

Conditional Probability

Colby Community College

Example 1

The `photo_classify` data set represents a machine learning algorithm classifying a sample of 1822 photos as either about fashion or not.

		truth		Total
		<i>fashion</i>	<i>not</i>	
mach_learn	<i>pred_fashion</i>	197	22	219
	<i>pred_not</i>	112	1491	1603
	Total	309	1513	1822

If a photo is actually about fashion, what is the chance the algorithm will correctly identify the photo as being about fashion?

Of the 309 fashion photos, the algorithm correctly classifies 197 of them.

$$P(\text{mach_learn is } \textit{pred_fashion} \text{ given truth is } \textit{fashion}) = \frac{197}{309} = 0.638$$

Example 2

Using the same data set as in Example 1.

		truth		Total
		<i>fashion</i>	<i>not</i>	
mach_learn	<i>pred_fashion</i>	197	22	219
	<i>pred_not</i>	112	1491	1603
	Total	309	1513	1822

If the algorithm predicts the photo as being about fashion, what is the probability is actually is?

Of the 1603 photos predicted to be about fashion, 112 we actually about fashion.

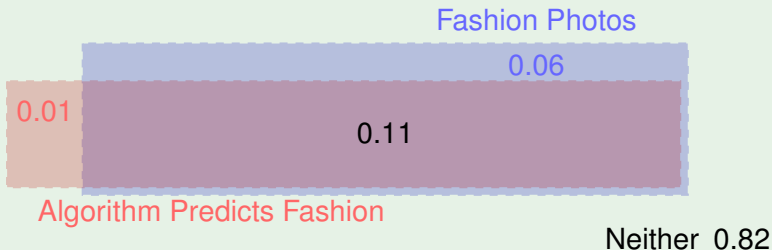
$$P(\text{truth is } \textit{fashion} \text{ given mach_learn is } \textit{pred_fashion}) = \frac{197}{219} = 0.900$$

Note

It can be helpful to draw Venn Diagrams of these contingency tables using rectangles.

Example 3

The Venn Diagram for Example 1 is:



Definition

A **marginal probability** is a probability based on a single variable without regard to other variables.

Example 4

$$P(\text{mach_learn is } \textit{pred_fashion}) = \frac{219}{1822} = 0.12$$

Definition

A probability of outcomes for two or more variables is called a **joint probability**.

Example 5

$$P(\text{mach_learn is } \textit{pred_fashion} \text{ and truth is } \textit{fashion}) = \frac{197}{1822} = 0.11$$

Note

Sometimes a comma is substituted for “and” in a joint probability.

$P(\text{mach_learn is } \textit{pred_fashion}, \text{truth is } \textit{fashion})$

means the same thing as

$P(\text{mach_learn is } \textit{pred_fashion} \text{ and truth is } \textit{fashion})$

Definition

A **table proportions** is a table that summarizes joint probabilities. The proportions are computed by dividing each count by table's total.

Example 6

The table proportions for `photo_classify` are:

	truth: <i>fashion</i>	truth: <i>not</i>	Total
<code>mach_learn: pred_fashion</code>	$\frac{197}{1822}$	$\frac{22}{1822}$	$\frac{219}{1822}$
<code>mach_learn: pred_not</code>	$\frac{112}{1822}$	$\frac{1491}{1822}$	$\frac{1603}{1822}$
Total	$\frac{309}{1822}$	$\frac{1513}{1822}$	$\frac{1822}{1822}$

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	truth: <i>fashion</i>	truth: <i>not</i>	Total
<code>mach_learn: pred_fashion</code>	0.1081	0.0121	0.1202
<code>mach_learn: pred_not</code>	0.0615	0.8183	0.8798
Total	0.1696	0.8304	1.0

Example 7

The table proportions from Example 6 make a probability distribution.

