Normal Distribution

Department of Mathematics

Salt Lake Community College

Example 1

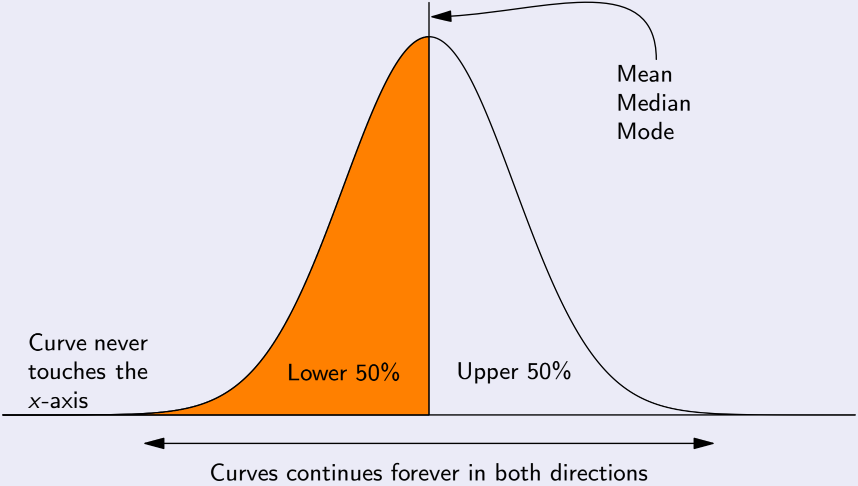
Consider making popcorn. You put some oil and corn kernels in a pan and start heating.

For the first few minutes nothing happens, then a few kernels start to pop. A little while later more and more start to pop. This goes one for a minute or so, and the popping gradually tappers off.

Most of the popping happens in that brief, noisy moment. This demonstrates a typical pattern that is part of many phenomena.

Definition

A **normal distribution** is a perfectly symmetric, mound-shaped distribution. It is also referred to as a **normal curve** or a **bell curve**.



Note

It is common to use the notation:

- $\mu =$ mean
- $\sigma =$ standard deviation

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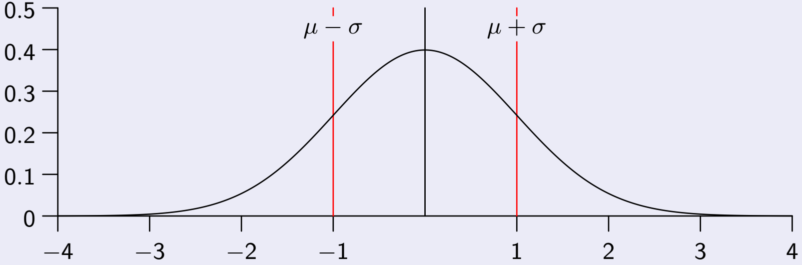
Definition

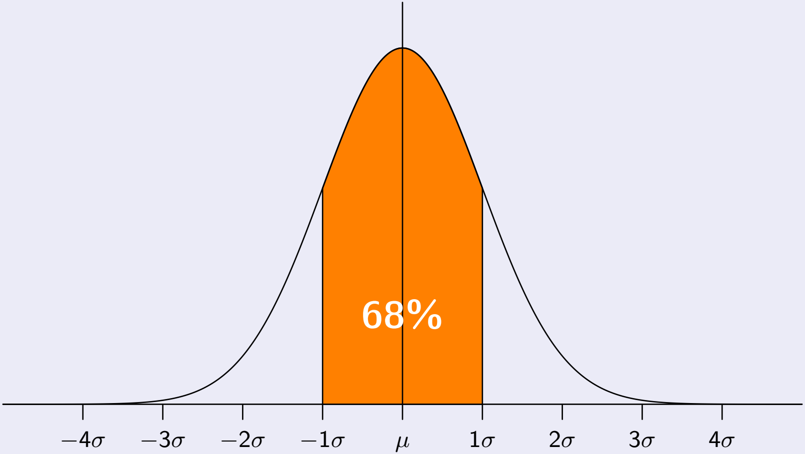
When $\mu = 0$ and $\sigma = 1$, the curve is called the **standard normal distribution**.



Definition

The **empirical rule**, or **68-95-99.7 rule**, states approximately how much of the area is contained when stepping one, two, or three standard deviations from the mean.

One standard deviation from the mean.



