

Complex Characteristic Roots

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Solution for Complex Characteristic Roots

For $\Delta < 0$, the characteristic roots of the DE

are

$$ay'' + by' + cy = 0$$

$$r_1 = \alpha + i\beta = -\frac{b}{2a} + i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$

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The functions $e^{\alpha t} \cos(\beta t)$ and $e^{\alpha t} \sin(\beta t)$ are linearly independent solutions, and the general solution is given by

$$y(t) = e^{\alpha t} (c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

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The set $\{e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)\}$ forms a basis for the solution space \mathbb{S} .

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Let us find the general solution of

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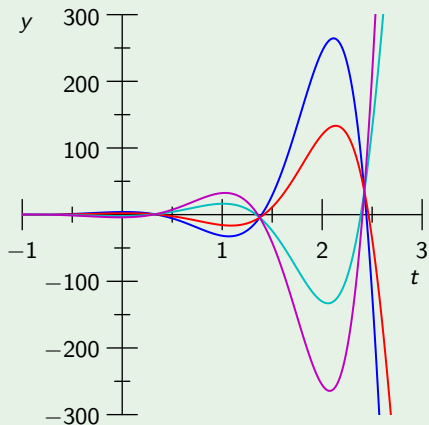
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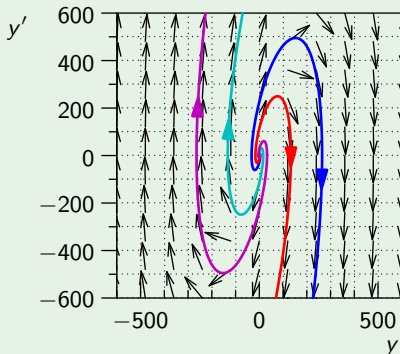
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(a) Time Series



(b) Phase Portrait

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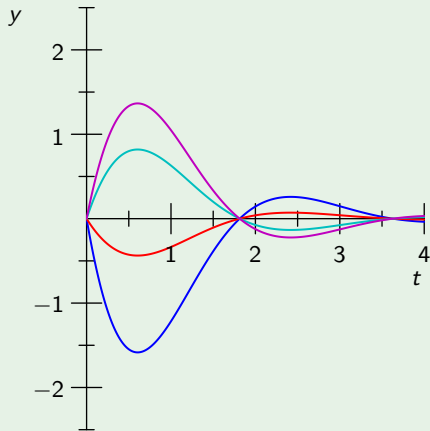
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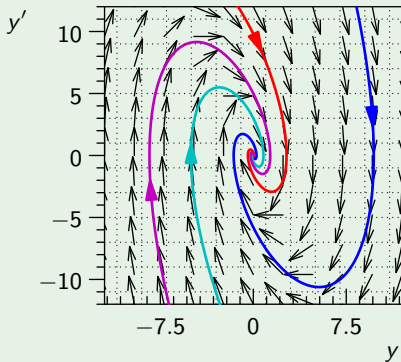
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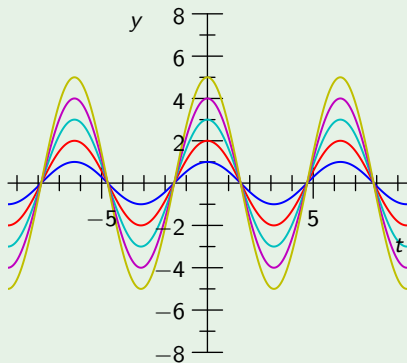
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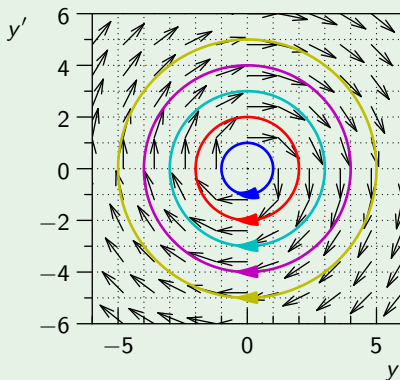
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Example 3



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(b) Phase Portrait

Underdamped Mass-Spring System

The motion of a mass-spring system is called **underdamped** when we have $\Delta = b^2 - 4mk < 0$. Both characteristic roots are complex and the solutions are given by

