The Harmonic Oscillator

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Newton's Dot Notation

Scientists and Engineers who work with many variables where the independent variables is always t commonly use dot notation:

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{dx^2}{d^2t} \quad \text{and} \quad \dddot{x} = \frac{dx^3}{d^3t} \quad \text{and} \quad \ddddot{x} = \frac{dx^4}{d^4t}$$

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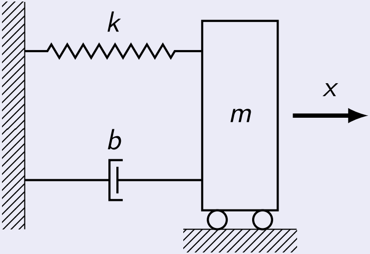
Definition 1

A very important DE is the second-order homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

where $m > 0$, b, and k are constants.
This models a class of phenomena called **damped harmonic oscillation**.

The Mass-Spring System



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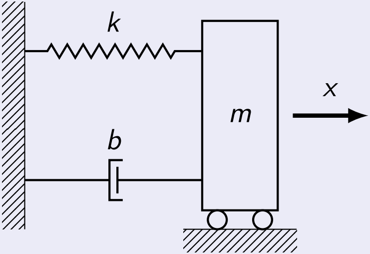

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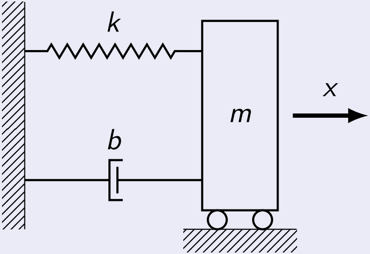

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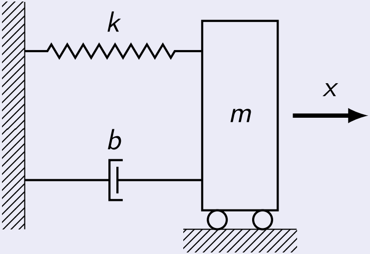

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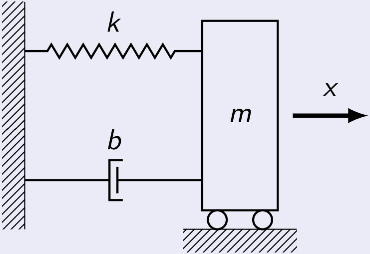

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Summing these forces gives:

$$\begin{array}{rlll} \text{mass} \times \text{acceleration} & = F_{\text{restoring}} & + F_{\text{damping}} & + F_{\text{external}} \\ m\ddot{x} & = -kx & \text{-}\ b\dot{x} & + f(t) \end{array}$$

Simple Harmonic Oscillator

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- When $b = 0$, the motion is called **undamped**; otherwise it is **damped**.
- If $f(t) = 0$ for all t, then the equation is homogeneous:

  $$m\ddot{x} + b\dot{x} + kx = 0$$

  and the motion is called **unforced**, **undriven**, or **free**; otherwise it is called **forced** or **driven**.

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We also measure the damping force of the object sliding on the table to be 0.5 newtons when the velocity is 0.25 meters per second.

$$b = \frac{0.5 \text{ newton}}{0.25 \; \frac{\text{meter}}{second}} = 2 \; \frac{\text{newton second}}{\text{meter}}$$

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Notice that a second-order DE requires **two** initial conditions.

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Gravity (Earth)	$9.8 \ \frac{\text{m}}{\text{s}^2}$	$980.665 \ \frac{\text{cm}}{\text{s}^2}$	$32 \ \frac{\text{ft}}{\text{s}^2}$

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Comparing

$$(\sin(\omega_0 t))'' = -\omega_0^2 \sin(\omega_0 t) \quad \text{and} \quad \ddot{x} = -\frac{k}{m}x$$

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Note

Another solutions is $x(t) = \cos(\omega_0 t)$.

Solution of the Undamped Unforced Oscillator

For the undamped unforced oscillator

$$m\ddot{x} + kx = 0$$

we know two solutions:

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The Superposition Principle tells us that any linear combination of these two solutions is itself a solution. Thus, for $c_1, c_2 \in \mathbb{R}$, the family of solutions is

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Note

We will see next section that these cover all solutions.

