# Conditional Probability

Colby Community College

## Example 1

The `photo_classify` data set represents a machine learning algorithm classifying a sample of 1822 photos as either about fashion or not.

| | | truth | | |
|---|---|---|---|---|
| | | *fashion* | *not* | Total |
| mach_learn | *pred_fashion* | 197 | 22 | 219 |
| | *pred_not* | 112 | 1491 | 1603 |
| | Total | 309 | 1513 | 1822 |

*If a photo is actually about fashion, what is the chance the algorithm will correctly identify the photo as being about fashion?*

Of the 309 fashion photos, the algorithm correctly classifies 197 of them.

$$P\left(\texttt{mach\_learn} \text{ is } \textit{pred\_fashion} \text{ given } \texttt{truth} \text{ is } \textit{fashion}\right) = \frac{197}{309} = 0.638$$

## Example 2

Using the same data set as in Example 1.

| | | truth | | |
|---|---|---|---|---|
| | | *fashion* | *not* | Total |
| mach_learn | *pred_fashion* | 197 | 22 | 219 |
| | *pred_not* | 112 | 1491 | 1603 |
| | Total | 309 | 1513 | 1822 |

*If the algorithm predicts the photo as being about fashion, what is the probability is actually is?*

Of the 1603 photos predicted to be about fashion, 112 we actually about fashion.

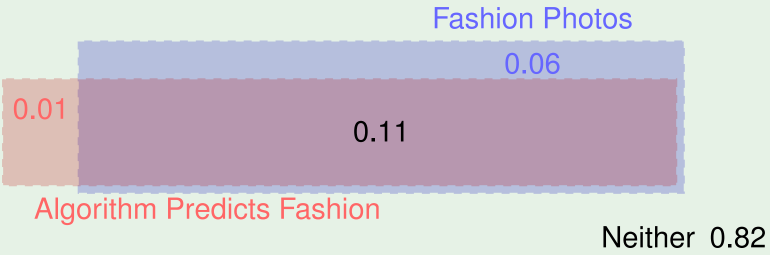
$$P\left(\texttt{truth} \text{ is } \mathit{fashion} \text{ given } \texttt{mach\_learn} \text{ is } \mathit{pred\_fashion}\right) = \frac{197}{219} = 0.900$$

## Note

It can be helpful to draw Venn Diagrams of these contingency tables using rectangles.

## Example 3

The Venn Diagram for Example 1 is:



## Definition

A **marginal probability** is a probability based on a single variable without regard to other variables.

## Example 4

$$P\left(\texttt{mach\_learn}\text{ is }\mathtt{\mathit{pred\_fashion}}\right)=\frac{219}{1822}=0.12$$

## Definition

A probability of outcomes for two or more variables is called a **joint probability**.

## Example 5

$$P\left(\texttt{mach\_learn}\text{ is }\mathtt{\mathit{pred\_fashion}}\text{ and }\texttt{truth}\text{ is }\mathtt{\mathit{fashion}}\right)=\frac{197}{1822}=0.11$$

### Note

Sometimes a comma is substituted for "and" in a joint probability.

$$P\left(\texttt{mach\_learn} \text{ is } \textit{pred\_fashion}, \texttt{truth} \text{ is } \textit{fashion}\right)$$

means the same thing as

$$P\left(\texttt{mach\_learn} \text{ is } \textit{pred\_fashion} \text{ and } \texttt{truth} \text{ is } \textit{fashion}\right)$$

## Definition

A **table proportions** is a table that summarizes joint probabilities. The proportions are computed by dividing each count by table's total.