$$x(t) = e^{-\frac{b}{2m}} (c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t) \cos(\omega_d t - \delta), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}t}$
- Phase angle δ
- Phase shift $\varphi = \frac{\delta}{\omega_d}$
- Circular quasi-frequency ω_d
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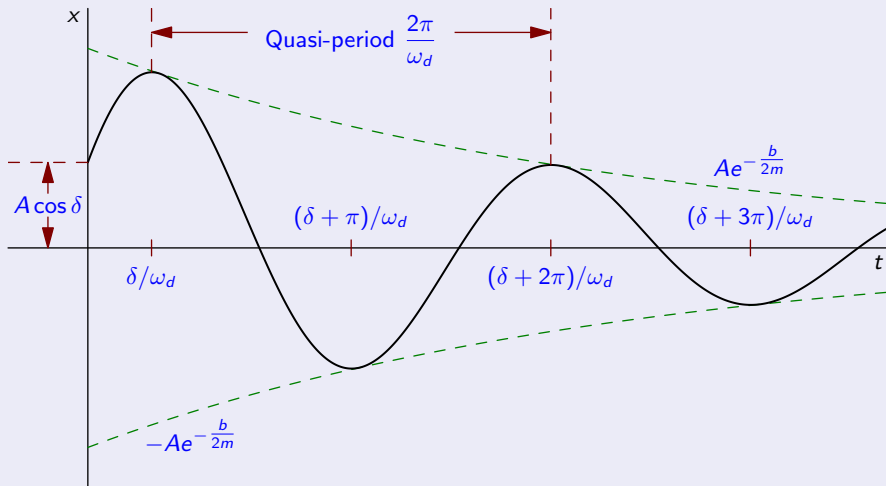
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Alternate Solution Visual Example



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Consider the Mass-Spring IVP where

$$\ddot{x} + \dot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0$$

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Which means $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$.

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The general solution is

$$x(t) = e^{-\frac{t}{2}} \left(c_1 \cos \left(\frac{\sqrt{3}}{2} t \right) + c_2 \sin \left(\frac{\sqrt{3}}{2} t \right) \right)$$

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We can then calculate

$$\dot{x}(t) = e^{-\frac{t}{2}} \left(\left(-\frac{c_1}{2} + \frac{c_2 \sqrt{3}}{2} \right) \cos \left(\frac{t \sqrt{3}}{2} \right) - \left(\frac{c_2}{2} + \frac{c_1 \sqrt{3}}{2} \right) \sin \left(\frac{t \sqrt{3}}{2} \right) \right)$$

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If we substitute in the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$,
we find that $c_1 = 1$ and $c_2 = \frac{1}{\sqrt{3}}$.

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In alternate polar form

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos \left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6} \right)$$

Where

$$A = \sqrt{1^2 + \left(\frac{1}{\sqrt{3}} \right)^2} = \frac{2}{\sqrt{3}} \quad \text{and} \quad \delta = \tan^{-1} \left(\frac{\frac{1}{\sqrt{3}}}{1} \right) = \frac{\pi}{6}$$

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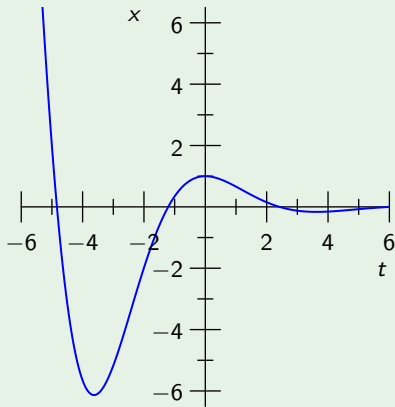
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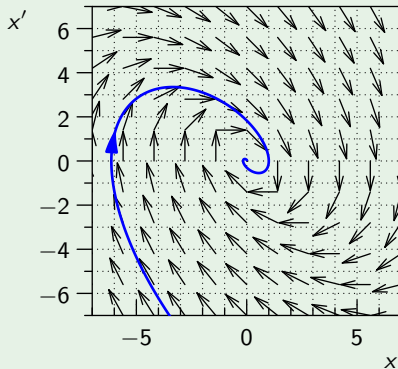
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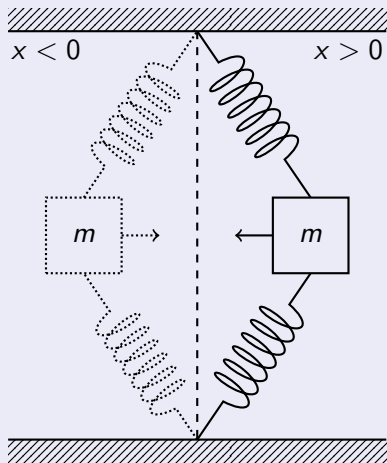
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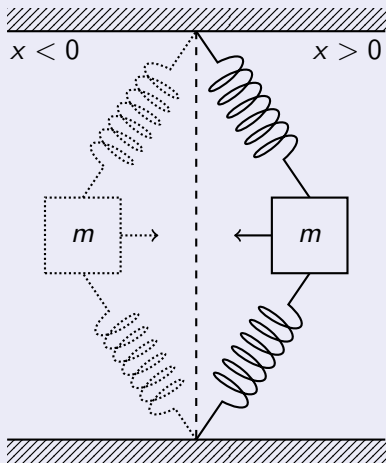
The Guitar String: A Qualitative Analysis

