Basis and Dimension

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Definition 1

For a vector space $\mathbb{V}$, a **linear combination** of vectors is:

$$c_1 \vec{\boldsymbol{v_1}} + c_2 \vec{\boldsymbol{v_2}} + \cdots + c_n \vec{\boldsymbol{v_k}}$$

where $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{v_i}} \in \mathbb{V}$

Definition 1

For a vector space $\mathbb{V}$, a **linear combination** of vectors is:

$$c_1 \vec{\boldsymbol{v_1}} + c_2 \vec{\boldsymbol{v_2}} + \cdots + c_n \vec{\boldsymbol{v_k}}$$

where $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{v_i}} \in \mathbb{V}$

Note

This is the only way to form new vectors, since in a vector space we can only add two vectors or multiply a vector by a scalar.

Definition 1

For a vector space $\mathbb{V}$, a **linear combination** of vectors is:

$$c_1\vec{\boldsymbol{v_1}} + c_2\vec{\boldsymbol{v_2}} + \cdots + c_n\vec{\boldsymbol{v_k}}$$

where $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{v_i}} \in \mathbb{V}$

Note

This is the only way to form new vectors, since in a vector space we can only add two vectors or multiply a vector by a scalar.

Definition 2

The **span** of a set $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}\}$ of vectors in a vector space $\mathbb{V}$ is the set of all linear combinations of these vectors. Denoted **span** $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}\}$

Definition 1

For a vector space $\mathbb{V}$, a **linear combination** of vectors is:

$$c_1 \vec{\boldsymbol{v_1}} + c_2 \vec{\boldsymbol{v_2}} + \cdots + c_n \vec{\boldsymbol{v_k}}$$

where $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{v_i}} \in \mathbb{V}$

Note

This is the only way to form new vectors, since in a vector space we can only add two vectors or multiply a vector by a scalar.

Definition 2

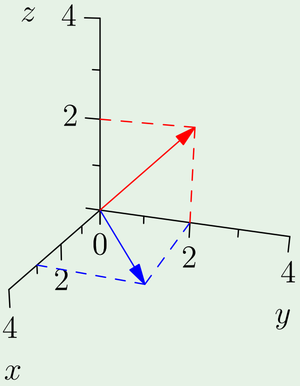
The **span** of a set $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}\}$ of vectors in a vector space $\mathbb{V}$ is the set of all linear combinations of these vectors. Denoted **span** $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}\}$

Note

If the **span** $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}\} = \mathbb{V}$ we say the set spans the vector space.

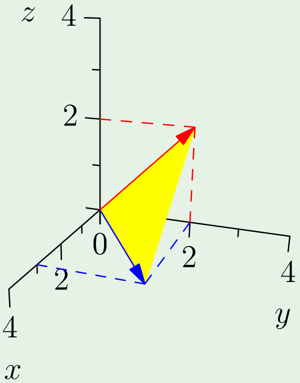
Example 3

Consider $\textbf{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.



Example 3

Consider **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.



This spanning set is the plane defined by these two vectors.

Example 4

Let up look closer at this spanning set. Where we give names to the two vectors:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Example 4

Let up look closer at this spanning set. Where we give names to the two vectors:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

We can then write a general vector in the spanning set as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{v}}$$

Example 4

Let up look closer at this spanning set. Where we give names to the two vectors:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

We can then write a general vector in the spanning set as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{v}} = a \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Example 4

Let up look closer at this spanning set. Where we give names to the two vectors:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

We can then write a general vector in the spanning set as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{v}} = a\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3a \\ 2a + 2b \\ 2b \end{bmatrix}$$

Example 4

We can write the vector equation as the system:

$$
\begin{aligned}
x &= 3a && \Rightarrow a = \frac{x}{3} \\
y &= 2a + 2b \\
z &= 2b && \Rightarrow b = \frac{z}{2}
\end{aligned}
$$

Example 4

We can write the vector equation as the system:

$$\begin{aligned} x &= 3a & &\Rightarrow a = \frac{x}{3} \\ y &= 2a + 2b & & \\ z &= 2b & &\Rightarrow b = \frac{z}{2} \end{aligned}$$

So, we can write y as

$$y = 2a + 2b$$

Example 4

We can write the vector equation as the system:

$$\begin{aligned} x &= 3a && \Rightarrow a = \frac{x}{3} \\ y &= 2a + 2b \\ z &= 2b && \Rightarrow b = \frac{z}{2} \end{aligned}$$

So, we can write y as

$$\begin{aligned} y &= 2a + 2b \\ &= 2\left(\frac{x}{3}\right) + 2\left(\frac{z}{2}\right) \end{aligned}$$

Example 4

We can write the vector equation as the system:

$$\begin{aligned} x &= 3a && \Rightarrow a = \frac{x}{3} \\ y &= 2a + 2b \\ z &= 2b && \Rightarrow b = \frac{z}{2} \end{aligned}$$

