Linear Transformations

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Linear Transformation

A **Linear Transformation** T on a vector space $\mathbb{V}$ to a vector space $\mathbb{W}$ is a function $T : \mathbb{V} \to \mathbb{W}$ that preserves *scalar multiplication* and *vector addition*. That is, for all $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{V}$ and $c \in \mathbb{R}$:

- $T(c\vec{\boldsymbol{u}}) = cT(\vec{\boldsymbol{u}})$
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Image of a Linear Transformation

The **image** of a linear transformation $T : \mathbb{V} \to \mathbb{W}$ is the set of vectors in $\mathbb{W}$ to which T maps the vectors in $\mathbb{V}$:

$$\mathbf{Im}(T) = \{\vec{\boldsymbol{w}} \in \mathbb{W} \mid \vec{\boldsymbol{w}} = T(\vec{\boldsymbol{v}}) \text{ for some } \vec{\boldsymbol{v}} \in \mathbb{V}\}$$

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Note

Linear transformations may map nonzero vectors from the domain into the zero vector of the codomain.

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Therefore, T is a linear transformation.

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Differentiation is a linear transformation. The **derivative operator** $D : \mathcal{C}^1[a,b] \to \mathcal{C}[a,b]$ is defined by

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Example 6

Similarly, we can confirm that the **integration operator** $I : \mathcal{C}[a,b] \to \mathbb{R}$, defined by

$$I(f) = \int_a^b f(t)dt$$

is a linear transformation.

Geometry of Matrix Linear Transformations

If $\boldsymbol{A}$ is an $m \times n$ matrix and $\vec{\boldsymbol{x}}$ is a column n-vector, then $\boldsymbol{A}\vec{\boldsymbol{x}}$ can be considered a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, where $T(\vec{\boldsymbol{x}}) = \boldsymbol{A}\vec{\boldsymbol{x}}$.

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$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

defines a **shear** of 1-unit in the x-direction.

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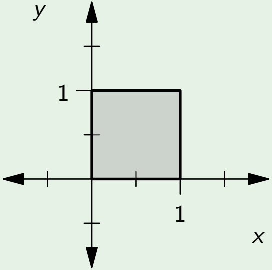

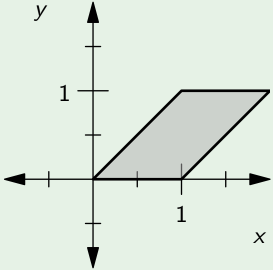

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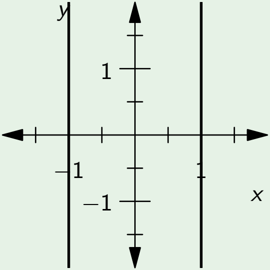

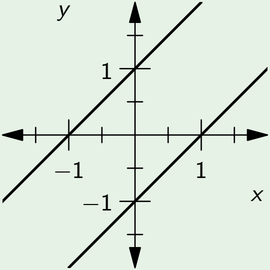

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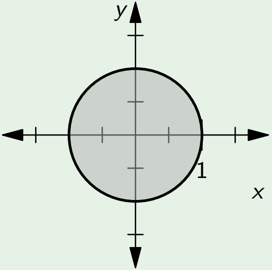

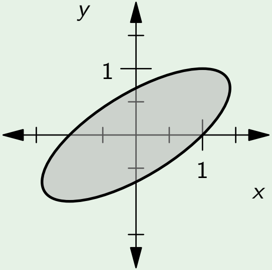

Example 8

The *Counterclockwise* rotation about the origin by angle θ is given by:

$$\boldsymbol{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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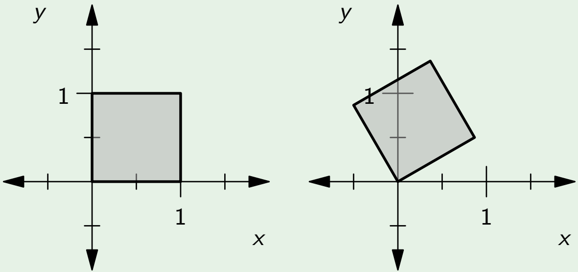