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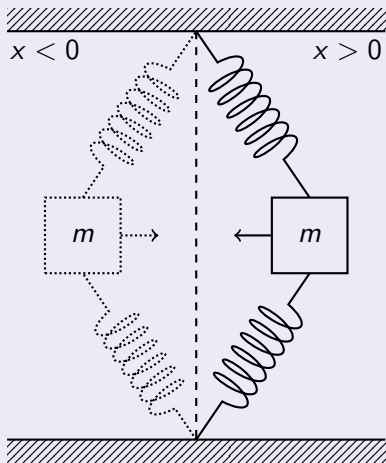
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$$\ddot{x} + \omega_0^2 x = 0$$

where ω_0 is the circular frequency at which the string vibrates. (In music, the frequency $f_0 = \frac{\omega_0}{2\pi}$ is often used. A middle C has 512 vibrations per second.)

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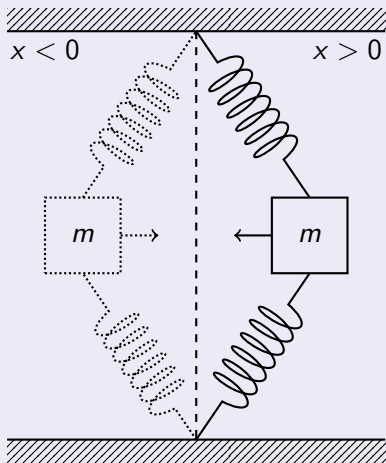
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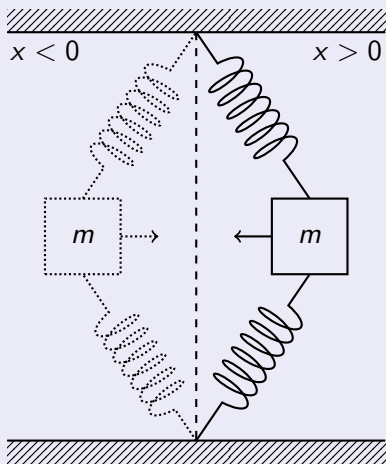
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Let us next consider a guitar string with damping.

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which has solutions $r = -1 \pm 5i$. So $\alpha = -1$ and $\beta = 5$.

Thus, the general solution is

$$x(t) = e^{-t} (c_1 \cos(5t) + c_2 \sin(5t))$$

Example 5

Consider the underdamped guitar string

$$\ddot{x} + 2\dot{x} + 26x = 0$$

First, let us write the characteristic equation:

$$0 = r^2 + 2r + 26$$

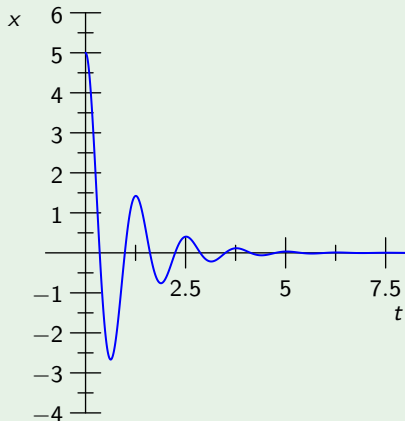
which has solutions $r = -1 \pm 5i$. So $\alpha = -1$ and $\beta = 5$.

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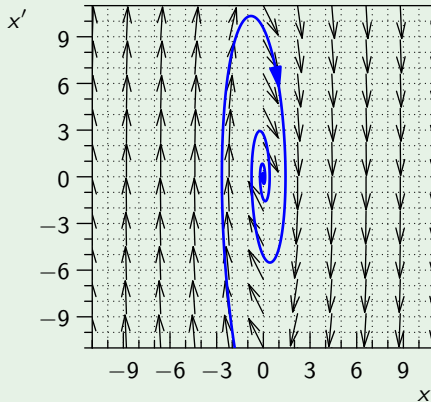
$$x(t) = e^{-t} (c_1 \cos(5t) + c_2 \sin(5t))$$

If we pluck the string, which means $x(0) = 5$ and $\dot{x}(0) = 0$, we find that $c_1 = 5$ and $c_2 = 1$.

Example 5



(a) Time Series



(b) Phase Portrait

Solutions to the Second-Order Linear DE with Constant Coefficients

The differential equation

$$ay'' + by' + cy = 0$$

has the characteristic equation

$$ar^2 + br + c = 0$$

The quadratic formula gives rise to three different general solutions, depending on the discriminant $\Delta = b^2 - 4ac$.

