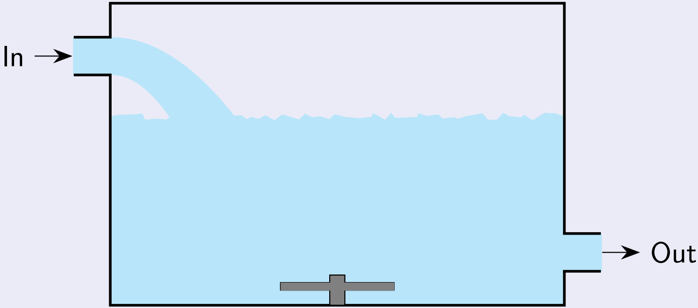
Linear Models: Mixing and Cooling

Department of Mathematics

Salt Lake Community College

Mixing Problems

A common problem consists of liquids being mixed in a tank. We will start with a simple system containing a single tank. Where some liquid flows into a tank, is mixed uniformly with the contents of the tank, and the resulting mixture flows out.



Mixing Model

If $x(t)$ is the amount of a dissolved substance, then

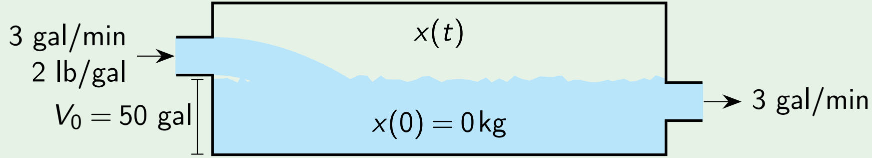
$$\frac{dx}{dt} = \text{Rate}_{\text{In}} - \text{Rate}_{\text{Out}}$$

where

$$\begin{aligned} \text{Rate}_{\text{In}} &= \text{Concentration}_{\text{In}} \cdot \text{Flow}_{\text{In}} \\ \text{Rate}_{\text{Out}} &= \text{Concentration}_{\text{Out}} \cdot \text{Flow}_{\text{Out}} \\ \uparrow \qquad & \qquad \uparrow \qquad\qquad\qquad \uparrow \\ [lb/min] \qquad & \qquad [lb/gal] \qquad\qquad [gal/min] \end{aligned}$$

Example 1

A tank initially contains 50 gal of pure water. A solution containing 2 lb/gal of salt is pumped into the tank at the rate of 3 gal/min. The mixture is stirred constantly and flows out at the same rate of 3 gal/min.



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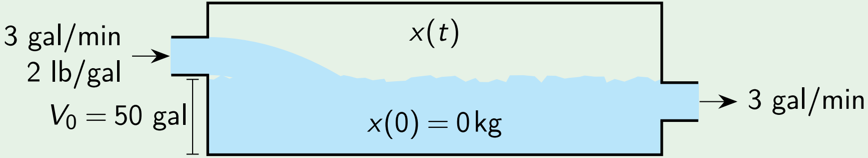

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To find the IVP, we need to determine the following:

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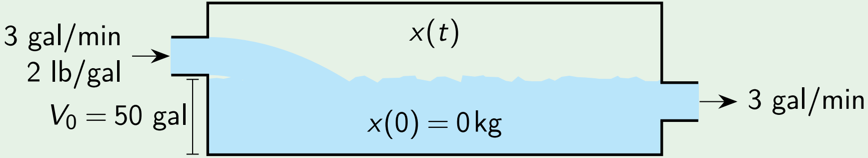

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To find the IVP, we need to determine the following:

$$\text{Rate}_{\text{In}} = (2 \text{ lb/gal})(3 \text{ gal/min})$$

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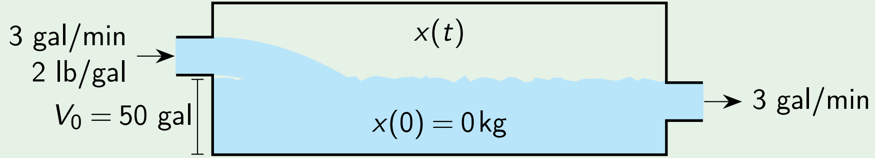

What IVP is satisfied by the amount of salt $x(t)$ in the tank at time t?