## Example 6

The table proportions for `photo_classify` are:

| | `truth`: *`fashion`* | `truth`: *`not`* | Total |
|---|---|---|---|
| `mach_learn`: *`pred_fashion`* | $\frac{197}{1822}$ | $\frac{22}{1822}$ | $\frac{219}{1822}$ |
| `mach_learn`: *`pred_not`* | $\frac{112}{1822}$ | $\frac{1491}{1822}$ | $\frac{1603}{1822}$ |
| Total | $\frac{309}{1822}$ | $\frac{1513}{1822}$ | $\frac{1822}{1822}$ |

$$\downarrow\downarrow\downarrow$$

| | `truth`: *`fashion`* | `truth`: *`not`* | Total |
|---|---|---|---|
| `mach_learn`: *`pred_fashion`* | 0.1081 | 0.0121 | 0.1202 |
| `mach_learn`: *`pred_not`* | 0.0615 | 0.8183 | 0.8798 |
| Total | 0.1696 | 0.8304 | 1.0 |

## Example 7

The table proportions from Example 6 make a probability distribution.

| Joint Outcome | Probability |
|---|---|
| `mach_learn` is *pred_fashion* and `truth` is *fashion* | 0.1081 |
| `mach_learn` is *pred_fashion* and `truth` is *not* | 0.0121 |
| `mach_learn` is *pred_not* and `truth` is *fashion* | 0.0615 |
| `mach_learn` is *pred_not* and `truth` is *not* | 0.8182 |

## Note

Joint probabilities can be used to calculate marginal probabilities in simple cases.

## Example 8

$$\begin{aligned} P\left(\texttt{truth} \text{ is } \textit{fashion}\right) &= P\left(\texttt{mac\_learn} \text{ is } \textit{pred\_fashion} \text{ and } \texttt{truth} \text{ is } \textit{fashion}\right) \\ &\quad + P\left(\texttt{mac\_learn} \text{ is } \textit{pred\_not} \text{ and } \texttt{truth} \text{ is } \textit{fashion}\right) \\ &= 0.1081 + 0.0615 = 0.1696 \end{aligned}$$

## Definition

A **conditional probability** is a probability computed under a condition.

## Example 9

$$P\left(\texttt{truth} \text{ is } \mathit{fashion} \text{ given } \texttt{mach\_learn} \text{ is } \mathit{pred\_fashion}\right) = \frac{197}{219} = 0.900$$

## Definition

There are two parts to a conditional probability, the **outcome of interest** and the **condition.**

$$P\left(\text{outcome of interest given condition}\right)$$

is the same as

$$P\left(\text{outcome of interest} \mid \text{condition}\right)$$

## Example 10

$$P\left(\texttt{truth} \text{ is } \mathit{fashion} \mid \texttt{mach\_learn} \text{ is } \mathit{pred\_fashion}\right) = \frac{197}{219} = 0.900$$

## Note

Conditional probabilities are computed as:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

## Example 11

$$\begin{aligned}
&P\left(\texttt{truth} \text{ is } \textit{fashion} \mid \texttt{mach_learn} \text{ is } \textit{pred_fashion}\right) \\
&\quad = \frac{P\left(\texttt{truth} \text{ is } \textit{fashion} \text{ and } \texttt{mach_learn} \text{ is } \textit{pred_fashion}\right)}{P\left(\texttt{mach_learn} \text{ is } \textit{pred_fashion}\right)} \\
&\quad = \frac{0.0615}{0.1696} \\
&\quad = 0.3626
\end{aligned}$$

## Example 12

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

| | | inoculated | | |
|---|---|---|---|---|
| | | *yes* | *no* | Total |
| result | *lived* | 238 | 5136 | 5374 |
| | *died* | 6 | 844 | 850 |
| | Total | 244 | 5980 | 6224 |

| inoculated | | |
|---|---|---|
| *yes* | *no* | Total |
| 0.0382 | 0.8252 | 0.8634 |
| 0.0010 | 0.1356 | 0.1366 |
| 0.0392 | 0.9608 | 1.0000 |

