Probability Distributions

Colby Community College

Definition

A **random variable** is a variable (typically represented by x) that has a single numerical value, determined by chance, for each outcome of a procedure.

Definition

A **random variable** is a variable (typically represented by x) that has a single numerical value, determined by chance, for each outcome of a procedure.

Example 1

Consider tossing a coin: We could get a heads or a tail.

If we let x be the number of tails we get in a single flip, then

$$x = \begin{cases} 0 & \text{Heads} \\ 1 & \text{Tails} \end{cases}$$

Definition

A **random variable** is a variable (typically represented by x) that has a single numerical value, determined by chance, for each outcome of a procedure.

Example 1

Consider tossing a coin: We could get a heads or a tail.

If we let x be the number of tails we get in a single flip, then

$$x = \begin{cases} 0 & \text{Heads} \\ 1 & \text{Tails} \end{cases}$$

Definition

A **probability distribution** is a description that gives the probability for each value of the random variable. It is often expressed in the format of a table, formula, or graph.

Definition

A **discrete random variable** has a collection of values that is finite or countable.

Definition

A **discrete random variable** has a collection of values that is finite or countable.

Note

If there are infinitely many values, the number of values are countable if it is possible to count them individually. Think counting the number of coin tosses.

Definition

A **discrete random variable** has a collection of values that is finite or countable.

Note

If there are infinitely many values, the number of values are countable if it is possible to count them individually. Think counting the number of coin tosses.

Definition

A **continuous random variable** has infinitely many values, and the collection of values is not countable.

Definition

A **discrete random variable** has a collection of values that is finite or countable.

Note

If there are infinitely many values, the number of values are countable if it is possible to count them individually. Think counting the number of coin tosses.

Definition

A **continuous random variable** has infinitely many values, and the collection of values is not countable.

Note

If there are infinitely many values, the number of values are not countable if some of them are on a continuous scale. Think body temperatures.

Definition

A **discrete random variable** has a collection of values that is finite or countable.

Note

If there are infinitely many values, the number of values are countable if it is possible to count them individually. Think counting the number of coin tosses.

Definition

A **continuous random variable** has infinitely many values, and the collection of values is not countable.

Note

If there are infinitely many values, the number of values are not countable if some of them are on a continuous scale. Think body temperatures.

Note

We will be dealing almost exclusively with discrete random variables in this chapter.

Note

We will be dealing almost exclusively with discrete random variables in this chapter.

Requirements

Every probability distribution must satisfy the following requirements.

1. There is a numerical random variable x, and its number values are associated with corresponding probabilities.

Note

We will be dealing almost exclusively with discrete random variables in this chapter.

Requirements

Every probability distribution must satisfy the following requirements.

1. There is a numerical random variable x, and its number values are associated with corresponding probabilities.
2. $\sum P(x) \approx 1$ where x assumes all possible values.

Note

We will be dealing almost exclusively with discrete random variables in this chapter.

Requirements

Every probability distribution must satisfy the following requirements.

1. There is a numerical random variable x, and its number values are associated with corresponding probabilities.
2. $\sum P(x) \approx 1$ where x assumes all possible values.
3. $0 \leq P(x) \leq 1$ for every individual value of the random variable x.

Example 2

Let's consider tossing two coins, with the following random variable:

$$x = \text{number of heads when two coins are tossed}$$

Example 2

Let's consider tossing two coins, with the following random variable:

$$x = \text{number of heads when two coins are tossed}$$

With two coins tossed, the number of heads can be 0, 1, or 2, so x is a discrete random variable. The table gives the probability distribution.

Number of Heads	$P(x)$
0	0.25
1	0.50
2	0.25

Example 2

Let's consider tossing two coins, with the following random variable:

$$x = \text{number of heads when two coins are tossed}$$

With two coins tossed, the number of heads can be 0, 1, or 2, so x is a discrete random variable. The table gives the probability distribution.

Number of Heads	$\boldsymbol{P(x)}$
0	0.25
1	0.50
2	0.25

We can check the requirements.

