The Determinant of a Matrix

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Determinant of a 2×2 Matrix

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Example 1

$$\begin{vmatrix} 3 & 8 \\ 5 & -1 \end{vmatrix} = 3 \cdot (-1) + 8 \cdot 5 = 37$$

Minors of a Matrix

For every element a_{ij} of a $n \times n$ matrix $\boldsymbol{A}$, the **minor** $\boldsymbol{M_{ij}}$ is an $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth column of $\boldsymbol{A}$.

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Cofactors of a Matrix

For every element a_{ij} of a $n \times n$ matrix $\boldsymbol{A}$, the **cofactor** of a_{ij} is the scalar

$$C_{ij} = (-1)^{(i+j)} \left| \boldsymbol{M_{ij}} \right|$$

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I recommend expanding across the first row.

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Compute the determinant:

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- If $\boldsymbol{A}$ is an diagonal, upper triangular, or lower triangular matrix, the determinant is the product of the diagonal elements:

$$|\boldsymbol{A}| = \prod_{i=1}^{m} a_{ii}$$

Cramer's Rule

Consider the matrix equation:

$$\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}} \quad \text{where} \quad |\boldsymbol{A}| \neq 0$$

The matrix $\boldsymbol{A_j}$ is obtained by replacing the jth column of $\boldsymbol{A}$ with $\vec{\boldsymbol{b}}$.

The jth solution is:

$$x_j = \frac{|\boldsymbol{A_j}|}{|\boldsymbol{A}|}$$

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Consider the system

$$\begin{array}{rcrcr} x & + & 2y & = & 5 \\ 2x & + & 3y & = & 8 \end{array}$$

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Let us solve this system using Cramer's Rule.
This means we need to calculate the following determinats.

$$|\boldsymbol{A}| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \qquad |\boldsymbol{A_x}| = \begin{vmatrix} 5 & 2 \\ 8 & 3 \end{vmatrix} \qquad |\boldsymbol{A_y}| = \begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix}$$

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We can now find x

$$x = \frac{|\boldsymbol{A_x}|}{|\boldsymbol{A}|}$$

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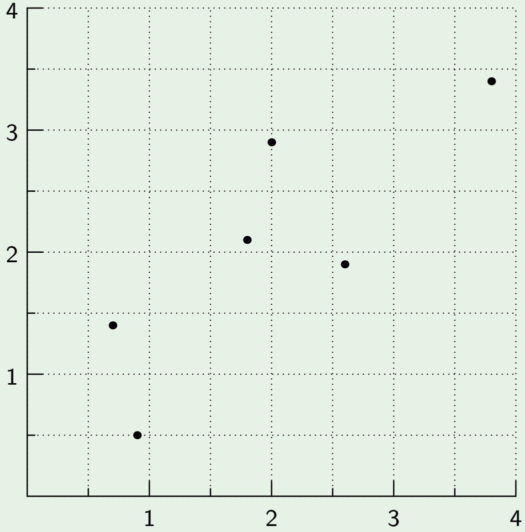
Example 5

The **line of best fit** is the line that gets "closest" to every point.



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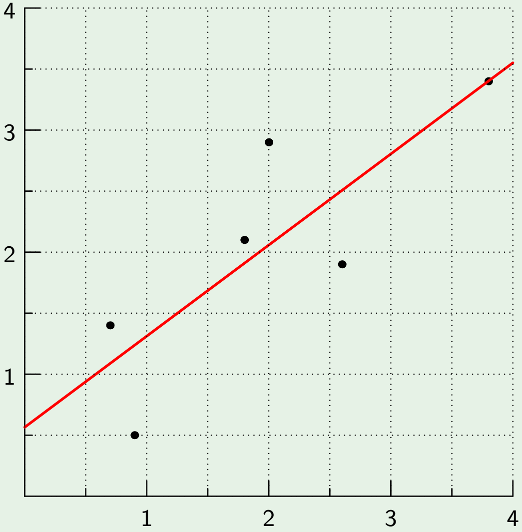

Least Squares Approximation

A general strategy for finding the line $y = mx + b$ that best describes a data set is to find b and m that minimizes the sums of the squares of the vertical distances between the data points and the line, given by $F(b, m)$

$$F(b, m) = \sum_{i=1}^{n} \left(y_i - (b + mx_i)\right)^2$$

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To find such a b and m, we need to solve the system:

$$\frac{\partial F}{\partial b} = 0 \quad \text{and} \quad \frac{\partial F}{\partial m} = 0$$