Joint Outcome	Probability
mach_learn is <i>pred_fashion</i> and truth is <i>fashion</i>	0.1081
mach_learn is <i>pred_fashion</i> and truth is <i>not</i>	0.0121
mach_learn is <i>pred_not</i> and truth is <i>fashion</i>	0.0615
mach_learn is <i>pred_not</i> and truth is <i>not</i>	0.8182

Note

Joint probabilities can be used to calculate marginal probabilities in simple cases.

Example 8

$$\begin{aligned}P(\text{truth is } \textit{fashion}) &= P(\text{mac_learn is } \textit{pred_fashion} \text{ and truth is } \textit{fashion}) \\&\quad + P(\text{mac_learn is } \textit{pred_not} \text{ and truth is } \textit{fashion}) \\&= 0.1081 + 0.0615 = 0.1696\end{aligned}$$

Definition

A **conditional probability** is a probability computed under a condition.

Example 9

$$P(\text{truth is } \textit{fashion} \text{ given mach_learn is } \textit{pred_fashion}) = \frac{197}{219} = 0.900$$

Definition

There are two parts to a conditional probability, the **outcome of interest** and the **condition**.

$P(\text{outcome of interest given condition})$
is the same as

$$P(\text{outcome of interest} \mid \text{condition})$$

Example 10

$$P(\text{truth is } \textit{fashion} \mid \text{mach_learn is } \textit{pred_fashion}) = \frac{197}{219} = 0.900$$

Note

Conditional probabilities are computed as:

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 11

$$\begin{aligned} &P(\text{truth is } \textit{fashion} \mid \text{mach_learn is } \textit{pred_fashion}) \\ &= \frac{P(\text{truth is } \textit{fashion} \text{ and } \text{mach_learn is } \textit{pred_fashion})}{P(\text{mach_learn is } \textit{pred_fashion})} \\ &= \frac{0.0615}{0.1696} \\ &= 0.3626 \end{aligned}$$

Example 12

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

		inoculated		Total	inoculated		Total
		yes	no		yes	no	
result	lived	238	5136	5374	0.0382	0.8252	0.8634
	died	6	844	850	0.0010	0.1356	0.1366
	Total	244	5980	6224	0.0392	0.9608	1.0000

What is the probability that a randomly selected person who was not inoculated died from smallpox?

$$\begin{aligned} &P(\text{result is died} \mid \text{inoculated is no}) \\ &= \frac{P(\text{result is died and inoculated is no})}{P(\text{inoculated is no})} \\ &= \frac{0.1356}{0.9608} \\ &= 0.1411 \approx 14.11\% \end{aligned}$$

Example 13

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

		inoculated		Total		inoculated		Total
		yes	no			yes	no	
result	lived	238	5136	5374		0.0382	0.8252	0.8634
	died	6	844	850		0.0010	0.1356	0.1366
	Total	244	5980	6224		0.0392	0.9608	1.0000

What is the probability that a randomly selected inoculated person died from smallpox?

$$\begin{aligned} &P(\text{result is died} \mid \text{inoculated is yes}) \\ &= \frac{P(\text{result is died and inoculated is yes})}{P(\text{inoculated is yes})} \\ &= \frac{0.0010}{0.0392} \\ &= 0.0255 \approx 2.55\% \end{aligned}$$

Example 14

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

The residents of Boston self-selected whether or not to be inoculated.

Is this study observational or experimental?

Observational

Can we infer any causal connection using this data?

No, the fact this is an observational study combined with the self-selecting bias means we cannot.

What are some potential confounding variables that might influence whether someone lived or died?

People die for many reasons, wealth determines level of medical care available, etc. . .

General Multiplication Rule

If A and B represent two outcomes or events, then

$$P(A \text{ and } B) = P(A | B) \cdot P(B)$$

Example 15

Suppose we are only given two pieces of information:

- 96.08% of Boston residents were not inoculated.
- 85.88% of Boston residents who were not inoculated ended up surviving.

