Complex Characteristic Roots

Colby Community College

Solution for Complex Characteristic Roots

For $\Delta < 0$, the characteristic roots of the DE

$$ay'' + by' + cy = 0$$

are

$$r_1 = \alpha + i\beta = -\frac{b}{2a} + i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$

$$r_2 = \alpha - i\beta = -\frac{b}{2a} - i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$

Solution for Complex Characteristic Roots

For $\Delta < 0$, the characteristic roots of the DE

$$ay'' + by' + cy = 0$$

are

$$r_1 = \alpha + i\beta = -\frac{b}{2a} + i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$
$$r_2 = \alpha - i\beta = -\frac{b}{2a} - i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$

The functions $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are linearly independent solutions, and the general solution is given by

$$y(t) = e^{\alpha t}\left(c_1\cos(\beta t) + c_2\sin(\beta t)\right)$$

where c_1 and c_2 are arbitrary constants determined by the initial conditions.

Solution for Complex Characteristic Roots

For $\Delta < 0$, the characteristic roots of the DE

$$ay'' + by' + cy = 0$$

are

$$r_1 = \alpha + i\beta = -\frac{b}{2a} + i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$
$$r_2 = \alpha - i\beta = -\frac{b}{2a} - i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$

The functions $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are linearly independent solutions, and the general solution is given by

$$y(t) = e^{\alpha t}\left(c_1\cos(\beta t) + c_2\sin(\beta t)\right)$$

where c_1 and c_2 are arbitrary constants determined by the initial conditions.

The set $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ forms a basis for the solution space $\mathbb{S}$.

Example 1

Let us find the general solution of

$$y'' - 4y' + 13y = 0$$

Example 1

Let us find the general solution of

$$y'' - 4y' + 13y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 - 4r + 13$$

Example 1

Let us find the general solution of

$$y'' - 4y' + 13y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 - 4r + 13$$

which has solutions $r = 2 \pm 3i$. So $\alpha = 2$ and $\beta = 3$.

Example 1

Let us find the general solution of

$$y'' - 4y' + 13y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 - 4r + 13$$

which has solutions $r = 2 \pm 3i$. So $\alpha = 2$ and $\beta = 3$.

Thus, the general solution is

$$y(t) = e^{2t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

Example 1

Let us find the general solution of

$$y'' - 4y' + 13y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 - 4r + 13$$

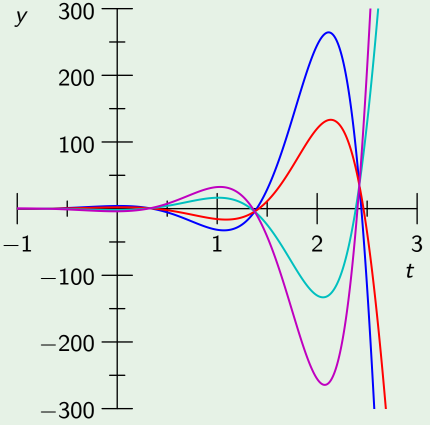
which has solutions $r = 2 \pm 3i$. So $\alpha = 2$ and $\beta = 3$.

Thus, the general solution is

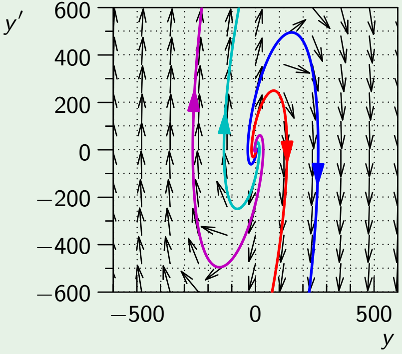
$$y(t) = e^{2t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

The set $\left\{e^{2t}\cos(3t), e^{2t}\sin(3t)\right\}$ is a basis of the solution space $\mathbb{S}$, and $\mathbf{dim}\,\mathbb{S} = 2$.

Example 1



(a) Time Series



(b) Phase Portrait

Example 2

Let us find the general solution of

$$y'' + 2y' + 4y = 0$$

Example 2

Let us find the general solution of

$$y'' + 2y' + 4y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 4$$

Example 2

Let us find the general solution of

$$y'' + 2y' + 4y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 4$$

which has solutions $r = -1 \pm i\sqrt{3}$. So $\alpha = -1$ and $\beta = \sqrt{3}$.

