

Conditional Probability

Colby Community College

Example 1

The `photo_classify` data set represents a machine learning algorithm classifying a sample of 1822 photos as either about fashion or not.

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		truth		Total
		<i>fashion</i>	<i>not</i>	
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	<i>pred_not</i>	112	1491	1603
	Total	309	1513	1822

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If a photo is actually about fashion, what is the chance the algorithm will correctly identify the photo as being about fashion?

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The `photo_classify` data set represents a machine learning algorithm classifying a sample of 1822 photos as either about fashion or not.

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	Total	309	1513	1822

If a photo is actually about fashion, what is the chance the algorithm will correctly identify the photo as being about fashion?

Of the 309 fashion photos, the algorithm correctly classifies 197 of them.

$$P(\text{mach_learn is } \textit{pred_fashion} \text{ given truth is } \textit{fashion}) = \frac{197}{309} = 0.638$$

Example 2

Using the same data set as in Example 1.

		truth		Total
		<i>fashion</i>	<i>not</i>	
mach_learn	<i>pred_fashion</i>	197	22	219
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If the algorithm predicts the photo as being about fashion, what is the probability is actually is?

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		<i>fashion</i>	<i>not</i>	
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	Total	309	1513	1822

If the algorithm predicts the photo as being about fashion, what is the probability is actually is?

Of the 1603 photos predicted to be about fashion, 112 we actually about fashion.

$$P(\text{truth is } \textit{fashion} \text{ given mach_learn is } \textit{pred_fashion}) = \frac{197}{219} = 0.900$$

Note

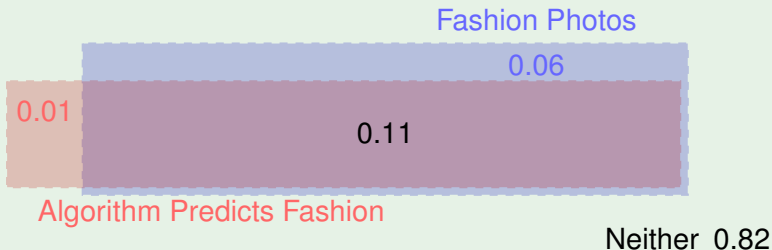
It can be helpful to draw Venn Diagrams of these contingency tables using rectangles.

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Example 3

The Venn Diagram for Example 1 is:



Definition

A **marginal probability** is a probability based on a single variable without regard to other variables.

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Example 4

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A probability of outcomes for two or more variables is called a **joint probability**.

Example 5

$$P(\text{mach_learn is } \textit{pred_fashion} \text{ and truth is } \textit{fashion}) = \frac{197}{1822} = 0.11$$

Note

Sometimes a comma is substituted for “and” in a joint probability.

$P(\text{mach_learn is } \textit{pred_fashion}, \text{truth is } \textit{fashion})$

means the same thing as

$P(\text{mach_learn is } \textit{pred_fashion} \text{ and truth is } \textit{fashion})$

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Example 6

The table proportions for `photo_classify` are:

	truth: <i>fashion</i>	truth: <i>not</i>	Total
mach_learn: <i>pred_fashion</i>	$\frac{197}{1822}$		
mach_learn: <i>pred_not</i>			
Total			
	↓ ↓ ↓		
	truth: <i>fashion</i>	truth: <i>not</i>	Total
mach_learn: <i>pred_fashion</i>	0.1081		
mach_learn: <i>pred_not</i>			
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Total			
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	truth: <i>fashion</i>	truth: <i>not</i>	Total
mach_learn: <i>pred_fashion</i>	0.1081	0.0121	
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mach_learn: <i>pred_not</i>			
Total			
	↓ ↓ ↓		
	truth: <i>fashion</i>	truth: <i>not</i>	Total
mach_learn: <i>pred_fashion</i>	0.1081	0.0121	0.1202
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<i>mach_learn: pred_not</i>	$\frac{112}{1822}$		
Total			
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	truth: <i>fashion</i>	truth: <i>not</i>	Total
<i>mach_learn: pred_fashion</i>	0.1081	0.0121	0.1202
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Total			
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	truth: <i>fashion</i>	truth: <i>not</i>	Total
<i>mach_learn: pred_fashion</i>	0.1081	0.0121	0.1202
<i>mach_learn: pred_not</i>	0.0615	0.8183	
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Total			

