

# Variation of Parameters

Department of Mathematics

Salt Lake Community College

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Just like with single order equations, we want to perturb the homogeneous solution into a particular solution to the nonhomogeneous DE.

## Variation of Parameters

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$$y_p' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2$$

So, we can choose  $v_1y_1' + v_2y_2' = 0$  as our auxiliary condition, which reduces  $y_p'$  to:

$$y_p' = v_1'y_1 + v_2'y_2$$

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$$y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

We then substitute  $y_p$ ,  $y_p'$ , and  $y_p''$  into  $L(y) = f$ .

$$(v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2') + p \cdot (v_1' y_1 + v_2' y_2) + q \cdot (v_1 y_1 + v_2 y_2) = f$$

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$$\begin{aligned}(v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2') + p \cdot (v_1' y_1 + v_2' y_2) + q \cdot (v_1 y_1 + v_2 y_2) &= f \\ v_1(y_1'' + p y_1' + q y_1) + v_2(y_2'' + p y_2' + q y_2) + (v_1' y_1' + v_2' y_2') &= f\end{aligned}$$

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So, we have the system

$$v_1' y_1' + v_2' y_2' = f$$

$$v_1' y_1 + v_2' y_2 = 0$$



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Using Cramer's Rule, the system

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has solution

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

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The denominator is just the Wronskian  $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$ , because  $y_1$  and  $y_2$  are linearly independent.

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$$v_1 = - \int \frac{y_2 f}{W(y_1, y_2)} \quad \text{and} \quad v_2 = \int \frac{y_1 f}{W(y_1, y_2)}$$

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$$y'' + y = \sec(t) \quad |t| < \frac{\pi}{2}$$

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## Note

This method can be extended to higher orders.

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So, using the Cramer's Rule formulas from before

$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} = -\frac{t}{t^2 + 1} \quad \text{and} \quad v_2' = \frac{y_1 f}{W(y_1, y_2)} = \frac{1}{t^2 + 1}$$

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## Example 4

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$$t^2 y'' - 2ty' + 2y = t \ln(t), \quad t > 0$$

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So, using Cramer's Rule

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The general solution is

$$y = c_1 t + c_2 t^2 - \frac{t}{2} \ln^2(t) - t \ln(t) - t$$