Systems of Linear Equations

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Linear Systems of Equations

System of Linear Equations

A $m \times n$ **system of linear equations** is a set of m equations in n variables $x_1, x_2, \ldots, x_n$ of the form

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & \ldots & + & a_{3n}x_n & = & b_3 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m
\end{array}$$

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A Matrix Equation (We will look at these in section 3.3)

As the matrix equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, where:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{\boldsymbol{b}}}$$

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- Add row j to row i (leaving row j unchanged):

$$R_i = r_i + r_j$$

Gaussian Elimination

Use row operations until in REF:

$$\left[\begin{array}{ccccc|c} 1 & c_{12} & c_{13} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & d_2 \\ 0 & 0 & 1 & \cdots & c_{3n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_m \end{array}\right]$$

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Then back solve the system:

$$\begin{aligned} x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n &= d_1 \\ x_2 + c_{23}x_3 + \cdots + c_{2n}x_n &= d_2 \\ &\vdots \\ x_n &= d_m \end{aligned}$$

Example

Consider the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ 2x & - & 3y & - & z & = & -8 \\ -x & + & 2y & + & 2z & = & 3 \end{array}$$

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We can write this as the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array}\right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

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Plug it into the second equation and solve for y:

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Plug both into the first equation and solve for x:

$$x + (4) + (-2) = 3 \quad \Rightarrow \quad x = 1$$

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Matrix Rank

Reduced Row Echelon Form

An augmented matrix is said to be in **Reduced Row Echelon Form** if:

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Rank

The **rank** r of a matrix is equal to how many 1's are in the diagonal of it's Reduced Row Echelon Form.

- If r equals the number of variables, there is a unique solution.
- If r is less than the number of variables, the solutions are not unique.