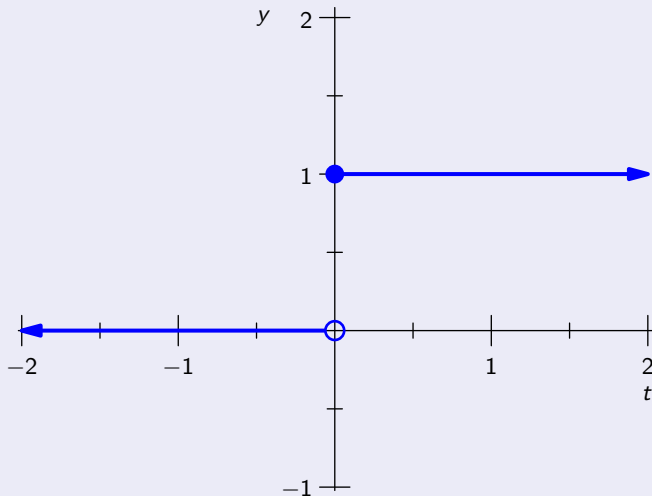


The Step Function and the Delta Function

Colby Community College

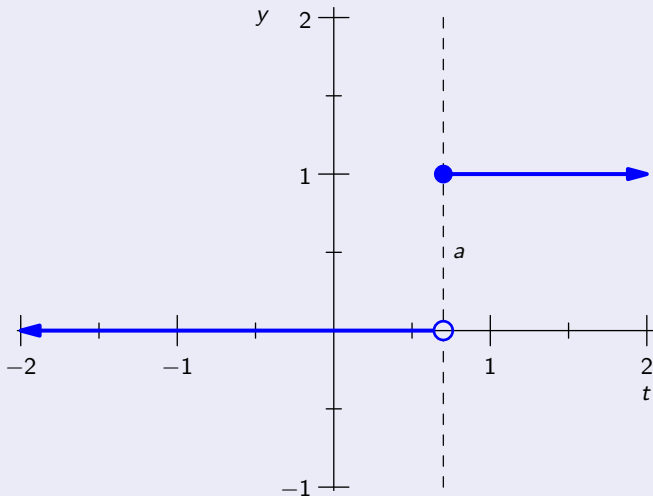
The Unit Step Function

$$\text{step}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



The Translated Step Function

$$\text{step}(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



Laplace Transform of the Step Function

$$\mathcal{L}\{\mathbf{step}(t - a)\} = \frac{e^{-as}}{s}$$

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$$\begin{aligned}\mathcal{L}\{\mathbf{step}(t - a)\} &= \int_0^{\infty} e^{-st} \mathbf{step}(t - a) dt \\ &= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} \cdot 1 dt\end{aligned}$$

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Example 1

Consider

$$f(t) = \begin{cases} 2 & \text{if } t < 3 \\ -4 & \text{if } 3 \leq t < 4 \\ 1 & \text{if } t \geq 4 \end{cases}$$

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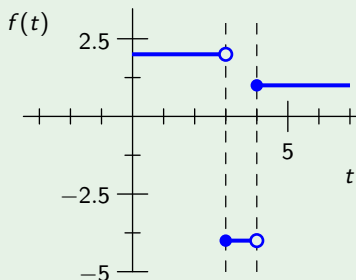
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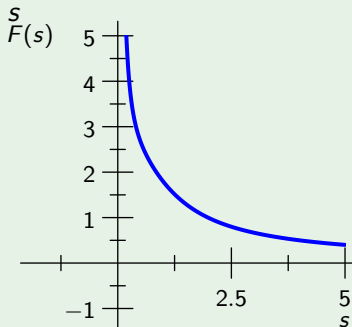
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$\xrightarrow{\mathcal{L}}$



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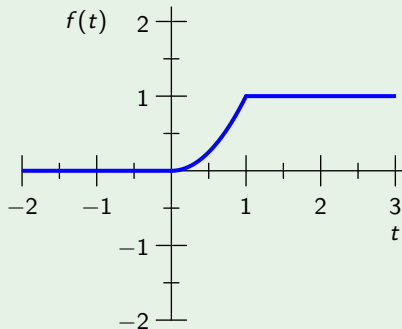
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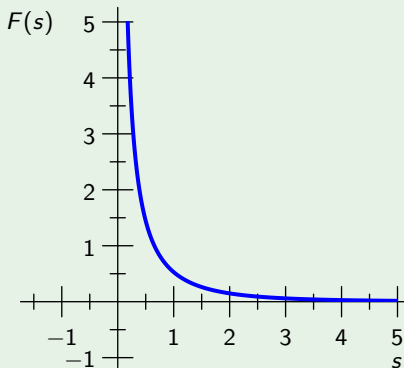
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\mathcal{L}



