

Linear Systems with Nonreal Eigenvalues

Department of Mathematics

Salt Lake Community College

Complex Eigenvalues and Eigenvectors

For a real matrix \mathbf{A} , nonreal eigenvalues come in complex conjugate pairs,

$$\lambda_1 = \alpha + \beta i \quad \text{and} \quad \lambda_2 = \alpha - \beta i$$

with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$.

The corresponding eigenvectors are also complex conjugate pairs and can be written

$$\vec{v}_1 = \vec{p} + \vec{q}i \quad \text{and} \quad \vec{v}_2 = \vec{p} - \vec{q}i$$

where \vec{p} and \vec{q} are real vectors.

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Note

We only need to find one eigenvalue/eigenvector pair.

Example 1

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

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Alternately, we can write

$$\vec{v} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{p}} \pm i \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{q}}$$

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$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2.$$

However, we want this solution in terms of the real vectors \vec{p} and \vec{q} .

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So, for eigenvalue $\lambda_1 = \alpha + \beta i$ and corresponding eigenvector $\vec{v}_1 = \vec{p} + \vec{q}i$ we get the solution

$$\vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1 = e^{\alpha + \beta i} (\vec{p} + \vec{q}i)$$

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So, for eigenvalue $\lambda_1 = \alpha + \beta i$ and corresponding eigenvector $\vec{v}_1 = \vec{p} + \vec{q}i$ we get the solution

$$\vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1 = e^{\alpha + \beta i} (\vec{p} + \vec{q}i)$$

Just like with second-order systems, we shall find that the real and imaginary parts of the complex solution above are both real and linearly independent solutions of the system.

Example 2

Consider the DE system:

$$\vec{x}' = \mathbf{A}\vec{x}$$

Suppose that

$$\vec{x}(t) = \vec{x}_{\text{Re}}(t) + i\vec{x}_{\text{Im}}(t)$$

is a complex vector solution to the system, with $\vec{x}_{\text{Im}} \neq \vec{0}$.

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$$\vec{x}'_{\text{Re}}(t) + i\vec{x}'_{\text{Im}}(t) = \mathbf{A}\vec{x}_{\text{Re}}(t) + i\mathbf{A}\vec{x}_{\text{Im}}(t)$$

Separately equating the real and imaginary parts, we get:

$$\vec{x}'_{\text{Re}}(t) = \mathbf{A}\vec{x}_{\text{Re}}(t) \quad \text{and} \quad \vec{x}'_{\text{Im}}(t) = \mathbf{A}\vec{x}_{\text{Im}}(t)$$

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Thus, $\vec{x}_{\text{Re}}(t)$ and $\vec{x}_{\text{Im}}(t)$ are separate real solutions to the system.

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For the complex solution

$$\vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1 = e^{\alpha + \beta i} (\vec{p} + \vec{q}i)$$

we can determine the real and imaginary parts by using Euler's formula:

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Since $\vec{x}_{\text{Re}}(t)$ and $\vec{x}_{\text{Im}}(t)$ are linearly independent solutions they must span the solution space. Thus, the general solution, for $c_1, c_2 \in \mathbb{R}$, is

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Any solutions derived from λ_2 and \vec{v}_2 will be linear combinations of $\vec{x}_{\text{Re}}(t)$ and $\vec{x}_{\text{Im}}(t)$.

Solving 2×2 DE System with Nonreal Eigenvalues

For the two-dimensional linear homogeneous differential equation $\vec{x}' = \mathbf{A}\vec{x}$ with real matrix \mathbf{A} , eigenvalues $\lambda_1, \lambda_2 = \alpha \pm i\beta$ ($\beta \neq 0$) the general solution can be found using the following steps:

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- ① For one eigenvalue λ_1 , find it's corresponding eigenvector \vec{v}_1 . The second eigenvalue λ_2 and it's eigenvector \vec{v}_2 are complex conjugates of the first. The eigenvectors are of the form $\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$.

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- 2 Construxt the linearly independent real (\vec{x}_{Re}) and imaginary (\vec{x}_{Im}) parts of the solutions as follows:

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- 3 The general solution is

$$\vec{x}(t) = c_1 \vec{x}_{\text{Re}}(t) + c_2 \vec{x}_{\text{Im}}(t)$$

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Thus

$$\vec{x}_{\text{Re}}(t) = e^{5t} \cos(2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{5t} \sin(2t) \begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t} \begin{bmatrix} \cos(2t) \\ \cos(2t) + 2 \sin(2t) \end{bmatrix}$$

$$\vec{x}_{\text{Im}}(t) = e^{5t} \sin(2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{5t} \cos(2t) \begin{bmatrix} 0 \\ -2 \end{bmatrix} = e^{5t} \begin{bmatrix} \sin(2t) \\ \sin(2t) - 2 \cos(2t) \end{bmatrix}$$

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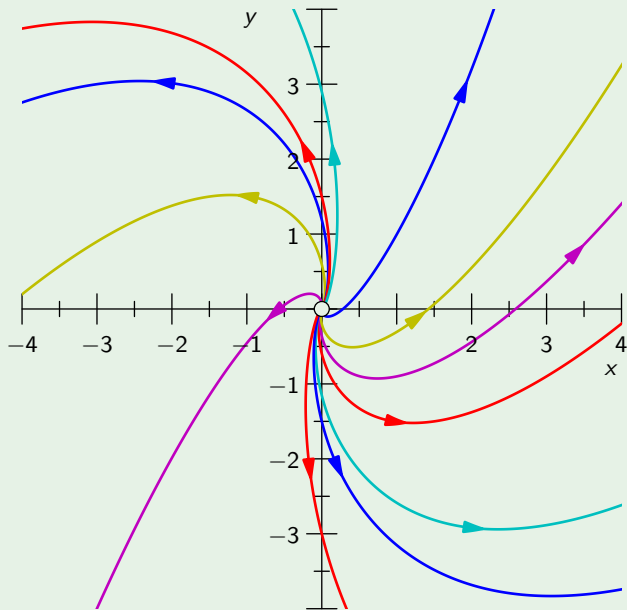
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And general solution

$$\vec{x}(t) = e^{5t} \left(c_1 \begin{bmatrix} \cos(2t) \\ \cos(2t) + 2 \sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ \sin(2t) - 2 \cos(2t) \end{bmatrix} \right)$$

