0.3581
0.5	1.0395	0.1922	0.1922	0.3348
0.6	1.0587	0.2257	0.2257	0.3068
0.7	1.0813	0.2563	0.2563	0.2743
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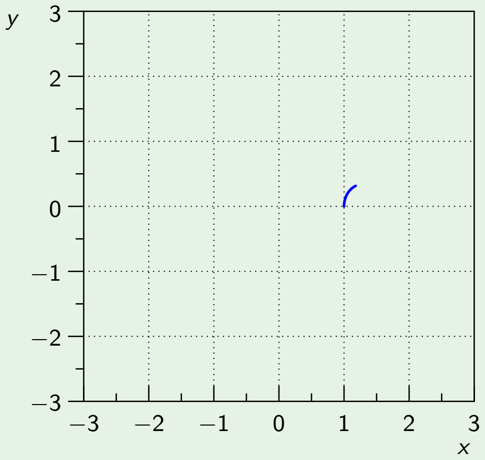

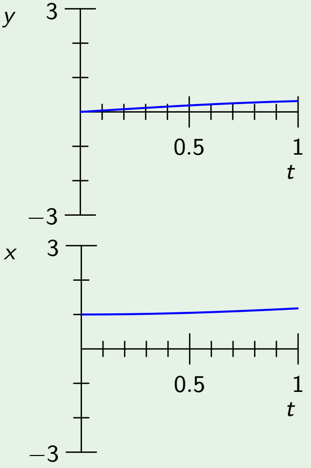

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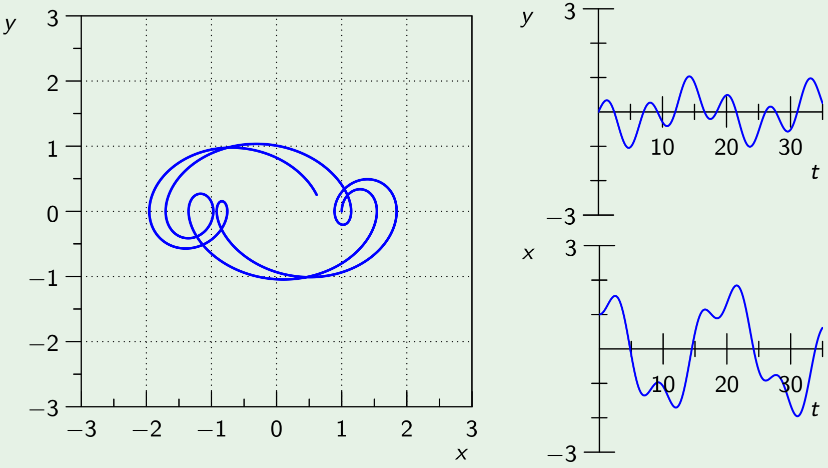

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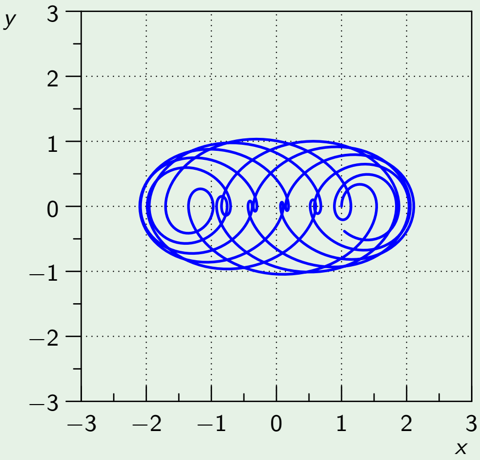

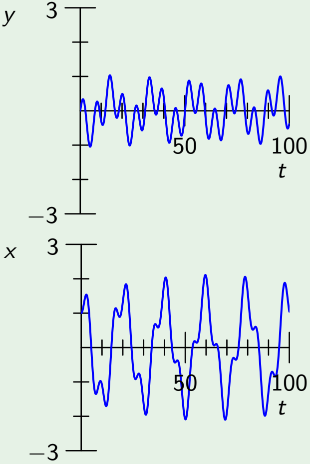

Graphical Properties of Uniqueness

- For a linear system of differential equations in $\mathbb{R}^n$, solutions so not cross in $t, x_1, x_2, \ldots, x_n$-space (i.e. $\mathbb{R}^{n+1}$).

Graphical Properties of Uniqueness

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- For an *autonomous* linear system in $\mathbb{R}^n$, trajectories *also* do not cross in $x_1, x_2, \ldots, x_n$-space (i.e. $\mathbb{R}^n$).