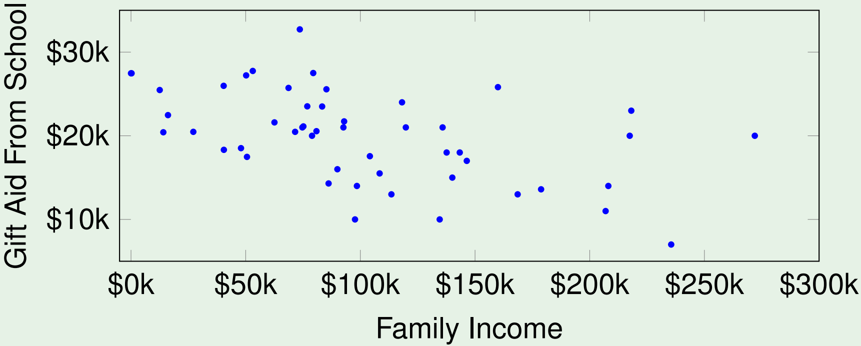
Least Squares Regression

Colby Community College

Example 1

Gift aid is financial aid that does not need to be paid back.

A sample of 50 random freshmen at Elmhurst College is shown, comparing the student's family income against gift aid received.



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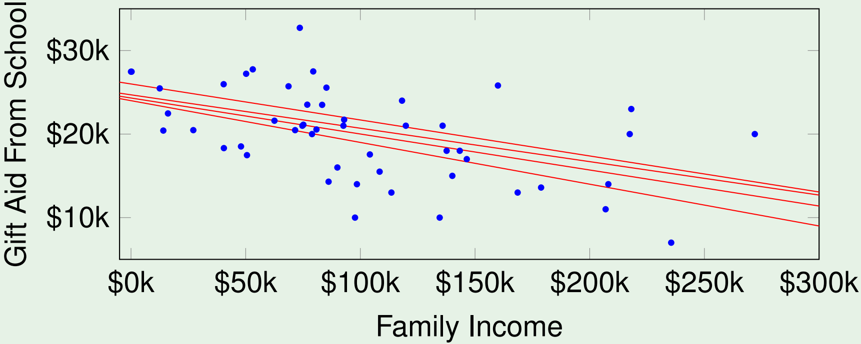

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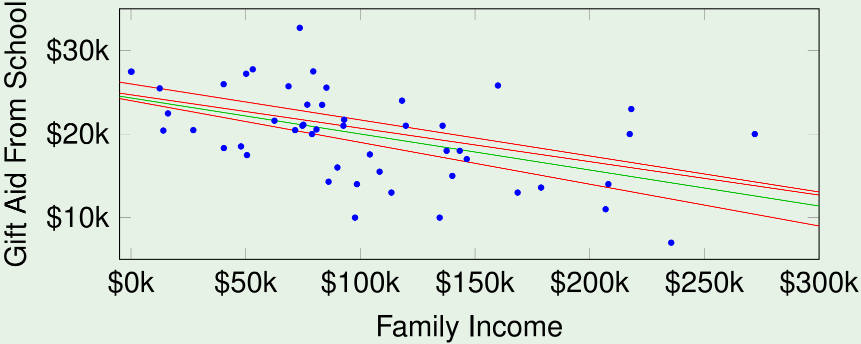

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Without an objective definition of measure of "best", the answer will vary from person to person.

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Note

In many applications, a residual twice as large as another is more than twice as bad. Squaring the residuals helps account for this discrepancy.

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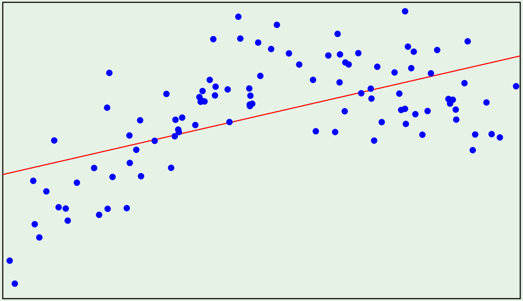
Independent Observations: Be careful about applying regression to **time series** data, which are sequential observations in time such as a stock price each day.