Least Squares Method

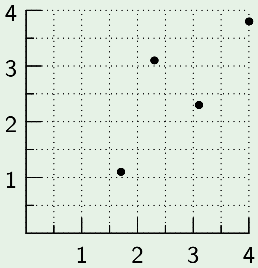
The best-fit straight line for n data points (x_i, y_i), $i = 1, 2, \ldots, n$, has y-intercept b and slope m as determined by the system

$$\begin{bmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Example 6

Consider the data comparing the high school and college GPA for four students.

i	x_i	y_i
1	1.7	1.1
2	2.3	3.1
3	3.1	2.3
4	4.0	3.8



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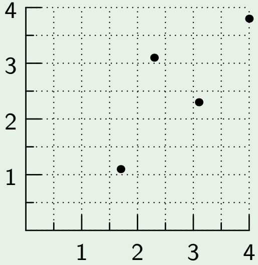

The Least Squares Method system for this dataset is:

$$\begin{bmatrix} 4 & 11.1 \\ 11.1 & 33.79 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 10.3 \\ 31.33 \end{bmatrix}$$

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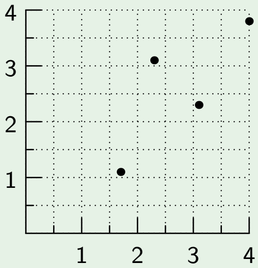

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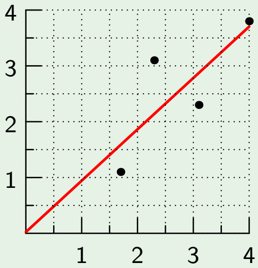

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So, the line of best fit is $y = 0.92x + 0.023$.

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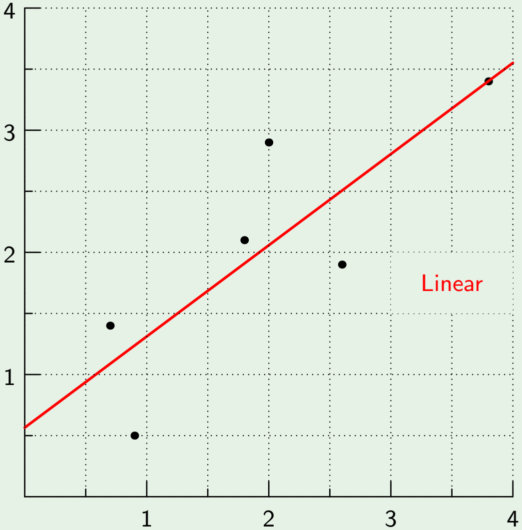
Let us consider some nonlinear approximations.



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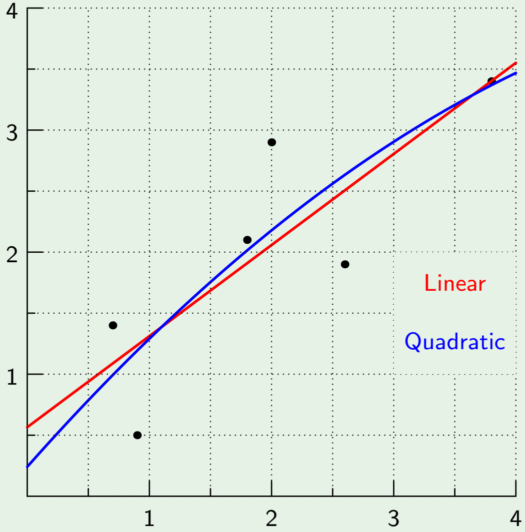

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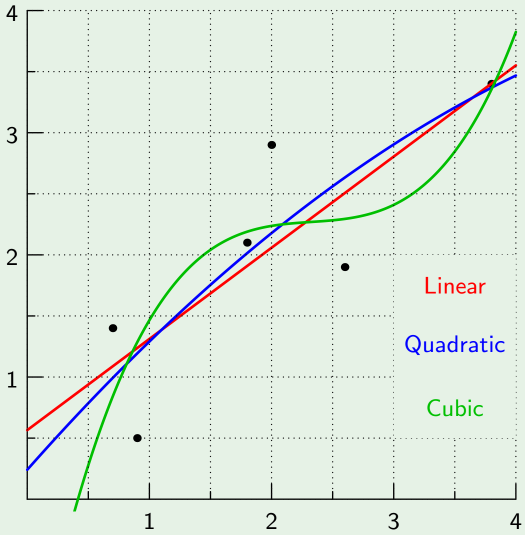

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