

# Systems of Linear Equations

Colby Community College

## System of Linear Equations

A  $m \times n$  **system of linear equations** is a set of  $m$  equations in  $n$  variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & \dots & + & a_{3n}x_n & = & b_3 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

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## Matrices

A **matrix** is a rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

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## An Augmented Matrix

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

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## A Matrix Equation (We will look at these in section 8.3)

As the matrix equation  $A\vec{x} = \vec{b}$ , where:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

## Example 1

Consider the system of linear equations:

$$3x - 4y = -6$$

$$2x - 3y = -5$$

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### Example 2

Consider the augmented matrix:

$$\left[ \begin{array}{cc|c} 5 & 2 & 13 \\ -3 & 1 & -10 \end{array} \right]$$

This matrix corresponds to the system of linear equations:

$$5x + 2y = 13$$

$$-3x + y = -10$$

### Example 3

Consider the system of linear equations:

$$2x - y + z = 0$$

$$x + z - 1 = 0$$

$$x + 2y - 8 = 0$$

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The system must be in standard form before we can write the augmented matrix.

$$2x - y + z = 0$$

$$x + 0y + z = 1$$

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Thus, the augmented matrix is:

$$\left[ \begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 8 \end{array} \right]$$



## Notation

- $r_i$  denotes row  $i$  *before* the row operation is applied
- $R_i$  denotes row  $i$  *after* the row operation is applied

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## Elementary Row Operations

- Swap row  $i$  and row  $j$ :

$$R_i \leftrightarrow R_j \quad (\text{or } R_i = r_j, R_j = r_i)$$

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- Multiply row  $i$  by a nonzero constant:

$$R_i = c \cdot r_i$$

- Add row  $j$  to row  $i$  (leaving row  $j$  unchanged):

$$R_i = r_i + r_j$$

### Example 4

Let us apply the row operation  $R_2 = -3r_1 + r_2$  to the matrix

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array} \right]$$

### Example 4

Let us apply the row operation  $R_2 = -3r_1 + r_2$  to the matrix

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array} \right]$$

We want to work one column at a time:

$$\left[ \begin{array}{cc|c} & 1 & -2 & 2 \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{ccc|c} & 1 & -2 & 2 \\ -3(1) & & & \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ -3(1) + 3 & & \end{array} \right]$$



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$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array} \right]$$

We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 0 & & \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 0 & (-3)(-2) & \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 1 \end{array} \right]$$

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We want to work one column at a time:

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array} \right]$$

### Example 5

Let us apply the row operation  $R_1 = 2r_2 + r_1$  to the matrix

$$\left[ \begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array} \right]$$



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$$\left[ \begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array} \right]$$

We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 2(3) & & \\ & 3 & 1 & 4 \end{array} \right]$$

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$$\left[ \begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array} \right]$$

We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 8 & & \\ 3 & 1 & 4 \end{array} \right]$$

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$$\left[ \begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array} \right]$$

We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 8 & 2(1) & \\ 3 & 1 & 4 \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 8 & 2(1) + (-2) & \\ 3 & 1 & 4 \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 8 & 0 & 2(4) \\ 3 & 1 & 4 \end{array} \right]$$



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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 8 & 0 & 2(4) + 1 \\ 3 & 1 & 4 \end{array} \right]$$

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We want to work one column at a time:

$$\left[ \begin{array}{cc|c} 8 & 0 & 9 \\ 3 & 1 & 4 \end{array} \right]$$

## Gaussian Elimination

Use row operations until in **Row Echelon Form**:

$$\left[ \begin{array}{ccccc|c} 1 & c_{12} & c_{13} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & d_2 \\ 0 & 0 & 1 & \cdots & c_{3n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_m \end{array} \right]$$

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Use row operations until in **Row Echelon Form**:

$$\left[ \begin{array}{cccc|c} 1 & c_{12} & c_{13} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & d_2 \\ 0 & 0 & 1 & \cdots & c_{3n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_m \end{array} \right]$$

Then back solve the system:

$$x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n = d_1$$

$$x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2$$

$$\vdots$$

$$x_n = d_m$$

## Example 6

Consider the system

$$\begin{array}{rcccccccl} x & + & y & + & z & = & 3 \\ 2x & - & 3y & - & z & = & -8 \\ -x & + & 2y & + & 2z & = & 3 \end{array}$$

