Growth and Decay Phenomena

Department of Mathematics

Salt Lake Community College

Growth and Decay Equation

The differential equation

$$\frac{dy}{dt} = ky$$

covers two phenomena, depending on whether k is positive or negative.

Growth and Decay Equation

The differential equation

$$\frac{dy}{dt} = ky$$

covers two phenomena, depending on whether k is positive or negative.

Growth: If $k > 0$, k is called the **growth constant** or **rate of growth** and the equation is called the **growth equation**.

Growth and Decay Equation

The differential equation

$$\frac{dy}{dt} = ky$$

covers two phenomena, depending on whether k is positive or negative.

Growth: If $k > 0$, k is called the **growth constant** or **rate of growth** and the equation is called the **growth equation**.

Decay: If $k < 0$, k is called the **decay constant** or **rate of decay** and the equation is called the **decay equation**.

Growth and Decay Equation

The differential equation

$$\frac{dy}{dt} = ky$$

covers two phenomena, depending on whether k is positive or negative.

- Growth: If $k > 0$, k is called the **growth constant** or **rate of growth** and the equation is called the **growth equation**.
- Decay: If $k < 0$, k is called the **decay constant** or **rate of decay** and the equation is called the **decay equation**.

Note

We last saw this equation in section 1.1 where Thomas Malthus used it to estimate global population growth.

Solving the Growth and Decay Equation

Let us start by rewriting the growth and decay equation in a familiar form:

$$y' - ky = 0$$

Solving the Growth and Decay Equation

Let us start by rewriting the growth and decay equation in a familiar form:

$$y' - ky = 0$$

We see that in fact it is a linear first-order homogeneous equation.

Solving the Growth and Decay Equation

Let us start by rewriting the growth and decay equation in a familiar form:

$$y' - ky = 0$$

We see that in fact it is a linear first-order homogeneous equation.

We could use the theories discussed last section to solve this equation, but the easiest route is to use Separation of Variables.

$$\frac{dy}{y} = k\ dt$$

Solving the Growth and Decay Equation

Let us start by rewriting the growth and decay equation in a familiar form:

$$y' - ky = 0$$

We see that in fact it is a linear first-order homogeneous equation.

We could use the theories discussed last section to solve this equation, but the easiest route is to use Separation of Variables.

$$\frac{dy}{y} = k \ dt$$
$$\ln|y| = kt + C$$

Solving the Growth and Decay Equation

Let us start by rewriting the growth and decay equation in a familiar form:

$$y' - ky = 0$$

We see that in fact it is a linear first-order homogeneous equation.

We could use the theories discussed last section to solve this equation, but the easiest route is to use Separation of Variables.

$$\begin{aligned} \frac{dy}{y} &= k\ dt \\ \ln|y| &= kt + C \\ |y| &= e^{kt+C} \end{aligned}$$

Solving the Growth and Decay Equation

Let us start by rewriting the growth and decay equation in a familiar form:

$$y' - ky = 0$$

We see that in fact it is a linear first-order homogeneous equation.

We could use the theories discussed last section to solve this equation, but the easiest route is to use Separation of Variables.

$$\begin{aligned} \frac{dy}{y} &= k\ dt \\ \ln|y| &= kt + C \\ |y| &= e^{kt+C} = e^{C}e^{kt} \end{aligned}$$

Solving the Growth and Decay Equation

Let us start by rewriting the growth and decay equation in a familiar form:

$$y' - ky = 0$$

We see that in fact it is a linear first-order homogeneous equation.

We could use the theories discussed last section to solve this equation, but the easiest route is to use Separation of Variables.

$$\begin{aligned} \frac{dy}{y} &= k\ dt \\ \ln|y| &= kt + C \\ |y| &= e^{kt+C} = e^{C}e^{kt} \end{aligned}$$

Note that $e^C > 0$ for all $C \in \mathbb{R}$. Thus, if we replace e^C with an arbitrary constant A, which could be negative, we can dispose of the absolute value bars.

$$y(t) = Ae^{kt}, \quad A \in \mathbb{R}$$

Growth and Decay

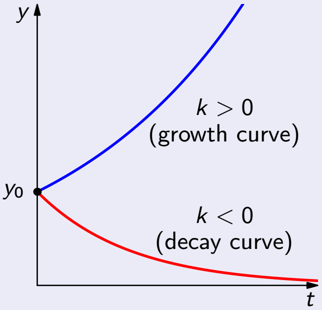
For each k, the solution of the IVP

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is given by

$$y(t) = y_0 e^{kt}$$

Solution curves are **growth curves** for $k > 0$ and **decay curves** for $k < 0$.



Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is the value of the decay constant k?

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is the value of the decay constant k?

Let us say that the initial amount of Carbon-14 is $Q(0) = Q_0$. So, after 5600 years, the half-life, there will be $\frac{1}{2}Q_0$ remaining. That is,

$$Q(5600) = Q_0 e^{5600k} = \frac{1}{2}Q_0$$

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is the value of the decay constant k?

