Addition Rule and Multiplication Rule

Colby Community College

Definition

Events A and B are **independent events** if the probability of B occuring is the same, whether or not A occurs.

Definition

Events A and B are **independent events** if the probability of B occuring is the same, whether or not A occurs.

Example 1

Are these events independent?

1. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

Definition

Events A and B are **independent events** if the probability of B occuring is the same, whether or not A occurs.

Example 1

Are these events independent?

1. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head. **Independent**

Definition

Events A and B are **independent events** if the probability of B occuring is the same, whether or not A occurs.

Example 1

Are these events independent?

1. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head. **Independent**
2. You draw a card from a deck, then without replacing the first, draw a second card?

Definition

Events A and B are **independent events** if the probability of B occuring is the same, whether or not A occurs.

Example 1

Are these events independent?

1. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head. **Independent**
2. You draw a card from a deck, then without replacing the first, draw a second card? **Dependent**

Definition

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Definition

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 2

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

Definition

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 2

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

The probability of getting tails on the coin and three on the die is

$$P(\text{T3}) = \frac{1}{12}$$

Definition

If events A and B are independent, then the probability of both A and B occurring is

$$P\left(A \text{ and } B\right) = P\left(A\right) \cdot P\left(B\right)$$

Example 2

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

The probability of getting tails on the coin and three on the die is

$$P\left(\text{T3}\right) = \frac{1}{12}$$

We could have also calculated

$$P\left(\text{H and 3}\right) = P\left(\text{H}\right) \cdot P\left(3\right) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Definition

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Definition

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 3

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

We want to calculate the probability of getting a head or a six.

Definition

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 3

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

We want to calculate the probability of getting a head or a six.

The outcomes are: H1, H2, H3, H4, H5, H6, T6. Giving $P(\text{H or } 6) = \frac{7}{12}$.

Definition

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 3

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

We want to calculate the probability of getting a head or a six.

The outcomes are: H1, H2, H3, H4, H5, H6, T6. Giving $P(\text{H or } 6) = \frac{7}{12}$.

Notice that $\frac{6}{12} = \frac{1}{2}$ of the outcomes have heads and $\frac{2}{12} = \frac{1}{6}$ have a six.

Definition

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 3

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

We want to calculate the probability of getting a head or a six.

The outcomes are: H1, H2, H3, H4, H5, H6, T6. Giving $P(\text{H or } 6) = \frac{7}{12}$.

Notice that $\frac{6}{12} = \frac{1}{2}$ of the outcomes have heads and $\frac{2}{12} = \frac{1}{6}$ have a six.

But, $\frac{1}{2} + \frac{1}{6} = \frac{8}{12}$, is wrong because we have double counted H6. Thus, we need to subtract $P(\text{H6}) = \frac{1}{12}$.

$$P(\text{H or } 6) = P(\text{H}) + P(6) - P(\text{H and } 6) = \frac{1}{2} + \frac{1}{6} - \frac{1}{12} = \frac{7}{12}$$

Example 4

Suppose we draw one card from a standard deck. Let us find the probability that we get a Queen or a King.

Example 4

Suppose we draw one card from a standard deck. Let us find the probability that we get a Queen or a King.

There are 4 Queens and 4 Kings in the deck, hence eight outcomes out of 52 possible outcomes. So, the probability is

$$P\left(\text{Q or K}\right) = \frac{8}{52}$$

Example 4

Suppose we draw one card from a standard deck. Let us find the probability that we get a Queen or a King.

There are 4 Queens and 4 Kings in the deck, hence eight outcomes out of 52 possible outcomes. So, the probability is

$$P(\text{Q or K}) = \frac{8}{52}$$

Since there are no cards that are both Kings and Queens, we have

$$P(\text{Q or K}) = P(\text{Q}) + P(K) - P(\text{Q and K}) = \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}$$

Example 4

Suppose we draw one card from a standard deck. Let us find the probability that we get a Queen or a King.

There are 4 Queens and 4 Kings in the deck, hence eight outcomes out of 52 possible outcomes. So, the probability is

$$P(\text{Q or K}) = \frac{8}{52}$$

Since there are no cards that are both Kings and Queens, we have

$$P(\text{Q or K}) = P(\text{Q}) + P(K) - P(\text{Q and K}) = \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}$$

Note

If two events are **mutually exclusive**, then $P(A \text{ or } B) = P(A) + P(B)$.

Example 5

Suppose we draw one card from a standard deck. Let us calculate the probability that we get a red card for a King.