Let us find the probability that a resident who was no inoculated and lived.

$$\begin{aligned} &P(\text{result is lived and inoculated is no}) \\ &= P(\text{result is lived} | \text{inoculated is no}) \cdot P(\text{inoculated is no}) \\ &= 0.8588 \cdot 0.9698 \\ &= 0.8251 \approx 82.51\% \end{aligned}$$

Example 16

Let's use

- $P(\text{inoculated is yes}) = 0.0392$
- $P(\text{result is lived} \mid \text{inoculated is yes}) = 0.9754$

to find the probability that a person was both inoculated and lived.

$$\begin{aligned} &P(\text{result is lived and inoculated is yes}) \\ &= P(\text{result is lived} \mid \text{inoculated is yes}) \cdot P(\text{inoculated is yes}) \\ &= 0.0382 \cdot 0.9754 \\ &= 0.0382 \approx 3.82\% \end{aligned}$$

Sum of Conditional Probabilities

Let A_1, A_2, \dots, A_k represent all the disjoint outcomes for a variable. Then if B is an event, possibly for another variable, we have:

$$P(A_1 | B) + P(A_2 | B) + \dots + P(A_k | B) = 1$$

Note

If an event and its complement are conditioned on the same information, then:

$$P(A | B) = 1 - P(A^C | B)$$

Example 17

If 97.54% of the inoculated people lived, what is the proportion of people that must have died?

There are only two outcomes: *lived* and *died*. Which means that $100\% - 97.54\% = 2.46\%$ people who were inoculated died.

Example 18

Let X and Y represent the outcomes of rolling two dice.

Let us compute the following:

$$\begin{aligned} P(Y = 1 \mid X = 1) &= \frac{P(Y = 1 \text{ and } X = 1)}{P(X = 1)} \\ &= \frac{P(Y = 1) \cdot P(X = 1)}{P(X = 1)} \\ &= \frac{P(Y = 1) \cdot \cancel{P(X = 1)}}{\cancel{P(X = 1)}} \\ &= P(Y = 1) \end{aligned}$$

Note

We have shown that if two events are independent, then knowing the outcome of one should provide no information about the other.

Example 19

Ron is watching a roulette table in a casino and notices that the last five outcomes were *black*. He figures that the chances of getting *black* six times in a row is very small and puts his paycheck on red.

Why is this a really bad idea?

Each spin of a roulette wheel is independent of any of the previous spins. The next spin has the exact same probability of getting black and any other.

Note

Posting the last several outcomes of a betting game is a real practice casinos use to trick people into believing the odds are in their favor. It's known as the **gambler's fallacy**.

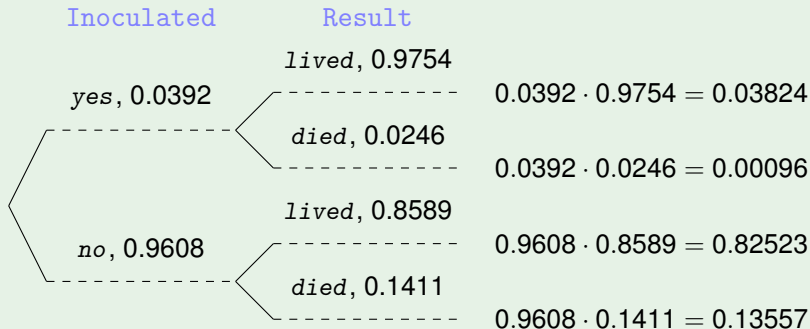
Definition

A **tree diagram** is a tool to organize outcomes and probabilities around the structure of the data.

They are most useful when two or more processes occur in a sequence and each process is conditioned on its predecessor.

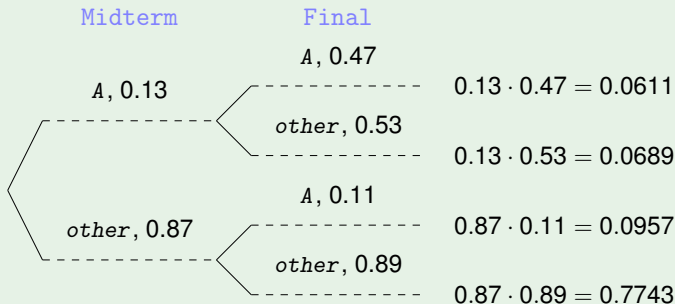
Example 20

Here is the tree diagram for `smallpox` dataset.