Example 2

Let us find the general solution of

$$y'' + 2y' + 4y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 4$$

which has solutions $r = -1 \pm i\sqrt{3}$. So $\alpha = -1$ and $\beta = \sqrt{3}$.

Thus, the general solution is

$$y(t) = e^{-t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right)\right)$$

Example 2

Let us find the general solution of

$$y'' + 2y' + 4y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 4$$

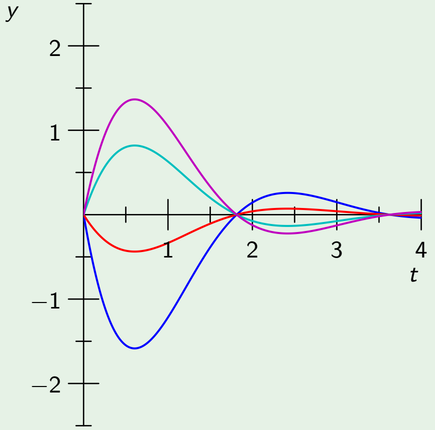
which has solutions $r = -1 \pm i\sqrt{3}$. So $\alpha = -1$ and $\beta = \sqrt{3}$.

Thus, the general solution is

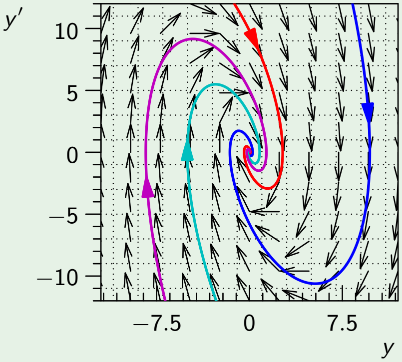
$$y(t) = e^{-t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right)\right)$$

The set $\left\{e^{-t}\cos\left(\sqrt{3}t\right), e^{-t}\sin\left(\sqrt{3}t\right)\right\}$ is a basis of the solution space $\mathbb{S}$, and $\mathbf{dim}\,\mathbb{S} = 2$.

Example 2



(a) Time Series



(b) Phase Portrait

Example 3

Let us find the general solution of

$$y'' + y = 0$$

Example 3

Let us find the general solution of

$$y'' + y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 1$$

Example 3

Let us find the general solution of

$$y'' + y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 1$$

which has solutions $r = \pm i$. So $\alpha = 0$ and $\beta = 1$.

Example 3

Let us find the general solution of

$$y'' + y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 1$$

which has solutions $r = \pm i$. So $\alpha = 0$ and $\beta = 1$.

Thus, the general solution is

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

Example 3

Let us find the general solution of

$$y'' + y = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 1$$

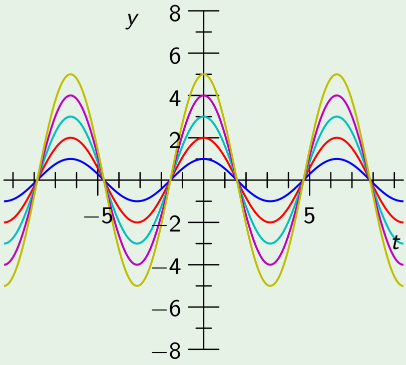
which has solutions $r = \pm i$. So $\alpha = 0$ and $\beta = 1$.

Thus, the general solution is

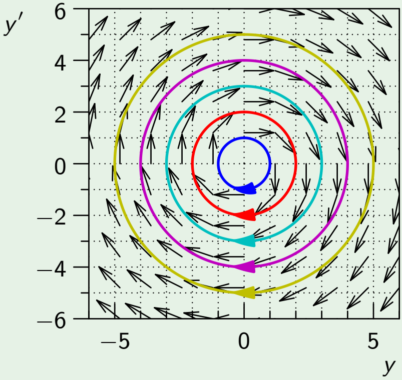
$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

The set $\{\cos(t), \sin(t)\}$ is a basis of the solution space $\mathbb{S}$, and $\mathbf{dim}\,\mathbb{S} = 2$.

