Eigenvalues and Eigenvectors

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Example 1

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{\boldsymbol{u}}) = \boldsymbol{A}\vec{\boldsymbol{u}}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

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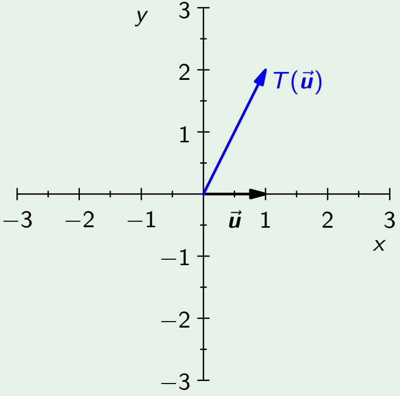
We can see how T maps a few vectors:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \longrightarrow$$



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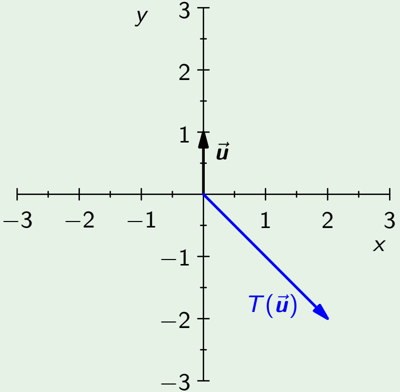

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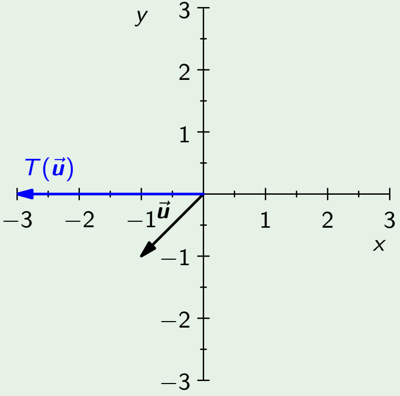
We can see how T maps a few vectors:

$$T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \qquad \longrightarrow$$



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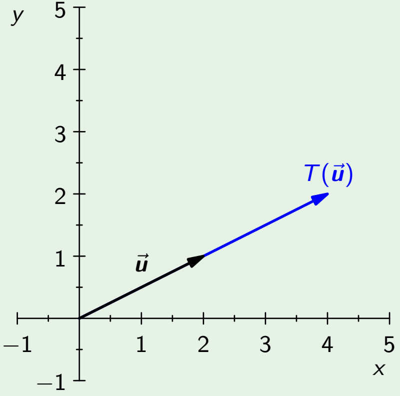
But something interesting happens for some special vectors.

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \longrightarrow$$



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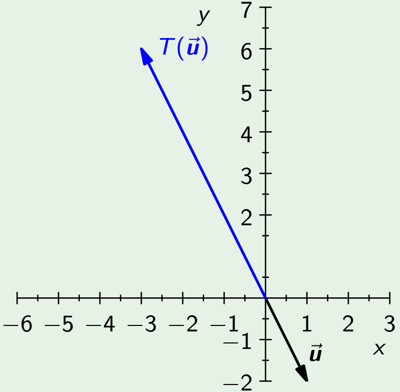
$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \longrightarrow$$



Eigenvalues and Eigenvectors

Let $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation from vector space $\mathbb{V}$ into itself. A scalar λ is a **eigenvalue** of T if there is a *nonzero* vector $\vec{\boldsymbol{v}} \in \mathbb{V}$ such that

$$T(\vec{\boldsymbol{v}}) = \lambda \vec{\boldsymbol{v}}$$

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If the linear transformation T is represented by an $n \times n$ matrix $\boldsymbol{A}$, where $\mathbb{V} = \mathbb{R}^n$ and $T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}}$, then λ and $\vec{\boldsymbol{v}}$ are characterized by the equation

$$\boldsymbol{A}\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}$$

Computing Eigenvalues and Eigenvectors

If $\boldsymbol{A}$ is a $n \times n$ matrix, and $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, then

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The polynomial in λ denoted by

$$p(\lambda) = |\boldsymbol{A} - \lambda\boldsymbol{I}_n|$$

is called the **characteristic polynomial** of $\boldsymbol{A}$.

