# Complex Characteristic Roots

Adam Wilson

Salt Lake Community College

# Complex Characteristic Roots ($\Delta < 0$)

## Solution for Complex Characteristic Roots

For $\Delta < 0$, the characteristic roots of the DE

$$ay'' + by' + cy = 0$$

are

$$r_1 = \alpha + i\beta = -\frac{b}{2a} + i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$

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The functions $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are linearly independent solutions, and the general solution is given by

$$y(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$$

where $c_1$ and $c_2$ are arbitrary constants determined by the initial conditions.

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The set $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ forms a basis for the solution space $\mathbb{S}$.

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The set $\left\{e^{2t}\cos(3t), e^{\alpha t}\sin(3t)\right\}$ is a basis of the solution space $\mathbb{S}$, and $\mathbf{dim}\,\mathbb{S} = 2$.

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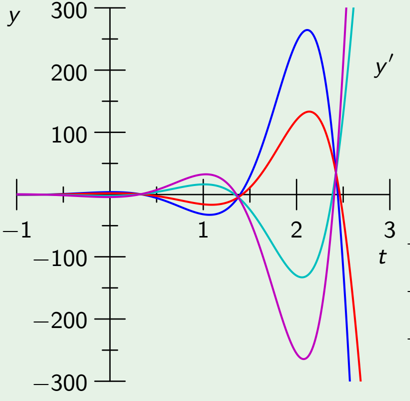
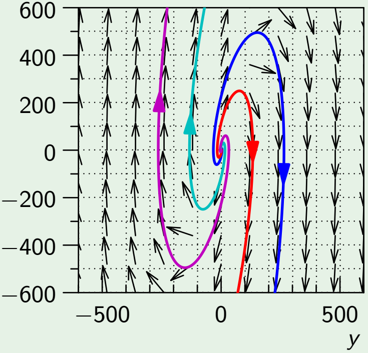



(a) Time Series




(b) Phase Portrait

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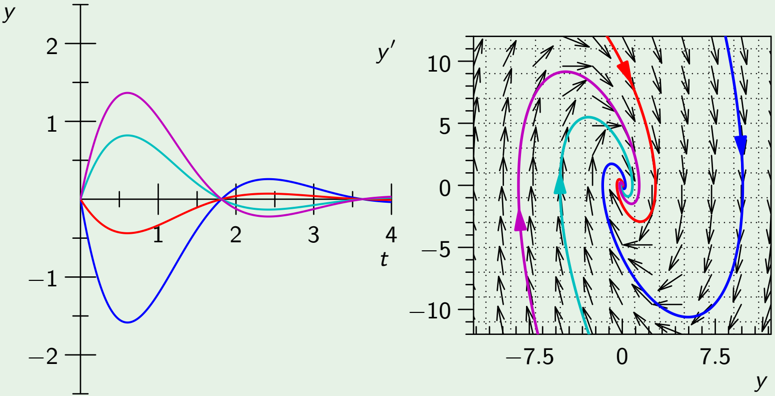


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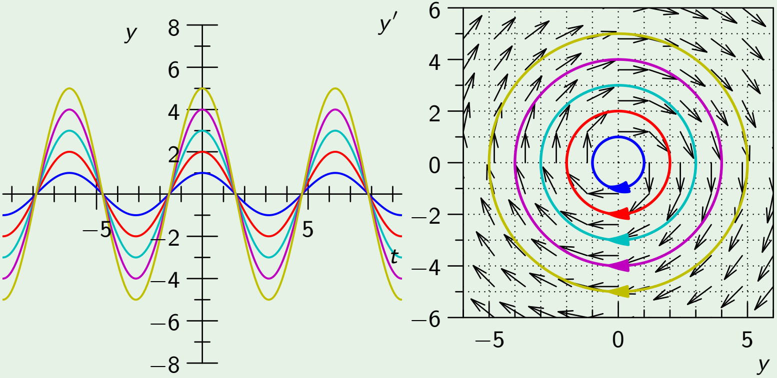


(a) Time Series (b) Phase Portrait

# Damped Mass-Spring Systems

## Underdamped Mass-Spring System

The motion of a mass-spring system is called **underdamped** when we have $\Delta = b^2 - 4mk < 0$. Both characteristic roots are complex and the solutions are given by

$$x(t) = e^{-\frac{b}{2m}} \left( c_1 \cos\left(\omega_d t\right) + c_2 \sin\left(\omega_d t\right)\right), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

