

Linear Systems with Real Eigenvalues

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Example 1

Let us first look at solutions for a 2×2 system, where \mathbf{A} is constant

$$\vec{x}' = \mathbf{A}\vec{x}$$

Example 1

Let us first look at solutions for a 2×2 system, where \mathbf{A} is constant

$$\vec{x}' = \mathbf{A}\vec{x}$$

We can start with what we have learned about the solutions for

$$ay'' + by' + cy = 0$$

Example 1

Let us first look at solutions for a 2×2 system, where \mathbf{A} is constant

$$\vec{x}' = \mathbf{A}\vec{x}$$

We can start with what we have learned about the solutions for

$$ay'' + by' + cy = 0$$

We know that when the second-order equation is converted to a system, the characteristic roots are the eigenvalues of the system. How can we use this fact?

Example 1

Let us first look at solutions for a 2×2 system, where \mathbf{A} is constant

$$\vec{x}' = \mathbf{A}\vec{x}$$

We can start with what we have learned about the solutions for

$$ay'' + by' + cy = 0$$

We know that when the second-order equation is converted to a system, the characteristic roots are the eigenvalues of the system. How can we use this fact?

If r_1 and r_2 are the characteristic roots, then solutions are build from $e^{r_1 t}$ and $e^{r_2 t}$. So, we need to find similar building blocks for the system.

Example 1

Given that solutions to the matrix-vector equation must be vectors:

$$\vec{x} = e^{\lambda t} \vec{v}$$

Example 1

Given that solutions to the matrix-vector equation must be vectors:

$$\vec{x} = e^{\lambda t} \vec{v}$$

Which we can substitute into the matrix-vector equation

$$\vec{x}' = \mathbf{A}\vec{x}$$

Example 1

Given that solutions to the matrix-vector equation must be vectors:

$$\vec{x} = e^{\lambda t} \vec{v}$$

Which we can substitute into the matrix-vector equation

$$\vec{x}' = \mathbf{A}\vec{x}$$

$$\lambda e^{\lambda t} \vec{v} = \mathbf{A} e^{\lambda t} \vec{v}$$

Example 1

Given that solutions to the matrix-vector equation must be vectors:

$$\vec{x} = e^{\lambda t} \vec{v}$$

Which we can substitute into the matrix-vector equation

$$\vec{x}' = \mathbf{A}\vec{x}$$

$$\lambda e^{\lambda t} \vec{v} = \mathbf{A} e^{\lambda t} \vec{v}$$

$$\vec{0} = e^{\lambda t} \mathbf{A} \vec{v} - \lambda e^{\lambda t} \vec{v}$$

Example 1

Given that solutions to the matrix-vector equation must be vectors:

$$\vec{x} = e^{\lambda t} \vec{v}$$

Which we can substitute into the matrix-vector equation

$$\vec{x}' = \mathbf{A}\vec{x}$$

$$\lambda e^{\lambda t} \vec{v} = \mathbf{A} e^{\lambda t} \vec{v}$$

$$\vec{0} = e^{\lambda t} \mathbf{A} \vec{v} - \lambda e^{\lambda t} \vec{v}$$

$$\vec{0} = e^{\lambda t} (\mathbf{A} - \lambda \mathbf{I}) \vec{v}$$

Example 1

Given that solutions to the matrix-vector equation must be vectors:

$$\vec{x} = e^{\lambda t} \vec{v}$$

Which we can substitute into the matrix-vector equation

$$\vec{x}' = \mathbf{A}\vec{x}$$

$$\lambda e^{\lambda t} \vec{v} = \mathbf{A} e^{\lambda t} \vec{v}$$

$$\vec{0} = e^{\lambda t} \mathbf{A} \vec{v} - \lambda e^{\lambda t} \vec{v}$$

$$\vec{0} = e^{\lambda t} (\mathbf{A} - \lambda \mathbf{I}) \vec{v}$$

Given that $e^{\lambda t} > 0$, we must find λ and \vec{v} that satisfy:

$$(\mathbf{A} - \lambda \mathbf{I}) \vec{v} = \vec{0}$$

Example 1

Given that solutions to the matrix-vector equation must be vectors:

