Binomial Distribution

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Example 1

Let us assume there is a 0.85 probability that a randomly chosen adult has heard of Twitter.

Four people are chosen at random:

Ariana (A), Brittany (B), Carlton (C), Damian (D)

We want the probability exactly one of them will have heard of Twitter.

There are four combinations possible:

$$P(A = \text{yes and } B = \text{no and } C = \text{no and } D = \text{no})$$
$$= 0.85 \cdot 0.15 \cdot 0.15 \cdot 0.15 = (0.85)^1 (0.15)^3 = 0.002869$$
$$P(A = \text{no and } B = \text{yes and } C = \text{no and } D = \text{no})$$
$$= 0.15 \cdot 0.85 \cdot 0.15 \cdot 0.15 = (0.85)^1 (0.15)^3 = 0.002869$$
$$P(A = \text{no and } B = \text{no and } C = \text{yes and } D = \text{no})$$
$$= 0.15 \cdot 0.15 \cdot 0.85 \cdot 0.15 = (0.85)^1 (0.15)^3 = 0.002869$$
$$P(A = \text{no and } B = \text{no and } C = \text{no and } D = \text{yes})$$
$$= 0.15 \cdot 0.15 \cdot 0.15 \cdot 0.85 = (0.85)^1 (0.15)^3 = 0.002869$$

So, the probability exactly one has heard of Twitter is

$$0.002869 + 0.002869 + 0.002869 + 0.002869 = 0.11475 = 11.475\%$$

Definition

The **binomial distribution** is used to describe the number of successes in a fixed number or trials.

Notation

p	The probability of a success.
$q = 1 - p$	The probability of a failure.
n	The fixed number of trials.
k	The number of successes.

Note

Example 1 is how to find a binomial distribution the hard way.

Note

If all the scenarios are independent of each other, then we can calculate the final probability as:

$$[\text{\# of scenarios}] \cdot P\,(\text{single scenario})$$

Definition

The **factorial**, for any positive integer n, is

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \cdot 1 = 2 \\ 3! &= 3 \cdot 2 \cdot 1 = 6 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ &\vdots \\ n! &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

Note

Factorials can be calculated iteratively. i.e.

$$(n+1)! = n! \cdot (n+1)$$

Definition

The **binomial coefficients** gives the number of ways to choose k successes in n trials.:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Read "n choose k."

Example 2

The number of ways to choose $k = 3$ successes in $n = 4$ trials:

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1} = \frac{4 \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot 1} = 4$$

Binomial Distribution

Suppose the probability of a single trial being a success is p. Then the probability of observing exactly k successes in n independent trials is given by

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

The mean, variance, and standard deviation of the number of observed successes are

$$\mu = np, \qquad \sigma^2 = np(1-p), \qquad \sigma = \sqrt{np(1-p)}$$

Is It Binomial?

Every binomial distribution has to satisfy the following:

- The trials are independent.
- The number of trials, n, is fixed.
- Each trial outcome can be classified as either a *success* or *failure*.
- The probability of a success, p, is the same for each trial.

Example 3

From Example 1 we have $p = 0.85$, $n = 4$, and $k = 1$.

$$
\begin{aligned}
P\,(\text{exactly one has heard of twitter}) &= \binom{n}{k} p^k (1-p)^{n-k} \\
&= \binom{4}{1} (0.85)^1 (1-0.85)^{4-1} \\
&= \frac{4!}{1!(4-1)!} (0.85)^1 (0.15)^3 \\
&= \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 3 \cdot 2 \cdot 1} (0.85)^1 (0.15)^3 \\
&= \frac{4 \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{1 \cdot \cancel{3} \cdot \cancel{2} \cdot 1} (0.85)^1 (0.15)^3 \\
&= 4 \cdot (0.85)^1 (0.15)^3 \\
&= 0.11475
\end{aligned}
$$

Example 4

Assume that 70% of customers won't exceed their car insurance deductible. Let's the probability that 5 of 8 randomly selected customers won't exceed their premium.

Start by identifying

$$p = 0.7, \qquad q = 1 - p = 0.3, \qquad n = 8, \qquad k = 5$$

$$\begin{aligned}
\binom{n}{k} p^k (1-p)^{n-k} &= \binom{8}{5} (0.7)^5 (0.3)^{8-5} \\
&= \frac{8!}{5!(8-5)!} (0.7)^5 (0.3)^3 = \frac{8!}{5!(3)!} (0.7)^5 (0.3)^3 \\
&= \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} (0.7)^5 (0.3)^3 \\
&= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot 3 \cdot 2 \cdot 1} (0.7)^5 (0.3)^3 \\
&= 56 \cdot (0.7)^5 (0.3)^3 \\
&= 0.254122
\end{aligned}$$

Example 5

Assume the probability that a smoker will develop a severe lung condition in their life time is 0.3.

If you have four friends who smoke, are the conditions for the binomial model satisfied?

It is likely that independence is not satisfied, since they probably all know each other.

Example 6

Suppose instead four people are randomly selected.

Is the binomial model appropriate to find the probability that none of them will develop a severe lung condition?

We are assuming that the four are randomly selected, yes.

$$\binom{n}{k}p^k(1-p)^{n-k} = \binom{4}{0}(0.3)^0(1-0.3)^{4-0} = \frac{4!}{0!(4-0)!}(0.3)^0(0.7)^4$$
$$= 1 \cdot 1 \cdot (0.7)^4 = 0.2401$$

Example 7

Let consider finding the probability than no more than one of the four people will develop a severe lung condition.

The events that "none of them develops a severe lung condition" and "exactly one develops a severe lung condition" are mutually exclusive.

$$
\begin{aligned}
P\,(\text{none}) + P\,(\text{exactly one})& \\
&= \binom{4}{0}(0.3)^0(1-0.3)^{4-0} + \binom{4}{1}(0.3)^1(1-0.3)^{4-1} \\
&= \frac{4!}{0!(4-0)!}(0.3)^0(0.7)^4 + \frac{4!}{1!(4-1)!}(0.3)^1(0.7)^3 \\
&= \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 4\cdot 3\cdot 2\cdot 1}(0.3)^0(0.7)^4 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 3\cdot 2\cdot 1}(0.3)^1(0.7)^3 \\
&= \frac{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{1\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}(0.3)^0(0.7)^4 + \frac{4\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{1\cdot \cancel{3}\cdot \cancel{2}\cdot 1}(0.3)^1(0.7)^3 \\
&= 0.2401 + 0.4116 \\
&= 0.6517 = 65.17\%
\end{aligned}
$$

Example 8

Lets consider finding the probability that at least two of the four people will develop a severe lung condition.

The complement of "at least two will develop a severe lung condition" is "no more than one will develop a severe lung condition."

We know from Example 7 that $P\,(\text{no more than one}) = 0.6517$.

So,

$$\begin{aligned} P\,(\text{at least two}) &= 1 - P\,(\text{no more than one}) \\ &= 1 - 0.6517 = 0.3483 = 34.83\% \end{aligned}$$

Example 9

Out of seven randomly selected smokers, how many would we expect to develop a severe lung condition?

The mean of the binomial model is

$$\mu = np = 7 \cdot 0.3 = 2.1$$

On average, we would expect 2.1 of 7 randomly chosen smokers to develop a severe lung condition.