Solving the First-Order Linear Differential Equation

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Euler-Lagrange Two-Stage Method

We saw last section that the general solution for the linear first-order differential equation

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has the form $y = y_h + y_p$.

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where $c \in \mathbb{R}$.

The second step is to find a particular solution, which we will accomplish using **variation of parameters**, which was developed by French mathematician Joseph Louis Lagrange.

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where the unknown function $v(t)$ is called the **varying parameter**.

Our goal is to find $v(t)$, to do so we need to substitute y_p into the DE.

$$\underbrace{\left(v'(t)e^{-\int p(t)dt} - p(t)v(t)e^{-\int p(t)dt}\right)}_{y_p'} + \underbrace{p(t)v(t)e^{-\int p(t)dt}}_{p(t)y_p} = f(t)$$

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Our goal is to find $v(t)$, to do so we need to substitute y_p into the DE.

$$v(t) = \int f(t)e^{\int p(t)dt}dt$$

Now that we have $v(t)$, we have determined a particular solution.

$$y_p(t) = v(t)e^{-\int p(t)dt} = e^{-\int p(t)dt} \int f(t)e^{\int p(t)dt}dt$$

Example 1

Consider the IVP

$$y' + \left(\frac{1}{t+1}\right) y = 2, \quad y(0) = 0, \quad t \geq 0$$

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Let us assume for the moment that $y \neq 0$ and use separation of variables.

$$\frac{dy}{y} = -\frac{dt}{t+1}$$

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$$\ln |y| \quad = \quad -\ln (t+1) + c$$

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$$|y| \quad = \quad e^c (t+1)^{-1}$$

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Let us assume for the moment that $y \neq 0$ and use separation of variables.

$$y_h \quad = \quad \frac{k}{t+1}$$

where $k = \pm e^c$

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$$y_p = \frac{v(t)}{t+1}$$

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$$y_p = \frac{v(t)}{t+1}$$

which gives:

$$v(t) = t^2 + 2t + c$$

But, we only need a single $v(t)$, so we can let $c = 0$, giving

$$y_p = \frac{t^2 + 2t}{t+1}$$

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$$y' + \left(\frac{1}{t+1}\right) y = 2, \quad y(0) = 0, \quad t \geq 0$$

Step 3. Thus, the general solution is:

$$y(t) = y_h + y_p = \frac{k}{t+1} + \frac{t^2 + 2t}{t+1}$$

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$$0 = y(0) = \frac{k}{(0)+1} + \frac{(0)^2 + 2(0)}{(0)+1} \Rightarrow k = 0$$

Which means that the solution to the IVP is

$$y(t) = \frac{t^2 + 2t}{t+1}$$

Euler Lagrange Method for Solving Linear First-Order DEs

To solve a linear differential equation

$$y' + p(t)y = f(t)$$

where p and f are continuous on a domain I, use the following steps.

Step 1. Solve the corresponding homogenous equation $y' + p(t)y = 0$ to obtain the one-parameter family.

$$y_h = ce^{-\int p(t)dt}$$

Step 2. Solve

$$v'(t)e^{-\int p(t)dt} = f(t)$$

for $v(t)$ to obtain a particular solution $y_p = v(t)e^{-\int p(t)dt}$.

Step 3. Combine the results of Step 1 and Step 2 to form the general solution

$$y(t) = y_h + y_p$$

Step 4. If you are solving an IVP, only after Step 3 can you plug in the initial condition.

Note

Variation of Parameters is a very powerful method, and we will see it again in our study of higher order differential equations later in the course. But, for first-order (and *only* first-order) equations we have a second method, called the **Integrating Factor Method** which may also be used.

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Variation of Parameters is a very powerful method, and we will see it again in our study of higher order differential equations later in the course. But, for first-order (and *only* first-order) equations we have a second method, called the **Integrating Factor Method** which may also be used.

For the differential equation

$$y' + p(t)y = f(t)$$

we will break this new method down into two cases:

- $p(t)$ is constant.
- $p(t)$ is variable.

