Linear Systems with Nonreal Eigenvalues

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Complex Eigenvalues and Eigenvectors

For a real matrix $\boldsymbol{A}$, nonreal eigenvalues come in complex conjugate pairs,

$$\lambda_1 = \alpha + \beta i \quad \text{and} \quad \lambda_2 = \alpha - \beta i$$

with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$.

The corresponding eigenvectors are also complex conjugate pairs and can be written

$$\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i \quad \text{and} \quad \vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{p}} - \vec{\boldsymbol{q}} i$$

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Note

We only need to find one eigenvalue/eigenvector pair.

Example

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

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$$\vec{\boldsymbol{v}}_{\mathbf{1}} = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\mathbf{2}} = \overline{\vec{\boldsymbol{v}}_{\mathbf{1}}} = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$$

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Alternately, we can write

$$\vec{\boldsymbol{v}} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{\boldsymbol{p}}} \pm i \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{\boldsymbol{q}}}$$

Example

Let us consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Which has nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and corresponding eigenvectors $\vec{\boldsymbol{v_1}}$ and $\vec{\boldsymbol{v_2}}$. We can then write:

$$\vec{\boldsymbol{x}} = c_1 e^{\lambda_1 t}\vec{\boldsymbol{v_1}} + c_2 e^{\lambda_2 t}\vec{\boldsymbol{v_2}}.$$

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So, for eigenvalue $\lambda_1 = \alpha + \beta i$ and corresponding eigenvector $\vec{\boldsymbol{v_1}} = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i$ we get the solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t} \vec{\boldsymbol{v_1}} = e^{\alpha + \beta i} \left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i \right)$$

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So, for eigenvalue $\lambda_1 = \alpha + \beta i$ and corresponding eigenvector $\vec{\boldsymbol{v_1}} = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i$ we get the solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha + \beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

Just like with second-order systems, we shall find that the real and imaginary parts of the complex solution above are both real and linearly independent solutions of the system.

Example

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\mathrm{Re}}(t) + \vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\mathrm{Im}} \neq \vec{\mathbf{0}}$.

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Separately equating the real and imaginary parts, we get:

$$\vec{\boldsymbol{x}}'_{\mathrm{Re}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) \quad \text{and} \quad \vec{\boldsymbol{x}}'_{\mathrm{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

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Thus, $\vec{\boldsymbol{x}}_{\text{Re}}(t)$ and $\vec{\boldsymbol{x}}_{\text{Im}}(t)$ are separate real solutions to the system.

Example

For the complex solution

$$\vec{\boldsymbol{x}}_{\mathbf{1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v}}_{\mathbf{1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

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$$\begin{aligned} e^{\lambda_1 t}\vec{\boldsymbol{v}}_{\mathbf{1}} &= e^{\alpha t+\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}e^{\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \end{aligned}$$

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&= \underbrace{e^{\alpha t}(\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q})}_{\vec{x}_{\text{Re}}(t)} + i\,\underbrace{e^{\alpha t}(\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q})}_{\vec{x}_{\text{Im}}(t)}
\end{aligned}$$

Example

Since $\vec{\boldsymbol{x}}_{\text{Re}}(t)$ and $\vec{\boldsymbol{x}}_{\text{Im}}(t)$ are linearly independent solutions they must span the solution space. Thus, the general solution, for $c_1, c_2 \in \mathbb{R}$, is

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Any solutions derived from λ_2 and $\vec{\boldsymbol{v}}_{\boldsymbol{2}}$ will be linear combinations of $\vec{\boldsymbol{x}}_{\text{Re}}(t)$ and $\vec{\boldsymbol{x}}_{\text{Im}}(t)$.

Solving 2×2 DE System with Nonreal Eigenvalues

For the two-dimensional linear homogeneous differential equation $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with real matrix $\boldsymbol{A}$, eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta$ $(\beta \neq 0)$ the general solution can be found using the following steps:

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1. For one eigenvector λ_1, find it's corresponding eigenvector $\vec{\boldsymbol{v_1}}$. The second eigenvalue λ_2 and it's eigenvector $\vec{\boldsymbol{v_2}}$ are complex conjugates of the first. The eigenvectors are of the form $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}} = \vec{\boldsymbol{p}} \pm i\vec{\boldsymbol{q}}$.

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2. Construxt the linearly independent real ($\vec{x}_{\mathrm{Re}}$) and imaginary ($\vec{x}_{\mathrm{Im}}$) parts of the solutions as follows:

$$\vec{x}_{\mathrm{Re}}(t) = e^{\alpha t}\left(\cos(\beta t)\,\vec{p} - \sin(\beta t)\,\vec{q}\right)$$
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3. The general solution is

$$\vec{\boldsymbol{x}}(t) = c_1\vec{\boldsymbol{x}}_{\text{Re}}(t) + c_2\vec{\boldsymbol{x}}_{\text{Im}}(t)$$

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Thus

$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = e^{5t}\cos(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{5t}\sin(2t)\begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t}\begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix}$$

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And general solution

$$\vec{\boldsymbol{x}}(t) = e^{5t}\left(c_1\begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{bmatrix}\right)$$

