Undetermined Coefficients

Adam Wilson

Salt Lake Community College

Superposition

If L is a linear differential operator defined by

$$L(y) = a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y$$

(where all functions of t are assumed to be defined over some interval I) then we can look at superposition for the DE $L(y) = f(t)$.

Superposition

If L is a linear differential operator defined by

$$L(y) = a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y$$

(where all functions of t are assumed to be defined over some interval I) then we can look at superposition for the DE $L(y) = f(t)$.

Superposition Principle for Nonhomogeneous Linear DEs

If $y_i(t)$ is a solution of $L(y) = f_i(t)$, for $i = 1, 2, \ldots, n$, and constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$, then

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

is a solution of

$$L(y) = c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)$$

Superposition

Nonhomogeneous Principle for Linear DEs

The general solution of the nonhomogeneous linear DE $L(y) = f$ is

$$y = y_h + y_p$$

where

- y_h is the general solution of $L(y) = 0$
- y_p is a particular solution of $L(y) = f$

Superposition

Nonhomogeneous Principle for Linear DEs

The general solution of the nonhomogeneous linear DE $L(y) = f$ is

$$y = y_h + y_p$$

where

- y_h is the general solution of $L(y) = 0$
- y_p is a particular solution of $L(y) = f$

This is just applying the superposition principle for $f_1(t) = 0$ and $f_2(t) = f$.

Superposition

Example

Consider the nonhomogeneous second-order DE

$$y'' - y' - 2y = 2t + 1 - 2e^t$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t + 1}_{f_1} \underbrace{-2e^t}_{f_2}$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t + 1}_{f_1}\underbrace{-2e^t}_{f_2}$$

We can verify the following following:

$$y_1 = -t \quad \text{is a solution to} \quad L(y) = f_1$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t + 1}_{f_1} \underbrace{-2e^t}_{f_2}$$

We can verify the following following:

$$\begin{array}{lll} y_1 = -t & \text{is a solution to} & L(y) = f_1 \\ y_2 = e^t & \text{is a solution to} & L(y) = f_2. \end{array}$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t + 1}_{f_1} \underbrace{-2e^t}_{f_2}$$

We can verify the following following:

$$\begin{array}{lll} y_1 = -t & \text{is a solution to} & L(y) = f_1 \\ y_2 = e^t & \text{is a solution to} & L(y) = f_2. \end{array}$$

We can then use superposition to build a particular solution

$$y_p = y_1 + y_2 = -t + e^t$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t + 1}_{f_1}\underbrace{-2e^t}_{f_2}$$

We can verify the following following:

$$\begin{aligned} y_1 &= -t \quad \text{is a solution to} \quad L(y) = f_1 \\ y_2 &= e^t \quad \text{is a solution to} \quad L(y) = f_2. \end{aligned}$$

We can then use superposition to build a particular solution

$$y_p = y_1 + y_2 = -t + e^t$$

Finally, we use characteristic roots to solve $L(y) = 0$

$$r^2 - r - 2 = 0$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t+1}_{f_1}\underbrace{-2e^t}_{f_2}$$

We can verify the following following:

$$\begin{array}{lll} y_1 = -t & \text{is a solution to} & L(y) = f_1 \\ y_2 = e^t & \text{is a solution to} & L(y) = f_2. \end{array}$$

We can then use superposition to build a particular solution

$$y_p = y_1 + y_2 = -t + e^t$$

Finally, we use characteristic roots to solve $L(y) = 0$

$$r^2 - r - 2 = 0 \to r_1 = 2,\ r_2 = -1$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t+1}_{f_1} \underbrace{-2e^t}_{f_2}$$

We can verify the following following:

$$\begin{array}{lll} y_1 = -t & \text{is a solution to} & L(y) = f_1 \\ y_2 = e^t & \text{is a solution to} & L(y) = f_2. \end{array}$$

We can then use superposition to build a particular solution

$$y_p = y_1 + y_2 = -t + e^t$$

Finally, we use characteristic roots to solve $L(y) = 0$

$$r^2 - r - 2 = 0 \rightarrow r_1 = 2,\ r_2 = -1 \rightarrow y_h = c_1 e^{2t} + c_2 e^{-t}$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{2t + 1}_{f_1} \underbrace{-2e^t}_{f_2}$$

