

Linear Transformations

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Salt Lake Community College

Linear Transformation

A **Linear Transformation** T on a vector space \mathbb{V} to a vector space \mathbb{W} is a function $T : \mathbb{V} \rightarrow \mathbb{W}$ that preserves *scalar multiplication* and *vector addition*. That is, for all $\vec{u}, \vec{v} \in \mathbb{V}$ and $c \in \mathbb{R}$:

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Image of a Linear Transformation

The **image** of a linear transformation $T : \mathbb{V} \rightarrow \mathbb{W}$ is the set of vectors in \mathbb{W} to which T maps the vectors in \mathbb{V} :

$$\text{Im}(T) = \{\vec{w} \in \mathbb{W} \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in \mathbb{V}\}$$

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Note: Linear transformations may map nonzero vectors from the domain into the zero vector of the codomain. We will talk more about this in the next section.

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Consider the mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(\langle x, y, z \rangle) = \langle x, y, 0 \rangle$.
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Therefore, T is a linear transformation.

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Differentiation is a linear transformation. The **derivative operator** $D : \mathcal{C}^1[a, b] \rightarrow \mathcal{C}[a, b]$ is defined by

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Similarly, we can confirm that the **integration operator** $I : \mathcal{C}[a, b] \rightarrow \mathbb{R}$, defined by

$$I(f) = \int_a^b f(t)dt$$

is a linear transformation.

Geometry of Matrix Linear Transformations

If \mathbf{A} is an $m \times n$ matrix and \vec{x} is a column n -vector, then $\mathbf{A}\vec{x}$ can be considered a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, where $T(\vec{x}) = \mathbf{A}\vec{x}$.

In this transformation, the matrix \mathbf{A} allows vectors to be dynamic.

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The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

defines a **shear** of 1-unit in the x -direction.

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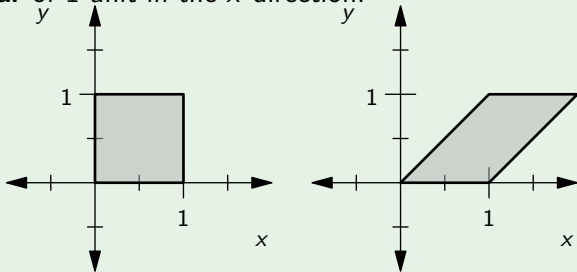
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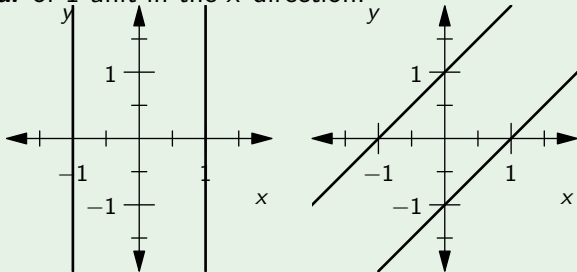
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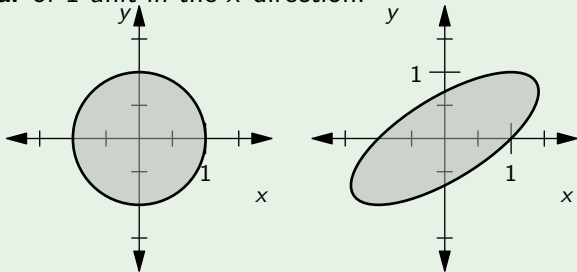
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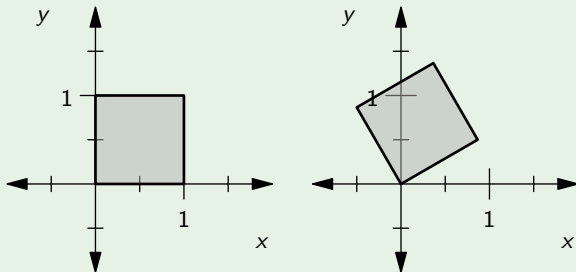
The *Counterclockwise* rotation about the origin by angle θ is given by:

