

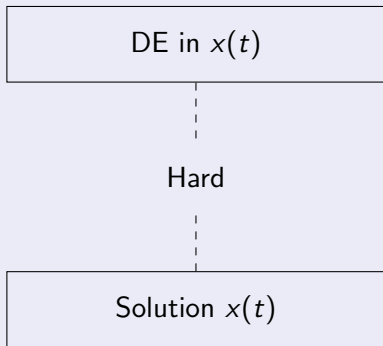
# Laplace Transforms

Department of Mathematics

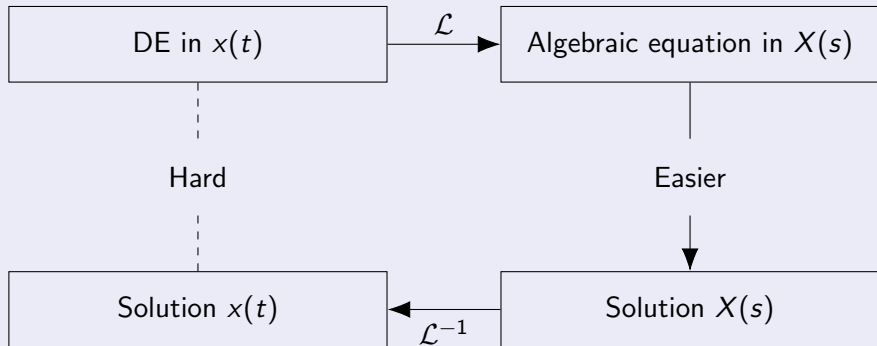
Salt Lake Community College

(Slides by Adam Wilson)

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## Laplace Transform

The **Laplace Transform**  $\mathcal{L}\{f(t)\}$  of a suitable function  $f(t)$  defined on  $[0, \infty)$  is the function  $F(s)$  given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} f(t) dt$$

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## Linearity of the Laplace Transform

If  $F(s) = \mathcal{L}\{f(t)\}$  and  $G(s) = \mathcal{L}\{g(t)\}$ , then by the properties of integrals

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s) \quad \text{for } a, b \in \mathbb{C}$$

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## Existence Theorem for Laplace Transform

If  $f(t)$  is piecewise continuous on  $[0, \infty)$  and of exponential order  $\alpha$ , then the Laplace transform  $F(s) = \mathcal{L}\{f(t)\}$  exists for  $s > \alpha$ .

## Example 1

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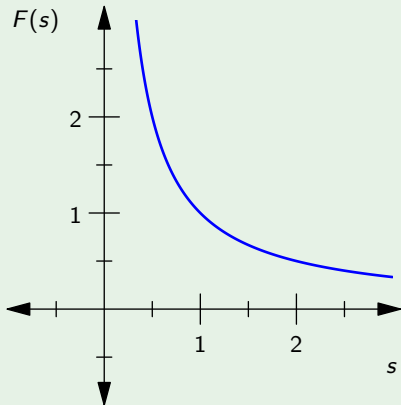
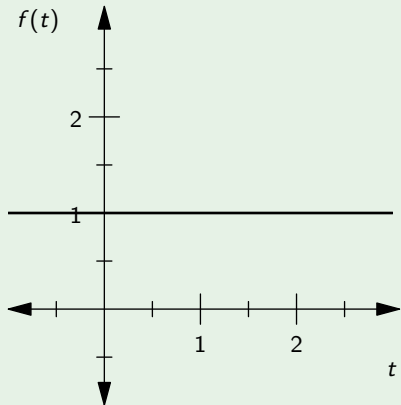
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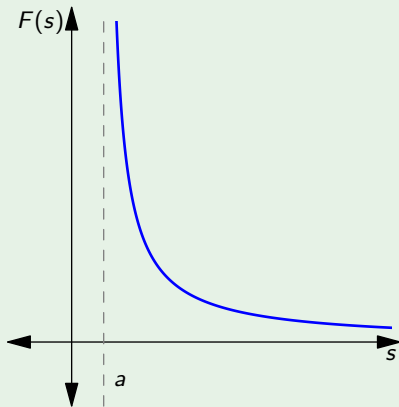
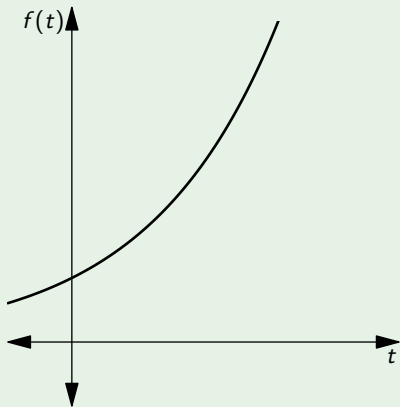
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So, if we equate the real and imaginary parts, we get:

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2} \quad \text{and} \quad \mathcal{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}$$

## Inverse Laplace Transform

A function  $f(t)$  whose transform is  $F(s)$  is called the **inverse Laplace transform** of  $F$ , and we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$



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Thus,

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## Some Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\}$	
1	$\frac{1}{s}$	$s > 0$
$t^n$	$\frac{n!}{s^{n+1}}$	$s > 0, n \in \mathbb{N}^+$
$e^{at}$	$\frac{1}{s-a}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a, n \in \mathbb{N}^+$
$\sin(bt)$	$\frac{b}{s^2 + b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2 + b^2}$	$s > 0$

## Some More Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\}$	
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$\sinh(bt)$	$\frac{b}{s^2 - b^2}$	$s >  b $
$\cosh(bt)$	$\frac{s}{s^2 - b^2}$	$s >  b $

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Now, if we let  $b = \sqrt{5}$ , then  $F$  becomes

$$F(s) = 2 \underbrace{\left( \frac{1}{s - 1} \right)}_{\mathcal{L}\{e^t\}} - \frac{1}{\sqrt{5}} \underbrace{\left( \frac{\sqrt{5}}{s^2 + (\sqrt{5})^2} \right)}_{\mathcal{L}\{\sin(\sqrt{5}t)\}}$$

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$$\text{Thus, } f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^t - \frac{1}{\sqrt{5}} \sin(\sqrt{5}t).$$

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Thus, by linearity we have  $f(t) = e^{-2t} \cos(3t) - \frac{1}{3} s^{-2t} \sin(3t)$ .