Example 5

Suppose we draw one card from a standard deck. Let us calculate the probability that we get a red card for a King.

Half the cards are red, so $P\left(\text{Red}\right) = \dfrac{26}{52}$.

Example 5

Suppose we draw one card from a standard deck. Let us calculate the probability that we get a red card for a King.

Half the cards are red, so $P\left(\text{Red}\right) = \dfrac{26}{52}$.

Four cards are Kings, so $P\left(\text{K}\right) = \dfrac{4}{52}$.

Example 5

Suppose we draw one card from a standard deck. Let us calculate the probability that we get a red card for a King.

Half the cards are red, so $P\left(\text{Red}\right) = \dfrac{26}{52}$.

Four cards are Kings, so $P\left(\text{K}\right) = \dfrac{4}{52}$.

Two cards are red kings, so $P\left(\text{Red and K}\right) = \dfrac{2}{52}$.

Example 5

Suppose we draw one card from a standard deck. Let us calculate the probability that we get a red card for a King.

Half the cards are red, so $P\left(\text{Red}\right) = \dfrac{26}{52}$.

Four cards are Kings, so $P\left(\text{K}\right) = \dfrac{4}{52}$.

Two cards are red kings, so $P\left(\text{Red and K}\right) = \dfrac{2}{52}$.

Thus,

$$P\left(\text{Red or K}\right) = P\left(\text{Red}\right) + P\left(\text{K}\right) - P\left(\text{Red and K}\right) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{7}{13}$$

Example 6

What is the probability that two cards drawn at random from a deck will both be aces?

Example 6

What is the probability that two cards drawn at random from a deck will both be aces?

You might guess that since there are four aces in the deck of 52 cards, the probability would be $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.

Example 6

What is the probability that two cards drawn at random from a deck will both be aces?

You might guess that since there are four aces in the deck of 52 cards, the probability would be $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.

The problem here is that the events are not independent. Once one card is drawn, there are only 51 cards remaining. Once an ace has been drawn, there are only three remaining.

Example 6

What is the probability that two cards drawn at random from a deck will both be aces?

You might guess that since there are four aces in the deck of 52 cards, the probability would be $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.

The problem here is that the events are not independent. Once one card is drawn, there are only 51 cards remaining. Once an ace has been drawn, there are only three remaining.

This means that the the probability of drawing two aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.

Example 6

What is the probability that two cards drawn at random from a deck will both be aces?

You might guess that since there are four aces in the deck of 52 cards, the probability would be $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.

The problem here is that the events are not independent. Once one card is drawn, there are only 51 cards remaining. Once an ace has been drawn, there are only three remaining.

This means that the the probability of drawing two aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.

Definition

The probability the event B occurs, given that event A has happened, is represented as $P(B \mid A)$. This is called a **conditional probability**.

Read as "the probability of B given A."

Example 7

Car color	Speeding ticket	No speeding ticket	Total
Red	15	135	150
Not red	45	470	515
Total	60	605	665

Find the probability someone has gotten a speeding ticket *given* they drive a red car.

Example 7

Car color	Speeding ticket	No speeding ticket	Total
Red	15	135	150
Not red	45	470	515
Total	60	605	665

Find the probability someone has gotten a speeding ticket *given* they drive a red car.

$$P\left(\text{ticket} \mid \text{red}\right) = \frac{15}{150} = \frac{1}{10} = 10\%$$

Example 7

Car color	Speeding ticket	No speeding ticket	Total
Red	15	135	150
Not red	45	470	515
Total	60	605	665

Find the probability someone has gotten a speeding ticket *given* they drive a red car.

$$P\left(\text{ticket} \mid \text{red}\right) = \frac{15}{150} = \frac{1}{10} = 10\%$$

Find the probability someone drives a red car *given* they have gotten a speeding ticker.

Example 7

Car color	Speeding ticket	No speeding ticket	Total
Red	15	135	150
Not red	45	470	515
Total	60	605	665

Find the probability someone has gotten a speeding ticket *given* they drive a red car.

$$P\left(\text{ticket} \mid \text{red}\right) = \frac{15}{150} = \frac{1}{10} = 10\%$$

Find the probability someone drives a red car *given* they have gotten a speeding ticker.

$$P\left(\text{red} \mid \text{ticket}\right) = \frac{15}{60} = \frac{1}{4} = 25\%$$

Example 7

Car color	Speeding ticket	No speeding ticket	Total
Red	15	135	150
Not red	45	470	515
Total	60	605	665

Find the probability someone has gotten a speeding ticket *given* they drive a red car.

$$P\left(\text{ticket} \mid \text{red}\right) = \frac{15}{150} = \frac{1}{10} = 10\%$$

Find the probability someone drives a red car *given* they have gotten a speeding ticker.

$$P\left(\text{red} \mid \text{ticket}\right) = \frac{15}{60} = \frac{1}{4} = 25\%$$

Note

In general $P\left(B \mid A\right) \neq P\left(A \mid B\right)$.