Integrating Factor Method (Constant Coefficient)

Let us look at the first-order linear differential equation

$$y' + ay = f(t), \quad a \in \mathbb{R}$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

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Let us start with the differential equation.

$$y' + ay = f(t)$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

We first multiply both sides of the equation by e^{at}.

$$e^{at}\left(y' + ay\right) = e^{at}f(t)$$

Integrating Factor Method (Constant Coefficient)

Let us look at the first-order linear differential equation

$$y' + ay = f(t), \quad a \in \mathbb{R}$$

This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

We then apply Euler's observation to the left-hand side.

$$\frac{d}{dt}\left(e^{at}y\right) = e^{at}f(t)$$

Integrating Factor Method (Constant Coefficient)

Let us look at the first-order linear differential equation

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

Next we integrate both sides.

$$e^{at}y = \int e^{at}f(t)dt + c$$

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$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

Solving for y gives:

$$y(t) = e^{-at}\int e^{at}f(t)dt + ce^{-at}$$

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Solving for y gives:

$$y(t) = e^{-at}\int e^{at}f(t)dt + ce^{-at}$$

Note

This is the same answer we got from Variation of Parameters, though achieved through a different route. We have obtained both y_h and y_p at the same time.

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

Integrating Factor Method (Variable Coefficient)

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$$y' + p(t)y = f(t)$$

We seek a function $\mu(t)$ that satisfies Euler's observation, i.e.

$$\mu(t) \cdot \left(y' + p(t)y\right) = \frac{d}{dt}\left(\mu(t) \cdot y\right)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

Let us carry out the differentiation on the right-hand side

$$\mu(t)y' + p(t)\mu(t)y = \mu'(t)y + \mu(t)y'$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

If we assume $y(t) \neq 0$, this simplifies to

$$\mu'(t) = p(t)\mu(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\ln|\mu(t)| = \int p(t)dt$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

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$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

We now know the integrating factor, and perform the same steps as before.

$$y' + p(t)y = f(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Multiply both sides by the integrating factor.

$$\mu(t) \cdot \left(y' + p(t)y\right) = \mu(t) \cdot f(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Apply the property $\mu(t) \cdot (y' + p(t)y) = (\mu(t) \cdot y)'$ to the left-hand side.

$$(\mu(t)y)' = \mu(t)f(t)$$

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Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Integrate both sides.

$$\mu(t)y(t) = \int \mu(t)f(t)dt + c$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Assuming $\mu(t) \neq 0$, we can solve for y.

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) f(t) dt + \frac{c}{\mu(t)}$$

Integrating Factor Method (Variable Coefficient)

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$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

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Assuming $\mu(t) \neq 0$, we can solve for y.

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) f(t) dt + \frac{c}{\mu(t)}$$

Note

We have again found y_h and y_p at the same time.

Integrating Factor Method for First-Order Linear DEs

To solve the linear first-order DE, where p and f are continuous on a domain I.

$$y' + p(t)y = f(t)$$

Step 1. Find the integrating factor $\mu(t) = e^{\int p(t)dt}$, where $\int p(t)dt$ represents *any* anti-derivative of $p(t)$.

Step 2. Multiply both sides of the DE by $mu(t)$, which always simplifies to:

$$\left(e^{\int p(t)dt}y(t)\right)' = e^{\int p(t)dt}f(t)$$

Step 3. Find the anti-derivative to get:

$$e^{\int p(t)dt}y(t) = \int e^{\int p(t)dt}f(t)dt + c$$

Step 4. Solve algebraically for y.

$$y = e^{-\int p(t)dt}\int e^{\int p(t)dt}f(t)dt + ce^{-\int p(t)dt}$$

Step 5. For IVPs, substitute the initial conditions in to find c.

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

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Step 1. Find the integrating factor:

$$\mu(t) = e^{\int p(t) dt}$$

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (-1) dt}$$

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (-1) dt} = e^{-t}$$

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int(-1)dt} = e^{-t}$$

Step 2. Multiply both sides of the DE by $\mu(t)$:

$$e^{-t}\left(y' - y\right) = e^{-t}$$

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int(-1)dt} = e^{-t}$$

Step 2. Multiply both sides of the DE by $\mu(t)$:

$$e^{-t}\left(y' - y\right) = e^{-t}$$

Which reduces to:

$$\left(e^{-t}y\right)' = te^{-t}$$

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt$$

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

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$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t}$$

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$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

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$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

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Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^{0}$$

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Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^{0} \Rightarrow c = 2$$

Example 2

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^{0} \Rightarrow c = 2$$

Thus, the solution to the IVP is $y(t) = -t - 1 + 2e^{t}$