Undetermined Coefficients

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Remember

If L is a linear differential operator defined by

$$L(y) = a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y$$

(where all functions of t are assumed to be defined over some interval I) then we can look at superposition for the DE $L(y) = f(t)$.

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Superposition Principle for Nonhomogeneous Linear DEs

If $y_i(t)$ is a solution of $L(y) = f_i(t)$, for $i = 1, 2, \ldots, n$, and constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$, then

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$$

is a solution of

$$L(y) = c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t)$$

Nonhomogeneous Principle for Linear DEs

The general solution of the nonhomogeneous linear DE $L(y) = f$ is

$$y = y_h + y_p$$

where

- y_h is the general solution of $L(y) = 0$
- y_p is a particular solution of $L(y) = f$

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- y_p is a particular solution of $L(y) = f$

Note

This is just applying the superposition principle for $f_1(t) = 0$ and $f_2(t) = f$.

Example 1

Consider the nonhomogeneous second-order DE

$$y'' - y' - 2y = 2t + 1 - 2e^t$$

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$$y_1 = -t \quad \text{is a solution to} \quad L(y) = f_1$$

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Finally, we use characteristic roots to solve $L(y) = 0$

$$r^2 - r - 2 = 0$$

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Thus, the general solution is

$$y = y_h + y_p = c_1 e^{2t} + c_2 e^{-t} - t + e^t$$

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Using the solutions found in the last example, we can use superposition to build a particular solution to this DE.

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Note

After accumulating some experience, a solution can be guessed by just "inspecting" the equation. By recognizing the patterns.

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Consider the second-order DE

$$ay'' + by' + cy = d$$

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Note

This idea works well for the nth-order equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = d$$

provided that $a_0 \neq 0$.

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Example 6

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$$y'' + y' - 3y = 9e^{3t}$$

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Note

There are a few limitations of this method:
It only works for linear differential equations with specific forcing terms.

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Forcing Terms That Work With Undetermined Coefficients

Any finite products or sums of:

- Polynomials in t.
- Exponentials e^{at}.
- Sinusoidal functions of the form $\cos(kt)$ and $\sin(kt)$.

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Note

Even with these limitations, undetermined coefficients is widely used, given that many functions are built from the above parts.

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$$y'' - y' - 2y = 3t^2 - 1$$

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Let us look for y_p in $\mathbb{P}_2$. Which means y_p will be of the form

$$y_p = At^2 + Bt + C$$

We can then calculate:

$$\begin{aligned} y_p' &= 2At + B \\ y_p'' &= 2A \end{aligned}$$

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Plugging these into the DE gives

$$2A - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1$$

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Plugging these into the DE gives

$$2A - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1$$
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So, equating both sides gives the system

$$-2A = 3, \quad -2A - 2B = 0, \quad 2A - B - 2C = -1$$

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Which has solution $A = -\dfrac{3}{2}$, $B = \dfrac{3}{2}$, and $C = -\dfrac{7}{4}$.

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The general solution is

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{3}{2}t^2 + \frac{3}{2}t + \frac{7}{4}$$

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We can then calculate:

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$$10A = 2 \quad \rightarrow \quad A = \frac{1}{5}$$

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The general solution is

$$y = c_1e^{2t} + c_2e^{-t} + \frac{1}{5}e^{-3t}$$

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We can then calculate:

$$y_p' = -3A\sin(3t) + 3B\cos(3t)$$
$$y_p'' = -9A\cos(3t) - 9B\sin(3t)$$

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$$y'' - y' - 2y = 2\cos(3t)$$

Plugging these into the DE gives

$$\begin{aligned}&(-9A\cos(3t) - 9\sin(3t))\\&\qquad -(-3A\sin(3t) + 3B\cos(3t))\\&\qquad\qquad -2(A\cos(3t) + B\sin(3t)) = 2\cos(3t)\end{aligned}$$

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&(-11A - 3B)\cos(3t) + (3A - 11B)\sin(3t) = 2\cos(3t)
\end{aligned}$$

So, equating both sides gives the system

$$-11A - 3B = 2, \quad 3A - 11B = 0$$

Which has solution $A = -\frac{11}{65}$ and $B = -\frac{3}{65}$.