So, we can write y as

$$\begin{aligned} y &= 2a + 2b \\ &= 2\left(\frac{x}{3}\right) + 2\left(\frac{z}{2}\right) \\ &= \frac{2}{3}x + z \end{aligned}$$

Example 4

We can write the vector equation as the system:

$$\begin{aligned} x &= 3a && \Rightarrow a = \frac{x}{3} \\ y &= 2a + 2b \\ z &= 2b && \Rightarrow b = \frac{z}{2} \end{aligned}$$

So, we can write y as

$$\begin{aligned} y &= 2a + 2b \\ &= 2\left(\frac{x}{3}\right) + 2\left(\frac{z}{2}\right) \\ &= \frac{2}{3}x + z \end{aligned}$$

Which is equivalent to $2x - 3y + 3z = 0$, the equation of the yellow plane.

Theorem 5

An additional vector only changes a spanning set if and only if it is not a linear combination of the original vectors in the set.

Theorem 5

An additional vector only changes a spanning set if and only if it is not a linear combination of the original vectors in the set.

Example 6

Consider adding $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Theorem 5

An additional vector only changes a spanning set if and only if it is not a linear combination of the original vectors in the set.

Example 6

Consider adding $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Since we can write

$$-1 \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

we see that this doesn't change to the spanning set.

Example 7

Consider adding $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Example 7

Consider adding $\begin{bmatrix}1\\1\\1\end{bmatrix}$ to **span** $\left\{\begin{bmatrix}3\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\2\end{bmatrix}\right\}$.

This would expand the spanning set.

Example 7

Consider adding $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

This would expand the spanning set.

To show this, let us try to find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Example 7

Consider adding $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

This would expand the spanning set.

To show this, let us try to find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Which is equivalent to solving the inconsistent system

$$\begin{aligned} 1 &= 3c_1 \\ 1 &= 2c_1 + 2c_2 \\ 1 &= 2c_2 \end{aligned}$$

Example 8

What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

Example 8

What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

It turns out the spanning set is $\mathbb{R}^3$.

Example 8

What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

It turns out the spanning set is $\mathbb{R}^3$.

To show this, we then need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example 8

What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

It turns out the spanning set is $\mathbb{R}^3$.

To show this, we then need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example 8

What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

It turns out the spanning set is $\mathbb{R}^3$.

To show this, we then need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Which has a unique solution for any $x, y, z \in \mathbb{R}$.

Theorem 9

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{R}^n$*, a vector* $\vec{b} \in \mathbb{R}^n$ *is in* **span** $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ *if and only if there is at least one solution to the matrix equation* $A\vec{x} = \vec{b}$*. Where* A *is formed from the column vectors* $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$.

Theorem 9

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{R}^n$*, a vector* $\vec{b} \in \mathbb{R}^n$ *is in* **span** $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ *if and only if there is at least one solution to the matrix equation* $A\vec{x} = \vec{b}$*. Where* A *is formed from the column vectors* $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$.

Note

We can write spanning sets using set builder notation.

Example 10

$$\textbf{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\} = \left\{c\begin{bmatrix}2\\1\end{bmatrix} \;\middle|\; c \in \mathbb{R}\right\}$$

Example 10

$$\textbf{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\} = \left\{c\begin{bmatrix}2\\1\end{bmatrix} \;\middle|\; c \in \mathbb{R}\right\}$$

Example 11

$$\textbf{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \left\{c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in \mathbb{R}\right\}$$

Example 10

$$\textbf{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\} = \left\{c\begin{bmatrix}2\\1\end{bmatrix} \;\middle|\; c \in \mathbb{R}\right\}$$

Example 11

$$\begin{aligned}\textbf{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} &= \left\{c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in \mathbb{R}\right\}\\ &= \left\{\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in \mathbb{R}\right\}\end{aligned}$$

Theorem 12

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{V}$, **span** $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ *is a subspace of* $\mathbb{V}$.

Theorem 12

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{V}$, $\textbf{span}\left\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\right\}$ *is a subspace of* $\mathbb{V}$.

Proof

The proof comes from the subspace theorem we saw last section.

Theorem 12

For $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k} \in \mathbb{V}$*,* $\textbf{span}\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ *is a subspace of* $\mathbb{V}$*.*

Proof

The proof comes from the subspace theorem we saw last section.
Let $\vec{u}$ and $\vec{w}$ be two vectors in the spanning set, which means there are scalars r_i and s_i such that

$$\vec{u} = r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_n\vec{v_k} \quad \text{and } \vec{w} \quad = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_k}$$

Theorem 12

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{V}$, $\textbf{span}\left\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\right\}$ *is a subspace of* $\mathbb{V}$*.*

Proof

The proof comes from the subspace theorem we saw last section.
Let $\vec{u}$ and $\vec{w}$ be two vectors in the spanning set, which means there are scalars r_i and s_i such that

$$\vec{u} = r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_n\vec{v_k} \quad \text{and } \vec{w} \quad = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_k}$$

So, for any $a, b \in \mathbb{R}$:

$$a\vec{u} + b\vec{w}$$

Theorem 12

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{V}$, $\textbf{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a subspace of $\mathbb{V}$.

