Systems of Linear Equations

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System of Linear Equations

A $m \times n$ **system of linear equations** is a set of m equations in n variables:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & \ldots & + & a_{3n}x_n & = & b_3 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m
\end{array}$$

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Matricies

A **matrix** is a rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

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An Augmented Matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

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A Matrix Equation (We will look at these in section 8.3)

As the matrix equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, where:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

Example

Consider the system of linear equations:

$$3x - 4y = -6$$
$$2x - 3y = -5$$

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Example

Consider the augmented matrix:

$$\left[\begin{array}{rr|r} 5 & 2 & 13 \\ -3 & 1 & -10 \end{array}\right]$$

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Example

Consider the augmented matrix:

$$\left[\begin{array}{rr|r} 5 & 2 & 13 \\ -3 & 1 & -10 \end{array}\right]$$

This matrix corresponds to the system of linear equations:

$$\begin{aligned} 5x + 2y &= 13 \\ -3x + y &= -10 \end{aligned}$$

Example

Consider the system of linear equations:

$$\begin{aligned} 2x - \ \ y + z &= 0 \\ x + \ \ z - 1 &= 0 \\ x + 2y - 8 &= 0 \end{aligned}$$

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The system must be in standard form before we can write the augmented matrix.

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The system must be in standard form before we can write the augmented matrix.

$$\begin{aligned} 2x - \ y + \ z &= 0 \\ x + 0y + \ z &= 1 \\ x + 2y + 0z &= 8 \end{aligned}$$

Thus, the augmented matrix is:

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 8 \end{array}\right]$$

Notation

- r_i denotes row i *before* the row operation is applied
- R_i denotes row i *after* the row operation is applied

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Elementary Row Operations

- Swap row i and row j:

$$R_i \leftrightarrow R_j \quad (\text{or } R_i = r_j,\ R_j = r_i)$$

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- Multiply row i by a nonzero constant:

$$R_i = c \cdot r_i$$

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Elementary Row Operations

- Swap row i and row j:

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- Multiply row i by a nonzero constant:

$$R_i = c \cdot r_i$$

- Add row j to row i (leaving row j unchanged):

$$R_i = r_i + r_j$$

Example

Let us apply the row operation $R_2 = -3r_1 + r_2$ to the matrix

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

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$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

We want to work one column at a time:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ & & \end{array}\right]$$

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We want to work one column at a time:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -3(1) & & \end{array}\right]$$

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$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -3(1)+3 & & \end{array}\right]$$

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We want to work one column at a time:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & & \end{array}\right]$$

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We want to work one column at a time:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & (-3)(-2) & \end{array}\right]$$

Example

Let us apply the row operation $R_2 = -3r_1 + r_2$ to the matrix

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

We want to work one column at a time:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & (-3)(-2)+(-5) & \end{array}\right]$$

Example

Let us apply the row operation $R_2 = -3r_1 + r_2$ to the matrix

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We want to work one column at a time:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & \end{array}\right]$$

Example

Let us apply the row operation $R_2 = -3r_1 + r_2$ to the matrix

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

We want to work one column at a time:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -3(2) \quad \end{array}\right]$$

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Let us apply the row operation $R_2 = -3r_1 + r_2$ to the matrix

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We want to work one column at a time:

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Example

Let us apply the row operation $R_1 = 2r_2 + r_1$ to the matrix

$$\left[\begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array}\right]$$

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We want to work one column at a time:

$$\left[\begin{array}{cc|c} 2(3) & & \\ 3 & 1 & 4 \end{array}\right]$$

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We want to work one column at a time:

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$$\left[\begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array}\right]$$

We want to work one column at a time:

$$\left[\begin{array}{cc|c} 8 & 2(1) + (-2) & \\ 3 & 1 & 4 \end{array}\right]$$

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$$\left[\begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array}\right]$$

