The Inverse of a Matrix

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Inverse Matrix

If there exists, for an $n \times n$ matrix $\boldsymbol{A}$, another matrix $\boldsymbol{A}^{-1}$ of the same order such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$$

then $\boldsymbol{A}^{-1}$ is called the **inverse** of matrix $\boldsymbol{A}$, and $\boldsymbol{A}$ is called **invertible**.

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Vocabulary

- A square matrix that is not invertible is called **singular**.
- A square matrix that is invertible is called **nonsingular**.

Properties of Invertible Matrices

Invertible Matrix Properties

- If $\boldsymbol{A}$ is invertible, then so is $\boldsymbol{A}^{-1}$ and

$$\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$$

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$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

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- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices of the same order, then their product $\boldsymbol{AB}$ is invertible. In fact,

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

- if $\boldsymbol{A}$ is invertible, then so is $\boldsymbol{A}^{\mathsf{T}}$, and

$$\left(\boldsymbol{A}^{\mathsf{T}}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{\mathsf{T}}$$

Inverses by Reduced Row Echelon Form

Inverses by Reduced Row Echelon Form

For an $n \times n$ matrix $\boldsymbol{A}$, the following process will calculate $\boldsymbol{A}^{-1}$, or show that $\boldsymbol{A}$ is not invertible.

Step 1: Form the $n \times 2n$ augmented matrix $\boldsymbol{M} = [\boldsymbol{A}|\boldsymbol{I}_n]$.

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Step 1: Form the $n \times 2n$ augmented matrix $\boldsymbol{M} = [\boldsymbol{A}|\boldsymbol{I}_n]$.

Step 2: Transform $\boldsymbol{M}$ into Reduced Row Echelon Form.

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For an $n \times n$ matrix $\boldsymbol{A}$, the following process will calculate $\boldsymbol{A}^{-1}$, or show that $\boldsymbol{A}$ is not invertible.

Step 1: Form the $n \times 2n$ augmented matrix $\boldsymbol{M} = [\boldsymbol{A}|\boldsymbol{I}_n]$.

Step 2: Transform $\boldsymbol{M}$ into Reduced Row Echelon Form.

Step 3:
- If the left hand side of $\boldsymbol{M}$ is the identity matrix, then the right hand side is $\boldsymbol{A}^{-1}$.
- Otherwise, $\boldsymbol{A}$ is a non-invertible matrix.

Inverses by Reduced Row Echelon Form

Example

Consider the matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $\boldsymbol{A}^{-1}$

Inverses by Reduced Row Echelon Form

Example

Consider the matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $\boldsymbol{A}^{-1}$

Start by building the augmented matrix

$$\boldsymbol{M_A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_A}$ into Reduced Row Echelon Form.

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \; R_3 = r_3 - r_1$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \; R_3 = r_3 - r_1$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 = -r_3 \\ R_3 = r_2 \end{array}$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 = -r_3 \\ R_3 = r_2 \end{array}$$

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$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \quad R_1 = r_1 - r_3$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \quad R_1 = r_1 - r_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \quad R_1 = r_1 - r_2$$

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$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \quad R_1 = r_1 - r_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Since the left hand side is $\boldsymbol{I}_3$, we know the right hand side is the inverse:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

Inverses by Reduced Row Echelon Form

Example

Consider the matrix:

$$\boldsymbol{B} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Find $\boldsymbol{B}^{-1}$

Inverses by Reduced Row Echelon Form

Example

Consider the matrix:

$$\boldsymbol{B} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Find $\boldsymbol{B}^{-1}$

Start by building the augmented matrix

$$\boldsymbol{M_B} = \left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_B}$ into Reduced Row Echelon Form.

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{rrr|rrr} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 = r_3 \\ \\ R_3 = r_1 \end{array}$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 = r_3 \\ \\ R_3 = r_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 = r_2 + r_1 \\ R_3 = r_2 - 3r_1 \end{array}$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 = r_2 + r_1 \\ R_3 = r_2 - 3r_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right]$$

Inverses by Reduced Row Echelon Form

Example

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Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} \\ R_2 = \frac{1}{3} r_2 \\ R_3 = r_3 + r_2 \end{array}$$

Inverses by Reduced Row Echelon Form

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} R_2 = \frac{1}{3}r_2 \\ R_3 = r_3 + r_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 \\ 0 & -3 & -3 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} \\ R_2 = \frac{1}{3}r_2 \\ R_3 = r_3 + r_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

This means that B is a non-invertible matrix.

Invertibility and Solutions

Invertibility and Solutions

Consider the matrix equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$.
Where $\boldsymbol{A}$ is an $n \times n$ matrix, and $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}$ are of length n.

- A unique solution exists if and only if $\boldsymbol{A}$ is invertible.

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Consider the matrix equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$.
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- A unique solution exists if and only if $\boldsymbol{A}$ is invertible.
- Otherwise there are either:
  - No solutions.
  - Infinitely many solutions.

  (Another method must be used to determine which.)

