Separation of Variables: Quantitative Analysis

Department of Mathematics

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(Slides by Adam Wilson)

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The method we just used is called **Separation of Variables**.

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Suppose $G(y)$ and $F(t)$ are antiderivatives of $\dfrac{1}{g(y)}$ and $f(t)$, respectively.

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This has shown that y is a solution to $y' = f(t)g(y)$ and explains why the previous example works.

Method of Separation of Variables

Step 1: Set $g(y) = 0$ and solve to find any equilibria.

Step 2: Now, assume that $g(y) \neq 0$. Rewrite the equation in separated form:

$$\frac{dy}{g(y)} = f(t)dt$$

Step 3: Integrate, if possible, each side:

$$\int \frac{dy}{g(y)} = \int f(t)dt$$

(obtaining the implicit one-parameter family of solutions.)

Step 4: If possible, solve for y in terms of t, getting the explicit solution $y = y(t)$

Step 5: If you have an IVP, use the initial condition to evaluate c.

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4. $\dfrac{dy}{dt} = t + y \quad \Rightarrow \quad$ (not separable)

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Note that because of the restrictions, each solution curve in the direction field will be a piecewise combination of several functions. A particular solution of an IVP for this DE would only be *one* of these.

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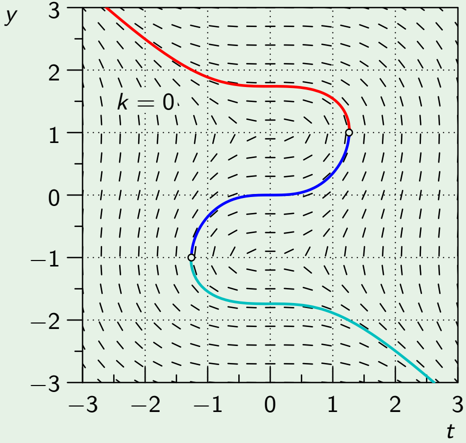

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Which give the solution:

$$\begin{aligned} y + y^2 &= t^3 + t + 2 \\ y &= \frac{1}{2}\left(-1 \pm \sqrt{4t^3+4t+9}\right) \end{aligned}$$

Again, the solution curve is made up of multiple parts.

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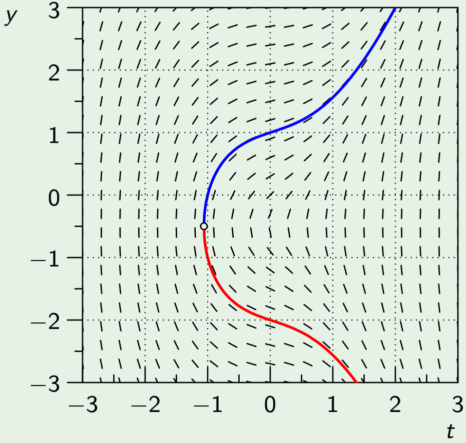

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We can then use the initial condition to find c.

$$\frac{1}{2} = -\frac{0^2}{2} + c \quad \Rightarrow \quad c = \frac{1}{2}$$

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Solve the IVP.

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We have two explicit solutions:

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