We want to work one column at a time:

$$\left[\begin{array}{cc|c} 8 & 0 & 2(4) \\ 3 & 1 & 4 \end{array}\right]$$

Example

Let us apply the row operation $R_2 = -3r_1 + r_2$ to the matrix

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

We want to work one column at a time:

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Example

Let us apply the row operation $R_1 = 2r_2 + r_1$ to the matrix

$$\left[\begin{array}{cc|c} 2 & -2 & 1 \\ 3 & 1 & 4 \end{array}\right]$$

We want to work one column at a time:

$$\left[\begin{array}{cc|c} 8 & 0 & 2(4)+1 \\ 3 & 1 & 4 \end{array}\right]$$

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$$\left[\begin{array}{cc|c} 8 & 0 & 9 \\ 3 & 1 & 4 \end{array}\right]$$

Gaussian Elimination

Use row operations until in **Row Echelon Form**:

$$\left[\begin{array}{ccccc|c} 1 & c_{12} & c_{13} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & d_2 \\ 0 & 0 & 1 & \cdots & c_{3n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_m \end{array}\right]$$

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Then back solve the system:

$$\begin{aligned} x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n &= d_1 \\ x_2 + c_{23}x_3 + \cdots + c_{2n}x_n &= d_2 \\ &\vdots \\ x_n &= d_m \end{aligned}$$

Example

Consider the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ 2x & - & 3y & - & z & = & -8 \\ -x & + & 2y & + & 2z & = & 3 \end{array}$$

Example

Consider the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ 2x & - & 3y & - & z & = & -8 \\ -x & + & 2y & + & 2z & = & 3 \end{array}$$

We can write this as the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array}\right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array}\right] \quad R_2 = r_2 + 2r_3$$

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array}\right] \quad R_2 = r_2 + 2r_3$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ -1 & 2 & 2 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ -1 & 2 & 2 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ -1 & 2 & 2 & 3 \end{array}\right] \; R_3 = r_1 + r_3$$

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ -1 & 2 & 2 & 3 \end{array}\right] \; R_3 = r_1 + r_3$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 3 & 6 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 3 & 6 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 3 & 6 \end{array}\right] R_3 = r_3 - 3r_2$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 3 & 6 \end{array}\right] \quad R_3 = r_3 - 3r_2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{array}\right] \; R_3 = -\frac{1}{6}r_3$$

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$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{array}\right] \; R_3 = -\frac{1}{6}r_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation: $z = -2$

Example

Now, back solve the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation: $z = -2$
Plug it into the second equation and solve for y:

$$y + 3(-2) = -2 \quad \Rightarrow \quad y = 4$$

Example

Now, back solve the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 3 \\ & & y & + & 3z & = & -2 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation: $z = -2$
Plug it into the second equation and solve for y:

$$y + 3(-2) = -2 \quad \Rightarrow \quad y = 4$$

Plug both into the first equation and solve for x:

$$x + (4) + (-2) = 3 \quad \Rightarrow \quad x = 1$$

Example

Consider the system

$$\begin{array}{rcrcrcr} 2x & + & 2y & & & = & 6 \\ x & + & y & + & z & = & 1 \\ 3x & + & 4y & - & z & = & 13 \end{array}$$

Example

Consider the system

$$\begin{array}{rcrcrcr} 2x & + & 2y & & & = & 6 \\ x & + & y & + & z & = & 1 \\ 3x & + & 4y & - & z & = & 13 \end{array}$$

We can write this as the augmented matrix:

$$\left[\begin{array}{rrr|r} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

Example

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right] \quad \begin{array}{l} R_1 = r_2 \\ R_2 = r_1 \end{array}$$

Example

$$\left[\begin{array}{rrr|r} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right] \quad \begin{array}{l} R_1 = r_2 \\ R_2 = r_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right] \quad R_2 = -2r_1 + r_2$$

