Variation of Parameters

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

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where c_1 and c_2 are arbitrary constants.

Just like with single order equations, we want to perturb the homogeneous solution into a particular solution to the nonhomogeneous DE.

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We do so by replacing the constants c_1 and c_2 with unknown functions.

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$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$

So, we can choose $v_1 y_1' + v_2 y_2' = 0$ as our auxiliary condition, which reduces y_p' to:

$$y_p' = v_1' y_1 + v_2' y_2$$

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We can then obtain

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$$y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

We then substitute y_p, y_p', and y_p'' into $L(y) = f$.

$$(v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2') + p \cdot (v_1' y_1 + v_2' y_2) + q \cdot (v_1 y_1 + v_2 y_2) = f$$

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So, we have the system

$$\begin{aligned}
v_1' y_1' + v_2' y_2' &= f \\
v_1' y_1 + v_2' y_2 &= 0
\end{aligned}$$

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Using Cramer's Rule, the system

$$\begin{aligned} v_1' y_1' + v_2' y_2' &= f \\ v_1' y_1 + v_2' y_2 &= 0 \end{aligned}$$

has solution

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

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The denominator is just the Wronskian $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$, because y_1 and y_2 are linearly independent.

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Thus, we can integrate to find v_1 and v_2.

$$v_1 = -\int \frac{y_2 f}{W(y_1, y_2)} \quad \text{and} \quad v_2 = \int \frac{y_1 f}{W(y_1, y_2)}$$

Example 1

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$$y'' + y = \sec(t) \quad |t| < \frac{\pi}{2}$$

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So,

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Note

This method can be extended to higher orders.

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So, using the Cramer's Rule formulas from before

$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} = -\frac{t}{t^2+1} \quad \text{and} \quad v_2' = \frac{y_1 f}{W(y_1, y_2)} = \frac{1}{t^2+1}$$

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$$v_1 = -\tfrac{1}{2} \ln\left(t^2+1\right) \quad \text{and} \quad v_2 = \tan^{-1}(t)$$

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Thus, the general solution is

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So, using Cramer's Rule

$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} = -\frac{\ln(t)}{t} \quad \rightarrow v_1 = -\frac{1}{2}\ln^2(t)$$
$$v_2' = \frac{y_1 f}{W(y_1, y_2)} = \frac{\ln(t)}{t^2} \qquad \rightarrow v_2 = -\frac{\ln(t) + 1}{t}$$

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$$t^2y'' - 2ty' + 2y = t\ln(t), \qquad t > 0$$
$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{\ln(t)}{t}$$

Two solutions of $L(y) = 0$ are $y_1 = t$ and $y_2 = t^2$.

The Wronskian is $W(y_1, y_2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$

So, using Cramer's Rule

$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} = -\frac{\ln(t)}{t} \quad \rightarrow v_1 = -\frac{1}{2}\ln^2(t)$$
$$v_2' = \frac{y_1 f}{W(y_1, y_2)} = \frac{\ln(t)}{t^2} \qquad \rightarrow v_2 = -\frac{\ln(t) + 1}{t}$$

The general solution is

$$y = c_1 t + c_2 t^2 - \tfrac{t}{2}\ln^2(t) - t\ln(t) - t$$