

Approximation Methods

Numerical Analysis

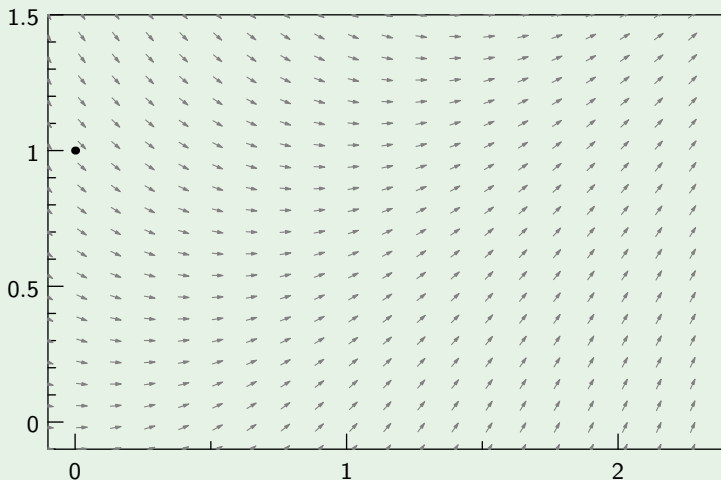
Department of Mathematics

Salt Lake Community College

Example 1

Consider the IVP

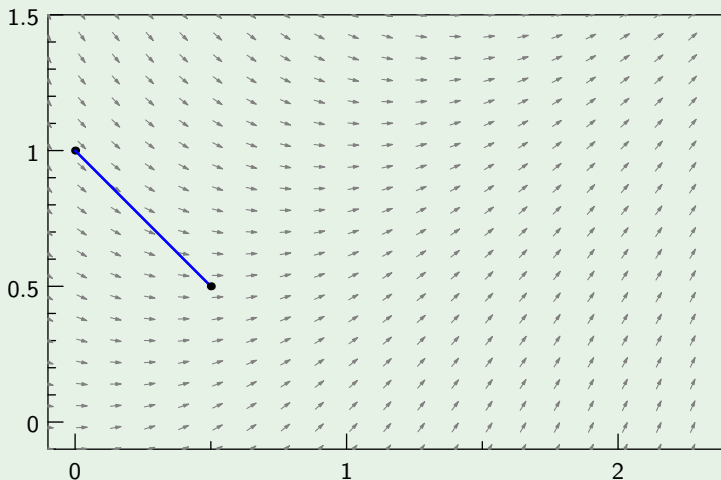
$$y' = t - y, \quad y(0) = 1$$



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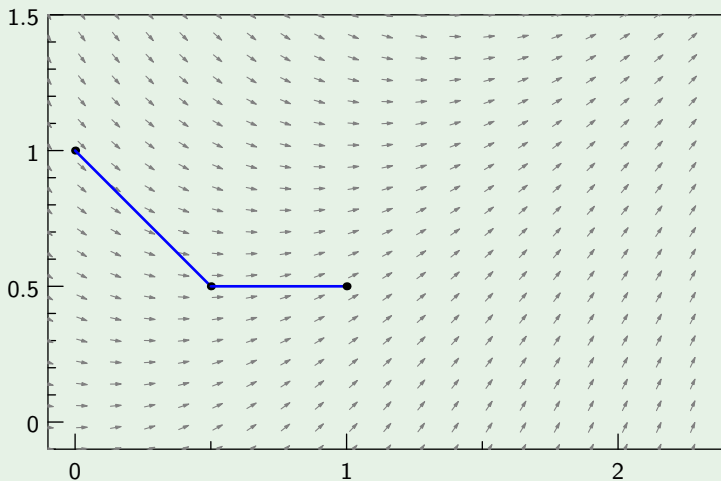
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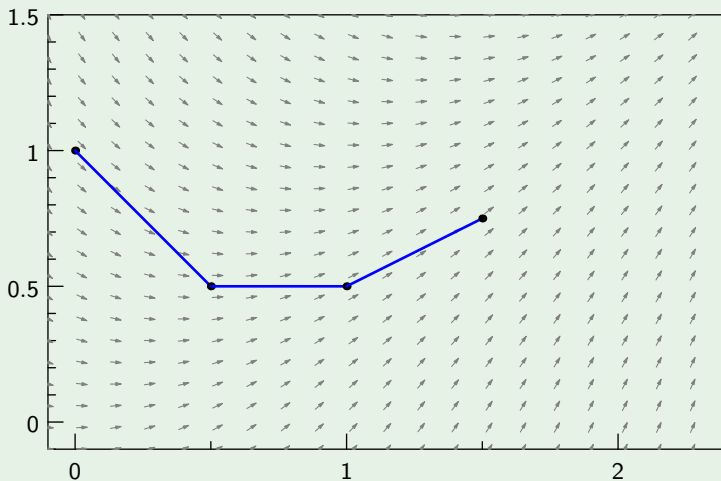
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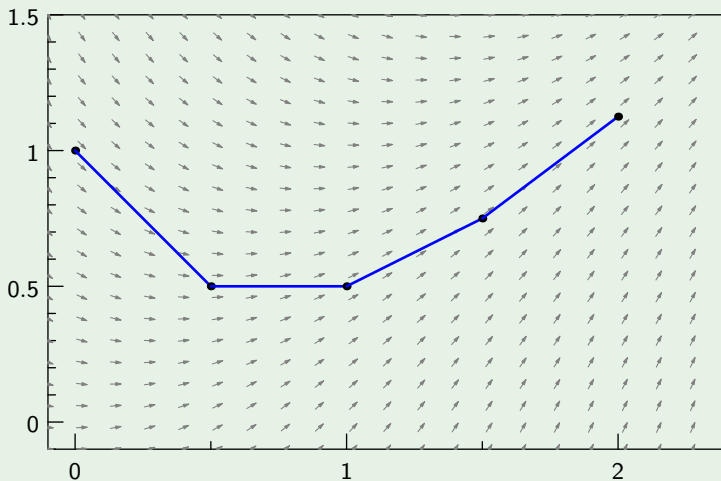
