

# The Step Function and the Delta Function

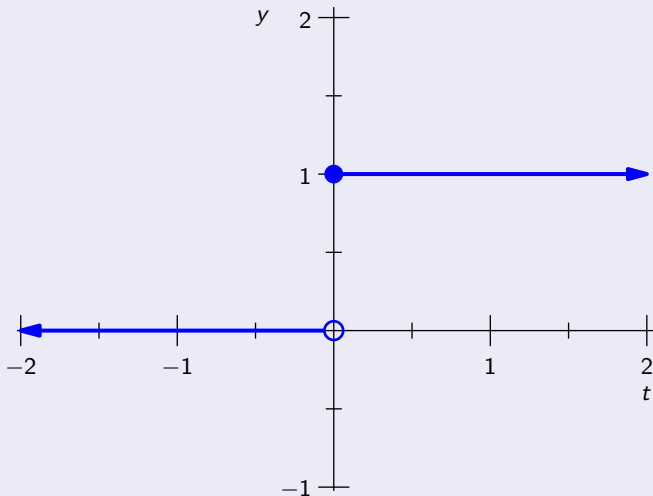
Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

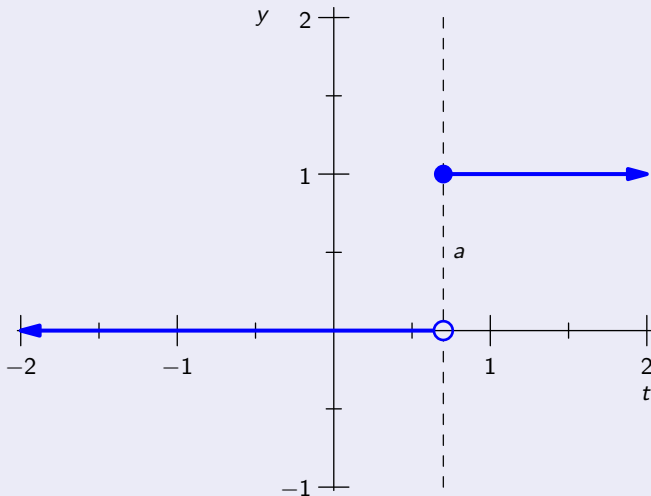
## The Unit Step Function

$$\text{step}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



## The Translated Step Function

$$\text{step}(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



## Laplace Transform of the Step Function

$$\mathcal{L}\{\mathbf{step}(t - a)\} = \frac{e^{-as}}{s}$$

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### Proof

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## Example 1

Consider

$$f(t) = \begin{cases} 2 & \text{if } t < 3 \\ -4 & \text{if } 3 \leq t < 4 \\ 1 & \text{if } t \geq 4 \end{cases}$$

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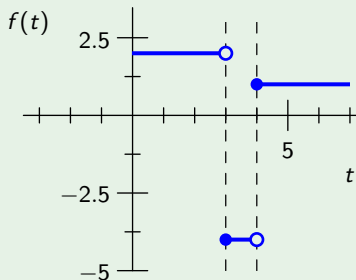
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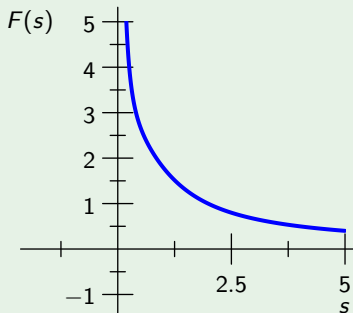
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$\xrightarrow{\mathcal{L}}$



## Example 2

Consider

$$g(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^2 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

## Example 2

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$$g(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^2 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t \geq 1 \end{cases} = t^2 \mathbf{step}(t) + (1 - t^2) \mathbf{step}(t - 1)$$



