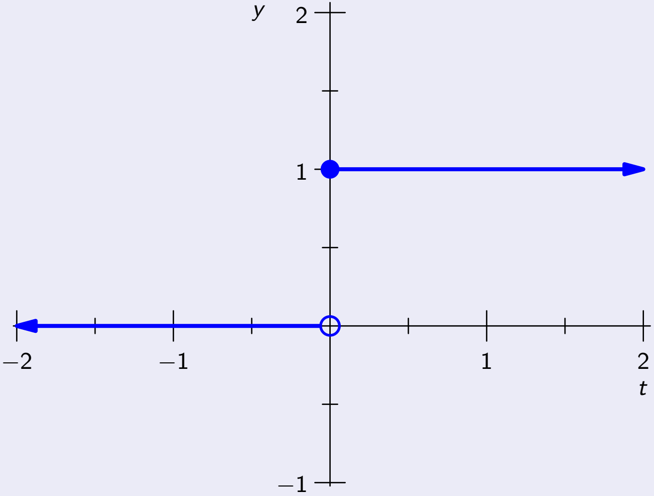
The Step Function and the Delta Function

Adam Wilson

Salt Lake Community College

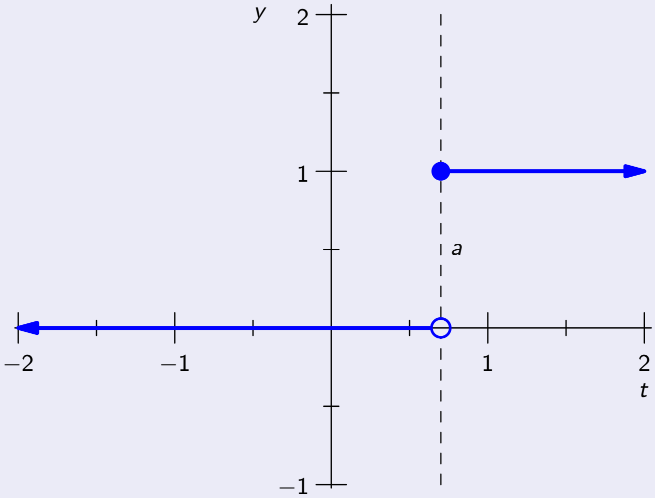
The Unit Step Function

$$\textbf{step}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



The Translated Step Function

$$\textbf{step}(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



Laplace Transform of the Step Function

$$\mathcal{L}\{\textbf{step}(t-a)\} = \frac{e^{-as}}{s}$$

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$$\begin{aligned}\mathcal{L}\{\textbf{step}(t-a)\} &= \int_0^\infty e^{-st}\,\textbf{step}(t-a)\ dt \\ &= \int_0^a e^{-st}\cdot 0\ dt + \int_a^\infty e^{-st}\cdot 1\ dt\end{aligned}$$

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&= \lim_{b\to\infty} -\frac{1}{s}\left[e^{-sb}-e^{-sa}\right] \\
&= \frac{e^{as}}{s}
\end{aligned}$$

Example

Consider

$$f(t) = \begin{cases} 2 & \text{if } t < 3 \\ -4 & \text{if } 3 \leq t < 4 \\ 1 & \text{if } t \geq 4 \end{cases}$$

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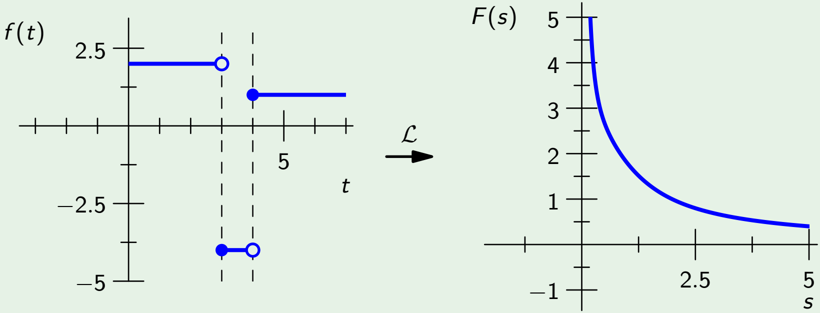