Example 3



(a) Time Series



(b) Phase Portrait

Underdamped Mass-Spring System

The motion of a mass-spring system is called **underdamped** when we have $\Delta = b^2 - 4mk < 0$. Both characteristic roots are complex and the solutions are given by

$$x(t) = e^{-\frac{b}{2m}}\left(c_1\cos\left(\omega_d t\right) + c_2\sin\left(\omega_d t\right)\right), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos\left(\omega_d t - \delta\right), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
- **Natural quasi-frequency** $f_d = \dfrac{\omega_d}{2\pi}$
- **Quasi-period** $T_d = \dfrac{1}{f_d} = \dfrac{2pi}{\omega_d}$
- **Time constant** $\tau = \dfrac{2m}{b}$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos(\omega_d t - \delta), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
- **Natural quasi-frequency** $f_d = \dfrac{\omega_d}{2\pi}$
- **Quasi-period** $T_d = \dfrac{1}{f_d} = \dfrac{2pi}{\omega_d}$
- **Time constant** $\tau = \dfrac{2m}{b}$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos(\omega_d t - \delta), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
- **Natural quasi-frequency** $f_d = \dfrac{\omega_d}{2\pi}$
- **Quasi-period** $T_d = \dfrac{1}{f_d} = \dfrac{2pi}{\omega_d}$
- **Time constant** $\tau = \dfrac{2m}{b}$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos(\omega_d t - \delta), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
- **Natural quasi-frequency** $f_d = \dfrac{\omega_d}{2\pi}$
- **Quasi-period** $T_d = \dfrac{1}{f_d} = \dfrac{2pi}{\omega_d}$
- **Time constant** $\tau = \dfrac{2m}{b}$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos\left(\omega_d t - \delta\right), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
- **Natural quasi-frequency** $f_d = \dfrac{\omega_d}{2\pi}$
- **Quasi-period** $T_d = \dfrac{1}{f_d} = \dfrac{2pi}{\omega_d}$
- **Time constant** $\tau = \dfrac{2m}{b}$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos(\omega_d t - \delta), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
- **Natural quasi-frequency** $f_d = \dfrac{\omega_d}{2\pi}$
- **Quasi-period** $T_d = \dfrac{1}{f_d} = \dfrac{2pi}{\omega_d}$
- **Time constant** $\tau = \dfrac{2m}{b}$

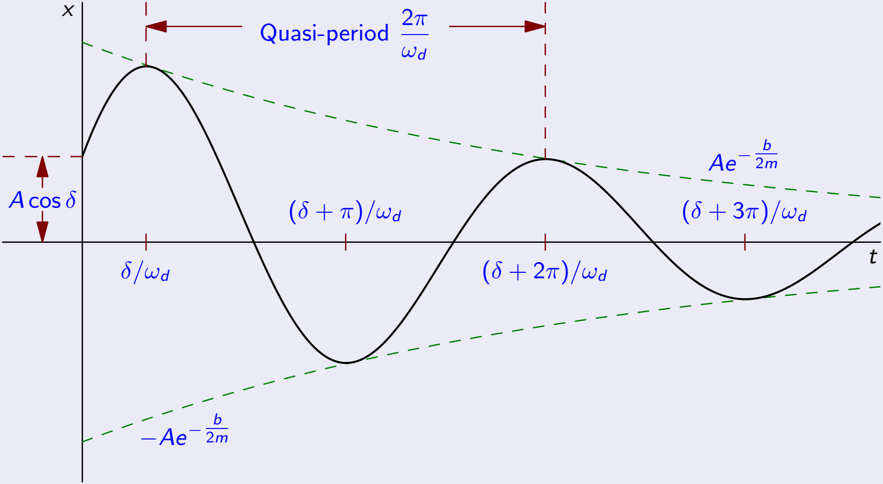
Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos\left(\omega_d t - \delta\right), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
- **Natural quasi-frequency** $f_d = \dfrac{\omega_d}{2\pi}$
- **Quasi-period** $T_d = \dfrac{1}{f_d} = \dfrac{2pi}{\omega_d}$
- **Time constant** $\tau = \dfrac{2m}{b}$