Definition

If A and B are not independent, then $P\,(A \text{ and } B) = P\,(A) \cdot P\,(B \mid A)$.

Definition

If A and B are not independent, then $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$.

Example 8

If you pull two cards out of a deck, find the probability that both are hearts.

Definition

If A and B are not independent, then $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$.

Example 8

If you pull two cards out of a deck, find the probability that both are hearts.

The probability that the first card is a heart is $P(1^{\text{st}} \heartsuit) = \dfrac{13}{52}$.

Definition

If A and B are not independent, then $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$.

Example 8

If you pull two cards out of a deck, find the probability that both are hearts.

The probability that the first card is a heart is $P\left(1^{\text{st}}\ \heartsuit\right) = \dfrac{13}{52}$.

The probability that the second card is a heart, given that the first card was a heart, is $P\left(2^{\text{nd}}\ \heartsuit \mid 1^{\text{st}}\ \heartsuit\right) = \dfrac{12}{51}$.

Definition

If A and B are not independent, then $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$.

Example 8

If you pull two cards out of a deck, find the probability that both are hearts.

The probability that the first card is a heart is $P(1^{\text{st}} \heartsuit) = \dfrac{13}{52}$.

The probability that the second card is a heart, given that the first card was a heart, is $P\left(2^{\text{nd}} \heartsuit \mid 1^{\text{st}} \heartsuit\right) = \dfrac{12}{51}$.

So, the probability that both are spades is

$$P(\text{both } \heartsuit) = P\left(1^{\text{st}} \heartsuit\right) \cdot P\left(2^{\text{nd}} \heartsuit \,\middle|\, 1^{\text{st}} \heartsuit\right) = \frac{13}{52} \cdot \frac{12}{52} = \frac{156}{2652} \approx 5.9\%$$

Example 9

If you draw two cards from a deck, find the probability that you will get the Ace of Diamonds and a black card.

Example 9

If you draw two cards from a deck, find the probability that you will get the Ace of Diamonds and a black card.

There are two events we have to consider:

Example 9

If you draw two cards from a deck, find the probability that you will get the Ace of Diamonds and a black card.

There are two events we have to consider:

Event A Drawing the Ace of Diamonds then a black card.

$$
\begin{aligned}
P(\text{A}\blacklozenge \text{ and Black}) &= P(\text{A}\blacklozenge) \cdot P(\text{Black} \mid \text{A}\blacklozenge) \\
&= \frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}
\end{aligned}
$$

Example 9

If you draw two cards from a deck, find the probability that you will get the Ace of Diamonds and a black card.

There are two events we have to consider:

Event A Drawing the Ace of Diamonds then a black card.

$$\begin{aligned} P(\text{A}\blacklozenge \text{ and Black}) &= P(\text{A}\blacklozenge) \cdot P(\text{Black} \mid \text{A}\blacklozenge) \\ &= \frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102} \end{aligned}$$

Event B Drawing a black card then the Ace of Diamonds.

$$\begin{aligned} P(\text{Black and A}\blacklozenge) &= P(\text{Black}) \cdot P(\text{A}\blacklozenge \mid \text{Black}) \\ &= \frac{26}{52} \cdot \frac{1}{51} = \frac{1}{102} \end{aligned}$$

Example 9

If you draw two cards from a deck, find the probability that you will get the Ace of Diamonds and a black card.

There are two events we have to consider:

Event A Drawing the Ace of Diamonds then a black card.

$$P(\text{A}\blacklozenge \text{ and Black}) = P(\text{A}\blacklozenge) \cdot P(\text{Black} \mid \text{A}\blacklozenge)$$
$$= \frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$$

Event B Drawing a black card then the Ace of Diamonds.

$$P(\text{Black and A}\blacklozenge) = P(\text{Black}) \cdot P(\text{A}\blacklozenge \mid \text{Black})$$
$$= \frac{26}{52} \cdot \frac{1}{51} = \frac{1}{102}$$

These events are independent and mutually exclusive, so

$$P(A \text{ or } B) = P(A) + P(B) = \frac{1}{102} + \frac{1}{102} = \frac{2}{102} \approx 1.96\%$$