We can verify the following following:

$$\begin{array}{lll} y_1 = -t & \text{is a solution to} & L(y) = f_1 \\ y_2 = e^t & \text{is a solution to} & L(y) = f_2. \end{array}$$

We can then use superposition to build a particular solution

$$y_p = y_1 + y_2 = -t + e^t$$

Finally, we use characteristic roots to solve $L(y) = 0$

$$r^2 - r - 2 = 0 \to r_1 = 2,\ r_2 = -1 \to y_h = c_1 e^{2t} + c_2 e^{-t}$$

Thus, the general solution is

$$y = y_h + y_p = c_1 e^{2t} + c_2 e^{-t} - t + e^t$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$y'' - y' - 2y = t + \frac{1}{2} + 8e^t$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{t + \frac{1}{2}}_{\frac{1}{2}f_1} + \underbrace{8e^t}_{-4f_2}$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{t + \tfrac{1}{2}}_{\frac{1}{2}f_1} + \underbrace{8e^t}_{-4f_2}$$

Using the solutions found in the last example, we can use superposition to build a particular solution to this DE.

$$y_p = \tfrac{1}{2}y_1 - 4y_2$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{t + \frac{1}{2}}_{\frac{1}{2}f_1} + \underbrace{8e^t}_{-4f_2}$$

Using the solutions found in the last example, we can use superposition to build a particular solution to this DE.

$$y_p = \frac{1}{2}y_1 - 4y_2 = -\frac{1}{2}t - 4e^t$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{t + \tfrac{1}{2}}_{\frac{1}{2}f_1} + \underbrace{8e^t}_{-4f_2}$$

Using the solutions found in the last example, we can use superposition to build a particular solution to this DE.

$$y_p = \tfrac{1}{2}y_1 - 4y_2 = -\tfrac{1}{2}t - 4e^t$$

Finally, we have already solved $L(y) = 0$. So, the general solution is

$$y = y_h + y_p = c_1e^{2t} + c_2e^{-t} - \tfrac{1}{2}t - 4e^t$$

Superposition

Example

Consider the nonhomogeneous second-order DE

$$\underbrace{y'' - y' - 2y}_{L(y)} = \underbrace{t + \tfrac{1}{2}}_{\frac{1}{2}f_1} + \underbrace{8e^t}_{-4f_2}$$

Using the solutions found in the last example, we can use superposition to build a particular solution to this DE.

$$y_p = \tfrac{1}{2}y_1 - 4y_2 = -\tfrac{1}{2}t - 4e^t$$

Finally, we have already solved $L(y) = 0$. So, the general solution is

$$y = y_h + y_p = c_1 e^{2t} + c_2 e^{-t} - \tfrac{1}{2}t - 4e^t$$

After accumulating some experience, a solution can be guessed by just "inspecting" the equation. By recognizing the patterns.

Method of Undetermined Coefficients

Example

Consider the second-order DE

$$ay'' + by' + cy = d$$

where all the coefficients and forcing term are constant.

Method of Undetermined Coefficients

Example

Consider the second-order DE

$$ay'' + by' + cy = d$$

where all the coefficients and forcing term are constant.
We can see that, when $c \neq 0$, $y_p = \frac{d}{c}$ is a particular solution.

Method of Undetermined Coefficients

Example

Consider the second-order DE

$$ay'' + by' + cy = d$$

where all the coefficients and forcing term are constant.
We can see that, when $c \neq 0$, $y_p = \frac{d}{c}$ is a particular solution.

This idea works well for the nth-order equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = d$$

provided that $a_0 \neq 0$.

Method of Undetermined Coefficients

Example

Inspection of

$$y'' + y' = 1$$

leads to the solution

Method of Undetermined Coefficients

Example

Inspection of

$$y'' + y' = 1$$

leads to the solution $y_p = t$.

Method of Undetermined Coefficients

Example

Inspection of

$$y'' + y' = 1$$

leads to the solution $y_p = t$.

