Linear Equations

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Definition

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$$\frac{dy}{dt} + P(t) \cdot y = Q(t)$$

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Integrating Factor Method (Constant Coefficient)

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$$y' + ay = f(t), \quad a \in \mathbb{R}$$

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$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at} y\right)$$

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Let us start with the differential equation.

$$y' + ay = f(t)$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at} y\right)$$

We first multiply both sides of the equation by e^{at}.

$$e^{at}\left(y' + ay\right) = e^{at} f(t)$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

We then apply Euler's observation to the left-hand side.

$$\frac{d}{dt}\left(e^{at}y\right) = e^{at}f(t)$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

Next we integrate both sides.

$$e^{at}y = \int e^{at}f(t)dt + c$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

Solving for y gives:

$$y(t) = e^{-at}\int e^{at}f(t)dt + ce^{-at}$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

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$$y' + p(t)y = f(t)$$

We seek a function $\mu(t)$ that satisfies Euler's observation, i.e.

$$\mu(t) \cdot \left(y' + p(t)y\right) = \frac{d}{dt}\left(\mu(t) \cdot y\right)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

Let us carry out the differentiation on the right-hand side

$$\mu(t)y' + p(t)\mu(t)y = \mu'(t)y + \mu(t)y'$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

If we assume $y(t) \neq 0$, this simplifies to

$$\mu'(t) = p(t)\mu(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\ln|\mu(t)| = \int p(t)dt$$

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Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

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$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

We now know the integrating factor, and perform the same steps as before.

$$y' + p(t)y = f(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Multiply both sides by the integrating factor.

$$\mu(t) \cdot \left(y' + p(t)y\right) = \mu(t) \cdot f(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Apply the property $\mu(t) \cdot (y' + p(t)y) = (\mu(t) \cdot y)'$ to the left-hand side.

$$(\mu(t)y)' = \mu(t)f(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Integrate both sides.

$$\mu(t)y(t) = \int \mu(t)f(t)dt + c$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Assuming $\mu(t) \neq 0$, we can solve for y.

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t)dt + \frac{c}{\mu(t)}$$

Integrating Factor Method for First-Order Linear DEs

To solve the linear first-order DE, where p and f are continuous on a domain I.

$$y' + p(t)y = f(t)$$

Step 1. Find the integrating factor $\mu(t) = e^{\int p(t)dt}$, where $\int p(t)dt$ represents *any* anti-derivative of $p(t)$.

Step 2. Multiply both sides of the DE by $mu(t)$, which always simplifies to:

$$\left(e^{\int p(t)dt}y(t)\right)' = e^{\int p(t)dt}f(t)$$

Step 3. Find the anti-derivative to get:

$$e^{\int p(t)dt}y(t) = \int e^{\int p(t)dt}f(t)dt + c$$

Step 4. Solve algebraically for y.

$$y = e^{-\int p(t)dt}\int e^{\int p(t)dt}f(t)dt + ce^{-\int p(t)dt}$$

Example 1

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

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$$\mu(t) = e^{\int (-1)dt}$$

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$$\mu(t) = e^{\int (-1) dt} = e^{-t}$$

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Step 1. Find the integrating factor:

$$\mu(t) = e^{\int(-1)dt} = e^{-t}$$

Step 2. Multiply both sides of the DE by $\mu(t)$:

$$e^{-t}\left(y' - y\right) = e^{-t}$$

Example 1

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int(-1)dt} = e^{-t}$$

Step 2. Multiply both sides of the DE by $\mu(t)$:

$$e^{-t}\left(y' - y\right) = e^{-t}$$

Which reduces to:

$$\left(e^{-t}y\right)' = te^{-t}$$

Example 1

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Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt$$

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$$y' - y = t, \quad y(0) = 1$$

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

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Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t}$$

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$$y' - y = t, \quad y(0) = 1$$

Step 3. Find the antiderivative:

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Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

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Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^{0}$$

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Step 3. Find the antiderivative:

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Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^{0} \Rightarrow c = 2$$

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Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^{0} \Rightarrow c = 2$$

Thus, the solution to the IVP is $y(t) = -t - 1 + 2e^{t}$

Example 2

Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

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Step 1. Find the integrating factor:

$$\mu(t) = e^{\int p(t)dt}$$

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Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (\frac{1}{t}) dt}$$

Example 2

Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (\frac{1}{t}) dt} = e^{\ln(t)} = t$$

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Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0), \quad y(1) = 3$$

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (\frac{1}{t}) dt} = e^{\ln(t)} = t$$

Step 2. Multiply both sides of the DE by $\mu(t)$:

$$t\left(y' - y\right) = \frac{1}{t^2} \cdot t$$

Example 2

Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (\frac{1}{t})dt} = e^{\ln(t)} = t$$

Step 2. Multiply both sides of the DE by $\mu(t)$:

$$t\left(y' - y\right) = \frac{1}{t^2} \cdot t$$

Which reduces to:

$$(t \cdot y)' = \frac{1}{t}$$

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Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

Step 3. Find the antiderivative:

$$ty = \int \frac{1}{t}dt$$

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Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

Step 3. Find the antiderivative:

$$ty = \int \frac{1}{t} dt = \ln(t) + c$$

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Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

Step 3. Find the antiderivative:

$$ty = \int \frac{1}{t} dt = \ln(t) + c$$

Step 4. Solve for y:

$$y(t) = \frac{\ln(t) + c}{t}$$

Example 2

Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad (\text{assume } t > 0), \quad y(1) = 3$$

Step 3. Find the antiderivative:

$$ty = \int \frac{1}{t}dt = \ln(t) + c$$

Step 4. Solve for y:

$$y(t) = \frac{\ln(t) + c}{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$3 = y(1) = \frac{\ln(1) + c}{1}$$

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Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

Step 3. Find the antiderivative:

$$ty = \int \frac{1}{t}dt = \ln(t) + c$$

Step 4. Solve for y:

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Step 5. Plug in the initial conditions to find the solution to the IVP:

$$3 = y(1) = \frac{\ln(1) + c}{1} \Rightarrow c = 3$$

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Consider the IVP

$$y' + \frac{1}{t}y = \frac{1}{t^2} \quad \text{(assume } t > 0\text{)}, \quad y(1) = 3$$

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$$ty = \int \frac{1}{t}dt = \ln(t) + c$$

Step 4. Solve for y:

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Step 5. Plug in the initial conditions to find the solution to the IVP:

$$3 = y(1) = \frac{\ln(1) + c}{1} \Rightarrow c = 3$$

Thus, the solution to the IVP is $y(t) = \dfrac{\ln(t) + 3}{t}$