Proof

The proof comes from the subspace theorem we saw last section.
Let $\vec{u}$ and $\vec{w}$ be two vectors in the spanning set, which means there are scalars r_i and s_i such that

$$\vec{u} = r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_n\vec{v_k} \quad \text{and } \vec{w} \quad = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_k}$$

So, for any $a, b \in \mathbb{R}$:

$$a\vec{u} + b\vec{w} = a(r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_n\vec{v_k}) + b(s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_k})$$

Theorem 12

For $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}} \in \mathbb{V}$, $\textbf{span}\left\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}\right\}$ *is a subspace of* $\mathbb{V}$*.*

Proof

The proof comes from the subspace theorem we saw last section.
Let $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{w}}$ be two vectors in the spanning set, which means there are scalars r_i and s_i such that

$$\vec{\boldsymbol{u}} = r_1\vec{\boldsymbol{v_1}} + r_2\vec{\boldsymbol{v_2}} + \cdots + r_n\vec{\boldsymbol{v_k}} \quad \text{and } \vec{\boldsymbol{w}} \quad = s_1\vec{\boldsymbol{v_1}} + s_2\vec{\boldsymbol{v_2}} + \cdots + s_n\vec{\boldsymbol{v_k}}$$

So, for any $a, b \in \mathbb{R}$:

$$\begin{aligned} a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{w}} &= a(r_1\vec{\boldsymbol{v_1}} + r_2\vec{\boldsymbol{v_2}} + \cdots + r_n\vec{\boldsymbol{v_k}}) + b(s_1\vec{\boldsymbol{v_1}} + s_2\vec{\boldsymbol{v_2}} + \cdots + s_n\vec{\boldsymbol{v_k}}) \\ &= (ar_1 + bs_2)\vec{\boldsymbol{v_1}} + (ar_2 + bs_2)\vec{\boldsymbol{v_2}} + \cdots + (ar_n + bs_n)\vec{\boldsymbol{v_k}} \end{aligned}$$

Theorem 12

For $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}} \in \mathbb{V}$, **span** $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}\}$ *is a subspace of $\mathbb{V}$.*

Proof

The proof comes from the subspace theorem we saw last section.
Let $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{w}}$ be two vectors in the spanning set, which means there are scalars r_i and s_i such that

$$\vec{\boldsymbol{u}} = r_1\vec{\boldsymbol{v_1}} + r_2\vec{\boldsymbol{v_2}} + \cdots + r_n\vec{\boldsymbol{v_k}} \quad \text{and } \vec{\boldsymbol{w}} \quad = s_1\vec{\boldsymbol{v_1}} + s_2\vec{\boldsymbol{v_2}} + \cdots + s_n\vec{\boldsymbol{v_k}}$$

So, for any $a, b \in \mathbb{R}$:

$$\begin{aligned} a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{w}} &= a(r_1\vec{\boldsymbol{v_1}} + r_2\vec{\boldsymbol{v_2}} + \cdots + r_n\vec{\boldsymbol{v_k}}) + b(s_1\vec{\boldsymbol{v_1}} + s_2\vec{\boldsymbol{v_2}} + \cdots + s_n\vec{\boldsymbol{v_k}}) \\ &= (ar_1 + bs_2)\vec{\boldsymbol{v_1}} + (ar_2 + bs_2)\vec{\boldsymbol{v_2}} + \cdots + (ar_n + bs_n)\vec{\boldsymbol{v_k}} \end{aligned}$$

Which means $a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{w}}$ is in the spanning set and we have closure.

Definition 13

For any $m \times n$ matrix $\boldsymbol{A}$, the **column space**, denoted **Col** $\boldsymbol{A}$, is the span of the column vectors of $\boldsymbol{A}$, and is a subspace of $\mathbb{R}^m$.

Definition 13

For any $m \times n$ matrix $\boldsymbol{A}$, the **column space**, denoted $\textbf{Col } \boldsymbol{A}$, is the span of the column vectors of $\boldsymbol{A}$, and is a subspace of $\mathbb{R}^m$.

Example 14

Consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

Definition 13

For any $m \times n$ matrix $\boldsymbol{A}$, the **column space**, denoted $\textbf{Col } \boldsymbol{A}$, is the span of the column vectors of $\boldsymbol{A}$, and is a subspace of $\mathbb{R}^m$.

Example 14

Consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

The column space of $\boldsymbol{B}$ is a subspace of $\mathbb{R}^2$ and defined:

$$\textbf{Col } \boldsymbol{B} = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 5 \end{bmatrix} \;\middle|\; c_1, \ldots, c_5 \in \mathbb{R} \right\}$$

Definition 15

A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of vectors in a vector space $\mathbb{V}$ is **linearly independent** if no vector of the set can be written as a linear combination of the others. Otherwise it is **linearly dependent**.

