

# Defining Probability

Colby Community College

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If the dice is fair, each side has the same chance of being rolled. So a 1 has a one-in-six chance, equivalently  $\frac{1}{6}$ .

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There are two outcomes, a 1 or a 2, and six faces on a die.

$$P(\text{roll 1 or 2}) = \frac{2}{6} = \frac{1}{3}$$

## Note

A standard deck of 52 playing cards consists of four **suits** in two colors: Hearts ♥, Spades ♠, Diamonds ♦, and Clubs ♣

Each suit contains 13 cards, each of a different **rank**:  
2 through 10, Jack, Queen, King, and Ace

The Jack, Queen, and King cards are called **face cards**.

The Jack, Queen, King, and Ace cards are called **honour cards**.

The cards numbered 2 to 10 are called **numerals**.

♣2	♣3	♣4	♣5	♣6	♣7	♣8	♣9	♣10	♣J	♣Q	♣K	♣A
♦2	♦3	♦4	♦5	♦6	♦7	♦8	♦9	♦10	♦J	♦Q	♦K	♦A
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An outcome with a probability of 1 is called **certain**.

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Probabilities are always between 0 and 1.



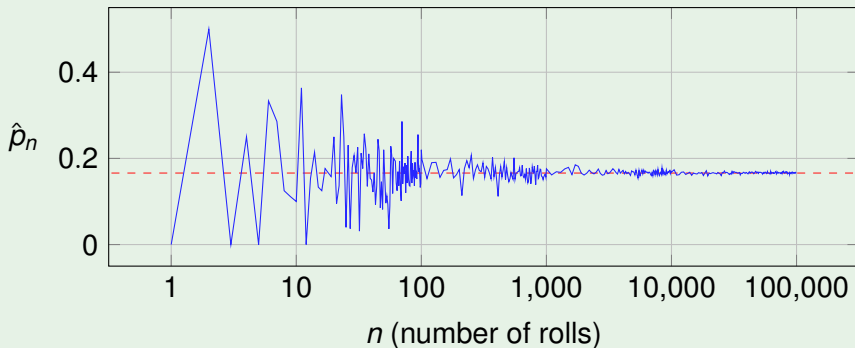
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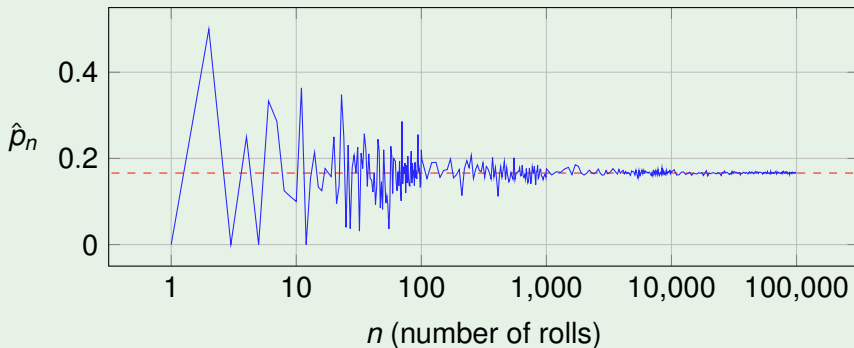
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## Note

It is not a coincidence that  $\hat{p}_n$  get closer to  $p$  as  $n$  increases.

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- If we know nothing about the likelihood of different possible outcomes, we should not assume that they are equally likely.
  - You should not think that the probability of passing the next exam is  $\frac{1}{2}$ , or 0.5. The actual probability depends on factors such as the amount of preparation and the difficulty of the exam.



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*Are the outcomes “draw an ace” and “draw a diamond” disjoint?*

No, the ♦A is both an ace and a diamond.

## Addition Rule of Disjoint Outcomes

If  $A_1$  and  $A_2$  represent two disjoint outcomes, then the probability that one of them occurs is given by:

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$



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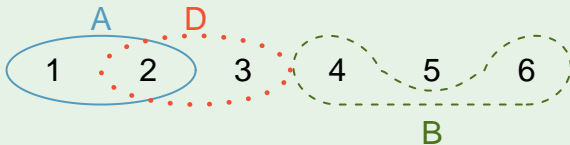
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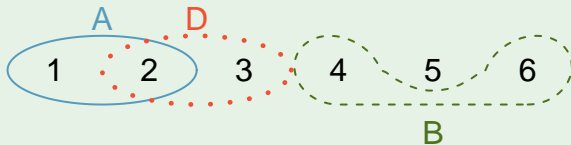
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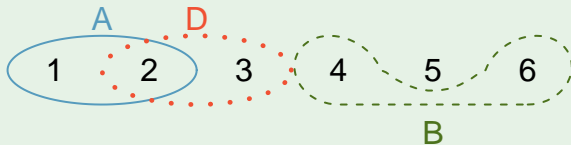
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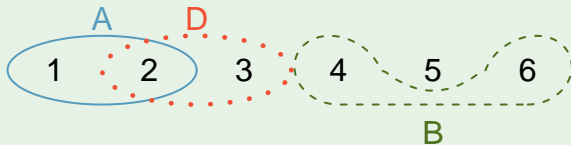
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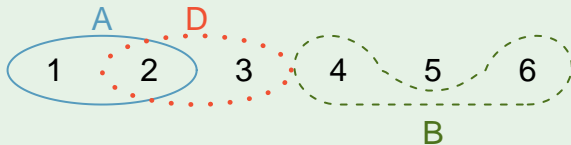
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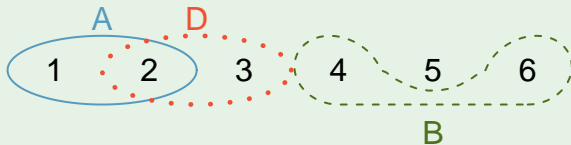
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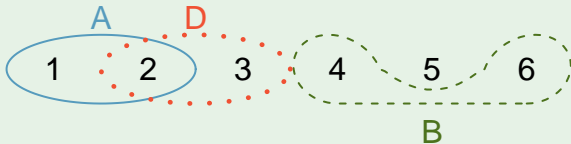
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## Example 14

Suppose we flip a fair coin and roll a fair die.

The list of all possible outcomes is:

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Meaning that  $P(\text{H or 6}) = \frac{7}{12}$ .



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The correct probability is:

$$P(\text{H or } 6) = P(\text{H}) + P(6) - P(\text{H and } 6)$$

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Meaning that  $P(\text{H or 6}) = \frac{7}{12}$ .

Notice that  $\frac{6}{12} = \frac{1}{2}$  of the outcomes have heads and  $\frac{2}{12} = \frac{1}{6}$  have a six.

But,  $\frac{1}{2} + \frac{1}{6} = \frac{8}{12}$ , is wrong because we have double counted H6.

The correct probability is:

$$P(\text{H or 6}) = P(\text{H}) + P(6) - P(\text{H and 6}) = \frac{1}{2} + \frac{1}{6} - \frac{1}{12}$$

## Example 14

Suppose we flip a fair coin and roll a fair die.

The list of all possible outcomes is:

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

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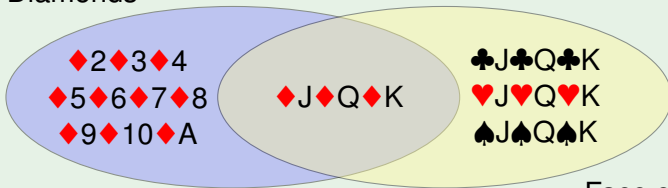
## Example 15

Let us consider the events “draw a diamond” and “draw a face card”.