Invertibility and Solutions

Example

Consider the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ & & 2y & + & z & = & -1 \\ x & & & + & z & = & 3 \end{array}$$

Invertibility and Solutions

Example

Consider the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ & & 2y & + & z & = & -1 \\ x & & & + & z & = & 3 \end{array}$$

We can can write this as the matrix equation:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

Invertibility and Solutions

Example

So, if $\boldsymbol{A}$ is invertible, then we can solve the matrix equation for $\vec{\boldsymbol{x}}$

$$\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$$

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So, if $\boldsymbol{A}$ is invertible, then we can solve the matrix equation for $\vec{\boldsymbol{x}}$

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$$\boldsymbol{A}^{-1}\boldsymbol{A}\vec{\boldsymbol{x}} = \boldsymbol{A}^{-1}\vec{\boldsymbol{b}}$$

Invertibility and Solutions

Example

So, if $\boldsymbol{A}$ is invertible, then we can solve the matrix equation for $\vec{\boldsymbol{x}}$

$$\begin{aligned}
\boldsymbol{A}\vec{\boldsymbol{x}} &= \vec{\boldsymbol{b}} \\
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\boldsymbol{I}_3\vec{\boldsymbol{x}} &= \boldsymbol{A}^{-1}\vec{\boldsymbol{b}}
\end{aligned}$$

Invertibility and Solutions

Example

So, if $\boldsymbol{A}$ is invertible, then we can solve the matrix equation for $\vec{\boldsymbol{x}}$

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So, if $\boldsymbol{A}$ is invertible, then we can solve the matrix equation for $\vec{\boldsymbol{x}}$

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So, if we can compute $\boldsymbol{A}^{-1}\vec{\boldsymbol{b}}$ we will have solved the system.

Example

$$
\begin{array}{c|c}
 & \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\
\hline
\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ \\ \end{bmatrix}
\end{array}
$$

Example

$$
\begin{array}{c|c}
 & \begin{bmatrix} \boxed{\begin{matrix} 2 \\ -1 \\ 0 \end{matrix}} \end{bmatrix} \\
\hline
\begin{bmatrix} 2 & -1 & -1 \\ \boxed{1 \quad 0 \quad -1} & & \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ \boxed{2} \\ \end{bmatrix}
\end{array}
$$

Invertibility and Solutions

Example

$$\begin{array}{c|c} & \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix} \end{array}$$

Invertibility and Solutions

Example

$$
\begin{array}{c|c}
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\hline
\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}
\end{array}
$$

So, we have

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}
$$

Singular Matrix Properties

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following are equivalent:

- $\boldsymbol{A}$ is an invertible matrix.

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- $\boldsymbol{A}^{\intercal}$ is an invertible matrix.

Singular Matrix Properties

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- $\boldsymbol{A}$ is an invertible matrix.
- $\boldsymbol{A}^{\mathsf{T}}$ is an invertible matrix.
- $\boldsymbol{A}$ is row equivalent to $\boldsymbol{I}_n$.
  (This means when you put $\boldsymbol{A}$ in RREF, you get $\boldsymbol{I}_n$)

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- The rank of $\boldsymbol{A}$ is n.

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  (This means when you put $\boldsymbol{A}$ in RREF, you get $\boldsymbol{I}_n$)
- The rank of $\boldsymbol{A}$ is n.
- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ has only the trivial solution $\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$.

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  (This means when you put $\boldsymbol{A}$ in RREF, you get $\boldsymbol{I}_n$)
- The rank of $\boldsymbol{A}$ is n.
- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ has only the trivial solution $\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$.
- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a unique solution for every $\vec{\boldsymbol{b}} \in \mathbb{R}^n$.

Third-Order Initial-Value Problem Example

Example

An engineering consultant finds that she must solve the following IVP:

$$y''' - 2y'' - y' + 2y = 0, \quad y(0) = b_1,\ y'(0) = b_2,\ y''(0) = b_3$$

She must solve this IVP for many different sets of initial conditions, and expects to do the same tomorrow.

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An engineering consultant finds that she must solve the following IVP:

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She must solve this IVP for many different sets of initial conditions, and expects to do the same tomorrow.
The general solution is:

$$y(t) = c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}$$

(We will talk about to solve this type of DE in Chapter 4.)

Third-Order Initial-Value Problem Example

Example

To determine c_1, c_2, and c_3, we must plug in each initial condition, giving the system:

$$\begin{aligned} y(0) &= \ \ c_1 + c_2 + c_3 &= b_1 \\ y'(0) &= 2c_1 + c_2 - c_3 &= b_2 \\ y''(0) &= 4c_1 + c_2 + c_3 &= b_3 \end{aligned}$$

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We can write this as the matrix equation:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

Third-Order Initial-Value Problem Example

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If we can find the inverse of $\boldsymbol{A}$, then we can compute the constants for any set of initial conditions $\vec{\boldsymbol{b}}$.

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If we can find the inverse of $\boldsymbol{A}$, then we can compute the constants for any set of initial conditions $\vec{\boldsymbol{b}}$.

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Third-Order Initial-Value Problem Example

Example

If we can find the inverse of $\boldsymbol{A}$, then we can compute the constants for any set of initial conditions $\vec{\boldsymbol{b}}$.

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Thus, the solution for any $\vec{\boldsymbol{b}}$ is:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$