

Eigenvalues and Eigenvectors

Department of Mathematics

Salt Lake Community College

Example 1

Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\vec{u}) = \mathbf{A}\vec{u}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

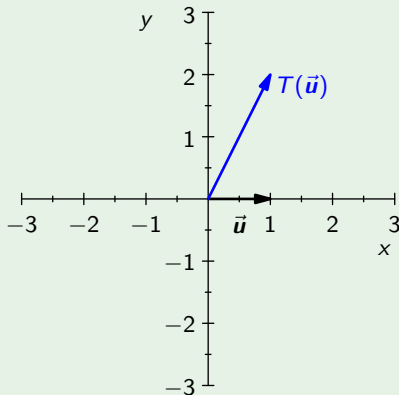
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We can see how T maps a few vectors:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \longrightarrow$$



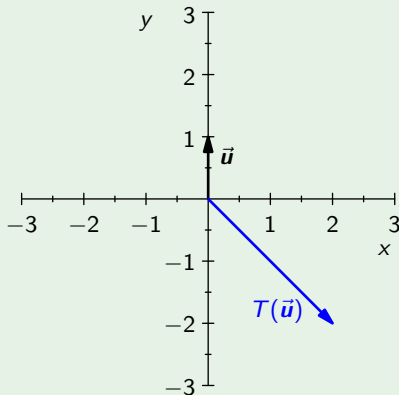
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We can see how T maps a few vectors:

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad \longrightarrow$$



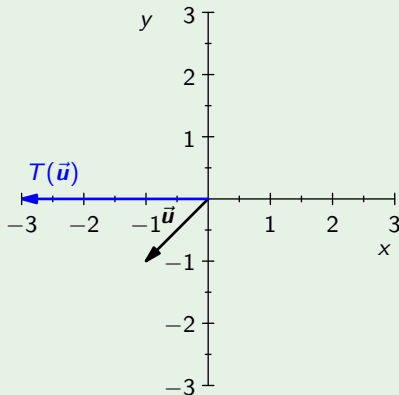
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We can see how T maps a few vectors:

$$T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \quad \longrightarrow$$



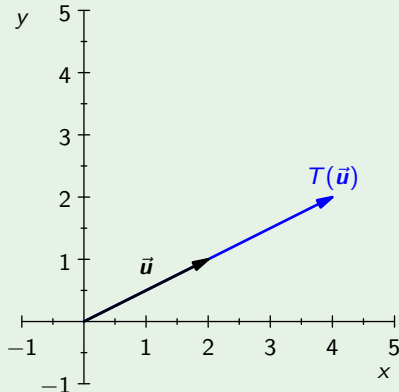
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But something interesting happens for some special vectors.

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow$$



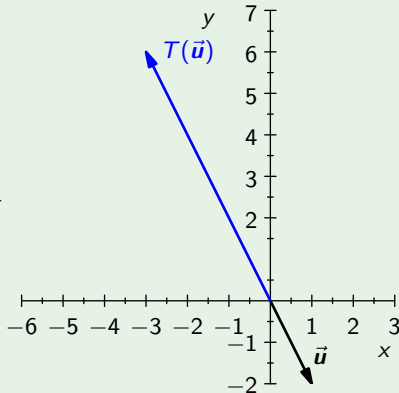
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$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \longrightarrow$$



Eigenvalues and Eigenvectors

Let $T : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation from vector space \mathbb{V} into itself. A scalar λ is a **eigenvalue** of T if there is a *nonzero* vector $\vec{v} \in \mathbb{V}$ such that

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If the linear transformation T is represented by an $n \times n$ matrix \mathbf{A} , where $\mathbb{V} = \mathbb{R}^n$ and $T(\vec{v}) = \mathbf{A}\vec{v}$, then λ and \vec{v} are characterized by the equation

$$\mathbf{A}\vec{v} = \lambda \vec{v}$$

Computing Eigenvalues and Eigenvectors

If \mathbf{A} is a $n \times n$ matrix, and \mathbf{I}_n is the $n \times n$ identity matrix, then

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The polynomial in λ denoted by

$$p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}_n|$$

is called the **characteristic polynomial** of \mathbf{A} .