Example 2

Scatter plot where the linearity condition fails:



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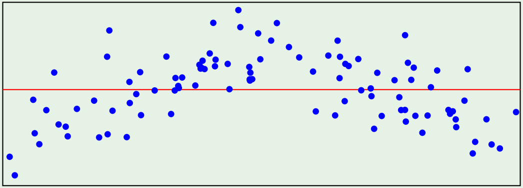
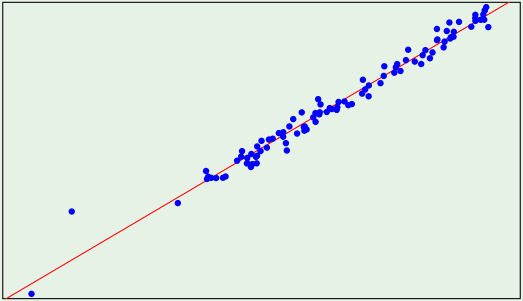


Example 3

Scatter plot where there are clear outliers:



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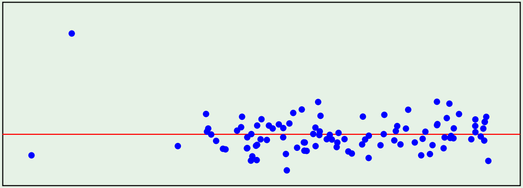
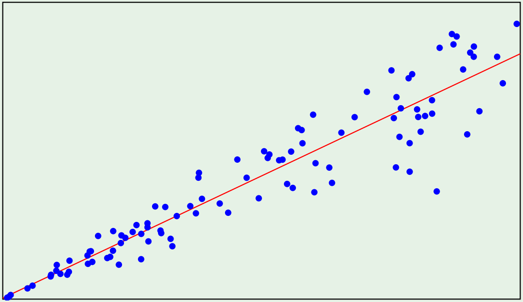


Example 4

Scatter plot where the variability around the line isn't constant:



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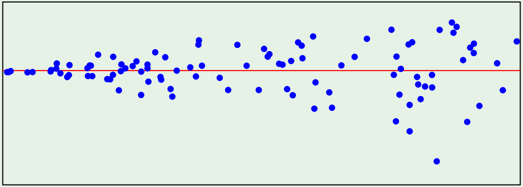
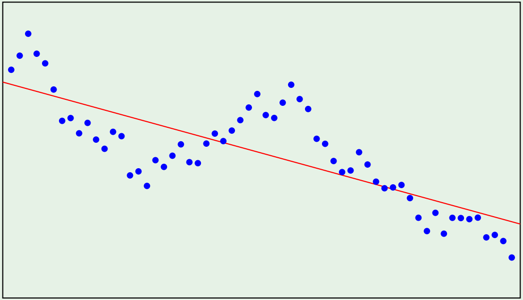


Example 5

Scatter plot using time series data:



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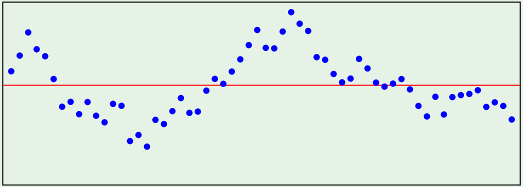

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Note

Recall from Algebra that if we know the slope, m, of a line and a point, (x_0, y_0), on that line, then:

$$y - y_0 = m(x - x_0)$$

Example 6

The summary statistics of the Elmhurst College data set are:

	Family Income (x)		Gift Aid (y)	
mean	$\bar{x} =$	\$101,780	$\bar{y} =$	\$19,940
std. dev.	$s_x =$	\$63,200	$s_y =$	\$5,460

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Process for estimating the least squares line

1. Estimate the slope parameter: $b_1 = \frac{s_y}{s_x} R$
2. Since $(\bar{x}, \bar{y})$ is on the least squares line, use $x_0 = \bar{x}$ and $y_0 = \bar{y}$
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The intercept, b_0, describes the average outcome of y if $x = 0$ and the linear model is valid all the way to $x = 0$.

Example 6 (Continued)

The slope, $b_1 = -0.0431$, means that for each \$1,000 family income, we would expect a student to receive a net difference of

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Note

We must be cautious about interpreting a causal connection between these variables because this data is observational, not experimental.

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The financial aid a school gives a student can never be less than zero!

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The R^2 of a linear model describes what percent of the variation in the response that is explained by the least squares line.

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Note that for this data set we have $R = -0.499$ and

$$R^2 = (-0.4999)^2 \approx 0.25$$