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Diamonds

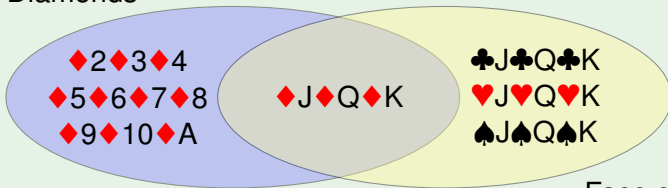


Face cards

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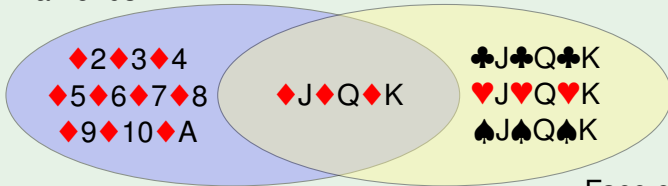
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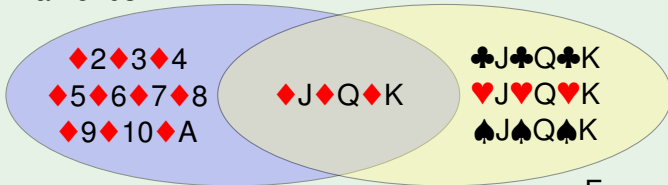
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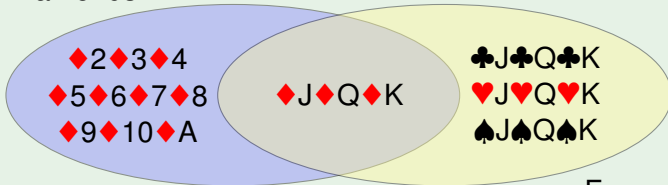
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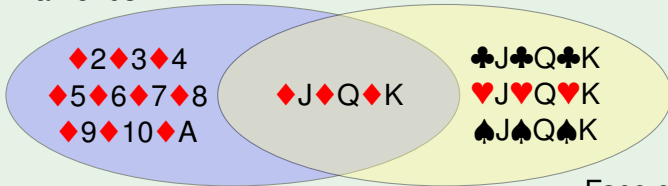
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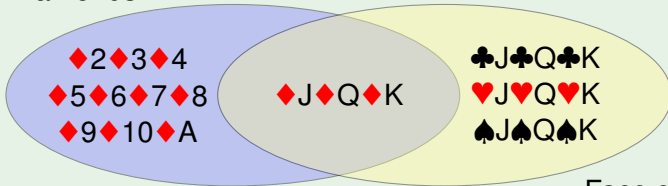
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$$\begin{aligned} P(\diamond \text{ and face}) &= P(\diamond) + P(\text{face}) - P(\diamond \text{ and face}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \end{aligned}$$

## Example 15

Let us consider the events “draw a diamond” and “draw a face card”. These outcomes are not disjoint, since three cards are both:

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Face cards

The Addition Rule for Disjoint Outcomes would count  $\diamondsuit J, \diamondsuit Q, \diamondsuit K$  twice!

$$\begin{aligned}P(\diamondsuit \text{ and face}) &= P(\diamondsuit) + P(\text{face}) - P(\diamondsuit \text{ and face}) \\&= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\&= \frac{22}{52} = \frac{11}{26}\end{aligned}$$

## General Addition Rule

If  $A$  and  $B$  are any two events, disjoint or not, then the probability that at least one of them will occur is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

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### Note

In statistics, when we write “or” what we mean is “and/or”, unless we explicitly say otherwise.

In other words, “ $A$  or  $B$ ” occurring means  $A$ ,  $B$ , or both  $A$  and  $B$  occur.

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### Note

If  $A$  and  $B$  are disjoint this means  $P(A \text{ and } B) = 0$ , and so we get the Addition Rule for Disjoint Outcomes:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= P(A) + P(B) - 0 \\ &= P(A) + P(B) \end{aligned}$$



## Example 16

Suppose we draw one card from a standard deck. Let us find the probability that we get a Queen or a King.

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There are 4 Queens and 4 Kings in the deck, hence eight outcomes out of 52 possible outcomes. So, the probability is

$$P(Q \text{ or } K) = \frac{8}{52}$$

Since there are no cards that are both Kings and Queens, we have

$$P(Q \text{ or } K) = P(Q) + P(K) - P(Q \text{ and } K) = \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}$$

## Definition

A **probability distribution** is a table of all disjoint outcomes and their associated probabilities.

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## Example 17

The probability distribution for the sum of two dies:

Dice sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

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## Rules for Probability Distributions

All probability distributions must satisfy the following rules:

- 1 The outcomes listed must be disjoint.
- 2 Each probability must be between 0 and 1.
- 3 The probabilities must sum to 1.

## Example 18

The follow table contain three possible distributions for household income in the United States.

Income Range	\$0-\$25k	\$25k-\$50k	\$50k-\$100k	\$100k+
(a)	0.18	0.39	0.33	0.16
(b)	0.38	-0.27	0.52	0.37
(c)	0.28	0.27	0.29	0.16

*Only one of the three is actually a probability distribution. Which one?*

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*Only one of the three is actually a probability distribution. Which one?*

- (a) sums to 2.5.
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- (c) is the real probability distribution.