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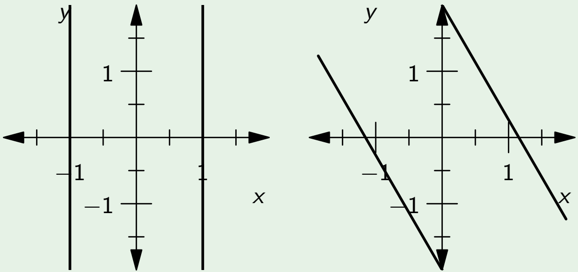

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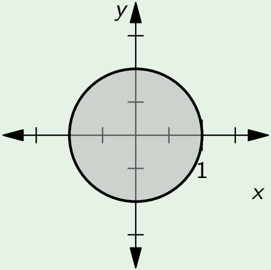
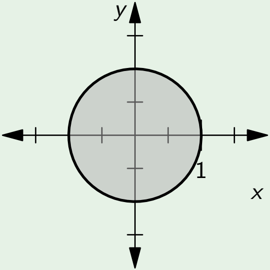
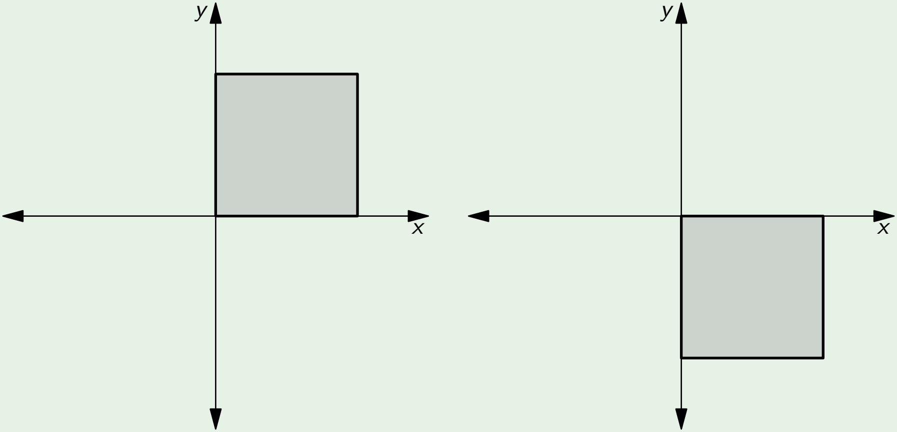




Example 9

Reflection about the x-axis

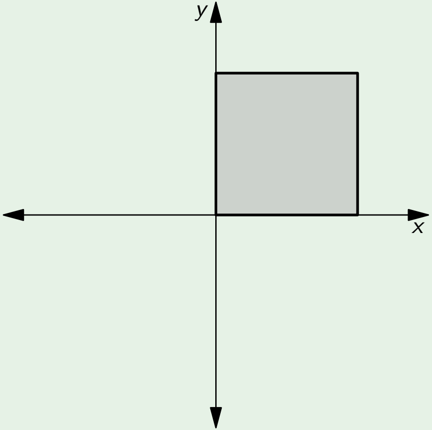
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

