Basic Concepts of Probability

Colby Community College

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Example 2

Drawing a hand for poker is an event, where the outcome is the hand of cards you drew.

This is not a simple event, since we could break it down to five individual card draws. Where each card draw would be a simple event.

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Classical Definition

Given that all out outcomes are equally likely, we can compute the probability of an event E using the formula

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of outcomes}}$$

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Note

Probabilities are ratios, and so can be reduced like fractions.

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Note

It is assumed that you cannot differentiate the cherries by touch. If, say, the sweet cherries were smaller than the sour ones, you could always pick a sweet cherry.

Relative Frequency Definition

Conduct an procedure and count the number of times that event A occurs. $P(A)$ is then *approximated* as follows:

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Relative frequency probabilities are often used in place of classical probabilities. The instruction "find the probability" often means "estimate the probability."

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We cannot use the classical definition of probability because the outcomes "dying" and "not dying" are not equally likely.

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  - Gamblers sometimes foolishly lose large sums of money by incorrectly thinking that a string of losses increases the chances of a win on the next bet, or that a string of wins is likely to continue.
- If we know nothing about the likelihood of different possible outcomes, we should not assume that they are equally likely.
  - You should not think that the probability of passing the next exam is $\frac{1}{2}$, or 0.5. The actual probability depends on factors such as the amount of preparation and the difficulty of the exam.

Definition

A standard deck of 52 playing cards consists of four **suits** in two colors: Hearts ♥, Spades ♠, Diamonds ♦, and Clubs ♣

Each suit contains 13 cards, each of a different **rank**:
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A common mistake is to use $\frac{3785}{4720}$ as the probability.

Don't make the common mistake finding a probability by mindlessly dividing a smaller number by a larger number. You need to think carefully about the numbers involved and what they represent.

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A certain event has a probability of 1.

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If a year is selected at random, find the probability that Thanksgiving Day in the United States will be on a Wednesday.

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Example 11

If a year is selected at random, find the probability that Thanksgiving Day in the United States will be on a Wednesday.

In the United States, Thanksgiving Day always falls on the fourth Thursday in November. This means it is impossible for Thanksgiving Day to fall on a Wednesday.

$$P(\text{Thanksgiving on Wednesday}) = 0$$
$$P(\text{Thanksgiving on Thursday}) = 1$$

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Example 12

If we roll a 6-sided die, and

$$P\left(\text{roll a six}\right) = \frac{1}{6}$$

The complement is

$$P\left(\text{roll not a six}\right) = P\left(\text{roll a one, two, three, four, or five}\right) = \frac{5}{6}$$

Note

Either an even occurs or it does not. This means the sum of $P(E)$ and $P(\bar{E})$ always has to equal 1.

In other words,

$$P(\bar{E}) = 1 - P(E) \quad \text{or} \quad P(E) = 1 - P(\bar{E})$$

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There are 13 hearts in a deck, giving $P(\text{heart}) = \dfrac{13}{52} = \dfrac{1}{4}$.

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This means the probability of not drawing a heart is

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$

The Rare Even Rule for Inferential Statistics

If, under a given assumption, the probability of a particular observed event is very small and the observed even occurs *significantly less than* or *significantly more than* what we typically expect with that assumption, we conclude that the assumption is probably not correct.

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Worded differently:

- x successes among n trails is a **significantly high** number of successes if the probability of x or more successes is unlikely with a probability of 0.05 or less, i.e if $P(x \text{ or more}) \leq 0.05$.
- x successes among n trails is a **significantly low** number of successes if the probability of x or fewer successes is unlikely with a probability of 0.05 or less, i.e if $P(x \text{ or fewer}) \leq 0.05$.

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The actual odds reflect the actual likelihood of an event. The payoff odds describe the payout amount determined by the casino, lottery, or racetrack operators. (i.e. based on how greedy the operators are.)

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- If the casino was not run for profit, the payoff odds would be the same as the actual odds in favor, 37 : 1. So, the profit would be \$185.