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	truth: <i>fashion</i>	truth: <i>not</i>	Total
<i>mach_learn: pred_fashion</i>	0.1081	0.0121	0.1202
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<i>mach_learn: pred_not</i>	$\frac{112}{1822}$	$\frac{1491}{1822}$	$\frac{1603}{1822}$
Total	$\frac{309}{1822}$		

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	truth: <i>fashion</i>	truth: <i>not</i>	Total
<i>mach_learn: pred_fashion</i>	0.1081	0.0121	0.1202
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Total	$\frac{309}{1822}$	$\frac{1513}{1822}$	

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	truth: <i>fashion</i>	truth: <i>not</i>	Total
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Total	0.1696	0.8304	

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Total	$\frac{309}{1822}$	$\frac{1513}{1822}$	$\frac{1822}{1822}$

↓ ↓ ↓

	truth: <i>fashion</i>	truth: <i>not</i>	Total
<i>mach_learn: pred_fashion</i>	0.1081	0.0121	0.1202
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Total	0.1696	0.8304	1.0

Example 7

The table proportions from Example 6 make a probability distribution.

Joint Outcome	Probability
<code>mach_learn</code> is <i>pred_fashion</i> and truth is <i>fashion</i>	0.1081
<code>mach_learn</code> is <i>pred_fashion</i> and truth is <i>not</i>	0.0121
<code>mach_learn</code> is <i>pred_not</i> and truth is <i>fashion</i>	0.0615
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Note

Joint probabilities can be used to calculate marginal probabilities in simple cases.

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Example 8

$$P(\text{truth is } \textit{fashion}) = P(\text{mach_learn is } \textit{pred_fashion} \text{ and truth is } \textit{fashion}) \\ + P(\text{mach_learn is } \textit{pred_not} \text{ and truth is } \textit{fashion})$$

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$$\begin{aligned}P(\text{truth is } \textit{fashion}) &= P(\text{mac_learn is } \textit{pred_fashion} \text{ and truth is } \textit{fashion}) \\&\quad + P(\text{mac_learn is } \textit{pred_not} \text{ and truth is } \textit{fashion}) \\&= 0.1081 + 0.0615\end{aligned}$$

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Example 9

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Example 9

$$P(\text{truth is } \textit{fashion} \text{ given mach_learn is } \textit{pred_fashion}) = \frac{197}{219} = 0.900$$

Definition

There are two parts to a conditional probability, the **outcome of interest** and the **condition**.

$P(\text{outcome of interest given condition})$
is the same as

$P(\text{outcome of interest} \mid \text{condition})$

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Example 10

$$P(\text{truth is } \textit{fashion} \mid \text{mach_learn is } \textit{pred_fashion}) = \frac{197}{219} = 0.900$$

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Conditional probabilities are computed as:

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Example 11

$$P(\text{truth is } \textit{fashion} \mid \text{mach_learn is } \textit{pred_fashion})$$

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$$\begin{aligned} &P(\text{truth is } \textit{fashion} \mid \text{mach_learn is } \textit{pred_fashion}) \\ &= \frac{P(\text{truth is } \textit{fashion} \text{ and } \text{mach_learn is } \textit{pred_fashion})}{P(\text{mach_learn is } \textit{pred_fashion})} \end{aligned}$$

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Example 12

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

		inoculated		Total	inoculated		Total
		yes	no		yes	no	
result	<i>lived</i>	238	5136	5374	0.0382	0.8252	0.8634
	<i>died</i>	6	844	850	0.0010	0.1356	0.1366
	Total	244	5980	6224	0.0392	0.9608	1.0000

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$$P(\text{result is died} \mid \text{inoculated is no})$$

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$$\begin{aligned} &P(\text{result is died} \mid \text{inoculated is no}) \\ &= \frac{P(\text{result is died and inoculated is no})}{P(\text{inoculated is no})} \\ &= \frac{0.1356}{0.9608} \\ &= 0.1411 \approx 14.11\% \end{aligned}$$

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$$P(\text{result is died} \mid \text{inoculated is yes})$$

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What is the probability that a randomly selected inoculated person died from smallpox?