## Example 6

Consider the system

$$\begin{array}{rrcrcl} x & + & y & + & z & = & 3 \\ 2x & - & 3y & - & z & = & -8 \\ -x & + & 2y & + & 2z & = & 3 \end{array}$$

We can write this as the augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array} \right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array} \right]$$

### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array} \right] R_2 = r_2 + 2r_3$$



### Example 6

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 2 & -3 & -1 & | & -8 \\ -1 & 2 & 2 & | & 3 \end{bmatrix} R_2 = r_2 + 2r_3$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ -1 & 2 & 2 & | & 3 \end{bmatrix}$$

### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ -1 & 2 & 2 & 3 \end{array} \right]$$

### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ -1 & 2 & 2 & 3 \end{array} \right] R_3 = r_1 + r_3$$

### Example 6

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ -1 & 2 & 2 & | & 3 \end{bmatrix} R_3 = r_1 + r_3$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ 0 & 3 & 3 & | & 6 \end{bmatrix}$$

### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 3 & 6 \end{array} \right]$$

### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 3 & 6 \end{array} \right] R_3 = r_3 - 3r_2$$

### Example 6

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ 0 & 3 & 3 & | & 6 \end{bmatrix} R_3 = r_3 - 3r_2$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ 0 & 0 & -6 & | & 12 \end{bmatrix}$$

### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{array} \right]$$



### Example 6

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{array} \right] R_3 = -\frac{1}{6}r_3$$

### Example 6

$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{array} \right] R_3 = -\frac{1}{6}r_3$$
$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

## Example 6

Now, back solve the system

$$\begin{array}{rcccccc} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

## Example 6

Now, back solve the system

$$\begin{array}{rclclcl} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation:  $z = -2$

## Example 6

Now, back solve the system

$$\begin{array}{rclcl} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation:  $z = -2$

Plug it into the second equation and solve for  $y$ :

$$y + 3(-2) = -2 \quad \Rightarrow \quad y = 4$$

## Example 6

Now, back solve the system

$$\begin{array}{rcrcrcrcrcrl} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation:  $z = -2$

Plug it into the second equation and solve for  $y$ :

$$y + 3(-2) = -2 \quad \Rightarrow \quad y = 4$$

Plug both into the first equation and solve for  $x$ :

$$x + (4) + (-2) = 3 \quad \Rightarrow \quad x = 1$$

## Example 7

Consider the system

$$2x + 2y = 6$$

$$x + y + z = 1$$

$$3x + 4y - z = 13$$

## Example 7

Consider the system

$$\begin{array}{rrcrcl} 2x & + & 2y & & = & 6 \\ x & + & y & + & z & = & 1 \\ 3x & + & 4y & - & z & = & 13 \end{array}$$

We can write this as the augmented matrix:

$$\left[ \begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array} \right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.



### Example 7

$$\left[ \begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array} \right]$$

### Example 7

$$\left[ \begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array} \right] \begin{array}{l} R_1 = r_2 \\ R_2 = r_1 \end{array}$$

### Example 7

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array} \right] \begin{array}{l} R_1 = r_2 \\ R_2 = r_1 \end{array} \\ \Rightarrow & \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array} \right] \end{aligned}$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array} \right]$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array} \right] R_2 = -2r_1 + r_2$$

### Example 7

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 2 & 2 & 0 & | & 6 \\ 3 & 4 & -1 & | & 13 \end{bmatrix} R_2 = -2r_1 + r_2$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & -2 & | & 4 \\ 3 & 4 & -1 & | & 13 \end{bmatrix}$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 3 & 4 & -1 & 13 \end{array} \right]$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 3 & 4 & -1 & 13 \end{array} \right] R_3 = -3r_1 + r_3$$



### Example 7

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & -2 & | & 4 \\ 3 & 4 & -1 & | & 13 \end{bmatrix} R_3 = -3r_1 + r_3$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & -2 & | & 4 \\ 0 & 1 & -4 & | & 10 \end{bmatrix}$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array} \right]$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array} \right] \begin{array}{l} \\ R_2 = r_3 \\ R_3 = r_2 \end{array}$$

### Example 7

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array} \right] \begin{array}{l} R_2 = r_3 \\ R_3 = r_2 \end{array} \\ \Rightarrow & \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array} \right] \end{aligned}$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array} \right]$$

