The Harmonic Oscillator

Adam Wilson

Salt Lake Community College

Damped Harmonic Oscillators

Newton's Dot Notation

Scientists and Engineers who work with many variables where the independent variables is always t commonly use the notation:

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{dx^2}{d^2t} \quad \text{and} \quad \dddot{x} = \frac{dx^3}{d^3t} \quad \text{and} \quad \ddddot{x} = \frac{dx^4}{d^4t}$$

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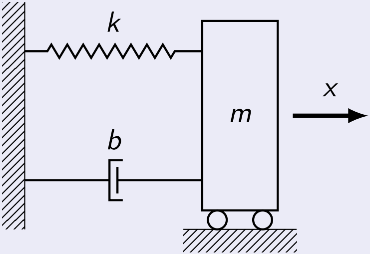
Definition

A very important DE is the second-order homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

where $m > 0$, b, and k are constants.
This models a class of phenomena called **damped harmonic oscillators**.

The Mass-Spring System



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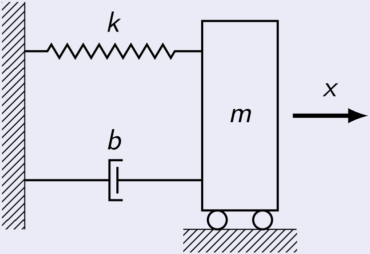

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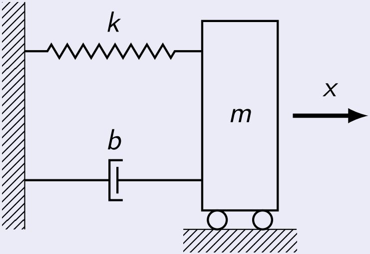

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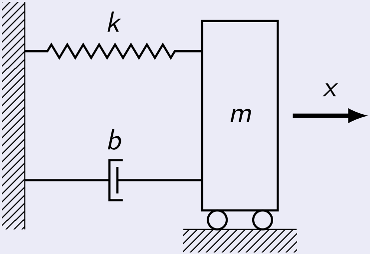

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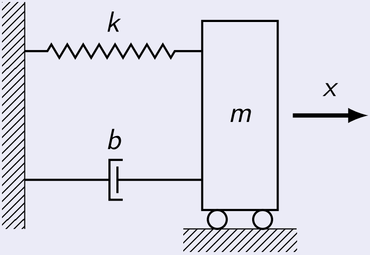

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Summing these forces gives:

$$\begin{aligned} \text{mass} \times \text{acceleration} &= F_{\text{restoring}} &+ F_{\text{damping}} &+ F_{\text{external}} \\ m\ddot{x} &= -kx &- b\dot{x} &+ f(t) \end{aligned}$$

Damped Harmonic Oscillators

Simple Harmonic Oscillator

The simple harmonic oscillator equation is

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with constant coefficients $m > 0$, $k > 0$, and $b \geq 0$.

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- When $b = 0$, the motion is called **undamped**; otherwise it is **damped**.
- If $f(t) = 0$ for all t, then the equation is homogeneous:

  $$m\ddot{x} + b\dot{x} + kx = 0$$

  and the motion is called **unforced**, **undriven**, or **free**; otherwise it is called **forced** or **driven**.

Damped Harmonic Oscillators

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Let us consider a mass of 1kg resting on a table that is attached to the wall by a spring.

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We also measure the damping force of the object sliding on the table to be 0.5 newtons when the velocity is 0.25 meters per second.

$$b = \frac{0.5 \text{ newton}}{0.25\frac{\text{meter}}{second}} = 2\frac{\text{newton second}}{\text{meter}}$$

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Notice that a second-order DE requires **two** initial conditions.

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Gravity (Earth)	$9.8\ \frac{m}{s^2}$	$980.665\ \frac{cm}{s^2}$	$32\ \frac{ft}{s^2}$

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$$(\sin(\omega_0 t))'' = -\omega_0^2 \sin(\omega_0 t) \quad \text{and} \quad \ddot{x} = -\frac{k}{m}x$$

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Another solutions is $x(t) = \cos(\omega_0 t)$.

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Solution of the Undamped Unforced Oscillator

For the undamped unforced oscillator

$$m\ddot{x} + kx = 0$$

we know two solutions:

$$x(t) = \cos(\omega_0 t) \quad \text{and} \quad x(t) = \sin(\omega_0 t) \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

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The Superposition Principle tells us that any linear combination of these two solutions is itself a solution. Thus, for $c_1, c_2 \in \mathbb{R}$, the family of solutions is

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We will see next section that these are all of the solutions.

Undamped Unforced Harmonic Oscillators

Alternate Form of the Undamped Unforced Oscillator Solution

Solutions to the undamped unforced oscillator may also be expressed as

$$x(t) = A\cos(\omega_0 t - \delta)$$

- A is the **amplitude**.
- δ is the **phase angle**, measured in radians.
- The motion has **circular frequency** $\omega_o = \sqrt{\frac{k}{m}}$, measured in $\frac{\text{radians}}{\text{second}}$.
- The **natural frequency** is $f_0 = \frac{\omega_0}{2\pi}$.
- The **period** is $\frac{1}{f_0} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$, measured in seconds.
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Converting from one form to the other

The translation is given by

$$A = \sqrt{c_1^2 + c_2^2}, \qquad \tan(\delta) = \frac{c_2}{c_1}$$

and

$$c_1 = A\cos(\delta), \qquad c_2 = A\sin(\delta)$$

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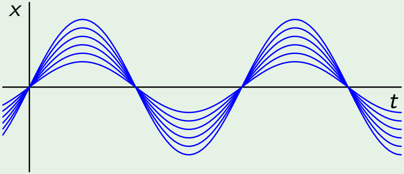
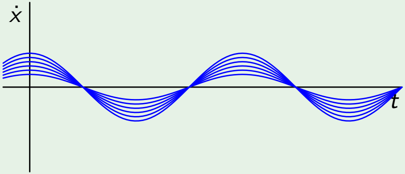
Substituting $t = 0$, $x(0) = 0$, and $\dot{x}(0) = 1$ into this system gives the solution $c_1 = 0$ and $c_2 = 1$.