Definition 15

A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of vectors in a vector space $\mathbb{V}$ is **linearly independent** if no vector of the set can be written as a linear combination of the others. Otherwise it is **linearly dependent**.

Testing for Linear Independence

To test for linear independence of a set of k vectors $\vec{v_i} \in \mathbb{R}^n$, we consider the system:

$$\begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_k} \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$$

The column vectors of A are linearly independent if and only if the solution $c_1 = c_2 = \cdots = c_k = 0$ is unique.

Example 16

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\3\\2\end{bmatrix}$ linearly independent?

Example 16

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\3\\2\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 3\\1 & -1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Example 16

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\3\\2\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 3\\1 & -1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Since $|\boldsymbol{A}| = 5$, we know that $\boldsymbol{A}$ is invertible and hence a unique solution exists. This means that these vectors are linearly independent.

Example 17

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\3\\2\end{bmatrix}$, and $\begin{bmatrix}5\\-1\\0\end{bmatrix}$ linearly independent?

Example 17

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\3\\2\end{bmatrix}$, and $\begin{bmatrix}5\\-1\\0\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 5\\1 & 1 & 3 & -1\\1 & -1 & 2 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Example 17

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\3\\2\end{bmatrix}$, and $\begin{bmatrix}5\\-1\\0\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 5\\1 & 1 & 3 & -1\\1 & -1 & 2 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

We have more columns than rows, which means there will be at least one free variable. Thus, the solution (if one exists) won't be unique, so these vectors are not linearly independent.

Example 18

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

Example 18

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 18

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

Example 18

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 18

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And thus, these vectors are not linearly independent.

Example 18

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And thus, these vectors are not linearly independent.

Moreover, since

$$\begin{bmatrix}1\\1\\1\end{bmatrix} + 3\begin{bmatrix}-1\\0\\1\end{bmatrix} - \begin{bmatrix}-2\\1\\4\end{bmatrix} = 0$$

we can see that any vector can be written as a combination of the others.

Definition 19

A set of vector functions $\{\vec{\boldsymbol{v_1}}(t), \vec{\boldsymbol{v_2}}(t), \ldots, \vec{\boldsymbol{v_k}}\}$ in a vector space $\mathbb{V}$ is **linearly independent** on an interval I if, for *all* $t \in I$, the equation

$$c_1\vec{\boldsymbol{v_1}}(t) + c_2\vec{\boldsymbol{v_2}}(t) + \cdots + c_k\vec{\boldsymbol{v_k}}(t) = \vec{\mathbf{0}} \quad (\text{where } c_i \in \mathbb{R})$$

has the only solution: $c_1 = c_2 = \cdots = c_k = 0$.

If for any value $t_0 \in I$ there is any solution with $c_i \neq 0$, the vector functions $\vec{\boldsymbol{v_1}}(t), \vec{\boldsymbol{v_2}}(t), \ldots, \vec{\boldsymbol{v_k}}(t)$ are **linearly dependent**.

Example 20

Are the vectors

$$\vec{v_1}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v_2}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v_3}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?

Example 20

Are the vectors

$$\vec{v_1}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v_2}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v_3}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?
We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0}$$

Example 20

Are the vectors

$$\vec{\boldsymbol{v}}_{\mathbf{1}}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{\boldsymbol{v}}_{\mathbf{2}}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{\boldsymbol{v}}_{\mathbf{3}}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?
We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{\boldsymbol{v}}_{\mathbf{1}}(t) + c_2 \vec{\boldsymbol{v}}_{\mathbf{2}}(t) + c_3 \vec{\boldsymbol{v}}_{\mathbf{3}}(t) = \vec{\mathbf{0}}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} e^{(0)} \\ 0 \\ 2e^{(0)} \end{bmatrix} + c_2 \begin{bmatrix} e^{-(0)} \\ 3e^{-(0)} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{2(0)} \\ e^{2(0)} \\ e^{2(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 20

Are the vectors

$$\vec{v_1}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v_2}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v_3}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?
We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{v_1}(t) + c_2 \vec{v_2}(t) + c_3 \vec{v_3}(t) = \vec{0}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 20

Are the vectors

$$\vec{v_1}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v_2}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v_3}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?
We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{v_1}(t) + c_2 \vec{v_2}(t) + c_3 \vec{v_3}(t) = \vec{0}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 20

Are the vectors

$$\vec{\boldsymbol{v_1}}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{\boldsymbol{v_2}}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{\boldsymbol{v_3}}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?
We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{\boldsymbol{v_1}}(t) + c_2 \vec{\boldsymbol{v_2}}(t) + c_3 \vec{\boldsymbol{v_3}}(t) = \vec{\boldsymbol{0}}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right]$$

Example 20

Are the vectors

$$\vec{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?