1. The variable x is a numerical random variable, and its values are associated with probabilities.

Example 2

Let's consider tossing two coins, with the following random variable:

$$x = \text{number of heads when two coins are tossed}$$

With two coins tossed, the number of heads can be 0, 1, or 2, so x is a discrete random variable. The table gives the probability distribution.

Number of Heads	$\boldsymbol{P(x)}$
0	0.25
1	0.50
2	0.25

We can check the requirements.

1. The variable x is a numerical random variable, and its values are associated with probabilities.
2. $\sum P(x) = 0.25 + 0.50 + 0.25 = 1$

Example 2

Let's consider tossing two coins, with the following random variable:

$$x = \text{number of heads when two coins are tossed}$$

With two coins tossed, the number of heads can be 0, 1, or 2, so x is a discrete random variable. The table gives the probability distribution.

Number of Heads	$\boldsymbol{P(x)}$
0	0.25
1	0.50
2	0.25

We can check the requirements.

1. The variable x is a numerical random variable, and its values are associated with probabilities.
2. $\sum P(x) = 0.25 + 0.50 + 0.25 = 1$
3. Each value of $P(x)$ is between 0 and 1.

Ways to display probability distributions.

A **probability table**:

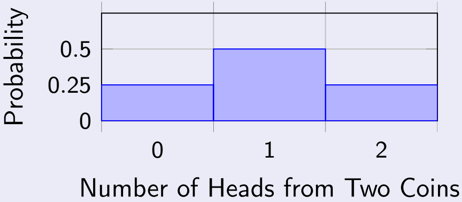
Number of Heads	$P(x)$
0	0.25
1	0.50
2	0.25

Ways to display probability distributions.

A **probability table**:

Number of Heads	$P(x)$
0	0.25
1	0.50
2	0.25

A **probability histogram**:

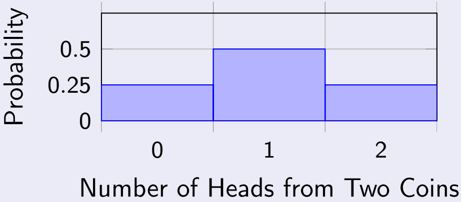


Ways to display probability distributions.

A **probability table**:

Number of Heads	$P(x)$
0	0.25
1	0.50
2	0.25

A **probability histogram**:



A **probability formula**, where x can be 0, 1, 2:

$$P(x) = \frac{1}{2(2-x)!x!}$$

Example 3

Hiring managers were asked to identify the biggest mistakes that job applicants make during an interview. The table is based on their responses.

x	$P(x)$
Inappropriate attire	0.50
Being late	0.44
Lack of Eye Contact	0.33
Checking phone or texting	0.30
Total	**1.57**

Does this table describe a probability distribution?

Example 3

Hiring managers were asked to identify the biggest mistakes that job applicants make during an interview. The table is based on their responses.

x	$P(x)$
Inappropriate attire	0.50
Being late	0.44
Lack of Eye Contact	0.33
Checking phone or texting	0.30
Total	**1.57**

Does this table describe a probability distribution?

We can check the requirements.

1. The variable x is not numerical. The values are categorical.

Example 3

Hiring managers were asked to identify the biggest mistakes that job applicants make during an interview. The table is based on their responses.

x	$P(x)$
Inappropriate attire	0.50
Being late	0.44
Lack of Eye Contact	0.33
Checking phone or texting	0.30
Total	**1.57**

Does this table describe a probability distribution?

We can check the requirements.

1. The variable x is not numerical. The values are categorical.
2. $\sum P(x) = 1.57 \neq 1$

Example 3

Hiring managers were asked to identify the biggest mistakes that job applicants make during an interview. The table is based on their responses.

x	$P(x)$
Inappropriate attire	0.50
Being late	0.44
Lack of Eye Contact	0.33
Checking phone or texting	0.30
Total	**1.57**

Does this table describe a probability distribution?

We can check the requirements.

1. The variable x is not numerical. The values are categorical.
2. $\sum P(x) = 1.57 \neq 1$
3. Each value of $P(x)$ is between 0 and 1.