$$R_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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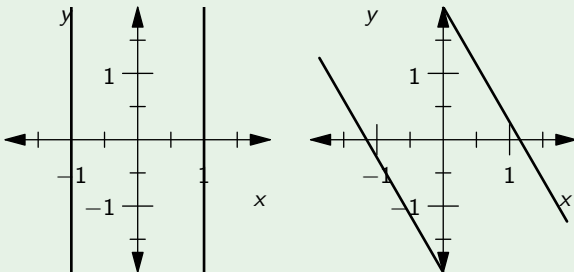
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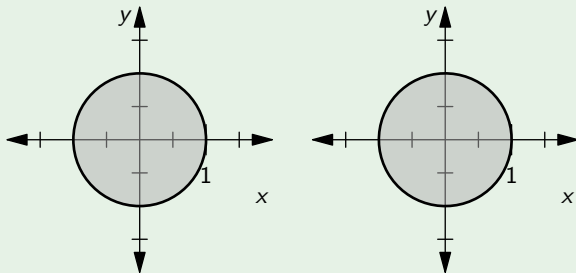
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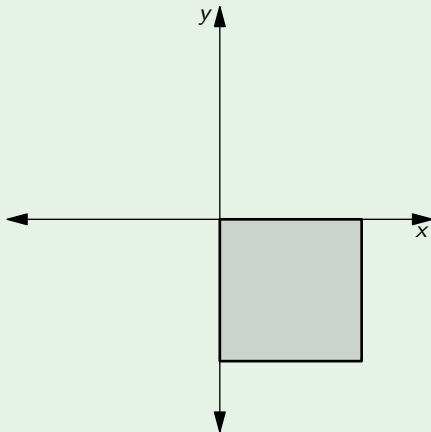
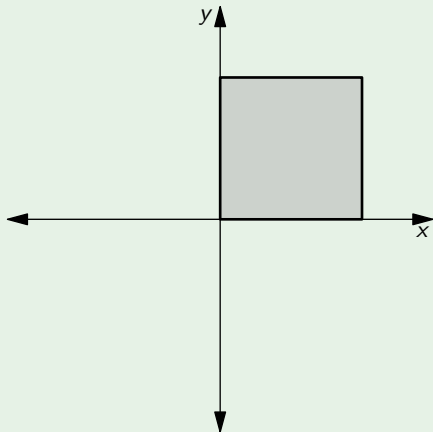
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Example

Reflection about the x -axis

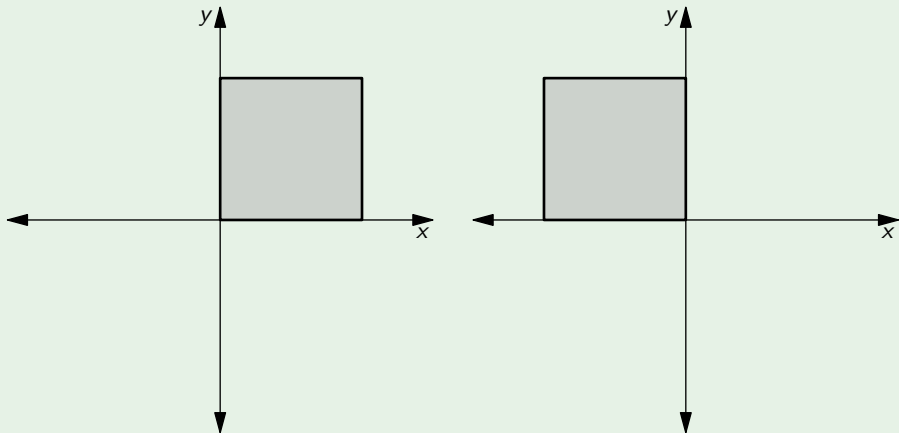
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Example

Reflection about the y-axis

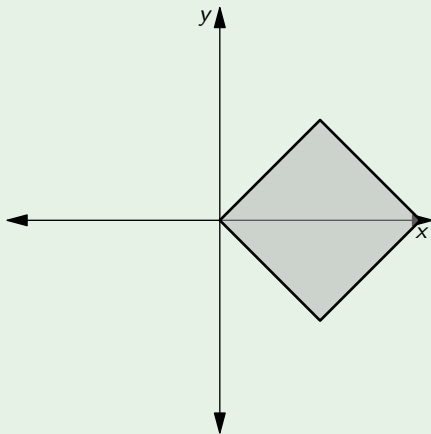
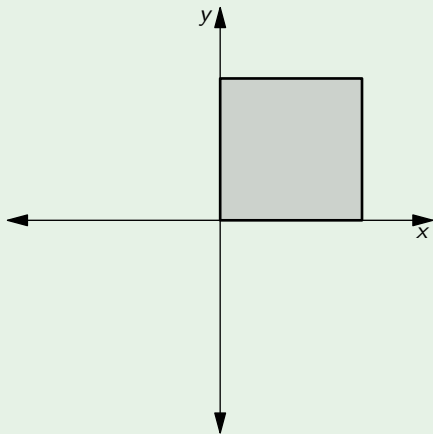
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