Example

Inspection of

$$y'' - y = \sin(t)$$

leads to the solution

Method of Undetermined Coefficients

Example

Inspection of

$$y'' + y' = 1$$

leads to the solution $y_p = t$.

Example

Inspection of

$$y'' - y = \sin(t)$$

leads to the solution $y_p = -\frac{1}{2}\sin(t)$

Method of Undetermined Coefficients

Example

Inspection of

$$y'' + y' = 1$$

leads to the solution $y_p = t$.

Example

Inspection of

$$y'' - y = \sin(t)$$

leads to the solution $y_p = -\frac{1}{2}\sin(t)$

Example

Inspection of

$$y'' + y' - 3y = 9e^{3t}$$

leads to the solution

Method of Undetermined Coefficients

Example

Inspection of

$$y'' + y' = 1$$

leads to the solution $y_p = t$.

Example

Inspection of

$$y'' - y = \sin(t)$$

leads to the solution $y_p = -\frac{1}{2}\sin(t)$

Example

Inspection of

$$y'' + y' - 3y = 9e^{3t}$$

leads to the solution $y_p = e^{3t}$

Method of Undetermined Coefficients

There are a few limitations of this method:
It only works for linear differential equations with specific forcing terms.

Method of Undetermined Coefficients

There are a few limitations of this method:
It only works for linear differential equations with specific forcing terms.

Forcing Terms That Work With Undetermined Coefficients

- Polynomials in t.
- Exponentials e^{at}.
- Sinusoidal functions of the form $\cos(kt)$ and $\sin(kt)$.
- Any finite products or sums of these functions.

Method of Undetermined Coefficients

There are a few limitations of this method:
It only works for linear differential equations with specific forcing terms.

Forcing Terms That Work With Undetermined Coefficients

- Polynomials in t.
- Exponentials e^{at}.
- Sinusoidal functions of the form $\cos(kt)$ and $\sin(kt)$.
- Any finite products or sums of these functions.

Even with these limitations, undetermined coefficients is widely used, given that many functions are built from the above parts.

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 3t^2 - 1$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 3t^2 - 1$$

Let us look for y_p in $\mathbb{P}_2$. Which means y_p will be in the form

$$y_p = At^2 + Bt + C$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 3t^2 - 1$$

Let us look for y_p in $\mathbb{P}_2$. Which means y_p will be in the form

$$y_p = At^2 + Bt + C$$

We can then calculate:

$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$2A - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$
\begin{aligned}
2A - (2At + B) - 2(At^2 + Bt + C) &= 3t^2 - 1 \\
(-2A)t^2 + (-2A - 2B)t + (2A - B - 2C) &= 3t^2 - 1
\end{aligned}
$$

Example

Plugging these into the DE gives

$$2A - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1$$
$$(-2A)t^2 + (-2A - 2B)t + (2A - B - 2C) = 3t^2 - 1$$

So, equating both sides gives the system

$$-2A = 3, \quad -2A - 2B = 0, \quad 2A - B - 2C = -1$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$2A - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1$$
$$(-2A)t^2 + (-2A - 2B)t + (2A - B - 2C) = 3t^2 - 1$$

So, equating both sides gives the system

$$-2A = 3, \quad -2A - 2B = 0, \quad 2A - B - 2C = -1$$

Which has solution $A = -\frac{3}{2}$, $B = \frac{3}{2}$, and $C = -\frac{7}{4}$.

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = -\frac{3}{2}t^2 + \frac{3}{2}t + \frac{7}{4}$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = -\frac{3}{2}t^2 + \frac{3}{2}t + \frac{7}{4}$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = -\frac{3}{2}t^2 + \frac{3}{2}t + \frac{7}{4}$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

The general solution is

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{3}{2}t^2 + \frac{3}{2}t + \frac{7}{4}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 2e^{-3t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 2e^{-3t}$$