Alternate Solution Visual Example



Example 4

Consider the Mass-Spring IVP where

$$\ddot{x} + \dot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0$$

Example 4

Consider the Mass-Spring IVP where

$$\ddot{x} + \dot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0$$

which has characteristic equation

$$r^2 + r + 1 = 0$$

Example 4

Consider the Mass-Spring IVP where

$$\ddot{x} + \dot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0$$

which has characteristic equation

$$r^2 + r + 1 = 0$$

and characteristic solutions

$$r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Example 4

Consider the Mass-Spring IVP where

$$\ddot{x} + \dot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0$$

which has characteristic equation

$$r^2 + r + 1 = 0$$

and characteristic solutions

$$r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Which means $\alpha = -\dfrac{1}{2}$ and $\beta = \dfrac{\sqrt{3}}{2}$.

Example 4

The general solution is

$$x(t) = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right)$$

Example 4

The general solution is

$$x(t) = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}}{2} t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} t\right)\right)$$

We can then calculate

$$\dot{x}(t) = e^{-\frac{t}{2}} \left(\left(-\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2}\right) \cos\left(\frac{t\sqrt{3}}{2}\right) - \left(\frac{c_2}{2} + \frac{c_1\sqrt{3}}{2}\right) \sin\left(\frac{t\sqrt{3}}{2}\right)\right)$$

Example 4

The general solution is

$$x(t) = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

We can then calculate

$$\dot{x}(t) = e^{-\frac{t}{2}}\left(\left(-\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2}\right)\cos\left(\frac{t\sqrt{3}}{2}\right) - \left(\frac{c_2}{2} + \frac{c_1\sqrt{3}}{2}\right)\sin\left(\frac{t\sqrt{3}}{2}\right)\right)$$

If we substitute in the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, we find that $c_1 = 1$ and $c_2 = \dfrac{1}{\sqrt{3}}$.

Example 4

Thus, the solution to the IVP is

$$x(t) = e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \right)$$

Example 4

Thus, the solution to the IVP is

$$x(t) = e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}}{2} t \right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t \right) \right)$$

In alternate polar form

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6} \right)$$

Where

$$A = \sqrt{1^2 + \left(\frac{1}{\sqrt{3}} \right)^2} = \frac{2}{\sqrt{3}} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{\frac{1}{\sqrt{3}}}{1} \right) = \frac{\pi}{6}$$

Example 4

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6}\right)$$

Has the following properties:

- time-varying amplitude $A(t) = \dfrac{2}{\sqrt{3}} e^{-\frac{t}{2}}$.
- circular quasi-frequency $\omega_d = \dfrac{\sqrt{3}}{2}$
- natural quasi-frequency $f_d = \dfrac{\omega_d}{2\pi} = \dfrac{\sqrt{3}}{4\pi}$ hertz
- quasi-period $T_d = \dfrac{2\pi}{\omega_d} = \dfrac{4\pi}{\sqrt{3}}$ seconds
- phase-shift $\varphi = \dfrac{\delta}{\omega_d} = \dfrac{\pi}{3\sqrt{3}}$ rad per second

Example 4

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6}\right)$$

Has the following properties:

- time-varying amplitude $A(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}}$.
- circular quasi-frequency $\omega_d = \frac{\sqrt{3}}{2}$
- natural quasi-frequency $f_d = \frac{\omega_d}{2\pi} = \frac{\sqrt{3}}{4\pi}$ hertz
- quasi-period $T_d = \frac{2\pi}{\omega_d} = \frac{4\pi}{\sqrt{3}}$ seconds
- phase-shift $\varphi = \frac{\delta}{\omega_d} = \frac{\pi}{3\sqrt{3}}$ rad per second

Example 4

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6}\right)$$

Has the following properties:

- time-varying amplitude $A(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}}$.
- circular quasi-frequency $\omega_d = \frac{\sqrt{3}}{2}$
- natural quasi-frequency $f_d = \frac{\omega_d}{2\pi} = \frac{\sqrt{3}}{4\pi}$ hertz
- quasi-period $T_d = \frac{2\pi}{\omega_d} = \frac{4\pi}{\sqrt{3}}$ seconds
- phase-shift $\varphi = \frac{\delta}{\omega_d} = \frac{\pi}{3\sqrt{3}}$ rad per second

Example 4

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6}\right)$$

Has the following properties:

