

# Linear Systems with Nonreal Eigenvalues

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

## Complex Eigenvalues and Eigenvectors

For a real matrix  $\mathbf{A}$ , nonreal eigenvalues come in complex conjugate pairs,

$$\lambda_1 = \alpha + \beta i \quad \text{and} \quad \lambda_2 = \alpha - \beta i$$

with  $\alpha, \beta \in \mathbb{R}$  and  $\beta \neq 0$ .

The corresponding eigenvectors are also complex conjugate pairs and can be written

$$\vec{v}_1 = \vec{p} + \vec{q}i \quad \text{and} \quad \vec{v}_2 = \vec{p} - \vec{q}i$$

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### Note

We only need to find one eigenvalue/eigenvector pair.

## Example 1

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Alternately, we can write

$$\vec{v} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{p}} \pm i \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{q}}$$



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$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2.$$

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So, for eigenvalue  $\lambda_1 = \alpha + \beta i$  and corresponding eigenvector  $\vec{v}_1 = \vec{p} + \vec{q}i$  we get the solution

$$\vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1 = e^{\alpha + \beta i} (\vec{p} + \vec{q}i)$$

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$$\vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1 = e^{\alpha + \beta i} (\vec{p} + \vec{q}i)$$

Just like with second-order systems, we shall find that the real and imaginary parts of the complex solution above are both real and linearly independent solutions of the system.

## Example 2

Consider the DE system:

$$\vec{x}' = \mathbf{A}\vec{x}$$

Suppose that

$$\vec{x}(t) = \vec{x}_{\text{Re}}(t) + \vec{x}_{\text{Im}}(t)$$

is a complex vector solution to the system, with  $\vec{x}_{\text{Im}} \neq \vec{0}$ .

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$$\vec{x}'_{\text{Re}}(t) + i\vec{x}'_{\text{Im}}(t) = \mathbf{A}\vec{x}_{\text{Re}}(t) + i\mathbf{A}\vec{x}_{\text{Im}}(t)$$

Separately equating the real and imaginary parts, we get:

$$\vec{x}'_{\text{Re}}(t) = \mathbf{A}\vec{x}_{\text{Re}}(t) \quad \text{and} \quad \vec{x}'_{\text{Im}}(t) = \mathbf{A}\vec{x}_{\text{Im}}(t)$$

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Thus,  $\vec{x}_{\text{Re}}(t)$  and  $\vec{x}_{\text{Im}}(t)$  are separate real solutions to the system.

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For the complex solution

$$\vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1 = e^{\alpha + \beta i} (\vec{p} + \vec{q}i)$$

we can determine the real and imaginary parts by using Euler's formula:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

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$$\begin{aligned} e^{\lambda_1 t} \vec{v}_1 &= e^{\alpha t + \beta t i} (\vec{p} + \vec{q}i) \\ &= e^{\alpha t} e^{\beta t i} (\vec{p} + \vec{q}i) \end{aligned}$$

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Since  $\vec{x}_{\text{Re}}(t)$  and  $\vec{x}_{\text{Im}}(t)$  are linearly independent solutions they must span the solution space. Thus, the general solution, for  $c_1, c_2 \in \mathbb{R}$ , is

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Any solutions derived from  $\lambda_2$  and  $\vec{v}_2$  will be linear combinations of  $\vec{x}_{\text{Re}}(t)$  and  $\vec{x}_{\text{Im}}(t)$ .

## Solving $2 \times 2$ DE System with Nonreal Eigenvalues

For the two-dimensional linear homogeneous differential equation  $\vec{x}' = \mathbf{A}\vec{x}$  with real matrix  $\mathbf{A}$ , eigenvalues  $\lambda_1, \lambda_2 = \alpha \pm \beta$  ( $\beta \neq 0$ ) the general solution can be found using the following steps:

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- ① For one eigenvalue  $\lambda_1$ , find its corresponding eigenvector  $\vec{v}_1$ . The second eigenvalue  $\lambda_2$  and its eigenvector  $\vec{v}_2$  are complex conjugates of the first. The eigenvectors are of the form  $\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$ .