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Thus

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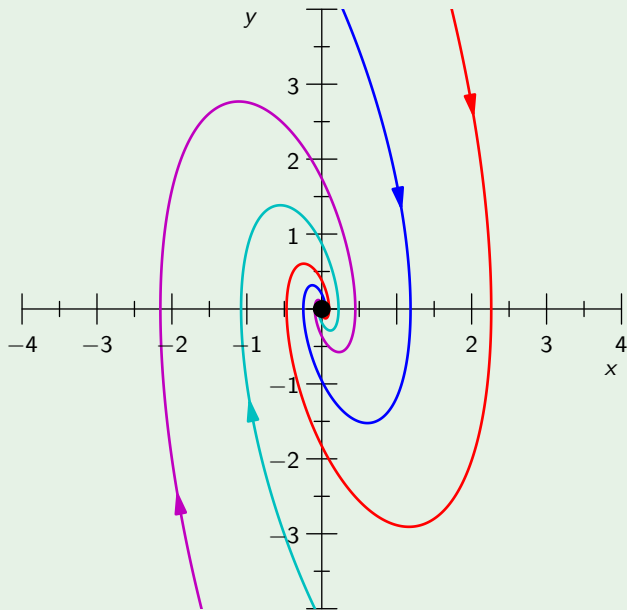
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And general solution

$$\vec{x}(t) = e^{-t} \left(c_1 \begin{bmatrix} \cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{bmatrix} \right)$$

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The eigenvalues are $\lambda_1, \lambda_2 = 0 \pm 3i$ and

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Thus

$$\vec{x}_{\text{Re}}(t) = \cos(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \cos(3t) \\ 4 \cos(3t) + 3 \sin(3t) \end{bmatrix}$$

$$\vec{x}_{\text{Im}}(t) = \sin(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \sin(3t) \\ 4 \sin(3t) - 3 \cos(3t) \end{bmatrix}$$

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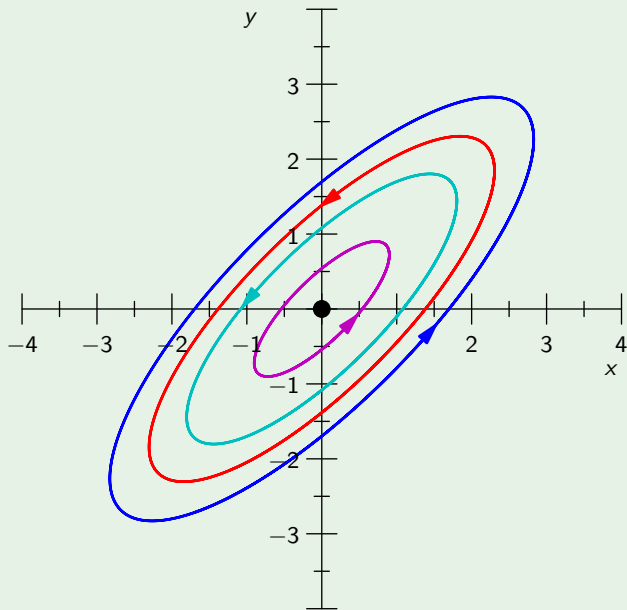
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$$\vec{x}_{\text{Im}}(t) = \sin(3t) \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \cos(3t) \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \sin(3t) \\ 4 \sin(3t) - 3 \cos(3t) \end{bmatrix}$$

And general solution

$$\vec{x}(t) = c_1 \begin{bmatrix} 5 \cos(3t) \\ 4 \cos(3t) + 3 \sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} 5 \sin(3t) \\ 4 \sin(3t) - 3 \cos(3t) \end{bmatrix}$$

Example 5



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- An **stable equilibrium** is one where the trajectories neither grow nor decay, they just circle in a periodic motion. (Since $\alpha = 0$.)

Nullclines for a DE System

For a two-dimensional system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

- The **v -nullcline** is the set of all points with vertical slope, which occur on the curve obtained by solving $x' = f(x, y) = 0$.
- The **h -nullcline** is the set of all points with horizontal slope, which occur on the curve obtained by solving $y' = g(x, y) = 0$.

When an h -nullcline and an v -nullcline intersect, an **equilibrium** occurs.

Interpreting the Solutions

For $\vec{x}' = \mathbf{A}\vec{x}$ with nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm \beta i$ and complex eigenvectors $\vec{v}_1, \vec{v}_2 = \vec{p} + \vec{q}i$, arrange the components of the solution as

$$\begin{bmatrix} \vec{x}_{\text{Re}} \\ \vec{x}_{\text{Im}} \end{bmatrix} = \underbrace{e^{\alpha t}}_{\text{expansion}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \vec{p} \\ \vec{q} \end{bmatrix}}_{\text{tilt and shape}}$$

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① The first factor $e^{\alpha t}$ determines *expansion or contraction*.

- If $\alpha > 0$, then trajectories spiral outward, representing unbound growth.
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- 3 The third factor, containing \vec{p} and \vec{q} , determines the *tilt* and *shape* of the *elliptical trajectories* that would result with $\alpha = 0$.