- time-varying amplitude $A(t) = \dfrac{2}{\sqrt{3}} e^{-\frac{t}{2}}$.
- circular quasi-frequency $\omega_d = \dfrac{\sqrt{3}}{2}$
- natural quasi-frequency $f_d = \dfrac{\omega_d}{2\pi} = \dfrac{\sqrt{3}}{4\pi}$ hertz
- quasi-period $T_d = \dfrac{2\pi}{\omega_d} = \dfrac{4\pi}{\sqrt{3}}$ seconds
- phase-shift $\varphi = \dfrac{\delta}{\omega_d} = \dfrac{\pi}{3\sqrt{3}}$ rad per second

Example 4

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6}\right)$$

Has the following properties:

- time-varying amplitude $A(t) = \dfrac{2}{\sqrt{3}} e^{-\frac{t}{2}}$.
- circular quasi-frequency $\omega_d = \dfrac{\sqrt{3}}{2}$
- natural quasi-frequency $f_d = \dfrac{\omega_d}{2\pi} = \dfrac{\sqrt{3}}{4\pi}$ hertz
- quasi-period $T_d = \dfrac{2\pi}{\omega_d} = \dfrac{4\pi}{\sqrt{3}}$ seconds
- phase-shift $\varphi = \dfrac{\delta}{\omega_d} = \dfrac{\pi}{3\sqrt{3}}$ rad per second

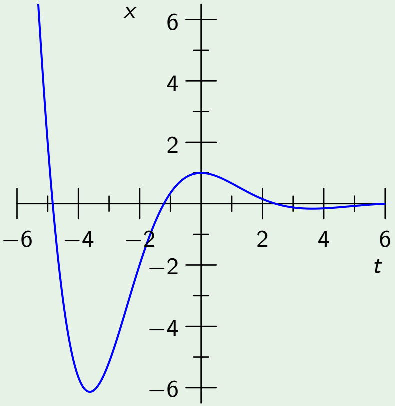
Example 4

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6}\right)$$

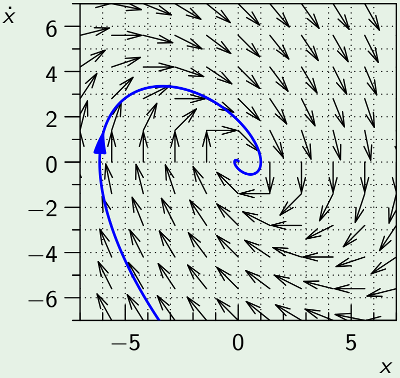
Has the following properties:

- time-varying amplitude $A(t) = \dfrac{2}{\sqrt{3}} e^{-\frac{t}{2}}$.
- circular quasi-frequency $\omega_d = \dfrac{\sqrt{3}}{2}$
- natural quasi-frequency $f_d = \dfrac{\omega_d}{2\pi} = \dfrac{\sqrt{3}}{4\pi}$ hertz
- quasi-period $T_d = \dfrac{2\pi}{\omega_d} = \dfrac{4\pi}{\sqrt{3}}$ seconds
- phase-shift $\varphi = \dfrac{\delta}{\omega_d} = \dfrac{\pi}{3\sqrt{3}}$ rad per second

Example 4



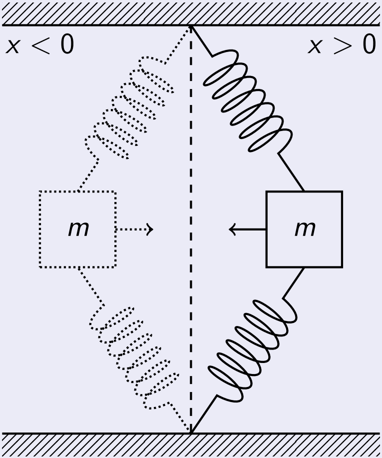
(a) Time Series



(b) Phase Portrait

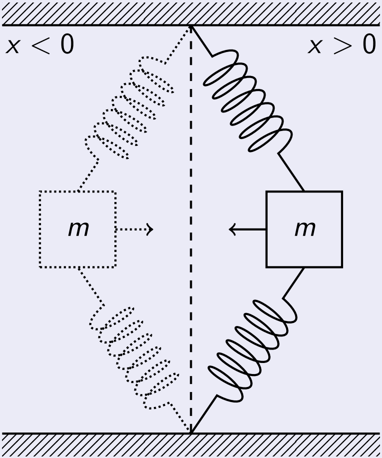
The Guitar String: A Qualitative Analysis

The vibration of a guitar string can be described as a damped oscillator.



