

# The Harmonic Oscillator

Department of Mathematics

Salt Lake Community College

## Newton's Dot Notation

Scientists and Engineers who work with many variables where the independent variable is always  $t$  commonly use dot notation:

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{dx^2}{d^2t} \quad \text{and} \quad \dddot{x} = \frac{dx^3}{d^3t} \quad \text{and} \quad \ddot{\ddot{x}} = \frac{dx^4}{d^4t}$$

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## Definition

A very important DE is the second-order homogeneous equation

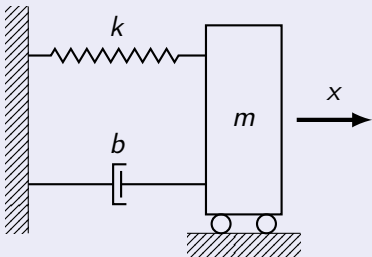
$$m\ddot{x} + b\dot{x} + kx = 0$$

where  $m > 0$ ,  $b$ , and  $k$  are constants.

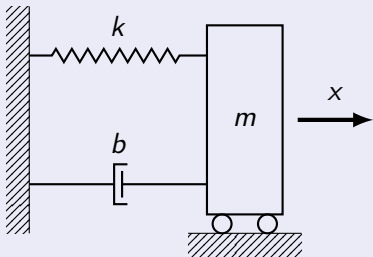
This models a class of phenomena called **damped harmonic oscillation**.

## The Mass-Spring System

We will model the Mass-Spring system using Newton's Second Law of Motion,  $F = m\ddot{x}$ , where  $F$  is the sum of the following forces:



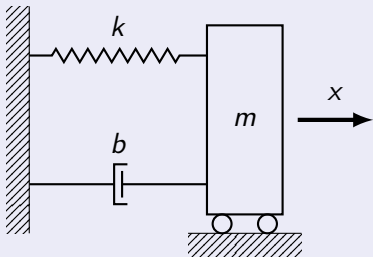
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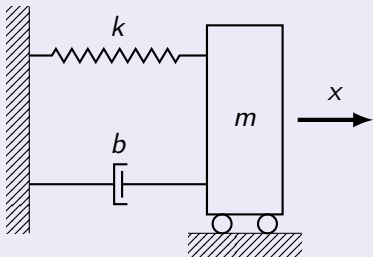
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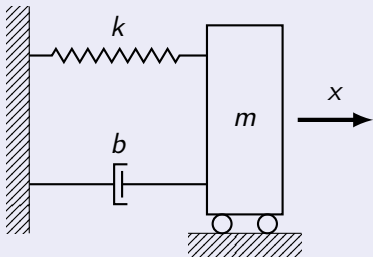
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- Any external "driving" forces.  
 $F_{\text{external}} = f(t)$  where  $f(t)$  is the sum of all external forces.

Summing these forces gives:

$$\begin{array}{rclclcl} \text{mass} \times \text{acceleration} & = & F_{\text{restoring}} & + & F_{\text{damping}} & + & F_{\text{external}} \\ m\ddot{x} & = & -kx & - & b\dot{x} & + & f(t) \end{array}$$



## Simple Harmonic Oscillator

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- When  $b = 0$ , the motion is called **undamped**; otherwise it is **damped**.
- If  $f(t) = 0$  for all  $t$ , then the equation is homogeneous:

$$m\ddot{x} + b\dot{x} + kx = 0$$

and the motion is called **unforced**, **undriven**, or **free**; otherwise it is called **forced** or **driven**.

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$$k = \frac{1 \text{ newton}}{0.25 \text{ meter}} = 4 \frac{\text{newton}}{\text{meter}}$$

We also measure the damping force of the object sliding on the table to be 0.5 newtons when the velocity is 0.25 meters per second.

$$b = \frac{0.5 \text{ newton}}{0.25 \frac{\text{meter}}{\text{second}}} = 2 \frac{\text{newton second}}{\text{meter}}$$

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$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= f(t) \\ \ddot{x} + 2\dot{x} + 4x &= 0 \quad x(0) = 0.5, \quad \dot{x}(0) = 0 \end{aligned}$$

Notice that a second-order DE requires **two** initial conditions.

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Energy	joule	erg	foot-pound (ft-lb)
Gravity (Earth)	$9.8 \frac{\text{m}}{\text{s}^2}$	$980.665 \frac{\text{cm}}{\text{s}^2}$	$32 \frac{\text{ft}}{\text{s}^2}$

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Comparing

$$(\sin(\omega_0 t))'' = -\omega_0^2 \sin(\omega_0 t) \quad \text{and} \quad \ddot{x} = -\frac{k}{m}x$$

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## Note

Another solutions is  $x(t) = \cos(\omega_0 t)$ .



