

Vector Spaces

Department of Mathematics

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Definition 1

A **vector space** \mathbb{V} is a nonempty collection of objects called **vectors** for which the following operations

- Vector addition, denoted $\vec{x} + \vec{y}$
- Scalar multiplication, denoted $c\vec{x}$

satisfy the nine properties on the following slide. (For all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$ and all $c, d \in \mathbb{R}$)

Closure

① $c\vec{x} + d\vec{y} \in \mathbb{V}$

Addition

- ② There exists a **zero vector** $\vec{0} \in \mathbb{V}$ such that $\vec{x} + \vec{0} = \vec{x}$
- ③ For all $\vec{x} \in \mathbb{V}$ there exists $-\vec{x} \in \mathbb{V}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
- ④ $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- ⑤ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

Scalar Multiplication

- ⑥ $1\vec{x} = \vec{x}$
- ⑦ $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
- ⑧ $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
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Example 2

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Example 4

Thinking back, we can see that the properties for addition and scalar multiplication of matrices we saw in section 3.1 satisfy all nine requirements to be a vector space.

Which means, for any $m, n \in \mathbb{R}$, \mathbb{M}_{mn} is a vector space.

Definition 5

A **function space** is a vector space where the “vectors” are functions defined on an interval I . The addition and scalar multiplication operations are defined in the usual way:

- $(f + g)(t) = f(t) + g(t)$, for all $t \in I$
- $(cf)(t) = cf(t)$, for all $t \in I$

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Note

Solutions to *linear homogeneous* DEs form a vector space.

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The set of all solutions of the first order linear homogeneous DE

$$y' + p(t)y = 0$$

(where p and y are defined on some interval I) is a vector space.

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For solutions $u(t)$ and $v(t)$, as well as scalars a and b , we need to verify that $a \cdot u(t) + b \cdot v(t)$ is a solution.

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The set of all solutions of the first-order linear (but not homogeneous) DE

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Example 9

Consider the collection of all polynomials of degree ≤ 3 . A vector in this space is given by

$$P(t) = a_3x^3 + a_2x^2 + a_1x + a_0$$

where $a_3, a_2, a_1, a_0 \in \mathbb{R}$.

This collection is a vector space, verifiable using basic algebra.

Prominent Vector Spaces

- \mathbb{R}^2 , the space of all real ordered pairs.
- \mathbb{R}^3 , the space of all real ordered triples.
- \mathbb{R}^n , the space of all real ordered n -tuples.
- \mathbb{C}^n , the space of all complex n -tuples.
- \mathbb{P} , the space of all polynomials.
- \mathbb{P}_n , the space of all polynomials of degree $\leq n$
- \mathbb{M}_{mn} , the space of all $m \times n$ matrices.
- $\mathcal{C}(I)$, the space of all continuous functions defined on the interval I .
- $\mathcal{C}^n(I)$, the space of all functions, defined on the interval I , having n continuous derivatives.
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A nonempty subset, \mathbb{W} , of a vector space \mathbb{V} is a **subspace** of \mathbb{V} if

- $\vec{u} + \vec{v} \in \mathbb{W}$ for all $\vec{u}, \vec{v} \in \mathbb{W}$
- $c\vec{u} \in \mathbb{W}$ for all $\vec{u} \in \mathbb{W}$ and $c \in \mathbb{R}$

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The definition of a subspace guarantees closure, everything else is inherited from the parent vector space.

For example, given $\vec{u}, \vec{v} \in \mathbb{W}$, consider $\vec{u} + \vec{v}$.

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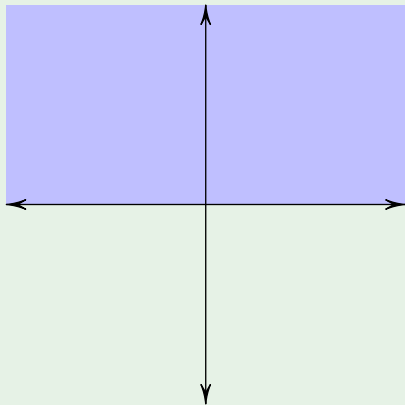
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Note

A vector space is a subspace of itself.

Example 11

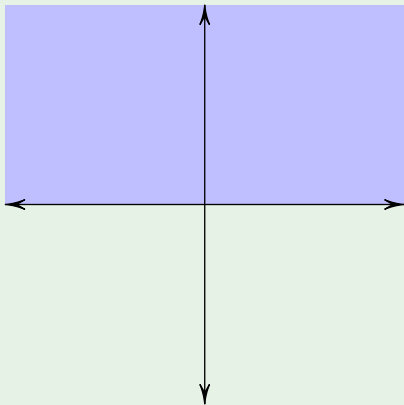
Is the upper half plane a subspace of \mathbb{R}^2 ?