Example 3

Hiring managers were asked to identify the biggest mistakes that job applicants make during an interview. The table is based on their responses.

x	$P(x)$
Inappropriate attire	0.50
Being late	0.44
Lack of Eye Contact	0.33
Checking phone or texting	0.30
Total	**1.57**

Does this table describe a probability distribution?

We can check the requirements.

1. The variable x is not numerical. The values are categorical.
2. $\sum P(x) = 1.57 \neq 1$
3. Each value of $P(x)$ is between 0 and 1.

Since not all three are satisfied, we see this table *does not* describe a probability distribution.

Note

A probability distribution describes a population instead of a sample.

Note

A probability distribution describes a population instead of a sample.

Mean μ for a Probability Distribution

$$\mu = \sum (x \cdot P(x))$$

Note

A probability distribution describes a population instead of a sample.

Mean μ for a Probability Distribution

$$\mu = \sum (x \cdot P(x))$$

Variance σ^2 for a Probability Distribution

$$\sigma^2 = \sum \left((x-\mu)^2 \cdot P(x)\right) = \sum \left(x^2 \cdot P(x)\right) - \mu^2$$

Note

A probability distribution describes a population instead of a sample.

Mean μ for a Probability Distribution

$$\mu = \sum (x \cdot P(x))$$

Variance σ^2 for a Probability Distribution

$$\sigma^2 = \sum \left((x-\mu)^2 \cdot P(x)\right) = \sum \left(x^2 \cdot P(x)\right) - \mu^2$$

Standard Deviation σ for a Probability Distribution

$$\sigma = \sqrt{\sum \left(x^2 \cdot P(x)\right) - \mu^2}$$

Note

A probability distribution describes a population instead of a sample.

Mean μ for a Probability Distribution

$$\mu = \sum (x \cdot P(x))$$

Variance σ^2 for a Probability Distribution

$$\sigma^2 = \sum \left((x - \mu)^2 \cdot P(x) \right) = \sum \left(x^2 \cdot P(x) \right) - \mu^2$$

Standard Deviation σ for a Probability Distribution

$$\sigma = \sqrt{\sum \left(x^2 \cdot P(x) \right) - \mu^2}$$

Round-Off Rule

Round results by carrying *one or more decimal place* than the number of decimal places used for the random variable x.

Example 4

In the casino game Roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the players bet \$1 on a single number. If that number is spun on the wheel, then they receive \$36. Otherwise, they lose their \$1.

On average, how much money should a player expect to win or lose if they play this game repeatedly?

Example 4

In the casino game Roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the players bet \$1 on a single number. If that number is spun on the wheel, then they receive \$36. Otherwise, they lose their \$1.

On average, how much money should a player expect to win or lose if they play this game repeatedly?

Any number you bet on will have the following probabilities:

Outcome	Probability
\$35 (win)	1/38
-\$1 (lose)	37/38

Example 4

In the casino game Roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the players bet \$1 on a single number. If that number is spun on the wheel, then they receive \$36. Otherwise, they lose their \$1.

On average, how much money should a player expect to win or lose if they play this game repeatedly?

Any number you bet on will have the following probabilities:

Outcome	Probability
\$35 (win)	1/38
-\$1 (lose)	37/38

So, **on average**, we will have a net change of

$$\$35 \cdot \frac{1}{38} + -\$1 \cdot \frac{37}{38} = \$0.9211 - \$0.9737 \approx -\$0.053$$

That is, **on average**, we will lose 5.3 cents per space we bet on.

Definition

The **expected value** of a discrete random variable x is denoted by E, and it is the mean value of the outcomes, so $E = \mu$.

$$E = \sum (x \cdot P(x))$$

Definition

The **expected value** of a discrete random variable x is denoted by E, and it is the mean value of the outcomes, so $E = \mu$.

$$E = \sum (x \cdot P(x))$$

Caution

An expected value need not be a whole number, even if the different possible values of x might all be whole numbers.

Example 5

Consider a lottery where balls numbered 1 through 48 are placed in a machine and six balls are drawn at random. If the six numbers drawn match the numbers that a player has chosen, that player wins $1,000,000. If they match five numbers, they win $1,000. A lottery ticket costs $1. Let us calculate the expected value.