$$\vec{x} = e^{\lambda t} \vec{v}$$

Which we can substitute into the matrix-vector equation

$$\vec{x}' = \mathbf{A}\vec{x}$$

$$\lambda e^{\lambda t} \vec{v} = \mathbf{A} e^{\lambda t} \vec{v}$$

$$\vec{0} = e^{\lambda t} \mathbf{A} \vec{v} - \lambda e^{\lambda t} \vec{v}$$

$$\vec{0} = e^{\lambda t} (\mathbf{A} - \lambda \mathbf{I}) \vec{v}$$

Given that $e^{\lambda t} > 0$, we must find λ and \vec{v} that satisfy:

$$(\mathbf{A} - \lambda \mathbf{I}) \vec{v} = \vec{0}$$

But, these are just the eigenvalues and eigenvectors of \mathbf{A} !

Solving Homogeneous Linear 2×2 DE Systems with Constant Coefficients

For a two-dimensional system of homogeneous linear differential equations

$$\vec{x}' = \mathbf{A}\vec{x}$$

where \mathbf{A} is a matrix of constants and has eigenvalues λ_1 and λ_2 with corresponding eigenvectors \vec{v}_1 and \vec{v}_2 . We can obtain the two solutions:

$$e^{\lambda_1 t} \vec{v}_1 \quad \text{and} \quad e^{\lambda_2 t} \vec{v}_2$$

- If $\lambda_1 \neq \lambda_2$, then these two solutions are linearly independent and form a basis for the solutions space. Thus, the general solutions, for $c_1, c_2 \in \mathbb{R}$, is

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

- If $\lambda_1 = \lambda_2$, then there may be only one linearly independent eigenvector. Additional tactics may be required to obtain a basis of two vectors for the solution space.

Solving Homogeneous Linear 2×2 DE Systems with Constant Coefficients

For a two-dimensional system of homogeneous linear differential equations

$$\vec{x}' = \mathbf{A}\vec{x}$$

where \mathbf{A} is a matrix of constants and has eigenvalues λ_1 and λ_2 with corresponding eigenvectors \vec{v}_1 and \vec{v}_2 . We can obtain the two solutions:

$$e^{\lambda_1 t} \vec{v}_1 \quad \text{and} \quad e^{\lambda_2 t} \vec{v}_2$$

- If $\lambda_1 \neq \lambda_2$, then these two solutions are linearly independent and form a basis for the solutions space. Thus, the general solutions, for $c_1, c_2 \in \mathbb{R}$, is

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

- If $\lambda_1 = \lambda_2$, then there may be only one linearly independent eigenvector. Additional tactics may be required to obtain a basis of two vectors for the solution space.

We will first consider some examples where the eigenvalues are distinct.

Example 2

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

Example 2

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 3$$

and eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 3$$

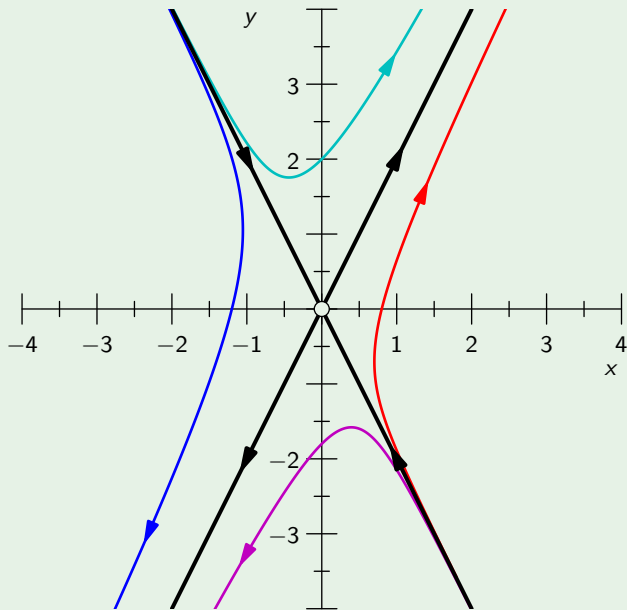
and eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thus, the general solution must be

