Complex Characteristic Roots

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Solution for Complex Characteristic Roots

For $\Delta < 0$, the characteristic roots of the DE

$$ay'' + by' + cy = 0$$

are

$$r_1 = \alpha + i\beta = -\frac{b}{2a} + i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$
$$r_2 = \alpha - i\beta = -\frac{b}{2a} - i\frac{\sqrt{-(b^2 - 4ac)}}{2a}$$

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The functions $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are linearly independent solutions, and the general solution is given by

$$y(t) = e^{\alpha t}\left(c_1\cos(\beta t) + c_2\sin(\beta t)\right)$$

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The set $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ forms a basis for the solution space $\mathbb{S}$.

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Let us find the general solution of

$$y'' - 4y' + 13y = 0$$

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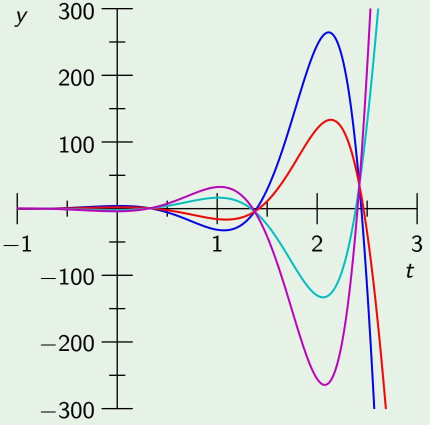
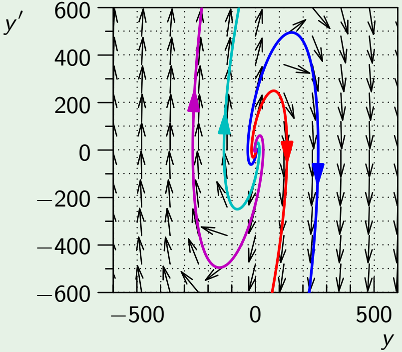


(a) Time Series



(b) Phase Portrait

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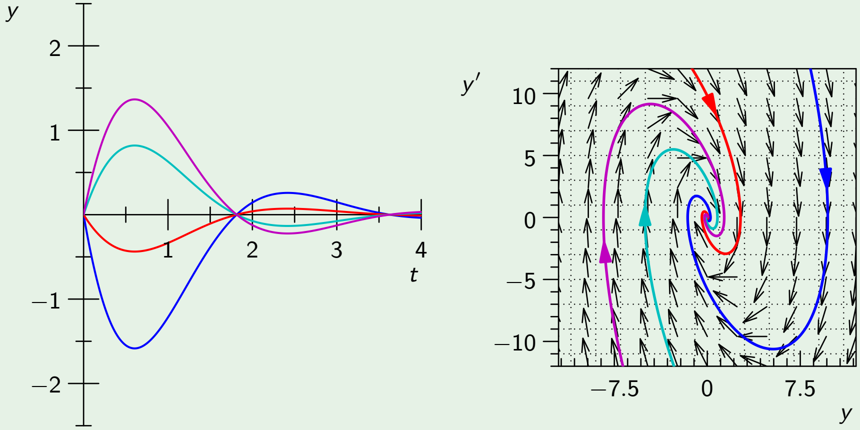

(a) Time Series (b) Phase Portrait

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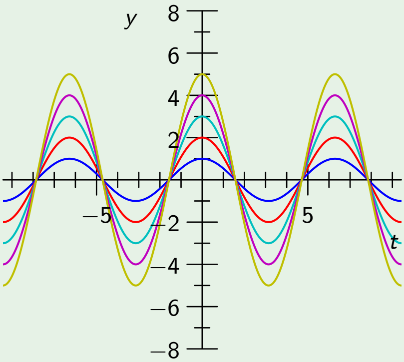