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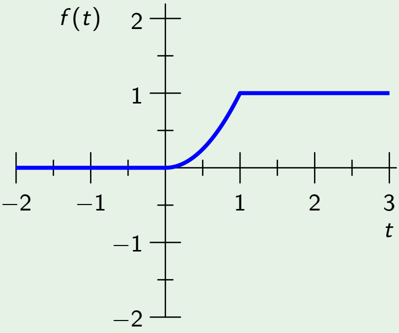

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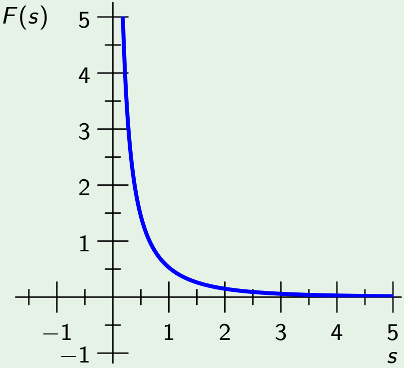

Delayed Function

For a given function $g(t)$, the **delayed function**

$$f(t) = \begin{cases} 0 & \text{if } t < c \\ g(t-c) & \text{if } t \geq c \end{cases}$$

shifts $g(t)$ to the right c units from the origin, and replaces it by zero to the left of $t = c$. Using the unit step function, the delayed function can also be written

$$f(t) = g(t-c)\,\textbf{step}(t-c)$$

Example

Consider the function $g(t) = \sqrt{t}$, which has the delayed function

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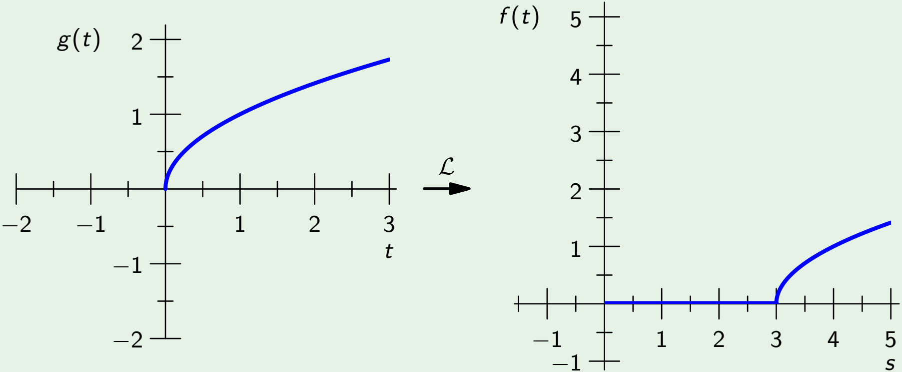

Let us now look at the Laplace transform of a function $f(t)$ that is delayed c units.

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Delay Theorem (or Shifting Theorem)

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&= e^{-s}\left(\frac{2}{s^2} + \frac{2}{s^2} + \frac{1}{s}\right)
\end{aligned}$$

Example

Let us find the inverse Laplace transform of

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$$\mathcal{L}^{-1}\{F(s)\} = t - \underbrace{(t-3)\,\mathbf{step}(t-3)}_{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}}$$

Chopper Function

$$\mathbf{step}(t-a) - \mathbf{step}(t-b) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$$

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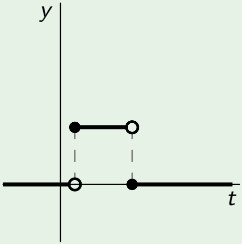
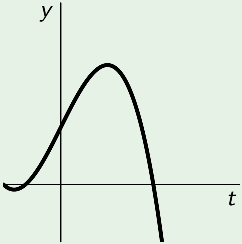


times



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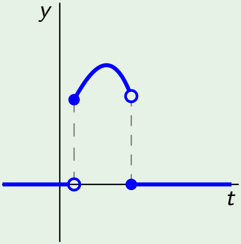

