Solution and Direction Fields: Qualitative Analysis

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

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Analytic Definition of a Solution

Analytically, $y(t)$ is a **solution** of a differential equation if substituting $y(t)$ for y reduced the equation to an identity:

$$y'(t) = f(t, y(t))$$

on an appropriate domain for t.

Example 1

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Similarly, we could show that

$$y(t) = 2e^{2t} \quad \text{and} \quad y(t) = \frac{-3}{2}e^{2t}$$

are also solutions. In fact, any constant multiple of e^{2t} is a solution.

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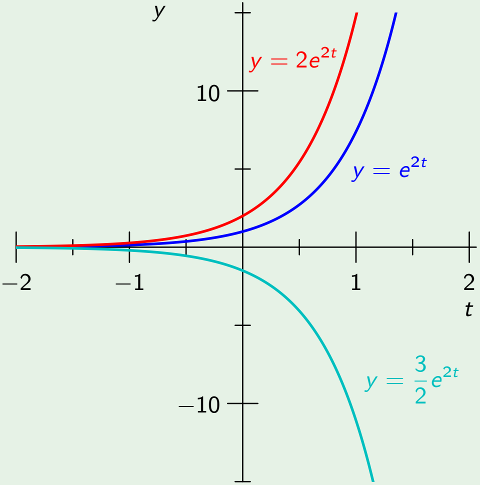

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Other solutions are of the form $y(t) = \sqrt{k-t^2}$.

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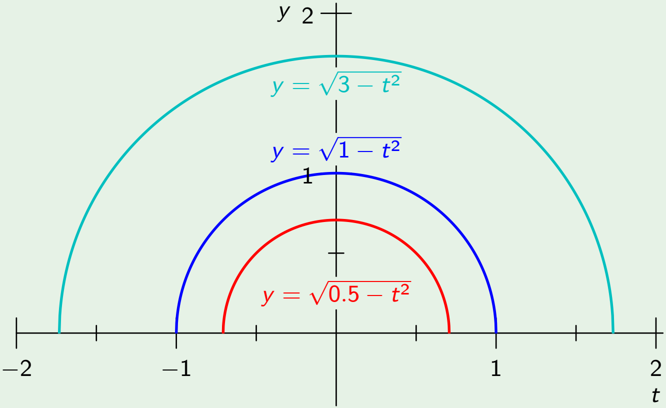

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Family of Solutions

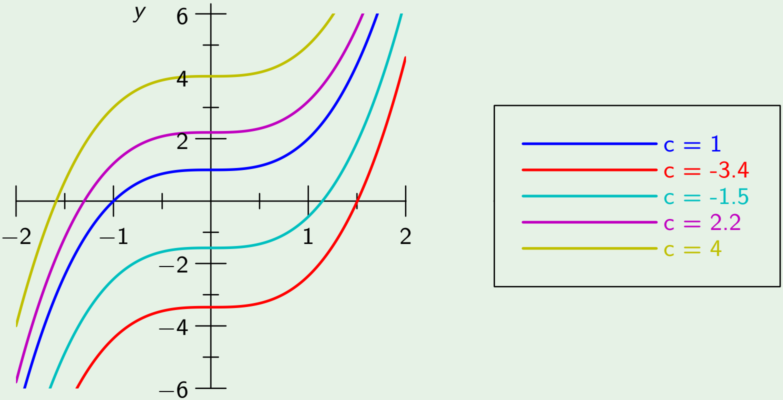
In general, all solutions of a first-order DE form a **family** of solutions expressed with a single parameter c. Such a family is called the **general solution**. A member of the family that results from a specific value of c is called a **particular solution**.

Example 4

The general solution of $y' = 3t^2$ is

$$y = t^3 + c$$

where c may be any real value.



Initial-Value Problem

The combination of a first-order differential equation and an **initial condition**

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

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Note

While a family of solutions for a DE contains multiple solutions, an IVP usually has only one solution. That is, the solution to an IVP is a particular solution to the DE.