Let us look for y_p of the form

$$y_p = Ae^{-3t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 2e^{-3t}$$

Let us look for y_p of the form

$$y_p = Ae^{-3t}$$

We can then calculate:

$$y_p' = -3Ae^{-3t}$$
$$y_p'' = 9Ae^{-3t}$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$9Ae^{-3t} + 3Ae^{-3t} - 2Ae^{-3t} = 2e^{-3t}$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned} 9Ae^{-3t} + 3Ae^{-3t} - 2Ae^{-3t} &= 2e^{-3t} \\ 10Ae^{-3t} &= 2e^{-3t} \end{aligned}$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned} 9Ae^{-3t} + 3Ae^{-3t} - 2Ae^{-3t} &= 2e^{-3t} \\ 10Ae^{-3t} &= 2e^{-3t} \end{aligned}$$

So, equating both sides gives

$$10A = 2 \quad \rightarrow \quad A = \frac{1}{5}$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = \frac{1}{5}e^{-3t}$$

Example

Thus, the particular solution is

$$y_p = \frac{1}{5}e^{-3t}$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r - 2)(r + 1) = 0$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = \frac{1}{5}e^{-3t}$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

The general solution is

$$y = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{5}e^{-3t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 2\cos(3t)$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 2\cos(3t)$$

Let us look for y_p of the form

$$y_p = A\cos(3t) + B\sin(3t)$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 2\cos(3t)$$

Let us look for y_p of the form

$$y_p = A\cos(3t) + B\sin(3t)$$

We can then calculate:

$$\begin{aligned} y_p' &= -3A\sin(3t) + 3B\cos(3t) \\ y_p'' &= -9A\cos(3t) - 9B\sin(3t) \end{aligned}$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned}(-9A\cos(3t) - 9\sin(3t)) \\ -(-3A\sin(3t) + 3B\cos(3t)) \\ -2(A\cos(3t) + B\sin(3t)) = 2\cos(3t)\end{aligned}$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$
\begin{aligned}
(-9A\cos(3t) - 9\sin(3t))& \\
-(-3A\sin(3t) + 3B\cos(3t))& \\
-2(A\cos(3t) + B\sin(3t)) &= 2\cos(3t) \\
(-11A - 3B)\cos(3t) + (3A - 11B)\sin(3t) &= 2\cos(3t)
\end{aligned}
$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$
\begin{aligned}
(-9A\cos(3t) - 9\sin(3t)) \\
-(-3A\sin(3t) + 3B\cos(3t)) \\
-2(A\cos(3t) + B\sin(3t)) &= 2\cos(3t) \\
(-11A - 3B)\cos(3t) + (3A - 11B)\sin(3t) &= 2\cos(3t)
\end{aligned}
$$

So, equating both sides gives the system

$$-11A - 3B = 2, \quad 3A - 11B = 0$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned}
(-9A\cos(3t) - 9\sin(3t)) \\
-(-3A\sin(3t) + 3B\cos(3t)) \\
-2(A\cos(3t) + B\sin(3t)) &= 2\cos(3t) \\
(-11A - 3B)\cos(3t) + (3A - 11B)\sin(3t) &= 2\cos(3t)
\end{aligned}$$

So, equating both sides gives the system

$$-11A - 3B = 2, \quad 3A - 11B = 0$$

Which has solution $A = -\dfrac{11}{65}$ and $B = -\dfrac{3}{65}$.

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = -\frac{11}{65}\cos(3t) - \frac{3}{65}\sin(3t)$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = -\frac{11}{65}\cos(3t) - \frac{3}{65}\sin(3t)$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = -\frac{11}{65}\cos(3t) - \frac{3}{65}\sin(3t)$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

The general solution is

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{11}{65}\cos(3t) - \frac{3}{65}\sin(3t)$$

Method of Undetermined Coefficients

Example

Consider
$$y'' - y' - 2y = t^2 e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = t^2 e^t$$

Let us look for y_p of the form

$$y_p = \left(At^2 + Bt + C\right) e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = t^2 e^t$$