Let us say that the initial amount of Carbon-14 is $Q(0) = Q_0$. So, after 5600 years, the half-life, there will be $\frac{1}{2}Q_0$ remaining. That is,

$$Q(5600) = Q_0 e^{5600k} = \frac{1}{2}Q_0 \Rightarrow k = -\frac{\ln(2)}{5600} \approx -0.00012378$$

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is $Q(t)$ at any time t?

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is $Q(t)$ at any time t?

Since we know k, we can plug it into the general solution:

$$Q(t) = Q_0 e^{-t\ln(2)/5600} \approx Q_0 e^{-0.00012378t}$$

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is the age of the charcoal, and hence the age of the paintings?

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is the age of the charcoal, and hence the age of the paintings?

We want to find the t-value that reduces Q_0 to 15% of it's initial amount. This means we need to solve:

$$Q_0 e^{-t\ln(2)/5600} = 0.15 Q_0$$

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$

What is the age of the charcoal, and hence the age of the paintings?

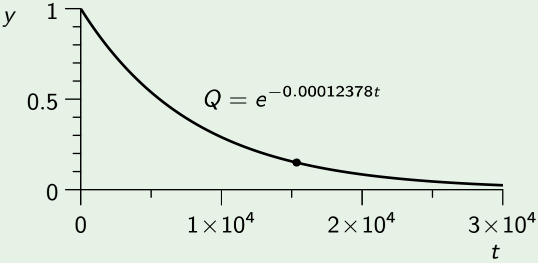
We want to find the t-value that reduces Q_0 to 15% of it's initial amount. This means we need to solve:

$$Q_0 e^{-t\ln(2)/5600} = 0.15 Q_0 \Rightarrow t = -\frac{5600\ln(0.15)}{\ln(2)} \approx 15,327 \text{ years}$$

Example 1

Radiocarbon Dating. Near Lascaux, France a cave was discovered containing multiple paintings on the walls, as well as the remains of a small fire. By chemical analysis it has been determined that the amount of Carbon-14 remaining in samples of the charcoal was 15% of the amount such trees would contain when living. The **half-life** of Carbon-14 is approximately 5600 years. The quantity Q of Carbon-14 in a charcoal sample satisfies the decay equation:

$$Q' = kQ$$



Compounded Interest

If an initial amount of A_0 dollars is deposited at an **annual interest rate** of r, **compounded** n times per year, the **future value** of the deposit at time t is given by

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

Compounded Interest

If an initial amount of A_0 dollars is deposited at an **annual interest rate** of r, **compounded** n times per year, the **future value** of the deposit at time t is given by

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

To earn the theoretical maximum interest, you would need the account to compounded at every instant of time. That is every hour, every minute, every second, and so on. We can represent this using the following limit:

$$A(t) = \lim_{n\to\infty} A_0\left(a + \frac{r}{n}\right)^{nt}$$

Compounded Interest

If an initial amount of A_0 dollars is deposited at an **annual interest rate** of r, **compounded** n times per year, the **future value** of the deposit at time t is given by

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

To earn the theoretical maximum interest, you would need the account to compounded at every instant of time. That is every hour, every minute, every second, and so on. We can represent this using the following limit:

$$A(t) = \lim_{n\to\infty} A_0\left(a + \frac{r}{n}\right)^{nt}$$

Using the calculus fact that $\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r$, we see that the limit reduces to:

$$A(t) = A_0 e^{rt}$$

Compounded Interest

If an initial amount of A_0 dollars is deposited at an **annual interest rate** of r, **compounded** n times per year, the **future value** of the deposit at time t is given by

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

To earn the theoretical maximum interest, you would need the account to compounded at every instant of time. That is every hour, every minute, every second, and so on. We can represent this using the following limit:

$$A(t) = \lim_{n\to\infty} A_0\left(a + \frac{r}{n}\right)^{nt}$$

Using the calculus fact that $\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r$, we see that the limit reduces to:

$$A(t) = A_0 e^{rt}$$

We call this **continuously compounded** interest.

Continuously Compounded Interest

If an initial amount of A_0 dollars is deposited at an annual interest rate of r, compounded continuously, the future value $A(t)$ of the deposit at time t satisfies the initial-value problem

$$\frac{dA}{dt} = rA, \quad A(0) = A_0$$

and is therefore given by

$$A(t) = A_0 e^{rt}$$

Example 2

Suppose you came across a time machine and decided to travel to London on 27 July 1694 and deposit £20 (about \$25.78 USD) during the Bank of England's grand opening. Let us suppose you got the founder's day savings account with an annual interest rate of 8% compounded continuously. How much money would be in the account on the bank's tricentennial in 1994?

