The Inverse of a Matrix

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Inverse Matrix

If there exists, for an $n \times n$ matrix $\boldsymbol{A}$, another matrix $\boldsymbol{A}^{-1}$ of the same order such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$$

then $\boldsymbol{A}^{-1}$ is called the **inverse** of matrix $\boldsymbol{A}$, and $\boldsymbol{A}$ is called **invertible**.

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Vocabulary

- A square matrix that is not invertible is called **singular**.
- A square matrix that is invertible is called **nonsingular**.

Invertible Matrix Properties

- If $\boldsymbol{A}$ is invertible, then so is $\boldsymbol{A}^{-1}$ and

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$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

- If $\boldsymbol{A}$ is invertible, then so is $\boldsymbol{A}^{\mathsf{T}}$, and

$$\left(\boldsymbol{A}^{\mathsf{T}}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{\mathsf{T}}$$

Inverses by Reduced Row Echelon Form

For an $n \times n$ matrix $\boldsymbol{A}$, the following process will calculate $\boldsymbol{A}^{-1}$, or show that $\boldsymbol{A}$ is not invertible.

Step 1: Form the $n \times 2n$ augmented matrix $\boldsymbol{M} = [\boldsymbol{A} | \boldsymbol{I}_n]$.

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Step 2: Transform $\boldsymbol{M}$ into Reduced Row Echelon Form.

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Step 2: Transform $\boldsymbol{M}$ into Reduced Row Echelon Form.

Step 3:
- If the left hand side of $\boldsymbol{M}$ is the identity matrix, then the right hand side is $\boldsymbol{A}^{-1}$.
- Otherwise, $\boldsymbol{A}$ is a non-invertible matrix.

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Note

This is far from the only method for calculating inverses, but it is the only one we will talk about. You are welcome to look up other methods on your own.

Example 1

Consider the matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $\boldsymbol{A}^{-1}$

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Start by building the augmented matrix

$$\boldsymbol{M_A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_A}$ into Reduced Row Echelon Form.

Example 1

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

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$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

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$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

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Since the left hand side is $\boldsymbol{I}_3$, we know the right hand side is the inverse:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

Example 2

Consider the matrix:

$$\boldsymbol{B} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

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Start by building the augmented matrix

$$\boldsymbol{M_B} = \left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Then transform $\boldsymbol{M_B}$ into Reduced Row Echelon Form.

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$$\left[\begin{array}{ccc|ccc} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

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This means that B is a non-invertible matrix.

Invertibility and Solutions

Consider the matrix equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$.
Where $\boldsymbol{A}$ is an $n \times n$ matrix, and $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}$ are of length n.

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- A unique solution exists if and only if $\boldsymbol{A}$ is invertible.
- Otherwise there are either:
    - No solutions.
    - Infinitely many solutions.

  (Another method must be used to determine which.)

Example 3

Consider the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ & & 2y & + & z & = & -1 \\ x & & & + & z & = & 3 \end{array}$$

Example 3

Consider the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ & & 2y & + & z & = & -1 \\ x & & & + & z & = & 3 \end{array}$$

We can can write this as the matrix equation:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

Example 3

We know from Example 1 that $\boldsymbol{A}$ is invertible.

This means we need solve the matrix equation for $\vec{\boldsymbol{x}}$

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$$\begin{aligned} \boldsymbol{A}\vec{\boldsymbol{x}} &= \vec{\boldsymbol{b}} \\ \boldsymbol{A}^{-1}\boldsymbol{A}\vec{\boldsymbol{x}} &= \boldsymbol{A}^{-1}\vec{\boldsymbol{b}} \\ \boldsymbol{I}_3\vec{\boldsymbol{x}} &= \boldsymbol{A}^{-1}\vec{\boldsymbol{b}} \end{aligned}$$

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\vec{\boldsymbol{x}} &= \boldsymbol{A}^{-1}\vec{\boldsymbol{b}}
\end{aligned}$$

So, if we can compute $\boldsymbol{A}^{-1}\vec{\boldsymbol{b}}$ we will have solved the system.

Example 3

$$
\begin{array}{c|c}
 & \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\
\hline
\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ \\ \end{bmatrix}
\end{array}
$$

Example 3

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Example 3

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\hline
\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}
\end{array}
$$

So, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$

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- $\boldsymbol{A}$ is row equivalent to $\boldsymbol{I}_n$.
  (This means when you put $\boldsymbol{A}$ in RREF, you get $\boldsymbol{I}_n$)

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- The rank of $\boldsymbol{A}$ is n.
- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ has only the trivial solution $\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$.

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- The rank of $\boldsymbol{A}$ is n.
- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ has only the trivial solution $\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$.
- The equation $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a unique solution for every $\vec{\boldsymbol{b}} \in \mathbb{R}^n$.

Example 4

An engineering consultant given the following IVP:

$$y''' - 2y'' - y' + 2y = 0, \quad y(0) = b_1,\ y'(0) = b_2,\ y''(0) = b_3$$

She must solve this IVP for many different sets of initial conditions, and expects to do the same tomorrow.

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The general solution is:

$$y(t) = c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}$$

(We will talk about how to solve this type of DE in Chapter 4.)

Example 4

To determine c_1, c_2, and c_3, we must plug in each initial condition, giving the system:

$$\begin{aligned} y(0) &= c_1 + c_2 + c_3 &= b_1 \\ y'(0) &= 2c_1 + c_2 - c_3 &= b_2 \\ y''(0) &= 4c_1 + c_2 + c_3 &= b_3 \end{aligned}$$

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We can write this as the matrix equation:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

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If we can find the inverse of $\boldsymbol{A}$, then we can compute the constants for any set of initial conditions $\vec{\boldsymbol{b}}$.

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

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Thus, the solution for any $\vec{\boldsymbol{b}}$ is:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$