Example 21

Suppose that 13% of students earned an A on the midterm. Of those students that earned an A, 47% received an A on the final and 11% of the student who earned a lower grade than an A on the midterm received an A on the final.



$$\begin{aligned}P(\text{midterm is A} \mid \text{final is A}) &= \frac{P(\text{midterm is A and final is A})}{P(\text{final is A})} \\&= \frac{0.0611}{0.1568} = 0.3897 \approx 38.97\%\end{aligned}$$

Definition

A **false negative** is when a test incorrectly gives a negative result.

Definition

A **false positive** is when a test incorrectly gives a positive result.

Example 22

In Canada, about 0.35% of women over 40 will develop breast cancer in any given year. A common screening test for cancer is the mammogram, but this test is not perfect.

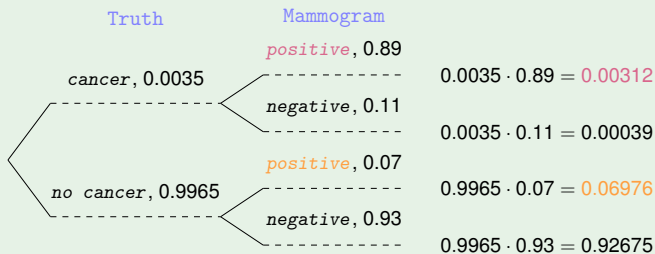
In about 11% of patients with breast cancer, the test gives a false negative. That is, the mammogram indicated the woman does not have breast cancer, when she really does have breast cancer.

In about 7% of patients that do not have breast cancer, the test gives a true negative. That is, the mammogram indicated the woman does have breast cancer, when she really doesn't have breast cancer.

Example 23

A question doctors and researchers have to answer is: If the mammogram comes back positive, what is the probability that the patient actually has breast cancer?

We have enough information to build a tree diagram:



So,

$$\begin{aligned} P(\text{has BC} \mid \text{test positive}) &= \frac{P(\text{has BC and test positive})}{P(\text{test positive})} \\ &= \frac{0.00312}{0.00312 + 0.06976} = 0.0428 \approx 4.28\% \end{aligned}$$

Note

There are times where we are given:

$$P(\text{statement about variable 1} \mid \text{statement about variable 2})$$

but we would rather know the inverted conditional probability:

$$P(\text{statement about variable 2} \mid \text{statement about variable 1})$$

Bayes' Theorem

Consider the following conditional probability for variable 1 and variable 2:

$$P(\text{outcome } A_1 \text{ of variable 1} \mid \text{outcome } B \text{ of variable 2})$$

Bayes' Theorem states that this conditional probability is the same as:

$$\frac{P(B \mid A_1) P(A_1)}{P(B \mid A_1) P(A_1) + P(B \mid A_2) P(A_2) + \cdots + P(B \mid A_k) P(A_k)}$$

where A_2, A_3, \dots, A_k represent all other possible outcomes of variable 1.

Example 24

In Example 23, we computed $P(\text{has BC} \mid \text{test positive})$ using a tree diagram.

We could have also used Bayes' Theorem:

$$\begin{aligned} &P(\text{has BC} \mid \text{test pos}) \\ &= \frac{P(\text{test pos} \mid \text{has BC}) P(\text{has BC})}{P(\text{test pos} \mid \text{has BC}) P(\text{has BC}) + P(\text{test pos} \mid \text{no BC}) P(\text{no BC})} \\ &= \frac{0.89 \cdot 0.0035}{0.89 \cdot 0.0035 + 0.07 \cdot 0.9965} = \frac{0.03115}{0.07287} = 0.0428 \approx 4.28\% \end{aligned}$$

Note

This strategy of updating beliefs using Bayes' Theorem is the foundation of an entire branch of statistics called **Bayesian statistics**.