## Note

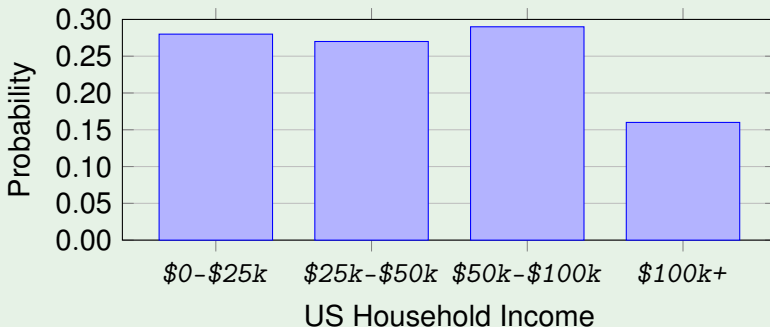
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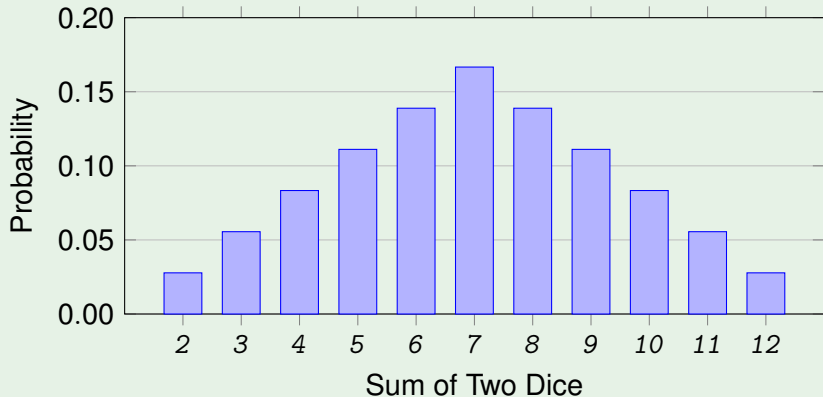
## Example 19

Here is the bar plot for the probability distribution in Example 18:



## Example 20

Here is the bar plot for the dice sum distribution in Example 17



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The set of all possible outcomes is called the **sample space**,  $S$ .

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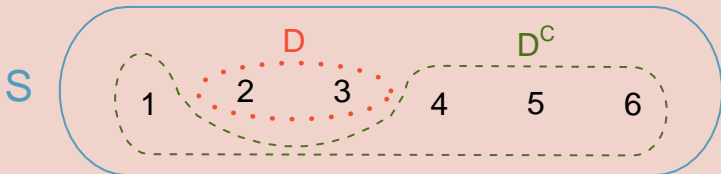
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The sample space of rolling a single die is:

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## Definition

The **complement** of an event  $D$  is all the outcomes in the sample space that are not in  $D$ . Denoted  $D^C$ .





## Note

$A$  and  $A^C$  are disjoint events.

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Since every outcome in the sample space is either in  $A$  or  $A^C$ :

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### Note

Finally,  $P(A^C) = 1 - P(A)$  and  $P(A) = 1 - P(A^C)$ .

## Example 22

Consider rolling two dice and summing the numbers.

*What is the complement of the event “the total is less than 12?”*

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## Note

If two events are not independent, they are called **dependent**.

## Multiplication Rule For Independent Events

If events  $A$  and  $B$  are independent, then the probability of both  $A$  and  $B$  occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

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The sample space is  $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$ .

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Since a coin and dice are independent, we could have used the Multiplication Rule:

$$P(T \text{ and } 3) = P(T) \cdot P(3) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

## Example 26

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The Airbus 310 airliner has three independent hydraulic systems, so if one system fails, full flight control is maintained. Let us assume the probability of a hydraulic system failing is 0.002.

- *If the Airbus 310 had only a single hydraulic system, what is the probability that the flight control system would work for an entire flight?*

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## Note

A good rule of thumb is that the sample size is very small if it is less than 5% of the total population size.

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$$\begin{aligned} P(\text{all use drugs}) &= P(1^{\text{st}} \text{ uses drugs and } 2^{\text{nd}} \text{ uses drugs and } 3^{\text{rd}} \text{ uses drugs}) \\ &= \left( \frac{33,337,369}{333,373,690} \right) \cdot \left( \frac{33,337,368}{333,373,689} \right) \cdot \left( \frac{33,337,367}{333,373,688} \right) \\ &= 0.000999999919 \end{aligned}$$

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Since 3 adults is a very small part of the total population, we can simplify the calculations considerably:

$$\begin{aligned} P(\text{all use drugs}) &= P(1^{\text{st}} \text{ uses drugs and } 2^{\text{nd}} \text{ uses drugs and } 3^{\text{rd}} \text{ uses drugs}) \\ &= 0.1 \cdot 0.1 \cdot 0.1 = 0.00100 \end{aligned}$$

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Since the equation holds, the two events must be independent.