The Step Function and the Delta Function

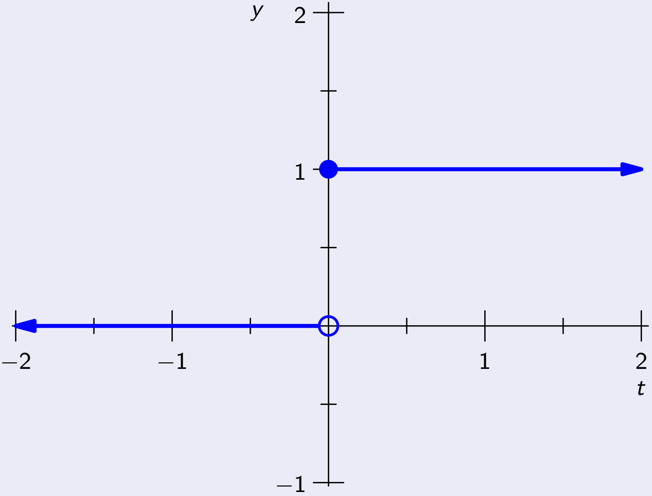
Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

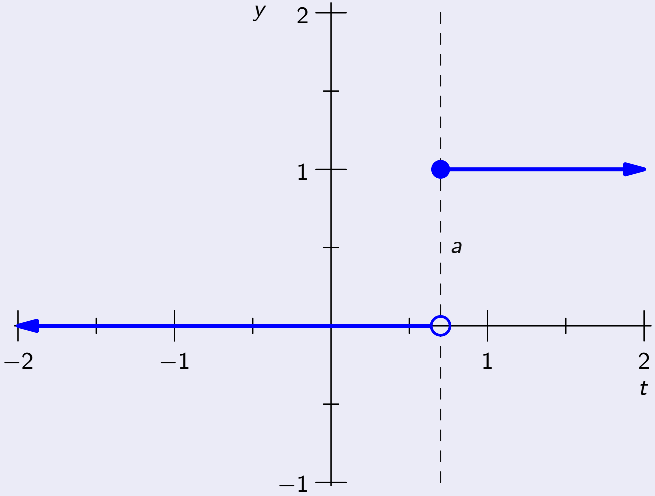
The Unit Step Function

$$\textbf{step}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



The Translated Step Function

$$\mathbf{step}(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



Laplace Transform of the Step Function

$$\mathcal{L}\{\textbf{step}(t-a)\} = \frac{e^{-as}}{s}$$

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&= \frac{e^{as}}{s}
\end{aligned}$$

Example 1

Consider

$$f(t) = \begin{cases} 2 & \text{if } t < 3 \\ -4 & \text{if } 3 \le t < 4 \\ 1 & \text{if } t \ge 4 \end{cases}$$

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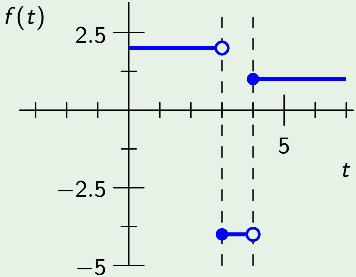

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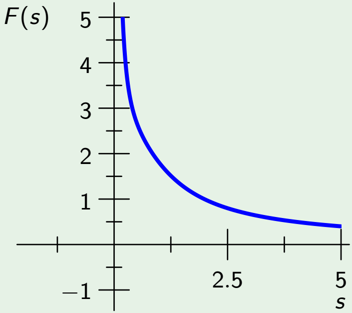

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$$g(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^2 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