## Solution of the Undamped Unforced Oscillator

For the undamped unforced oscillator

$$m\ddot{x} + kx = 0$$

we know two solutions:

$$x(t) = \cos(\omega_0 t) \quad \text{and} \quad x(t) = \sin(\omega_0 t) \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

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The Superposition Principle tells us that any linear combination of these two solutions is itself a solution. Thus, for  $c_1, c_2 \in \mathbb{R}$ , the family of solutions is

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### Note

We will see next section that these cover all solutions.

## Alternate Form of the Undamped Unforced Oscillator Solution

Solutions to the undamped unforced oscillator may also be expressed as

$$x(t) = A \cos(\omega_0 t - \delta)$$

- $A$  is the **amplitude**.
- $\delta$  is the **phase angle**, measured in radians.
- The motion has **circular frequency**  $\omega_o = \sqrt{\frac{k}{m}}$ , measured in  $\frac{\text{radians}}{\text{second}}$ .
- The **natural frequency** is  $f_0 = \frac{\omega_0}{2\pi}$ .
- The **period** is  $\frac{1}{f_0} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$ , measured in seconds.
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## Converting Between the Two Forms

The translation is given by

$$A = \sqrt{c_1^2 + c_2^2}, \quad \tan(\delta) = \frac{c_2}{c_1}$$

and

$$c_1 = A \cos(\delta), \quad c_2 = A \sin(\delta)$$

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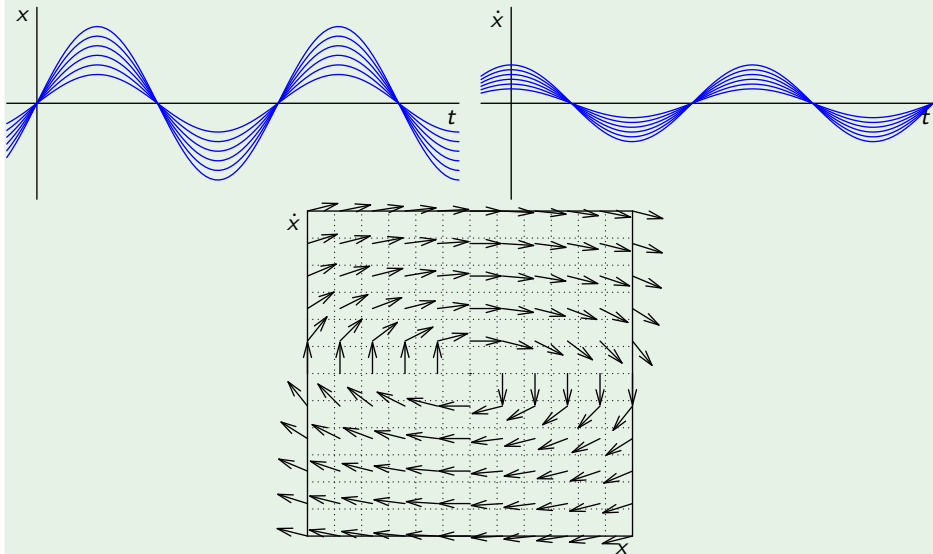
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Substituting  $t = 0$ ,  $x(0) = 0$ , and  $\dot{x}(0) = 1$  into this system gives the solution  $c_1 = 0$  and  $c_2 = 1$ .



## Example 4

Let us look at some plots concerning  $\ddot{x} + 0.25x = 0$ :



## Phase Portraits

For any autonomous second-order differential equation

$$\ddot{x} = F(x, \dot{x})$$

the **phase plane** is the two-dimensional graph with  $x$  and  $\dot{x}$  axes.

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## Note

Phase portraits can be graphed *without* solving the DE.

## Definition

The second-order differential equation

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is equivalent to the system of first-order equations:

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To draw the vector field for  $\ddot{x} + 0.25x = 0$  we can build the system:

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## Definition

The second-order differential equation

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

is equivalent to the system of first-order equations:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \ddot{x} = \frac{f(t)}{m} - \frac{k}{m}x - \frac{b}{m}y\end{aligned}$$

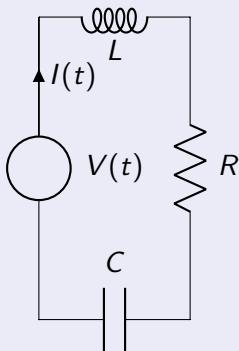
## Example 5

To draw the vector field for  $\ddot{x} + 0.25x = 0$  we can build the system:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -0.25x\end{aligned}$$

Then, pplane may be used to plot the phase portrait.