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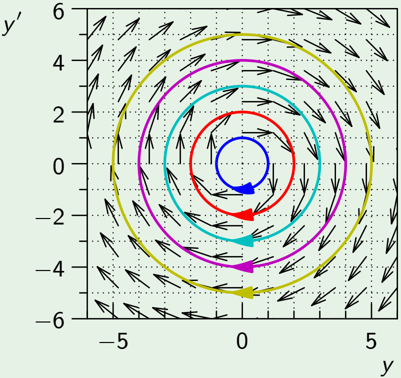

(b) Phase Portrait

Underdamped Mass-Spring System

The motion of a mass-spring system is called **underdamped** when we have $\Delta = b^2 - 4mk < 0$. Both characteristic roots are complex and the solutions are given by

$$x(t) = e^{-\frac{b}{2m}} \left(c_1 \cos\left(\omega_d t\right) + c_2 \sin\left(\omega_d t\right) \right), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Alternate Solution to the Underdamped Unforced Oscillator

$$x(t) = A(t)\cos(\omega_d t - \delta), \quad \omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

Where A and δ are determined by initial conditions, the following hold:

- **Time-varying amplitude** $A(t) = Ae^{-\frac{b}{2m}}$
- **Phase angle** δ
- **Phase shift** $\varphi = \dfrac{\delta}{\omega_d}$
- **Circular quasi-frequency** ω_d
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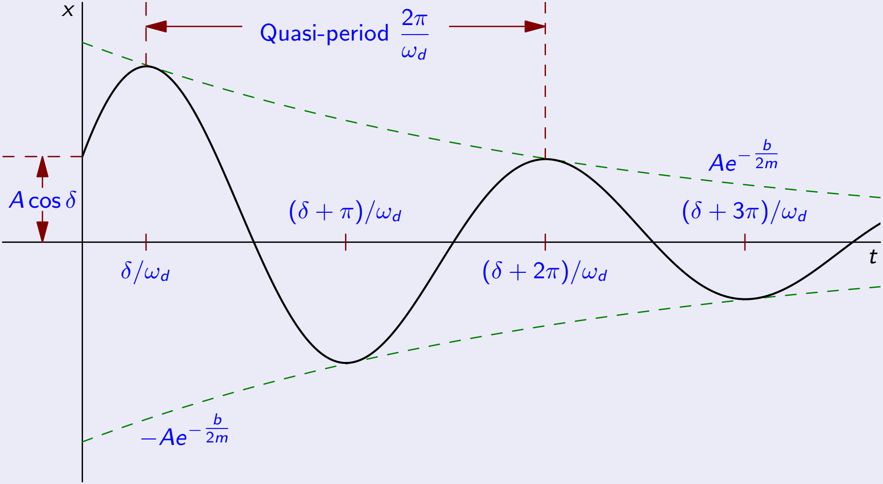
Alternate Solution Visual Example



Example 4

Consider the Mass-Spring IVP where

$$\ddot{x} + \dot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0$$

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Which means $\alpha = -\dfrac{1}{2}$ and $\beta = \dfrac{\sqrt{3}}{2}$.

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The general solution is

$$x(t) = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right)$$

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We can then calculate

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If we substitute in the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, we find that $c_1 = 1$ and $c_2 = \dfrac{1}{\sqrt{3}}$.

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In alternate polar form

$$x(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6} \right)$$

Where

$$A = \sqrt{1^2 + \left(\frac{1}{\sqrt{3}} \right)^2} = \frac{2}{\sqrt{3}} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{\frac{1}{\sqrt{3}}}{1} \right) = \frac{\pi}{6}$$

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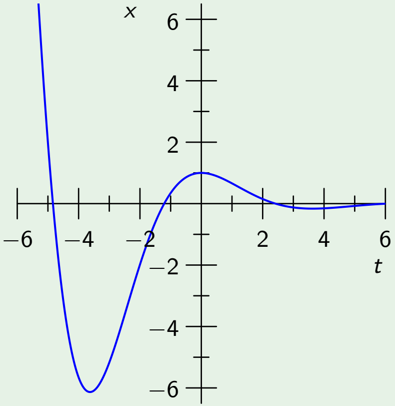