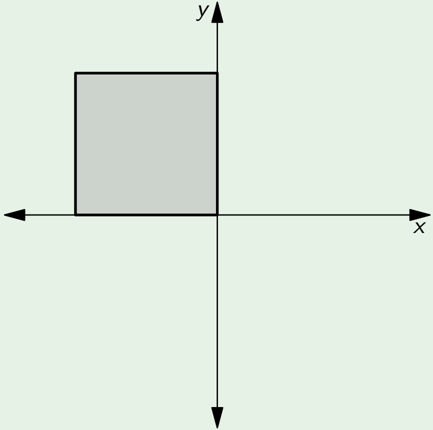


Example 9

Reflection about the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

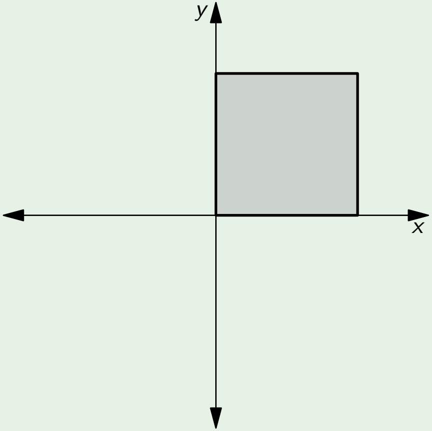


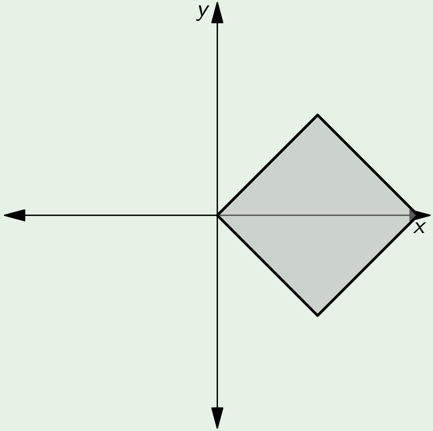


Example 9

Rotation clockwise about the origin of $\frac{\pi}{4}$

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ -\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}$$





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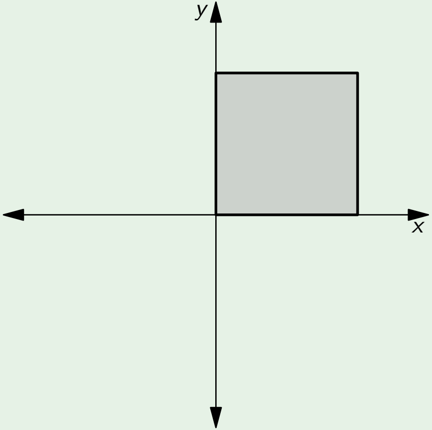
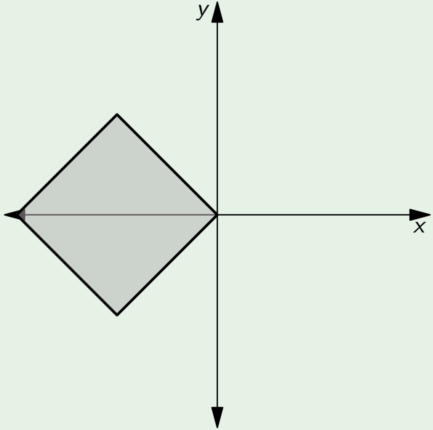
Rotation counterclockwise about the origin of $\frac{3\pi}{4}$

$$\begin{bmatrix} \cos\left(\frac{3\pi}{4}\right) & -\sin\left(\frac{3\pi}{4}\right) \\ \sin\left(\frac{3\pi}{4}\right) & \cos\left(\frac{3\pi}{4}\right) \end{bmatrix}$$