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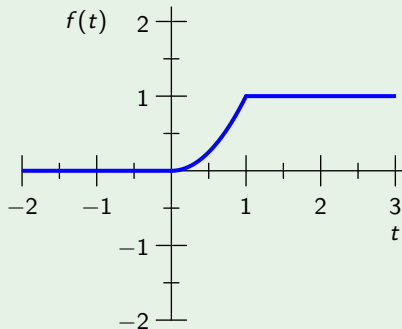
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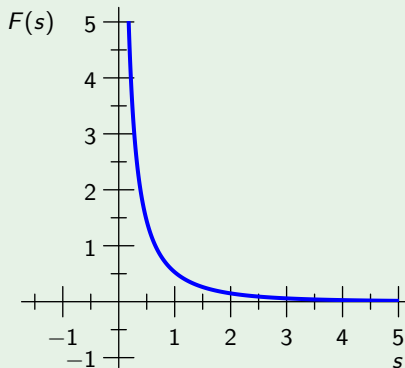
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$\mathcal{L}$



## Delayed Function

For a given function  $g(t)$ , the **delayed function**

$$f(t) = \begin{cases} 0 & \text{if } t < c \\ g(t - c) & \text{if } t \geq c \end{cases}$$

shifts  $g(t)$  to the right  $c$  units from the origin, and replaces it by zero to the left of  $t = c$ . Using the unit step function, the delayed function can also be written

$$f(t) = g(t - c) \mathbf{step}(t - c)$$

### Example 3

Consider the function  $g(t) = \sqrt{t}$ , which has the delayed function

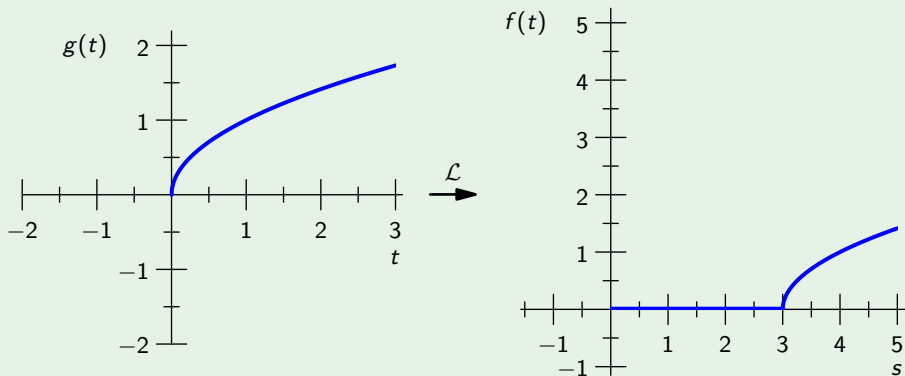
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## Closer Look

Consider the Laplace transform of a function  $f(t)$  that is delayed  $c$  units.

$$\mathcal{L}\{f(t - c) \mathbf{step}(t - c)\}$$

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let  $w = t - c$



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## Delay Theorem (or Shifting Theorem)

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Let us find the inverse Laplace transform of

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$$\mathcal{L}^{-1}\{F(s)\} = t - \underbrace{(t - 3)\text{step}(t - 3)}_{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}}$$

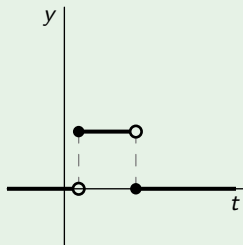
## Chopper Function

$$\mathbf{step}(t - a) - \mathbf{step}(t - b) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$$

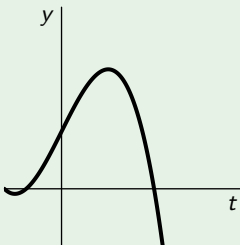
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$$\text{step}(t - a) - \text{step}(t - b) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$$

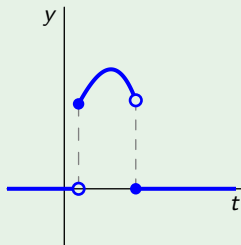
### Example 6



times



gives

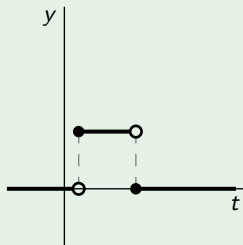




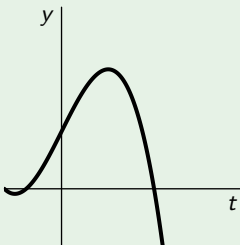
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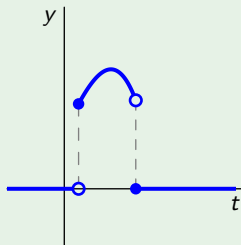
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## Laplace Transform of Chopper Function

$$\mathcal{L}\{f(t) \cdot [\mathbf{step}(t - a) - \mathbf{step}(t - b)]\} = e^{-as} \mathcal{L}\{f(t+a)\} - e^{-bs} \mathcal{L}\{f(t+b)\}$$

