# Linear Systems with Real Eigenvalues

Department of Mathematics

Salt Lake Community College

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If $r_1$ and $r_2$ are the characteristic roots, then solutions are build from $e^{r_1 t}$ and $e^{r_2 t}$. So, we need to find similar building blocks for the system.

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But, these are just the eigenvalues and eigenvectors of $\boldsymbol{A}$!

## Solving Homogeneous Linear $2 \times 2$ DE Systems with Constant Coefficients

For a two-dimensional system of homogeneous linear differential equations

$$\vec{x'} = A\vec{x}$$

where $A$ is a matrix of constants and has eigenvalues $\lambda_1$ and $\lambda_2$ with corresponding eigenvectors $\vec{v_1}$ and $\vec{v_2}$. We can obtain the two solutions:

$$e^{\lambda_1 t}\vec{v_1} \quad \text{and} \quad e^{\lambda_2 t}\vec{v_2}$$

- If $\lambda_1 \neq \lambda_2$, then these two solutions are linearly independent and form a basis for the solutions space. Thus, the general solutions, for $c_1, c_2 \in \mathbb{R}$, is
$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$$
- If $\lambda_1 = \lambda_2$, then there may be only one linearly independent eigenvector. Additional tactics may be required to obtain a basis of two vectors for the solution space.

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We will first consider some examples where the eigenvalues are distinct.

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Consider the system

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The matrix $\boldsymbol{A}$ has eigenvalues

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and eigenvectors

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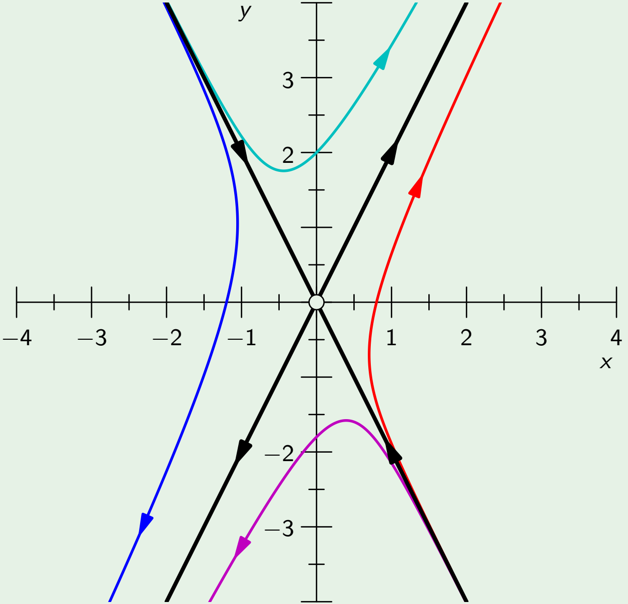

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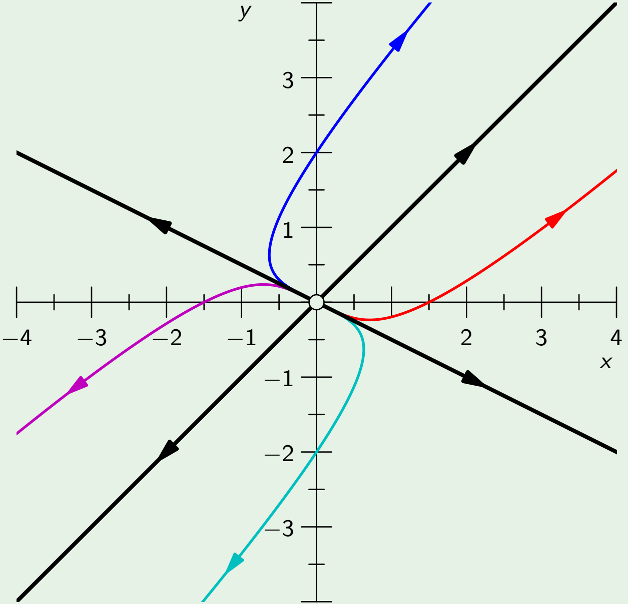