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2



Example 3

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$$

Example 3

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has eigenvalues

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = 1$$

and eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Example 3

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has eigenvalues

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = 1$$

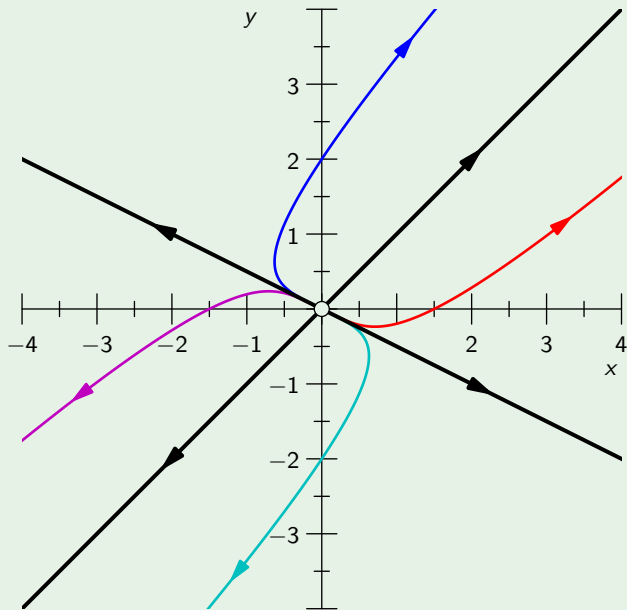
and eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Thus, the general solution must be

$$\vec{x} = c_1 e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Example 3



Example 4

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x}$$

Example 4

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -3$$

and eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example 4

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -3$$

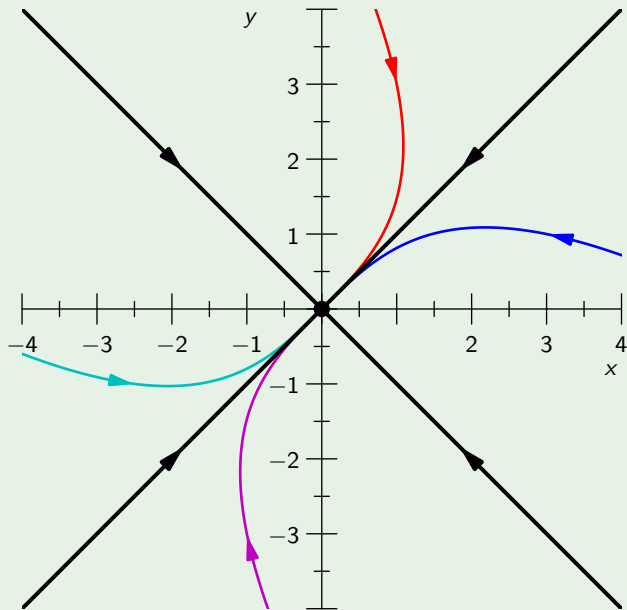
and eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, the general solution must be

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example 4



Behavior of Solutions

- If all trajectories tend towards the origin as $t \rightarrow \infty$, then the origin is a **stable** equilibrium. (Both eigenvalues are negative.)

Behavior of Solutions

- If all trajectories tend towards the origin as $t \rightarrow \infty$, then the origin is a **stable** equilibrium. (Both eigenvalues are negative.)
- If all trajectories tend away from the origin as $t \rightarrow \infty$, then the origin is an **unstable** equilibrium. (Both eigenvalues are positive.)

Behavior of Solutions

- If all trajectories tend towards the origin as $t \rightarrow \infty$, then the origin is a **stable** equilibrium. (Both eigenvalues are negative.)
- If all trajectories tend away from the origin as $t \rightarrow \infty$, then the origin is an **unstable** equilibrium. (Both eigenvalues are positive.)
- If only some solutions tend towards the origin as $t \rightarrow \infty$, then the origin is a **saddle** equilibrium. (Eigenvalues have different signs.)

