Growth and Decay Phenomena

Colby Community College

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Note

We last saw this equation in section 1.1 where Thomas Malthus used it to estimate global population growth.

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Note that $e^C > 0$ for all $C \in \mathbb{R}$. Thus, if we replace e^C with an arbitrary constant A, which could be negative, we can dispose of the absolute value bars.

$$y(t) = Ae^{kt}, \quad A \in \mathbb{R}$$

Growth and Decay

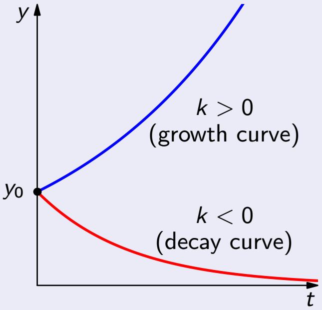
For each k, the solution of the IVP

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is given by

$$y(t) = y_0 e^{kt}$$

Solution curves are **growth curves** for $k > 0$ and **decay curves** for $k < 0$.



Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is the value of the decay constant k?

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Let us say that the initial amount of Carbon-14 is $Q(0) = Q_0$. So, after 5600 years, the half-life, there will be $\frac{1}{2}Q_0$ remaining. That is,

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What is $Q(t)$ at any time t?

Since we know k, we can plug it into the general solution:

$$Q(t) = Q_0 e^{-t\ln(2)/5600} \approx Q_0 e^{-0.00012378t}$$

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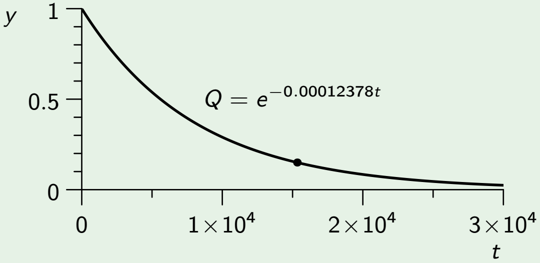

Compounded Interest

If an initial amount of A_0 dollars is deposited at an **annual interest rate** of r, **compounded** n times per year, the **future value** of the deposit at time t is given by

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To earn the theoretical maximum interest, you would need the account to compounded at every instant of time. That is every hour, every minute, every second, and so on. We can represent this using the following limit:

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We call this **continuously compounded** interest.

Continuously Compounded Interest

If an initial amount of A_0 dollars is deposited at an annual interest rate of r, compounded continuously, the future value $A(t)$ of the deposit at time t satisfies the initial-value problem

$$\frac{dA}{dt} = rA, \quad A(0) = A_0$$

and is therefore given by

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Example 2

Suppose you came across a time machine and decided to travel to London on 27 July 1694 and deposit £20 (about \$25.78 USD) during the Bank of England's grand opening. Let us suppose you got the founder's day savings account with an annual interest rate of 8% compounded continuously. How much money would be in the account on the bank's tricentennial in 1994?

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So we would have the queenly sum of £529,782,442,597 pounds (which is about \$682,836,581,952 USD).

The Savings Problem

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It is left as an exercise to the student that we can use the methods from earlier in the chapter solve this nonhomogenous linear IVP, obtaining:

$$A(t) = A_0 e^{rt} + \frac{a}{r}(e^{rt} - 1)$$

Here the first term represents the accumulation due to the starting deposit, and the second term the accumulation due to the subsequent deposits and the interest that they earn.

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Ravi has just entered college at age 18 and has decided to improve his health and save money by quitting smoking. He figures he can save \$30 per week in this way. If he deposits the amount in an account paying 10% annual interest compounded continuously how much will he have in the account when he retires at age 65?

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After $65 - 18 = 47$ years, Ravi will have

$$A(47) = 15600(e^{0.1(47)} - 1) \approx \$1,699,575.89$$