# Solving DEs and IVPs with Laplace Transforms

Colby Community College

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## Derivative Theorem for Laplace Transforms

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$, $f^{(n)}$ is piecewise continuous on $[0, \infty)$, and $f, f', \ldots, f^{(n)}$ are of exponential order $\alpha$, then for $s > a$, and $n = 1, 2, \ldots$

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

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## Strategy to Solve DEs with Laplace Transforms

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## Strategy to Solve DEs with Laplace Transforms

1. Using the Laplace transform, transform the IVP with unknown function $y(t)$ into an algebraic problem with unknown function $Y(s)$.
2. Solve the algebraic problem for $Y(s)$.
3. Manipulating $Y(s)$ algebraically if necessary, use the inverse Laplace transform to transform $Y(s)$ into the IVP solution $y(t)$.

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Which means

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 4e^{-t} - 2e^{3t}$$

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Which means

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}e^{-3t} - 2e^{-2t} + \frac{3}{2}e^{-t}$$

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(s^2+1)Y(s) &= \frac{s^2+2}{s^2+1} \\
Y(s) &= \frac{s^2+2}{(s^2+1)(s^2+4)}
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s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} &= \mathcal{L}\{\sin(t)\} \\
s^2 Y(s) - 1 + 4Y(s) &= \frac{1}{s^2+1} \\
(s^2+1)Y(s) &= \frac{s^2+2}{s^2+1} \\
Y(s) &= \frac{s^2+2}{(s^2+1)(s^2+4)} \\
&= \frac{\frac{1}{3}}{s^2+1} + \frac{\frac{2}{3}}{s^2+4}
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## Example 4

Consider

$$y'' + 4y = \sin(t) \quad \text{where} \quad y(0) = 0,\ y'(0) = 1$$

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Thus, the solution is

$$y(t) = \frac{1}{3}\sin(t) + \frac{1}{3}\sin(2t)$$

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## Translation Property for Multiplication by $e^{at}$

If the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \text{for } s > a + \alpha$$

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&= e^{-t}\left(3\cos(2t) - 2\sin(2t)\right)
\end{aligned}$$

## Multiplication by $t^n$ Rule for the Laplace Transform

If $f(t)$ is a piecewise continuos function on $[0, \infty)$ and is of exponential order $\alpha$, then for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s) \quad \text{where} \quad n \in \mathbb{N}^+$$

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This process can be repeated for an arbitrary $n$.

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