To find the IVP, we need to determine the following:

$$\text{Rate}_{\text{In}} = 6 \text{ lb/min}$$

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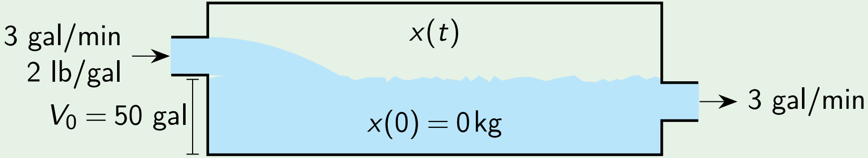

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To find the IVP, we need to determine the following:

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$$\mathrm{Rate_{Out}} = \left(\frac{x}{50} \text{ gal/min}\right)(3 \text{ gal/min})$$

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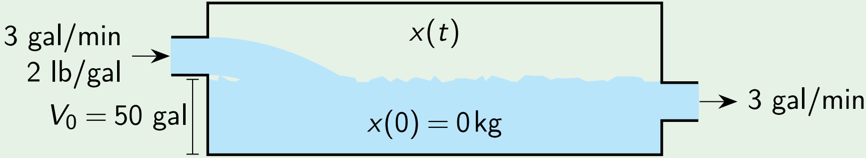

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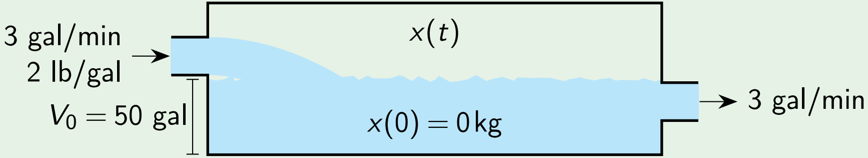

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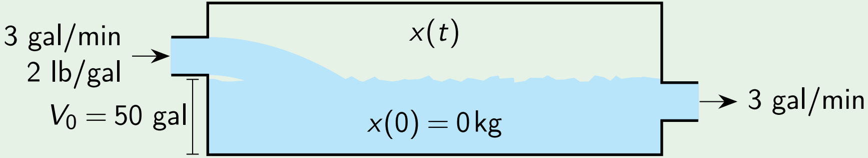

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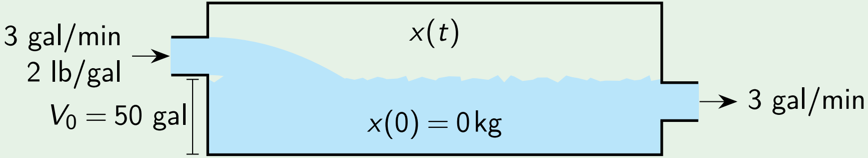

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$$x' + 0.06x = 6, \quad x(0) = 0$$

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What is the actual amount of salt in the tank at time t?

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This is a linear nonhomogeneous equation which has solution:

$$x(t) = 100\left(1 - e^{-0.06t}\right)$$

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How much salt is in the tank after 20 minutes?

We need to plug $t = 20$ into the above solution:

$$x(20) = 100\left(1 - e^{-0.06(2)}\right) \approx 69.9 \text{ lb}$$

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How much salt is in the tank after a very long time?

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This is the same as asking what is the behavior, as $t \to \infty$, of

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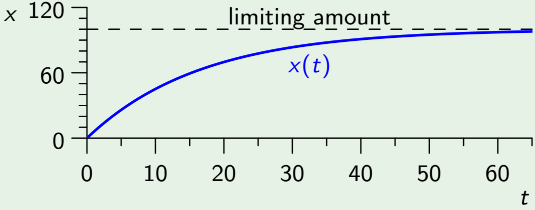
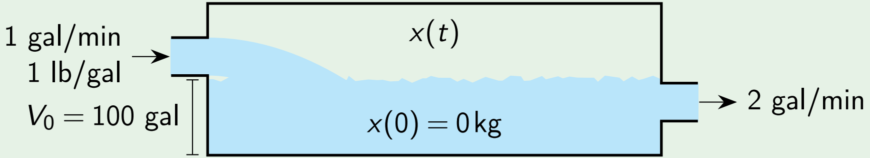
Note that $e^{-0.06t} \to 0$ as $t \to \infty$, which means that 100 is the limiting amount.



Example 2

A tank initially contains 100 gal of pure water. A brine containing 1 lb/gal of salt is pumped into the tank at the rate of 1 gal/min. The mixture is stirred constantly and flows out at the same rate of 2 gal/min.