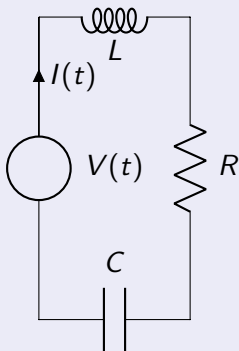
## Electrical Circuits



The current  $I$  in a wire, measured in *amperes*, is the flow of charge  $Q$ . That is, the current is the rate of change of the charge

$$I(t) = \dot{Q}(t)$$

## Electrical Circuits

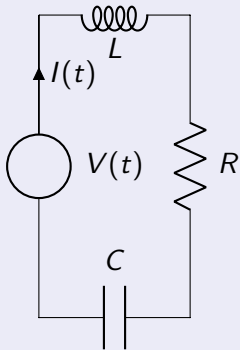


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**Kirchoff's Voltage Law** tell us that the input voltage  $V(t)$  is the sum of voltage drops around the circuit. In our circuit, we have three such voltage drops.

## Electrical Circuits

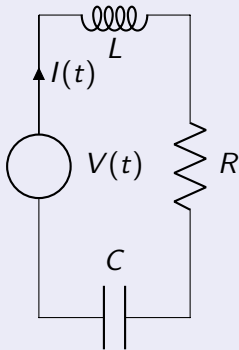


**Drop across a Resistor:** By **Ohm's Law**, the voltage drop across a resistor is proportional to the current passing through it.

$$V_R(t) = R \cdot I(t)$$

Where  $R$  is the **resistance** of the resistor and is measured in *ohms*.

## Electrical Circuits

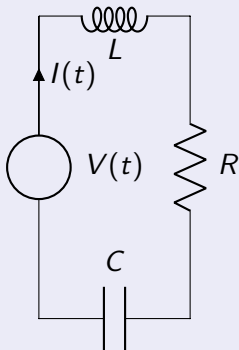


**Drop across an Inductor:** By Faraday's Law, the voltage drop across an inductor is proportional to the time rate of change of the current passing through it.

$$V_L(t) = L \cdot \dot{I}(t)$$

where  $L$  is the **inductance** and is measured in *henries*.

## Electrical Circuits

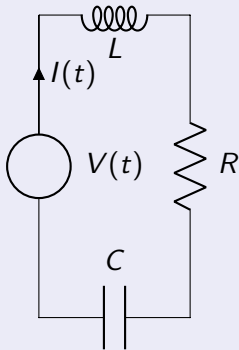


**Drop across a Capacitor:** The voltage drop across a capacitor is proportional to the charge  $Q(t)$  on the capacitor.

$$V_C(t) = \frac{1}{C} Q(t) = \frac{1}{C} \int I(t) dt$$

where  $C$  is the **capacitance** of the capacitor and is measured in *farads*.

## Electrical Circuits



Thus, the voltage drop across the circuit is

$$V(t) = R \cdot I + L \cdot i + \frac{1}{C} \int I(t) dt$$

This is called an **integro-differential equation** because it contains both a derivative and an integral.

Using the fact that  $I(t) = \dot{Q}(t)$  we can build the following equations.



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$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)$$

If there is no voltage source ( $V(t) = 0$ ), then

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0$$

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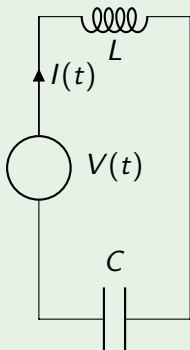
### Series Circuit Equation (Current)

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}(t)$$

If there is no voltage source ( $V(t) = 0$ ), then

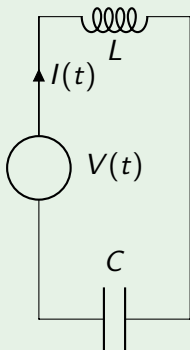
$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0$$

## Example 6



Consider a circuit composed of a capacitor and inductor hooked up in series. Suppose that at  $t = 0$  a charge  $Q_0$  is put on the capacitor.

## Example 6

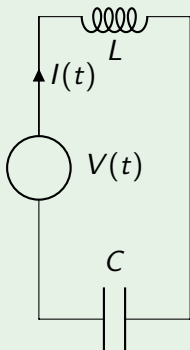


Consider a circuit composed of a capacitor and inductor hooked up in series. Suppose that at  $t = 0$  a charge  $Q_0$  is put on the capacitor.

The IVP is

$$L\ddot{Q} + \frac{1}{C}Q = 0, \quad Q(0) = Q_0, \quad \dot{Q}(0) = 0$$

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Thus, the solution is

$$Q(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

where

$$\omega_0 = \sqrt{\frac{1}{LC}}$$