Example 11

Is the upper half plane a subspace of \mathbb{R}^2 ?

No, points in the upper half plane are not closed under scalar multiplication.

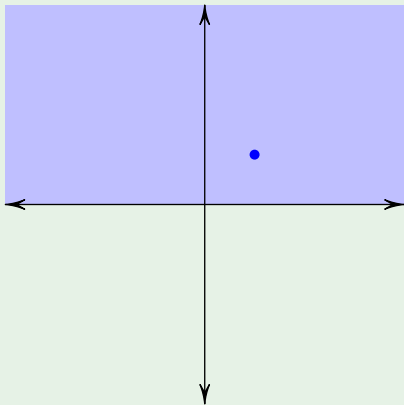


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No, points in the upper half plane are not closed under scalar multiplication.

Consider $(1, 1)$.



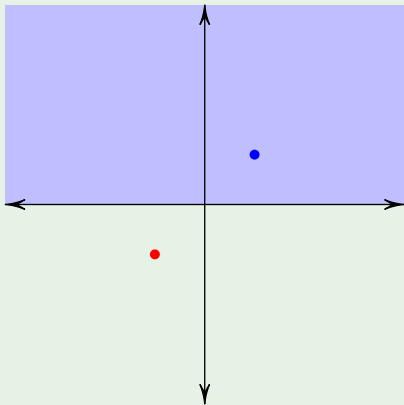
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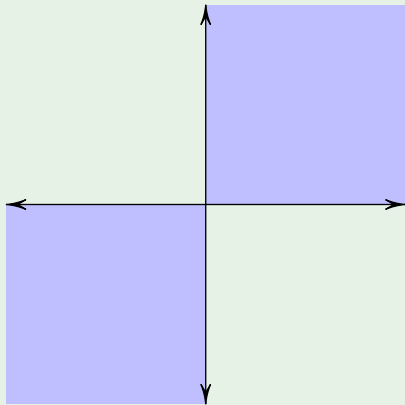
Consider $(1, 1)$.

Multiplying by the scalar -1 gives $(-1 \cdot 1, -1 \cdot 1) = (-1, -1)$, a point in Q3.



Example 12

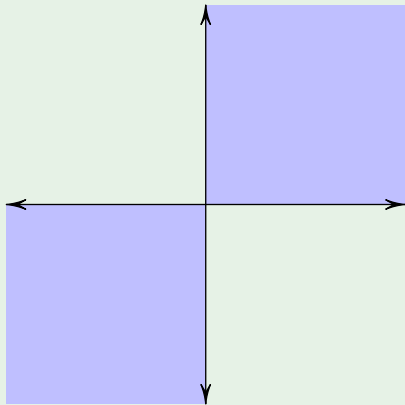
Is the set containing Q1 and Q3 a subspace of \mathbb{R}^2 ?



Example 12

Is the set containing $Q1$ and $Q3$ a subspace of \mathbb{R}^2 ?

No, points in the set containing $Q1$ and $Q3$ are not closed under addition.

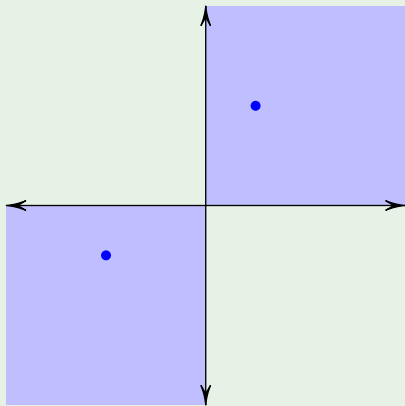


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No, points in the set containing $Q1$ and $Q3$ are not closed under addition.

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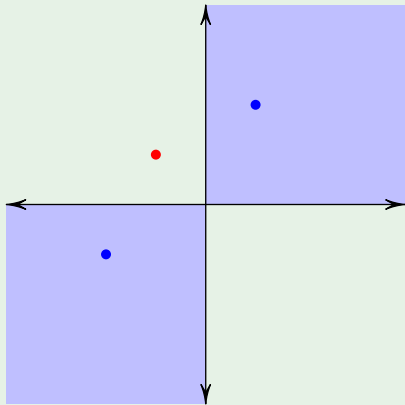
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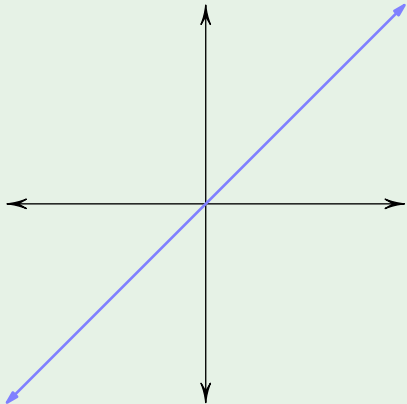
Consider $(1, 2)$ and $(-2, -1)$.

