

# Linear Equations: The Nature of Their Solutions

Department of Mathematics

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(Slides by Adam Wilson)

## Definition

An equation  $F(x_1, x_2, \dots, x_n) = C$  is **linear** if it is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = C$$

where  $a_1, a_2, \dots, a_n$  and  $C$  are constants.

If  $C = 0$ , the equation is said to be **homogeneous**.

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Which of the following are linear equations?

$$4x - 3e^x = 15$$

$$4x - 2y + 3\sqrt{z} = 12$$

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## First and Second Order Notation

It is common to write first-order differential equations as

$$y' + p(t)y = f(t)$$

and second-order differential equations as

$$y'' + p(t)y' + q(t)y = f(t)$$



## Example 2

Let us classify the following differential equations.

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## Notation

We will use a **vector** notation to represent a whole set of variables:

Linear Algebraic Equations:

$$\vec{x} = [x_1, x_2, \dots, x_n]$$

Linear Differential Equations:

$$\vec{y} = [y^{(n)}, y^{(n-1)}, \dots, y', y]$$

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A **linear operator**  $L$  is an entire operation performed on a set of variables.

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Linear Differential Equations:

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### Linear Operator Properties

$$\begin{aligned}L(k\vec{u}) &= kL(\vec{u}), \quad k \in \mathbb{R} \\L(\vec{u} + \vec{w}) &= L(\vec{u}) + L(\vec{w})\end{aligned}$$

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### Proof

The properties can be proved directly for algebraic operators.

For differential operators, the proof follows from the derivative properties:

- $(kf)' = kf'$
- $(f + g)' = f' + g'$

## Superposition Principle for Linear Homogeneous Equations

Let  $\vec{u}_1$  and  $\vec{u}_2$  be any solutions of the *homogeneous linear* equation

$$L(\vec{u}) = 0$$

- The sum  $\vec{u} = \vec{u}_1 + \vec{u}_2$  is also a solution.
- For any constant  $k$ ,  $\vec{u} = k\vec{u}_1$  is also a solution.

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### Proof

The proof of the Superposition Principle follows directly from the properties of linear operators from the previous slides.

$$L(\vec{u}) = L(\vec{u}_1 + \vec{u}_2) = L(\vec{u}_1) + L(\vec{u}_2) = 0 + 0 = 0$$

$$L(\vec{u}) = L(k\vec{u}_1) = kL(\vec{u}_1) = k \cdot 0 = 0$$

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## Nonhomogeneous Principle

Let  $\vec{u}_p$  be any solution (called a particular solution) to *linear nonhomogeneous* equation

$$L(\vec{u}) = C \quad (\text{algebraic})$$

or

$$L(\vec{u}) = f(t) \quad (\text{differential})$$

Then,

$$\vec{u} = \vec{u}_h + \vec{u}_p$$

is also a solution, here  $\vec{u}_h$  is a solution to the **associated homogeneous** equation

$$L(\vec{u}) = 0$$

Furthermore, *every solution of the nonhomogeneous equation must be of the form  $\vec{u} = \vec{u}_h + \vec{u}_p$ .*



## Proof

It is easy to show that  $\vec{u} = \vec{u}_h + \vec{u}_p$  is a solution.

$$L(\vec{u}) = L(\vec{u}_h + \vec{u}_p) = L(\vec{u}_h) + L(\vec{u}_p) = 0 + f(t) = f(t)$$

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To show that every solution has to be of this form, suppose that  $\vec{u}_q$  is any solution. Note that  $\vec{u}_q = \vec{u}_p + (\vec{u}_q - \vec{u}_p)$ .

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We can then show that  $\vec{u}_q - \vec{u}_p$  is also a solution to  $L(\vec{u}) = 0$ :

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We can then show that  $\vec{u}_q - \vec{u}_p$  is also a solution to  $L(\vec{u}) = 0$ :

$$L(\vec{u}_q - \vec{u}_p) = L(\vec{u}_q) + L(-\vec{u}_p)$$

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To show that every solution has to be of this form, suppose that  $\vec{u}_q$  is any solution. Note that  $\vec{u}_q = \vec{u}_p + (\vec{u}_q - \vec{u}_p)$ .

We can then show that  $\vec{u}_q - \vec{u}_p$  is also a solution to  $L(\vec{u}) = 0$ :

$$\begin{aligned} L(\vec{u}_q - \vec{u}_p) &= L(\vec{u}_q) + L(-\vec{u}_p) \\ &= L(\vec{u}_q) - L(\vec{u}_p) \end{aligned}$$

## Proof

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## Process for Solving Nonhomogeneous Linear Equations

Step 1: Find all solutions  $\vec{u}_h$  of  $L(\vec{u}) = 0$ .

Step 2: Find any solution  $\vec{u}_p$  of  $L(\vec{u}) = f$ .

Step 3: Add  $\vec{u}_h + \vec{u}_p = \vec{u}$  to find all solutions of  $L(\vec{u}) = f$ .

## Example 6

Consider

$$y' - y = t$$

To solve using superposition we need to complete three steps.

Step 1:

Step 2:

Step 3:



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**Step 1:** Solve the associated homogeneous equation  $y' - y = 0$ , or  $y' = y$ . (Note: first-order homogeneous linear differential equations are always separable.)

$$y_h = ce^t, \quad \text{for any } c \in \mathbb{R}$$

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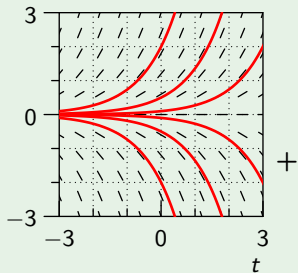
$$y = y_h + y_p = ce^t - t - 1$$

is a solution for any  $c \in \mathbb{R}$ .

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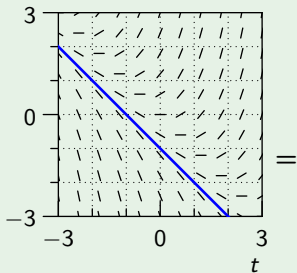
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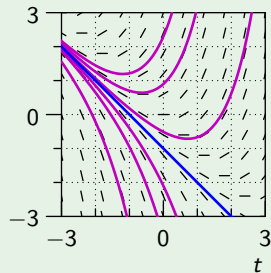
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+



$y_p$

=



$\{y_h\} + y_p$

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But, sometimes a particular solution may be staring us in the face.

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Let us solve

$$y' + ay = b$$

where  $a$  and  $b$  are constants.

Step 1:

Step 2:

Step 3:



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Alternatively, we can look for a horizontal line in the direction field.

