Dynamical Systems: Modeling

Department of Mathematics

Salt Lake Community College

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The study of multidimensional systems will be aided by the study of **linear algebra** in chapters 3 and 5.

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But, if all we care about is the temperature of the coffee, we can use a limited model called Newton's Law of Cooling, which incorporates the surrounding temperature to give an accurate description of the coffee's temperature.

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Note

We will only be studying ODE's in this class.

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- $\dfrac{\partial^2 y}{\partial x^2} + \dfrac{\partial^2 z}{\partial t^2} = xyz$ is a second-order PDE with independent variables x and t and dependent variables y and z.

Constants of Proportionality

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- The rate of change of y is **directly proportional** to y^2 and **inversely proportional** to $\sqrt{t}$:

$$\frac{dy}{dt} = k\frac{y^2}{\sqrt{t}}$$

Example 3

Exponential Growth The population P is growing at a rate proportional to the population at any time t:

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Example 4

Exponential Decay Let A be the amount of radioactive material in a sample at any time t. The amount A is decreasing at a rate proportional to the amount at any time t:

$$\frac{dA}{dt} = kA, \quad k < 0$$

Example 5

Newton's Law of Cooling or Heating The rate of change of temperature T of an object is proportional to the difference between the temperature M of the surroundings and the temperature of the object:

$$\frac{dT}{dt} = k(M - T), \quad k > 0$$

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Example 6

Logistic Growth The rate a which a disease is spread (i.e. the rate of increase of the number N of people infected) in a fixed population L is proportional to the product of the number of people infected and the number of people not yet infected:

$$\frac{dN}{dt} = kN(L - N), \quad k > 0$$

Example 7

Voltage Across an Inductor The voltage drop V is proportional to the rate of current I in the inductor:

$$V = L\frac{dI}{dt}$$

(The proportionality constant is this instance is written as L (instead of k) and is called the **inductance**.)

The Malthus Model for Population Growth

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Malthus assumed that the rate of increase of the worlds population, $y(t)$, was proportional to it's size. We can state this as a DE:

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In 1798, the population was about 0.9 million people. Malthus assumed the growth rate was a 3% annual increase. Giving the DE:

$$\frac{dy}{dt} = 0.03y, \quad y(0) = 0.9$$

Accuracy of the Malthus Model

Year	t	Malthus	Actual	Year	t	Malthus	Actual
1800	0	0.90	0.9	1910	110	24.42	1.8
1810	10	1.21	0.9	1920	120	32.98	1.9
1820	20	1.64	1.0	1930	130	44.52	2.1
1830	30	2.21	1.0	1940	140	60.10	2.3
1840	40	2.99	1.1	1950	150	81.13	2.7
1850	50	4.03	1.2	1960	160	109.53	3.0
1860	60	5.45	1.3	1970	170	147.87	3.5
1870	70	7.35	1.4	1980	180	199.62	4.2
1880	80	9.93	1.5	1990	190	269.49	5.1
1890	90	13.40	1.6	2000	200	363.81	6.0
1900	100	18.09	1.7				

Example 8

Hooke's Law The restoring force on a spring is proportional to the displacement x but opposite in direction:

$$F_{res} = -kx, \quad k > 0$$

If friction is negligible, we can assume Newton's First Law of Motion:

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Example 9

Hooke's Law as a System If we substitute $dx/dt = y$ into the previous example, we can convert it to an equivalent system of first-order equations:

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\frac{k}{m}x \end{aligned}$$