Complements, Conditional Probability, and Bayes's Theorem

Colby Community College

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Example 1

The following are the same event:

- "Not getting at least 1 girl in 10 births."
- "Getting no girls in 10 births."
- "Getting 10 boys in 10 births."

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4. Thus $P\left(A\right) = 1 - P\left(\bar{A}\right) = 1 - 0.290 = 0.710$.

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Formal Approach

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(A)}$$

Example 3

	Positive Test Result	**Negative Test Result**
Uses Drugs	45 (True Positive)	5 (False Negative)
Doesn't Use Drugs	25 (False Positive)	480 (True Negative)

Find the following probabilities:

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Note

In general $P\left(B \mid A\right) \neq P\left(A \mid B\right)$.

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A cancer test has the following performance characteristics:

- $P(C) = 0.01$
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- Among the 10 subject with cancer, 8 will get a positive result.
- Among the 10 subjects with cancer, 2 will get a negative result.

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We can summarize the results in the following table.

	Positive Test Result	**Negative Test Result**	**Total**
Has Cancer	8 (True Positive)	2 (False Negative)	10
No Cancer	99 (False Positive)	891 (True Negative)	990
Total	107	893	

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We can also use the following formula:

$$P(C \mid \text{positive}) = \frac{P(C) \cdot P(\text{positive} \mid C)}{(P(C) \cdot P(\text{positive} \mid C)) + \left(P\left(\bar{C}\right) \cdot P\left(\text{positive} \mid \bar{C}\right)\right)}$$

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Bayes' Theorem

$$P(A \mid B) = \frac{P(A)\,P(B \mid A)}{P(A)} = \frac{P(A) \cdot P(B \mid A)}{(P(A) \cdot P(B \mid A)) + \left(P\left(\bar{A}\right) \cdot P\left(B \mid \bar{A}\right)\right)}$$

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A **posterior probability** is a probability value that has been revised by using additional information that is later obtained.