We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example 20

Are the vectors

$$\vec{\boldsymbol{v_1}}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{\boldsymbol{v_2}}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{\boldsymbol{v_3}}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?
We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{\boldsymbol{v_1}}(t) + c_2 \vec{\boldsymbol{v_2}}(t) + c_3 \vec{\boldsymbol{v_3}}(t) = \vec{\boldsymbol{0}}$$

Since this equation has to hold for all t, it has to hold for $t = 0$:

Since the unique solution is $c_1 = c_2 = c_3 = 0$, these vectors are linearly independent.

Example 21

Are the following functions linearly independent?

$$\vec{\boldsymbol{v_1}}(t) = e^{t}, \quad \vec{\boldsymbol{v_2}}(t) = 5e^{-t}, \quad \vec{\boldsymbol{v_3}}(t) = e^{3t}$$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \begin{bmatrix} e^{t} \end{bmatrix}, \quad \vec{v_2}(t) = \begin{bmatrix} 5e^{-t} \end{bmatrix}, \quad \vec{v_3}(t) = \begin{bmatrix} e^{3t} \end{bmatrix}$$

We can think of each of these as a one-dimensional vector.

Example 21

Are the following functions linearly independent?

$$\vec{\boldsymbol{v}}_{\mathbf{1}}(t) = \begin{bmatrix} e^t \end{bmatrix}, \quad \vec{\boldsymbol{v}}_{\mathbf{2}}(t) = \begin{bmatrix} 5e^{-t} \end{bmatrix}, \quad \vec{\boldsymbol{v}}_{\mathbf{3}}(t) = \begin{bmatrix} e^{3t} \end{bmatrix}$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \vec{\boldsymbol{v}}_{\mathbf{1}}(t) + c_2 \vec{\boldsymbol{v}}_{\mathbf{2}}(t) + c_3 \vec{\boldsymbol{v}}_{\mathbf{3}}(t) = \vec{\mathbf{0}} \quad (\text{for all } t \in \mathbb{R})$$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^t\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

For $t =$ 0: $\quad c_1 \cdot 5e^{(0)} + \quad c_2 \cdot e^{-(0)} + \quad c_3 \cdot e^{3(0)} = 0$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^{t}\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

$$\begin{array}{lllll} \text{For } t = & 0: & c_1 \cdot 5e^{(0)} + & c_2 \cdot e^{-(0)} + & c_3 \cdot e^{3(0)} = 0 \\ \text{For } t = & 1: & c_1 \cdot 5e^{(1)} + & c_2 \cdot e^{-(1)} + & c_3 \cdot e^{3(1)} = 0 \end{array}$$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^{t}\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

$$\begin{array}{lrrr}
\text{For } t = 0: & c_1 \cdot 5e^{(0)} + & c_2 \cdot e^{-(0)} + & c_3 \cdot e^{3(0)} = 0 \\
\text{For } t = 1: & c_1 \cdot 5e^{(1)} + & c_2 \cdot e^{-(1)} + & c_3 \cdot e^{3(1)} = 0 \\
\text{For } t = -1: & c_1 \cdot 5e^{(-1)} + & c_2 \cdot e^{-(-1)} + & c_3 \cdot e^{3(-1)} = 0
\end{array}$$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^t\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad \text{(for all } t \in \mathbb{R})$$

$$\begin{array}{lrcrcrl} \text{For } t = 0: & c_1 & + & 5c_2 & + & c_3 & = 0 \\ \text{For } t = 1: & ec_1 & + & \frac{5}{e}c_2 & + & e^3c_3 & = 0 \\ \text{For } t = -1: & \frac{1}{e}c_1 & + & ec_2 & + & \frac{1}{e^3}c_3 & = 0 \end{array}$$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^t\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad \text{(for all } t \in \mathbb{R})$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ e & \frac{5}{e} & e^3 & 0 \\ \frac{1}{e} & e & \frac{1}{e^3} & 0 \end{array}\right]$$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^t\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad \text{(for all } t \in \mathbb{R})$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example 21

Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^t\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since we have a unique solution, $c_1 = c_2 = c_3 = 0$, these functions are linearly independent.

Definition 22

The **Wronskian** of functions $f_1, f_2, \ldots, f_k$ on interval I is the determinant:

$$W[f_1, f_2, \ldots, f_k](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{vmatrix}$$

Definition 22

The **Wronskian** of functions $f_1, f_2, \ldots, f_k$ on interval I is the determinant:

$$W[f_1, f_2, \ldots, f_k](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{vmatrix}$$

Theorem 23

If $W[f_1, f_2, \ldots, f_k](t) \neq 0$ for all $t \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is a linearly independent set of functions on I.

Definition 22

The **Wronskian** of functions $f_1, f_2, \ldots, f_k$ on interval I is the determinant:

$$W[f_1, f_2, \ldots, f_k](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{vmatrix}$$

Theorem 23

If $W[f_1, f_2, \ldots, f_k](t) \neq 0$ for all $t \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is a linearly independent set of functions on I.

