Matrix Algebra

Colby Community College

Matrix

A **matrix** is a rectangular array of **elements** or **entries** (numbers or functions) arranged in **rows** (horizontal) and **columns** (vertical).

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

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Equal Matrices

Two matrices of the same order are **equal** if their corresponding entries are equal. If matrices $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [a_{ij}]$ are both $m \times n$, then

$$\boldsymbol{A} = \boldsymbol{B} \Leftrightarrow a_{ij} = b_{ij}, \quad 1 \leq i \leq m,\, 1 \leq j \leq n$$

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$$\boldsymbol{D} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mn} \end{bmatrix}$$

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- The $n \times n$ **identity matrix**, denoted $\boldsymbol{I}_n$ is:

$$\boldsymbol{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix Addition

Two matrices of the same order are added (or subtracted) by adding (or subtracting) corresponding entries and recording the results in a matrix of the same size. Using matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$.

$$\boldsymbol{A} + \boldsymbol{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$
$$\boldsymbol{A} - \boldsymbol{B} = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$$

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Multiplication by a Scalar

To find the product of a matrix and a scalar (a complex number), multiply each entry of the matrix by that number. This is called **multiplication by a scalar**. Using matrix notation, if $\boldsymbol{A} = [a_{ij}]$, then

$$c \cdot \boldsymbol{A} = [c \cdot a_{ij}] = [a_{ij} \cdot c] = \boldsymbol{A} \cdot c$$

Example 1

Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix}$$

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$$\begin{bmatrix} 3\cdot 3 & 3\cdot 1 & 3\cdot 5 \\ 3\cdot(-2) & 3\cdot 0 & 3\cdot 6 \end{bmatrix} - \begin{bmatrix} 2\cdot 4 & 2\cdot 1 & 2\cdot 0 \\ 2\cdot 8 & 2\cdot 1 & 2\cdot(-3) \end{bmatrix}$$

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$$\begin{bmatrix} 9 & 3 & 15 \\ -6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 8 & 2 & 0 \\ 16 & 2 & -6 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 15 \\ -22 & -2 & 24 \end{bmatrix}$$

Properties of Matrix Addition and Scalar Multiplication

Suppose $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are $m \times n$ matrices and c and k are scalars. Then the following properties hold:

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Vectors

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Vector addition and Scalar Multiplication

Let

$$\vec{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

be vectors in $\mathbb{R}^n$ and c be any scalar. Then, we have:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c \cdot x_1 \\ \vdots \\ c \cdot x_n \end{bmatrix}$$

Properties of Vector Addition and Multiplication

For vectors $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{w}}$ in $\mathbb{R}^n$ and scalars c and k.

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Dot Product

The **dot product** of a row vector $\vec{\boldsymbol{x}}$ and a column vector $\vec{\boldsymbol{y}}$ of equal length n is the result of adding the products of the corresponding entries as follows:

$$\begin{aligned}\vec{\boldsymbol{x}} \bullet \vec{\boldsymbol{y}} &= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \bullet \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\ &= x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n\end{aligned}$$

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Consider

$$\vec{\boldsymbol{r}} = \begin{bmatrix} 3 & -5 & 2 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{c}} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & -5 & 2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = 3 \cdot 3 + (-5) \cdot 4 + 2 \cdot (-5)$$

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$$\begin{bmatrix} 3 & -5 & 2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = 3 \cdot 3 + (-5) \cdot 4 + 2 \cdot (-5) = 9 - 20 - 10$$

Dot Product

The **dot product** of a row vector $\vec{\boldsymbol{x}}$ and a column vector $\vec{\boldsymbol{y}}$ of equal length n is the result of adding the products of the corresponding entries as follows:

$$\begin{aligned}\vec{\boldsymbol{x}} \bullet \vec{\boldsymbol{y}} &= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \bullet \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\ &= x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n\end{aligned}$$

Example 2

Consider

$$\vec{\boldsymbol{r}} = \begin{bmatrix} 3 & -5 & 2 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{c}} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$$

What is $\vec{\boldsymbol{r}} \bullet \vec{\boldsymbol{c}}$?

$$\begin{bmatrix} 3 & -5 & 2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = 3 \cdot 3 + (-5) \cdot 4 + 2 \cdot (-5) = 9 - 20 - 10 = -21$$

Matrix Product

The **matrix product** of a $m \times r$ matrix $\boldsymbol{A}$ and a $r \times n$ matrix $\boldsymbol{B}$ is denoted

$$\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{AB}$$

where the ijth entry of $\boldsymbol{C}$ is the dot product of the ith row vector of A and the jth column vector of $\boldsymbol{B}$:

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{2j} & \cdots & a_{ir} \end{bmatrix} \bullet \begin{bmatrix} b_{1j} \\ \vdots \\ b_{rj} \end{bmatrix}$$

The matrix $\boldsymbol{C}$ has order $m \times n$.

