Defining Probability

Colby Community College

Definition

The result of **random process** is called an **outcome**.

Note

A **die**, the singular of **dice**, is a cube with six sides numbered 1 to 6.

Example 1

Assume we roll a die and get a 1.

What is the random process?

Rolling the die.

What is the outcome?

The 1 that was rolled.

What is the chance of rolling a 1 on this die?

If the dice is fair, each side has the same chance of being rolled. So a 1 has a one-in-six chance, equivalently $\frac{1}{6}$.

Definition

A **probability** of an outcome is the proportion of times the outcome would occur if we observed the random process an infinite number of times.

Note

$$P(X) = \frac{\text{Number of outcomes corresponding to } X}{\text{Total number of outcomes}}$$

Example 2

What is the probability of rolling a 1 or 2 on a die?

There are two outcomes, a 1 or a 2, and six faces on a die.

$$P(\text{roll 1 or 2}) = \frac{2}{6} = \frac{1}{3}$$

Note

A standard deck of 52 playing cards consists of four **suits** in two colors: Hearts ♥, Spades ♠, Diamonds ♦, and Clubs ♣

Each suit contains 13 cards, each of a different **rank**:
2 through 10, Jack, Queen, King, and Ace

The Jack, Queen, and King cards are called **face cards.**

The Jack, Queen, King, and Ace cards are called **honour cards.**

The cards numbered 2 to 10 are called **numerals.**

♣2	♣3	♣4	♣5	♣6	♣7	♣8	♣9	♣10	♣J	♣Q	♣K	♣A
♦2	♦3	♦4	♦5	♦6	♦7	♦8	♦9	♦10	♦J	♦Q	♦K	♦A
♥2	♥3	♥4	♥5	♥6	♥7	♥8	♥9	♥10	♥J	♥Q	♥K	♥A
♠2	♠3	♠4	♠5	♠6	♠7	♠8	♠9	♠10	♠J	♠Q	♠K	♠A

Example 3

What is the probability of drawing a single card from a deck and getting an Ace?

There are four aces in a deck of 52 cards. Which gives the probability

$$P\left(\text{Ace}\right) = \frac{4}{52} = \frac{1}{13} = 0.0769 = 7.67\%$$

Example 4

What is the probability of rolling a 1, 2, 3, 4, 5, or 6 on a die?

Every side of the die is listed, so

$$P\left(\text{roll 1 or 2 or 3 or 4 or 5 or 6}\right) = \frac{6}{6} = 1 = 100\%$$

Definition

An outcome with a probability of 1 is called **certain**.

Example 5

If a year is selected at random, what is the probability that Thanksgiving Day (in the United States) will be on a Wednesday?

In the United States, Thanksgiving Day always falls on the fourth Thursday in November.

This means it is impossible for Thanksgiving Day to fall on a Wednesday.

$$P\left(\text{Thanksgiving on a Wednesday}\right) = 0 = 0\%$$
$$P\left(\text{Thanksgiving on a Thursday}\right) = 1 = 100\%$$

Definition

An outcome with a probability of 0 is called **impossible**.

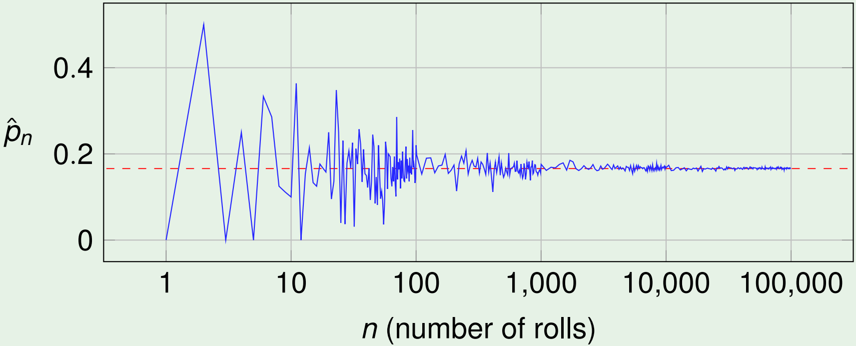
Note

Probabilities are always between 0 and 1.

Example 6

The probability of rolling a 1 on a die is $p = 1/6 \approx 0.167$, but if we roll six dice, we may get no 1's or multiple 1's.

Let $\hat{p}_n$ be the proportion of number of 1's rolled after n rolls.



Note

It is not a coincidence that $\hat{p}_n$ get closer to p as n increases.

Law of Large Numbers

As more observations are collected, the proportion $\hat{p}_n$ of occurrences with a particular outcome converges to the probability p of that outcome.

Cautions

- The law of large numbers applies to behavior over a large number of trails, and it does not apply to any one individual outcome.
  - Gamblers sometimes foolishly loose large sums of money by incorrectly thinking that a string of losses increases the chances of a win on the next bet, or that a string of wins is likely to continue.
- If we know nothing about the likelihood of different possible outcomes, we should not assume that they are equally likely.
  - You should not think that the probability of passing the next exam is $\frac{1}{2}$, or 0.5. The actual probability depends on factors such as the amount of preparation and the difficulty of the exam.

Definition

Outcomes A and B are **disjoint** (or **mutually exclusive**) if they cannot occur at the same time.