$$\begin{aligned} &P(\text{result is died} \mid \text{inoculated is yes}) \\ &= \frac{P(\text{result is died and inoculated is yes})}{P(\text{inoculated is yes})} \end{aligned}$$

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$$\begin{aligned} &P(\text{result is died} \mid \text{inoculated is yes}) \\ &= \frac{P(\text{result is died and inoculated is yes})}{P(\text{inoculated is yes})} \\ &= \frac{0.0010}{0.0392} \end{aligned}$$

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What is the probability that a randomly selected inoculated person died from smallpox?

$$\begin{aligned} &P(\text{result is died} \mid \text{inoculated is yes}) \\ &= \frac{P(\text{result is died and inoculated is yes})}{P(\text{inoculated is yes})} \\ &= \frac{0.0010}{0.0392} \\ &= 0.0255 \approx 2.55\% \end{aligned}$$

Example 14

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

The residents of Boston self-selected whether or not to be inoculated.

Is this study observational or experimental?

Example 14

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People die for many reasons, wealth determines level of medical care available, etc. . .

General Multiplication Rule

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$$P(A \text{ and } B) = P(A | B) \cdot P(B)$$

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Let's use

- $P(\text{inoculated is yes}) = 0.0392$
- $P(\text{result is lived} \mid \text{inoculated is yes}) = 0.9754$

to find the probability that a person was both inoculated and lived.

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Sum of Conditional Probabilities

Let A_1, A_2, \dots, A_k represent all the disjoint outcomes for a variable. Then if B is an event, possibly for another variable, we have:

$$P(A_1 | B) + P(A_2 | B) + \cdots + P(A_k | B) = 1$$

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If 97.54% of the inoculated people lived, what is the proportion of people that must have died?

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There are only two outcomes: *lived* and *died*. Which means that $100\% - 97.54\% = 2.46\%$ people who were inoculated died.

Example 18

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Let us compute the following:

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Note

We have shown that if two events are independent, then knowing the outcome of one should provide no information about the other.

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Ron is watching a roulette table in a casino and notices that the last five outcomes were *black*. He figures that the chances of getting *black* six times in a row is very small and puts his paycheck on red.

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Note

Posting the last several outcomes of a betting game is a real practice casinos use to trick people into believing the odds are in their favor. It's known as the **gambler's fallacy**.

Definition

A **tree diagram** is a tool to organize outcomes and probabilities around the structure of the data.

They are most useful when two or more processes occur in a sequence and each process is conditioned on its predecessor.

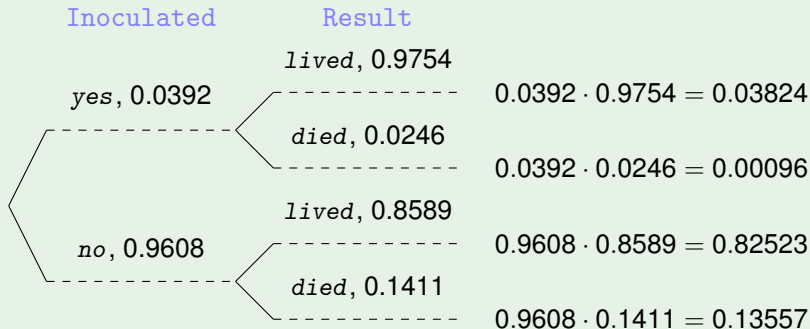
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Here is the tree diagram for `smallpox` dataset.

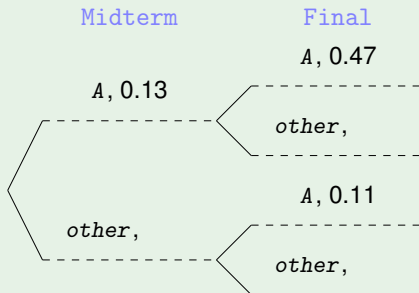


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Suppose that 13% of students earned an A on the midterm. Of those students that earned an A, 47% received an A on the final and 11% of the student who earned a lower grade than an A on the midterm received an A on the final.