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Which means

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&= \int_0^1 t^2\ e^{-st}\ dt + \int_1^\infty e^{-st}\ dt \\
&= \frac{2}{s} - e^{-st}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}\right) + \frac{1}{s}e^{-s}
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&= \frac{2}{s} - e^{-st} \left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} \right) + \frac{1}{s} e^{-s} \\
&= \frac{2}{s^2} - 2e^{-s} \left(\frac{1}{s^2} + \frac{1}{s^3} \right)
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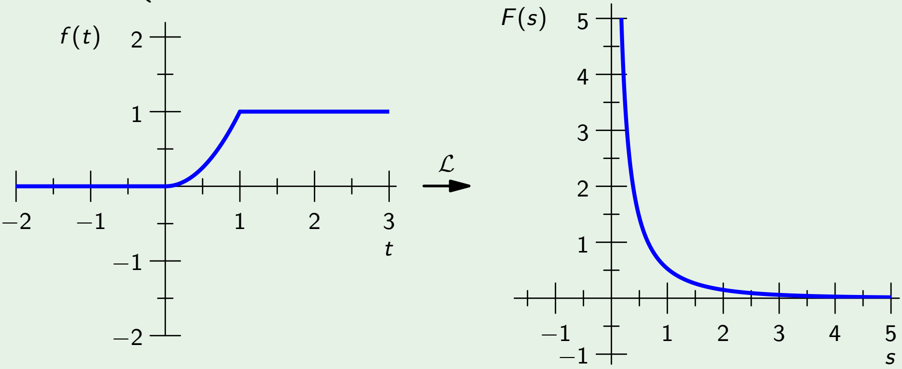

Delayed Function

For a given function $g(t)$, the **delayed function**

$$f(t) = \begin{cases} 0 & \text{if } t < c \\ g(t-c) & \text{if } t \geq c \end{cases}$$

shifts $g(t)$ to the right c units from the origin, and replaces it by zero to the left of $t = c$. Using the unit step function, the delayed function can also be written

$$f(t) = g(t-c)\,\textbf{step}(t-c)$$

Example 3

Consider the function $g(t) = \sqrt{t}$, which has the delayed function

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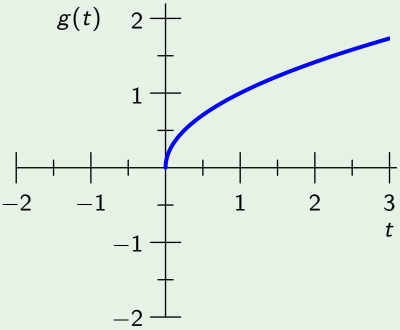

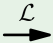

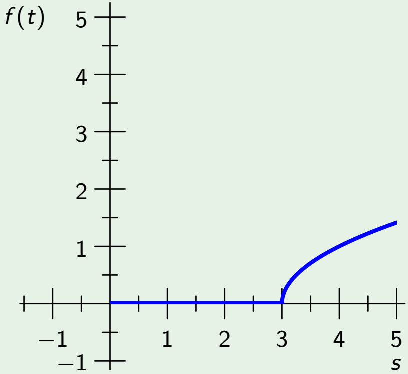

Closer Look

Consider the Laplace transform of a function $f(t)$ that is delayed c units.

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We may assume $b > c$, since $b \to \infty$.

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Furthermore, $\mathbf{step}(t-c) = 0$ for $t < c$ and $\mathbf{step}(t-c) = 1$ for $t \geq c$.

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Delay Theorem (or Shifting Theorem)

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If we let $c = 1$ and $g(t) = t^2$, then by the Delay theorem we have

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&= e^{-s}\left(\frac{2}{s^2} + \frac{2}{s^2} + \frac{1}{s}\right)
\end{aligned}$$

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Let us find the inverse Laplace transform of

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We can treat $\dfrac{e^{-3s}}{s^2}$ as the transform of a delay function.

$$\mathcal{L}^{-1}\{F(s)\} = t - \underbrace{(t-3)\,\mathbf{step}(t-3)}_{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}}$$

Chopper Function

$$\textbf{step}(t-a) - \textbf{step}(t-b) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a \le t < b \\ 0 & \text{if } t \ge b \end{cases}$$

