Linear Systems with Nonreal Eigenvalues

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Complex Eigenvalues and Eigenvectors

For a real matrix $\boldsymbol{A}$, nonreal eigenvalues come in complex conjugate pairs,

$$\lambda_1 = \alpha + \beta i \quad \text{and} \quad \lambda_2 = \alpha - \beta i$$

with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$.

The corresponding eigenvectors are also complex conjugate pairs and can be written

$$\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i \quad \text{and} \quad \vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{p}} - \vec{\boldsymbol{q}} i$$

where $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{q}}$ are real vectors.

Complex Eigenvalues and Eigenvectors

For a real matrix $\boldsymbol{A}$, nonreal eigenvalues come in complex conjugate pairs,

$$\lambda_1 = \alpha + \beta i \quad \text{and} \quad \lambda_2 = \alpha - \beta i$$

with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$.

The corresponding eigenvectors are also complex conjugate pairs and can be written

$$\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i \quad \text{and} \quad \vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{p}} - \vec{\boldsymbol{q}} i$$

where $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{q}}$ are real vectors.

Note

We only need to find one eigenvalue/eigenvector pair.

Example 1

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

Example 1

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

The characteristic equation is:

$$(6-\lambda)(4-\lambda)+5=0$$

Example 1

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

The characteristic equation is:

$$(6-\lambda)(4-\lambda)+5=0 \quad \rightarrow \quad \lambda = 5 \pm 2i$$

Example 1

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

The characteristic equation is:

$$(6-\lambda)(4-\lambda)+5=0 \quad \rightarrow \quad \lambda = 5 \pm 2i$$

The eigenvectors are

$$\vec{\boldsymbol{v}}_{\mathbf{1}} = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\mathbf{2}} = \overline{\vec{\boldsymbol{v}}_{\mathbf{1}}} = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$$

Example 1

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

The characteristic equation is:

$$(6-\lambda)(4-\lambda) + 5 = 0 \quad \rightarrow \quad \lambda = 5 \pm 2i$$

The eigenvectors are

$$\vec{\boldsymbol{v}}_{\mathbf{1}} = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\mathbf{2}} = \overline{\vec{\boldsymbol{v}}_{\mathbf{1}}} = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$$

Alternately, we can write

$$\vec{\boldsymbol{v}} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{\boldsymbol{p}}} \pm i \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{\boldsymbol{q}}}$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Which has nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and corresponding eigenvectors $\vec{\boldsymbol{v_1}}$ and $\vec{\boldsymbol{v_2}}$. We can then write:

$$\vec{\boldsymbol{x}} = c_1 e^{\lambda_1 t}\vec{\boldsymbol{v_1}} + c_2 e^{\lambda_2 t}\vec{\boldsymbol{v_2}}.$$

However, we want this solution in terms of the real vectors $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{q}}$.

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Which has nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and corresponding eigenvectors $\vec{\boldsymbol{v_1}}$ and $\vec{\boldsymbol{v_2}}$. We can then write:

$$\vec{\boldsymbol{x}} = c_1 e^{\lambda_1 t}\vec{\boldsymbol{v_1}} + c_2 e^{\lambda_2 t}\vec{\boldsymbol{v_2}}.$$

However, we want this solution in terms of the real vectors $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{q}}$.

So, for eigenvalue $\lambda_1 = \alpha + \beta i$ and corresponding eigenvector $\vec{\boldsymbol{v_1}} = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i$ we get the solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha + \beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

Example 2

Consider the DE system:

$$\vec{x}' = A\vec{x}$$

Which has nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and corresponding eigenvectors $\vec{v_1}$ and $\vec{v_2}$. We can then write:

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}.$$

However, we want this solution in terms of the real vectors $\vec{p}$ and $\vec{q}$.

So, for eigenvalue $\lambda_1 = \alpha + \beta i$ and corresponding eigenvector $\vec{v_1} = \vec{p} + \vec{q}i$ we get the solution

$$\vec{x_1}(t) = e^{\lambda_1 t}\vec{v_1} = e^{\alpha + \beta i}\left(\vec{p} + \vec{q}i\right)$$

Just like with second-order systems, we shall find that the real and imaginary parts of the complex solution above are both real and linearly independent solutions of the system.

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\text{Re}}(t) + \vec{\boldsymbol{x}}_{\text{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\text{Im}} \neq \vec{\boldsymbol{0}}$.