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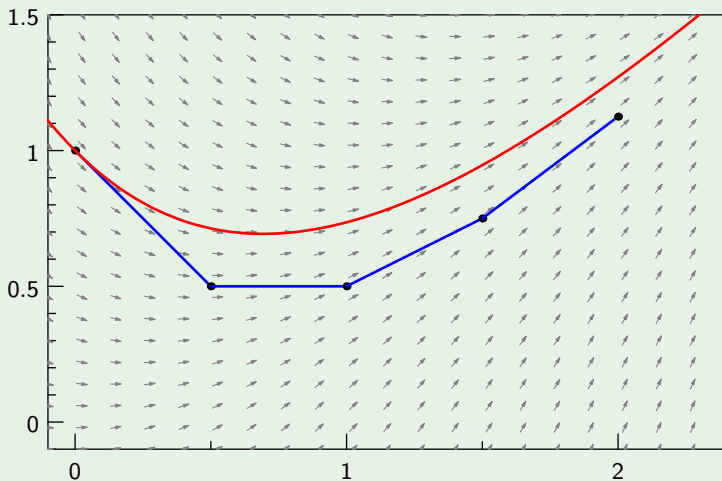
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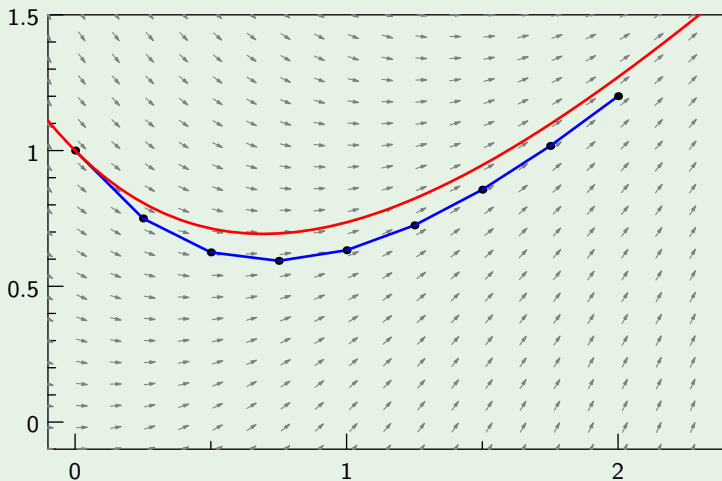
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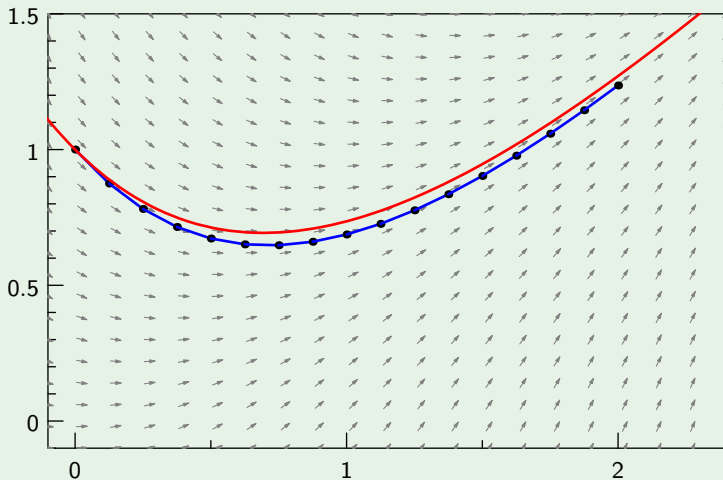
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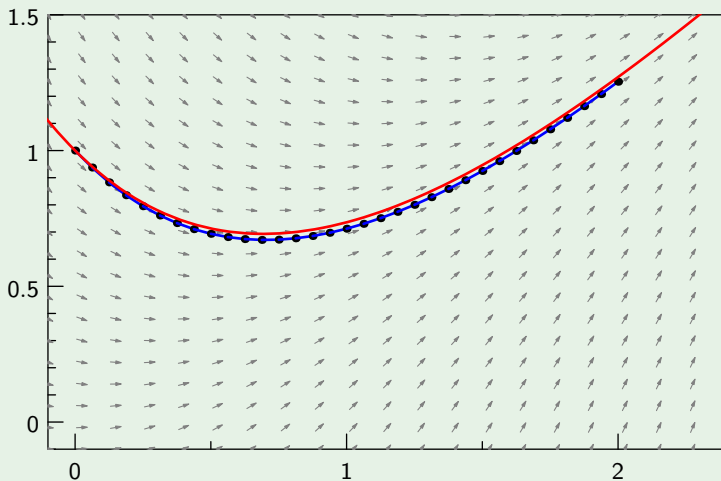
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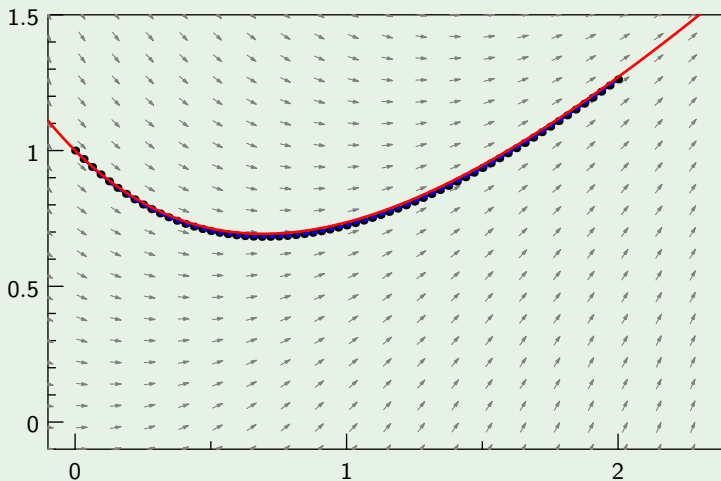
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$$y' = f(t, y), \quad y(t_0) = y_0$$