Summary of Steps for Finding Eigenvalues and Eigenvectors

1. Write the characteristic equation $|\boldsymbol{A} - \lambda \boldsymbol{I}_n| = 0$.
2. Solve the characteristic equation for λ.
3. For each eigenvalue λ_i, find the corresponding eigenvector $\vec{\boldsymbol{v}}_{\boldsymbol{i}}$ by solving the system of equations

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Eigenvectors are *not* unique. An eigenvector is just a direction, any nonzero multiple of $\vec{\boldsymbol{v}_i}$ works just as well.

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For large matrices these steps become cumbersome, so computer algebra systems are often employed.

Example 2

In our first example we saw two eigenvectors, let us verify these using the characteristic equation.

$$\left| \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

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$$\begin{bmatrix} 1-(3) & 1 \\ 4 & 1-(3) \end{bmatrix} \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}} \rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{\boldsymbol{v}}_{\boldsymbol{1}} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 3

Let us find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

The characteristic equation is

$$|\boldsymbol{A} - \lambda \boldsymbol{I}_2| = 0 \quad \to \quad \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \quad \to \quad (1-\lambda)^2 - 4 = 0$$

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To find the eigenvector for λ_2 we need to solve

$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

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Let us find the eigenvalues and eigenvectors for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

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Which simplifies to:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

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Which simplifies to:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$
$$(\lambda - 2)(\lambda - 1)(\lambda + 1) = 0$$

So, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$.

Example 4

Let us find the eigenvalues and eigenvectors for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

To find the eigenvector for $\lambda_1 = 2$ we need to solve the system:

$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, we have $v_1 = v_3$ and $v_2 = 3v_3$. Replacing v_3 with parameter s gives

$$\vec{\boldsymbol{v_1}} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Example 4

Let us find the eigenvalues and eigenvectors for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

To find the eigenvector for $\lambda_2 = 1$ we need to solve the system:

$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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To find the eigenvector for $\lambda_2 = 1$ we need to solve the system:

$$\begin{bmatrix} 1-(1) & 1 & -2 \\ -1 & 2-(1) & 1 \\ 0 & 1 & -1-(1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_1 = 3v_3$ and $v_2 = 2v_3$. Replacing v_3 with parameter s gives

$$\vec{\boldsymbol{v_2}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

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To find the eigenvector for $\lambda_3 = -1$ we need to solve the system:

$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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To find the eigenvector for $\lambda_3 = -1$ we need to solve the system:

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To find the eigenvector for $\lambda_3 = -1$ we need to solve the system:

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So, we have $v_1 = v_3$ and $v_2 = 0$. Replacing v_3 with parameter s gives

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Special Cases

Triangular Matrices: The eigenvalues of an upper (or lower) triangular matrix appear on the main diagonal.

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2×2 Matricies: For

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The eigenvalues are the solutions to

$$\lambda^2 - (\mathbf{tr}\,\boldsymbol{A})\lambda + |\boldsymbol{A}| = 0$$

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Trace

The **trace** of a matrix, $\mathbf{tr}\,\boldsymbol{A}$, is the sum of all elements in the diagonal.

Eigenspace Theorem for Linear Transformations

For each eigenvalue λ of a linear transformations $T : \mathbb{V} \to \mathbb{V}$, the **eigenspace**, defined by

$$\mathbb{E}_\lambda = \{\vec{\boldsymbol{v}} \in \mathbb{V} \mid T(\vec{\boldsymbol{v}}) = \lambda \vec{\boldsymbol{v}}\}$$

is a subspace of $\mathbb{V}$.

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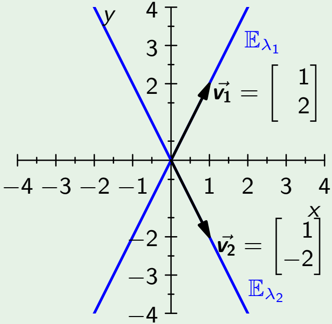
Example 5



Example 6

For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

we had the following eigenvectors:

$$\lambda_1 = 2 \qquad \vec{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \qquad \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -1 \qquad \vec{\boldsymbol{v}}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

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$$\lambda_2 = 1 \qquad \vec{v_2} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbb{E}_{\lambda_2} = \textbf{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_3 = -1 \qquad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbb{E}_{\lambda_3} = \textbf{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Distinct Eigenvalue Theorem

Let $\boldsymbol{A}$ be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_p$, then $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_p\}$ is a set of linearly independent vectors.