Summary of Steps for Finding Eigenvalues and Eigenvectors

- 1 Write the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}_n| = 0$.
- 2 Solve the characteristic equation for λ .
- 3 For each eigenvalue λ_i , find the corresponding eigenvector \vec{v}_i by solving the system of equations

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For large matrices these steps become cumbersome, so computer algebra systems are often employed.

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In our first example we saw two eigenvectors, let us verify these using the characteristic equation.

$$\left| \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

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Let us find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

The characteristic equation is

$$|\mathbf{A} - \lambda \mathbf{I}_2| = 0 \rightarrow \begin{vmatrix} 1 - \lambda & 1 \\ 4 & 1 - \lambda \end{vmatrix} = 0 \rightarrow (1 - \lambda)^2 - 4 = 0$$

Which has solutions $\lambda_1 = 3$ and $\lambda_2 = -1$.

To find the eigenvector for λ_1 we need to solve

$$\begin{bmatrix} 1 - (3) & 1 \\ 4 & 1 - (3) \end{bmatrix} \vec{v} = \vec{0} \rightarrow \begin{bmatrix} -2 & 1 & | & 0 \\ 4 & -2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow \vec{v}_1 = s \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find the eigenvector for λ_2 we need to solve

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Which simplifies to:

$$\begin{aligned} \lambda^3 - 2\lambda^2 - \lambda + 2 &= 0 \\ (\lambda - 2)(\lambda - 1)(\lambda + 1) &= 0 \end{aligned}$$

Example 4

Let us find the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The characteristic equation is:

$$|\mathbf{A} - \lambda \mathbf{I}_3| = \begin{vmatrix} 1 - \lambda & 1 & -2 \\ -1 & 2 - \lambda & 1 \\ 0 & 1 & -1 - \lambda \end{vmatrix} = 0$$

Which simplifies to:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1)(\lambda + 1) = 0$$

So, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$.

Example 4

Let us find the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

To find the eigenvector for $\lambda_1 = 2$ we need to solve the system:

$$\begin{bmatrix} 1 - \lambda & 1 & -2 \\ -1 & 2 - \lambda & 1 \\ 0 & 1 & -1 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_1 = v_3$ and $v_2 = 3v_3$. Replacing v_3 with parameter s gives

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Example 4

Let us find the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

To find the eigenvector for $\lambda_2 = 1$ we need to solve the system:

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To find the eigenvector for $\lambda_2 = 1$ we need to solve the system:

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So, we have $v_1 = 3v_3$ and $v_2 = 2v_3$. Replacing v_3 with parameter s gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

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To find the eigenvector for $\lambda_3 = -1$ we need to solve the system:

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So, we have $v_1 = v_3$ and $v_2 = 0$. Replacing v_3 with parameter s gives

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Special Cases

Triangular Matrices: The eigenvalues of an upper (or lower) triangular matrix appear on the main diagonal.

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Trace

The **trace** of a matrix, **tr \mathbf{A}** , is the sum of all elements in the diagonal.

Eigenspace Theorem for Linear Transformations

For each eigenvalue λ of a linear transformations $T : \mathbb{V} \rightarrow \mathbb{V}$, the **eigenspace**, defined by

$$\mathbb{E}_\lambda = \{\vec{v} \in \mathbb{V} \mid T(\vec{v}) = \lambda\vec{v}\}$$

is a subspace of \mathbb{V} .