*What is the probability that a randomly selected person who was not inoculated died from smallpox?*

$$P\left(\texttt{result} \text{ is } \textit{died} \mid \texttt{inoculated} \text{ is } \textit{no}\right)$$

$$= \frac{P\left(\texttt{result} \text{ is } \textit{died} \text{ and } \texttt{inoculated} \text{ is } \textit{no}\right)}{P\left(\texttt{inoculated} \text{ is } \textit{no}\right)}$$

$$= \frac{0.1356}{0.9608}$$

$$= 0.1411 \approx 14.11\%$$

## Example 13

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

| | | inoculated | | |
|---|---|---|---|---|
| | | *yes* | *no* | Total |
| result | *lived* | 238 | 5136 | 5374 |
| | *died* | 6 | 844 | 850 |
| | Total | 244 | 5980 | 6224 |

| inoculated | | |
|---|---|---|
| *yes* | *no* | Total |
| 0.0382 | 0.8252 | 0.8634 |
| 0.0010 | 0.1356 | 0.1366 |
| 0.0392 | 0.9608 | 1.0000 |

*What is the probability that a randomly selected inoculated person died from smallpox?*

$$P\left(\texttt{result is } \textit{died} \;\middle|\; \texttt{inoculated is } \textit{yes}\right)$$

$$= \frac{P\left(\texttt{result is } \textit{died} \text{ and } \texttt{inoculated is } \textit{yes}\right)}{P\left(\texttt{inoculated is } \textit{yes}\right)}$$

$$= \frac{0.0010}{0.0392}$$

$$= 0.0255 \approx 2.55\%$$

## Example 14

The `smallpox` data set provides a sample of 6,224 individuals from the year 1721 who were exposed to smallpox in Boston.

The residents of Boston self-selected whether or not to be inoculated.

*Is this study observational or experimental?*

Observational

*Can we infer any causal connection using this data?*

No, the fact this is an observational study combined with the self-selecting bias means we cannot.

*What are some potential confounding variables that might influence whether someone `lived` or `died`?*

People die for many reasons, wealth determines level of medical care available, etc...

## General Multiplication Rule

If $A$ and $B$ represent two outcomes or events, then

$$P(A \text{ and } B) = P(A \mid B) \cdot P(B)$$

## Example 15

Suppose we are only given two pieces of information:

- 96.08% of Boston residents were not inoculated.
- 85.88% of Boston residents who were not inoculated ended up surviving.

Let us find the probability that a resident who was no inoculated and lived.

$$\begin{aligned} &P(\texttt{result} \text{ is } \texttt{\textit{lived}} \text{ and } \texttt{inoculated} \text{ is } \texttt{\textit{no}}) \\ &\qquad = P(\texttt{result} \text{ is } \texttt{\textit{lived}} \mid \texttt{inoculated} \text{ is } \texttt{\textit{no}}) \cdot P(\texttt{inoculated} \text{ is } \texttt{\textit{no}}) \\ &\qquad = 0.8588 \cdot 0.9698 \\ &\qquad = 0.8251 \approx 82.51\% \end{aligned}$$

## Example 16

Let's use

- $P\left(\texttt{inoculated} \text{ is } \mathit{yes}\right) = 0.0392$
- $P\left(\texttt{result} \text{ is } \mathit{lived} \mid \texttt{inoculated} \text{ is } \mathit{yes}\right) = 0.9754$

to find the probability that a person was both inoculated and lived.

$$\begin{aligned}
&P\left(\texttt{result} \text{ is } \mathit{lived} \text{ and } \texttt{inoculated} \text{ is } \mathit{yes}\right)\\
&\qquad = P\left(\texttt{result} \text{ is } \mathit{lived} \mid \texttt{inoculated} \text{ is } \mathit{yes}\right) \cdot P\left(\texttt{inoculated} \text{ is } \mathit{yes}\right)\\
&\qquad = 0.0382 \cdot 0.9754\\
&\qquad = 0.0382 \approx 3.82\%
\end{aligned}$$

## Sum of Conditional Probabilities

Let $A_1$, $A_2$, ..., $A_k$ represent all the disjoint outcomes for a variable. Then if $B$ is an event, possibly for another variable, we have:

$$P(A_1 \mid B) + P(A_2 \mid B) + \cdots + P(A_k \mid B) = 1$$

## Note

If an event and it's complement are conditioned on the same information, then:

$$P(A \mid B) = 1 - P\left(A^C \mid B\right)$$

## Example 17

*If 97.54% of the inoculated people lived, what is the proportion of people that must have died?*

There are only two outcomes: `lived` and `died`. Which means that $100\% - 97.54\% = 2.46\%$ people who were inoculated died.

