Properties of Linear Transformations

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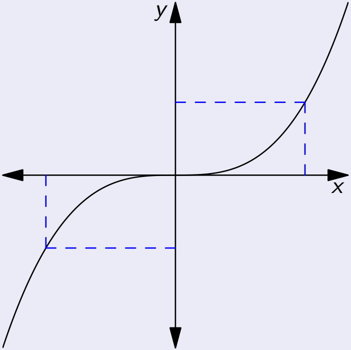
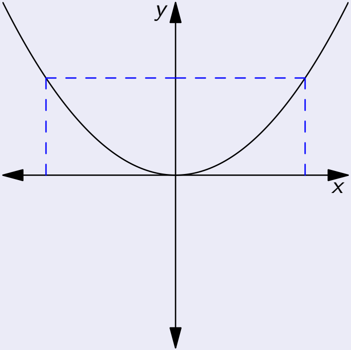


(a) $f(x) = x^3$ is injective



(b) $g(x) = x^2$ is not injective

Surjectivity

The set of output values of a function $f : \mathbb{X} \to \mathbb{Y}$ is a subset of the codomain $\mathbb{Y}$ and is called the **image** of the function. If the image is all of $\mathbb{Y}$, the function f is said to map **onto** $\mathbb{Y}$, or to be **surjective**.

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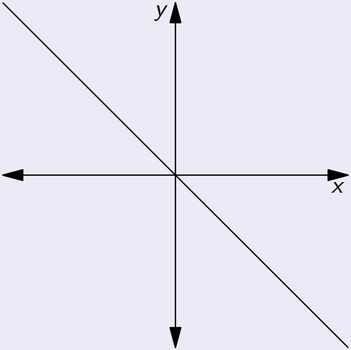
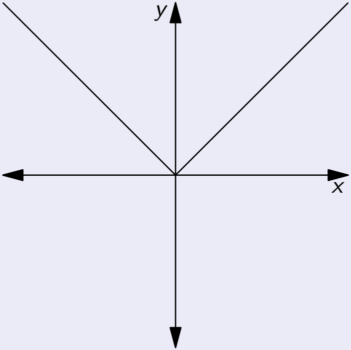


(a) $f(x) = -x$ is surjective



(b) $g(x) = |x|$ is not surjective

Example

The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$$

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$$u_1 = v_1 + v_2, \quad u_2 = v_1 - v_2, \quad u_3 = 2v_1 + v_2$$

Eliminating v_1 and v_2 gives us

$$3u_1 + u_2 - 2u_3 = 0$$

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Rank of a Linear Transformation

The dimension of the image of a linear transformation T is called its **rank**

$$\mathbf{rank}(T) = \mathbf{dim}(\mathbf{Im}(T))$$

Example

For $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$T(\vec{v}) = \boldsymbol{A}\vec{v} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \vec{w}$$

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We can write

$$\vec{\boldsymbol{w}} = v_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + v_2 \begin{bmatrix} -4 \\ 2 \end{bmatrix} + v_3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + v_4 \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

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Which means

$$\textbf{Im}(T) = \textbf{span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, $\mathbf{rank}(T) = \mathbf{dim}(\mathbf{Im}(T)) = \mathbf{dim}(\mathbf{Col}\ \boldsymbol{A}) = 2$.

Rank of a Matrix Multiplication Operator

For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{\boldsymbol{x}}) = \boldsymbol{A}\vec{\boldsymbol{x}}$, where $\boldsymbol{A} \in \mathbb{M}_{mn}$ and $\vec{\boldsymbol{x}} \in \mathbb{V}$, the image of T is the column space of A. (Thats is, $\mathbf{Im}(T) = \mathbf{Col}\ \boldsymbol{A}$.)

The pivot columns of A form a basis for $\mathbf{Im}(T)$.

Consequently,

$$\begin{aligned}\mathbf{rank}(T) &= \mathbf{dim}(\mathbf{Im}(T))\\ &= \mathbf{dim}(\mathbf{Col}\ \boldsymbol{A})\\ &= \text{The number of pivot columns in } \boldsymbol{A}.\end{aligned}$$

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Reminder

The basis of **Col** $\boldsymbol{A}$ must come from $\boldsymbol{A}$, *not* from the RREF of $\boldsymbol{A}$.

Recall that a linear transformation $T : \mathbb{V} \to \mathbb{W}$ must map the zero vector of $\mathbb{V}$ to the zero vector of $\mathbb{W}$.

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Kernel of a Linear Transformation

The **kernel** (or **nullspace**) of a linear transformation $T : \mathbb{V} \rightarrow \mathbb{W}$, denoted $\mathbf{ker}(T)$, is the set of vectors in $\mathbb{V}$ that are mapped by T to the zero vector of $\mathbb{W}$.

$$\mathbf{ker}(T) = \left\{ \vec{\boldsymbol{v}} \in \mathbb{V} \;\middle|\; T(\vec{\boldsymbol{v}}) = \vec{\mathbf{0}} \right\}$$

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Example

Consider the projection $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x, y, 0)$$

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$$\mathbf{ker}(T) = \{(0, 0, z) \mid z \in \mathbb{R}\}$$

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Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 + 2v_3 \\ 2v_1 + 3v_2 + 5v_3 \end{bmatrix}$$

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To find the vectors that are mapped to $\vec{\mathbf{0}}$, we have to solve the system:

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Thus, the kernel is any scalar multiple of $< -1, -1, \;\; 1 >$:

$$\mathbf{ker}(T) = \mathbf{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

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Thus,

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Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

What is the kernel of T?

To find the kernel we need to solve $\boldsymbol{A\vec{v}} = \vec{\mathbf{0}}$:

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In this case we have $v_1 = -\frac{1}{2}v_2$ and so, if we let our parameter be $v_2 = s$ we have

$$\mathbf{ker}(T) = \left\{ s \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

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These examples seem to suggest that the kernel of the linear transformation $T : \mathbb{V} \to \mathbb{W}$ is a subspace of $\mathbb{W}$.

Kernel Theorem

Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation from vector space $\mathbb{V}$ to vector space $\mathbb{W}$ with kernel $\mathbf{ker}(T)$.

Then,

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It is straightforward to verify 1, so let us prove 2.

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Thus, $\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}$ is in the kernel, which means $\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}}$ and thus T is injective.

Example

Consider the transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}}$, where

$$\boldsymbol{A} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix}$$

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So, we see that $v_1 = 2v_2 - 3v_4$ and $v_3 = 0$. If we let $v_2 = r$ and $v_4 = s$,

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The dimension of the kernel of T is 2.

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The dimension of the kernel of T is 2.
(Remember that the dimension of the image of T was 2.)

Dimension Theorem

Let $T : \mathbb{V} \rightarrow \mathbb{W}$ be a linear transformation from a finite vector space $\mathbb{V}$.

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- $D_2 : \mathbb{P}_3 \to \mathbb{P}_1$ had

$$\textbf{ker}(D_2) = \{cx + d\} \quad \text{and} \quad \textbf{Im}(D_2) = \{6ax + 2b\}$$

$$\textbf{dim}(\textbf{ker}(D_2)) + \textbf{dim}(\textbf{Im}(D_2)) = 2 + 2 = 4 = \textbf{dim}(\mathbb{P}_3)$$