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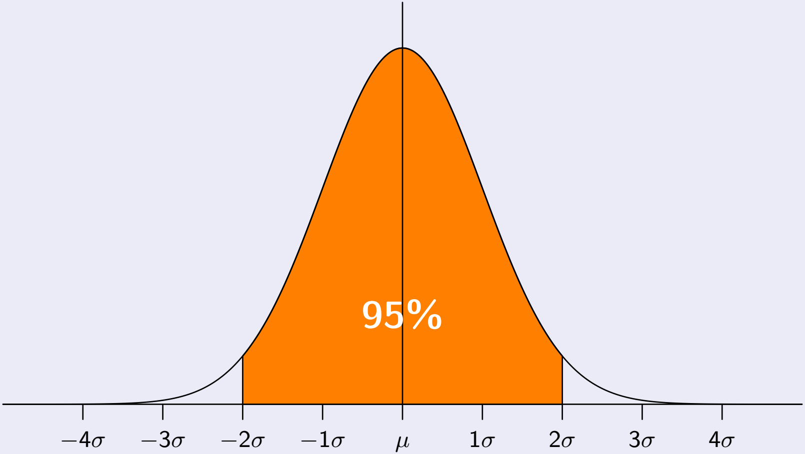
Two standard deviations from the mean.



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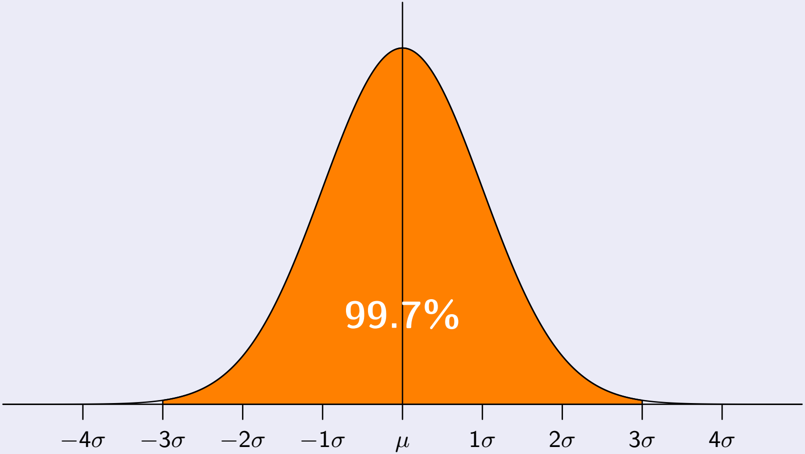

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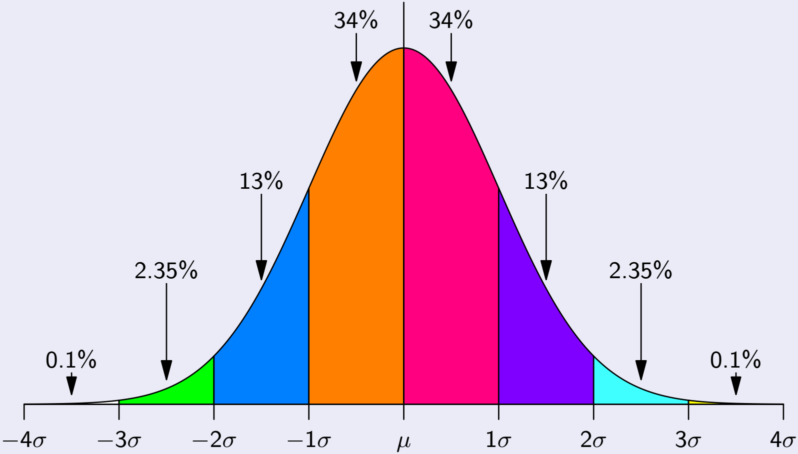

Definition

A **z-score** is a measure of the number of standard deviations a particular data point is away from the mean. It is calculated with:

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We know from the empirical rule that roughly 68% of the scores fall within one standard deviation of the mean. This means that 68% of the students scored between 75 and 89.

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Moreover, we know that roughly 95% of the scores fall within two standard deviations of the mean. Which means that $95\% - 68\% = 32\%$ of the scores are more than one standard deviation from the mean, but less than two.

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Since the curve is symmetric, we know that 16% of the students scored between 89 and 96, as well as 16% between 68 and 75

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Visual representations of data, such as histograms can tell you if data is roughly normal or not.