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The function $y(t) = t^3 + 1$ is a solution to the IVP

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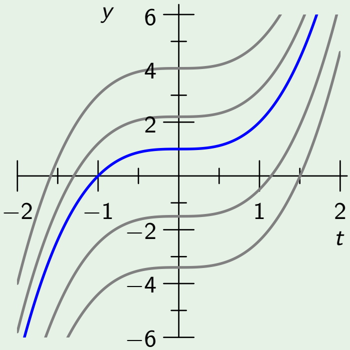
Differentiating $y(t)$ confirms that

$$y'(t) = (t^3+1)' = 3t^2, \quad \text{and} \quad y(0) = 0^3 + 1 = 1$$



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Let us look again at the Malthusian population problem

$$\frac{dy}{dt} = 0.03y, \quad y(0) = 0.9$$

Where we have the one-parameter family of solutions

$$y(t) = ce^{0.03t}$$

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So, the solution to the IVP is

$$y(t) = 0.9e^{0.03t}$$

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- Obtaining an appropriate **numerical approximation** to $y(t)$.
- Informally, refer to a study of a **geometrical representation**.

Quantitative Analysis

Historically, the study of differential equations was **quantitative**, to find explicit formulas or power series representations of solutions. This type of analysis dominated the thinking of the seventeenth and eighteenth centuries, and the work of Isaac Newton, Gottfried Leibniz, Leonhard Euler, and Joseph Lagrange.

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Qualitative Analysis

In the late nineteenth century, the French mathematician Henri Poincaré, while working on problems in celestial mechanics, started investigating the behavior of solutions. His new approach, now called **qualitative** theory, focuses on the properties of the solutions, instead of the search for an explicit formula. In this way, we are able to demonstrate the existence of constant or periodic solutions, as well as describe the long term behavior.

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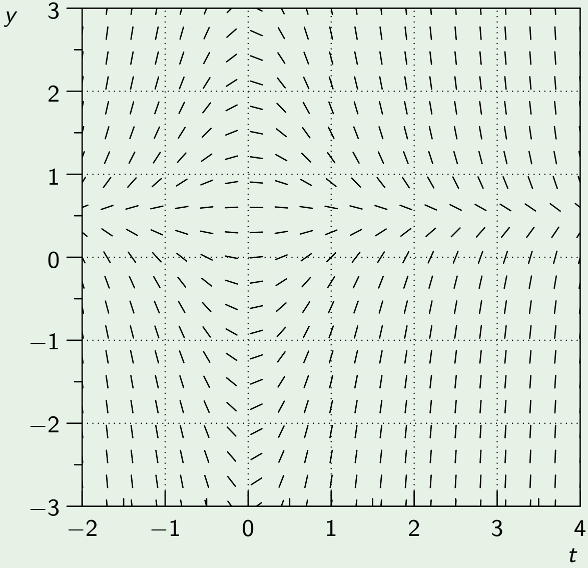
Direction Fields

We can see what solution curves look like by, on regular intervals, draw short line segments with slope determined by the DE for that point. The collection of these segments are called **direction field** (or a slope field).

Example 7

$$y' = -2ty + t$$



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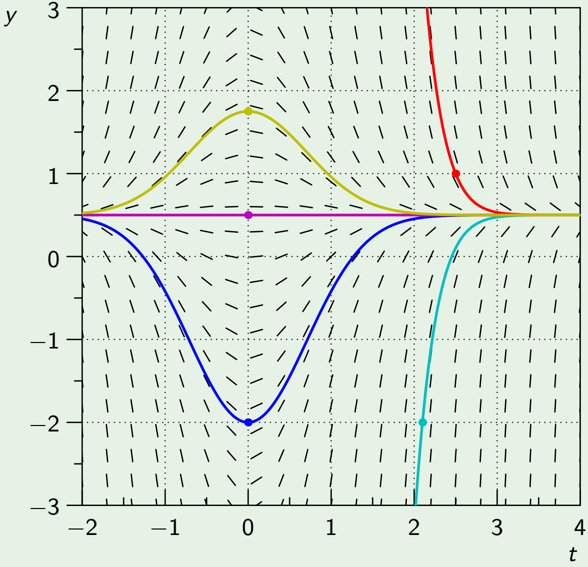