Example

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$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 3 & 4 & -1 & 13 \end{array}\right] \quad R_3 = -3r_1 + r_3$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 3 & 4 & -1 & 13 \end{array}\right] \quad R_3 = -3r_1 + r_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 = r_3 \\ R_3 = r_2 \end{array}$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 = r_3 \\ R_3 = r_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right] \quad R_3 = -\frac{1}{2}r_3$$

Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right] \quad R_3 = -\frac{1}{2}r_3$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation: $z = -2$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation: $z = -2$
Plug it into the second equation and solve for y:

$$y - 4(-2) = 10 \quad \Rightarrow \quad y = 2$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ & & y & - & 4z & = & 10 \\ & & & & z & = & -2 \end{array}$$

Start with the third equation: $z = -2$
Plug it into the second equation and solve for y:

$$y - 4(-2) = 10 \quad \Rightarrow \quad y = 2$$

Plug both into the first equation and solve for x:

$$x + (2) + (-2) = 1 \quad \Rightarrow \quad x = 1$$

Example

Consider the system

$$\begin{array}{rcrcrcr} 6x & - & y & - & z & = & 4 \\ -12x & + & 2y & + & 2z & = & -8 \\ 5x & + & y & - & z & = & 3 \end{array}$$

Example

Consider the system

$$\begin{array}{rcrcrcr} 6x & - & y & - & z & = & 4 \\ -12x & + & 2y & + & 2z & = & -8 \\ 5x & + & y & - & z & = & 3 \end{array}$$

We can write this as the augmented matrix:

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

Example

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \quad R_1 = -r_3 + r_1$$

Example

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \quad R_1 = -r_3 + r_1$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \quad R_2 = 12r_1 + r_2$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \quad R_2 = 12r_1 + r_2$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 5 & 1 & -1 & 3 \end{array}\right] R_3 = -5r_1 + r_3$$

Example

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 5 & 1 & -1 & 3 \end{array}\right] \; R_3 = -5r_1 + r_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right]$$

Example

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right] \quad R_2 = -\frac{1}{22} r_2$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right] \quad R_2 = -\frac{1}{22} r_2$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right]$$

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right] \; R_3 = -11r_2 + r_3$$

Example

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right] \quad R_3 = -11r_2 + r_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & - & 2y & & & = & 1 \\ & & y & - & \frac{1}{11}z & = & -\frac{2}{11} \\ & & & & 0 & = & 0 \end{array}$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & - & 2y & & & = & 1 \\ & & y & - & \frac{1}{11}z & = & -\frac{2}{11} \\ & & & & 0 & = & 0 \end{array}$$

This system of equations has an infinite number of solutions.

Example

Now, back solve the system

$$\begin{array}{ccccccc} x & - & 2y & & & = & 1 \\ & & y & - & \frac{1}{11}z & = & -\frac{2}{11} \\ & & & & 0 & = & 0 \end{array}$$

This system of equations has an infinite number of solutions.

For any choice of z, we can calculate values for x and y that work:

$$x = \frac{2}{11}z + \frac{7}{11}$$
$$y = \frac{1}{11}z - \frac{2}{11}$$

Example

Consider the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ 2x & - & y & - & z & = & 3 \\ x & + & 2y & + & 2z & = & 0 \end{array}$$

Example

Consider the system

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 6 \\ 2x & - & y & - & z & = & 3 \\ x & + & 2y & + & 2z & = & 0 \end{array}$$

We can write this as the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

Example

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$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right] \; R_2 = -2r_1 + r_2$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

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Example

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array}\right] \begin{array}{l} \\ R_2 = r_3 \\ R_3 = r_2 \end{array}$$

Example

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$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & -27 \end{array}\right]$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ & & y & + & z & = & -6 \\ & & & & 0 & = & -27 \end{array}$$

Example

Now, back solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ & & y & + & z & = & -6 \\ & & & & 0 & = & -27 \end{array}$$

This system of equations has no solutions.

Existence and Uniqueness of Solutions

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- If a system has at least one solution, it is called **consistent**.
  - A system with exactly one solution is called **independent**.
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