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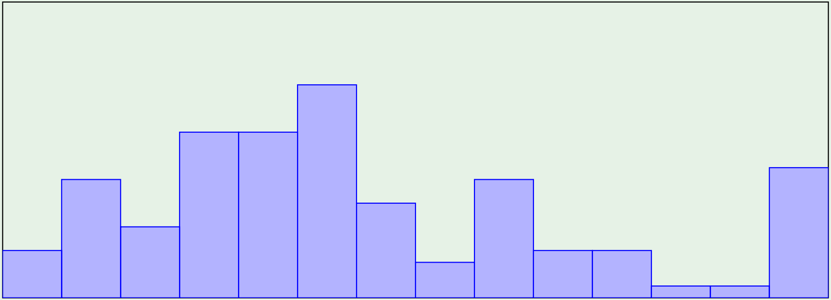
This data is not a normal distribution.



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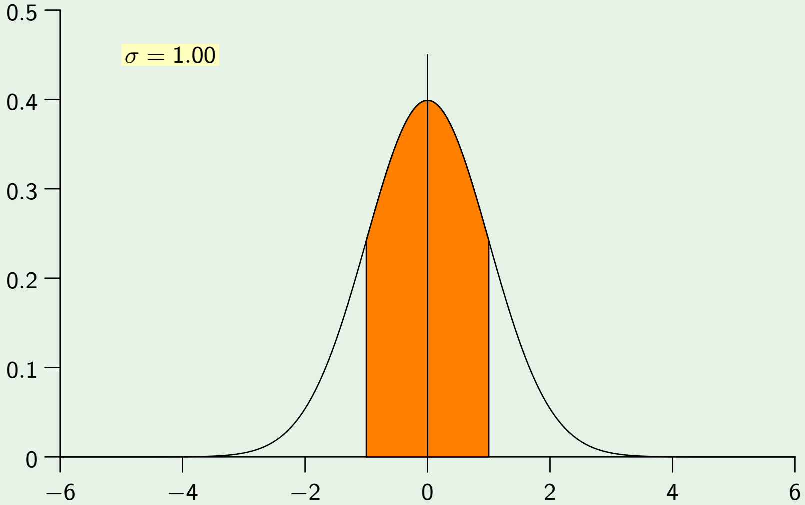
We know from the Empirical rule that roughly two-thirds of the data in a normal distribution falls within one standard deviation of the mean.

With a normal density curve, this means that about 68% of the total area under the curve is within z-scores of ± 1.

Example 6

Notice that as the standard deviation increases, the curve gets flatter. This is because 68% of the area always falls within a single standard deviation of the mean.



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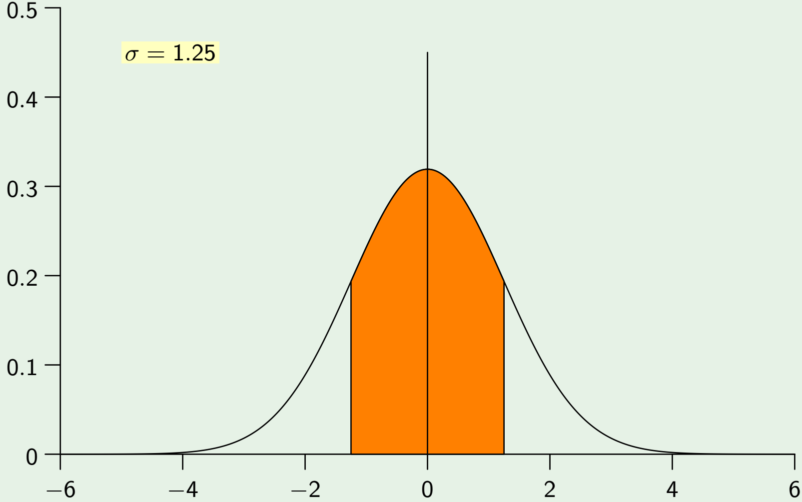

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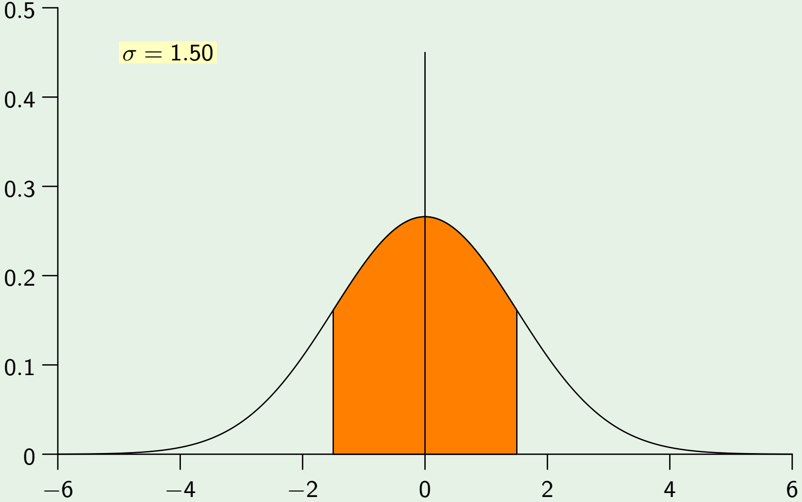