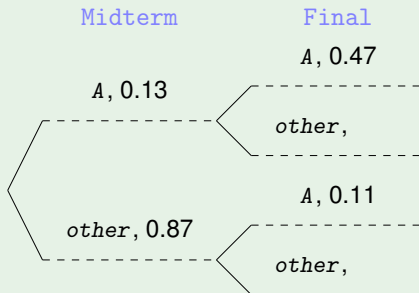
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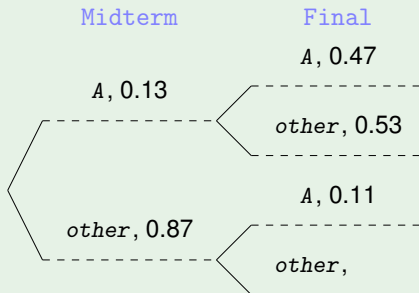
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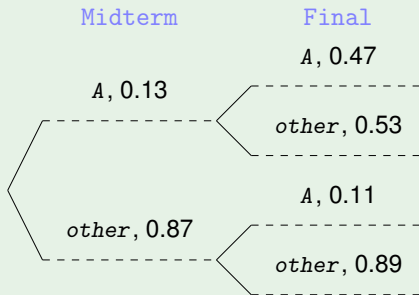
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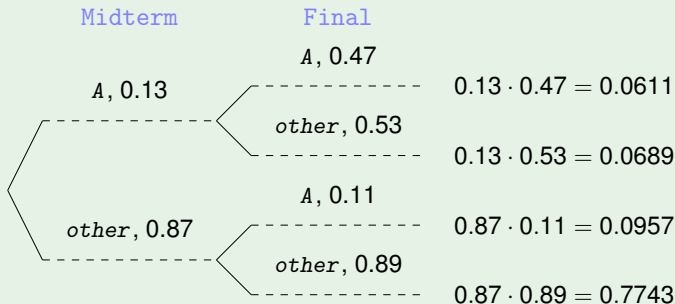
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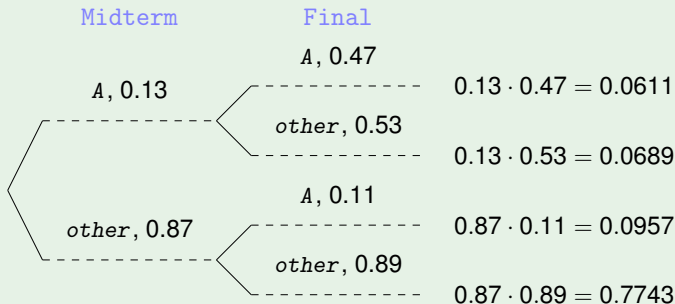
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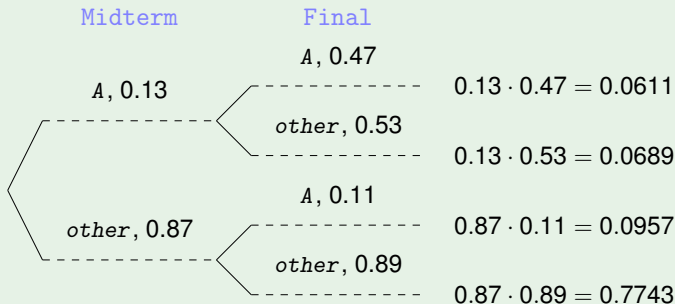
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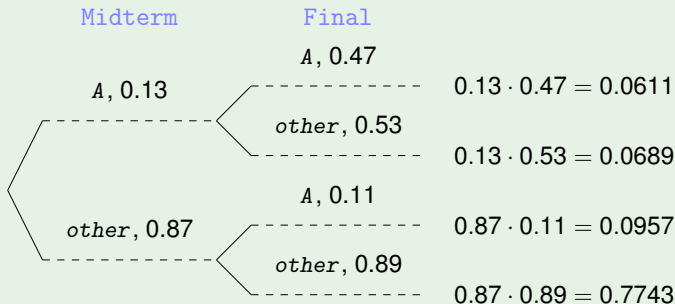
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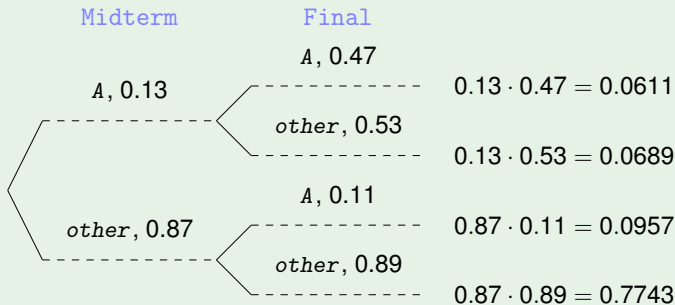
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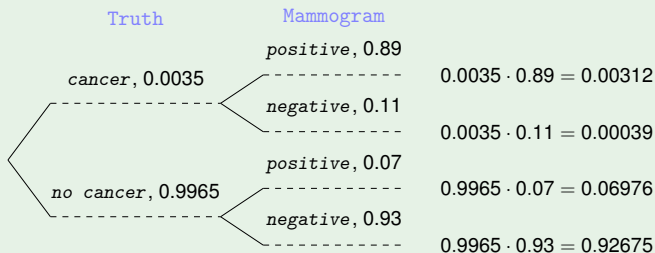
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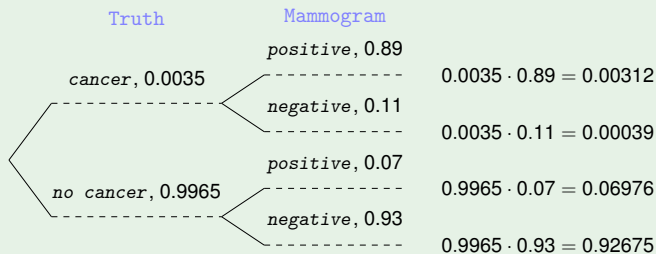
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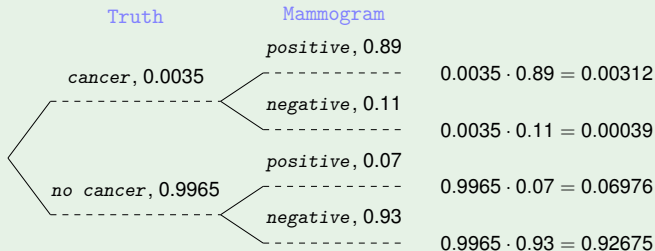
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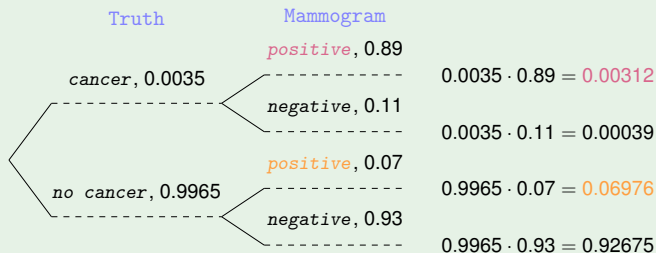
