Laplace Transforms

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

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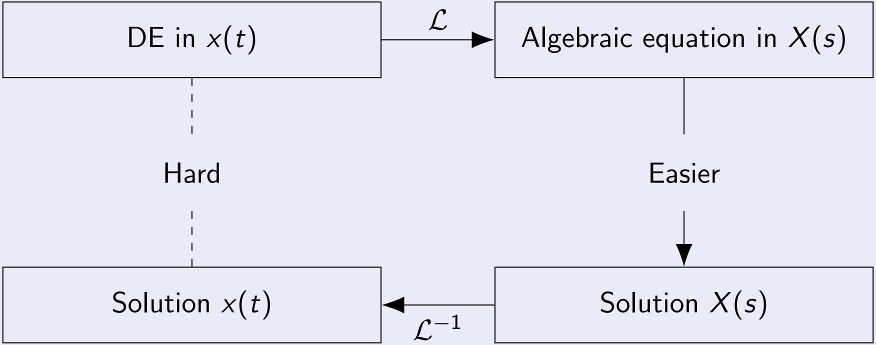
DE in $x(t)$

Hard

Solution $x(t)$

In this chapter we shall study the Laplace transformation, which will allow us to reframe some types of hard problems into equivalent easier problems.



Laplace Transform

The **Laplace Transform** $\mathcal{L}\{f(t)\}$ of a suitable function $f(t)$ defined on $[0, \infty)$ is the function $F(s)$ given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t)dt$$

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Linearity of the Laplace Transform

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then by the properties of integrals

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(t) \quad \text{for } a, b \in \mathbb{C}$$

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Existence Theorem for Laplace Transform

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

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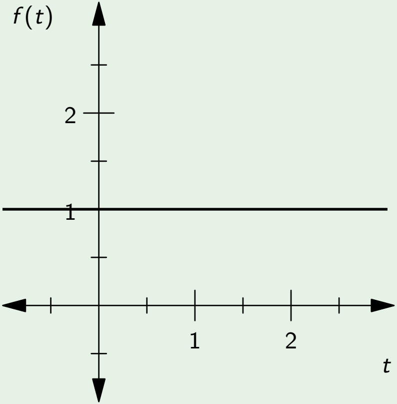

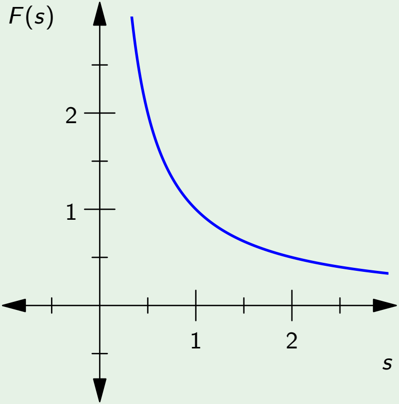

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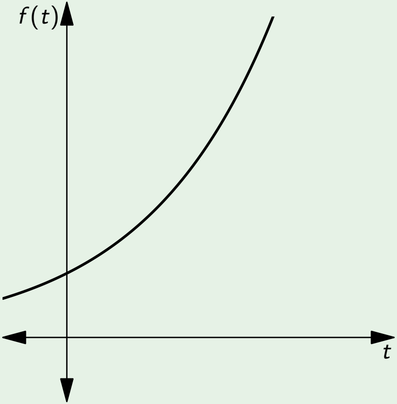

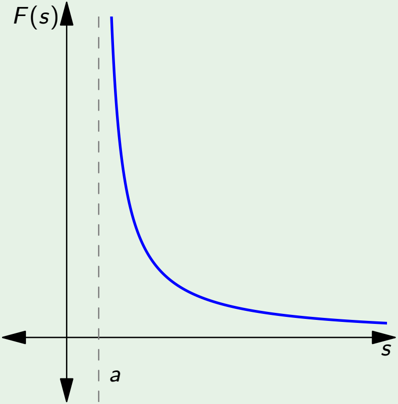

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&= \frac{2s+10}{s(s+2)}
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So, if we equate the real and imaginary parts, we get:

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \quad \text{and} \quad \mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

Inverse Laplace Transform

A function $f(t)$ whose transform if $F(s)$ is called the **inverse Laplace transform** of F, and we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

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Thus,

$$f(t) = 4e^{-t} - 2e^{3t}$$

Some Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\}$	
1	$\dfrac{1}{s}$	$s > 0$
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0, \ n \in \mathbb{N}^+$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a, \ n \in \mathbb{N}^+$
$\sin(bt)$	$\dfrac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\dfrac{s}{s^2+b^2}$	$s > 0$

Some More Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\}$	
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$	$s>a$
$\sinh(bt)$	$\dfrac{b}{s^2-b^2}$	$s>\lvert b\rvert$
$\cosh(bt)$	$\dfrac{s}{s^2-b^2}$	$s>\lvert b\rvert$

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$$F(s) = 2\underbrace{\left(\frac{1}{s-1}\right)}_{\mathcal{L}\{e^t\}} - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{\sqrt{5}}{s^2 + \left(\sqrt{5}\right)^2}\right)}_{\mathcal{L}\{\sin\left(\sqrt{5}t\right)}$$

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Thus, $f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^t - \dfrac{1}{\sqrt{5}}\sin\left(\sqrt{5}t\right)$.

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&= \mathcal{L}\{e^{-2t}\cos(3t)\} - \frac{1}{3}\mathcal{L}\{s^{-2t}\sin(3t)\}
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&= \mathcal{L}\{e^{-2t}\cos(3t)\} - \frac{1}{3}\mathcal{L}\{s^{-2t}\sin(3t)\}
\end{aligned}$$

Thus, by linearity we have $f(t) = e^{-2t}\cos(3t) - \frac{1}{3}s^{-2t}\sin(3t)$.