Real Characteristic Roots

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Salt Lake Community College

Constant Coefficient Second-Order DE

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Because the range of e^{rt} is $(0, \infty)$ this will be satisfied only when

$$ar^2 + br + c = 0$$

We call this the **characteristic equation** of the DE and is key to finding the solutions that form a basis of the solution space.

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- $\Delta < 0$: Two conjugate complex roots. (Section 4.3.)

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These roots are called **characaristic roots** or **eigenvalues**.
(The term *eigenvalue* is from Linear Algebra and we will talk more about them later in the semester.)

Real Unequal Characteristic Roots ($\Delta > 0$)

Solution for Distinct Real Characteristic Roots

For $\Delta > 0$, the characteristic roots of the DE

$$ay'' + by' + cy = 0$$

are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

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The functions $e^{r_1 t}$ and $e^{r_2 t}$ are linearly independent solutions, and the general solution is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where c_1 and c_2 are arbitrary constants determined by the initial conditions.

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The set $\{e^{r_1 t}, e^{r_2 t}\}$ forms a basis for the solution space $\mathbb{S}$.

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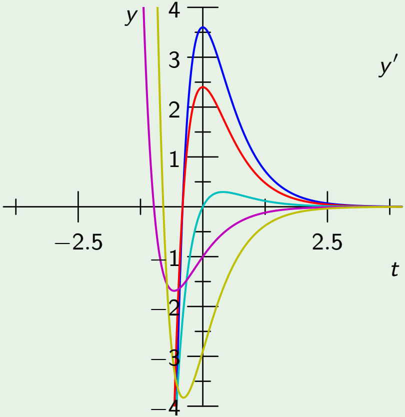
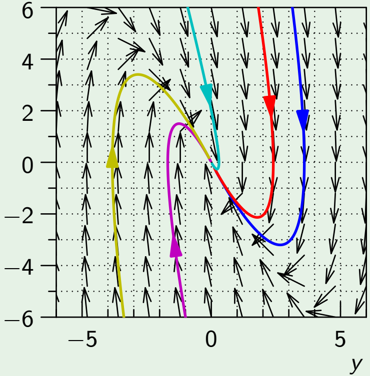


(a) Time Series



(b) Phase Portrait

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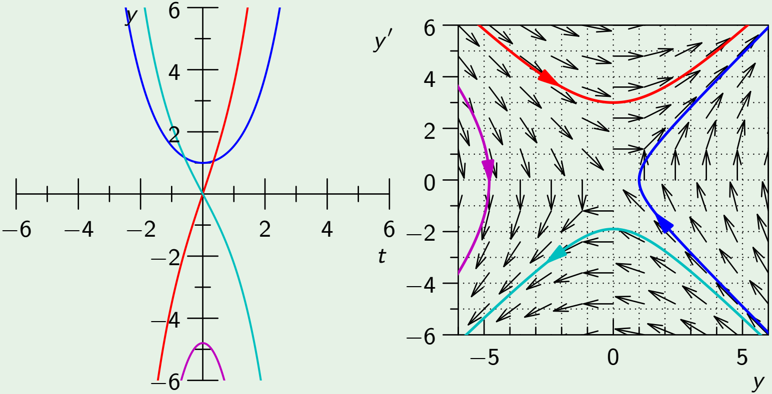

(a) Time Series (b) Phase Portrait

Real Equal Characteristic Roots ($\Delta = 0$)

Solution for Equal Real Characteristic Roots

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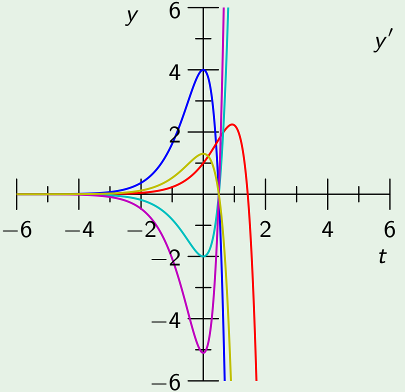

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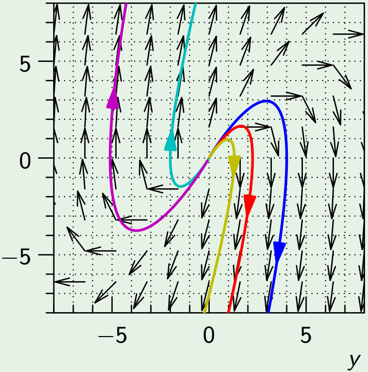

(b) Phase Portrait

Damped Mass-Spring Systems

Overdamped Mass-Spring System

The motion of a mass-spring system is called **overdamped** when we have $\Delta > 0$. Both characteristic roots are negative and the solutions

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

tend towards zero with oscillation, crossing the t-axis at most once.