# Damped Mass-Spring Systems

## Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos\left(\omega_d t - \delta\right), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where $A$ and $\delta$ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** $\delta$
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** $\omega_d$
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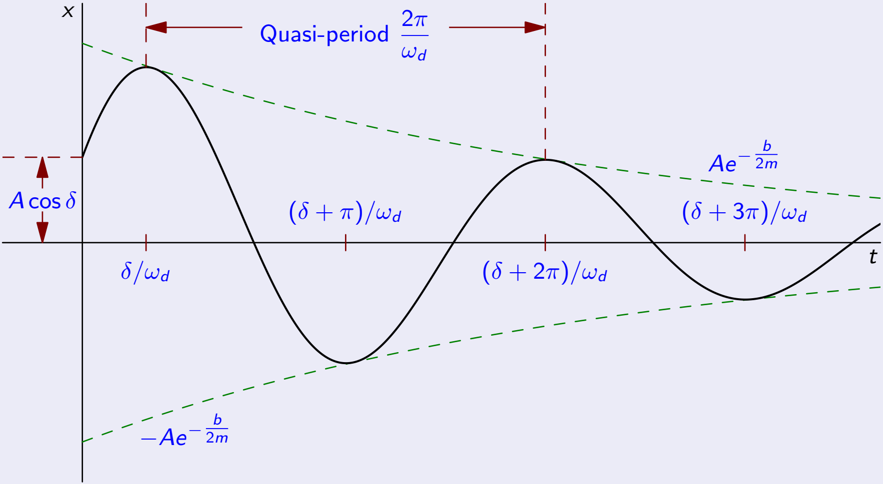

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Which means $\alpha = -\dfrac{1}{2}$ and $\beta = \dfrac{\sqrt{3}}{2}$.

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The general solution is

$$x(t) = e^{-\frac{t}{2}} \left( c_1 \cos\left( \frac{\sqrt{3}}{2} t \right) + c_2 \sin\left( \frac{\sqrt{3}}{2} t \right) \right)$$

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We can then calculate

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If we substitute in the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, we find that $c_1 = 1$ and $c_2 = \dfrac{1}{\sqrt{3}}$.

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Thus, the solution to the IVP is

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In alternate polar form

$$x(t) = \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)$$

Where

$$A = \sqrt{1^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{\frac{1}{\sqrt{3}}}{1}\right) = \frac{\pi}{6}$$

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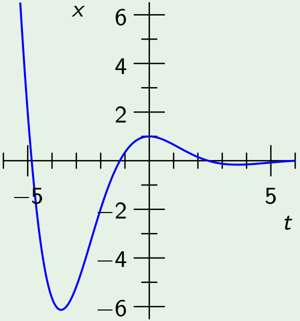


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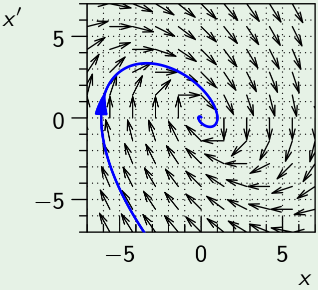
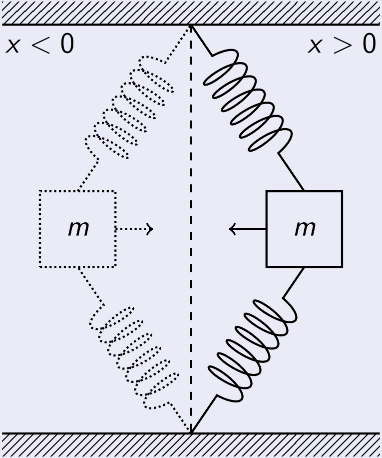



(b) Phase Portrait

## The Guitar String: A Qualitative Analysis

The vibration of a guitar string can be described as a damped oscillator.



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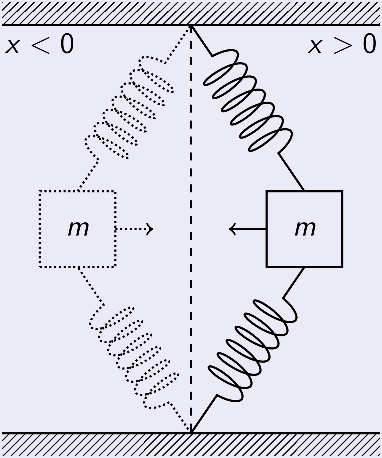


The motion of this spring is given by

$$\ddot{x} + \omega_0^2 x = 0$$

where $\omega_0$ is the circular frequency at which the string vibrates. (In music, the frequency $f_0 = \frac{\omega_o}{2\pi}$ is often used. A middle C has 512 vibrations per second.)

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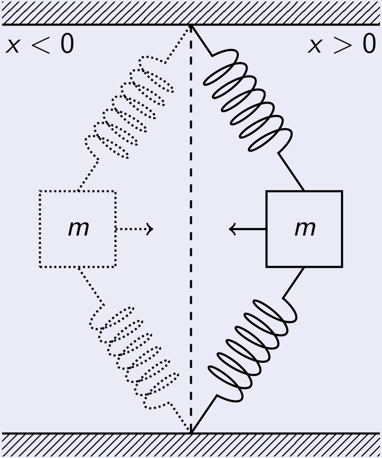


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# The Guitar String: A Qualitative Analysis