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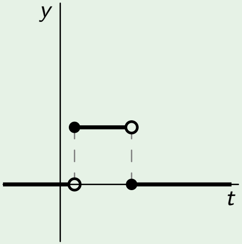
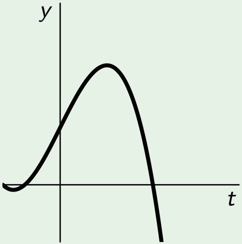
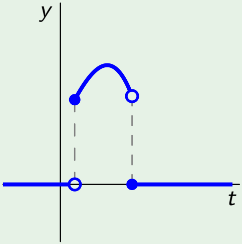
Example 6



times



gives



Chopper Function

$$\textbf{step}(t-a)-\textbf{step}(t-b)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } a\le t<b\\ 0 & \text{if } t\ge b\end{cases}$$

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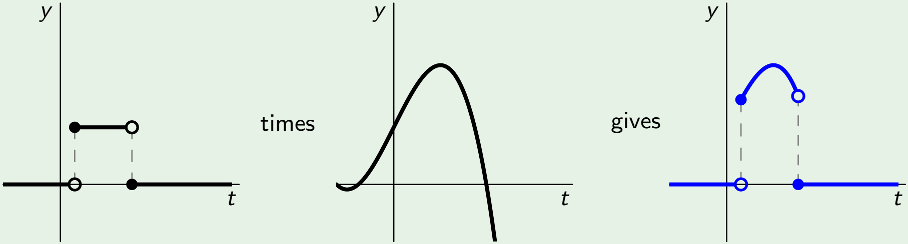

Laplace Transform of Chopper Function

$$\mathcal{L}\left\{f(t)\cdot[\textbf{step}(t-a)-\textbf{step}(t-b)]\right\}=e^{-as}\mathcal{L}\{f(t+a)\}-e^{-bs}\mathcal{L}\{f(t+b)\}$$

Example 7

Let us find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ -\sin(\pi t) & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases}$$

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$$= -\sin(\pi t) \cdot [\mathbf{step}(t-1) - \mathbf{step}(t-2)]$$

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Thus,

$$\mathcal{L}\{f(t)\} = -e^{-s}\mathcal{L}\{-\sin(\pi(t+1))\} + e^{2s}\mathcal{L}\{\sin(\pi(t+2))\}$$

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$$+ e^{2s}\mathcal{L}\{\sin(\pi t)\cos(2\pi) + \cos(\pi t)\sin(2\pi)\}$$

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$$\begin{aligned} \mathcal{L}\{f(t)\} &= -e^{-s}\mathcal{L}\{-\sin(\pi(t+1))\} + e^{2s}\mathcal{L}\{\sin(\pi(t+2))\} \\ \mathcal{L}\{f(t)\} &= -e^{-s}\mathcal{L}\{-\sin(\pi t)\cos(\pi) - \cos(\pi t)\sin(\pi)\} \\ &\quad + e^{2s}\mathcal{L}\{\sin(\pi t)\cos(2\pi) + \cos(\pi t)\sin(2\pi)\} \\ &= -e^{-s}\mathcal{L}\{\sin(\pi t)\} + e^{2s}\mathcal{L}\{\sin(\pi t)\} \end{aligned}$$

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&= -e^{-s}\mathcal{L}\{\sin(\pi t)\} + e^{2s}\mathcal{L}\{\sin(\pi t)\} \\
&= \mathcal{L}\{\sin(\pi t)\}(e^{-s} + e^{-2s}) \\
&= \frac{\pi}{s^2+\pi^2}(e^{-s} + e^{-2s})
\end{aligned}$$

Example 8

Consider the IVP

$$x'' + x = f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases} \qquad \text{with } x(0) = 0,\ x'(0) = 0$$

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We can rewrite this DE using a step function

$$x'' + x = 1 - \textbf{step}(t - \pi) \qquad \text{with } x(0) = 0,\ x'(0) = 0$$

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$$s^2 X(s) + X(s) = \mathcal{L}\{1 - \textbf{step}(t - \pi)\}$$

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$$s^2 X(s) + X(s) = \mathcal{L}\{1 - \textbf{step}(t - \pi)\}$$

We can then use the Delay Theorem on the RHS

$$s^2 X(s) + X(s) = \frac{1}{s} + \frac{e^{-\pi s}}{s}$$

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We can now solve for $X(s)$. (As well as rearrange for the inverse.)