Note

If $\{f_1, f_2, \ldots, f_k\}$ are linearly dependent, then $W[f_1, f_2, \ldots, f_k](t) = 0$ for all $t \in I$. Thus, to show independence we only need to find a single t that makes the Wronskian nonzero.

Example 24

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

Example 24

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

$$W(t) = \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

Example 24

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

$$\begin{aligned} W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \end{aligned}$$

Example 24

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

$$\begin{aligned}
W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\
&= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \\
&= (t^2+1)(0-4) - (t^2-1)(0-4) + (2t+5)(4t-4t)
\end{aligned}$$

Example 24

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

$$\begin{aligned}
W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\
&= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \\
&= (t^2+1)(0-4) - (t^2-1)(0-4) + (2t+5)(4t-4t) \\
&= -4t^2 - 4 + 4t^2 - 4
\end{aligned}$$

Example 24

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

$$\begin{aligned}
W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\
&= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \\
&= (t^2+1)(0-4) - (t^2-1)(0-4) + (2t+5)(4t-4t) \\
&= -4t^2 - 4 + 4t^2 - 4 = -8
\end{aligned}$$

Since $W(t) = -8 \neq 0$, this is a set of linearly independent functions.

Example 25

Let us consider the converse:

Does the Wronskian being zero imply dependence?

Example 25

Let us consider the converse:

Does the Wronskian being zero imply dependence?

In general, the answer is no.

Example 25

Let us consider the converse:

Does the Wronskian being zero imply dependence?

In general, the answer is no.

Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

Example 25

Let us consider the converse:

Does the Wronskian being zero imply dependence?

In general, the answer is no.

Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

Example 25

Let us consider the converse:

Does the Wronskian being zero imply dependence?

In general, the answer is no.

Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

So,

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

Definition 26

The set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a **basis** for vector space $\mathbb{V}$, provided that

- $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a linearly independent set
- **span**$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\} = \mathbb{V}$

Definition 26

The set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a **basis** for vector space $\mathbb{V}$, provided that

- $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a linearly independent set
- $\textbf{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\} = \mathbb{V}$

Example 27

The vectors

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for $\mathbb{R}^3$

Definition 26

The set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a **basis** for vector space $\mathbb{V}$, provided that

- $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a linearly independent set
- **span**$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\} = \mathbb{V}$

Example 27

The vectors

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for $\mathbb{R}^3$

We saw earlier that these vectors span $\mathbb{R}^3$.

Definition 26

The set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a **basis** for vector space $\mathbb{V}$, provided that

- $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a linearly independent set
- $\textbf{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\} = \mathbb{V}$

Example 27

The vectors

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for $\mathbb{R}^3$

We saw earlier that these vectors span $\mathbb{R}^3$.

It's easy to see that $c_1\vec{i} + c_2\vec{j} + c_3\vec{k} = \vec{0}$ has the unique solution $c_1 = c_2 = c_3 = 0$.

Definition 28

The **standard basis** for $\mathbb{R}^n$ is $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$ where

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \cdots, \ \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are the column vectors of the identity matrix $\boldsymbol{I}_n$.

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

We will do so by arbitrarily choosing values for $x_1 = a$, $x_2 = b$, and $x_3 = c$, we can then determine x_4 using the equation of the hyperplane.

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

We will do so by arbitrarily choosing values for $x_1 = a$, $x_2 = b$, and $x_3 = c$, we can then determine x_4 using the equation of the hyperplane.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

We will do so by arbitrarily choosing values for $x_1 = a$, $x_2 = b$, and $x_3 = c$, we can then determine x_4 using the equation of the hyperplane.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 2a + 3b - 4c \end{bmatrix} =$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

We will do so by arbitrarily choosing values for $x_1 = a$, $x_2 = b$, and $x_3 = c$, we can then determine x_4 using the equation of the hyperplane.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 2a + 3b - 4c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

We will do so by arbitrarily choosing values for $x_1 = a$, $x_2 = b$, and $x_3 = c$, we can then determine x_4 using the equation of the hyperplane.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 2a + 3b - 4c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

Since $a, b, c \in \mathbb{R}$ were arbitrary, we see these three vectors span the hyperplane.

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Now, we need to show that the vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 \end{array}\right]$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The (unique) solution is $c_1 = c_2 = c_3 = 0$, thus these vectors are linearly independent.

Example 29

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

So, we see that the hyperplane has a basis of three vectors.

It looks like this hyperplane is a three-dimensional subspace of a four-dimensional space.

Example 30

It is possible for a vector space to have multiple bases.

Example 30

It is possible for a vector space to have multiple bases.
For $\mathbb{R}^2$, one is the standard basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Example 30

It is possible for a vector space to have multiple bases.
For $\mathbb{R}^2$, one is the standard basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

but, another basis is given by

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Theorem 31

The number of vectors in a basis is always the same for a particular vector space

Theorem 31

The number of vectors in a basis is always the same for a particular vector space

Proof

The proof is in Appendix LT of your textbook, on page 602.