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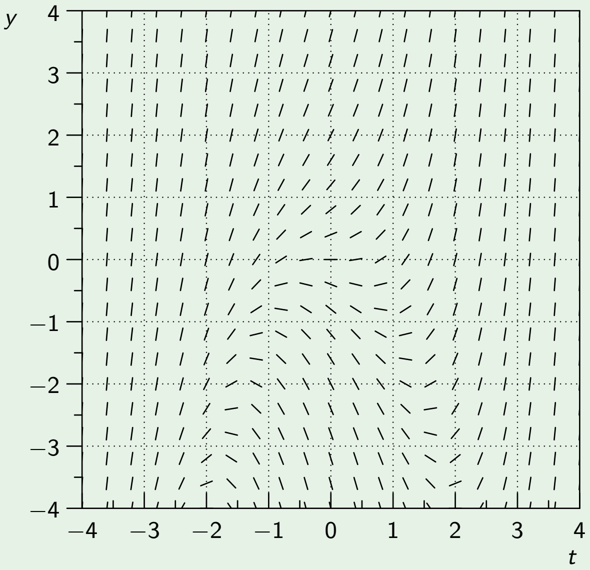

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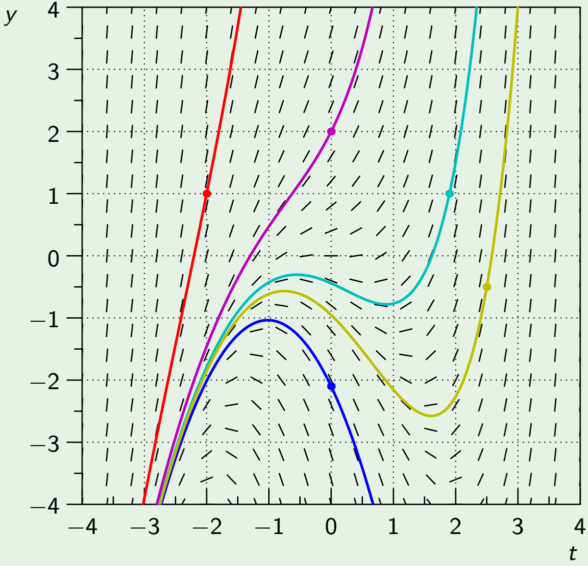

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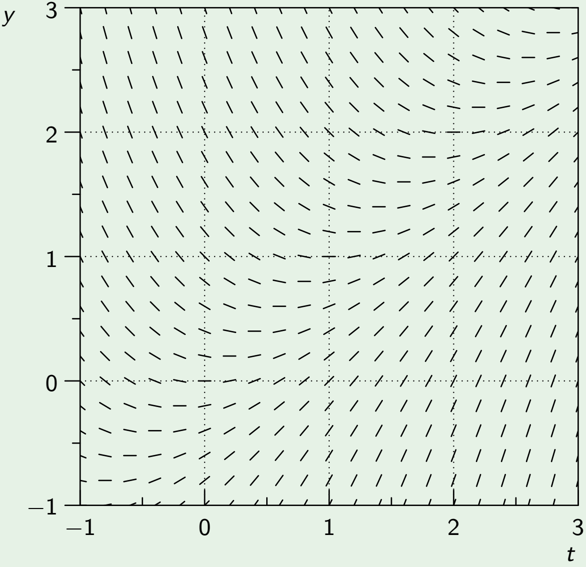

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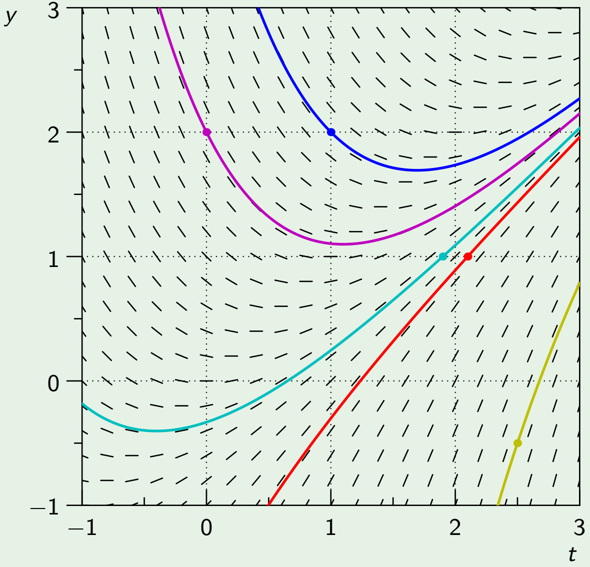
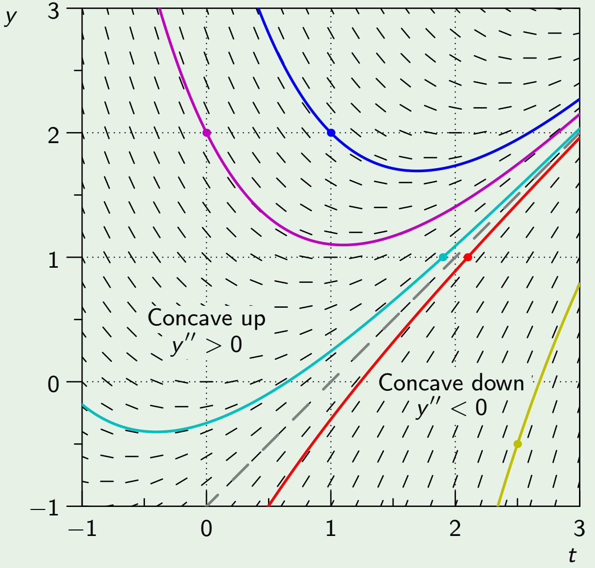