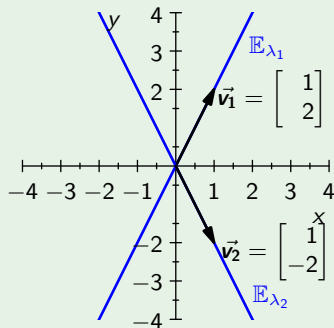
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Example 5



Example 6

For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

we had the following eigenvectors:

$$\begin{array}{ll} \lambda_1 = 2 & \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\ \lambda_2 = 1 & \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \\ \lambda_3 = -1 & \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

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Distinct Eigenvalue Theorem

Let \mathbf{A} be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \dots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is a set of linearly independent vectors.

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Proof (sketch)

If we have two eigenvalues with $\lambda_1 \neq \lambda_2$, then if the associated eigenvectors \vec{v}_1 and \vec{v}_2 were linearly dependent, we have

$$\vec{v}_2 = c\vec{v}_1 \quad \text{where } c \neq 0$$

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$$\begin{aligned}\vec{v}_2 &= c\vec{v}_1 \quad \text{where } c \neq 0 \\ \lambda_2 \vec{v}_2 &= c\lambda_2 \vec{v}_1\end{aligned}$$

But, we could also have multiplied by \mathbf{A}

$$\mathbf{A}\vec{v}_2 = c\mathbf{A}\vec{v}_1$$

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But, we could also have multiplied by \mathbf{A}

$$\begin{aligned}\mathbf{A}\vec{v}_2 &= c\mathbf{A}\vec{v}_1 \\ \lambda_2 \vec{v}_2 &= c\lambda_1 \vec{v}_1\end{aligned}$$

Which would imply that $\lambda_1 = \lambda_2$,

Example 7

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

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So, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -3$. (Note that -3 is a repeated root.)

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To find the eigenvector for $\lambda_1 = 0$ we need to solve the system:

$$\begin{bmatrix} -2 - \lambda & 1 & 1 \\ 1 & -2 - \lambda & 1 \\ 1 & 1 & -2 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_1 = v_3$ and $v_2 = v_3$. Replacing v_3 with parameter s gives

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So, we have $v_1 = -v_2 - v_3$. This means we need two parameters, $v_2 = r$ and $v_3 = s$. Which means we have two linearly independent eigenvectors.

$$\vec{v}_2 = \begin{bmatrix} -r - s \\ 1 \\ 1 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

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This means the eigenspace is

$$\mathbb{E}_{\lambda_2} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Any linear combination of these two vectors is also an eigenvector, which means that the eigenspace is a plane.

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Since this is an upper triangular matrix, we know that the eigenvalue is $\lambda = 1$, with multiplicity of 3.

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$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the eigenvector for $\lambda = 1$ we need to solve the system:

$$\begin{bmatrix} 1 - \lambda & 1 & 1 \\ 0 & 1 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Which means the eigenspace has dimension 1.

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We can continue in the same way and find that the eigenvectors are

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} -1 \\ -i \end{bmatrix}$$

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Consider the rotation transformation

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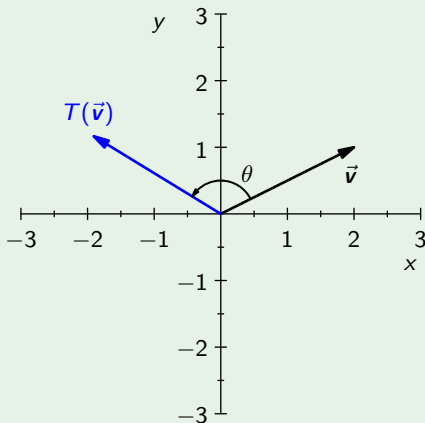
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Which means these eigenvalues rotate a vector, instead of scaling it.

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- \mathbf{A} and \mathbf{A}^T have the same characteristic polynomials and the same eigenvalues.
- If λ is an eigenvalue of an invertible matrix \mathbf{A} , then $\frac{1}{\lambda}$ is an eigenvalue of \mathbf{A}^{-1} .

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Characteristic roots of a linear homogeneous DEs are eigenvalues.

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Note

We will explore the connection between eigenvalues and solutions to differential equations next chapter.