Behavior of Solutions

- If all trajectories tend towards the origin as $t \rightarrow \infty$, then the origin is a **stable** equilibrium. (Both eigenvalues are negative.)
- If all trajectories tend away from the origin as $t \rightarrow \infty$, then the origin is an **unstable** equilibrium. (Both eigenvalues are positive.)
- If only some solutions tend towards the origin as $t \rightarrow \infty$, then the origin is a **saddle** equilibrium. (Eigenvalues have different signs.)

Phase Plane Role of Real Eigenvalues and Eigenvectors

For an autonomous and homogeneous two-dimensional DE system:

Behavior of Solutions

- If all trajectories tend towards the origin as $t \rightarrow \infty$, then the origin is a **stable** equilibrium. (Both eigenvalues are negative.)
- If all trajectories tend away from the origin as $t \rightarrow \infty$, then the origin is an **unstable** equilibrium. (Both eigenvalues are positive.)
- If only some solutions tend towards the origin as $t \rightarrow \infty$, then the origin is a **saddle** equilibrium. (Eigenvalues have different signs.)

Phase Plane Role of Real Eigenvalues and Eigenvectors

For an autonomous and homogeneous two-dimensional DE system:

- Trajectories move towards or away from the equilibrium according to the signs of the eigenvalues.

Behavior of Solutions

- If all trajectories tend towards the origin as $t \rightarrow \infty$, then the origin is a **stable** equilibrium. (Both eigenvalues are negative.)
- If all trajectories tend away from the origin as $t \rightarrow \infty$, then the origin is an **unstable** equilibrium. (Both eigenvalues are positive.)
- If only some solutions tend towards the origin as $t \rightarrow \infty$, then the origin is a **saddle** equilibrium. (Eigenvalues have different signs.)

Phase Plane Role of Real Eigenvalues and Eigenvectors

For an autonomous and homogeneous two-dimensional DE system:

- Trajectories move towards or away from the equilibrium according to the signs of the eigenvalues.
- Along each eigenvector is a unique trajectory called a **separatrix** that separates trajectories curving one way from those curving another way.

Behavior of Solutions

- If all trajectories tend towards the origin as $t \rightarrow \infty$, then the origin is a **stable** equilibrium. (Both eigenvalues are negative.)
- If all trajectories tend away from the origin as $t \rightarrow \infty$, then the origin is an **unstable** equilibrium. (Both eigenvalues are positive.)
- If only some solutions tend towards the origin as $t \rightarrow \infty$, then the origin is a **saddle** equilibrium. (Eigenvalues have different signs.)

Phase Plane Role of Real Eigenvalues and Eigenvectors

For an autonomous and homogeneous two-dimensional DE system:

- Trajectories move towards or away from the equilibrium according to the signs of the eigenvalues.
- Along each eigenvector is a unique trajectory called a **separatrix** that separates trajectories curving one way from those curving another way.
- The equilibrium occurs at the origin, and the phase portrait is symmetric about this point.

Recall

The magnitude of a vector is defined as:

$$\|\vec{v}\| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

Recall

The magnitude of a vector is defined as:

$$\|\vec{v}\| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

Speed and Shape of Trajectories

Recall

The magnitude of a vector is defined as:

$$\|\vec{v}\| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

Speed and Shape of Trajectories

- The “speed” along a trajectory in the direction of an eigenvector depends on the magnitude of the associated eigenvalue:

Recall

The magnitude of a vector is defined as:

$$\|\vec{v}\| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

Speed and Shape of Trajectories

- The “speed” along a trajectory in the direction of an eigenvector depends on the magnitude of the associated eigenvalue:
 - “fast” for the eigenvalue with the largest magnitude.

Recall

The magnitude of a vector is defined as:

$$\|\vec{v}\| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

Speed and Shape of Trajectories

- The “speed” along a trajectory in the direction of an eigenvector depends on the magnitude of the associated eigenvalue:
 - “fast” for the eigenvalue with the largest magnitude.
 - “slow” for the eigenvalue with the smallest magnitude.

