Inverses of Matrices and Matrix Equations

Department of Mathematics

Salt Lake Community College

Inverse Matrix

If there exists, for an $n \times n$ matrix $\boldsymbol{A}$, another matrix $\boldsymbol{A}^{-1}$ of the same order such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$$

then $\boldsymbol{A}^{-1}$ is called the **inverse** of matrix $\boldsymbol{A}$, and $\boldsymbol{A}$ is called **invertible**.

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- If $\boldsymbol{A}$ is invertible, then so is $\boldsymbol{A}^{-1}$ and $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$
- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices of the same order, then their product $\boldsymbol{AB}$ is invertible. In fact, $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Inverse of a 2×2 Matrix

If

$$\boldsymbol{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

then,

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

If $ad - bc = 0$, then $\boldsymbol{A}$ is not invertable.

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For an $n \times n$ matrix $\boldsymbol{A}$, the following process will calculate $\boldsymbol{A}^{-1}$, or show that $\boldsymbol{A}$ is not invertible.

Step 1: Form the $n \times 2n$ augmented matrix $\boldsymbol{M} = [\boldsymbol{A}|\boldsymbol{I}_n]$.

Step 2: Transform $\boldsymbol{M}$ into Reduced Row Echelon Form.

Step 3:
- If the left hand side of $\boldsymbol{M}$ is the identity matrix, then the right hand side is $\boldsymbol{A}^{-1}$.
- Otherwise, $\boldsymbol{A}$ is a non-invertible matrix.

Example 1

Consider the matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

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Start by building the augmented matrix

$$\boldsymbol{M_A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_A}$ into Reduced Row Echelon Form.

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Since the left hand side is $\boldsymbol{I}_3$, we know the right hand side is the inverse:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

Example 2

Consider the matrix:

$$\boldsymbol{B} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

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$$\boldsymbol{M_B} = \left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_B}$ into Reduced Row Echelon Form.

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$$\left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

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This means that B is a non-invertible matrix.

Invertibility and Solutions

Consider the matrix equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$.
Where $\boldsymbol{A}$ is an $n \times n$ matrix, and $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}$ are of length n.

- A unique solution exists if and only if $\boldsymbol{A}$ is invertible.

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Where $\boldsymbol{A}$ is an $n \times n$ matrix, and $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}$ are of length n.

- A unique solution exists if and only if $\boldsymbol{A}$ is invertible.
- Otherwise there are either:
  - No solutions.
  - Infinitely many solutions.

  (Another method must be used to determine which.)

Example 3

Consider the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ & & 2y & + & z & = & -1 \\ x & & & + & z & = & 3 \end{array}$$

Example 3

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$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ & & 2y & + & z & = & -1 \\ x & & & + & z & = & 3 \end{array}$$

We can can write this as the matrix equation:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

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So, if $\boldsymbol{A}$ is invertible, then we can solve the matrix equation for $\vec{\boldsymbol{x}}$

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$$\begin{aligned} \boldsymbol{A}\vec{\boldsymbol{x}} &= \vec{\boldsymbol{b}} \\ \boldsymbol{A}^{-1}\boldsymbol{A}\vec{\boldsymbol{x}} &= \boldsymbol{A}^{-1}\vec{\boldsymbol{b}} \\ \boldsymbol{I}_3\vec{\boldsymbol{x}} &= \boldsymbol{A}^{-1}\vec{\boldsymbol{b}} \end{aligned}$$

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So, if we can compute $\boldsymbol{A}^{-1}\vec{\boldsymbol{b}}$ we will have solved the system.

Example 3

$$
\begin{array}{c|c}
 & \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\
\hline
\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ \\ \end{bmatrix}
\end{array}
$$

Example 3

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\end{array}
$$

So, we have

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}
$$

Invertible Matrix Characterization

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  (This means when you put $\boldsymbol{A}$ in RREF, you get $\boldsymbol{I}_n$)

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- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ has only the trivial solution $\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$.
- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a unique solution for every $\vec{\boldsymbol{b}} \in \mathbb{R}^n$.