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\mathrm{Re}}(t) + \vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\mathrm{Im}} \neq \vec{\boldsymbol{0}}$.

Then

$$\vec{\boldsymbol{x}}'$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\mathrm{Re}}(t) + \vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\mathrm{Im}} \neq \vec{\mathbf{0}}$.

Then

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}}_{\mathrm{Re}}'(t) + i\vec{\boldsymbol{x}}_{\mathrm{Im}}'(t)$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\text{Re}}(t) + \vec{\boldsymbol{x}}_{\text{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\text{Im}} \neq \vec{\boldsymbol{0}}$.

Then

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}}'_{\text{Re}}(t) + i\vec{\boldsymbol{x}}'_{\text{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Re}}(t) + i\boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Im}}(t)$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\text{Re}}(t) + \vec{\boldsymbol{x}}_{\text{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\text{Im}} \neq \vec{\boldsymbol{0}}$.

Then

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}}'_{\text{Re}}(t) + i\vec{\boldsymbol{x}}'_{\text{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Re}}(t) + i\boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}(t)$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\text{Re}}(t) + \vec{\boldsymbol{x}}_{\text{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\text{Im}} \neq \vec{\boldsymbol{0}}$.

Then

$$\vec{\boldsymbol{x}}'_{\text{Re}}(t) + i\vec{\boldsymbol{x}}'_{\text{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Re}}(t) + i\boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Im}}(t)$$

Separately equating the real and imaginary parts, we get:

$$\vec{\boldsymbol{x}}'_{\text{Re}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Re}}(t) \quad \text{and} \quad \vec{\boldsymbol{x}}'_{\text{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\text{Im}}(t)$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Suppose that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_{\mathrm{Re}}(t) + \vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

is a complex vector solution to the system, with $\vec{\boldsymbol{x}}_{\mathrm{Im}} \neq \vec{\boldsymbol{0}}$.

Then

$$\vec{\boldsymbol{x}}'_{\mathrm{Re}}(t) + i\vec{\boldsymbol{x}}'_{\mathrm{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) + i\boldsymbol{A}\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

Separately equating the real and imaginary parts, we get:

$$\vec{\boldsymbol{x}}'_{\mathrm{Re}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) \quad \text{and} \quad \vec{\boldsymbol{x}}'_{\mathrm{Im}}(t) = \boldsymbol{A}\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

Thus, $\vec{\boldsymbol{x}}_{\mathrm{Re}}(t)$ and $\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$ are separate real solutions to the system.

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

For the complex solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

For the complex solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos\left(\theta\right) + i\sin\left(\theta\right)$$

to write:

$$e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha t+\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i\right)$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

For the complex solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos\left(\theta\right) + i\sin\left(\theta\right)$$

to write:

$$\begin{aligned} e^{\lambda_1 t}\vec{\boldsymbol{v_1}} &= e^{\alpha t+\beta ti}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}e^{\beta ti}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \end{aligned}$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

For the complex solution

$$\vec{\boldsymbol{x}}_{\mathbf{1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v}}_{\mathbf{1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos\left(\theta\right) + i\sin\left(\theta\right)$$

to write:

$$\begin{aligned} e^{\lambda_1 t}\vec{\boldsymbol{v}}_{\mathbf{1}} &= e^{\alpha t+\beta ti}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}e^{\beta ti}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \end{aligned}$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

For the complex solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

to write:

$$\begin{aligned} e^{\lambda_1 t}\vec{\boldsymbol{v_1}} &= e^{\alpha t+\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}e^{\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}\left(\cos(\beta t)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) + i\sin(\beta t)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)\right) \end{aligned}$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

For the complex solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos\left(\theta\right) + i\sin\left(\theta\right)$$

to write:

$$\begin{aligned} e^{\lambda_1 t}\vec{\boldsymbol{v_1}} &= e^{\alpha t+\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}e^{\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}\left(\cos\left(\beta t\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) + i\sin\left(\beta t\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)\right) \\ &= e^{\alpha t}\left(\cos\left(\beta t\right)\vec{\boldsymbol{p}} - \sin\left(\beta t\right)\vec{\boldsymbol{q}}\right) + ie^{\alpha t}\left(\sin\left(\beta t\right)\vec{\boldsymbol{p}} + \cos\left(\beta t\right)\vec{\boldsymbol{q}}\right) \end{aligned}$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