Theorem 31

The number of vectors in a basis is always the same for a particular vector space

Proof

The proof is in Appendix LT of your textbook, on page 602.

Definition 32

The **dimension** of a vector space $\mathbb{V}$ is the number of vectors in any basis of $\mathbb{V}$.

Theorem 31

The number of vectors in a basis is always the same for a particular vector space

Proof

The proof is in Appendix LT of your textbook, on page 602.

Definition 32

The **dimension** of a vector space $\mathbb{V}$ is the number of vectors in any basis of $\mathbb{V}$.

Definition 33

If a vector space is so large that cannot be spanned by a finite set of vectors, it is called **infinite-dimensional**.

Example 34

The solution to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

Example 34

The solution to the system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

Writing this system in RREF gives

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 5x_3 & - & x_4 & = & 0 \\ & & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}$$

Example 34

The solution to the system

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

Writing this system in RREF gives

$$\begin{array}{rcrcrcrcl} x_1 & & & - & 5x_3 & - & x_4 & = & 0 \\ & & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}$$

The two free variables tell us that the solution to this system will be a two-parameter family.

Example 34

The solution to the system

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

To find a basis, let $x_3 = a$ and $x_4 = b$, be arbitrary real numbers.

Example 34

The solution to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

To find a basis, let $x_3 = a$ and $x_4 = b$, be arbitrary real numbers.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5a + b \\ -2a - b \\ a \\ b \end{bmatrix}$$

Example 34

The solution to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

To find a basis, let $x_3 = a$ and $x_4 = b$, be arbitrary real numbers.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5a + b \\ -2a - b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Example 34

The solution to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

The two vectors

$$\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

span the subspace.

Example 34

The solution to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

The two vectors

$$\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

span the subspace.

They are also linearly independent, thus this a basis of the subspace.

Example 34

The solution to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

The two vectors

$$\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

span the subspace.

They are also linearly independent, thus this a basis of the subspace.

Which means the dimension is 2.

Properties of the Column Space of a Matrix

1. The pivot columns of a matrix $\boldsymbol{A}$ form a basis for Col $\boldsymbol{A}$.
   - A pivot column is a column of $\boldsymbol{A}$ that corresponds to a column in the RREF of $\boldsymbol{A}$ with a leading 1.

Properties of the Column Space of a Matrix

1. The pivot columns of a matrix $\boldsymbol{A}$ form a basis for Col $\boldsymbol{A}$.
   - A pivot column is a column of $\boldsymbol{A}$ that corresponds to a column in the RREF of $\boldsymbol{A}$ with a leading 1.
2. The dimension of the column space of $\boldsymbol{A}$ is the rank of $\boldsymbol{A}$.
   - $\textbf{rank}(\boldsymbol{A}) = \textbf{dim}(\textbf{Col } \boldsymbol{A})$

Properties of the Column Space of a Matrix

1. The pivot columns of a matrix $\boldsymbol{A}$ form a basis for Col $\boldsymbol{A}$.
   - A pivot column is a column of $\boldsymbol{A}$ that corresponds to a column in the RREF of $\boldsymbol{A}$ with a leading 1.
2. The dimension of the column space of $\boldsymbol{A}$ is the rank of $\boldsymbol{A}$.
   - $\textbf{rank}(\boldsymbol{A}) = \textbf{dim}(\textbf{Col } \boldsymbol{A})$

Example 35

Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 & 8 \end{bmatrix}$$

Properties of the Column Space of a Matrix

1. The pivot columns of a matrix $\boldsymbol{A}$ form a basis for Col $\boldsymbol{A}$.
   - A pivot column is a column of $\boldsymbol{A}$ that corresponds to a column in the RREF of $\boldsymbol{A}$ with a leading 1.
2. The dimension of the column space of $\boldsymbol{A}$ is the rank of $\boldsymbol{A}$.
   - $\mathbf{rank}(\boldsymbol{A}) = \mathbf{dim}(\mathbf{Col}\ \boldsymbol{A})$

Example 35

Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Properties of the Column Space of a Matrix

1. The pivot columns of a matrix $\boldsymbol{A}$ form a basis for Col $\boldsymbol{A}$.
   - A pivot column is a column of $\boldsymbol{A}$ that corresponds to a column in the RREF of $\boldsymbol{A}$ with a leading 1.
2. The dimension of the column space of $\boldsymbol{A}$ is the rank of $\boldsymbol{A}$.
   - $\textbf{rank}(\boldsymbol{A}) = \textbf{dim}(\textbf{Col } \boldsymbol{A})$

Example 35

Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

The pivot columns are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which means $\textbf{rank}(\boldsymbol{A}) = 2$ and thus the dimension of the column space is 2.