### Example 7

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array} \right] R_3 = -\frac{1}{2}r_3$$

### Example 7

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & -4 & | & 10 \\ 0 & 0 & -2 & | & 4 \end{bmatrix} R_3 = -\frac{1}{2}r_3$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & -4 & | & 10 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

## Example 7

Now, back solve the system

$$\begin{array}{rcccccc} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$



## Example 7

Now, back solve the system

$$\begin{array}{rclclcl} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation:  $z = -2$

### Example 7

Now, back solve the system

$$\begin{array}{rcrcrcrcrcrl} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation:  $z = -2$

Plug it into the second equation and solve for  $y$ :

$$y - 4(-2) = 10 \quad \Rightarrow \quad y = 2$$

### Example 7

Now, back solve the system

$$\begin{array}{rclcl} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation:  $z = -2$

Plug it into the second equation and solve for  $y$ :

$$y - 4(-2) = 10 \quad \Rightarrow \quad y = 2$$

Plug both into the first equation and solve for  $x$ :

$$x + (2) + (-2) = 1 \quad \Rightarrow \quad x = 1$$

## Example 8

Consider the system

$$\begin{array}{rcccccccl} 6x & - & y & - & z & = & 4 \\ -12x & + & 2y & + & 2z & = & -8 \\ 5x & + & y & - & z & = & 3 \end{array}$$

## Example 8

Consider the system

$$\begin{array}{rrcrcl} 6x & - & y & - & z & = & 4 \\ -12x & + & 2y & + & 2z & = & -8 \\ 5x & + & y & - & z & = & 3 \end{array}$$

We can write this as the augmented matrix:

$$\left[ \begin{array}{ccc|c} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array} \right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

### Example 8

$$\left[ \begin{array}{ccc|c} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array} \right]$$

### Example 8

$$\left[ \begin{array}{ccc|c} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array} \right] R_1 = -r_3 + r_1$$

### Example 8

$$\begin{bmatrix} 6 & -1 & -1 & | & 4 \\ -12 & 2 & 2 & | & -8 \\ 5 & 1 & -1 & | & 3 \end{bmatrix} R_1 = -r_3 + r_1$$
$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & | & 1 \\ -12 & 2 & 2 & | & -8 \\ 5 & 1 & -1 & | & 3 \end{bmatrix}$$



### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array} \right]$$

### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array} \right] R_2 = 12r_1 + r_2$$

### Example 8

$$\begin{bmatrix} 1 & -2 & 0 & | & 1 \\ -12 & 2 & 2 & | & -8 \\ 5 & 1 & -1 & | & 3 \end{bmatrix} R_2 = 12r_1 + r_2$$
$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & | & 1 \\ 0 & -22 & 2 & | & 4 \\ 5 & 1 & -1 & | & 3 \end{bmatrix}$$

### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 5 & 1 & -1 & 3 \end{array} \right]$$

### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 5 & 1 & -1 & 3 \end{array} \right] R_3 = -5r_1 + r_3$$

### Example 8

$$\begin{bmatrix} 1 & -2 & 0 & | & 1 \\ 0 & -22 & 2 & | & 4 \\ 5 & 1 & -1 & | & 3 \end{bmatrix} R_3 = -5r_1 + r_3$$
$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & | & 1 \\ 0 & -22 & 2 & | & 4 \\ 0 & 11 & -1 & | & -2 \end{bmatrix}$$

### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array} \right]$$

### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array} \right] R_2 = -\frac{1}{22}r_2$$



### Example 8

$$\begin{bmatrix} 1 & -2 & 0 & | & 1 \\ 0 & -22 & 2 & | & 4 \\ 0 & 11 & -1 & | & -2 \end{bmatrix} R_2 = -\frac{1}{22}r_2$$
$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & | & 1 \\ 0 & 1 & -\frac{1}{11} & | & -\frac{2}{11} \\ 0 & 11 & -1 & | & -2 \end{bmatrix}$$

### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array} \right]$$

### Example 8

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array} \right] R_3 = -11r_2 + r_3$$

### Example 8

$$\begin{bmatrix} 1 & -2 & 0 & \left| & 1 \right. \\ 0 & 1 & -\frac{1}{11} & \left| & -\frac{2}{11} \right. \\ 0 & 11 & -1 & \left| & -2 \right. \end{bmatrix} R_3 = -11r_2 + r_3$$
$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & \left| & 1 \right. \\ 0 & 1 & -\frac{1}{11} & \left| & -\frac{2}{11} \right. \\ 0 & 0 & 0 & \left| & 0 \right. \end{bmatrix}$$