Example 5

Consider a lottery where balls numbered 1 through 48 are placed in a machine and six balls are drawn at random. If the six numbers drawn match the numbers that a player has chosen, that player wins \$1,000,000. If they match five numbers, they win \$1,000. A lottery ticket costs \$1. Let us calculate the expected value.

The following table gives the values and probabilities.

Outcome	Value	Probability
Match all six	\$999,999	$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12272512}$
Match five	\$999	$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12272512}$
Match four for fewer	-\$1	$1 - \frac{253}{12272512} = \frac{12271259}{12272512}$

Example 5

Consider a lottery where balls numbered 1 through 48 are placed in a machine and six balls are drawn at random. If the six numbers drawn match the numbers that a player has chosen, that player wins \$1,000,000. If they match five numbers, they win \$1,000. A lottery ticket costs \$1. Let us calculate the expected value.

The following table gives the values and probabilities.

Outcome	Value	Probability
Match all six	\$999,999	$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12272512}$
Match five	\$999	$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12272512}$
Match four for fewer	-\$1	$1 - \frac{253}{12272512} = \frac{12271259}{12272512}$

The expected value is then:

$$(\$999,999) \cdot \frac{1}{12272512} + (\$999) \cdot \frac{252}{12272512} + (-\$1) \cdot \frac{12271259}{12272512} \approx -\$0.898$$

So, on average, a player can expect to lose about 90 cents on a ticket.

Note

It is generally a bad idea to play a game with a negative expected value.

A game is called **fair** if it's expected value is zero.

Note

It is generally a bad idea to play a game with a negative expected value.

A game is called **fair** if it's expected value is zero.

Example 6

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. Should you play this game?

Note

It is generally a bad idea to play a game with a negative expected value.

A game is called **fair** if it's expected value is zero.

Example 6

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. Should you play this game?

Suppose you roll the first die. The probabilities are:

$$P\left(\text{no match}\right) = \frac{\left({}_6C_1\right)\left({}_5C_1\right)\left({}_4C_1\right)}{\left({}_6C_1\right)\left({}_6C_1\right)\left({}_6C_1\right)} = \frac{6\cdot 5\cdot 4}{6\cdot 6\cdot 6} = \frac{5}{9} \quad (\approx 55.5\%)$$

$$P\left(\text{at least two match}\right) = 1 - P\left(\text{no match}\right) = 1 - \frac{5}{9} = \frac{4}{9} \qquad (\approx 44.4\%)$$

Note

It is generally a bad idea to play a game with a negative expected value.

A game is called **fair** if it's expected value is zero.

Example 6

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. Should you play this game?

Suppose you roll the first die. The probabilities are:

$$P\left(\text{no match}\right) = \frac{\left({}_6C_1\right)\left({}_5C_1\right)\left({}_4C_1\right)}{\left({}_6C_1\right)\left({}_6C_1\right)\left({}_6C_1\right)} = \frac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \frac{5}{9} \quad (\approx 55.5\%)$$

$$P\left(\text{at least two match}\right) = 1 - P\left(\text{no match}\right) = 1 - \frac{5}{9} = \frac{4}{9} \qquad (\approx 44.4\%)$$

The expected value is:

$$\$2 \cdot \frac{4}{9} - \$1 \cdot \frac{5}{9} = \frac{1}{3} \approx \$0.33$$

You will, on average, win 33 cents per play.

Example 7

Suppose an individual has a 0.242% risk of dying during the next year. An insurance company charges \$275 for a life-insurance policy that pays \$100,000 death benefit. What is the expected value for the person buying the insurance?

Example 7

Suppose an individual has a 0.242% risk of dying during the next year. An insurance company charges \$275 for a life-insurance policy that pays \$100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and values for the two outcomes are:

Value	Probability
\$100,000-\$275 = \$99,725	0.00242
-\$275	1-0.00242 = 0.99758

Example 7

Suppose an individual has a 0.242% risk of dying during the next year. An insurance company charges \$275 for a life-insurance policy that pays \$100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and values for the two outcomes are:

Value	Probability
\$100,000-\$275 = \$99,725	0.00242
-\$275	1-0.00242 = 0.99758

The expected value is $\$99,725 \cdot 0.00242 - \$275 \cdot 0.99758 = -\$33$.