## Example 7

Let us find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ -\sin(\pi t) & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases}$$

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Let us find the Laplace transform of

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## Example 8

Consider the IVP

$$x'' + x = f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases} \quad \text{with } x(0) = 0, \quad x'(0) = 0$$

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$$s^2 X(s) + X(s) = \mathcal{L}\{1 - \mathbf{step}(t - \pi)\}$$

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$$s^2 X(s) + X(s) = \mathcal{L}\{1 - \mathbf{step}(t - \pi)\}$$

We can then use the Delay Theorem on the RHS

$$s^2 X(s) + X(s) = \frac{1}{s} + \frac{e^{-\pi s}}{s}$$

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$$\begin{aligned} s^2 X(s) + X(s) &= \frac{1}{s} + \frac{e^{-\pi s}}{s} \\ (s^2 + 1)X(s) &= \frac{1 - e^{-\pi s}}{s} \end{aligned}$$

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$$= \frac{1}{s(s^2 + 1)} - e^{-\pi s} \frac{1}{s(s^2 + 1)}$$

$$= \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) - e^{-\pi s} \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right)$$

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So, we can use the Delay Theorem again to find  $x(t)$ .

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = (1 - \cos(t)) - (1 - \cos(t - \pi)) \mathbf{step}(t - \pi)$$

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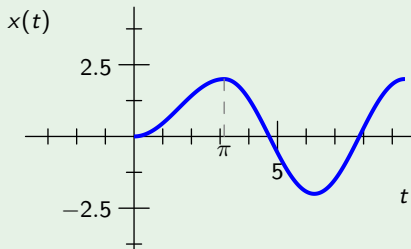
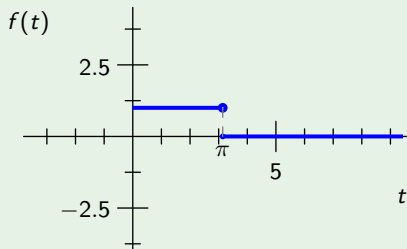
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Physical systems often involve impulsive forces, which act over very short spans of time. To model these forces, the physicist Paul Dirac invented a “function-like” object.

Let us first look at a special function

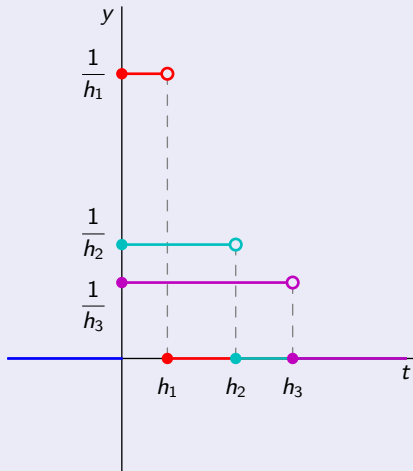
$$f_h(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{h} & \text{if } 0 \leq t < h \\ 0 & \text{if } t \geq h \end{cases}$$

such that

$$\int_{-\infty}^{\infty} f_h(t) dt = 1$$

Dirac suggested that

$$\delta(t) = \lim_{h \rightarrow 0} f_h(t)$$



## Dirac Delta Function

The **Dirac Delta function** or **unit impulse function**  $\delta(t)$  is defined by two conditions:

①

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \lim_{h \rightarrow 0} \left( \frac{1}{h} \right) & \text{if } t = 0 \end{cases}$$

②

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

## Finding the Laplace Transform

To find that Laplace transform of  $\delta(t)$ , we will first calculate the transform of  $f_h(t)$ .

$$\mathcal{L}\{f_h(t)\} = \int_0^{\infty} e^{-st} f_h(t) dt$$



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## Laplace Transform of the Delta Function

$$\mathcal{L}\{\delta(t)\} = 1 \quad \text{and} \quad \mathcal{L}\{\delta(t - a)\} = e^{-as}$$