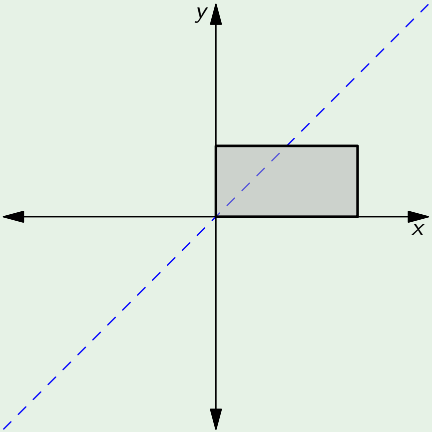


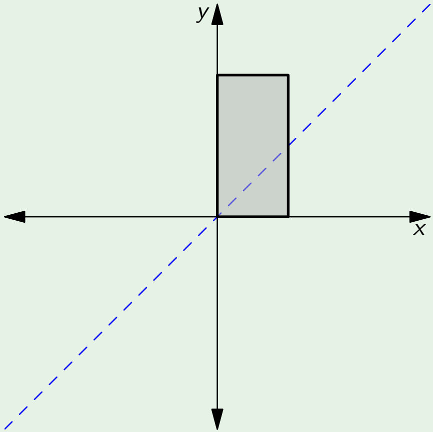


Example 9

Reflection about the line $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

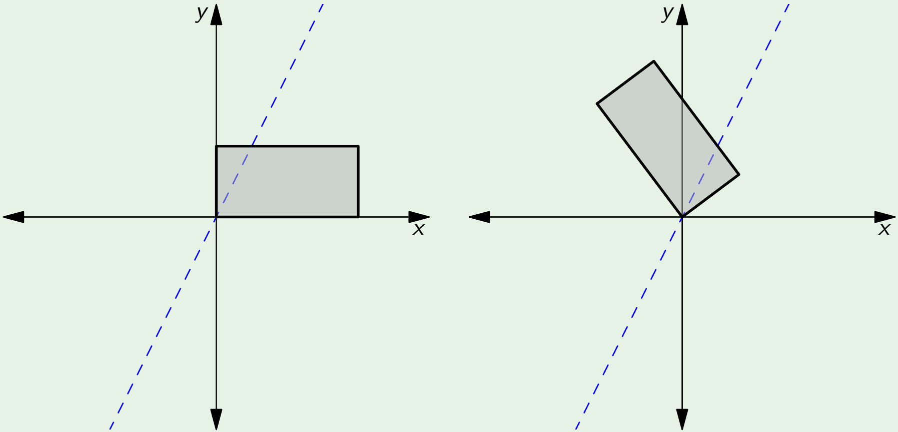




Example 9

Reflection about the line $y = 2x$

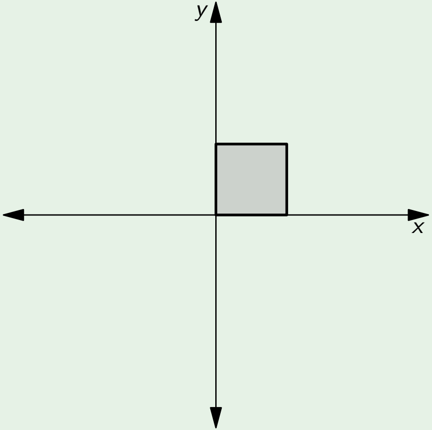
$$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

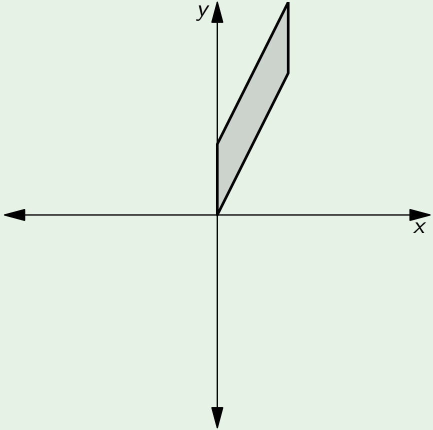


Example 9

Shear of 2 in the y-direction

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$





Example 10

Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \vec{\boldsymbol{v}}$$

maps

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 + 2v_3 \\ 2v_1 + 3v_2 + 5v_3 \end{bmatrix}$$

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What is the image of T?

A typical vector in the range is

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It can be easily verified that $[1, 2]$ and $[1, 3]$ are independent in $\mathbb{R}^2$. Which means the image must contain their span, which is exactly $\mathbb{R}^2$.

The Standard Matrix for a Linear Transform

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. The **standard matrix** associated with T is defined by

$$\boldsymbol{A} = [\,T(\vec{\boldsymbol{e_1}})|\,T(\vec{\boldsymbol{e_2}})|\cdots|\,T(\vec{\boldsymbol{e_n}})]$$

where the columns $T(\vec{\boldsymbol{e_j}})$ are the images under T of the standard basis vectors $\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}, \ldots, \vec{\boldsymbol{e_n}}$.

Proof

We can check that this matrix satisfies $T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}}$ by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) =$$

Proof

We can check that this matrix satisfies $T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}}$ by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = T\big(v_1\vec{\boldsymbol{e_1}} + v_2\vec{\boldsymbol{e_2}} + \cdots + v_n\vec{\boldsymbol{e_n}}\big)$$

Proof

We can check that this matrix satisfies $T(\vec{v}) = \boldsymbol{A}\vec{v}$ by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = T\left(v_1\vec{\boldsymbol{e_1}} + v_2\vec{\boldsymbol{e_2}} + \cdots + v_n\vec{\boldsymbol{e_n}}\right)$$

$$= v_1 T(\vec{\boldsymbol{e_1}}) + v_2 T(\vec{\boldsymbol{e_2}}) + \cdots + v_n T(\vec{\boldsymbol{e_n}})$$

