Point Estimates and Sampling Variability

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Definition

A **point estimate** is a single value used to estimate a parameter.

Note

The sample proportion, $\hat{p}$, the best point estimate of the population proportion p. But, is it a *good* estimate?

Definition

The difference between a point estimate and the true parameter is called the **error** in the estimate.

Note

In general, there are two sources of error: sampling error and bias.

Definition

Sampling error describes how much an estimate will tend to vary from one sample to the next.

Example 1

One sample may have $\hat{p} = 1\%$ and another sample may have $\hat{p} = 3\%$.

Note

Much of statistics is focused on understanding and quantifying sampling error.

Definition

Bias describes a systematic tendency to over-estimate or under-estimate the true population value.

Example 2

If a university took a student poll asking about support for a new stadium, they'd get a biased response if they asked:

"Do you support your school by supporting funding for the new stadium?"

Note

We try to minimize bias by using thoughtful data collection procedures.

Example 3

Suppose the proportion of American adults who support the expansion of solar energy is $p = 0.88$.

If we take a poll of 1000 American adults on this topic, the estimate would not be perfect. But, how close can we expect $\hat{p}$ to be to p?

We can simulate such a sample:

1. As of 2021, there are about 258 million adults in America. Let us get 258 million slips of paper and write "support" on 88% of them and "not" on the remaining 12%.
2. Mix up the slips and pull out 1000, to represent our sample.
3. Compute the fraction of the sample that says "support".

Note

While this method seems silly, a compute can do these steps in a short amount of time.

Example 4

I wrote a short program to run this simulation, but one simulation isn't enough to get a sense of the distribution of the point estimates.

So, I ran nine simulations:

$\hat{p}$	Error	$\hat{p}$	Error	$\hat{p}$	Error
0.867	-0.013	0.876	-0.004	0.883	0.003
0.889	0.009	0.887	0.007	0.874	-0.006
0.896	0.016	0.898	0.018	0.874	-0.006

Notice that they are all kinda close to $p = 0.88$, but there is variation.

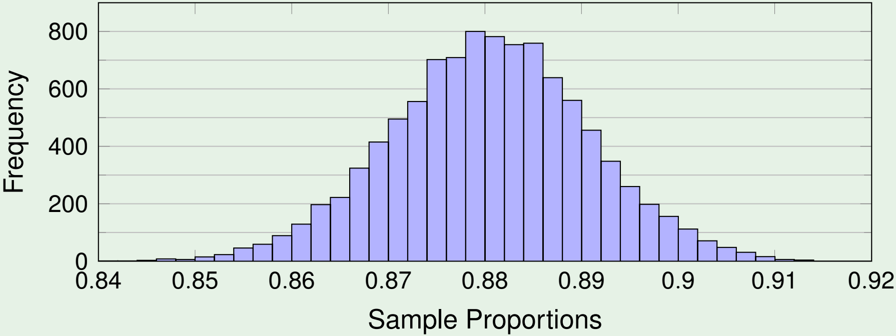
The mean of all these $\hat{p}$ values is 0.8827, which is pretty close to p.

Definition

The **sampling distribution** is the distribution of sample proportions.

Example 5

Running the simulation 10,000 times gives the histogram:



Center : The center of this distribution is $\bar{x}_{\hat{p}} = 0.8799$, which is very close to $p = 0.88$.

Spread : The standard deviation of this distribution is $s_{\hat{p}} = 0.0102$. This is often called the **standard error**.

Shape : This distribution is approximately normal.

Example 6

What if we used a much smaller sample size of $n = 50$?

- **Center**: The center of this distribution is $\bar{x}_{\hat{p}} = 0.8791$, which is still very close to $p = 0.88$.
- **Spread**: The standard deviation of this distribution is $s_{\hat{p}} = 0.0462$, which is much bigger.

Note

This highlights an important property: a bigger sample tends to provide a more precise point estimate than a smaller sample.

Note

In real-world applications, we never actually observe the sampling distribution, yet it is useful to always think of a point estimate as coming from a hypothetical distribution.

Central Limit Theorem

When observations are independent and the sample size, n, is sufficiently large, the sample proportions $\hat{p}$ will tend to follow a normal distribution with the following mean and standard deviation:

$$\mu_{\hat{p}} = p \qquad \text{and} \qquad \sigma_{\hat{p}} = SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

In order for the Central Limit Theorem to hold, the sample size is typically considered sufficiently large when

$$np \geq 10 \qquad \text{and} \qquad n(1-p) \geq 10$$

which is called the **success-failure conditions**.