For the complex solution

$$\vec{\boldsymbol{x_1}}(t) = e^{\lambda_1 t}\vec{\boldsymbol{v_1}} = e^{\alpha+\beta i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos\left(\theta\right) + i\sin\left(\theta\right)$$

to write:

$$\begin{aligned} e^{\lambda_1 t}\vec{\boldsymbol{v_1}} &= e^{\alpha t+\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}e^{\beta t i}\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) \\ &= e^{\alpha t}\left(\cos\left(\beta t\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right) + i\sin\left(\beta t\right)\left(\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i\right)\right) \\ &= \underbrace{e^{\alpha t}\left(\cos\left(\beta t\right)\vec{\boldsymbol{p}} - \sin\left(\beta t\right)\vec{\boldsymbol{q}}\right)}_{\vec{\boldsymbol{x}}_{\mathrm{Re}}(t)} + i\,\underbrace{e^{\alpha t}\left(\sin\left(\beta t\right)\vec{\boldsymbol{p}} + \cos\left(\beta t\right)\vec{\boldsymbol{q}}\right)}_{\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)} \end{aligned}$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Since $\vec{\boldsymbol{x}}_{\text{Re}}(t)$ and $\vec{\boldsymbol{x}}_{\text{Im}}(t)$ are linearly independent solutions they must span the solution space. Thus, the general solution, for $c_1, c_2 \in \mathbb{R}$, is

$$\vec{\boldsymbol{x}}(t) = c_1\vec{\boldsymbol{x}}_{\text{Re}}(t) + c_2\vec{\boldsymbol{x}}_{\text{Im}}(t)$$

Example 2

Consider the DE system:

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$$

Since $\vec{\boldsymbol{x}}_{\mathrm{Re}}(t)$ and $\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$ are linearly independent solutions they must span the solution space. Thus, the general solution, for $c_1, c_2 \in \mathbb{R}$, is

$$\vec{\boldsymbol{x}}(t) = c_1\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) + c_2\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$$

Any solutions derived from λ_2 and $\vec{\boldsymbol{v}}_2$ will be linear combinations of $\vec{\boldsymbol{x}}_{\mathrm{Re}}(t)$ and $\vec{\boldsymbol{x}}_{\mathrm{Im}}(t)$.

Solving 2×2 DE System with Nonreal Eigenvalues

For the two-dimensional linear homogeneous differential equation $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with real matrix $\boldsymbol{A}$, eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta$ $(\beta \neq 0)$ the general solution can be found using the following steps:

Solving 2×2 DE System with Nonreal Eigenvalues

For the two-dimensional linear homogeneous differential equation $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with real matrix $\boldsymbol{A}$, eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta$ $(\beta \neq 0)$ the general solution can be found using the following steps:

1. For one eigenvector λ_1, find it's corresponding eigenvector $\vec{\boldsymbol{v_1}}$. The second eigenvalue λ_2 and it's eigenvector $\vec{\boldsymbol{v_2}}$ are complex conjugates of the first. The eigenvectors are of the form $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}} = \vec{\boldsymbol{p}} \pm i\vec{\boldsymbol{q}}$.

Solving 2×2 DE System with Nonreal Eigenvalues

For the two-dimensional linear homogeneous differential equation $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with real matrix $\boldsymbol{A}$, eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta$ ($\beta \neq 0$) the general solution can be found using the following steps:

1. For one eigenvector λ_1, find it's corresponding eigenvector $\vec{\boldsymbol{v_1}}$. The second eigenvalue λ_2 and it's eigenvector $\vec{\boldsymbol{v_2}}$ are complex conjugates of the first. The eigenvectors are of the form $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}} = \vec{\boldsymbol{p}} \pm i\vec{\boldsymbol{q}}$.
2. Construxt the linearly independent real ($\vec{\boldsymbol{x}}_{\text{Re}}$) and imaginary ($\vec{\boldsymbol{x}}_{\text{Im}}$) parts of the solutions as follows:

$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = e^{\alpha t}\left(\cos\left(\beta t\right)\vec{\boldsymbol{p}} - \sin\left(\beta t\right)\vec{\boldsymbol{q}}\right)$$
$$\vec{\boldsymbol{x}}_{\text{Im}}(t) = e^{\alpha t}\left(\sin\left(\beta t\right)\vec{\boldsymbol{p}} + \cos\left(\beta t\right)\vec{\boldsymbol{q}}\right)$$