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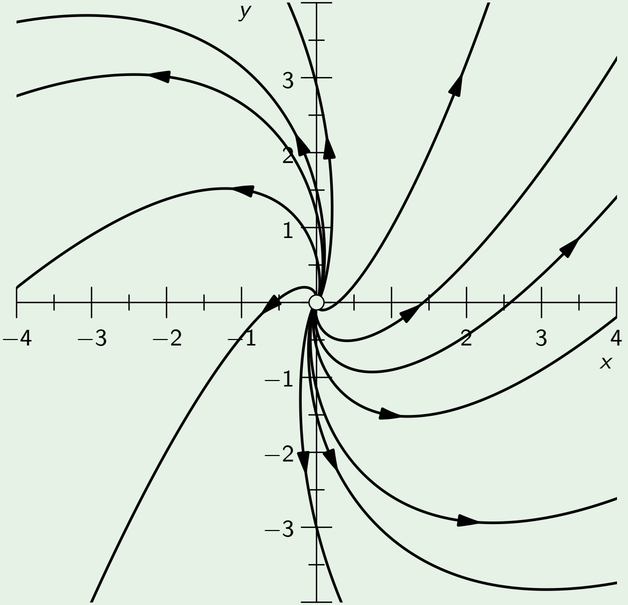

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Thus

$$\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) = e^{-t}\cos(2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-t}\sin(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

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And general solution

$$\vec{\boldsymbol{x}}(t) = e^{-t}\left(c_1\begin{bmatrix} \cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{bmatrix}\right)$$

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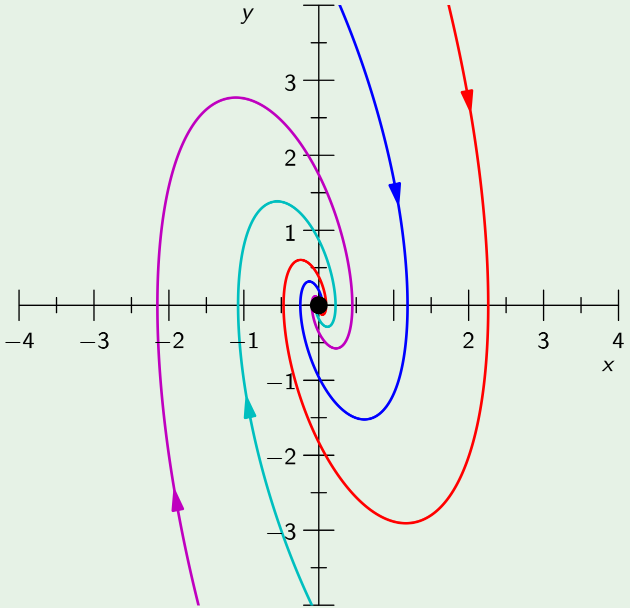

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Thus

$$\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) = \cos(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{bmatrix}$$

$$\vec{\boldsymbol{x}}_{\mathrm{Im}}(t) = \sin(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \end{bmatrix}$$

Example

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = 0 \pm 3i$ and

$$\vec{\boldsymbol{v}}_{\boldsymbol{1}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Thus

$$\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) = \cos(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{bmatrix}$$

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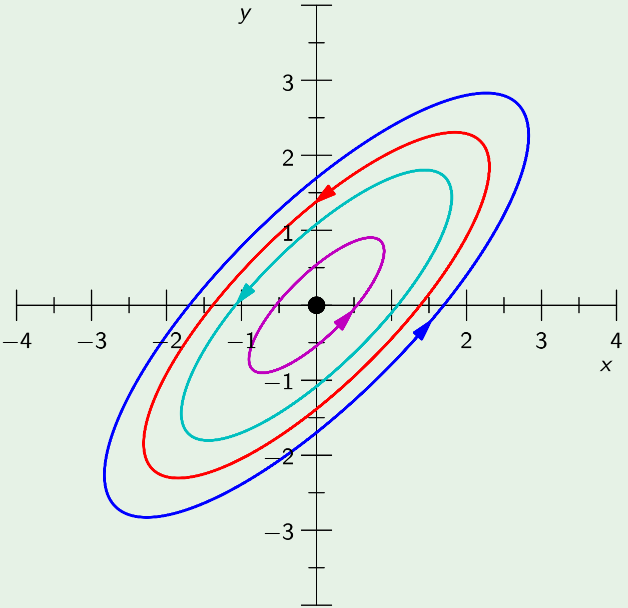
And general solution

$$\vec{\boldsymbol{x}}(t) = c_1 \begin{bmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \end{bmatrix}$$

Example



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- An **stable equilibrium** is one where the trajectories neither grow nor decay, they just circle in a periodic motion. (Since $\alpha = 0$.)

Nullclines for a DE System

For a two-dimensional system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

- The v-**nullcline** is the set of all points with vertical slope, which occur on the curve obtained by solving $x' = f(x,y) = 0$.
- The h-**nullcline** is the set of all points with horizontal slope, which occur on the curve obtained by solving $y' = g(x,y) = 0$.

When an h-nullcline and an v-nullcline intersect, an **equilibrium** occurs.

Interpreting the Solutions

For $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and comple eigenvectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i$, arrange the components of the solution as

$$\begin{bmatrix} \vec{\boldsymbol{x}}_{\text{Re}} \\ \vec{\boldsymbol{x}}_{\text{Im}} \end{bmatrix} = \underbrace{e^{\alpha t}}_{\text{expansion}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{q}} \end{bmatrix}}_{\text{tilt and shape}}$$

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1. The first factor $e^{\alpha t}$ determines *expansion* or *contraction*.
   - If $\alpha > 0$, then trajectories spiral outward, representing unbound growth.
   - If $\alpha < 0$, then trajectories spiral inward, decay to zero.
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3. The third factor, containing $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{q}}$, determines the *tilt* and *shape* of the *elliptical trajectories* that would result with $\alpha = 0$.