

Solving DEs and IVPs with Laplace Transforms

Department of Mathematics

Salt Lake Community College

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Consider the second-order IVP.

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Derivative Theorem for Laplace Transforms

If $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$, $f^{(n)}$ is piecewise continuous on $[0, \infty)$, and $f, f', \dots, f^{(n)}$ are of exponential order α , then for $s > \alpha$, and $n = 1, 2, \dots$

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

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$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

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Strategy to Solve DEs with Laplace Transforms

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- 2 Solve the algebraic problem for $Y(s)$.
- 3 Manipulating $Y(s)$ algebraically if necessary, use the inverse Laplace transform to transform $Y(s)$ into the IVP solution $y(t)$.

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Which means

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 4e^{-t} - 2e^{3t}$$

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$$Y(s) = \frac{s^2 + 2}{(s^2 + 1)(s^2 + 4)}$$

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$$\begin{aligned} Y(s) &= \frac{s^2 + 2}{(s^2 + 1)(s^2 + 4)} \\ &= \frac{\frac{1}{3}}{s^2 + 1} + \frac{\frac{2}{3}}{s^2 + 4} \end{aligned}$$

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$$s^2 Y(s) - 1 + 4Y(s) = \frac{1}{s^2 + 1}$$

$$(s^2 + 1)Y(s) = \frac{s^2 + 2}{s^2 + 1}$$

$$\begin{aligned} Y(s) &= \frac{s^2 + 2}{(s^2 + 1)(s^2 + 4)} \\ &= \frac{\frac{1}{3}}{s^2 + 1} + \frac{\frac{2}{3}}{s^2 + 4} \end{aligned}$$

Thus, the solution is

$$y(t) = \frac{1}{3} \sin(t) + \frac{1}{3} \sin(2t)$$

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Consider

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Translation Property for Multiplication by e^{at}

If the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a) \quad \text{for } s > a + \alpha$$

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Multiplication by t^n Rule for the Laplace Transform

If $f(t)$ is a piecewise continuous function on $[0, \infty)$ and is of exponential order α , then for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s) \quad \text{where } n \in \mathbb{N}^+$$

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This process can be repeated for an arbitrary n .

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