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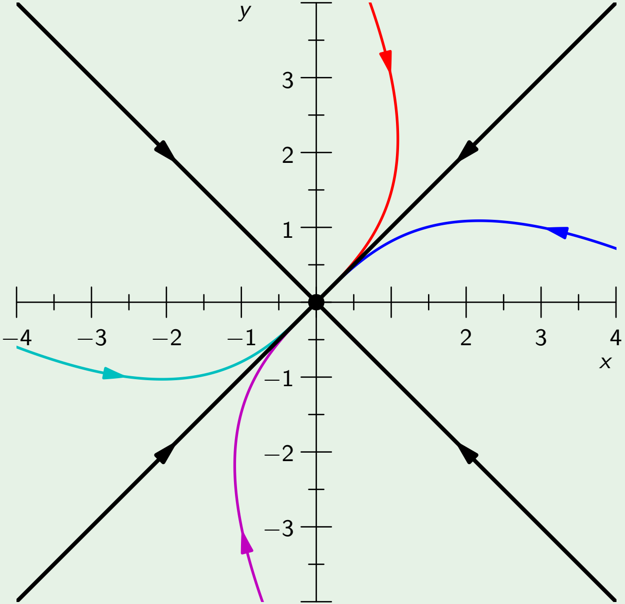

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- The equilibrium occurs at the origin, and the phase portrait is symmetric about this point.

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  - "fast" for the eigenvalue with the largest magnitude.
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- Trajectories become parallel to the fast eigenvectors further away from the origin, and tangent to the slow eigenvectors. (closer to the origin for sources or sinks, further from the origin for a saddle.)

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The matrix $\boldsymbol{A}$ has a (repeated) eigenvalue

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and two linearly independent eigenvectors

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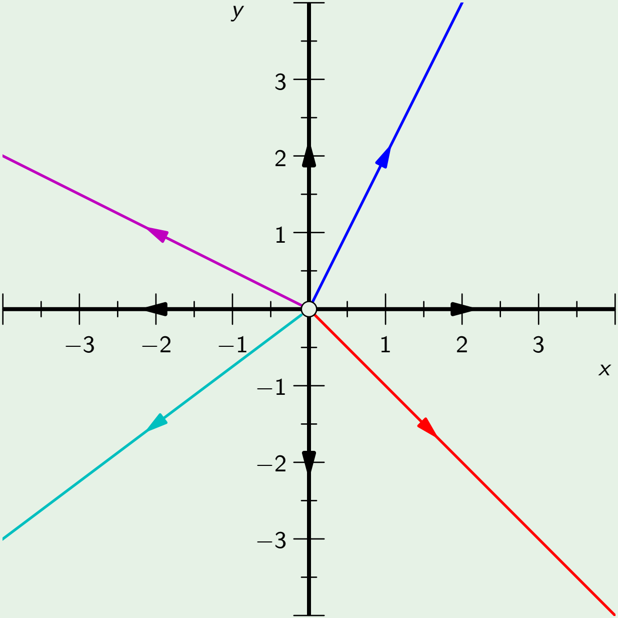

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Which is true if and only if $\vec{v} = \vec{0}$. But this contradicts that $\vec{v}$ is an eigenvector. We will need to try something else.

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Equating coefficients gives the system or equations:

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$$\begin{aligned} \left(te^{4t}\vec{v} + e^{4t}\vec{u}\right)' &= \boldsymbol{A}\left(te^{4t}\vec{v} + e^{4t}\vec{u}\right) \\ e^{4t}\vec{v} + 4te^{4t}\vec{v} + 4e^{4t}\vec{u} &= \boldsymbol{A}\left(te^{4t}\vec{v} + e^{4t}\vec{u}\right) \end{aligned}$$

Equating coefficients gives the system or equations:

$$\begin{array}{rcl} 4\vec{v} & = & \boldsymbol{A}\vec{v} \\ \vec{v} + 4\vec{u} & = & \boldsymbol{A}\vec{u} \end{array} \quad \text{or} \quad \begin{array}{rcl} (\boldsymbol{A} - 4\boldsymbol{I})\vec{v} & = & \vec{0} \\ (\boldsymbol{A} - 4\boldsymbol{I})\vec{u} & = & \vec{v} \end{array}$$

Since $\vec{v}$ is an eigenvector, we only need to find a solution for $\vec{u}$.