Delayed Function

For a given function $g(t)$, the **delayed function**

$$f(t) = \begin{cases} 0 & \text{if } t < c \\ g(t - c) & \text{if } t \geq c \end{cases}$$

shifts $g(t)$ to the right c units from the origin, and replaces it by zero to the left of $t = c$. Using the unit step function, the delayed function can also be written

$$f(t) = g(t - c) \mathbf{step}(t - c)$$

Example 3

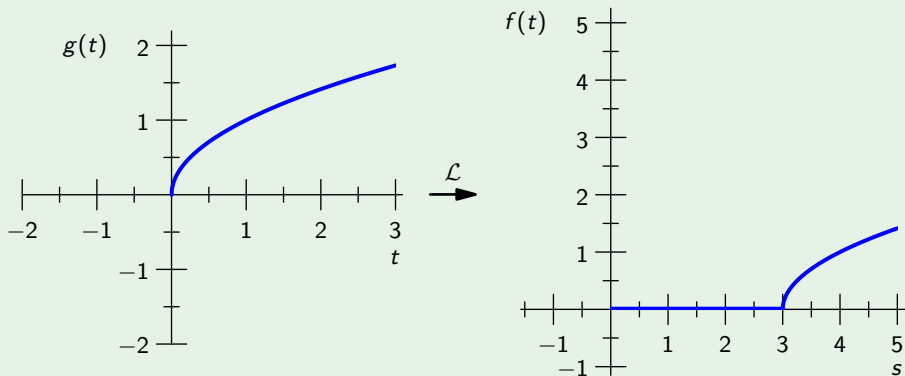
Consider the function $g(t) = \sqrt{t}$, which has the delayed function

$$f(t) = \begin{cases} 0 & \text{if } t < 3 \\ \sqrt{t-3} & \text{if } t \geq 3 \end{cases}$$

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Closer Look

Consider the Laplace transform of a function $f(t)$ that is delayed c units.

$$\mathcal{L}\{f(t - c) \mathbf{step}(t - c)\}$$

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$$\mathcal{L}\{f(t-c)\mathbf{step}(t-c)\} = \int_0^{\infty} e^{-st} f(t-c)\mathbf{step}(t-c) dt$$

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Consider the Laplace transform of a function $f(t)$ that is delayed c units.

$$\begin{aligned}\mathcal{L}\{f(t-c)\mathbf{step}(t-c)\} &= \int_0^{\infty} e^{-st} f(t-c)\mathbf{step}(t-c) dt \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-st} f(t-c)\mathbf{step}(t-c) dt\end{aligned}$$

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We may assume $b > c$, since $b \rightarrow \infty$.

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Furthermore, $\mathbf{step}(t-c) = 0$ for $t < c$ and $\mathbf{step}(t-c) = 1$ for $t \geq c$.

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let $w = t - c$

Closer Look

Consider the Laplace transform of a function $f(t)$ that is delayed c units.

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Consider the Laplace transform of a function $f(t)$ that is delayed c units.

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Closer Look

Consider the Laplace transform of a function $f(t)$ that is delayed c units.

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Closer Look

Consider the Laplace transform of a function $f(t)$ that is delayed c units.

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Delay Theorem (or Shifting Theorem)

$$\mathcal{L}\{f(t-c)\mathbf{step}(t-c)\} = e^{-cs} F(s) \quad \text{where } c > 0$$

Delay Theorem (or Shifting Theorem)

$$\mathcal{L}\{f(t - c) \mathbf{step}(t - c)\} = e^{-cs} F(s) \quad \text{where } c > 0$$

Alternate Form

$$\mathcal{L}\{g(t) \mathbf{step}(t - c)\} = e^{-cs} \mathcal{L}\{g(t + c)\}$$

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Example 4

Consider

$$h(t) = t^2 \mathbf{step}(t - 1)$$

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$$h(t) = t^2 \mathbf{step}(t - 1)$$

If we let $c = 1$ and $g(t) = t^2$, then by the Delay theorem we have

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Delay Theorem (or Shifting Theorem)

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Example 5

Let us find the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-3s}}{s^2} = \frac{1}{s^2} - \frac{e^{-3s}}{s^2}$$

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We can treat $\frac{e^{-3s}}{s^2}$ as the transform of a delay function.

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$$F(s) = \frac{1 - e^{-3s}}{s^2} = \frac{1}{s^2} - \frac{e^{-3s}}{s^2}$$

We can treat $\frac{e^{-3s}}{s^2}$ as the transform of a delay function.

$$\mathcal{L}^{-1}\{F(s)\} = t - \underbrace{(t - 3)\text{step}(t - 3)}_{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}}$$

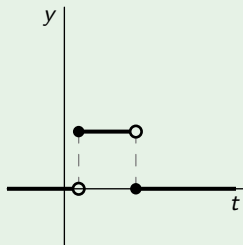
Chopper Function

$$\mathbf{step}(t - a) - \mathbf{step}(t - b) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$$