The Guitar String: A Qualitative Analysis

The vibration of a guitar string can be described as a damped oscillator.



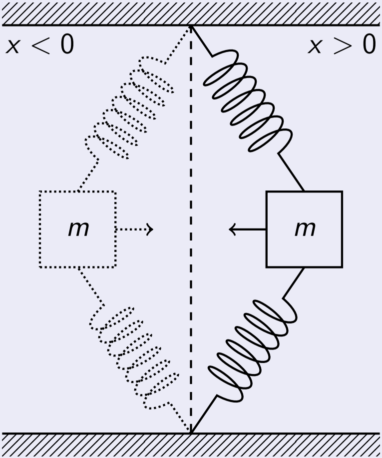
The motion of this spring is given by

$$\ddot{x} + \omega_0^2 x = 0$$

where ω_0 is the circular frequency at which the string vibrates. (In music, the frequency $f_0 = \frac{\omega_o}{2\pi}$ is often used. A middle C has 512 vibrations per second.)

The Guitar String: A Qualitative Analysis

The vibration of a guitar string can be described as a damped oscillator.



The motion of this spring is given by

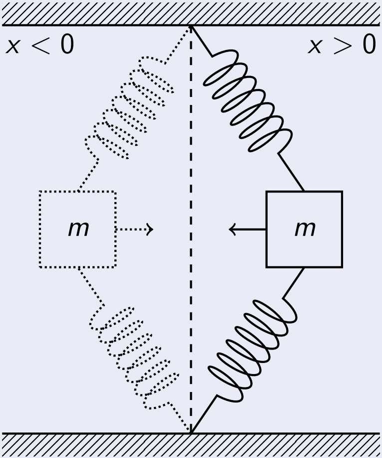
$$\ddot{x} + \omega_0^2 x = 0$$

where ω_0 is the circular frequency at which the string vibrates. (In music, the frequency $f_0 = \frac{\omega_o}{2\pi}$ is often used. A middle C has 512 vibrations per second.)

Because there is no damping in this model, the sound will last forever.

The Guitar String: A Qualitative Analysis

The vibration of a guitar string can be described as a damped oscillator.



The motion of this spring is given by

$$\ddot{x} + \omega_0^2 x = 0$$

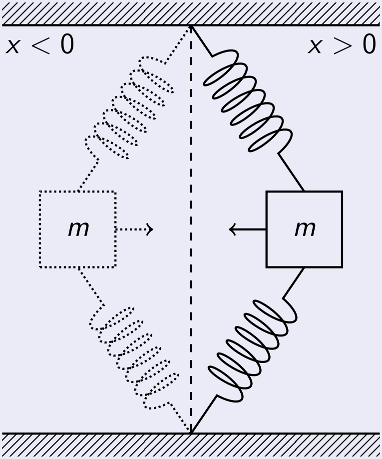
where ω_0 is the circular frequency at which the string vibrates. (In music, the frequency $f_0 = \frac{\omega_o}{2\pi}$ is often used. A middle C has 512 vibrations per second.)

Because there is no damping in this model, the sound will last forever.

When a guitar string is plucked, it has the $x(0) = x_0$ and $\dot{x}(0) = 0$ for initial conditions.

The Guitar String: A Qualitative Analysis

The vibration of a guitar string can be described as a damped oscillator.



The motion of this spring is given by

$$\ddot{x} + \omega_0^2 x = 0$$

where ω_0 is the circular frequency at which the string vibrates. (In music, the frequency $f_0 = \frac{\omega_o}{2\pi}$ is often used. A middle C has 512 vibrations per second.)

Because there is no damping in this model, the sound will last forever.

When a guitar string is plucked, it has the $x(0) = x_0$ and $\dot{x}(0) = 0$ for initial conditions.

Let use next consider a guitar string with damping.