Example

Rotation clockwise about the origin of $\frac{\pi}{4}$

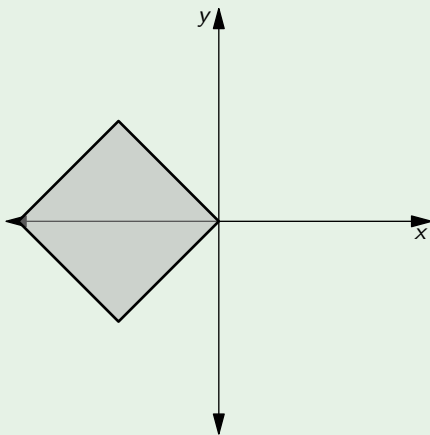
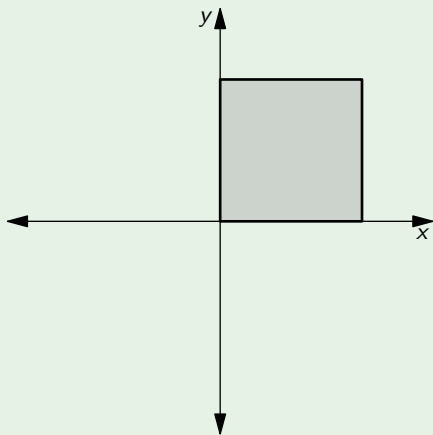
$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ -\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}$$



Example

Rotation counterclockwise about the origin of $\frac{3\pi}{4}$

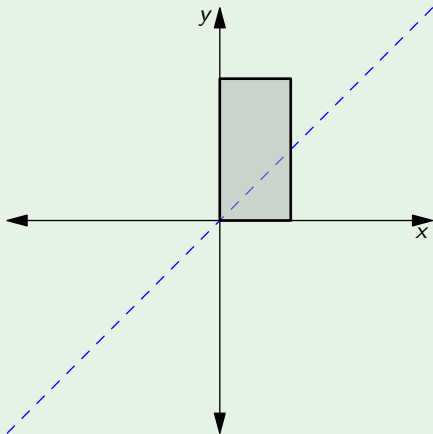
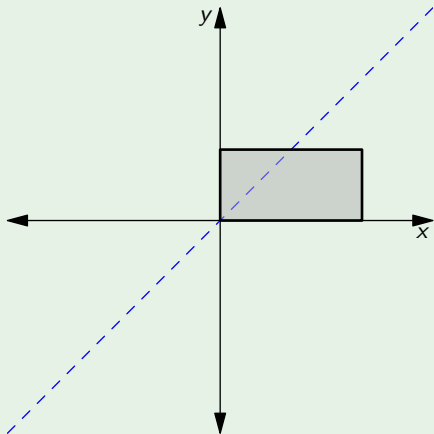
$$\begin{bmatrix} \cos\left(\frac{3\pi}{4}\right) & -\sin\left(\frac{3\pi}{4}\right) \\ \sin\left(\frac{3\pi}{4}\right) & \cos\left(\frac{3\pi}{4}\right) \end{bmatrix}$$



Example

Reflection about the line $y = x$

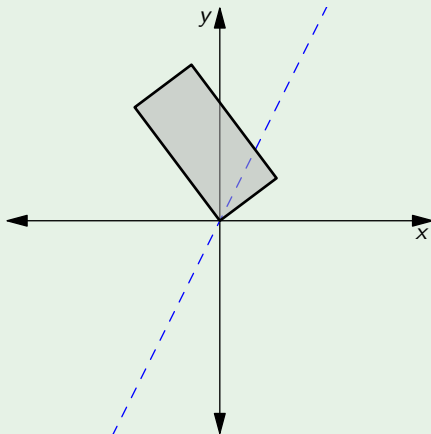
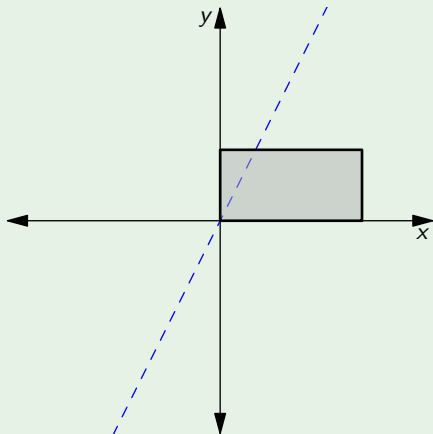
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Example