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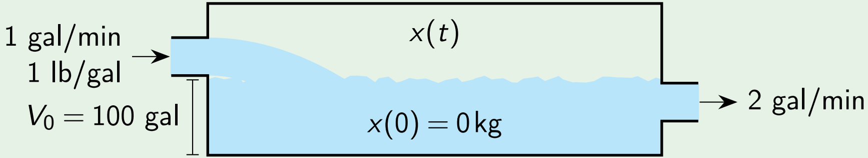

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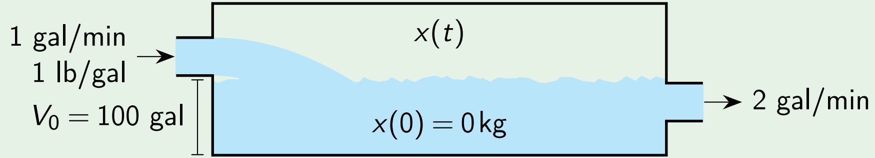

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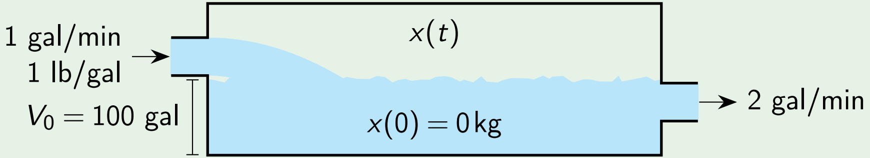

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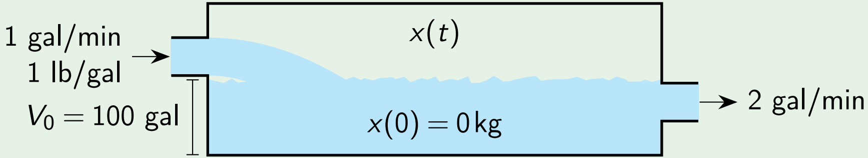

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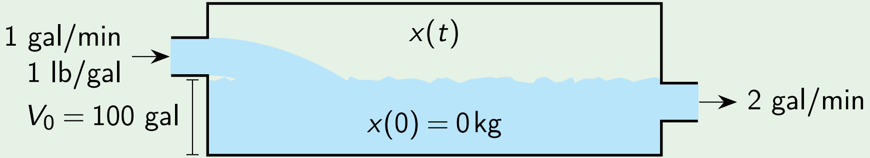

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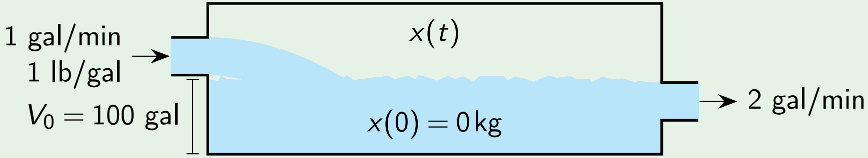

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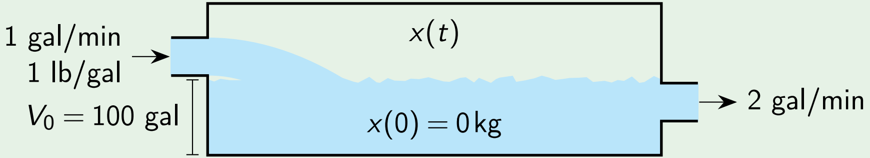

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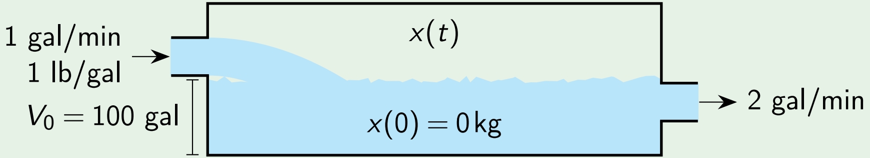

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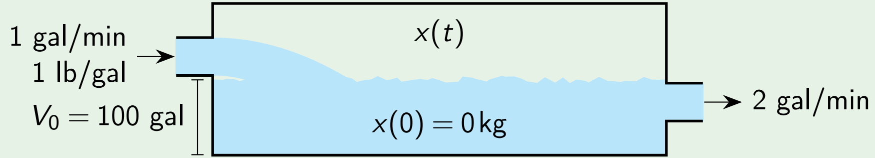

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The IVP is:

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Example 2

A tank initially contains 100 gal of pure water. A brine containing 1 lb/gal of salt is pumped into the tank at the rate of 1 gal/min. The mixture is stirred constantly and flows out at the same rate of 2 gal/min.

What is the formula for this amount of salt?