Recall

The magnitude of a vector is defined as:

$$\|\vec{v}\| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

Speed and Shape of Trajectories

- The “speed” along a trajectory in the direction of an eigenvector depends on the magnitude of the associated eigenvalue:
 - “fast” for the eigenvalue with the largest magnitude.
 - “slow” for the eigenvalue with the smallest magnitude.
- Trajectories become parallel to the fast eigenvectors further away from the origin, and tangent to the slow eigenvectors. (closer to the origin for sources or sinks, further from the origin for a saddle.)

Example 5

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \vec{x}$$

Example 5

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has a (repeated) eigenvalue

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 3$$

and two linearly independent eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 5

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has a (repeated) eigenvalue

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 3$$

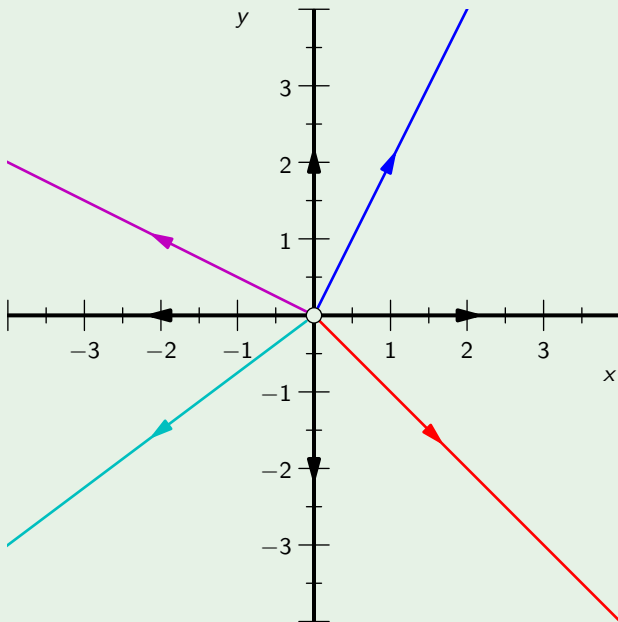
and two linearly independent eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, the general solution must be

$$\vec{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 5



Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has a (repeated) eigenvalue

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = 4$$

and only one eigenvector

$$\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has a (repeated) eigenvalue

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = 4$$

and only one eigenvector

$$\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, one solution is

$$\vec{x}_1 = e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

The matrix \mathbf{A} has a (repeated) eigenvalue

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = 4$$

and only one eigenvector

$$\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, one solution is

$$\vec{x}_1 = e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

But, we need two solutions to form a basis of the solution space!

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us try the same trick we used for second-order systems:

$$\vec{x}_2 = te^{4t}\vec{v}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us try the same trick we used for second-order systems:

$$\vec{x}_2 = te^{4t}\vec{v}$$

Substituting into the differential equation gives:

$$(te^{4t}\vec{v})' = \mathbf{A}(te^{4t}\vec{v})$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us try the same trick we used for second-order systems:

$$\vec{x}_2 = te^{4t}\vec{v}$$

Substituting into the differential equation gives:

$$\begin{aligned} (te^{4t}\vec{v})' &= \mathbf{A}(te^{4t}\vec{v}) \\ e^{4t}\vec{v} + 4te^{4t}\vec{v} &= te^{4t}\mathbf{A}\vec{v} \end{aligned}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us try the same trick we used for second-order systems:

$$\vec{x}_2 = te^{4t}\vec{v}$$

Substituting into the differential equation gives:

$$\begin{aligned} (te^{4t}\vec{v})' &= \mathbf{A}(te^{4t}\vec{v}) \\ e^{4t}\vec{v} + 4te^{4t}\vec{v} &= te^{4t}\mathbf{A}\vec{v} \end{aligned}$$

Which is true if and only if $\vec{v} = \vec{0}$. But this contradicts that \vec{v} is an eigenvector. We will need to try something else.