We want to compute approximate values for $y(t_n)$ at the (finite) set of points $t_1, t_2, t_3, \dots, t_k$.

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to find the approximate solution $(t_1, y(t_1))$:

$$y_1 = y_0 + h \cdot f(t_0, y_0)$$

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We can extend this process to find all k points.

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The resulting piecewise-linear function (i.e. play connect-the-dots) is called the **Euler-approximate** solution.

Euler's Method

For the Initial-value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

use the formulas

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

to iteratively compute the points, using step size h ,

$$(t_1, y_1), (t_2, y_2), \dots, (t_k, y_k).$$

The piecewise-linear function connecting these points is the Euler approximation to the solution $y(t)$ of the IVP for $t_0 \leq t \leq t_k$.

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Let us obtain the Euler-approximate solution of the IVP

$$y' = -2ty + t, \quad y(0) = -1$$

with step size 0.1 on $[0, 0.4]$.

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In other words:

$$f(t, y) = -2ty + t = t(1 - 2y)$$

$$t_0 = 0$$

$$y_0 = -1$$

$$h = 0.1$$

$$k = 1, 2, 3, 4$$

Example 2

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y_0 + h \cdot f(t_0, y_0) = -1 + (0.1)(0)(1 - 2(-1)) = -1$$

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$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2$$

$$\begin{aligned} y_2 &= y_1 + h \cdot f(t_1, y_1) \\ &= -1 + (0.1)(0.1)(1 - 2(-1)) = -0.97 \end{aligned}$$

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$$t_3 = t_2 + h = 0.2 + 0.1 = 0.3$$

$$\begin{aligned} y_3 &= y_0 + h \cdot f(t_2, y_2) \\ &= -0.97 + (0.1)(0.2)(1 - 2(-0.97)) = -0.9112 \end{aligned}$$

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$$t_4 = t_3 + h = 0.3 + 0.1 = 0.4$$

$$\begin{aligned} y_4 &= y_3 + h \cdot f(t_3, y_3) \\ &= -0.9112 + (0.1)(0.3)(1 - 2(-0.9112)) = -0.82652 \end{aligned}$$

Example 2

How does this compare to the exact solution $y(t) = 0.5 - 1.5e^{-t^2}$?