Example 3

Perform $\boldsymbol{AB}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 3 & 1 \\ 2 & -4 \\ -1 & 0 \end{bmatrix}$$

Example 3

Perform $\boldsymbol{AB}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 3 & 1 \\ 2 & -4 \\ -1 & 0 \end{bmatrix}$$

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$$\begin{array}{c|c} & \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 2 \\ -1 \end{matrix}} & \begin{matrix} 1 \\ -4 \\ 0 \end{matrix} \end{bmatrix} \\ \hline \begin{bmatrix} \boxed{1 \quad -1 \quad 3} \\ 0 \quad\;\; 4 \quad\;\; 2 \end{bmatrix} & \begin{bmatrix} \boxed{-2} & \\ & \end{bmatrix} \end{array}$$

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Example 4

Perform $\boldsymbol{AB}$ where

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

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$$\begin{array}{c|c} & \begin{bmatrix} 2 & \boxed{5} & 1 & 4 \\ 4 & \boxed{8} & 0 & 6 \\ -3 & \boxed{1} & -2 & -1 \end{bmatrix} \\ \hline \begin{bmatrix} 2 & 4 & -1 \\ \boxed{5 \quad 8 \quad 0} \end{bmatrix} & \begin{bmatrix} 23 & 41 & 4 & 33 \\ 42 & \boxed{89} & & \end{bmatrix} \end{array}$$

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Properties of Matrix Multiplication

- $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$ (Associativity)
- $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ (Distributivity)
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Properties of Identity Matrices

For a $m \times n$ matrix $\boldsymbol{A}$:

- $\boldsymbol{A} \cdot I_n = \boldsymbol{A}$ and $I_m \cdot \boldsymbol{A} = \boldsymbol{A}$
- $\boldsymbol{A} \cdot \mathbf{0}_n = \mathbf{0}_{mn}$ and $\mathbf{0}_m \cdot \boldsymbol{A} = \mathbf{0}_{mn}$

Inverse Matrix

If there exists, for an $n \times n$ matrix $\boldsymbol{A}$, another matrix $\boldsymbol{A}^{-1}$ of the same order such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$$

then $\boldsymbol{A}^{-1}$ is called the **inverse** of matrix $\boldsymbol{A}$, and $\boldsymbol{A}$ is called **invertible**.

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Vocabulary

- A square matrix that is not invertible is called **singular**.
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Invertible Matrix Properties

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- A square matrix that is not invertible is called **singular**.
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Invertible Matrix Properties

- If $\boldsymbol{A}$ is invertible, then so is $\boldsymbol{A}^{-1}$ and $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$
- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices of the same order, then their product $\boldsymbol{AB}$ is invertible. In fact, $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Inverses by Reduced Row Echelon Form

For an $n \times n$ matrix $\boldsymbol{A}$, the following process will calculate $\boldsymbol{A}^{-1}$, or show that $\boldsymbol{A}$ is not invertible.

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Step 2: Transform $\boldsymbol{M}$ into Reduced Row Echelon Form.

Step 3:
- If the left hand side of $\boldsymbol{M}$ is the identity matrix, then the right hand side is $\boldsymbol{A}^{-1}$.
- Otherwise, $\boldsymbol{A}$ is a non-invertible matrix.

Example 5

Consider the matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $\boldsymbol{A}^{-1}$

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Find $\boldsymbol{A}^{-1}$

Start by building the augmented matrix

$$\boldsymbol{M_A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_A}$ into Reduced Row Echelon Form.

Example 5

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Example 5

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_3 = r_3 - r_1$$

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$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_3 = r_3 - r_1$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Example 5

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] R_1 = r_1 - r_3$$

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$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] R_1 = r_1 - r_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] R_1 = r_1 - r_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Example 5

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Since the left hand side is $\boldsymbol{I}_3$, we know the right hand side is the inverse:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

Example 6

Consider the matrix:

$$\boldsymbol{B} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Find $\boldsymbol{B}^{-1}$

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$$\boldsymbol{B} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Find $\boldsymbol{B}^{-1}$

Start by building the augmented matrix

$$\boldsymbol{M_B} = \left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_B}$ into Reduced Row Echelon Form.

Example 6

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{rrr|rrr} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 = r_3 \\ \\ R_3 = r_1 \end{array}$$

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$$\left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 = r_3 \\ \\ R_3 = r_1 \end{array}$$

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$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 = r_2 + r_1 \\ R_3 = r_2 - 3r_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right]$$

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$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} \\ R_2 = \frac{1}{3} r_2 \\ R_3 = r_3 + r_2 \end{array}$$

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$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} R_2 = \frac{1}{3}r_2 \\ R_3 = r_3 + r_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Example 6

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} \\ R_2 = \frac{1}{3}r_2 \\ R_3 = r_3 + r_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

This means that B is a non-invertible matrix.

Invertibility and Solutions

Consider the matrix equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$.
Where $\boldsymbol{A}$ is an $n \times n$ matrix, and $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}$ are of length n.

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- A unique solution exists if and only if $\boldsymbol{A}$ is invertible.
- Otherwise there are either:
  - No solutions.
  - Infinitely many solutions.

  (Another method must be used to determine which.)

Example 7

Consider the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ & & 2y & + & z & = & -1 \\ x & & & + & z & = & 3 \end{array}$$

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We can can write this as the matrix equation:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

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So, if we can compute $\boldsymbol{A}^{-1}\vec{\boldsymbol{b}}$ we will have solved the system.

Example 7

$$\begin{array}{c|c} & \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ \\ \end{bmatrix} \end{array}$$

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$$\begin{array}{c|c} & \begin{bmatrix} \boxed{\begin{matrix} 2 \\ -1 \\ 0 \end{matrix}} \end{bmatrix} \\ \hline \begin{bmatrix} 2 & -1 & -1 \\ \boxed{1 \quad 0 \quad -1} \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ \boxed{2} \\ \end{bmatrix} \end{array}$$

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Example 7

$$
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 & \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\
\hline
\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}
\end{array}
$$

So, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following are equivalent:

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- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a unique solution for every $\vec{\boldsymbol{b}} \in \mathbb{R}^n$.