Let us look for y_p of the form

$$y_p = \left(At^2 + Bt + C\right) e^t$$

We can then calculate:

$$y_p' = \left(At^2 + (2A + B)t + (B + C)\right) e^t$$
$$y_p'' = \left(At^2 + (4A + B)t + (2A + 2B + C)\right) e^t$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned}\left(At^2 + (4A+B)t + (2A+2B+C)\right)e^t& \\ -\left(At^2 + (2A+B)t + (B+C)\right)e^t& \\ +2\left(At^2 + Bt + C\right)e^t &= t^2e^t\end{aligned}$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned}\left(At^2 + (4A+B)t + (2A+2B+C)\right)e^t& \\ -\left(At^2 + (2A+B)t + (B+C)\right)e^t& \\ +2\left(At^2 + Bt + C\right)e^t &= t^2e^t \\ \left((-2A)t^2 + (2A-2B)t + (2A+B-2C)\right)e^t &= t^2e^t\end{aligned}$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned}
&\left(At^2 + (4A+B)t + (2A+2B+C)\right)e^t \\
&\qquad - \left(At^2 + (2A+B)t + (B+C)\right)e^t \\
&\qquad\qquad +2\left(At^2 + Bt + C\right)e^t = t^2e^t \\
&\left((-2A)t^2 + (2A-2B)t + (2A+B-2C)\right)e^t = t^2e^t
\end{aligned}$$

So, equating both sides gives the system

$$-2A = 1, \quad 2A - 2B = 0, \quad 2A + B - 2C = 0$$

Method of Undetermined Coefficients

Example

Plugging these into the DE gives

$$\begin{aligned}\left(At^2 + (4A+B)t + (2A+2B+C)\right)e^t \\ -\left(At^2 + (2A+B)t + (B+C)\right)e^t \\ +2\left(At^2 + Bt + C\right)e^t &= t^2e^t \\ \left((-2A)t^2 + (2A-2B)t + (2A+B-2C)\right)e^t &= t^2e^t\end{aligned}$$

So, equating both sides gives the system

$$-2A = 1, \quad 2A - 2B = 0, \quad 2A + B - 2C = 0$$

Which has solution $A = -\dfrac{1}{2}$, $B = -\dfrac{1}{2}$, and $C = -\dfrac{3}{4}$.

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = \left(-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}\right)e^t$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = \left(-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}\right)e^t$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

Method of Undetermined Coefficients

Example

Thus, the particular solution is

$$y_p = \left(-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}\right)e^t$$

Since the homogeneous equation has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

The general solution is

$$y = c_1 e^{2t} + c_2 e^{-t} + \left(-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}\right)e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ae^{2t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ae^{2t}$$

We can then calculate:

$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ae^{2t}$$

We can then calculate:

$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$

Substituting into the DE gives

$$4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} = 5e^{2t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ae^{2t}$$

We can then calculate:

$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$

Substituting into the DE gives

$$\begin{aligned} 4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} &= 5e^{2t} \\ 0 &= 5e^{2t} \end{aligned}$$

Method of Undetermined Coefficients

Example

Consider
$$y'' - y' - 2y = 5e^{2t}$$
Let us look for y_p of the form
$$y_p = Ae^{2t}$$
We can then calculate:
$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$
Substituting into the DE gives
$$4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} = 5e^{2t}$$
$$0 = 5e^{2t}$$
Thats not good. We'll have to try something else.

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ate^{2t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ate^{2t}$$

We can then calculate:

$$y_p' = (2At + A)e^{2t}$$
$$y_p'' = (4A + 4A)e^{2t}$$

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$(4A + 4A)e^{2t} - 2Ae^{2t} - 2Ate^{2t} = 5e^{2t}$$

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$\begin{aligned}(4A + 4A)e^{2t} - 2Ae^{2t} - 2Ate^{2t} &= 5e^{2t} \\ 3Ae^{2t} &= 5e^{2t}\end{aligned}$$

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$\begin{aligned}(4A + 4A)e^{2t} - 2Ae^{2t} - 2Ate^{2t} &= 5e^{2t}\\ 3Ae^{2t} &= 5e^{2t}\end{aligned}$$

When we equate both sides we get $3A = 5$ and so $A = \dfrac{5}{3}$.