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent:

- $\boldsymbol{A}$ is invertible.
- The column vectors of $\boldsymbol{A}$ are linearly independent.
- Every column of $\boldsymbol{A}$ is a pivot column.
- The column vectors of $\boldsymbol{A}$ form a basis for $\mathbf{Col}\ \boldsymbol{A}$
- $\mathbf{rank}\ \boldsymbol{A} = n$

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent:

- $\boldsymbol{A}$ is invertible.
- The column vectors of $\boldsymbol{A}$ are linearly independent.
- Every column of $\boldsymbol{A}$ is a pivot column.
- The column vectors of $\boldsymbol{A}$ form a basis for **Col** $\boldsymbol{A}$
- **rank** $\boldsymbol{A} = n$

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent:

- $\boldsymbol{A}$ is invertible.
- The column vectors of $\boldsymbol{A}$ are linearly independent.
- Every column of $\boldsymbol{A}$ is a pivot column.
- The column vectors of $\boldsymbol{A}$ form a basis for **Col** $\boldsymbol{A}$
- **rank** $\boldsymbol{A} = n$

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent:

- $\boldsymbol{A}$ is invertible.
- The column vectors of $\boldsymbol{A}$ are linearly independent.
- Every column of $\boldsymbol{A}$ is a pivot column.
- The column vectors of $\boldsymbol{A}$ form a basis for **Col** $\boldsymbol{A}$
- **rank** $\boldsymbol{A} = n$

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent:

- $\boldsymbol{A}$ is invertible.
- The column vectors of $\boldsymbol{A}$ are linearly independent.
- Every column of $\boldsymbol{A}$ is a pivot column.
- The column vectors of $\boldsymbol{A}$ form a basis for **Col** $\boldsymbol{A}$
- **rank** $\boldsymbol{A} = n$

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$\text{For } x = -1: \qquad c_3(-1)^2 + \quad c_2(-1) + \quad c_1 = \quad 0$$

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$\begin{array}{lrrrr} \text{For } x = -1: & c_3(-1)^2 + & c_2(-1) + & c_1 = & 0 \\ \text{For } x = 0: & c_3(0)^2 + & c_2(0) + & c_1 = & 0 \end{array}$$

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$
\begin{array}{lrrrr}
\text{For } x = -1: & c_3(-1)^2 + & c_2(-1) + & c_1 = & 0 \\
\text{For } x = 0: & c_3(0)^2 + & c_2(0) + & c_1 = & 0 \\
\text{For } x = 1: & c_3(1)^2 + & c_2(1) + & c_1 = & 0
\end{array}
$$

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$\begin{array}{lrcr}
\text{For } x = -1: & c_3 - \; c_2 + \; c_1 & = & 0 \\
\text{For } x = \;\;\, 0: & c_1 & = & 0 \\
\text{For } x = \;\;\, 1: & c_3 + \; c_2 + \; c_1 & = & 0
\end{array}$$

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So, since we have the unique solution $c_1 = c_2 = c_3 = 0$, these functions are linearly independent and thus form a basis of $\mathbb{P}_2$.

Example 36

The set of polynomials $\{x^2, x, 1\}$ span $\mathbb{P}_2$, since a typical vector is of the form $ax^2 + bx + c$.

So, if this set is linearly independent, then it is a basis of $\mathbb{P}_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So, since we have the unique solution $c_1 = c_2 = c_3 = 0$, these functions are linearly independent and thus form a basis of $\mathbb{P}_2$.

Which means $\mathbf{dim}\,\mathbb{P}_2 = 3$.

Example 37

What is the dimension of $\mathbb{P}$, the space of all polynomials?

Example 37

What is the dimension of $\mathbb{P}$, the space of all polynomials?

Suppose we have a possible finite basis of $\mathbb{P}$, where the highest degree polynomial in this basis is k.

Example 37

What is the dimension of $\mathbb{P}$, the space of all polynomials?

Suppose we have a possible finite basis of $\mathbb{P}$, where the highest degree polynomial in this basis is k.

But, there would be no way, using the basis to generate:

$$t^{k+1}$$

Example 37

What is the dimension of $\mathbb{P}$, the space of all polynomials?

Suppose we have a possible finite basis of $\mathbb{P}$, where the highest degree polynomial in this basis is k.

But, there would be no way, using the basis to generate:

$$t^{k+1}$$

Thus, $\mathbb{P}$ is infinite-dimensional. $(\mathbf{dim}(\mathbb{P}) = \infty)$.

Example 37

What is the dimension of $\mathbb{P}$, the space of all polynomials?

Suppose we have a possible finite basis of $\mathbb{P}$, where the highest degree polynomial in this basis is k.

But, there would be no way, using the basis to generate:

$$t^{k+1}$$

Thus, $\mathbb{P}$ is infinite-dimensional. ($\mathbf{dim}(\mathbb{P}) = \infty$).

Note

There are many infinite-dimensional spaces.

We have seen $\mathbb{P}$, $\mathcal{C}(I)$, and $\mathcal{C}^n(I)$.