Example 9

$$y' = t - y \quad \text{and} \quad y'' = 1 - y' = 1 - t + y$$



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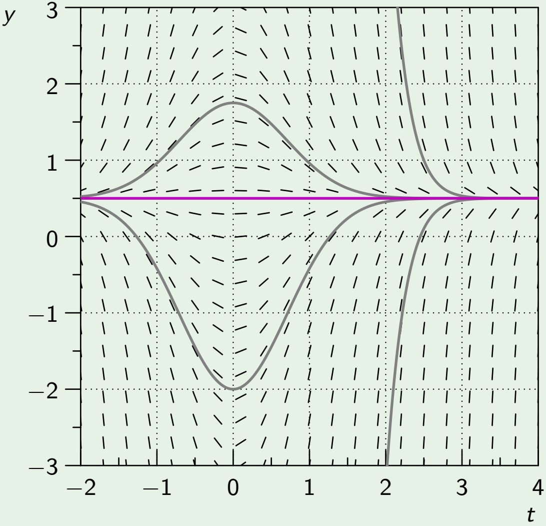
Note

A equilibrium solution is often called **semistable** if it is stable on one side and unstable on the other.

Example 10

The DE $y' = -2ty + t$ has the constant solution $y(t) = \dfrac{1}{2}$.



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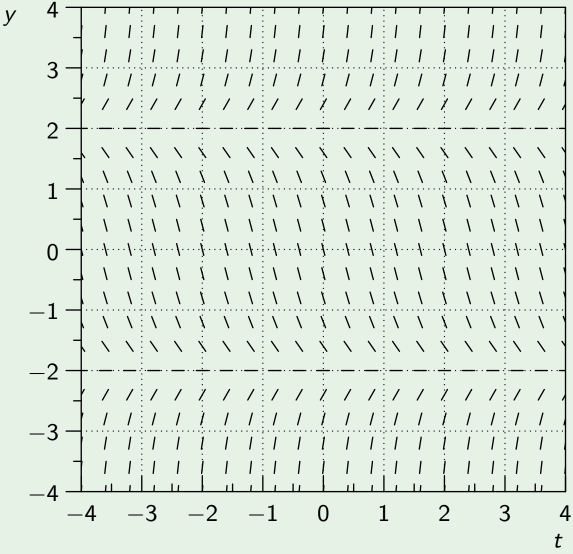
$$y' = y^2 - 4$$



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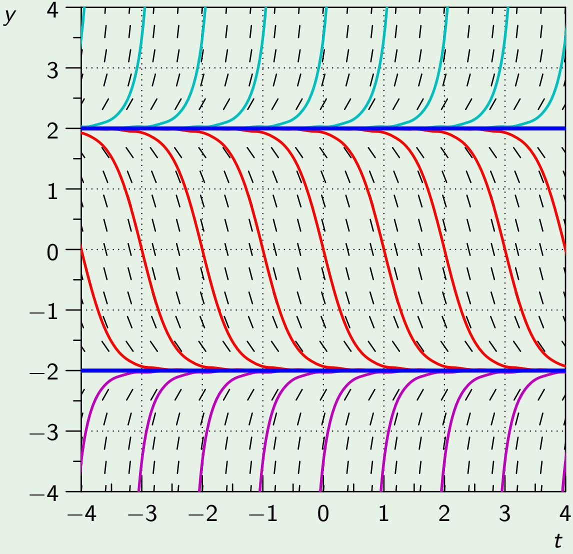

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Method of Isoclines

Drawing multiple isoclines forms a handy guide to the slopes of solutions. Though, they rarely coincide with solutions.