Example 7

Suppose an individual has a 0.242% risk of dying during the next year. An insurance company charges \$275 for a life-insurance policy that pays \$100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and values for the two outcomes are:

Value	Probability
\$100,000-\$275 = \$99,725	0.00242
-\$275	1-0.00242 = 0.99758

The expected value is $\$99,725 \cdot 0.00242 - \$275 \cdot 0.99758 = -\$33$.

Note

It makes sense that a insurance policy would have a negative expected value, otherwise the insurance company couldn't stay in business.

The benefit for the consumer is the security that the policy provides.

Example 8

A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of $350. If they offer a 2 year extended warranty for $48, what is the company's expected value?

Example 8

A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$350. If they offer a 2 year extended warranty for \$48, what is the company's expected value?

The probabilities and values for the two outcomes are:

Value	Probability
-\$350 + \$48 = -\$302	0.007
\$48	1-0.007 = 0.993

Example 8

A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$350. If they offer a 2 year extended warranty for \$48, what is the company's expected value?

The probabilities and values for the two outcomes are:

Value	Probability
-\$350 + \$48 = -\$302	0.007
\$48	1-0.007 = 0.993

The expected value is $-\$302 \cdot 0.007 + \$48 \cdot 0.993 = \$45.55$.

The company makes, on average, \$45.55 on each extended warranty.

Example 8

A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$350. If they offer a 2 year extended warranty for \$48, what is the company's expected value?

The probabilities and values for the two outcomes are:

Value	Probability
-\$350 + \$48 = -\$302	0.007
\$48	1-0.007 = 0.993

The expected value is $-\$302 \cdot 0.007 + \$48 \cdot 0.993 = \$45.55$.

The company makes, on average, \$45.55 on each extended warranty.

Note

The expected value for the consumer may be different. The consumer is likely to pay the more to repair or replace the item out of warranty. (The company pays manufacturing cost, consumer has to pay retail cost.)

Identifying Significant Results with Probabilities

- x successes among n trials is a *significantly high* number of successes if the probability of x or more successes is 0.05 or less.
- x successes among n trials is a *significantly low* number of successes if the probability of x or fewer successes is 0.05 or less.

Identifying Significant Results with Probabilities

- x successes among n trials is a *significantly high* number of successes if the probability of x or more successes is 0.05 or less.
- x successes among n trials is a *significantly low* number of successes if the probability of x or fewer successes is 0.05 or less.

The Rare Event Rule for Inferential Statistics

If under a given assumption, the probability of a particular outcome is very small and the outcome occurs *significantly less than* or *significantly more than* what we expect with that assumption, we conclude that the assumption is probably not correct.

Example 9

Is 252 heads in 460 coin tosses a significantly high number of heads?

Example 9

Is 252 heads in 460 coin tosses a significantly high number of heads?

A result of 252 heads in 460 coin tosses is greater than we expect with random chance, but we need to determine whether 252 heads is significantly high.

Example 9

Is 252 heads in 460 coin tosses a significantly high number of heads?

A result of 252 heads in 460 coin tosses is greater than we expect with random chance, but we need to determine whether 252 heads is significantly high.

Here, the relevant probability is the probability of getting 252 or more heads in 460 coin tosses.

$$P\left(255 \text{ or more heads in } 460 \text{ coins}\right) = \frac{1}{2^{460}} \sum_{i=252}^{460} {}_{460}C_i = 0.0224$$

Example 9

Is 252 heads in 460 coin tosses a significantly high number of heads?

A result of 252 heads in 460 coin tosses is greater than we expect with random chance, but we need to determine whether 252 heads is significantly high.

Here, the relevant probability is the probability of getting 252 or more heads in 460 coin tosses.

$$P\left(255 \text{ or more heads in 460 coins}\right) = \frac{1}{2^{460}} \sum_{i=252}^{460} {}_{460}C_i = 0.0224$$

Since $0.0224 < 0.05$, 252 heads is significantly high.