

Properties of Linear Transformations

Department of Mathematics

Salt Lake Community College

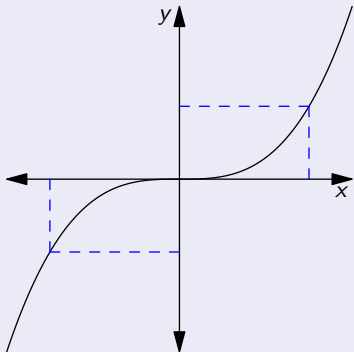
(Slides by Adam Wilson)

Injectivity

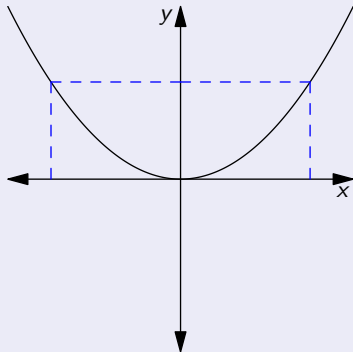
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(a) $f(x) = x^3$ is injective



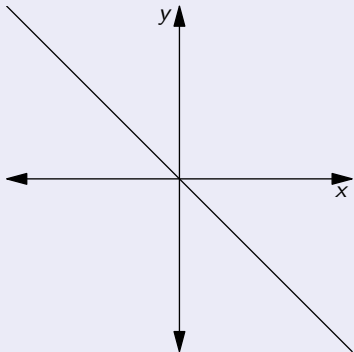
(b) $g(x) = x^2$ is not injective

Surjectivity

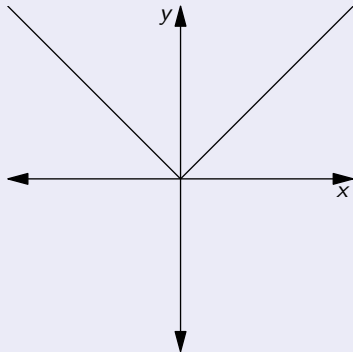
The set of output values of a function $f : \mathbb{X} \rightarrow \mathbb{Y}$ is a subset of the codomain \mathbb{Y} and is called the **image** of the function. If the image is all of \mathbb{Y} , the function f is said to map **onto** \mathbb{Y} , or to be **surjective**.

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(a) $f(x) = -x$ is surjective



(b) $g(x) = |x|$ is not surjective

Example 1

The linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(\vec{v}) = \mathbf{A}\vec{v}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$$

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So, **Col A**, the column space of \mathbf{A} , is the image of T .

We can derive the equation of this plane by looking at the system

$$u_1 = v_1 + v_2, \quad u_2 = v_1 - v_2, \quad u_3 = 2v_1 + v_2$$

Eliminating v_1 and v_2 gives us

$$3u_1 + u_2 - 2u_3 = 0$$

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Rank of a Linear Transformation

The dimension of the image of a linear transformation T is called its **rank**

$$\mathbf{rank}(T) = \mathbf{dim}(\mathbf{Im}(T))$$

Example 2

For $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ defined by

$$T(\vec{v}) = \mathbf{A}\vec{v} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \vec{w}$$

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We can write

$$\vec{w} = v_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + v_2 \begin{bmatrix} -4 \\ 2 \end{bmatrix} + v_3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + v_4 \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

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Which means

$$\text{Im}(T) = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, $\text{rank}(T) = \text{dim}(\text{Im}(T)) = \text{dim}(\text{Col } \mathbf{A}) = 2$.

Rank of a Matrix Multiplication Operator

For any linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(\vec{x}) = \mathbf{A}\vec{x}$, where $\mathbf{A} \in \mathbb{M}_{mn}$ and $\vec{x} \in \mathbb{V}$, the image of T is the column space of A . (That's is, $\mathbf{Im}(T) = \mathbf{Col} \mathbf{A}$.)

The pivot columns of A form a basis for $\mathbf{Im}(T)$.

Consequently,

$$\begin{aligned}\mathbf{rank}(T) &= \mathbf{dim}(\mathbf{Im}(T)) \\ &= \mathbf{dim}(\mathbf{Col} \mathbf{A}) \\ &= \text{The number of pivot columns in } \mathbf{A}.\end{aligned}$$

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Reminder

The basis of $\mathbf{Col} \mathbf{A}$ must come from \mathbf{A} , *not* from the RREF of \mathbf{A} .

Recall

A linear transformation $T : \mathbb{V} \rightarrow \mathbb{W}$ must map the zero vector of \mathbb{V} to the zero vector of \mathbb{W} .

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Kernel of a Linear Transformation

The **kernel** (or **nullspace**) of a linear transformation $T : \mathbb{V} \rightarrow \mathbb{W}$, denoted $\ker(T)$, is the set of vectors in \mathbb{V} that are mapped by T to the zero vector of \mathbb{W} .

$$\ker(T) = \left\{ \vec{v} \in \mathbb{V} \mid T(\vec{v}) = \vec{0} \right\}$$

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Note

The kernel always contains at least one element.

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$$\mathbf{ker}(T) = \{(0, 0, z) \mid z \in \mathbb{R}\}$$

Example 4

Consider the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(\vec{v}) = \mathbf{A}\vec{v} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 + 2v_3 \\ 2v_1 + 3v_2 + 5v_3 \end{bmatrix}$$

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To find the vectors that are mapped to $\vec{0}$, we have to solve the system:

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So, we see that v_3 is a free variable and if $v_3 = s$ is a parameter, we have $v_1 = -s$, $v_2 = -s$, and $v_3 = s$.

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Thus, the kernel is any scalar multiple of $\langle -1, -1, 1 \rangle$:

$$\ker(T) = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

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Thus,

$$\ker(T) = \{\vec{0}\}$$

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In this case we have $v_1 = -\frac{1}{2}v_2$ and so, if we let our parameter be $v_2 = s$ we have

$$\ker(T) = \left\{ s \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

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Consider the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by the matrix

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Note

These examples seem to suggest that the kernel of the linear transformation $T : \mathbb{V} \rightarrow \mathbb{W}$ is a subspace of \mathbb{W} .

Kernel Theorem

Let $T : \mathbb{V} \rightarrow \mathbb{W}$ be a linear transformation from vector space \mathbb{V} to vector space \mathbb{W} with kernel $\mathbf{ker}(T)$.

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Thus, $\vec{u} - \vec{v}$ is in the kernel, which means $\vec{u} - \vec{v} = \vec{0}$ and thus T is injective.

Example 8

Consider the transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ defined by $T(\vec{v}) = \mathbf{A}\vec{v}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix}$$

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So, we see that $v_1 = 2v_2 - 3v_4$ and $v_3 = 0$. If we let $v_2 = r$ and $v_4 = s$,

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(Remember that the dimension of the image of T was 2.)

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Let $T : \mathbb{V} \rightarrow \mathbb{W}$ be a linear transformation from a finite vector space \mathbb{V} .

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$$\mathbf{dim}(\ker(T)) + \mathbf{dim}(\mathbf{Im}(T)) = \mathbf{dim}(\mathbb{V})$$

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- $D_2 : \mathbb{P}_3 \rightarrow \mathbb{P}_1$ had

$$\mathbf{\ker(D_2) = \{cx + d\} \quad \text{and} \quad \text{Im}(D_2) = \{6ax + 2b\}}$$

$$\mathbf{\dim(\ker(D_2)) + \dim(\text{Im}(D_2)) = 2 + 2 = 4 = \dim(\mathbb{P}_3)}$$