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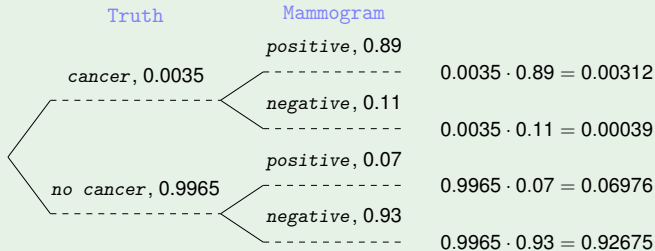
So,

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Note

There are times where we are given:

$$P(\text{statement about variable 1} \mid \text{statement about variable 2})$$

but we would rather know the inverted conditional probability:

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Bayes' Theorem

Consider the following conditional probability for variable 1 and variable 2:

$$P(\text{outcome } A_1 \text{ of variable 1} \mid \text{outcome } B \text{ of variable 2})$$

Bayes' Theorem states that this conditional probability is the same as:

$$\frac{P(B \mid A_1) P(A_1)}{P(B \mid A_1) P(A_1) + P(B \mid A_2) P(A_2) + \cdots + P(B \mid A_k) P(A_k)}$$

where A_2, A_3, \dots, A_k represent all other possible outcomes of variable 1.

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Note

This strategy of updating beliefs using Bayes' Theorem is the foundation of an entire branch of statistics called **Bayesian statistics**.