Alternate Form of the Undamped Unforced Oscillator Solution

Solutions to the undamped unforced oscillator may also be expressed as

$$x(t) = A\cos(\omega_0 t - \delta)$$

- A is the **amplitude**.
- δ is the **phase angle**, measured in radians.
- The motion has **circular frequency** $\omega_o = \sqrt{\frac{k}{m}}$, measured in $\frac{\text{radians}}{\text{second}}$.
- The **natural frequency** is $f_0 = \frac{\omega_0}{2\pi}$.
- The **period** is $\frac{1}{f_0} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$, measured in seconds.
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Converting Between the Two Forms

The translation is given by

$$A = \sqrt{c_1^2 + c_2^2}, \qquad \tan(\delta) = \frac{c_2}{c_1}$$

and

$$c_1 = A\cos(\delta), \qquad c_2 = A\sin(\delta)$$

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$$\ddot{x} + x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

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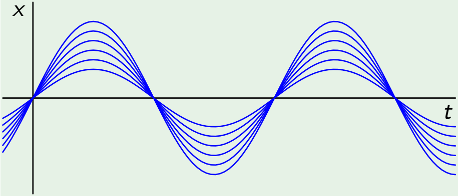
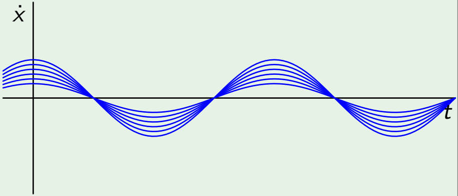
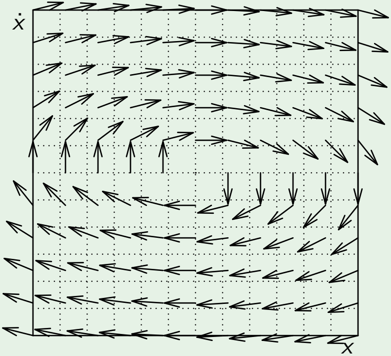
Substituting $t = 0$, $x(0) = 0$, and $\dot{x}(0) = 1$ into this system gives the solution $c_1 = 0$ and $c_2 = 1$.

Example 5

Let us look at some plots concerning $\ddot{x} + 0.25x = 0$:







Phase Portraits

For any autonomous second-order differential equation

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the **phase plane** is the two-dimensional graph with x and $\dot{x}$ axes.

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The phase plane has a **vector field** specified by the DE, which at any point in the phase plane gives a direction vector with

$$\begin{aligned} \text{horizontal component} \quad & dx/dt = \dot{x} \\ \text{vertical component} \quad & d\dot{x}/dt = \ddot{x} \end{aligned}$$

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Note

Phase portraits can be graphed *without* solving the DE.

Definition 6

The second-order differential equation

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

is equivalent to the system of first-order equations:

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The second-order differential equation

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

is equivalent to the system of first-order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \ddot{x} = \frac{f(t)}{m} - \frac{k}{m}x - \frac{b}{m}y \end{aligned}$$

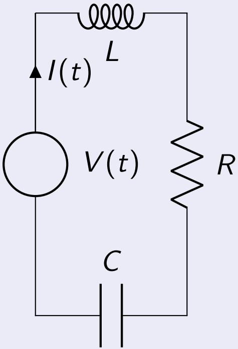
Example 7

To draw the vector field for $\ddot{x} + 0.25x = 0$ we can build the system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 0.25x \end{aligned}$$

Then, pplane may be used to plot the phase portrait.

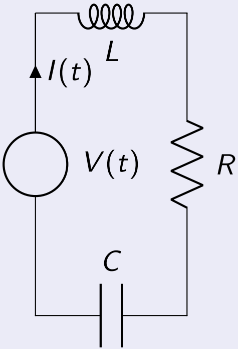
Electrical Circuits



The current I in a wire, measured in *amperes*, is the flow of charge Q. That is, the current is the rate of change of the charge

$$I(t) = \dot{Q}(t)$$

Electrical Circuits



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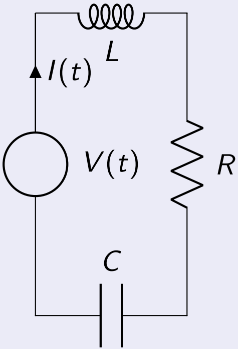
Kirchoff's Voltage Law tell us that the input voltage $V(t)$ is the sum of voltage drops around the circuit. In our circuit, we have three such voltage drops.