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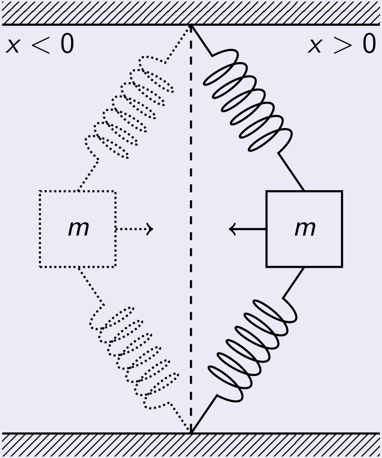


The motion of this spring is given by

$$\ddot{x} + \omega_0^2 x = 0$$

where $\omega_0$ is the circular frequency at which the string vibrates. (In music, the frequency $f_0 = \frac{\omega_o}{2\pi}$ is often used. A middle C has 512 vibrations per second.) Because there is no damping in this model, the sound will last forever. When a guitar string is plucked, it has the $x(0) = x_0$ and $\dot{x}(0) = 0$ for initial conditions.

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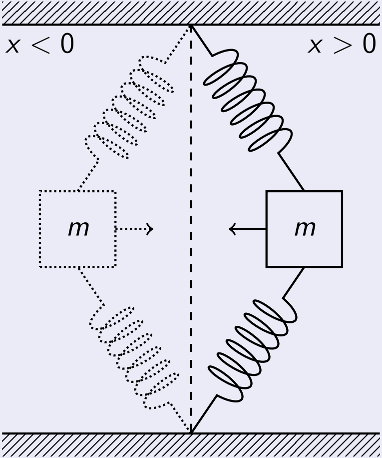


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Let use next consider a guitar string with damping.

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If we pluck the string, which means $x(0) = 5$ and $\dot{x}(0) = 0$, we find that $c_1 = 5$ and $c_2 = 1$.

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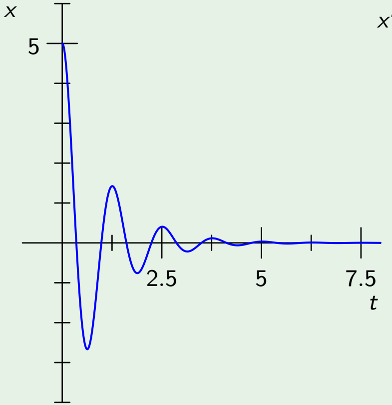
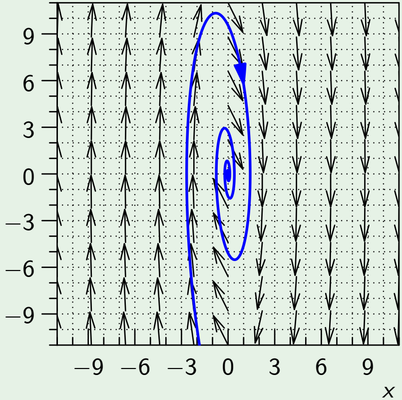



(a) Time Series




(b) Phase Portrait

# Real and Complex Characteristic Roots

## Solutions to the Second-Order Linear DE with Constant Coefficients

The differential equation

$$ay'' + by' + cy = 0$$

has the characteristic equation

$$ar^2 + br + c = 0$$

The quadratic formula gives rise to three different general solutions, depending on the discriminant $\Delta = b^2 - 4ac$.

| | Characteristic Roots | General Solution |
|---|---|---|
| $\Delta > 0$ | $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ |
| $\Delta = 0$ | $r = -\dfrac{b}{2a}$ | $y = c_1 e^{rt} + c_2 t e^{rt}$ |
| $\Delta < 0$ | $r_1, r_2 = \alpha \pm \beta$ | $y = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$ |

$$\alpha = -\frac{b}{2a},\ \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

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$$r_1 = 2, \quad r_2 = -2, \quad r_3 = 2i, \quad r_4 = -2i$$

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Thus, $\{e^{2t}, e^{-2t}, \cos(2t), \sin(2t)\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos(2t) + c_4 \sin(2t)$$

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Thus, $\{e^t, te^t, e^{-3t}\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = c_1 e^t + c_2 te^t + c_3 e^{-3t}$$

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Consider the fifth-order DE

$$\frac{d^5y}{dt^5} + 3\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = 0$$

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Which has the characteristic solutions

$$r_1 = -1, \quad r_2 = -1, \quad r_3 = -1, \quad r_4 = 0, \quad r_5 = 0$$

Thus, $\{e^{-t}, te^{-t}, t^2e^{-t}, 1, t\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = \underbrace{(c_1 + c_2 t + c_3 t^2)}_{\text{for triple root}} e^{-t} + \underbrace{(c_4 + c_5 t)}_{\text{for double root}}$$

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Thus, $\{\cos(2t), t\cos(2t), \sin(2t), t\sin(2t)\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = \underbrace{(c_1 + c_2 t)}_{\text{for double root}}\cos(2t) + \underbrace{(c_3 + c_4 t)}_{\text{for double root}}\sin(2t)$$