

# Forced Oscillations

Department of Mathematics

Salt Lake Community College

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and

$$x = c_1 \cos(t) + c_2 \sin(t) - \frac{1}{4} \cos(3t)$$

## General Solution

We can now look at the general solution for the undamped system

$$m\ddot{x} + kx = F_0 \cos(\omega_f t)$$

Where  $\omega_f$  is the **forcing frequency** and  $F_0$  is the **forcing amplitude**.

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This leaves two separate cases for  $x_p$ :

- 1 The frequencies  $\omega_f$  and  $\omega_0$  are different.
- 2 The frequencies  $\omega_f$  and  $\omega_0$  are the same.

## Unmatched Frequencies ( $\omega_f \neq \omega_0$ )

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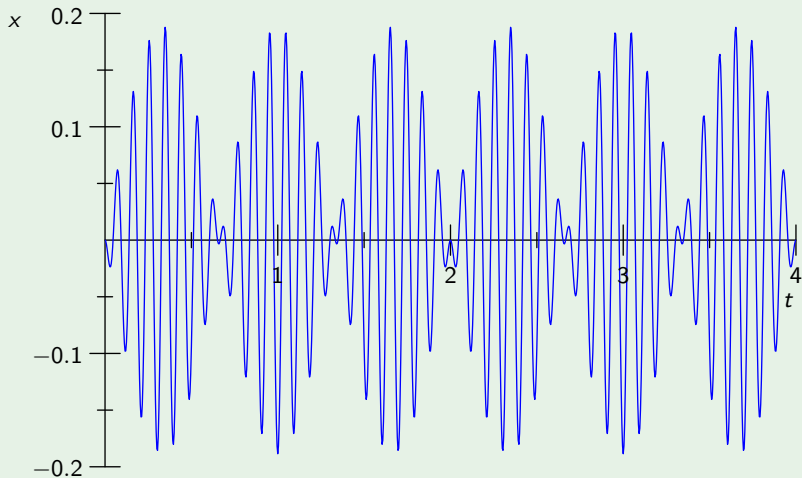
So, where  $c_1$  and  $c_2$  are determined by initial conditions, we have

$$\begin{aligned} x(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega_f^2)} \cos(\omega_f t) \\ &= C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega_f^2)} \cos(\omega_f t) \end{aligned}$$

where  $C = \sqrt{c_1^2 + c_2^2}$  and  $\tan(\delta) = \frac{c_2}{c_1}$ .

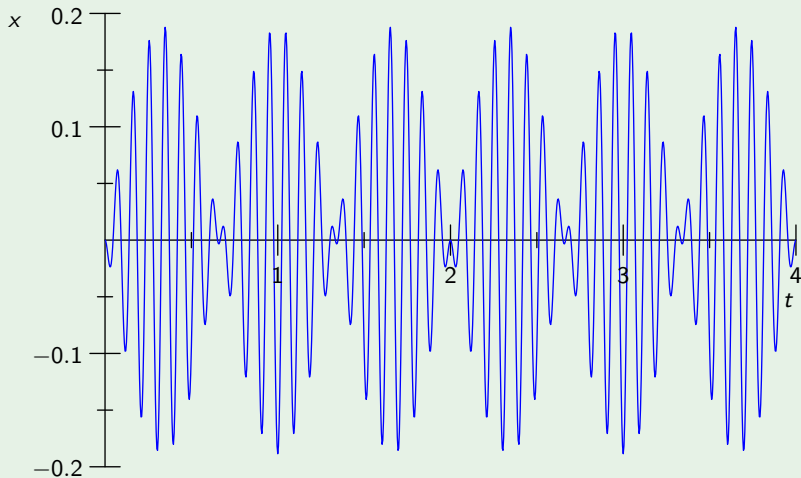
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The regular periodic patterns are called **beats**.