## Example 6

Consider the system

$$\vec{x'} = \boldsymbol{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Which means we need to solve:

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

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So, we have $u_1 + \frac{1}{2}u_2 = -\frac{1}{2}$. Letting $u_1 = k$ and $u_2 = -2k - 1$ we get

$$\vec{\boldsymbol{u}} = \begin{bmatrix} k \\ -2k - 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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And,

$$\vec{x_2} = te^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + ke^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = te^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

## Example 6

Consider the system

$$\vec{\boldsymbol{x}'} = \boldsymbol{A}\vec{\boldsymbol{x}} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{\boldsymbol{x}}$$

Thus, the general solution is

$$\vec{\boldsymbol{x}} = c_1 e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \left( te^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

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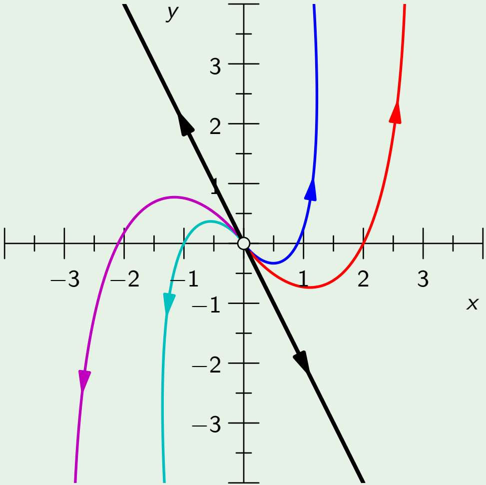

## Creating a Generalized Eigenvector for a System with Insufficient Eigenvectors

If a homogeneous linear $2 \times 2$ system of first-order DEs has repeated eigenvalue $\lambda$ with only a single eigenvector, a second linearly independent solution can be created as follows:

1. Find an eigenvector $\vec{\boldsymbol{v}}$ corresponding to $\lambda$.
2. Find a nonzero vector $\vec{\boldsymbol{u}}$ such that

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}}$$

3. Then the general solution is

$$\vec{\boldsymbol{x}}(t) = c_1 e^{\lambda t}\vec{\boldsymbol{v}} + c_2 e^{\lambda t}(t\vec{\boldsymbol{v}} + \vec{\boldsymbol{u}})$$

The vector $\vec{\boldsymbol{u}}$ is called a **generalized eigenvector** of $\boldsymbol{A}$ corresponding to $\lambda$.

### Solving $n \times n$ Homogeneous Linear DE Systems with Constant Coefficients

For an $n$-dimensional system of homogeneous linear differential equations $\vec{\boldsymbol{x}'} = \boldsymbol{A}\vec{\boldsymbol{x}}$ where $\boldsymbol{A}$ is a matrix of constants that has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_n}}$, we obtain solutions

$$e^{\lambda_1 t}\vec{\boldsymbol{v_1}}, e^{\lambda_2 t}\vec{\boldsymbol{v_2}}, \ldots, e^{\lambda_n t}\vec{\boldsymbol{v_n}}$$

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If all eigenvalues are distinct, then these solutions are linearly independent and form a basis of the solution space. Thus, the general solution, for $c_1, c_2, \ldots, c_n \in \mathbb{R}$, is

$$\vec{\boldsymbol{x}} = c_1 e^{\lambda_1 t}\vec{\boldsymbol{v_1}} + c_2 e^{\lambda_2 t}\vec{\boldsymbol{v_2}} + \cdots + c_n e^{\lambda_n t}\vec{\boldsymbol{v_n}}$$

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The case where eigenvalues are repeated will require either independent eigenvectors or generalized eigenvectors.