Solving 2×2 DE System with Nonreal Eigenvalues

For the two-dimensional linear homogeneous differential equation $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with real matrix $\boldsymbol{A}$, eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta$ ($\beta \neq 0$) the general solution can be found using the following steps:

1. For one eigenvector λ_1, find it's corresponding eigenvector $\vec{\boldsymbol{v_1}}$. The second eigenvalue λ_2 and it's eigenvector $\vec{\boldsymbol{v_2}}$ are complex conjugates of the first. The eigenvectors are of the form $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}} = \vec{\boldsymbol{p}} \pm i\vec{\boldsymbol{q}}$.
2. Construxt the linearly independent real ($\vec{\boldsymbol{x}}_{\text{Re}}$) and imaginary ($\vec{\boldsymbol{x}}_{\text{Im}}$) parts of the solutions as follows:

$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = e^{\alpha t}\left(\cos(\beta t)\,\vec{\boldsymbol{p}} - \sin(\beta t)\,\vec{\boldsymbol{q}}\right)$$
$$\vec{\boldsymbol{x}}_{\text{Im}}(t) = e^{\alpha t}\left(\sin(\beta t)\,\vec{\boldsymbol{p}} + \cos(\beta t)\,\vec{\boldsymbol{q}}\right)$$

3. The general solution is

$$\vec{\boldsymbol{x}}(t) = c_1\vec{\boldsymbol{x}}_{\text{Re}}(t) + c_2\vec{\boldsymbol{x}}_{\text{Im}}(t)$$

Example 3

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{\boldsymbol{x}}$$

Example 3

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = 5 \pm 2i$ and

$$\vec{\boldsymbol{v}_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Example 3

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = 5 \pm 2i$ and

$$\vec{\boldsymbol{v_1}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Thus

$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = e^{5t}\cos(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{5t}\sin(2t)\begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t}\begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix}$$

$$\vec{\boldsymbol{x}}_{\text{Im}}(t) = e^{5t}\sin(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{5t}\cos(2t)\begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t}\begin{bmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{bmatrix}$$

Example 3

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = 5 \pm 2i$ and

$$\vec{\boldsymbol{v}_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Thus

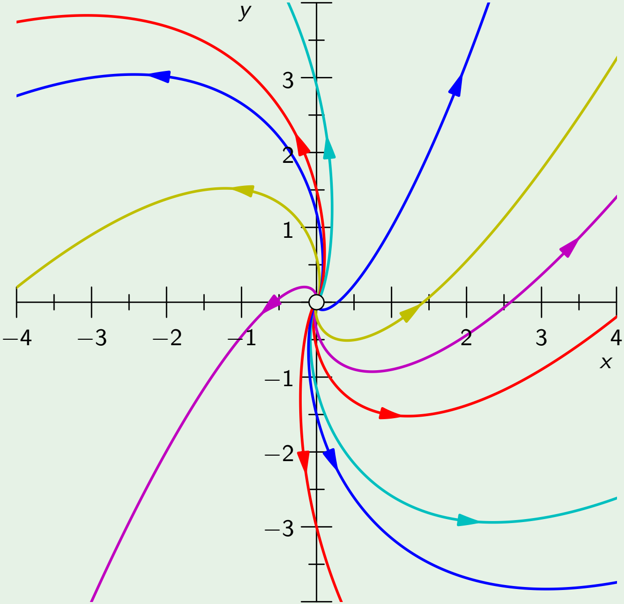
$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = e^{5t}\cos(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{5t}\sin(2t)\begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t}\begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix}$$

$$\vec{\boldsymbol{x}}_{\text{Im}}(t) = e^{5t}\sin(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{5t}\cos(2t)\begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t}\begin{bmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{bmatrix}$$

And general solution

$$\vec{\boldsymbol{x}}(t) = e^{5t}\left(c_1\begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{bmatrix}\right)$$

Example 3



Example 4

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \vec{\boldsymbol{x}}$$

Example 4

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = -1 \pm 2i$ and

$$\vec{\boldsymbol{v_1}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Example 4

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = -1 \pm 2i$ and

$$\vec{\boldsymbol{v}_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Thus

$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = e^{-t} \cos(2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-t} \sin(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\vec{\boldsymbol{x}}_{\text{Im}}(t) = e^{-t} \sin(2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-t} \cos(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Example 4