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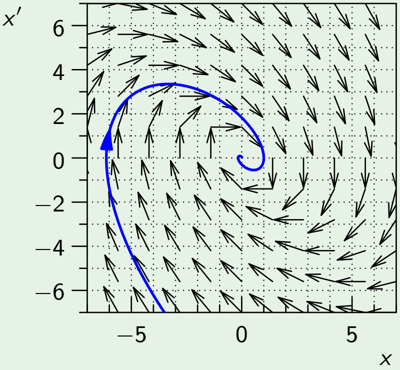
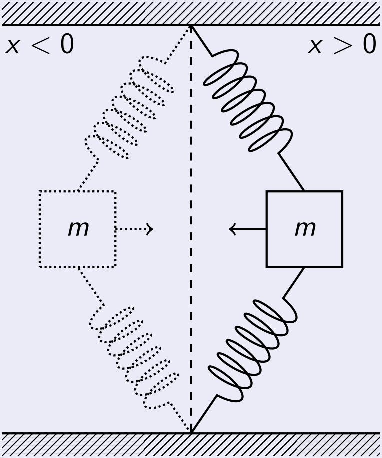


(b) Phase Portrait

The Guitar String: A Qualitative Analysis

The vibration of a guitar string can be described as a damped oscillator.



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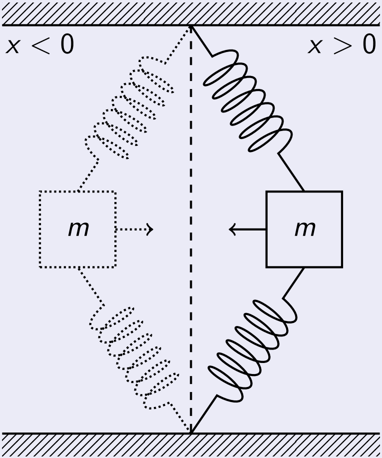

The motion of this spring is given by

$$\ddot{x} + \omega_0^2 x = 0$$

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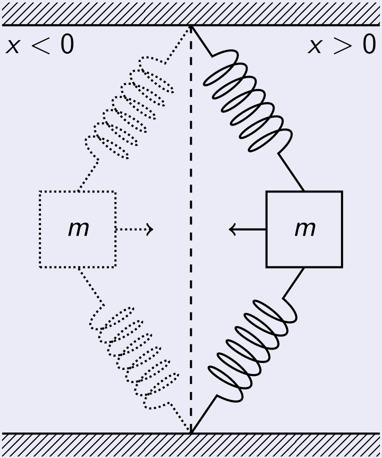

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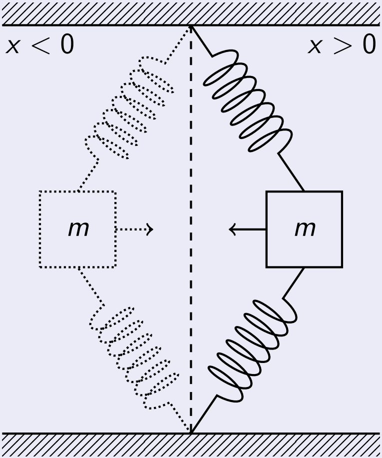

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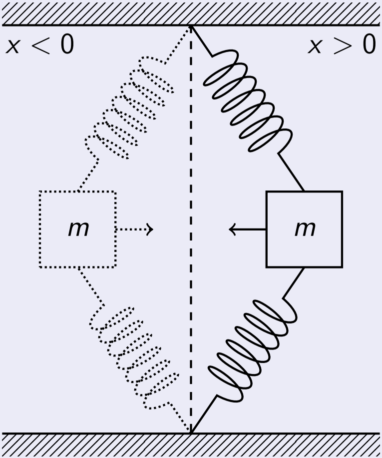

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Let use next consider a guitar string with damping.

