# Laplace Transforms

Department of Mathematics

Salt Lake Community College

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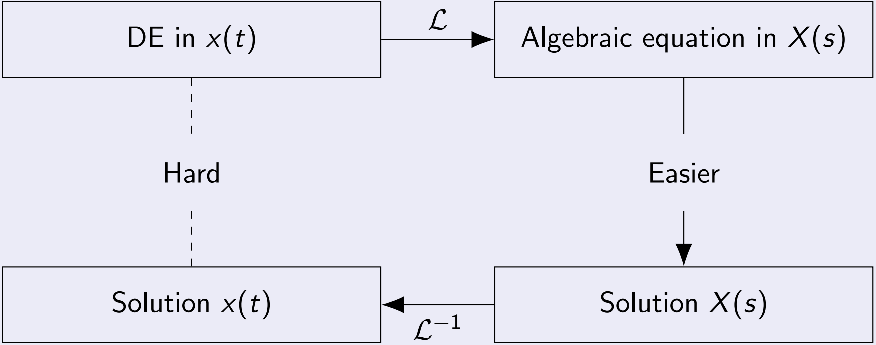
DE in $x(t)$

Hard

Solution $x(t)$

In this chapter we shall study the Laplace transformation, which will allow us to reframe some types of hard problems into equivalent easier problems.



## Laplace Transform

The **Laplace Transform** $\mathcal{L}\{f(t)\}$ of a suitable function $f(t)$ defined on $[0, \infty)$ is the function $F(s)$ given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)dt$$

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## Linearity of the Laplace Transform

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then by the properties of integrals

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(t) \quad \text{for } a, b \in \mathbb{C}$$

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## Existence Theorem for Laplace Transform

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $\alpha$, then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

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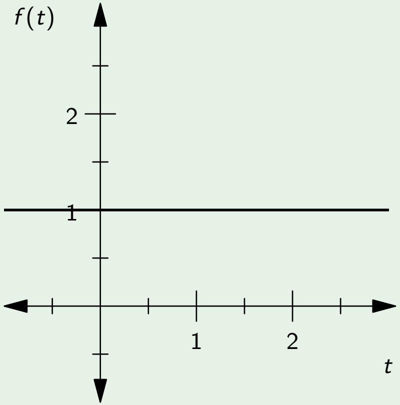


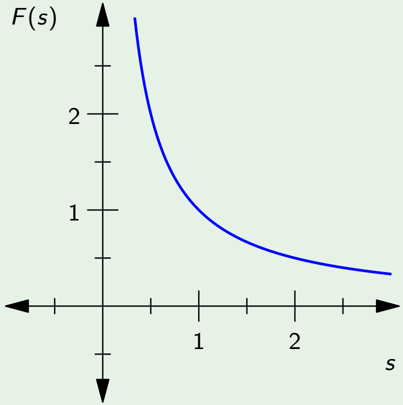

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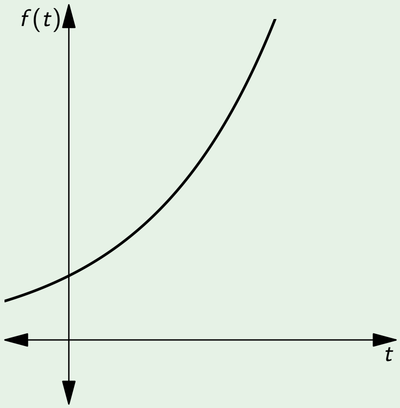


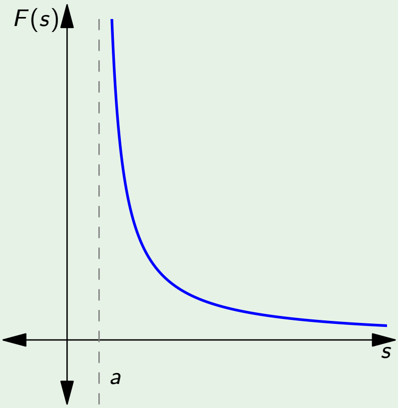

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&= \frac{2s+10}{s(s+2)}
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So, if we equate the real and imaginary parts, we get:

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \quad \text{and} \quad \mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

### Inverse Laplace Transform

A function $f(t)$ whose transform if $F(s)$ is called the **inverse Laplace transform** of $F$, and we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

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Thus,

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## Some Laplace Transforms

| $f(t)$ | $\mathcal{L}\{f(t)\}$ | |
|---|---|---|
| $1$ | $\dfrac{1}{s}$ | $s > 0$ |
| $t^n$ | $\dfrac{n!}{s^{n+1}}$ | $s > 0, \ n \in \mathbb{N}^+$ |
| $e^{at}$ | $\dfrac{1}{s-a}$ | $s > a$ |
| $t^n e^{at}$ | $\dfrac{n!}{(s-a)^{n+1}}$ | $s > a, \ n \in \mathbb{N}^+$ |
| $\sin(bt)$ | $\dfrac{b}{s^2+b^2}$ | $s > 0$ |
| $\cos(bt)$ | $\dfrac{s}{s^2+b^2}$ | $s > 0$ |

## Some More Laplace Transforms

| $f(t)$ | $\mathcal{L}\{f(t)\}$ | |
|---|---|---|
| $e^{at}\sin(bt)$ | $\dfrac{b}{(s-a)^2+b^2}$ | $s > a$ |
| $e^{at}\cos(bt)$ | $\dfrac{s-a}{(s-a)^2+b^2}$ | $s > a$ |
| $\sinh(bt)$ | $\dfrac{b}{s^2-b^2}$ | $s > \lvert b\rvert$ |
| $\cosh(bt)$ | $\dfrac{s}{s^2-b^2}$ | $s > \lvert b\rvert$ |

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Now, if we let $b = \sqrt{5}$, then $F$ becomes

$$F(s) = 2\underbrace{\left(\frac{1}{s-1}\right)}_{\mathcal{L}\{e^t\}} - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{\sqrt{5}}{s^2+\left(\sqrt{5}\right)^2}\right)}_{\mathcal{L}\{\sin\left(\sqrt{5}t\right)}$$

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Thus, $f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^t - \dfrac{1}{\sqrt{5}}\sin\left(\sqrt{5}t\right)$.

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&= \mathcal{L}\{e^{-2t}\cos(3t)\} - \frac{1}{3}\mathcal{L}\{s^{-2t}\sin(3t)\}
\end{aligned}$$

Thus, by linearity we have $f(t) = e^{-2t}\cos(3t) - \frac{1}{3}s^{-2t}\sin(3t)$.