## Example 18

Let $X$ and $Y$ represent the outcomes of rolling two dice.

Let us compute the following:

$$\begin{aligned} P(Y=1 \mid X=1) &= \frac{P(Y=1 \text{ and } X=1)}{P(X=1)} \\ &= \frac{P(Y=1)\cdot P(X=1)}{P(X=1)} \\ &= \frac{P(Y=1)\cdot \cancel{P(X=1)}}{\cancel{P(X=1)}} \\ &= P(Y=1) \end{aligned}$$

## Note

We have shown that if two events are independent, then knowing the outcome of one should provide no information about the other.

## Example 19

Ron is watching a roulette table in a casino and notices that the last five outcomes were `black`. He figures that the chances of getting `black` six times in a row is very small and puts his paycheck on red.

*Why is this a really bad idea?*

Each spin of a roulette wheel is independent of any of the previous spins. The next spin has the exact same probability of getting black and any other.

## Note

Posting the last several outcomes of a betting game is a real practice casinos use to trick people into believing the odds are in their favor. It's known as the **gambler's fallacy**.

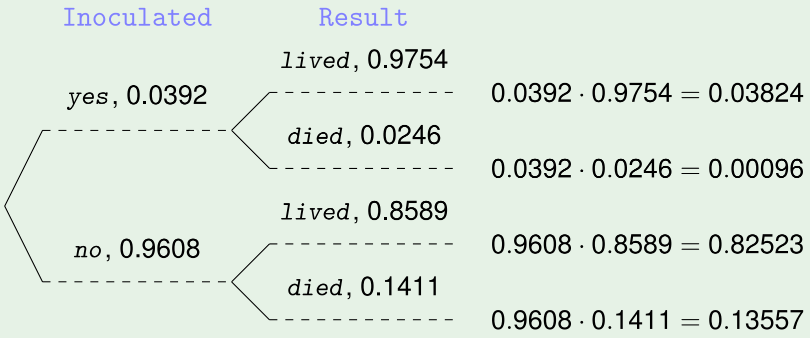
## Definition

A **tree diagram** is a tool to organize outcomes and probabilities around the structure of the data.

They are most useful when two or more processes occur in a sequence and each process is conditioned on its predecessor.

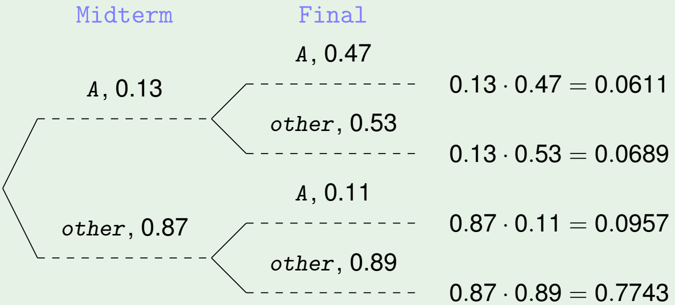
## Example 20

Here is the tree diagram for `smallpox` dataset.



