

# Solving DEs and IVPs with Laplace Transforms

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

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Consider the second-order IVP.

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## Derivative Theorem for Laplace Transforms

If  $f, f', \dots, f^{(n-1)}$  are continuous on  $[0, \infty)$ ,  $f^{(n)}$  is piecewise continuous on  $[0, \infty)$ , and  $f, f', \dots, f^{(n)}$  are of exponential order  $\alpha$ , then for  $s > \alpha$ , and  $n = 1, 2, \dots$

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## Strategy to Solve DEs with Laplace Transforms

- 1 Using the Laplace transform, transform the IVP with unknown function  $y(t)$  into an algebraic problem with unknown function  $Y(s)$ .

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- 2 Solve the algebraic problem for  $Y(s)$ .
- 3 Manipulating  $Y(s)$  algebraically if necessary, use the inverse Laplace transform to transform  $Y(s)$  into the IVP solution  $y(t)$ .

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Which means

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$$\begin{aligned} Y(s) &= \frac{s^2 + 2}{(s^2 + 1)(s^2 + 4)} \\ &= \frac{\frac{1}{3}}{s^2 + 1} + \frac{\frac{2}{3}}{s^2 + 4} \end{aligned}$$

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$$y'' + 4y = \sin(t) \quad \text{where} \quad y(0) = 0, \quad y'(0) = 1$$

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Thus, the solution is

$$y(t) = \frac{1}{3} \sin(t) + \frac{1}{3} \sin(2t)$$

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## Translation Property for Multiplication by $e^{at}$

If the Laplace transform  $F(s) = \mathcal{L}\{f(t)\}$  exists for  $s > a$ , then

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a) \quad \text{for } s > a + \alpha$$

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## Multiplication by $t^n$ Rule for the Laplace Transform

If  $f(t)$  is a piecewise continuous function on  $[0, \infty)$  and is of exponential order  $\alpha$ , then for  $s > \alpha$ ,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s) \quad \text{where } n \in \mathbb{N}^+$$

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This process can be repeated for an arbitrary  $n$ .

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