Electrical Circuits

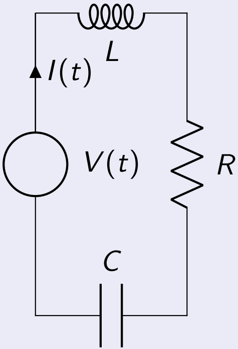


Drop across a Resistor: By **Ohm's Law**, the voltage drop across a resistor is proportional to the current passing through it.

$$V_R(t) = R \cdot I(t)$$

Where R is the **resistance** of the resistor and is measured in *ohms*.

Electrical Circuits

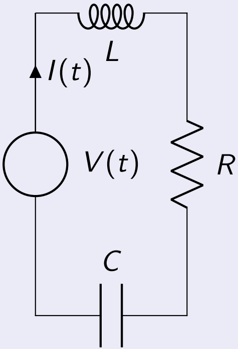


Drop across an Inductor: By Faraday's Law, the voltage drop across an inductor is proportional to the time rate of change of the current passing through it.

$$V_L(t) = L \cdot \dot{I}(t)$$

where L is the **inductance** and is measured in *henries*.

Electrical Circuits

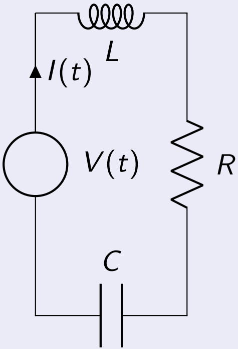


Drop across a Capacitor: The voltage drop across a capacitor is proportional to the charge $Q(t)$ on the capacitor.

$$V_C(t) = \frac{1}{C}Q(t) = \frac{1}{C}\int I(t)dt$$

where C is the **capacitance** of the capacitor and is measured in *farads*.

Electrical Circuits



Thus, the voltage drop across the circuit is

$$V(t) = R \cdot I + L \cdot \dot{I} + \frac{1}{C} \int I(t)dt$$

This is called an **integro-differential equation** because it contains both a derivative and an integral.

Using the fact that $I(t) = \dot{Q}(t)$ we can build the following equations.

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Series Circuit Equation (Charge)

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)$$

If there is no voltage source $(V(t) = 0)$, then

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0$$

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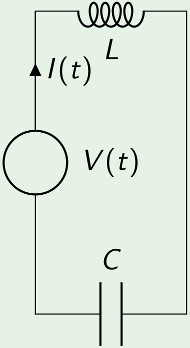
Series Circuit Equation (Current)

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}(t)$$

If there is no voltage source ($V(t) = 0$), then

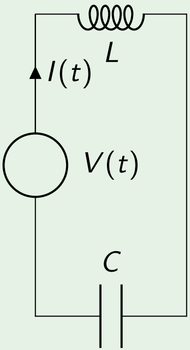
$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0$$

Example 8



Consider a circuit composed of a capacitor and inductor hooked up in series. Suppose that at $t = 0$ a charge Q_0 is put on the capacitor.

Example 8



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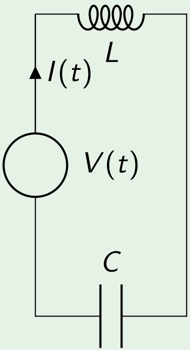
The IVP is

$$L\ddot{Q} + \frac{1}{C} = 0, \quad Q(0) = Q_0, \quad \dot{Q}(0) = 0$$

Example 8



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The IVP is

$$L\ddot{Q} + \frac{1}{C} = 0, \quad Q(0) = Q_0, \quad \dot{Q}(0) = 0$$

Thus, the solution is

$$Q(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

where

$$\omega_0 = \sqrt{\frac{1}{LC}}$$