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- 2 Construct the linearly independent real ( $\vec{x}_{\text{Re}}$ ) and imaginary ( $\vec{x}_{\text{Im}}$ ) parts of the solutions as follows:

$$\begin{aligned}\vec{x}_{\text{Re}}(t) &= e^{\alpha t} (\cos(\beta t) \vec{p} - \sin(\beta t) \vec{q}) \\ \vec{x}_{\text{Im}}(t) &= e^{\alpha t} (\sin(\beta t) \vec{p} + \cos(\beta t) \vec{q})\end{aligned}$$

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- 3 The general solution is

$$\vec{x}(t) = c_1 \vec{x}_{\text{Re}}(t) + c_2 \vec{x}_{\text{Im}}(t)$$



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Thus

$$\vec{x}_{\text{Re}}(t) = e^{5t} \cos(2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{5t} \sin(2t) \begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t} \begin{bmatrix} \cos(2t) \\ \cos(2t) + 2 \sin(2t) \end{bmatrix}$$

$$\vec{x}_{\text{Im}}(t) = e^{5t} \sin(2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{5t} \cos(2t) \begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t} \begin{bmatrix} \sin(2t) \\ \sin(2t) - 2 \cos(2t) \end{bmatrix}$$

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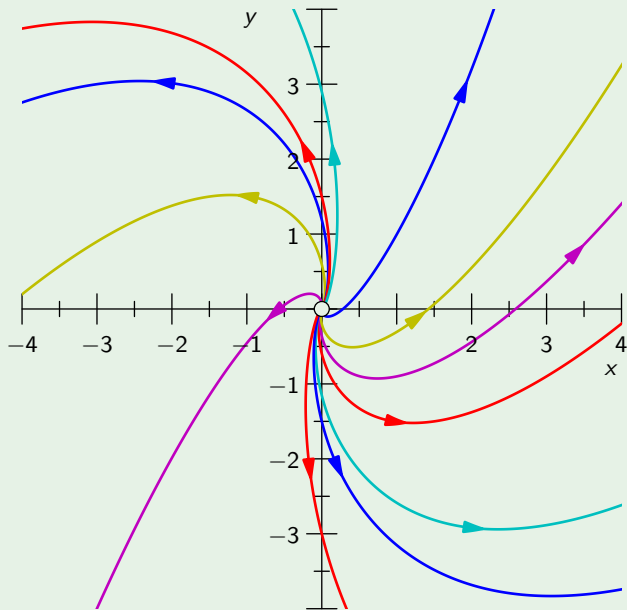
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$$\vec{x}_{\text{Im}}(t) = e^{5t} \sin(2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{5t} \cos(2t) \begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t} \begin{bmatrix} \sin(2t) \\ \sin(2t) - 2 \cos(2t) \end{bmatrix}$$

And general solution

$$\vec{x}(t) = e^{5t} \left( c_1 \begin{bmatrix} \cos(2t) \\ \cos(2t) + 2 \sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ \sin(2t) - 2 \cos(2t) \end{bmatrix} \right)$$

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Thus

$$\vec{x}_{\text{Re}}(t) = e^{-t} \cos(2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-t} \sin(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\vec{x}_{\text{Im}}(t) = e^{-t} \sin(2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-t} \cos(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



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Thus

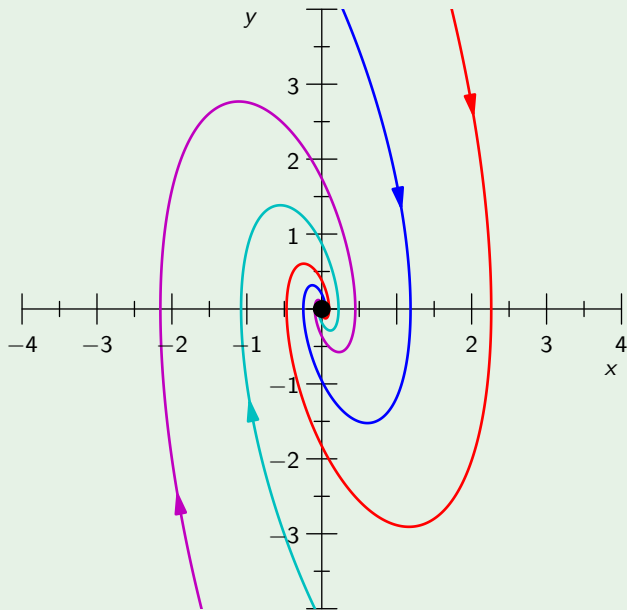
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And general solution

$$\vec{x}(t) = e^{-t} \left( c_1 \begin{bmatrix} \cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{bmatrix} \right)$$