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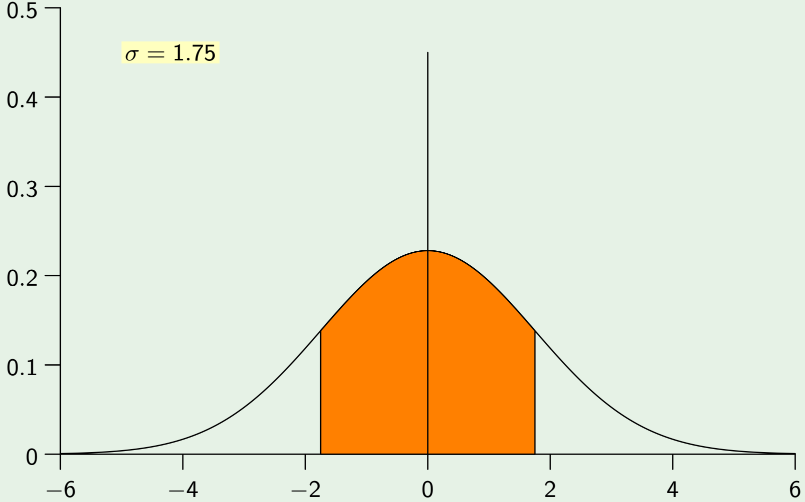

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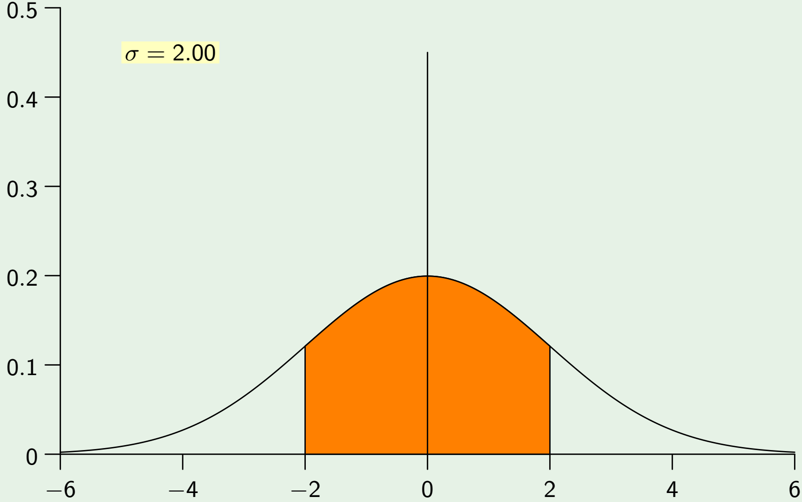

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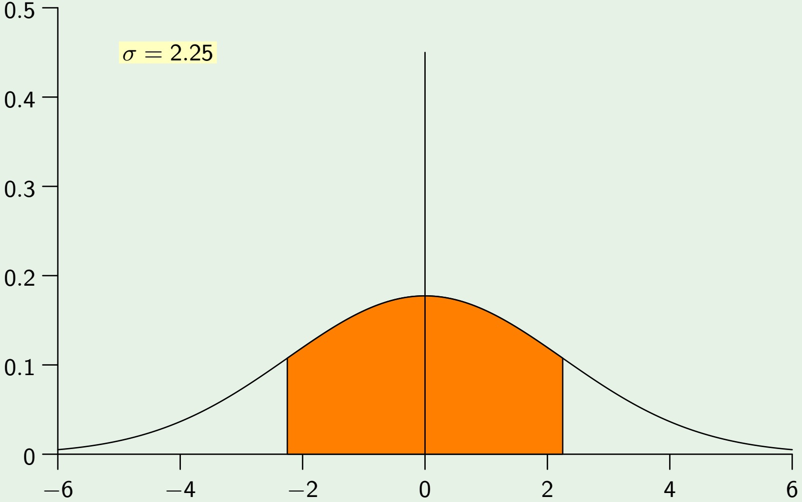