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The general solution is

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{11}{65}\cos(3t) - \frac{3}{65}\sin(3t)$$

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$$\begin{aligned} y_p' &= \left(At^2 + (2A + B)t + (B + C)\right) e^t \\ y_p'' &= \left(At^2 + (4A + B)t + (2A + 2B + C)\right) e^t \end{aligned}$$

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So, equating both sides gives the system

$$-2A = 1, \quad 2A - 2B = 0, \quad 2A + B - 2C = 0$$

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$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ae^{2t}$$

We can then calculate:

$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$

Substituting into the DE gives

$$4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} = 5e^{2t}$$

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We can then calculate:

$$\begin{aligned} y_p' &= 2Ae^{2t} \\ y_p'' &= 4Ae^{2t} \end{aligned}$$

Substituting into the DE gives

$$\begin{aligned} 4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} &= 5e^{2t} \\ 0 &= 5e^{2t} \end{aligned}$$

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Let us look for y_p of the form

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Substituting into the DE gives

$$4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} = 5e^{2t}$$
$$0 = 5e^{2t}$$

Thats not good. We'll have to try something else.

Example 11

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ate^{2t}$$

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$$y'' - y' - 2y = 5e^{2t}$$

Let us look for y_p of the form

$$y_p = Ate^{2t}$$

We can then calculate:

$$y_p' = (2At + A)e^{2t}$$
$$y_p'' = (4A + 4A)e^{2t}$$

Example 11

Consider

$$y'' - y' - 2y = 5e^{2t}$$

Substituting into the DE gives

$$(4A + 4A)e^{2t} - 2Ae^{2t} - 2Ate^{2t} = 5e^{2t}$$

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$$y'' - y' - 2y = 5e^{2t}$$

Substituting into the DE gives

$$\begin{aligned}(4A + 4A)e^{2t} - 2Ae^{2t} - 2Ate^{2t} &= 5e^{2t}\\ 3Ae^{2t} &= 5e^{2t}\end{aligned}$$

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When we equate both sides we get $3A = 5$ and so $A = \dfrac{5}{3}$.

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And so, the particular solution is

$$y_p = \frac{5}{3}te^{2t}$$

Example 12

Consider

$$y'' - 2y' + y = 3e^t$$

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$$y'' - 2y' + y = 3e^t$$

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$$y_p = Ae^t$$

We can then calculate:

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Substituting into the DE gives

$$Ae^t - 2Ae^t + Ae^{2t} = 3e^t$$

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$$y'' - 2y' + y = 3e^t$$

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$$y_p = Ae^t$$

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Substituting into the DE gives

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Thats not good. We'll have to try something else.

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Let us look for y_p of the form

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$$y'' - 2y' + y = 3e^t$$

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Substituting into the DE gives

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Substituting into the DE gives

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This too is a problem. We'll have to try something else.

Example 12

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = At^2 e^t$$

Example 12

Consider

$$y'' - 2y' + y = 3e^t$$

Let us look for y_p of the form

$$y_p = At^2e^t$$

We can then calculate:

$$y_p' = 2Ate^t + At^2e^t$$
$$y_p'' = 2Ae^t + 4Ate^t + At^2e^t$$

Example 12

Consider

$$y'' - 2y' + y = 3e^t$$

Substituting into the DE gives

$$2Ae^t + 4Ate^t + At^2e^t - 2\left(2Ate^t + At^2e^t\right) + At^2e^t = 5e^{2t}$$

Example 12

Consider

$$y'' - 2y' + y = 3e^t$$

Substituting into the DE gives

$$\begin{aligned} 2Ae^t + 4Ate^t + At^2e^t - 2\left(2Ate^t + At^2e^t\right) + At^2e^t &= 5e^{2t} \\ 2Ae^t &= 5e^{2t} \end{aligned}$$

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$$y'' - 2y' + y = 3e^t$$

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When we equate both sides we get $2A = 5$ and so $A = \dfrac{5}{2}$.

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When we equate both sides we get $2A = 5$ and so $A = \dfrac{5}{2}$.

And so, the particular solution is

$$y_p = \frac{5}{2}te^{2t}$$