Draw the isoclines with dashed lines, so you don't confuse the two.

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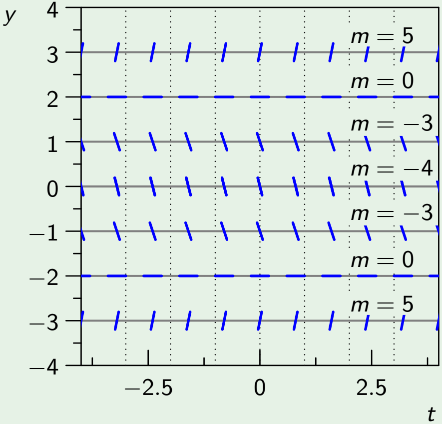
The differential equation

$$y' = y^2 - 4$$

has isoclines

$$y^2 - 4 = m \quad \Rightarrow \quad y = \pm\sqrt{m+4}$$



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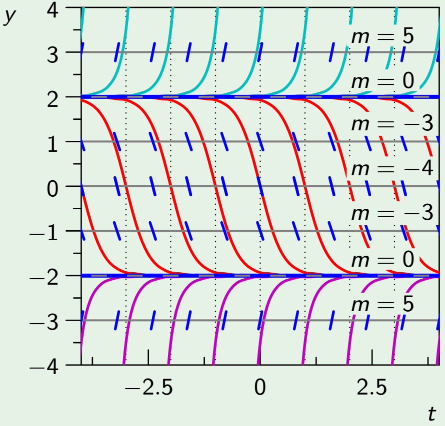

Example 13

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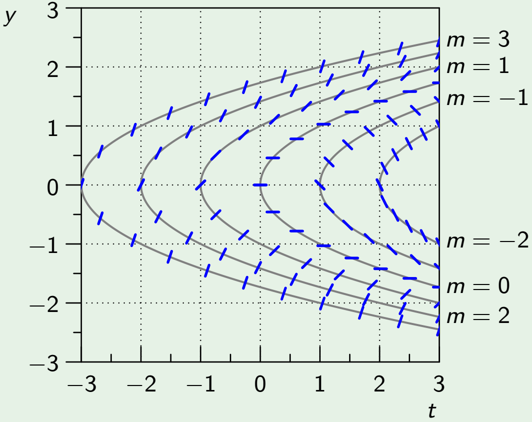
$$y^2 - t = m \quad \to \quad t = y^2 - m$$



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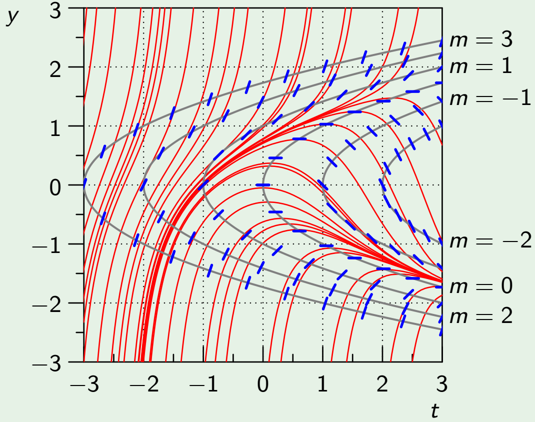

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6. What is the pattern of the isoclines?
7. Are there any periodic solutions?

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We can usually answer the following questions for DE $y' = f(t, y)$.

1. Is the field well defined? That is, are there any points (t, y) such that $f(t, y)$ does not exist?
2. does there appear to be a unique solution curve passing through each point of the plane?
3. Are there equilibrium solutions? Are they stable, unstable, or semistable?
4. What is the concavity of solutions?
5. Do any solutions appear to “blow up”? That is, do there appear to be any vertical asymptotes?
6. What is the pattern of the isoclines?
7. Are there any periodic solutions?
8. What is the long term behavior or solutions as $t \to \infty$ and $t \to -\infty$?

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9. Does the field have any symmetries?