Example 5

Consider the underdamped guitar string

$$\ddot{x} + 2\dot{x} + 26x = 0$$

Example 5

Consider the underdamped guitar string

$$\ddot{x} + 2\dot{x} + 26x = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 26$$

Example 5

Consider the underdamped guitar string

$$\ddot{x} + 2\dot{x} + 26x = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 26$$

which has solutions $r = -1 \pm 5i$. So $\alpha = -1$ and $\beta = 5$.

Example 5

Consider the underdamped guitar string

$$\ddot{x} + 2\dot{x} + 26x = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 26$$

which has solutions $r = -1 \pm 5i$. So $\alpha = -1$ and $\beta = 5$.

Thus, the general solution is

$$x(t) = e^{-t}\left(c_1 \cos(5t) + c_2 \sin(5t)\right)$$

Example 5

Consider the underdamped guitar string

$$\ddot{x} + 2\dot{x} + 26x = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 26$$

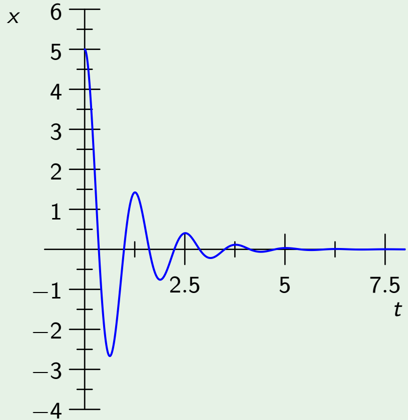
which has solutions $r = -1 \pm 5i$. So $\alpha = -1$ and $\beta = 5$.

Thus, the general solution is

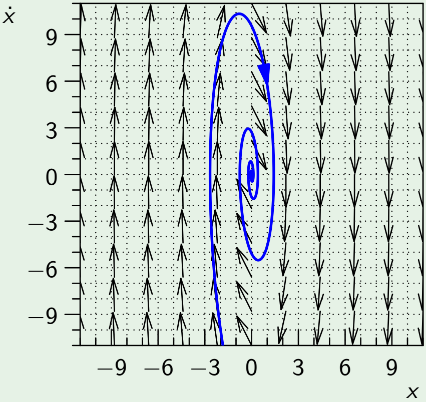
$$x(t) = e^{-t}\left(c_1 \cos(5t) + c_2 \sin(5t)\right)$$

If we pluck the string, which means $x(0) = 5$ and $\dot{x}(0) = 0$, we find that $c_1 = 5$ and $c_2 = 1$.

Example 5



(a) Time Series



(b) Phase Portrait

Solutions to the Second-Order Linear DE with Constant Coefficients

The differential equation

$$ay'' + by' + cy = 0$$

has the characteristic equation

$$ar^2 + br + c = 0$$

The quadratic formula gives rise to three different general solutions, depending on the discriminant $\Delta = b^2 - 4ac$.

	Characteristic Roots	General Solution
$\Delta > 0$	$r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$\Delta = 0$	$r = -\dfrac{b}{2a}$	$y = c_1 e^{rt} + c_2 t e^{rt}$
$\Delta < 0$	$r_1, r_2 = \alpha \pm \beta$	$y = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$
	$\alpha = -\dfrac{b}{2a}$, $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$	

Example 6

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} - 16y = 0$$

Example 6

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} - 16y = 0$$

It's characteristic equation is

$$0 = r^4 - 16$$

Example 6

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} - 16y = 0$$

It's characteristic equation is

$$0 = r^4 - 16 = (r^2 - 4)(r^2 + 4)$$

Example 6

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} - 16y = 0$$

It's characteristic equation is

$$0 = r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r + 2)(r - 2)(r^2 + 4)$$

Example 6

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} - 16y = 0$$

It's characteristic equation is

$$0 = r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r + 2)(r - 2)(r^2 + 4)$$

Which has the characteristic solutions

$$r_1 = 2, \quad r_2 = -2, \quad r_3 = 2i, \quad r_4 = -2i$$

Example 6

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} - 16y = 0$$

It's characteristic equation is

$$0 = r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r + 2)(r - 2)(r^2 + 4)$$

Which has the characteristic solutions

$$r_1 = 2, \quad r_2 = -2, \quad r_3 = 2i, \quad r_4 = -2i$$

Thus, $\{e^{2t}, e^{-2t}, \cos(2t), \sin(2t)\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos(2t) + c_4 \sin(2t)$$