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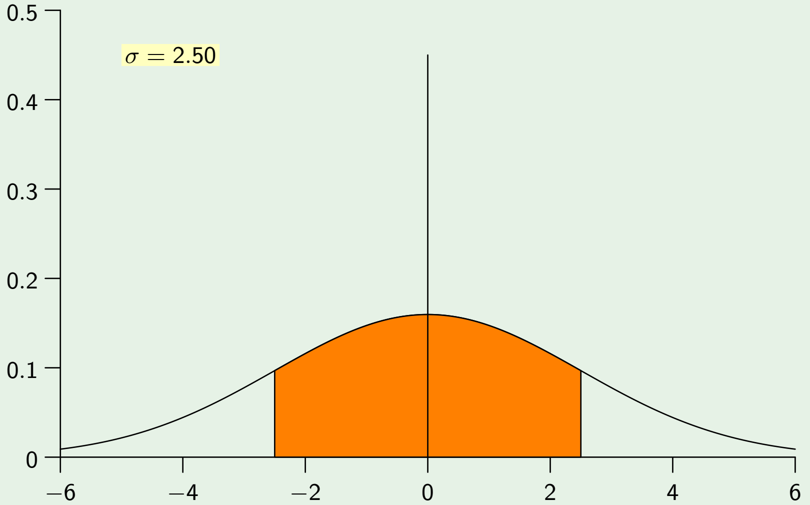

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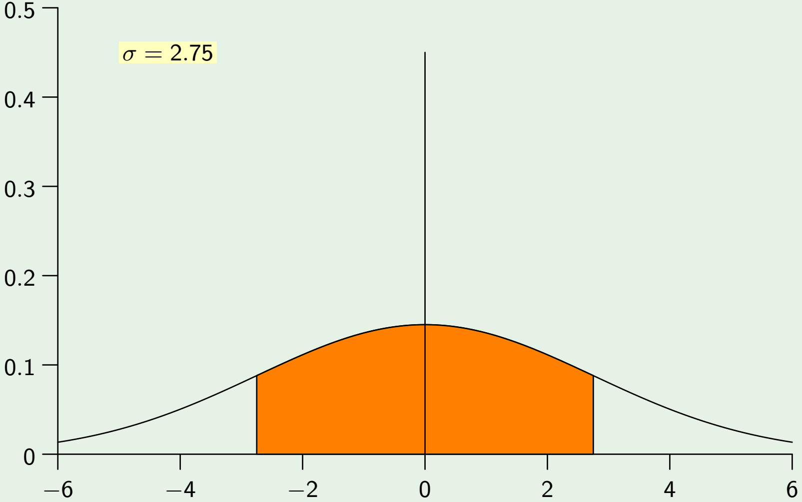

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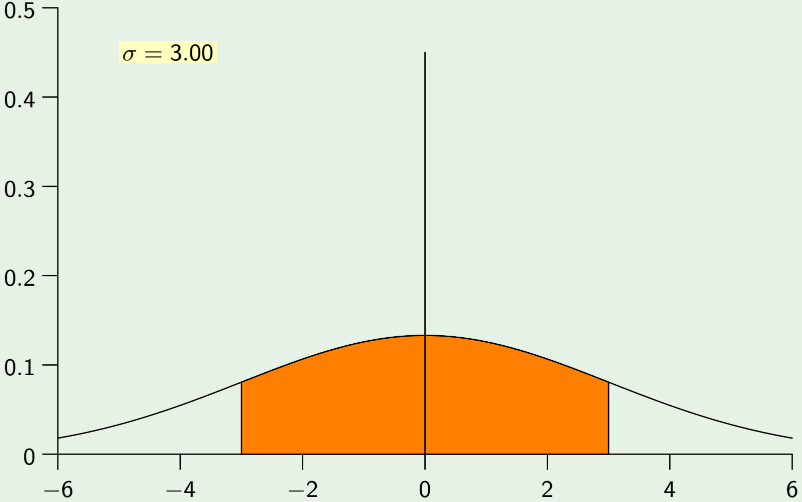