Characteristic Roots

General Solution

$$\Delta > 0 \quad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$\Delta = 0 \quad r = -\frac{b}{2a}$$

$$y = c_1 e^{rt} + c_2 t e^{rt}$$

$$\Delta < 0 \quad r_1, r_2 = \alpha \pm \beta$$
$$\alpha = -\frac{b}{2a}, \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

$$y = e^{\alpha t} (c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Example 6

Consider the fourth-order DE

$$\frac{d^4 y}{dy^4} - 16y = 0$$

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Which has the characteristic solutions

$$r_1 = 2, \quad r_2 = -2, \quad r_3 = 2i, \quad r_4 = -2i$$

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$$r_1 = 2, \quad r_2 = -2, \quad r_3 = 2i, \quad r_4 = -2i$$

Thus, $\{e^{2t}, e^{-2t}, \cos(2t), \sin(2t)\}$ form a basis of \mathbb{S} and the general solution is

$$y = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos(2t) + c_4 \sin(2t)$$

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Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

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$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3 = (r - 1)(r^2 + 2r - 3) = (r - 1)^2(r + 3)$$

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$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3 = (r - 1)(r^2 + 2r - 3) = (r - 1)^2(r + 3)$$

Which has the characteristic solutions

$$r_1 = 1, \quad r_2 = 1, \quad r_3 = -3$$

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Consider the third-order DE

$$y''' + y'' - 5y' + 3y = 0$$

It's characteristic equation is

$$0 = r^3 + r^2 - 5r + 3 = (r - 1)(r^2 + 2r - 3) = (r - 1)^2(r + 3)$$

Which has the characteristic solutions

$$r_1 = 1, \quad r_2 = 1, \quad r_3 = -3$$

Thus, $\{e^t, te^t, e^{-3t}\}$ form a basis of \mathbb{S} and the general solution is

$$y = c_1 e^t + c_2 t e^t + c_3 e^{-3t}$$

Example 8

Consider the fifth-order DE

$$\frac{d^5 y}{dt^5} + 3\frac{d^4 y}{dt^4} + 3\frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} = 0$$

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Consider the fifth-order DE

$$\frac{d^5 y}{dt^5} + 3\frac{d^4 y}{dt^4} + 3\frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} = 0$$

It's characteristic equation is

$$0 = r^5 + 3r^4 + 3r^3 + r^2$$

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It's characteristic equation is

$$0 = r^5 + 3r^4 + 3r^3 + r^2 = (r + 1)^3 r^2$$

Which has the characteristic solutions

$$r_1 = -1, \quad r_2 = -1, \quad r_3 = -1, \quad r_4 = 0, \quad r_5 = 0$$

Example 8

Consider the fifth-order DE

$$\frac{d^5 y}{dt^5} + 3 \frac{d^4 y}{dt^4} + 3 \frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} = 0$$

It's characteristic equation is

$$0 = r^5 + 3r^4 + 3r^3 + r^2 = (r + 1)^3 r^2$$

Which has the characteristic solutions

$$r_1 = -1, \quad r_2 = -1, \quad r_3 = -1, \quad r_4 = 0, \quad r_5 = 0$$

Thus, $\{e^{-t}, te^{-t}, t^2e^{-t}, 1, t\}$ form a basis of \mathbb{S} and the general solution is

$$y = \underbrace{(c_1 + c_2 t + c_3 t^2)}_{\text{for triple root}} e^{-t} + \underbrace{(c_4 + c_5 t)}_{\text{for double root}}$$

Example 9

Consider the fourth-order DE

$$\frac{d^4 y}{dt^4} + 8 \frac{d^2 y}{dt^2} + 16y = 0$$

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$$\frac{d^4 y}{dt^4} + 8\frac{d^2 y}{dt^2} + 16y = 0$$

It's characteristic equation is

$$0 = r^4 + 8r^2 + 16 = (r^2 + 4)^2$$

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Which has the characteristic solutions

$$r_1 = 2i, \quad r_2 = 2i, \quad r_3 = -2i, \quad r_4 = -2i$$

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Consider the fourth-order DE

$$\frac{d^4 y}{dt^4} + 8 \frac{d^2 y}{dt^2} + 16y = 0$$

It's characteristic equation is

$$0 = r^4 + 8r^2 + 16 = (r^2 + 4)^2$$

Which has the characteristic solutions

$$r_1 = 2i, \quad r_2 = 2i, \quad r_3 = -2i, \quad r_4 = -2i$$

Thus, $\{\cos(2t), t \cos(2t), \sin(2t), t \sin(2t)\}$ form a basis of \mathbb{S} and the general solution is

$$y = \underbrace{(c_1 + c_2 t)}_{\text{for double root}} \cos(2t) + \underbrace{(c_3 + c_4 t)}_{\text{for double root}} \sin(2t)$$