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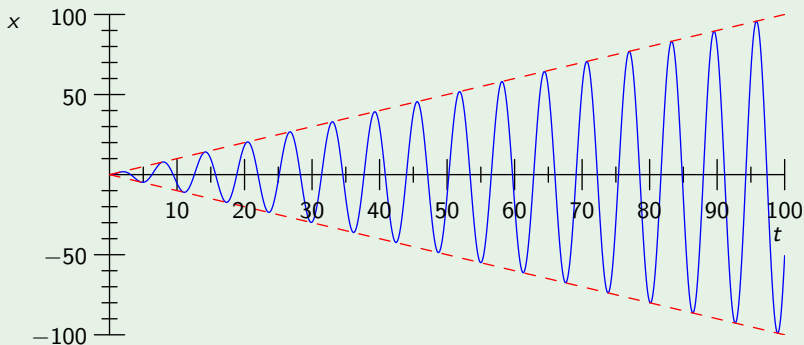
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If the system is initially at rest ( $x(0) = 0$  and  $\dot{x}(0) = 0$ ) then the solution is

$$x(t) = -\frac{F_0}{m(\omega_0^2 - \omega_f^2)} \cos(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega_f^2)} \cos(\omega_f t)$$

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So, we can simplify using the trigonometric identity

$$\cos(u) - \cos(v) = -2 \sin\left(\frac{u-v}{2}\right) \sin\left(\frac{u+v}{2}\right)$$

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The two curves

$$\pm \frac{2F_0}{m(\omega_0^2 - \omega_f^2)} \sin\left(\frac{\omega_0 - \omega_f}{2}t\right)$$

form an envelope of the more rapid oscillation and is called the **sinusoidal amplitude**.



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If the system starts from rest ( $x(0) = 0$  and  $\dot{x}(0) = 0$ ), the solution can be written as

$$x(t) = \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega_f^2)} \sin\left(\frac{\omega_0 - \omega_f}{2} t\right)}_{\text{sinusoidal amplitude}} \underbrace{\sin\left(\frac{\omega_0 + \omega_f}{2} t\right)}_{\text{rapid oscillation within beats}}$$

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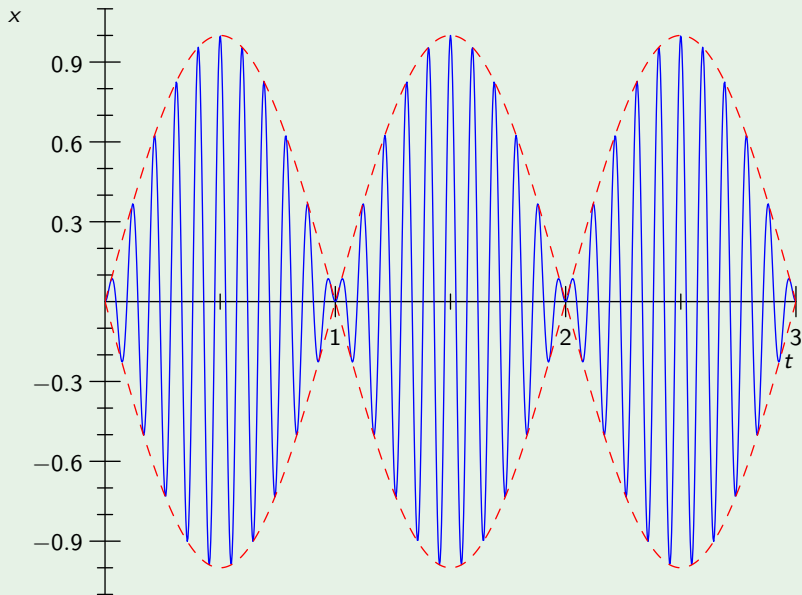
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Thus, envelope curves are

$$y = \pm 1 \cdot \sin(\pi t)$$

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We can use the method of undetermined coefficients:

$$\begin{aligned} x(t) &= A \cos(3t) + B \sin(3t) \\ \dot{x}(t) &= -3A \sin(3t) + 3B \cos(3t) \\ \ddot{x}(t) &= -9A \cos(3t) - 9B \sin(3t) \end{aligned}$$

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Substituting into the DE gives

$$\begin{aligned} &(-9A \cos(3t) - 9B \sin(3t)) \\ &\quad + 4(-3A \sin(3t) + 3B \cos(3t)) \\ &\quad + 5(A \cos(3t) + B \sin(3t)) = 10 \cos(3t) \end{aligned}$$