Note

The Central Limit Theorem is incredibly important, and provides a foundation for much of statistics.

Example 7

In Example 3, we had a sample size of $n = 1000$ and $p = 0.88$.

Before we can apply the Central Limit Theorem, we need to check the success-failure conditions:

$$np = 1000 \cdot 0.88 = 880 \geq 10 \checkmark$$
$$n(1-p) = 1000(1-0.88) = 1000 \cdot 0.12 = 120 \geq 10 \checkmark$$

Applying the Central Limit Theorem gives:

$$\mu_{\hat{p}} = p = 0.88$$
$$SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.88(1-0.12)}{1000}} = 0.0103$$

This is very close to the observed standard error, 0.0102.

How to Verify Sample Observations are Independent

- Subjects in an experiment are considered independent if they undergo random assignment to the treatment groups.
- If the observations are from a simple random sample, then they are independent.
- If a sample is from a seemingly random process, e.g. an occasional error on an assembly line, checking independence is more difficult. In this case, use your best judgment.

Note

If a sample is larger than 10% of the population, the methods we discuss tend to overestimate the sampling error slightly. In these cases more advanced methods are needed.

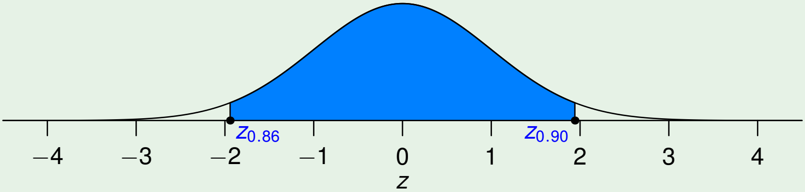
Example 8

Using $n = 1000$ and $p = 0.88$ from Example 3, let us find out how often $\hat{p}$ is within 0.02 (2%) of the population value $p = 0.88$.

In Example 7, we applied the Central Limit Theorem, getting $\mu_{\hat{p}} = 0.88$ and $SE_{\hat{p}} = 0.0103$.

We start by calculating the z-values:

$$z_{0.86} = \frac{0.86 - 0.88}{0.0103} = -1.942 \quad \text{and} \quad z_{0.90} = \frac{0.90 - 0.88}{0.0103} = 1.942$$



Using technology gives:

$$P(-1.94175 \leq z \leq 1.94175) = 0.947833$$

We expect $\hat{p}$ to be within 0.02 of 0.88 about 94.78% of the time.

Example 9

We do not actually know the population proportion unless we conduct a full census of the entire population.

The value $p = 0.88$ was based on a Pew Research poll of 1000 adults that found $\hat{p} = 0.887$ of them favored expanding solar energy.

A question the researchers might have asked is:

"Does the sample proportion from the poll approximately follow a normal distribution?"

Independence Pew Research is a well known non-profit think tank, so we can believe that the poll is a simple random sample, and hence the observations are independent.

Success-Failure Conditions Since we don't actually know p, the next best thing we have is $\hat{p}$.

$$n\hat{p} = 1000 \cdot 0.887 = 887 \geq 10 \checkmark$$

$$n(1 - \hat{p}) = 1000(1 - 0.887) = 1000 \cdot 0.113 = 113 \geq 10 \checkmark$$

Because $n\hat{p}$ and $n(1 - \hat{p})$ are both well above 10, we can conclude that $\hat{p}$ is a reasonable substitute for p.

Substitution Approximation

When np and $n(1-p)$ are much larger than 10, we can use $\hat{p}$ in place of p and the Centeral Limit Theorem becomes:

$$\mu_{\hat{p}} = p \approx \hat{p}$$

$$SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Example 10

For $n = 1000$, $p - 0.88$, $\hat{p} = 0.887$:

$$SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.88(1-0.88)}{1000}} = 0.010276$$

$$SE_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.887(1-0.887)}{1000}} = 0.010012$$

These values are the same to three decimal places, so using the substitution approximation won't make a major difference.

Trends in the Sampling Distribution

- The smaller either np or $n(1-p)$ is, the more discrete the sampling distribution is.
- When np or $n(1-p)$ is smaller than 10, the skew in the distribution is more pronounced.
  - If p is close to 0, the distribution will be more right skewed.
  - If p is close to 1, the distribution will be more left skewed.
  - if p is close to 0.5, the distribution will be more symmetric.
- The larger both np and $n(1-p)$ are, the more normal the sampling distribution is.
  - This may be harder to see for larger sample sizes, as the variability also becomes smaller.
- When np and $n(1-p)$ are both very large, the distributions discreteness is hardly evident, and the distribution looks much more like a normal distribution.