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = -1 \pm 2i$ and

$$\vec{\boldsymbol{v_1}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Thus

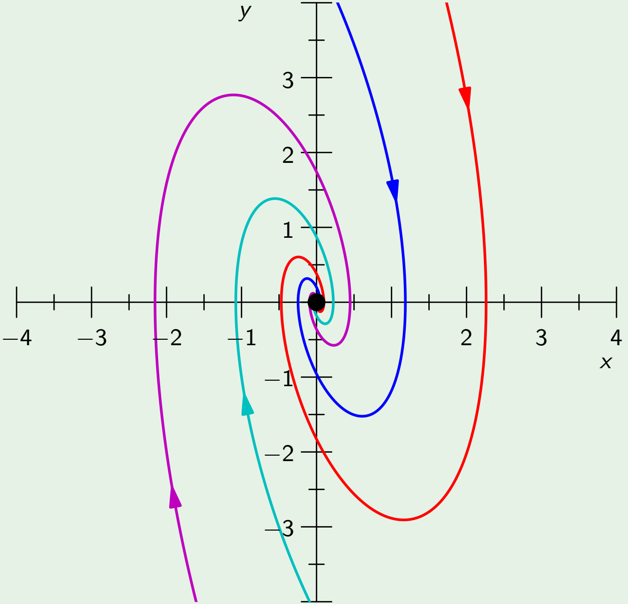
$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = e^{-t}\cos(2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-t}\sin(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\vec{\boldsymbol{x}}_{\text{Im}}(t) = e^{-t}\sin(2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-t}\cos(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

And general solution

$$\vec{\boldsymbol{x}}(t) = e^{-t}\left(c_1\begin{bmatrix} \cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{bmatrix}\right)$$

Example 4



Example 5

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{\boldsymbol{x}}$$

Example 5

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = 0 \pm 3i$ and

$$\vec{\boldsymbol{v}_1} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Example 5

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = 0 \pm 3i$ and

$$\vec{\boldsymbol{v}_1} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Thus

$$\vec{\boldsymbol{x}}_{\mathrm{Re}}(t) = \cos(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{bmatrix}$$

$$\vec{\boldsymbol{x}}_{\mathrm{Im}}(t) = \sin(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \end{bmatrix}$$

Example 5

Consider the system

$$\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{\boldsymbol{x}}$$

The eigenvalues are $\lambda_1, \lambda_2 = 0 \pm 3i$ and

$$\vec{\boldsymbol{v}_1} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Thus

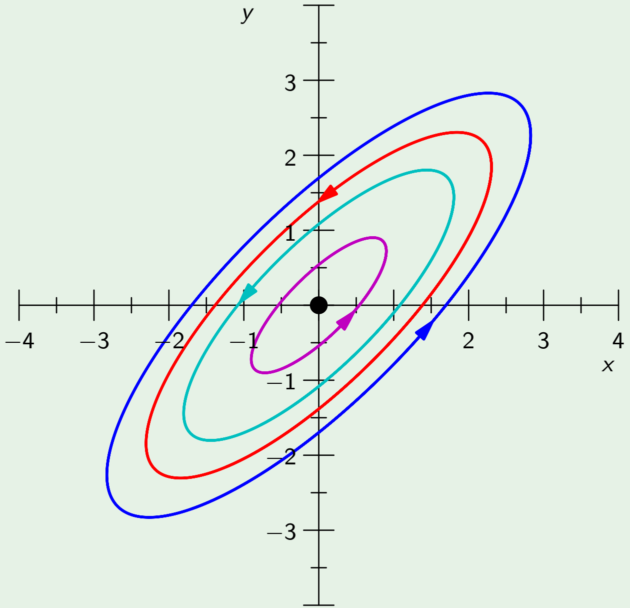
$$\vec{\boldsymbol{x}}_{\text{Re}}(t) = \cos(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{bmatrix}$$

$$\vec{\boldsymbol{x}}_{\text{Im}}(t) = \sin(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \end{bmatrix}$$

And general solution

$$\vec{\boldsymbol{x}}(t) = c_1 \begin{bmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \end{bmatrix}$$

Example 5



Behavior of Solutions

- An **unstable equilibrium** is one where all solutions spiral away from the origin. (Since $\alpha > 0$.)