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The eigenvalues are  $\lambda_1, \lambda_2 = 0 \pm 3i$  and

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

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Thus

$$\vec{x}_{\text{Re}}(t) = \cos(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \cos(3t) \\ 4 \cos(3t) + 3 \sin(3t) \end{bmatrix}$$

$$\vec{x}_{\text{Im}}(t) = \sin(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \sin(3t) \\ 4 \sin(3t) - 3 \cos(3t) \end{bmatrix}$$

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Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{x}$$

The eigenvalues are  $\lambda_1, \lambda_2 = 0 \pm 3i$  and

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Thus

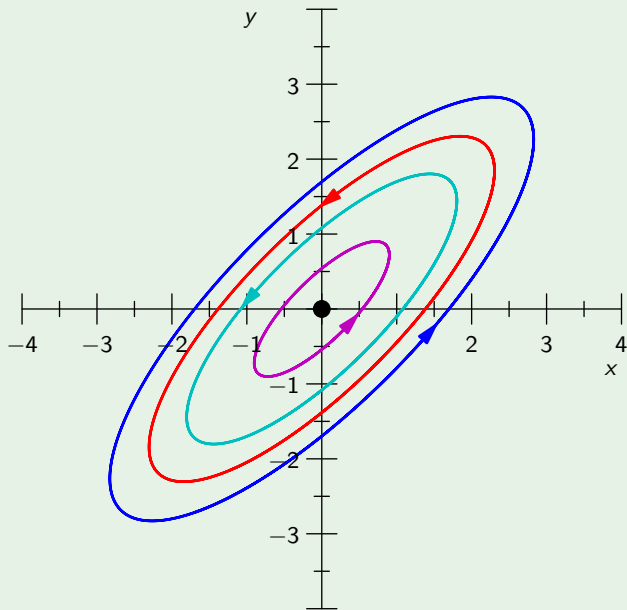
$$\vec{x}_{\text{Re}}(t) = \cos(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \cos(3t) \\ 4 \cos(3t) + 3 \sin(3t) \end{bmatrix}$$

$$\vec{x}_{\text{Im}}(t) = \sin(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \sin(3t) \\ 4 \sin(3t) - 3 \cos(3t) \end{bmatrix}$$

And general solution

$$\vec{x}(t) = c_1 \begin{bmatrix} 5 \cos(3t) \\ 4 \cos(3t) + 3 \sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} 5 \sin(3t) \\ 4 \sin(3t) - 3 \cos(3t) \end{bmatrix}$$

## Example 5



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- An **stable equilibrium** is one where the trajectories neither grow nor decay, they just circle in a periodic motion. (Since  $\alpha = 0$ .)

## Nullclines for a DE System

For a two-dimensional system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

- The  **$v$ -nullcline** is the set of all points with vertical slope, which occur on the curve obtained by solving  $x' = f(x, y) = 0$ .
- The  **$h$ -nullcline** is the set of all points with horizontal slope, which occur on the curve obtained by solving  $y' = g(x, y) = 0$ .

When an  $h$ -nullcline and an  $v$ -nullcline intersect, an **equilibrium** occurs.

## Interpreting the Solutions

For  $\vec{x}' = \mathbf{A}\vec{x}$  with nonreal eigenvalues  $\lambda_1, \lambda_2 = \alpha \pm \beta i$  and complex eigenvectors  $\vec{v}_1, \vec{v}_2 = \vec{p} + \vec{q}i$ , arrange the components of the solution as

$$\begin{bmatrix} \vec{x}_{\text{Re}} \\ \vec{x}_{\text{Im}} \end{bmatrix} = \underbrace{e^{\alpha t}}_{\text{expansion}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \vec{p} \\ \vec{q} \end{bmatrix}}_{\text{tilt and shape}}$$

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① The first factor  $e^{\alpha t}$  determines *expansion or contraction*.

- If  $\alpha > 0$ , then trajectories spiral outward, representing unbound growth.
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- 3 The third factor, containing  $\vec{p}$  and  $\vec{q}$ , determines the *tilt* and *shape* of the *elliptical trajectories* that would result with  $\alpha = 0$ .