## Example 8

Now, back solve the system

$$\begin{array}{rclcl} x & - & 2y & & = & 1 \\ & & y & - & \frac{1}{11}z & = & -\frac{2}{11} \\ & & & & 0 & = & 0 \end{array}$$

### Example 8

Now, back solve the system

$$\begin{array}{rclcl} x & - & 2y & & = & 1 \\ & & y & - & \frac{1}{11}z & = & -\frac{2}{11} \\ & & & & 0 & = & 0 \end{array}$$

This system of equations has an infinite number of solutions.

### Example 8

Now, back solve the system

$$\begin{array}{rclcrcl} x & - & 2y & & = & 1 \\ & & y & - & \frac{1}{11}z & = & -\frac{2}{11} \\ & & & & 0 & = & 0 \end{array}$$

This system of equations has an infinite number of solutions.

For any choice of  $z$ , we can calculate values for  $x$  and  $y$  that work:

$$x = \frac{2}{11}z + \frac{7}{11}$$

$$y = \frac{1}{11}z - \frac{2}{11}$$

## Example 9

Consider the system

$$\begin{array}{rcccccc} x & + & y & + & z & = & 6 \\ 2x & - & y & - & z & = & 3 \\ x & + & 2y & + & 2z & = & 0 \end{array}$$



## Example 9

Consider the system

$$\begin{array}{rrcrcl} x & + & y & + & z & = & 6 \\ 2x & - & y & - & z & = & 3 \\ x & + & 2y & + & 2z & = & 0 \end{array}$$

We can write this as the augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array} \right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

### Example 9

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array} \right]$$

### Example 9

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array} \right] R_2 = -2r_1 + r_2$$

### Example 9

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 2 & -1 & -1 & | & 3 \\ 1 & 2 & 2 & | & 0 \end{bmatrix} R_2 = -2r_1 + r_2$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -3 & -3 & | & -9 \\ 1 & 2 & 2 & | & 0 \end{bmatrix}$$

### Example 9

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 1 & 2 & 2 & 0 \end{array} \right]$$

### Example 9

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 1 & 2 & 2 & 0 \end{array} \right] R_3 = -r_1 + r_3$$

### Example 9

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -3 & -3 & | & -9 \\ 1 & 2 & 2 & | & 0 \end{bmatrix} R_3 = -r_1 + r_3$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -3 & -3 & | & -9 \\ 0 & 1 & 1 & | & -6 \end{bmatrix}$$

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$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array} \right]$$



### Example 9

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array} \right] \begin{array}{l} \\ R_2 = r_3 \\ R_3 = r_2 \end{array}$$

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$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -3 & -3 & | & -9 \\ 0 & 1 & 1 & | & -6 \end{bmatrix} \begin{array}{l} \\ R_2 = r_3 \\ R_3 = r_2 \end{array}$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 1 & | & -6 \\ 0 & -3 & -3 & | & -9 \end{bmatrix}$$

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$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{array} \right] R_3 = 3r_2 + r_3$$

### Example 9

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 1 & | & -6 \\ 0 & -3 & -3 & | & -9 \end{bmatrix} R_3 = 3r_2 + r_3$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 1 & | & -6 \\ 0 & 0 & 0 & | & -27 \end{bmatrix}$$

## Example 9

Now, back solve the system

$$\begin{array}{rcccccc} x & + & y & + & z & = & 6 \\ & & y & + & z & = & -6 \\ & & & & 0 & = & -27 \end{array}$$

## Example 9

Now, back solve the system

$$\begin{array}{rcccccc} x & + & y & + & z & = & 6 \\ & & y & + & z & = & -6 \\ & & & & 0 & = & -27 \end{array}$$

This system of equations has no solutions.

# Existence and Uniqueness of Solutions

During Gaussian Elimination:



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  - A system with exactly one solution is called **independent**.

# Existence and Uniqueness of Solutions

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Some vocabulary:

- If a system has no solutions, it is called **inconsistent**.
- If a system has at least one solution, it is called **consistent**.
  - A system with exactly one solution is called **independent**.
  - A system with more than one solution is called **dependent**.