Probability Distributions

Colby Community College

Example 1

In the casino game Roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the players bet \$1 on a single number. If that number is spun on the wheel, then they receive \$36. Otherwise, they lose their \$1.

On average, how much money should a player expect to win or lose if they play this game repeatedly?

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So, **on average**, we will have a net change of

$$\$35 \cdot \frac{1}{38} + -\$1 \cdot \frac{37}{38} = \$0.9211 - \$0.9737 \approx -\$0.053$$

That is, **on average**, we will lose 5.3 cents per space we bet on.

Definition

The **expected value** is the average gain or loss of an event if the procedure is repeated many times.

Expected values is calculated using the following formula.

$$EV = V(O_1) \cdot P(O_1) + V(O_2) \cdot P(O_2) + \cdots + V(O_n) \cdot P(O_n)$$

where $V(O_i)$ is the value of the ith outcome, and $P(O_i)$ is the probability of the ith outcome.

Example 2

Consider a lottery where balls numbered 1 through 48 are placed in a machine and six balls are drawn at random. If the six numbers drawn match the numbers that a player has chosen, that player wins \$1,000,000. If they match five numbers, they win \$1,000. A lottery ticket costs \$1. Let us calculate the expected value.

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The following table gives the values and probabilities.

Outcome	Value	Probability
Match all six	\$999,999	$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12272512}$
Match five	\$999	$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12272512}$
Match four for fewer	-\$1	$1 - \frac{253}{12272512} = \frac{12271259}{12272512}$

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The expected value is then:

$$(\$999,999) \cdot \frac{1}{12272512} + (\$999) \cdot \frac{252}{12272512} + (-\$1) \cdot \frac{12271259}{12272512} \approx -\$0.898$$

So, on average, a player can expect to lose about 90 cents on a ticket.

Note

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A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. Should you play this game?

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$$P\left(\text{no match}\right) = \frac{\left({}_6C_1\right)\left({}_5C_1\right)\left({}_4C_1\right)}{\left({}_6C_1\right)\left({}_6C_1\right)\left({}_6C_1\right)} = \frac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \frac{5}{9} \quad (\approx 55.5\%)$$

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The expected value is:

$$\$2\cdot\frac{4}{9} - \$1\cdot\frac{5}{9} = \frac{1}{3} \approx \$0.33$$

You will, on average, win 33 cents per play.

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Note

It makes sense that a insurance policy would have a negative expected value, otherwise the insurance company couldn't stay in business.

The benefit for the consumer is the security that the policy provides.

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The expected value for the consumer may be different. The consumer is likely to pay the more to repair or replace the item out of warranty. (The company pays manufacturing cost, consumer has to pay retail cost.)