Definition

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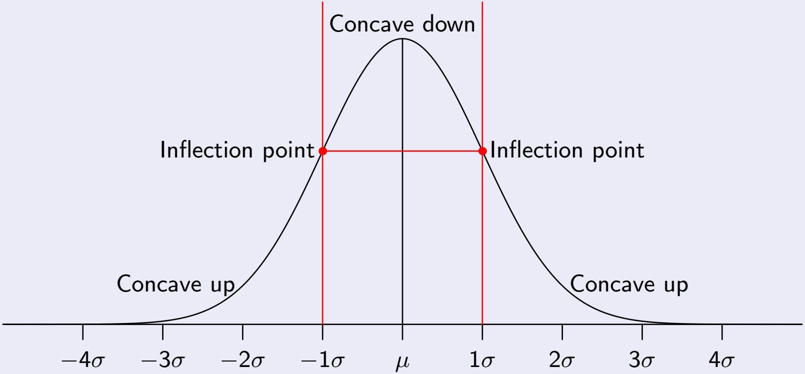
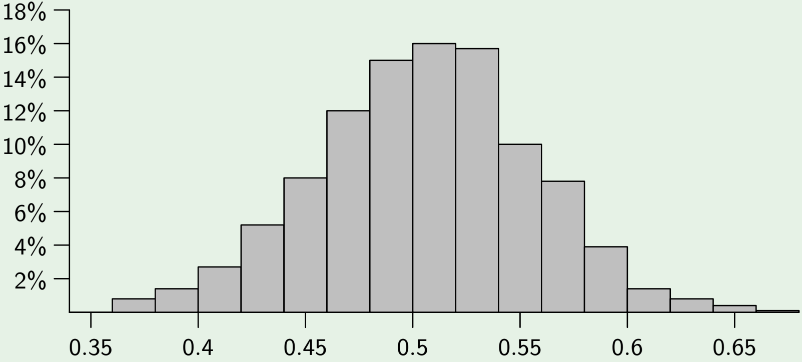
A normal density curve always has two inflection points, each one standard deviation from the mean.



Example 7

Estimate the standard deviation of the distribution represented by the histogram.



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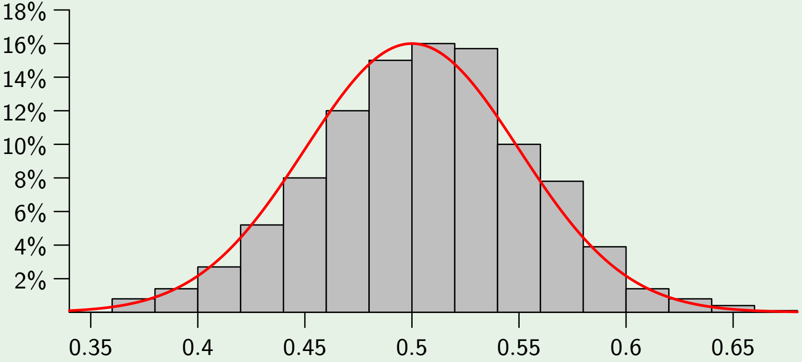

First, draw a density curve that fits the histogram fairly well.

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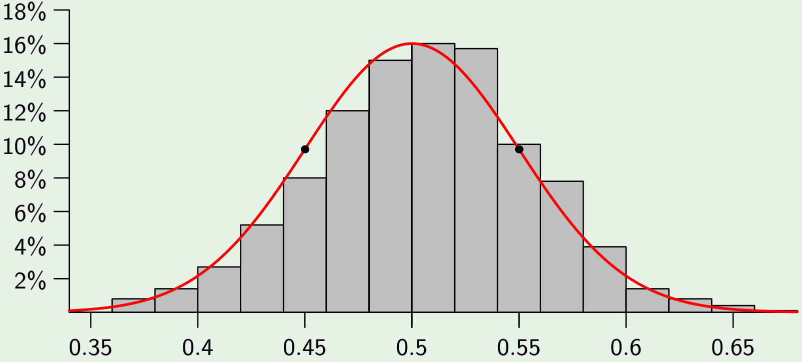

Next, estimate the inflections points, as well as the mean.

In this case, we have $\mu \approx 0.5$ and $\sigma \approx 0.05$.

(The actual statistics are: mean $= 0.04988$ and std. dev. $= 0.4997$.)

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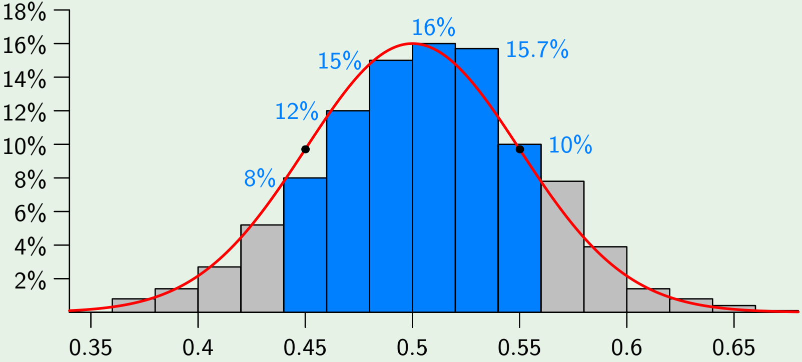