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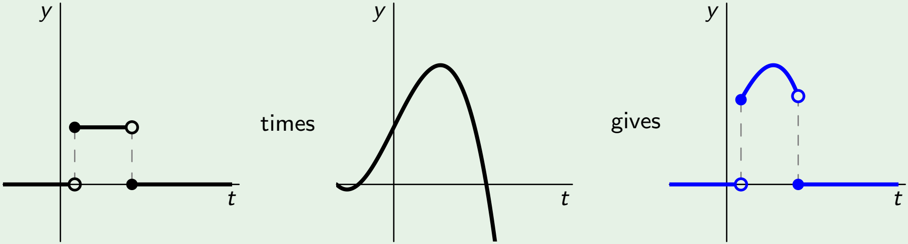

Laplace Transform of Chopper Function

$$\mathcal{L}\left\{f(t)\cdot[\textbf{step}(t-a)-\textbf{step}(t-b)]\right\} = e^{-as}\mathcal{L}\{f(t+a)\} - e^{-bs}\mathcal{L}\{f(t+b)\}$$

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$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ -\sin(\pi t) & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases}$$

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\end{aligned}$$

Example

Consider the IVP

$$x'' + x = f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases} \qquad \text{with } x(0) = 0,\ x'(0) = 0$$

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We can then use the Delay Theorem on the RHS

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So, we can use the Delay Theorem again to find $x(t)$.

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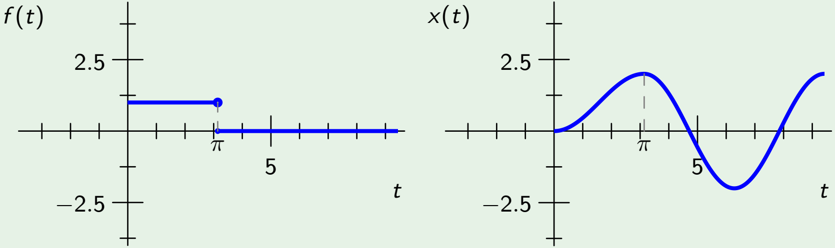


Physical systems often involved impulsive forces, which act over very short spans of time. To model these forces, the physicist Paul Dirac invented a "function-like" object.

Let us first look at a special function

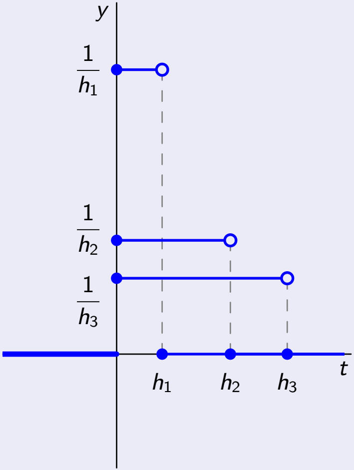
$$f_h(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{h} & \text{if } 0 \le t < h \\ 0 & \text{if } y \ge h \end{cases}$$

such that

$$\int_{-\infty}^{\infty} f_h(t)\, dt = 1$$

Dirac suggested that

$$\delta(t) = \lim_{b \to 0} f_h(t)$$



Dirac Delta Function

The **Dirac Delta function** or **unit impulse function** $\delta(t)$ is defined by two condtitions:

1. $$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \lim_{h\to 0}\left(\dfrac{1}{h}\right) & \text{if } t = 0 \end{cases}$$

2. $$\int_{-\infty}^{\infty} \delta(t)\ dt = 1$$

To find that Laplace transform of $\delta(t)$, we will first calculate the transform of $f_h(t)$.

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We can then use l'Hôpital's rule to find

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Laplace Transform of the Delta Function

$$\mathcal{L}\{\delta(t)\} = 1 \qquad \text{and} \qquad \mathcal{L}\{\delta(t-a)\} = e^{-as}$$