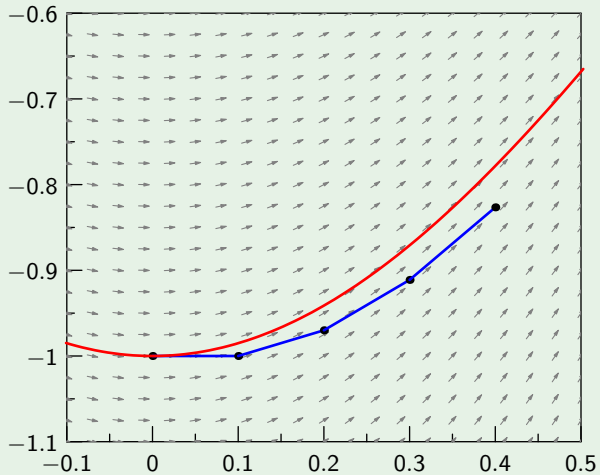
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| n | t_n | y_n | $y(t_n)$ | Error |
|-----|-------|-----------|-----------|------------------|
| 0 | 0.0 | -1.000000 | -1.000000 | 0.000000 |
| 1 | 0.1 | -1.000000 | -0.985075 | -0.014925 |
| 2 | 0.2 | -0.970000 | -0.941184 | -0.028815 |
| 3 | 0.3 | -0.911200 | -0.870897 | -0.040303 |
| 4 | 0.4 | -0.826528 | -0.778216 | -0.048312 |

Notice how the error grows rapidly.

Example 2



Example 3

Find the Euler-approximation of

$$y' = -2ty, \quad y(0) = 1$$

using a step size of 0.2 over the range of $[0, 2]$.

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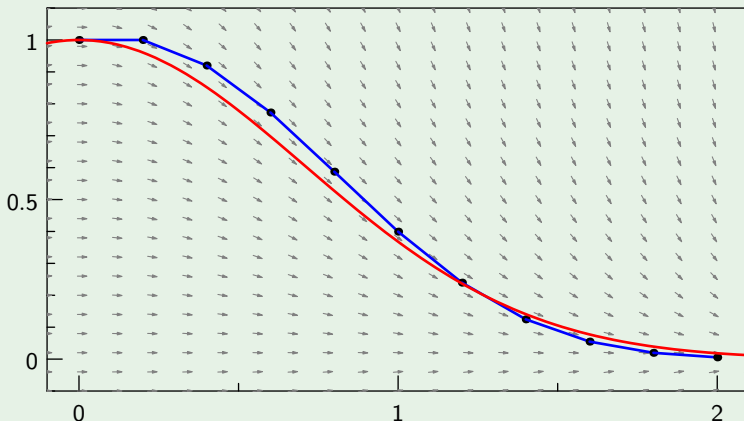
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It can be shown, using Taylor series expansions, that the error is proportional to the square of the step size.

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Where the constant C depends on the size of the second derivative of the exact solution.

We call this error the **local discretization error** because it estimates the error for a single step only. After n steps, we have n times the error. Which we call the **global discretization error**.

Global Discretization Error in Euler's Method

If the solution of the IVP $y' = f(t, y)$, $y(t_0) = y_0$ has a continuous second derivative on the interval $[t_0, t_k]$, and y_n is the value of the Euler approximation at t_n , $t_0 < t_1 < \cdots < t_n \cdots < t_k$, then there exists a constant C such that

$$|y_n - y(t_n)| \leq C \cdot h, \quad n = 1, 2, \dots, k.$$

where step size $h = t_n - t_{n-1}$.

Second-Order Runge-Kutta Method

For the IVP $y' = f(t, y)$, $y(t_0) = y_0$, use the following formulas to compute the points $(t_1, y_1), (t_2, y_2), \dots$ of the approximate solution, using step size h :

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + h \cdot k_{n1}$$

where

$$k_{n1} = f(t_n, y_n)$$

$$k_{n2} = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot k_{n1}\right)$$

Fourth-Order Runge-Kutta Method

For the IVP $y' = f(t, y)$, $y(t_0) = y_0$, use the following formulas to compute the points $(t_1, y_1), (t_2, y_2), \dots$ of the approximate solution, using step size h :

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})$$

where

$$k_{n1} = f(t_n, y_n)$$

$$k_{n2} = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot k_{n1}\right)$$

$$k_{n3} = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot k_{n2}\right)$$

$$k_{n4} = f(t_n + h, y_n + h \cdot k_{n3})$$