## Example 21

Suppose that 13% of students earned an A on the midterm. Of those students that earned an A, 47% received an A on the final and 11% of the student who earned a lower grade than an A on the midterm received an A on the final.




$$P\left(\texttt{midterm}\text{ is }\texttt{A} \;\middle|\; \texttt{final}\text{ is }\texttt{A}\right) = \frac{P\left(\texttt{midterm}\text{ is }\texttt{A}\text{ and }\texttt{final}\text{ is }\texttt{A}\right)}{P\left(\texttt{final}\text{ is }\texttt{A}\right)}$$

$$= \frac{0.0611}{0.1568} = 0.3897 \approx 38.97\%$$

## Definition

A **false negative** is when a test incorrectly gives a negative result.

## Definition

A **false positive** is when a test incorrectly gives a positive result.

## Example 22

In Canada, about 0.35% of women over 40 will develop breast cancer in any given year. A common screening test for cancer is the mammogram, but this test is not perfect.

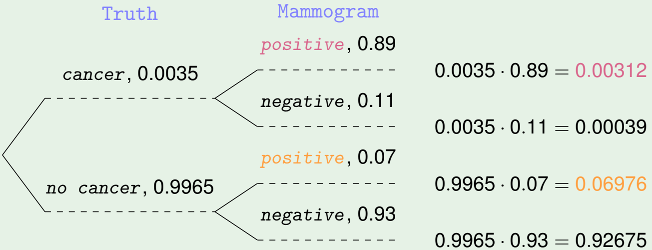
In about 11% of patients with breast cancer, the test gives a false negative. That is, the mammogram indicated the woman does not have breast cancer, when she really does have breast cancer.

In about 7% of patients that do not have breast cancer, the test gives a true negative. That is, the mammogram indicated the woman does have breast cancer, when she really doesn't have breast cancer.

## Example 23

A question doctors and researchers have to answer is: If the mammogram comes back positive, what is the probability that the patient actually has breast cancer?

We have enough information to build a tree diagram:




So,

$$\begin{aligned} P\left(\text{has BC} \mid \text{test positive}\right) &= \frac{P\left(\text{has BC and test positive}\right)}{P\left(\text{test positive}\right)} \\ &= \frac{0.00312}{0.00312 + 0.06976} = 0.0428 \approx 4.28\% \end{aligned}$$

## Note

There are times where we are given:

$$P\left(\text{statement about } \texttt{variable 1|} \,\middle|\, \text{statement about } \texttt{variable 2}\right)$$

but we would rather know the inverted conditional probability:

$$P\left(\text{statement about } \texttt{variable 2} \,\middle|\, \text{statement about } \texttt{variable 1}\right)$$

## Bayes' Theorem

Consider the following conditional probability for `variable 1` and `variable 2`:

$$P\left(\text{outcome } A_1 \text{ of } \texttt{variable 1} \,\middle|\, \text{outcome } B \text{ of } \texttt{variable 2}\right)$$

Bayes' Theorem states that this conditional probability is the same as:

$$\frac{P(B \mid A_1)\,P(A_1)}{P(B \mid A_1)\,P(A_1) + P(B \mid A_2)\,P(A_2) + \cdots + P(B \mid A_k)\,P(A_k)}$$

where $A_2, A_3, \ldots, A_k$ represent all other possible outcomes of `variable 1`.

## Example 24

In Example 23, we computed $P\,(\text{has BC} \mid \text{test positive})$ using a tree diagram.

We could have also used Bayes' Theorem:

$$
\begin{aligned}
&P\,(\text{has BC} \mid \text{test pos})\\
&\quad= \frac{P\,(\text{test pos} \mid \text{has BC})\,P\,(\text{has BC})}{P\,(\text{test pos} \mid \text{has BC})\,P\,(\text{has BC}) + P\,(\text{test pos} \mid \text{no BC})\,P\,(\text{no BC})}\\
&\quad= \frac{0.89 \cdot 0.0035}{0.89 \cdot 0.0035 + 0.07 \cdot 0.9965} = \frac{0.03115}{0.07287} = 0.0428 \approx 4.28\%
\end{aligned}
$$

## Note

This strategy of updating beliefs using Bayes' Theorem is the foundation of an entire branch of statistics called **Bayesian statistics**.