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Thus,

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The general solution is

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$$\begin{aligned}x(0) = 0 &\Rightarrow c_1 - \frac{1}{4} = 0 \\ \dot{x}(0) = 0 &\Rightarrow c_2 + \frac{7}{4} = 0\end{aligned}$$

### Example 5

The general solution is

$$x = e^{-2t} (c_1 \cos(t) + c_2 \sin(t)) - \frac{1}{4} \cos(3t) + \frac{3}{4} \sin(3t)$$

To solve the IVP, we need to calculate

$$\begin{aligned}\dot{x} &= -2e^{-2t} (c_1 \cos(t) + c_2 \sin(t)) \\ &\quad + e^{-2t} (-c_1 \sin(t) + c_2 \cos(t)) \\ &\quad + \frac{1}{4} \sin(3t) - \frac{3}{4} \cos(3t)\end{aligned}$$

Next, we need to plug in the initial conditions

$$\begin{aligned}x(0) = 0 &\Rightarrow c_1 - \frac{1}{4} = 0 &\Rightarrow c_1 = \frac{1}{4} \\ \dot{x}(0) = 0 &\Rightarrow c_2 + \frac{7}{4} = 0 &\Rightarrow c_2 = -\frac{7}{4}\end{aligned}$$

## Example 5

The solution to the IVP is

$$x = e^{-2t} \left( \frac{1}{4} \cos(t) - \frac{7}{4} \sin(t) \right) - \frac{1}{4} \cos(3t) + \frac{3}{4} \sin(3t)$$

## Example 5

The solution to the IVP is

$$x = e^{-2t} \underbrace{\left( \frac{1}{4} \cos(t) - \frac{7}{4} \sin(t) \right)}_{\text{Transient}} \underbrace{- \frac{1}{4} \cos(3t) + \frac{3}{4} \sin(3t)}_{\text{Steady-State}}$$

We call  $x_h$  **transient**, because for  $b > 0$  the solution tends towards zero.

## Example 5

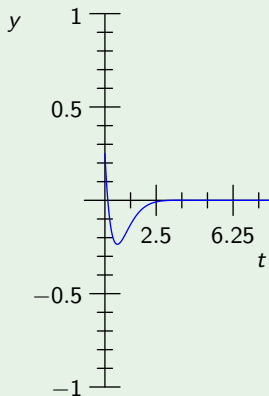
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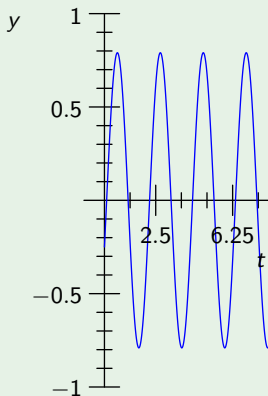
We call  $x_h$  **transient**, because for  $b > 0$  the solution tends towards zero.

The particular solution  $x_p$  may either be constant or a periodic **steady-state** solution.

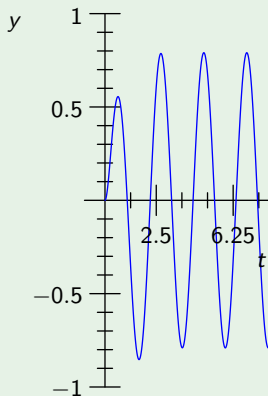
## Example 5



(a) Transient Solution



(b) Steady-State



(c) IVP Solution

## Particular Solution $x_p$ of a Damped Mass-Spring System

The damped mass-spring system

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_f t)$$

has particular solution

$$x_p = A \cos(\omega_f t) + B \sin(\omega_f t)$$

with

$$A = \frac{m(\omega_0^2 - \omega_f^2) F_0}{m^2(\omega_0^2 - \omega_f^2)^2 + (b\omega_f)^2} \quad \text{and} \quad B = \frac{b\omega_f F_0}{m^2(\omega_0^2 - \omega_f^2)^2 + (b\omega_f)^2}$$

with natural circular frequency  $\omega_0 = \sqrt{\frac{k}{m}}$ .



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### Note

You will verify this in the homework.