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us instead try to introduce a second vector \vec{u} , that multiplies the troublesome e^{4t} term:

$$\vec{x}_2 = te^{4t}\vec{v} + e^{4t}\vec{u}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us instead try to introduce a second vector \vec{u} , that multiplies the troublesome e^{4t} term:

$$\vec{x}_2 = te^{4t}\vec{v} + e^{4t}\vec{u}$$

Let us plug \vec{x}_2 into the differential equation:

$$(te^{4t}\vec{v} + e^{4t}\vec{u})' = \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u})$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us instead try to introduce a second vector \vec{u} , that multiplies the troublesome e^{4t} term:

$$\vec{x}_2 = te^{4t}\vec{v} + e^{4t}\vec{u}$$

Let us plug \vec{x}_2 into the differential equation:

$$\begin{aligned} (te^{4t}\vec{v} + e^{4t}\vec{u})' &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \\ e^{4t}\vec{v} + 4te^{4t}\vec{v} + 4e^{4t}\vec{u} &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \end{aligned}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us instead try to introduce a second vector \vec{u} , that multiplies the troublesome e^{4t} term:

$$\vec{x}_2 = te^{4t}\vec{v} + e^{4t}\vec{u}$$

Let us plug \vec{x}_2 into the differential equation:

$$\begin{aligned} (te^{4t}\vec{v} + e^{4t}\vec{u})' &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \\ e^{4t}\vec{v} + 4te^{4t}\vec{v} + 4e^{4t}\vec{u} &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \end{aligned}$$

Equating coefficients gives the system or equations:

$$\begin{aligned} 4\vec{v} &= \mathbf{A}\vec{v} \\ \vec{v} + 4\vec{u} &= \mathbf{A}\vec{u} \end{aligned}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us instead try to introduce a second vector \vec{u} , that multiplies the troublesome e^{4t} term:

$$\vec{x}_2 = te^{4t}\vec{v} + e^{4t}\vec{u}$$

Let us plug \vec{x}_2 into the differential equation:

$$\begin{aligned} (te^{4t}\vec{v} + e^{4t}\vec{u})' &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \\ e^{4t}\vec{v} + 4te^{4t}\vec{v} + 4e^{4t}\vec{u} &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \end{aligned}$$

Equating coefficients gives the system or equations:

$$\begin{aligned} 4\vec{v} &= \mathbf{A}\vec{v} & \text{or} & & (\mathbf{A} - 4\mathbf{I})\vec{v} &= \vec{0} \\ \vec{v} + 4\vec{u} &= \mathbf{A}\vec{u} & & & (\mathbf{A} - 4\mathbf{I})\vec{u} &= \vec{v} \end{aligned}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Let us instead try to introduce a second vector \vec{u} , that multiplies the troublesome e^{4t} term:

$$\vec{x}_2 = te^{4t}\vec{v} + e^{4t}\vec{u}$$

Let us plug \vec{x}_2 into the differential equation:

$$\begin{aligned} (te^{4t}\vec{v} + e^{4t}\vec{u})' &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \\ e^{4t}\vec{v} + 4te^{4t}\vec{v} + 4e^{4t}\vec{u} &= \mathbf{A}(te^{4t}\vec{v} + e^{4t}\vec{u}) \end{aligned}$$

Equating coefficients gives the system or equations:

$$\begin{array}{rcl} 4\vec{v} & = & \mathbf{A}\vec{v} \\ \vec{v} + 4\vec{u} & = & \mathbf{A}\vec{u} \end{array} \quad \text{or} \quad \begin{array}{rcl} (\mathbf{A} - 4\mathbf{I})\vec{v} & = & \vec{0} \\ (\mathbf{A} - 4\mathbf{I})\vec{u} & = & \vec{v} \end{array}$$

Since \vec{v} is an eigenvector, we only need to find a solution for \vec{u} .