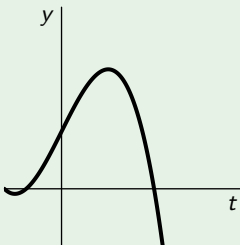
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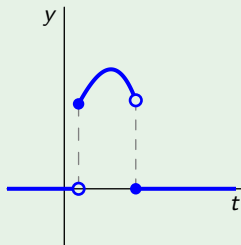
Example 6



times



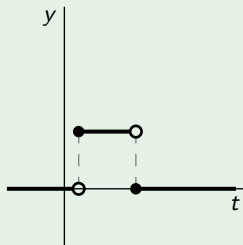
gives



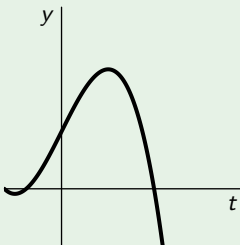
Chopper Function

$$\mathbf{step}(t - a) - \mathbf{step}(t - b) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$$

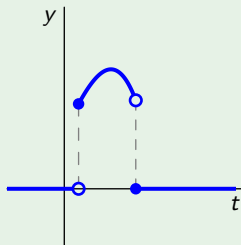
Example 6



times



gives



Laplace Transform of Chopper Function

$$\mathcal{L}\{f(t) \cdot [\mathbf{step}(t - a) - \mathbf{step}(t - b)]\} = e^{-as} \mathcal{L}\{f(t+a)\} - e^{-bs} \mathcal{L}\{f(t+b)\}$$

Example 7

Let us find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ -\sin(\pi t) & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases}$$

Example 7

Let us find the Laplace transform of

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Thus,

$$\mathcal{L}\{f(t)\} = -e^{-s}\mathcal{L}\{-\sin(\pi(t+1))\} + e^{-2s}\mathcal{L}\{\sin(\pi(t+2))\}$$

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Example 8

Consider the IVP

$$x'' + x = f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases} \quad \text{with } x(0) = 0, \quad x'(0) = 0$$

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We can rewrite this DE using a step function

$$x'' + x = 1 - \mathbf{step}(t - \pi) \quad \text{with } x(0) = 0, \quad x'(0) = 0$$

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$$s^2 X(s) + X(s) = \mathcal{L}\{1 - \mathbf{step}(t - \pi)\}$$

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We can then use the Delay Theorem on the RHS

$$s^2 X(s) + X(s) = \frac{1}{s} + \frac{e^{-\pi s}}{s}$$

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We can now solve for $X(s)$. (As well as rearrange for the inverse.)

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$$\begin{aligned} s^2 X(s) + X(s) &= \frac{1}{s} + \frac{e^{-\pi s}}{s} \\ (s^2 + 1)X(s) &= \frac{1 - e^{-\pi s}}{s} \end{aligned}$$

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So, we can use the Delay Theorem again to find $x(t)$.

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = (1 - \cos(t)) - (1 - \cos(t - \pi)) \mathbf{step}(t - \pi)$$

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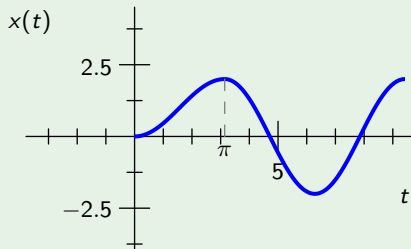
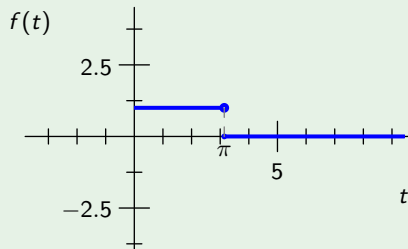
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Physical systems often involve impulsive forces, which act over very short spans of time. To model these forces, the physicist Paul Dirac invented a “function-like” object.

Let us first look at a special function

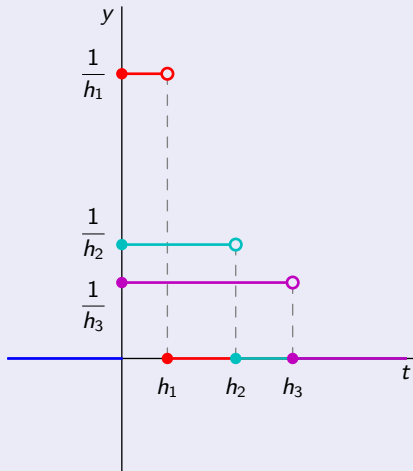
$$f_h(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{h} & \text{if } 0 \leq t < h \\ 0 & \text{if } t \geq h \end{cases}$$

such that

$$\int_{-\infty}^{\infty} f_h(t) dt = 1$$

Dirac suggested that

$$\delta(t) = \lim_{h \rightarrow 0} f_h(t)$$



Dirac Delta Function

The **Dirac Delta function** or **unit impulse function** $\delta(t)$ is defined by two conditions:

1

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) & \text{if } t = 0 \end{cases}$$

2

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Finding the Laplace Transform

To find that Laplace transform of $\delta(t)$, we will first calculate the transform of $f_h(t)$.

$$\mathcal{L}\{f_h(t)\} = \int_0^{\infty} e^{-st} f_h(t) dt$$

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We can then use l'Hôpital's rule to find that

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Laplace Transform of the Delta Function

$$\mathcal{L}\{\delta(t)\} = 1 \quad \text{and} \quad \mathcal{L}\{\delta(t - a)\} = e^{-as}$$