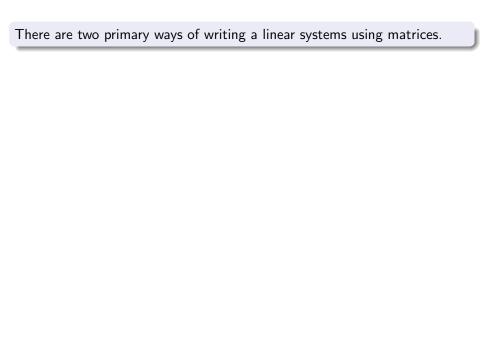
# Systems of Linear Equations

Colby Community College

# System of Linear Equations

A  $m \times n$  system of linear equations is a set of m equations in n variables  $x_1, x_2, \dots, x_n$  of the form



There are two primary ways of writing a linear systems using matrices.

### An Augmented Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

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# A Matrix Equation (We will look at these in section 3.3)

As the matrix equation  $A\vec{x} = \vec{b}$ , where:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

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$$R_i \leftrightarrow R_j$$
 (or  $R_i = r_j$ ,  $R_j = r_i$ )

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Add row j to row i (leaving row j unchanged):

$$R_i = r_i + r_j$$

#### Gaussian Elimination

Use row operations until the augmented matrix is in Row Echelon Form:

$$\begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & d_2 \\ 0 & 0 & 1 & \cdots & c_{3n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_m \end{bmatrix}$$

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Then back solve the system:

$$x_1 + c_{12}x_2 + c_{13}x_3 + \dots + c_{1n}x_n = d_1$$
  
 $x_2 + c_{23}x_3 + \dots + c_{2n}x_n = d_2$   
 $\vdots$   
 $x_n = d_m$ 

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We can write this as the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{bmatrix}$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

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$$\left[ egin{array}{ccc|c} 1 & 1 & 1 & 3 \ 0 & 1 & 3 & -2 \ -1 & 2 & 2 & 3 \ \end{array} 
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$$egin{bmatrix} 1 & 1 & 1 & 3 \ 0 & 1 & 3 & -2 \ -1 & 2 & 2 & 3 \end{bmatrix} R_3 = r_1 + r_3$$

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Now, back solve the system

$$\begin{array}{rclcrcr}
x & + & y & + & z & = & 3 \\
y & + & 3z & = & -2 \\
z & = & -2
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$$y + 3(-2) = -2 \quad \Rightarrow \quad y = 4$$

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$$y + 3(-2) = -2 \Rightarrow y = 4$$

Plug both into the first equation and solve for x:

$$x + (4) + (-2) = 3 \implies x = 1$$

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#### Reduced Row Echelon Form

An augmented matrix is said to be in Reduced Row Echelon Form if:

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#### Rank

The **rank** r of a matrix is equal to how many 1's are in the diagonal of it's Reduced Row Echelon Form.

- If r equals the number of variables, there is a unique solution.
- If r is less than the number of variables, the solutions are not unique.