Damped Mass-Spring Systems

Example

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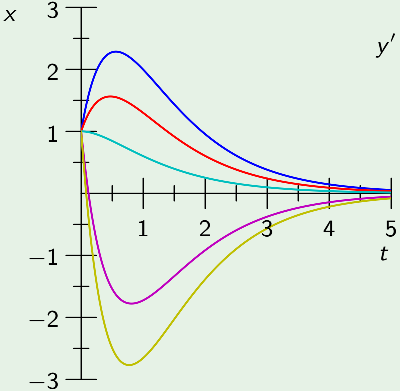

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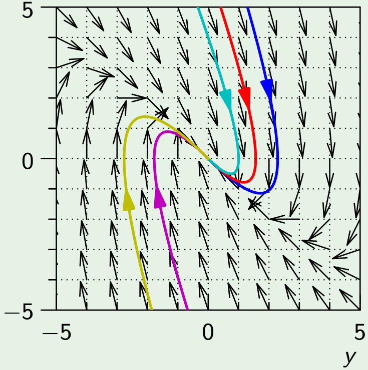

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Damped Mass-Spring Systems

Critically Damped Mass-Spring System

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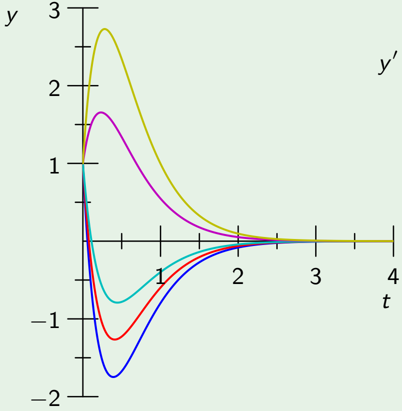

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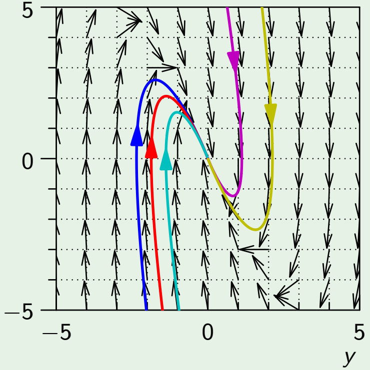

(b) Phase Portrait

Theoretical Considerations

Existence and Uniqueness Theorem (Second-Order)

Let $p(t)$ and $q(t)$ be continuous on the open interval (a, b) contaning t_0. For *any* $A, B \in \mathbb{R}$, there exists a unique solution $y(t)$ defined on (a, b) to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = A, \quad y'(t_0) = B$$

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The solution space $\mathbb{S}$ for a second-order homogeneous differential equation has dimension 2.

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Proof

See Page 217 in your textbook

Theoretical Considerations

Solutions of Homogeneous Linear DE (Second-Order)

For any linear second-order homogeneous DE on (a, b),

$$y'' + p(t)y' + q(t)y = 0$$

for which p and q are continuous on (a, b), *any* two linearly independent solutions $\{y_1, y_2\}$ form a basis of the solution space $\mathbb{S}$, and *every* solution y on (a, b) can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some $c_1, c_2 \in \mathbb{R}$.

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Existence and Uniqueness Theorem (nth-Order)

Let $p_1(t), p_2(t), \ldots, p_n(t)$ be continuous on the open interval (a, b) contaning t_0. For any initial conditions $A_0, A_1, \ldots, A_{n-1} \in \mathbb{R}$, there exists a unique solution $y(t)$ defined on (a, b) to the IVP

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-3)} + \cdots + p_n(t)y = 0$$

where

$$y(t_0) = A_0, \quad y'(t_0) = A_1, \ldots, \quad y^{(n-1)}(t_0) = A_{n-1}$$

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Solution Space Theorem (*n*th-Order)

The solution space $\mathbb{S}$ for a *n*th-order linear homogeneous differential equation has dimension n.

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Solutions of Homogeneous Linear DE (*n*th-Order)

For any linear *n*th-order homogeneous DE on (a, b),

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

for which $p_1(t), p_2(t), \ldots, p_n(t)$ are continuous on (a, b), *any* n linearly independent solutions $\{y_1, y_2, \ldots, y_2\}$ form a basis of the solution space $\mathbb{S}$, and *every* solution y on (a, b) can be written as

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A Wronskian conveys more information in the test for linear independence when the functions are solutions to the same nth-order linear homogeneous DE.

The Wronskian Test for Linear Independence of DE Solutions

Suppose $\{y_1, y_2, \ldots, y_n\}$ is a set of solutions on (a, b) of a nth-order linear homogeneous DE,

$$L(y) = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1(t)\frac{d^1 y}{dt^1} + a_0 y = 0$$

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Proof

See page 220 in your textbook

Example

Consider the set of solutions $A = \{2, t-1, t^2, t^3 + t\}$ to $\dfrac{d^4y}{dy^4} = 0$ on $\mathbb{R}$.

Theoretical Considerations

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$$W = \begin{vmatrix} 2 & t-1 & t^2 & t^3+t \\ 0 & 1 & 2t & 3t^2+1 \\ 0 & 0 & 2 & 6t \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

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&= 2\begin{vmatrix} 2 & 6t \\ 0 & 6 \end{vmatrix} \\
&= 24
\end{aligned}$$

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A is linearly independent and hence a basis of $\mathbb{S}$.

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Example

Consider the set of solutions $B = \{t, t+1, t^2-1, t^2\}$ to $\dfrac{d^4y}{dy^4} = 0$ on $\mathbb{R}$.

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B is linearly dependent.

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B is linearly dependent. (For example, $t = (t+1) + (t^2-1) - (t^2)$.)

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&= 6t^2
\end{aligned}
$$

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W is not identically zero, so we know C is a linearly independent set.

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W is not identically zero, so we know C is a linearly independent set. But the strong conclusion of the Wronskian test did not occur here because C contains only three solutions for a fourth-order DE.