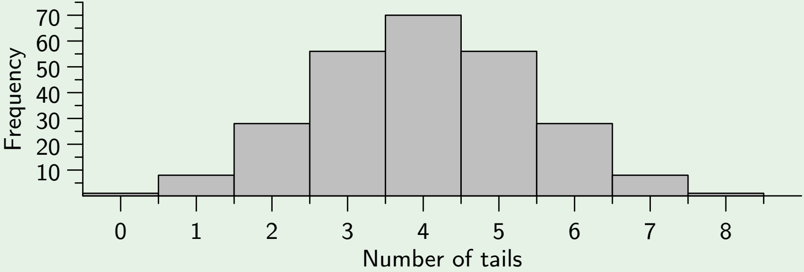
We can estimate how much of the graph is within one standard deviation by estimating the heights of the colored bins.

$$\frac{8}{2} + 12 + 15 + 16 + 15.75 + \frac{10}{2} = 67.7$$

This is remarkably close to the 68% the Empirical rule gives.

Example 8

Suppose you flip 8 coins and record the results. The histogram shows the results of performing this experiment 256 times.



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Suppose you flip 8 coins and record the results. The histogram shows the results of performing this experiment 256 times.

One way to calculate the probability that exactly three tails are flipped is:

$$P\left(3 \text{ tails}\right) = \frac{{}_8C_3}{2^8} = \frac{56}{256} \approx 0.2188$$

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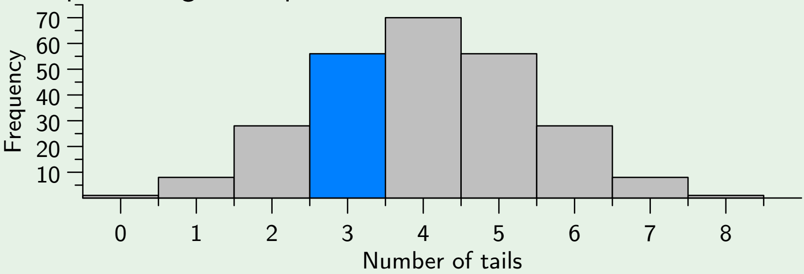

A second way is to approximate the heights of each box and divide the area of the colored box by the areas of all the boxes:

$$P\,(3 \text{ tails}) = \frac{55}{1+8+28+55+70+55+28+8+1} = \frac{55}{254} \approx 0.2165$$

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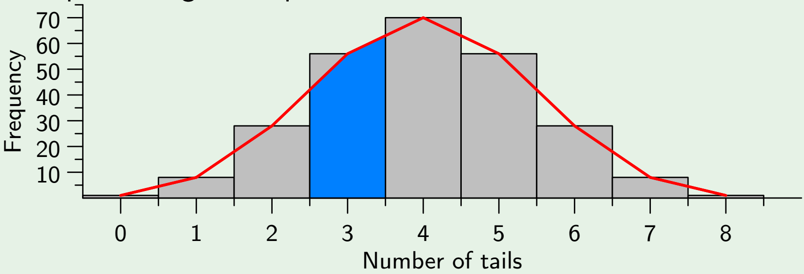

A third way is to draw an approximate normal curve. Then, the area from 2.5 to 3.5 relates to the probability of the event.

$$P\,(3 \text{ tails}) = \frac{\text{Area from 2.5 to 3.5}}{\text{Total area under curve}} = \frac{61.25}{324} \approx 0.1890$$

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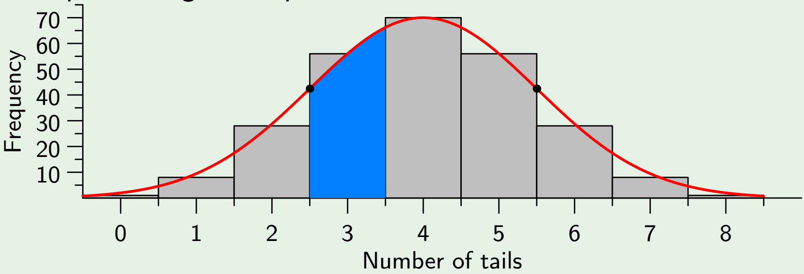

The better we draw the density curve, the better our approximation.

$$P\,(3\text{ tails}) = \frac{\text{Area from 2.5 to 3.5}}{\text{Total area under curve}} = \frac{55.478}{262.075} \approx 0.2117$$

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Example 9

Here is a section of a Standard Normal Probability table.

z	0.00	0.01	0.02	0.03	0.04	0.05
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678

What is the probability that $z \leq 1.63$?