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Which means we need to solve:

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Which means we need to solve:

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \rightarrow \quad \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 4 & 2 & -2 \end{array} \right]$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Which means we need to solve:

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 4 & 2 & -2 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{array} \right]$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Which means we need to solve:

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 4 & 2 & -2 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{array} \right]$$

So, we have $u_1 + \frac{1}{2}u_2 = -\frac{1}{2}$. Letting $u_1 = k$ and $u_2 = -2k - 1$ we get

$$\vec{u} = \begin{bmatrix} k \\ -2k - 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

Which means we need to solve:

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 4 & 2 & -2 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{array} \right]$$

So, we have $u_1 + \frac{1}{2}u_2 = -\frac{1}{2}$. Letting $u_1 = k$ and $u_2 = -2k - 1$ we get

$$\vec{u} = \begin{bmatrix} k \\ -2k - 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

And,

$$\vec{x}_2 = te^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + ke^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = te^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$

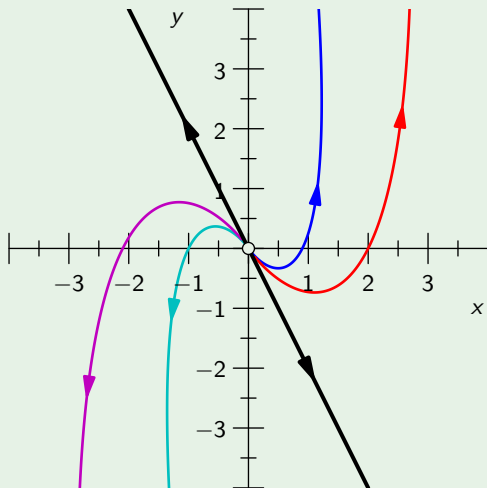
Thus, the general solution is

$$\vec{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \left(t e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

Example 6

Consider the system

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix} \vec{x}$$



Creating a Generalized Eigenvector for a System with Insufficient Eigenvectors

If a homogeneous linear 2×2 system of first-order DEs has repeated eigenvalue λ with only a single eigenvector, a second linearly independent solution can be created as follows:

- 1 Find an eigenvector \vec{v} corresponding to λ .
- 2 Find a nonzero vector \vec{u} such that

$$(\mathbf{A} - \lambda \mathbf{I})\vec{u} = \vec{v}$$

- 3 Then the general solution is

$$\vec{x}(t) = c_1 e^{\lambda t} \vec{v} + c_2 e^{\lambda t} (t \vec{v} + \vec{u})$$

The vector \vec{u} is called a **generalized eigenvector** of \mathbf{A} corresponding to λ .

Solving $n \times n$ Homogeneous Linear DE Systems with Constant Coefficients

For an n -dimensional system of homogeneous linear differential equations $\vec{x}' = \mathbf{A}\vec{x}$ where \mathbf{A} is a matrix of constants that has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, we obtain solutions

$$e^{\lambda_1 t} \vec{v}_1, e^{\lambda_2 t} \vec{v}_2, \dots, e^{\lambda_n t} \vec{v}_n$$

Solving $n \times n$ Homogeneous Linear DE Systems with Constant Coefficients

For an n -dimensional system of homogeneous linear differential equations $\vec{x}' = \mathbf{A}\vec{x}$ where \mathbf{A} is a matrix of constants that has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, we obtain solutions

$$e^{\lambda_1 t} \vec{v}_1, e^{\lambda_2 t} \vec{v}_2, \dots, e^{\lambda_n t} \vec{v}_n$$

If all eigenvalues are distinct, then these solutions are linearly independent and form a basis of the solution space. Thus, the general solution, for $c_1, c_2, \dots, c_n \in \mathbb{R}$, is

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \dots + c_n e^{\lambda_n t} \vec{v}_n$$

Solving $n \times n$ Homogeneous Linear DE Systems with Constant Coefficients

For an n -dimensional system of homogeneous linear differential equations $\vec{x}' = \mathbf{A}\vec{x}$ where \mathbf{A} is a matrix of constants that has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, we obtain solutions

$$e^{\lambda_1 t} \vec{v}_1, e^{\lambda_2 t} \vec{v}_2, \dots, e^{\lambda_n t} \vec{v}_n$$

If all eigenvalues are distinct, then these solutions are linearly independent and form a basis of the solution space. Thus, the general solution, for $c_1, c_2, \dots, c_n \in \mathbb{R}$, is

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \dots + c_n e^{\lambda_n t} \vec{v}_n$$

The case where eigenvalues are repeated will require either independent eigenvectors or generalized eigenvectors.