Adding these points gives
 $(1 + (-2), 2 + (-1)) = (-1, 1)$, a point in Q2.



Example 13

Is the line $y = x$ a subspace of \mathbb{R}^2 ?

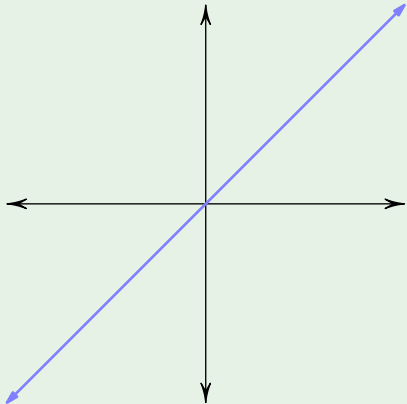


Example 13

Is the line $y = x$ a subspace of \mathbb{R}^2 ?

Yes. Given (s, s) and (t, t) , two points on the line, then

$$a \cdot (s, s) + b \cdot (t, t) =$$

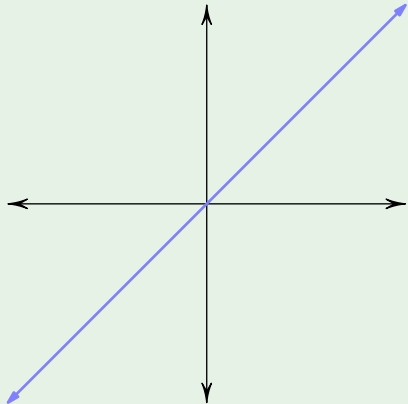


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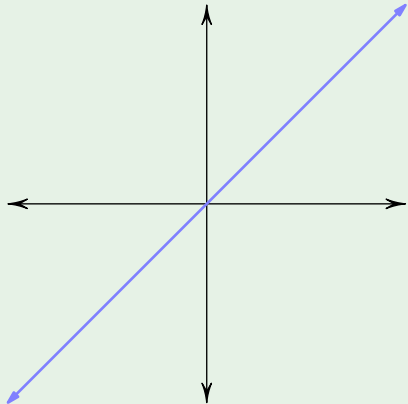


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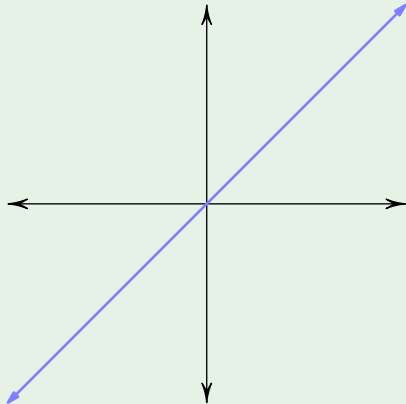
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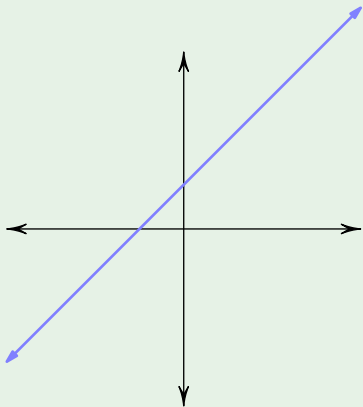
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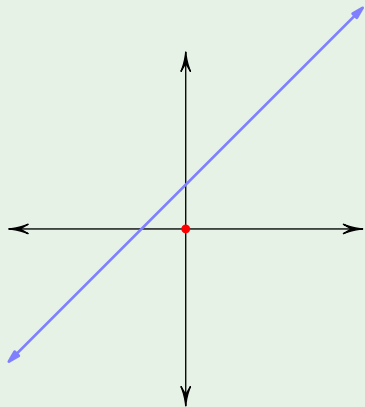
Is the line $y = x + 1$ a subspace of \mathbb{R}^2 ?



Example 14

Is the line $y = x + 1$ a subspace of \mathbb{R}^2 ?

No, the zero vector, $(0, 0)$ is not on the line.



Corollary 15

The only subspaces of \mathbb{R}^2 are

- *The zero subspace $(0, 0)$*
- *Any line passing through the origin*
- \mathbb{R}^2

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The set of solutions of the linear system $A\vec{x} = \vec{0}$ is a subspace of \mathbb{R}^m , where A is a $m \times n$ matrix and $\vec{x} \in \mathbb{R}^m$, is a subspace of \mathbb{R}^m .

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Since solutions to $A\vec{x} = \vec{0}$ are vectors in \mathbb{R}^m , the remaining properties are inherited from \mathbb{R}^m