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Proof (sketch)

If we have two eigenvalues with $\lambda_1 \neq \lambda_2$, then if the associated eigenvectors $\vec{\boldsymbol{v_1}}$ and $\vec{\boldsymbol{v_2}}$ were linearly dependent, we have

$$\vec{\boldsymbol{v_2}} = c\vec{\boldsymbol{v_1}} \quad \text{where } c \neq 0$$

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But, we could also have multiplied by $\boldsymbol{A}$

$$\boldsymbol{A}\vec{\boldsymbol{v}}_2 = c\boldsymbol{A}\vec{\boldsymbol{v}}_1$$

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Distinct Eigenvalue Theorem

Let $\boldsymbol{A}$ be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_p}}$, then $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_p}}\}$ is a set of linearly independent vectors.

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$$\boldsymbol{A}\vec{\boldsymbol{v_2}} = c\boldsymbol{A}\vec{\boldsymbol{v_1}}$$
$$\lambda_2\vec{\boldsymbol{v_2}} = c\lambda_1\vec{\boldsymbol{v_1}}$$

Which would imply that $\lambda_1 = \lambda_2$,

Example 7

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

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The characteristic equation is

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$
$$\lambda(\lambda + 3)^2 = 0$$

So, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -3$. (Note that -3 is a repeated root.)

Example 7

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

To find the eigenvector for $\lambda_1 = 0$ we need to solve the system:

$$\begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

To find the eigenvector for $\lambda_1 = 0$ we need to solve the system:

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So, we have $v_1 = v_3$ and $v_2 = v_3$. Replacing v_3 with parameter s gives

$$\vec{\boldsymbol{v_1}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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To find the eigenvector for $\lambda_2 = -3$ we need to solve the system:

$$\begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_1 = -v_2 - v_3$ This means we need two parameters, $v_2 = r$ and $v_3 = s$. Which means we have two linearly independent eigenvectors.

$$\vec{\boldsymbol{v_2}} = \begin{bmatrix} -r - s \\ 1 \\ 1 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

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This means the eigenspace is

$$\mathbb{E}_{\lambda_2} = \textbf{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Any linear combination of these two vectors is also an eigenvector, which means that the eigenspace is a plane.

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Since this is an upper diagonal matrix, we know that the eigenvalue is $\lambda = 1$, with multiplicity of 3.

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$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the eigenvector for $\lambda = 1$ we need to solve the system:

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_2 + v_3 = 0$ and $v_3 = 0$. Replacing v_1 with parameter s gives

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Which means the eigenspace has dimension 1.

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We can continue in the same way and find that the eigenvectors are

$$\vec{\boldsymbol{v}}_{\boldsymbol{1}} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\boldsymbol{2}} = \begin{bmatrix} -1 \\ -i \end{bmatrix}$$

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Consider the rotation transformation

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Which means these eigenvalues rotate a vector, instead of scaling it.

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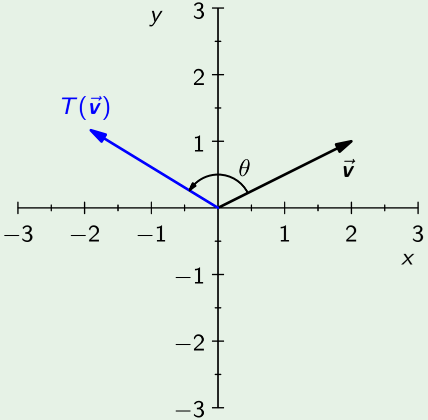

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Let $\boldsymbol{A}$ be an $n \times n$ matrix

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- If λ is an eigenvalue of an invertible matrix $\boldsymbol{A}$, then $\dfrac{1}{\lambda}$ is an eigenvalue of $\boldsymbol{A}^{-1}$.

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Characteristic roots of a linear homogeneous DEs are eigenvalues.

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Note

We will explore the connection between eigenvalues and solutions to differential equations next chapter.