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$\begin{aligned}(4A+4A)e^{2t} - 2Ae^{2t} - 2Ate^{2t} &= 5e^{2t}\\ 3Ae^{2t} &= 5e^{2t}\end{aligned}$$

When we equate both sides we get $3A = 5$ and so $A = \dfrac{5}{3}$.
And so, the particular solution is

$$y_p = \frac{5}{3}te^{2t}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ae^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ae^t$$

We can then calculate:

$$y_p' = Ae^t$$
$$y_p'' = Ae^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ae^t$$

We can then calculate:

$$y_p' = Ae^t$$
$$y_p'' = Ae^t$$

Substituting into the DE gives

$$Ae^t - 2Ae^t + Ae^{2t} = 3e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ae^t$$

We can then calculate:

$$y_p' = Ae^t$$
$$y_p'' = Ae^t$$

Substituting into the DE gives

$$Ae^t - 2Ae^t + Ae^{2t} = 3e^t$$
$$0 = 3e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ae^t$$

We can then calculate:

$$y_p' = Ae^t$$
$$y_p'' = Ae^t$$

Substituting into the DE gives

$$Ae^t - 2Ae^t + Ae^{2t} = 3e^t$$
$$0 = 3e^t$$

Thats not good. We'll have to try something else.

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ate^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ate^t$$

We can then calculate:

$$y_p' = Ae^t + Ate^t$$
$$y_p'' = 2Ae^t + Ate^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ate^t$$

We can then calculate:

$$y_p' = Ae^t + Ate^t$$
$$y_p'' = 2Ae^t + Ate^t$$

Substituting into the DE gives

$$2Ae^t + Ate^t - 2\left(Ae^t + Ate^t\right) + Ate^t = 3e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ate^t$$

We can then calculate:

$$\begin{aligned} y_p' &= Ae^t + Ate^t \\ y_p'' &= 2Ae^t + Ate^t \end{aligned}$$

Substituting into the DE gives

$$\begin{aligned} 2Ae^t + Ate^t - 2\left(Ae^t + Ate^t\right) + Ate^t &= 3e^t \\ 0 &= 3e^t \end{aligned}$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = Ate^t$$

We can then calculate:

$$\begin{aligned} y_p' &= Ae^t + Ate^t \\ y_p'' &= 2Ae^t + Ate^t \end{aligned}$$

Substituting into the DE gives

$$\begin{aligned} 2Ae^t + Ate^t - 2\left(Ae^t + Ate^t\right) + Ate^t &= 3e^t \\ 0 &= 3e^t \end{aligned}$$

This is also a problem. We'll have to try something else.

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 3e^t$$

Let us look for y_p of the form

$$y_p = At^2e^t$$

Method of Undetermined Coefficients

Example

Consider

$$y'' - y' - 2y = 3e^t$$

Let us look for y_p of the form

$$y_p = At^2e^t$$

We can then calculate:

$$y_p' = 2Ate^t + At^2e^t$$
$$y_p'' = 2Ae^t + 4Ate^t + At^2e^t$$

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$2Ae^t + 4Ate^t + At^2e^t - 2\left(2Ate^t + At^2e^t\right) + At^2e^t = 5e^{2t}$$

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$
\begin{aligned}
2Ae^t + 4Ate^t + At^2e^t - 2\left(2Ate^t + At^2e^t\right) + At^2e^t &= 5e^{2t} \\
2Ae^t &= 5e^{2t}
\end{aligned}
$$

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$\begin{aligned} 2Ae^t + 4Ate^t + At^2e^t - 2\left(2Ate^t + At^2e^t\right) + At^2e^t &= 5e^{2t} \\ 2Ae^t &= 5e^{2t} \end{aligned}$$

When we equate both sides we get $2A = 5$ and so $A = \dfrac{5}{2}$.

Method of Undetermined Coefficients

Example

Substituting into the DE gives

$$\begin{aligned} 2Ae^t + 4Ate^t + At^2e^t - 2\left(2Ate^t + At^2e^t\right) + At^2e^t &= 5e^{2t} \\ 2Ae^t &= 5e^{2t} \end{aligned}$$

When we equate both sides we get $2A = 5$ and so $A = \dfrac{5}{2}$.
And so, the particular solution is

$$y_p = \frac{5}{2}te^{2t}$$