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A tank initially contains 100 gal of pure water. A brine containing 1 lb/gal of salt is pumped into the tank at the rate of 1 gal/min. The mixture is stirred constantly and flows out at the same rate of 2 gal/min.

What is the formula for this amount of salt?

The solution to the associated homogenous equation is:

$$x_h(t) = c(100 - t)^2$$

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Thus, the general solution is:

$$x(t) = x_h(t) + x_p(t) = c(100 - t)^2 + (100 - t)$$

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When $x = 0$ and $t = 0$ we find that $c = -0.01$. Thus the IVPs solution is:

$$x(t) = x_h(t) + x_p(t) = -0.01(100 - t)^2 + (100 - t)$$

Temperature Problems

We will next look at how an object, say a cup of coffee, changes temperature when left sitting in a room.

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Intuitively, we know that the rate of change of the temperature of the object is proportional to the difference in temperature between the object and the surroundings. (i.e. very hot things cool rapidly to start, but take a while to go from warm to room temperature.)

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Newton's Law of Cooling

The rate of change in the temperature T of an object placed in surroundings of uniform temperature M is proportional to the difference between the temperature of the object and the temperature of the surroundings.

Mathematically,

$$\frac{dT}{dt} = k(M - T)$$

where $k > 0$ is a constant of proportionality.

Solving Newton's Law of Cooling

Consider an object with initial temperature T_0 placed into surroundings of temperature M. Then $T(t)$ satisfies the IVP:

$$\frac{dT}{dt} = k(M - T), \quad T(0) = T_0$$

Solving Newton's Law of Cooling

Consider an object with initial temperature T_0 placed into surroundings of temperature M. Then $T(t)$ satisfies the IVP:

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Which is a linear nonhomogeneous differential equation:

$$T' + kT = kM$$

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Consider an object with initial temperature T_0 placed into surroundings of temperature M. Then $T(t)$ satisfies the IVP:

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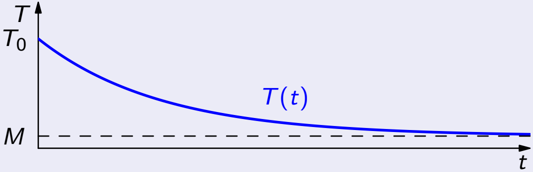
Which is a linear nonhomogeneous differential equation:

$$T' + kT = kM$$

We know from section 2.1 that the solution is:

$$T(t) = T_0 e^{-kt} + M(1 - e^{-kt})$$



Example 3

At midnight, with the temperature inside the house at 70°F and the outside temperature at 20°F, the furnace breaks. Two hours later the temperature inside the house has fallen to 50°F. We will assume that the outside temperature remains at 20°F.

What IVP is satisfied by the temperature inside the house?

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Using the Newton's Law of Cooling, we get:

$$T' = k(20 - T), \quad T(0) = 70$$

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What formula gives the inside temperature?

Using the general solution from the previous slide we get:

$$T(t) = 70e^{-kt} + 20(1 - e^{-kt})$$

Example 3

At midnight, with the temperature inside the house at 70°F and the outside temperature at 20°F, the furnace breaks. Two hours later the temperature inside the house has fallen to 50°F. We will assume that the outside temperature remains at 20°F.

What IVP is satisfied by the temperature inside the house?

Using the Newton's Law of Cooling, we get:

$$T' = k(20 - T), \quad T(0) = 70$$

What formula gives the inside temperature?

Using the general solution from the previous slide we get:

$$T(t) = 20 + 50e^{-kt}$$

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Now we need to find k. We know that $T(2) = 50$, which allows us to find $k = -\ln(0.6)/2 \approx 0.255$. So, we have:

$$T(t) = 20 + 50e^{t\ln(0.6)/2}$$

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At what time will the temperature inside be $70^{\circ}F$?

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This is the same as asking what t-value gives $T(t) = 40$.

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This is the same as asking what t-value gives $T(t) = 40$.

Thus, we need to solve the equation:

$$40 = 20 + 50e^{t\ln(0.6)/2}$$

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At what time will the temperature inside be 70°F?

This is the same as asking what t-value gives $T(t) = 40$.

Thus, we need to solve the equation:

$$40 = 20 + 50e^{t\ln(0.6)/2}$$

Which has solution:

$$t = \frac{2\ln(0.4)}{\ln(0.6)} \approx 3.592$$

So, it takes about 3 hours and 35 minutes to cool to 40°F, at 4:35am.