Example 7

Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

Example 7

Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3$$

Example 7

Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3 = (r-1)(r^2 + 2r - 3)$$

Example 7

Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3 = (r-1)(r^2 + 2r - 3) = (r-1)^2(r+3)$$

Example 7

Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3 = (r-1)(r^2 + 2r - 3) = (r-1)^2(r+3)$$

Which has the characteristic solutions

$$r_1 = 1, \quad r_2 = 1, \quad r_3 = -3$$

Example 7

Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3 = (r-1)(r^2 + 2r - 3) = (r-1)^2(r+3)$$

Which has the characteristic solutions

$$r_1 = 1, \quad r_2 = 1, \quad r_3 = -3$$

Thus, $\{e^t, te^t, e^{-3t}\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = c_1 e^t + c_2 te^t + c_3 e^{-3t}$$

Example 8

Consider the fifth-order DE

$$\frac{d^5y}{dt^5} + 3\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = 0$$

Example 8

Consider the fifth-order DE

$$\frac{d^5y}{dt^5} + 3\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = 0$$

It's characteristic equation is

$$0 = r^5 + 3r^4 + 3r^3 + r^2$$

Example 8

Consider the fifth-order DE

$$\frac{d^5y}{dt^5} + 3\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = 0$$

It's characteristic equation is

$$0 = r^5 + 3r^4 + 3r^3 + r^2 = (r+1)^3r^2$$

Example 8

Consider the fifth-order DE

$$\frac{d^5y}{dt^5} + 3\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = 0$$

It's characteristic equation is

$$0 = r^5 + 3r^4 + 3r^3 + r^2 = (r+1)^3 r^2$$

Which has the characteristic solutions

$$r_1 = -1, \quad r_2 = -1, \quad r_3 = -1, \quad r_4 = 0, \quad r_5 = 0$$

Example 8

Consider the fifth-order DE

$$\frac{d^5y}{dt^5} + 3\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = 0$$

It's characteristic equation is

$$0 = r^5 + 3r^4 + 3r^3 + r^2 = (r+1)^3 r^2$$

Which has the characteristic solutions

$$r_1 = -1, \quad r_2 = -1, \quad r_3 = -1, \quad r_4 = 0, \quad r_5 = 0$$

Thus, $\{e^{-t}, te^{-t}, t^2e^{-t}, 1, t\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = \underbrace{(c_1 + c_2 t + c_3 t^2)e^{-t}}_{\text{for triple root}} + \underbrace{(c_4 + c_5 t)}_{\text{for double root}}$$

Example 9

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} + 8\frac{d^2y}{dt^2} + 16y = 0$$

Example 9

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} + 8\frac{d^2y}{dt^2} + 16y = 0$$

It's characteristic equation is

$$0 = r^4 + 8r^2 + 16$$

Example 9

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} + 8\frac{d^2y}{dt^2} + 16y = 0$$

It's characteristic equation is

$$0 = r^4 + 8r^2 + 16 = \left(r^2 + 4\right)^2$$

Example 9

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} + 8\frac{d^2y}{dt^2} + 16y = 0$$

It's characteristic equation is

$$0 = r^4 + 8r^2 + 16 = \left(r^2 + 4\right)^2$$

Which has the characteristic solutions

$$r_1 = 2i, \quad r_2 = 2i, \quad r_3 = -2i, \quad r_4 = -2i$$

Example 9

Consider the fourth-order DE

$$\frac{d^4y}{dt^4} + 8\frac{d^2y}{dt^2} + 16y = 0$$

It's characteristic equation is

$$0 = r^4 + 8r^2 + 16 = \left(r^2 + 4\right)^2$$

Which has the characteristic solutions

$$r_1 = 2i, \quad r_2 = 2i, \quad r_3 = -2i, \quad r_4 = -2i$$

Thus, $\{\cos(2t), t\cos(2t), \sin(2t), t\sin(2t)\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = \underbrace{(c_1 + c_2 t)}_{\text{for double root}}\cos(2t) + \underbrace{(c_3 + c_4 t)}_{\text{for double root}}\sin(2t)$$