Proof

We can check that this matrix satisfies $T(\vec{v}) = \boldsymbol{A}\vec{v}$ by

$$\begin{aligned}
T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) &= T(v_1\vec{\boldsymbol{e_1}} + v_2\vec{\boldsymbol{e_2}} + \cdots + v_n\vec{\boldsymbol{e_n}}) \\
&= v_1 T(\vec{\boldsymbol{e_1}}) + v_2 T(\vec{\boldsymbol{e_2}}) + \cdots + v_n T(\vec{\boldsymbol{e_n}}) \\
&= [T(\vec{\boldsymbol{e_1}}) | T(\vec{\boldsymbol{e_2}}) | \cdots | T(\vec{\boldsymbol{e_n}})] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
\end{aligned}$$

Proof

We can check that this matrix satisfies $T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}}$ by

$$
\begin{aligned}
T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) &= T\left(v_1\vec{\boldsymbol{e_1}} + v_2\vec{\boldsymbol{e_2}} + \cdots + v_n\vec{\boldsymbol{e_n}}\right) \\
&= v_1 T(\vec{\boldsymbol{e_1}}) + v_2 T(\vec{\boldsymbol{e_2}}) + \cdots + v_n T(\vec{\boldsymbol{e_n}}) \\
&= [T(\vec{\boldsymbol{e_1}}) | T(\vec{\boldsymbol{e_2}}) | \cdots | T(\vec{\boldsymbol{e_n}})] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= \boldsymbol{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
\end{aligned}
$$

Example 11

Find the standard matrix that will describe the transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \\ 2x \end{bmatrix}$$

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$$\boldsymbol{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \\ 2x \end{bmatrix}$$

Thus, for dimensions in the product to match, $\boldsymbol{A}$ must be a 3×2 matrix. Which means:

$$\boldsymbol{A} = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \middle| T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Example 12

Let $D_2 : \mathbb{P}_3 \to \mathbb{P}_1$ be the second-derivative operator. So, for a typical cubic polynomial:

$$D_2(ax^3 + bx^2 + cx + d) = 6ax + 2b$$

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The standard matrix is made up of

$$\left[D_2\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) \middle| D_2\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) \middle| D_2\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) \middle| D_2\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}6 & 0 & 0 & 0\\0 & 2 & 0 & 0\end{bmatrix}$$

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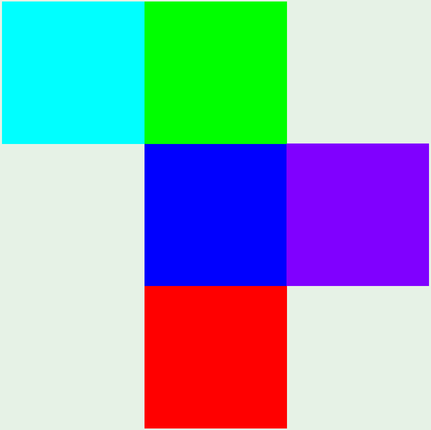
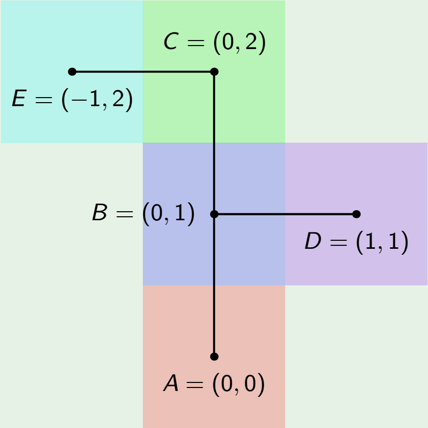
Which gives us a matrix that satisfies:

$$\begin{bmatrix}6 & 0 & 0 & 0\\0 & 2 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}6a\\2b\end{bmatrix}$$

Example 13

Linear transforms are used extensively in computer graphics, where images or models are just collections of points and line segments. Let us look at a simple example, where we can think of each pixel as a point at it's center:





Example 14

We can rotate this image 90° clockwise with the matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Example 14

We can rotate this image 90° clockwise with the matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

When applying this linear transformation to our image we get

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

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We can rotate this image 90° clockwise with the matrix

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