Behavior of Solutions

- An **unstable equilibrium** is one where all solutions spiral away from the origin. (Since $\alpha > 0$.)
- A **asymptotically stable equilibrium** is one where all solutions spiral towards the origin. (Since $\alpha < 0$.) Technically they never reach zero, because the origin is a separate, fixed-point solution.

Behavior of Solutions

- An **unstable equilibrium** is one where all solutions spiral away from the origin. (Since $\alpha > 0$.)
- A **asymptotically stable equilibrium** is one where all solutions spiral towards the origin. (Since $\alpha < 0$.) Technically they never reach zero, because the origin is a separate, fixed-point solution.
- An **stable equilibrium** is one where the trajectories neither grow nor decay, they just circle in a periodic motion. (Since $\alpha = 0$.)

Nullclines for a DE System

For a two-dimensional system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

- The *v*-**nullcline** is the set of all points with vertical slope, which occur on the curve obtained by solving $x' = f(x,y) = 0$.
- The *h*-**nullcline** is the set of all points with horizontal slope, which occur on the curve obtained by solving $y' = g(x,y) = 0$.

When an *h*-nullcline and an *v*-nullcline intersect, an **equilibrium** occurs.

Interpreting the Solutions

For $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and comple eigenvectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i$, arrange the components of the solution as

$$\begin{bmatrix} \vec{\boldsymbol{x}}_{\text{Re}} \\ \vec{\boldsymbol{x}}_{\text{Im}} \end{bmatrix} = \underbrace{e^{\alpha t}}_{\text{expansion}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{q}} \end{bmatrix}}_{\text{tilt and shape}}$$

Interpreting the Solutions

For $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and comple eigenvectors $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}} = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i$, arrange the components of the solution as

$$\begin{bmatrix} \vec{\boldsymbol{x}}_{\text{Re}} \\ \vec{\boldsymbol{x}}_{\text{Im}} \end{bmatrix} = \underbrace{e^{\alpha t}}_{\text{expansion}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{q}} \end{bmatrix}}_{\text{tilt and shape}}$$

1. The first factor $e^{\alpha t}$ determines *expansion* or *contraction*.
   - If $\alpha > 0$, then trajectories spiral outward, representing unbound growth.
   - If $\alpha < 0$, then trajectories spiral inward, decay to zero.
   - If $\alpha = 0$, then trajectories are closed loops, representing periodic solutions.

Interpreting the Solutions

For $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and comple eigenvectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}}i$, arrange the components of the solution as

$$\begin{bmatrix} \vec{\boldsymbol{x}}_{\text{Re}} \\ \vec{\boldsymbol{x}}_{\text{Im}} \end{bmatrix} = \underbrace{e^{\alpha t}}_{\text{expansion}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{q}} \end{bmatrix}}_{\text{tilt and shape}}$$

1. The first factor $e^{\alpha t}$ determines *expansion* or *contraction*.
   - If $\alpha > 0$, then trajectories spiral outward, representing unbound growth.
   - If $\alpha < 0$, then trajectories spiral inward, decay to zero.
   - If $\alpha = 0$, then trajectories are closed loops, representing periodic solutions.
2. The second factor is the rotation matrix, which describes the spiral. The angle of rotation βt is ever growing.

Interpreting the Solutions

For $\vec{\boldsymbol{x}}' = \boldsymbol{A}\vec{\boldsymbol{x}}$ with nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and comple eigenvectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{p}} + \vec{\boldsymbol{q}} i$, arrange the components of the solution as

$$\begin{bmatrix} \vec{\boldsymbol{x}}_{\text{Re}} \\ \vec{\boldsymbol{x}}_{\text{Im}} \end{bmatrix} = \underbrace{e^{\alpha t}}_{\text{expansion}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{q}} \end{bmatrix}}_{\text{tilt and shape}}$$

1. The first factor $e^{\alpha t}$ determines *expansion* or *contraction*.
   - If $\alpha > 0$, then trajectories spiral outward, representing unbound growth.
   - If $\alpha < 0$, then trajectories spiral inward, decay to zero.
   - If $\alpha = 0$, then trajectories are closed loops, representing periodic solutions.
2. The second factor is the rotation matrix, which describes the spiral. The angle of rotation βt is ever growing.
3. The third factor, containing $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{q}}$, determines the *tilt* and *shape* of the *elliptical trajectories* that would result with $\alpha = 0$.