Example 2

Suppose you came across a time machine and decided to travel to London on 27 July 1694 and deposit £20 (about \$25.78 USD) during the Bank of England's grand opening. Let us suppose you got the founder's day savings account with an annual interest rate of 8% compounded continuously. How much money would be in the account on the bank's tricentennial in 1994?

Since it's the tricentennial, we have $t = 300$. This allows use to calculate the future value:

$$A(300) = 20e^{0.08(300)}$$

Example 2

Suppose you came across a time machine and decided to travel to London on 27 July 1694 and deposit £20 (about $25.78 USD) during the Bank of England's grand opening. Let us suppose you got the founder's day savings account with an annual interest rate of 8% compounded continuously. How much money would be in the account on the bank's tricentennial in 1994?

Since it's the tricentennial, we have $t = 300$. This allows use to calculate the future value:

$$A(300) = 20e^{0.08(300)} \approx 529,782,442,596.869$$

Example 2

Suppose you came across a time machine and decided to travel to London on 27 July 1694 and deposit £20 (about \$25.78 USD) during the Bank of England's grand opening. Let us suppose you got the founder's day savings account with an annual interest rate of 8% compounded continuously. How much money would be in the account on the bank's tricentennial in 1994?

Since it's the tricentennial, we have $t = 300$. This allows use to calculate the future value:

$$A(300) = 20e^{0.08(300)} \approx 529,782,442,596.869$$

So we would have the queenly sum of £529,782,442,597 pounds (which is about \$682,836,581,952 USD).

The Savings Problem

The more interesting type of account is one where in addition to interest being earned, regular deposits are made, such as with retirement accounts and pension funds.

The Savings Problem

The more interesting type of account is one where in addition to interest being earned, regular deposits are made, such as with retirement accounts and pension funds.

We now have the case where we have an annual interest rate of r and a annual deposit amount of a dollars. So, if the account starts with A_0 dollars, then it satisfies the initial-value problem:

$$\frac{dA}{dt} = rA + a, \quad A(0) = A_0$$

The Savings Problem

The more interesting type of account is one where in addition to interest being earned, regular deposits are made, such as with retirement accounts and pension funds.

We now have the case where we have an annual interest rate of r and a annual deposit amount of a dollars. So, if the account starts with A_0 dollars, then it satisfies the initial-value problem:

$$\frac{dA}{dt} = rA + a, \quad A(0) = A_0$$

It is left as an exercise to the student that we can use the methods from earlier in the chapter solve this nonhomogenous linear IVP, obtaining:

$$A(t) = A_0 e^{rt} + \frac{a}{r}(e^{rt} - 1)$$

Here the first term represents the accumulation due to the starting deposit, and the second term the accumulation due to the subsequent deposits and the interest that they earn.

Example 3

Ravi has just entered college at age 18 and has decided to improve his health and save money by quitting smoking. He figures he can save \$30 per week in this way. If he deposits the amount in an account paying 10% annual interest compounded continuously how much will he have in the account when he retires at age 65?

Example 3

Ravi has just entered college at age 18 and has decided to improve his health and save money by quitting smoking. He figures he can save \$30 per week in this way. If he deposits the amount in an account paying 10% annual interest compounded continuously how much will he have in the account when he retires at age 65?

We start by noting that \$30 per week represents a rate of \$1,560 per year.

Example 3

Ravi has just entered college at age 18 and has decided to improve his health and save money by quitting smoking. He figures he can save \$30 per week in this way. If he deposits the amount in an account paying 10% annual interest compounded continuously how much will he have in the account when he retires at age 65?

We start by noting that \$30 per week represents a rate of \$1,560 per year.

The IVP describing the value of his account is:

$$\frac{dA}{dt} = 0.10A + 1560, \quad A(0) = 0$$

Example 3

Ravi has just entered college at age 18 and has decided to improve his health and save money by quitting smoking. He figures he can save \$30 per week in this way. If he deposits the amount in an account paying 10% annual interest compounded continuously how much will he have in the account when he retires at age 65?

We start by noting that \$30 per week represents a rate of \$1,560 per year.

The IVP describing the value of his account is:

$$\frac{dA}{dt} = 0.10A + 1560, \quad A(0) = 0$$

and the solution, by Euler-Lagrange, is

$$A(t) = 15600(e^{0.1t} - 1)$$

Example 3

Ravi has just entered college at age 18 and has decided to improve his health and save money by quitting smoking. He figures he can save \$30 per week in this way. If he deposits the amount in an account paying 10% annual interest compounded continuously how much will he have in the account when he retires at age 65?

We start by noting that \$30 per week represents a rate of \$1,560 per year.

The IVP describing the value of his account is:

$$\frac{dA}{dt} = 0.10A + 1560, \quad A(0) = 0$$

and the solution, by Euler-Lagrange, is

$$A(t) = 15600(e^{0.1t} - 1)$$

After $65 - 18 = 47$ years, Ravi will have

$$A(47) = 15600(e^{0.1(47)} - 1) \approx \$1,699,575.89$$