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1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678

What is the probability that $z \leq 1.63$? $P(z \leq 1.63) = 0.9484$

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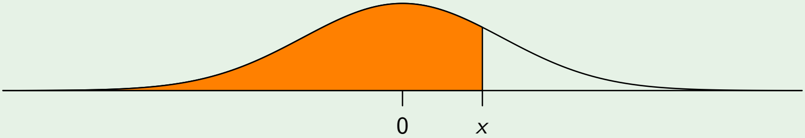

The area of the shaded region is the probability that a z score is less than or equal to x, $P(z \leq x)$.

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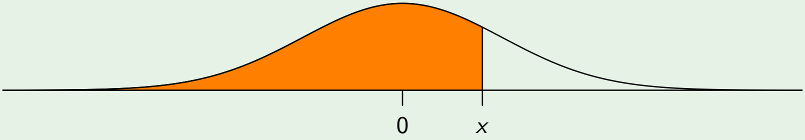

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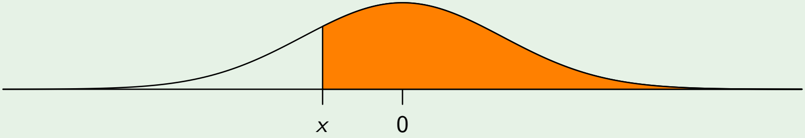

The area of the shaded region is the probability that a z score is greater than or equal to x, $P(z \geq x)$. Can also be calculated as $1 - P(z \leq x)$.

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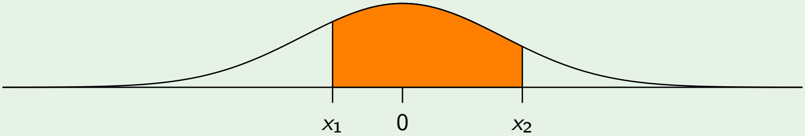

The area of the shaded region is the probability that a z score lies between x_1 and x_2, $P\left(x_1 \leq z \leq x_2\right)$.

Can also be calculated as $P\left(x \leq x_2\right) - P\left(z \leq x_1\right)$.

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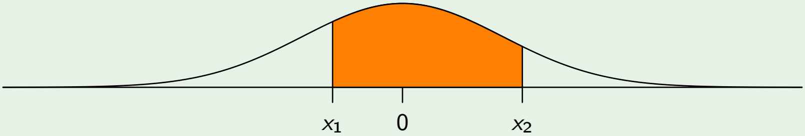

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Can also be calculated as $P(x \leq x_2) - P(z \leq x_1)$.

Note

It is always a good idea to sketch a picture of the normal density curve you are dealing with, so that you make sure you're finding the proper area.

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To use such a table on an arbitary density function, you need to **standardize** the distribution by calculating z-scores.

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Note

In statistics, the term "unusual" often refers to an event with probability less than 5% ($p < 0.05$).

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$$\begin{aligned} P\left(-1 \leq z \leq -0.3333\right) &= P\left(z \leq -0.3333\right) - P\left(z \leq -1\right) \\ &= 0.369441 - 0.158655 \\ &= 0.210786 \end{aligned}$$

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Note

It's not at all unusual to flip eight coins and get three tails.

Example 14

A candy company sells small bags of candy. While they attempt to keep the number of pieces in each bag the same, small differences occur because of variation in the packaging process. A quality control expert in the factory determined that the mean number of pieces in each bag is normally distributed, with a mean of 57.3 and a standard deviation of 1.6.

Let us find the probability that we will purchase a bag with 55 or fewer pieces of candy.

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Thus,

$$P\left(z \leq -1.4375\right) = 0.075288$$

A roughly 8% chance, it would be rare, but not unusual.

Example 15

Suppose that a fictional scientist measured the resting body temperature for a large number of people. She found that the mean is 98.25°F and the standard deviation is 0.73°F.

Johnny "Hothead" Rico has a resting body temperature of 100°F. Would you consider this unusual?

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So, the probability that someone has a resting body temperature ≥ 100 is

$$P\left(z \geq 2.3972\right) = 1 - P\left(z \leq 2.3972\right) = 1 - 0.99174 = 0.00826$$

Johnny is pretty unusual, but don't let him hear you say that.