Example 5

Consider the underdamped guitar string

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If we pluck the string, which means $x(0) = 5$ and $\dot{x}(0) = 0$, we find that $c_1 = 5$ and $c_2 = 1$.

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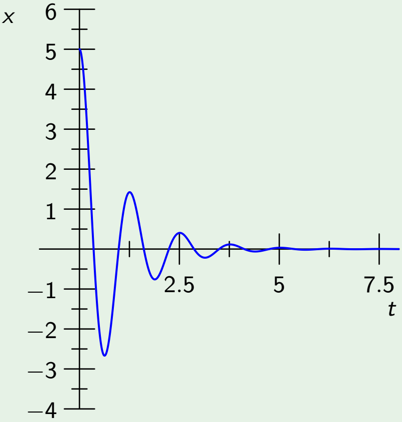
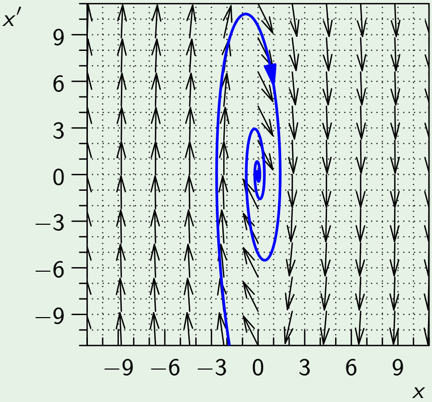


(a) Time Series



(b) Phase Portrait

Solutions to the Second-Order Linear DE with Constant Coefficients

The differential equation

$$ay'' + by' + cy = 0$$

has the characteristic equation

$$ar^2 + br + c = 0$$

The quadratic formula gives rise to three different general solutions, depending on the discriminant $\Delta = b^2 - 4ac$.

	Characteristic Roots	General Solution
$\Delta > 0$	$r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$\Delta = 0$	$r = -\dfrac{b}{2a}$	$y = c_1 e^{rt} + c_2 t e^{rt}$
$\Delta < 0$	$r_1, r_2 = \alpha \pm \beta$	$y = e^{\alpha t}\left(c_1 \cos\left(\beta t\right) + c_2 \sin\left(\beta t\right)\right)$
	$\alpha = -\dfrac{b}{2a}$, $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$	

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Which has the characteristic solutions

$$r_1 = 2, \quad r_2 = -2, \quad r_3 = 2i, \quad r_4 = -2i$$

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Thus, $\{e^{2t}, e^{-2t}, \cos(2t), \sin(2t)\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos(2t) + c_4 \sin(2t)$$

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Which has the characteristic solutions

$$r_1 = 1, \quad r_2 = 1, \quad r_3 = -3$$

Thus, $\{e^t, te^t, e^{-3t}\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = c_1 e^t + c_2 te^t + c_3 e^{-3t}$$

Example 8

Consider the fifth-order DE

$$\frac{d^5y}{dt^5} + 3\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = 0$$

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$$r_1 = -1, \quad r_2 = -1, \quad r_3 = -1, \quad r_4 = 0, \quad r_5 = 0$$

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Which has the characteristic solutions

$$r_1 = -1, \quad r_2 = -1, \quad r_3 = -1, \quad r_4 = 0, \quad r_5 = 0$$

Thus, $\{e^{-t}, te^{-t}, t^2e^{-t}, 1, t\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = \underbrace{(c_1 + c_2 t + c_3 t^2)}_{\text{for triple root}} e^{-t} + \underbrace{(c_4 + c_5 t)}_{\text{for double root}}$$

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Thus, $\{\cos(2t), t\cos(2t), \sin(2t), t\sin(2t)\}$ form a basis of $\mathbb{S}$ and the general solution is

$$y = \underbrace{(c_1 + c_2 t)}_{\text{for double root}}\cos(2t) + \underbrace{(c_3 + c_4 t)}_{\text{for double root}}\sin(2t)$$