Reflection about the line $y = 2x$

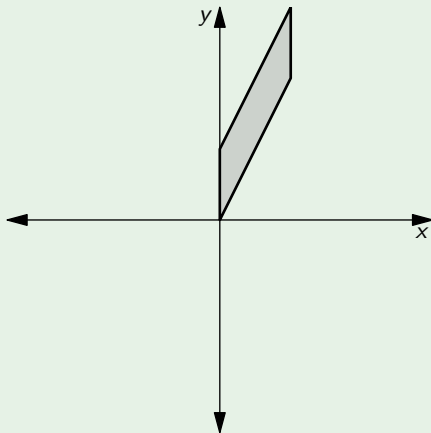
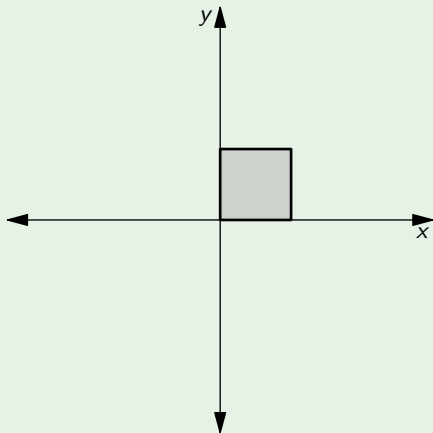
$$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$



Example

Shear of 2 in the y-direction

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$



Example

Consider the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(\vec{v}) = \mathbf{A}\vec{v} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \vec{v}$$

maps

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 + 2v_3 \\ 2v_1 + 3v_2 + 5v_3 \end{bmatrix}$$

Example

Consider the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(\vec{v}) = \mathbf{A}\vec{v} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \vec{v}$$

maps

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 + 2v_3 \\ 2v_1 + 3v_2 + 5v_3 \end{bmatrix}$$

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It can be easily verified that $[1, 2]$ and $[1, 3]$ are independent in \mathbb{R}^2 . Which means the image must contain their span, which is exactly \mathbb{R}^2 .

The Standard Matrix for a Linear Transform

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. The **standard matrix** associated with T is defined by

$$\mathbf{A} = [T(\vec{e}_1) | T(\vec{e}_2) | \cdots | T(\vec{e}_n)]$$

where the columns $T(\vec{e}_j)$ are the images under T of the standard basis vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.

Proof

We can check that this matrix satisfies $T(\vec{v}) = \mathbf{A}\vec{v}$ by

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$$\begin{aligned} T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) &= T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n) \\ &= v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2) + \cdots + v_n T(\vec{e}_n) \end{aligned}$$

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Example

Find the standard matrix that will describe the transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \\ 2x \end{bmatrix}$$

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Thus, for dimensions in the product to match, **A** must be a 3×2 matrix. Which means:

$$\mathbf{A} = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \mid T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Example

Let $D_2 : \mathbb{P}_3 \rightarrow \mathbb{P}_1$ be the second-derivative operator. So, for a typical cubic polynomial:

$$D_2(ax^3 + bx^2 + cx + d) = 6ax + 2b$$

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The standard matrix is made up of

$$\left[D_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| D_2 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \middle| D_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \middle| D_2 \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \right] = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

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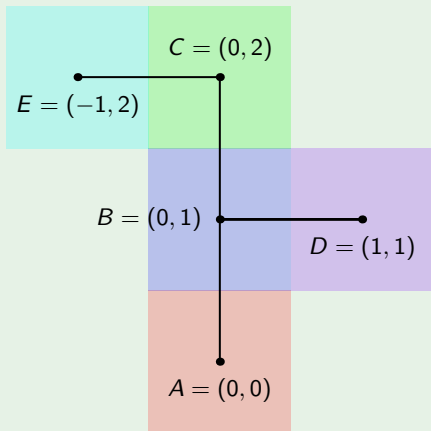
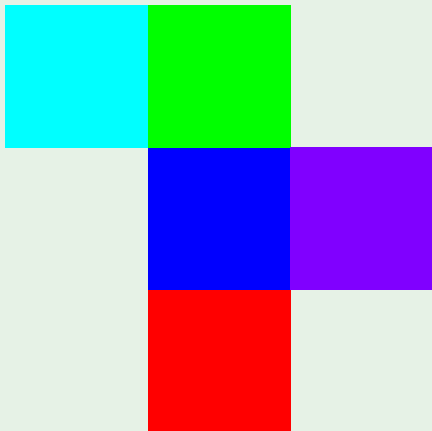
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Which gives us a matrix that satisfies:

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 6a \\ 2b \end{bmatrix}$$

Example

Linear transforms are used extensively in computer graphics, where images or models are just collections of points and line segments. Let us look at a simple example, where we can think of each pixel as a point at its center:



Example

We can rotate this image 90° clockwise with the matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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When applying this linear transformation to our image we get

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