

# Separation of Variables: Quantitative Analysis

Department of Mathematics

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(Slides by Adam Wilson)

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The method we just used is called **Separation of Variables**.

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Suppose  $G(y)$  and  $F(t)$  are antiderivatives of  $\frac{1}{g(y)}$  and  $f(t)$ , respectively.

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$$y'(t) = f(t)g(y(t))$$

This has shown that  $y$  is a solution to  $y' = f(t)g(y)$  and explains why the previous example works.

## Method of Separation of Variables

**Step 1:** Set  $g(y) = 0$  and solve to find any equilibria.

**Step 2:** Now, assume that  $g(y) \neq 0$ . Rewrite the equation in separated form:

$$\frac{dy}{g(y)} = f(t)dt$$

**Step 3:** Integrate, if possible, each side:

$$\int \frac{dy}{g(y)} = \int f(t)dt$$

(obtaining the implicit one-parameter family of solutions.)

**Step 4:** If possible, solve for  $y$  in terms of  $t$ , getting the explicit solution  $y = y(t)$

**Step 5:** If you have an IVP, use the initial condition to evaluate  $c$ .

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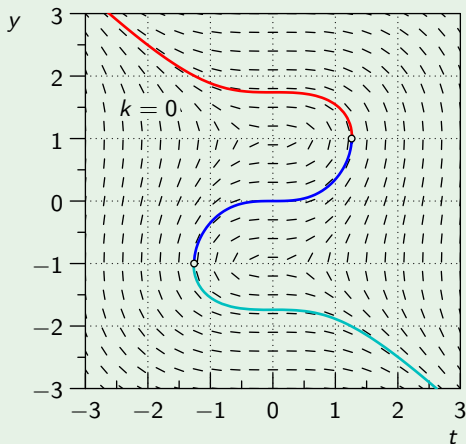
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Note that because of the restrictions, each solution curve in the direction field will be a piecewise combination of several functions. A particular solution of an IVP for this DE would only be *one* of these.

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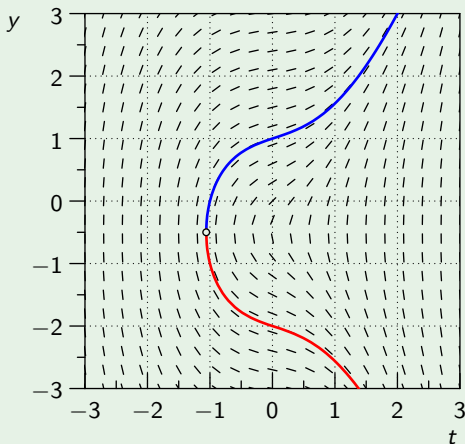
$$y + y^2 = t^3 + t + 2$$
$$y = \frac{1}{2} \left( -1 \pm \sqrt{4t^3 + 4t + 9} \right)$$

Again, the solution curve is made up of multiple parts.

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We can then use the initial condition to find  $c$ .

$$\frac{1}{2} = -\frac{0^2}{2} + c \quad \Rightarrow \quad c = \frac{1}{2}$$

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We have two explicit solutions:

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