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$$\begin{aligned} s^2 X(s) + X(s) &= \frac{1}{s} + \frac{e^{-\pi s}}{s} \\ (s^2+1)X(s) &= \frac{1 - e^{-\pi s}}{s} \end{aligned}$$

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So, we can use the Delay Theorem again to find $x(t)$.

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = (1 - \cos(t)) - (1 - \cos(t - \pi))\,\textbf{step}(t - \pi)$$

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Which, when written as a piecewise function gives

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$$\begin{aligned} x(t) &= \begin{cases} 1 - \cos(t) & \text{if } 0 \le t < \pi \\ 1 - \cos(t) - (1 - \cos(t - \pi)) & \text{if } t \ge \pi \end{cases} \\ &= \begin{cases} 1 - \cos(t) & \text{if } 0 \le t < \pi \\ -2\cos(t) & \text{if } t \ge \pi \end{cases} \end{aligned}$$

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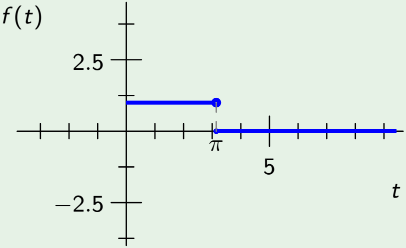
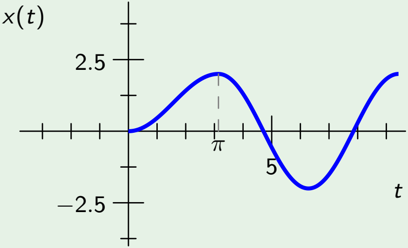
Consider the IVP

$$x'' + x = f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases} \qquad \text{with } x(0) = 0,\ x'(0) = 0$$





Physical systems often involve impulsive forces, which act over very short spans of time. To model these forces, the physicist Paul Dirac invented a "function-like" object.

Let us first look at a special function

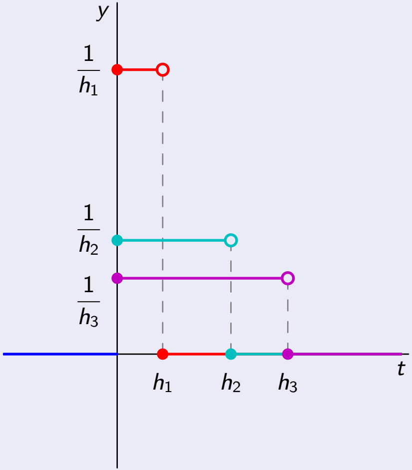
$$f_h(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{h} & \text{if } 0 \le t < h \\ 0 & \text{if } y \ge h \end{cases}$$

such that

$$\int_{-\infty}^{\infty} f_h(t)\; dt = 1$$

Dirac suggested that

$$\delta(t) = \lim_{b \to 0} f_h(t)$$



Dirac Delta Function

The **Dirac Delta function** or **unit impulse function** $\delta(t)$ is defined by two condtitions:

1. $$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \lim_{h\to 0}\left(\frac{1}{h}\right) & \text{if } t = 0 \end{cases}$$

2. $$\int_{-\infty}^{\infty} \delta(t)\; dt = 1$$

Finding the Laplace Transform

To find that Laplace transform of $\delta(t)$, we will first calculate the transform of $f_h(t)$.

$$\mathcal{L}\{f_h(t)\} = \int_0^\infty e^{-st} f_h(t)\, dt$$

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&= \int_0^h e^{-st} f_h(t)\ dt \\
&= \frac{1}{h}\int_0^h e^{-st}\ dt \\
&= \frac{1-e^{-hs}}{hs}
\end{aligned}$$

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We can then use l'Hôpital's rule to find that

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Laplace Transform of the Delta Function

$$\mathcal{L}\{\delta(t)\} = 1 \qquad \text{and} \qquad \mathcal{L}\{\delta(t-a)\} = e^{-as}$$