

# Systems of Linear Equations

Department of Mathematics

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(Slides by Adam Wilson)

## System of Linear Equations

A  $m \times n$  **system of linear equations** is a set of  $m$  equations in  $n$  variables  $x_1, x_2, \dots, x_n$  of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & \dots & + & a_{3n}x_n & = & b_3 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

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## A Matrix Equation (We will look at these in section 3.3)

As the matrix equation  $A\vec{x} = \vec{b}$ , where:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

## Row Operation Notation

- $r_i$  denotes row  $i$  *before* the row operation is applied
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- Add row  $j$  to row  $i$  (leaving row  $j$  unchanged):

$$R_i = r_i + r_j$$

## Gaussian Elimination

Use row operations until the augmented matrix is in **Row Echelon Form**:

$$\left[ \begin{array}{ccccc|c} 1 & c_{12} & c_{13} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & d_2 \\ 0 & 0 & 1 & \cdots & c_{3n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_m \end{array} \right]$$

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Then back solve the system:

$$x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n = d_1$$

$$x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2$$

$$\vdots$$

$$x_n = d_m$$

## Example 1

Consider the system

$$\begin{array}{rcccccc} x & + & y & + & z & = & 3 \\ 2x & - & 3y & - & z & = & -8 \\ -x & + & 2y & + & 2z & = & 3 \end{array}$$

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We can write this as the augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{array} \right]$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

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Plug both into the first equation and solve for  $x$ :

$$x + (4) + (-2) = 3 \quad \Rightarrow \quad x = 1$$

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  - A system with more than one solution is called **dependent**.

## Reduced Row Echelon Form

An augmented matrix is said to be in **Reduced Row Echelon Form** if:

$$\left[ \begin{array}{ccc|c} 1 & \cdots & 0 & k_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & k_m \end{array} \right]$$

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## Rank

The **rank**  $r$  of a matrix is equal to how many 1's are in the diagonal of it's Reduced Row Echelon Form.

- If  $r$  equals the number of variables, there is a unique solution.
- If  $r$  is less than the number of variables, the solutions are not unique.