Example 7

Are the outcomes "roll a 1" and "roll a 2" disjoint?

Yes, it is impossible to roll two different numbers at the same time.

Example 8

Are the outcomes "roll a 1" and "roll an odd number" disjoint?

No, 1 is an odd number.

Example 9

Are the outcomes "draw an ace" and "draw a diamond" disjoint?

No, the ♦A is both an ace and a diamond.

Addition Rule of Disjoint Outcomes

If A_1 and A_2 represent two disjoint outcomes, then the probability that one of them occurs is given by:

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$

Example 10

The probability of rolling a 1 or rolling a 2 on a die can be calculated two ways:

Directly: There are two outcomes that are either a 1 or a 2, so the probability is:

$$P(\text{roll a 1 or roll a 2}) = \frac{2}{6} = \frac{1}{3}$$

Addition Rule: Rolling a 1 and rolling a 2 are disjoint, so:

$$\begin{aligned} P(\text{roll a 1 or roll a 2}) &= P(\text{roll a 1}) + P(\text{roll a 2}) \\ &= \frac{1}{6} + \frac{1}{6} = \frac{1+1}{6} = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Example 11

Let's calculate the probability of rolling a 1, 2, or 3.

$$\begin{aligned} P\,(\text{roll a 1 or roll a 2 or roll a 3}) &= P\,(\text{roll a 1}) + P\,(\text{roll a 2}) + P\,(\text{roll a 3}) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{3}{6} = \frac{1}{2} \end{aligned}$$

Example 12

Let's calculate the probability of rolling a 1, 2, 3 or 4.

$$\begin{aligned} P\,(\text{roll a 1 or 2 or 3 or 4}) &= P\,(1) + P\,(2) + P\,(3) + P\,(4) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{4}{6} = \frac{2}{3} \end{aligned}$$

Definition

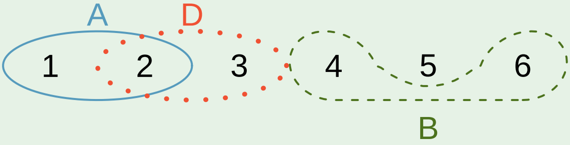
A **event** is a collection of outcomes.

Note

If two events have no elements in common, they are disjoint.

Example 13

Consider the following events.



Are A and B disjoint? Yes.

Are A and D disjoint? No, 2 is in both.

Are B and D disjoint? Yes.

Example 14

Suppose we flip a fair coin and roll a fair die.

The list of all possible outcomes is:

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

We want to calculate the probability of getting a head or a six.

The relevant outcomes are: H1, H2, H3, H4, H5, H6, T6.

Meaning that $P\,(\text{H or 6}) = \frac{7}{12}$.

Notice that $\frac{6}{12} = \frac{1}{2}$ of the outcomes have heads and $\frac{2}{12} = \frac{1}{6}$ have a six.

But, $\frac{1}{2} + \frac{1}{6} = \frac{8}{12}$, is wrong because we have double counted H6.

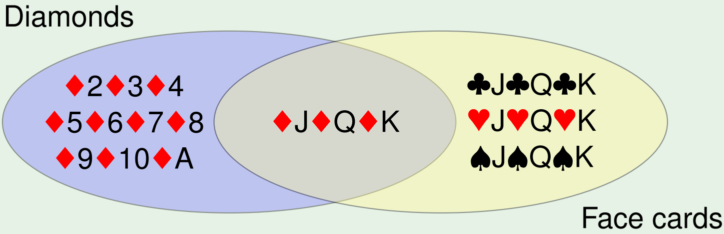
The correct probability is:

$$P\,(\text{H or 6}) = P\,(\text{H}) + P\,(6) - P\,(\text{H and 6}) = \frac{1}{2} + \frac{1}{6} - \frac{1}{12} = \frac{7}{12}$$

Example 15

Let us consider the events "draw a diamond" and "draw a face card".

These outcomes are not disjoint, since three cards are both:



The Addition Rule for Disjoint Outcomes would count ♦J♦Q♦K twice!

$$P(\diamondsuit \text{ and face}) = P(\diamondsuit) + P(\text{face}) - P(\diamondsuit \text{ and face})$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$$

$$= \frac{22}{52} = \frac{11}{26}$$

General Addition Rule

If A and B are any two events, disjoint or not, then the probability that at least one of them will occur is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Note

In statistics, when we write "or" what we mean is "and/or", unless we explicitly say otherwise.

In other words, "A or B" occurring means A, B, or both A and B occur.

Note

If A and B are disjoint this means $P(A \text{ and } B) = 0$, and so we get the Addition Rule for Disjoint Outcomes:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= P(A) + P(B) - 0 \\ &= P(A) + P(B) \end{aligned}$$

Example 16

Suppose we draw one card from a standard deck. Let us find the probability that we get a Queen or a King.

There are 4 Queens and 4 Kings in the deck, hence eight outcomes out of 52 possible outcomes. So, the probability is

$$P\,(\text{Q or K}) = \frac{8}{52}$$

Since there are no cards that are both Kings and Queens, we have

$$P\,(\text{Q or K}) = P\,(\text{Q}) + P\,(K) - P\,(\text{Q and K}) = \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}$$