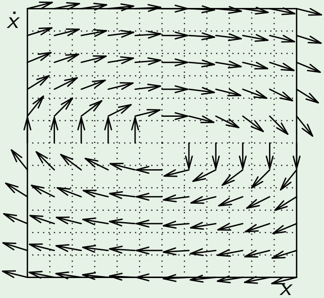
Plane Description

Example

Let us look at some plots concerning $\ddot{x} + 0.25x = 0$:







Plane Description

Phase Portraits

For any autonomous second-order differential equation

$$\ddot{x} = F(x, \dot{x})$$

the **phase plane** is the two-dimensional graph with x and $\dot{x}$ axes.

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$$\begin{aligned} \text{horizontal component} \quad & dx/dt = \dot{x} \\ \text{vertical component} \quad & d\dot{x}/dt = \ddot{x} \end{aligned}$$

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Note: Phase portraits can be graphed *without* solving the DE.

Second-Order Linear DE

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The second-order differential equation

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is equivalent to the system of first-order equations:

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Example

To draw the vector field for $\ddot{x} + 0.25x = 0$ we can build the system:

Second-Order Linear DE

Definition

The second-order differential equation

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

is equivalent to the system of first-order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \ddot{x} = \frac{f(t)}{m} - \frac{k}{m}x - \frac{b}{m}y \end{aligned}$$

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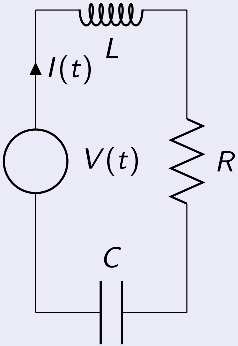
Then, pplane may be used to plot the phase portrait.

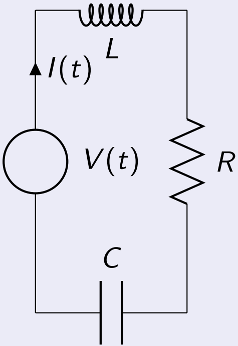
Electrical Circuits



The current I in a wire, measured in *amperes*, is the flow of charge Q. That is, the current is the rate of change of the charge

$$I(t) = \dot{Q}(t)$$

Electrical Circuits



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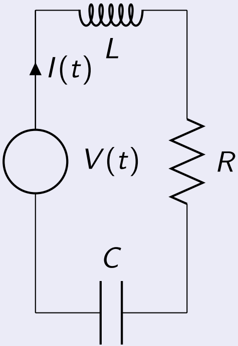
Kirchoff's Voltage Law tell us that the input voltage $V(t)$ is the sum of voltage drops around the circuit. In our circuit, we have three such voltage drops.

Modeling Electrical Circuits

Electrical Circuits



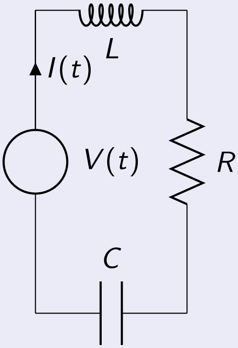
Drop across a Resistor: By **Ohm's Law**, the voltage drop across a resistor is proportional to the current passing through it.

$$V_R(t) = RI(t)$$

Where R is the **resistance** of the resistor and is measured in *ohms*.

Modeling Electrical Circuits

Electrical Circuits



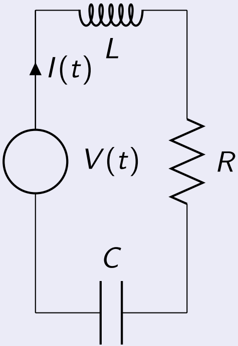
Drop across an Inductor: By Faraday's Law, the voltage drop across an inductor is proportional to the time rate of change of the current passing through it.

$$V_L(t) = L\dot{I}(t)$$

where L is the **inductance** and is measured in *henries*.

Modeling Electrical Circuits

Electrical Circuits



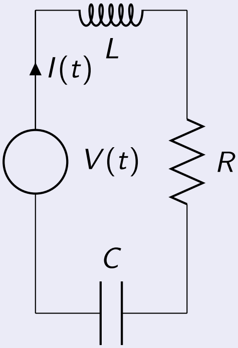
Drop across a Capacitor: The voltage drop across a capacitor is proportional to the charge $Q(t)$ on the capacitor.

$$V_C(t) = \frac{1}{C}Q(t) = \frac{1}{C}\int I(t)dt$$

where C is the **capacitance** of the capacitor and is measured in *farads*.

Modeling Electrical Circuits

Electrical Circuits



Thus, the voltage drop across the circuit is

$$V(t) = RI + L\dot{I} + \frac{1}{C}\int I(t)dt$$

This is called an **integro-differential equation** because it contains both a derivative and an integral.

Modeling Electrical Circuits

Using the fact that $I(t) = \dot{Q}(t)$ we can build the following equations.

Modeling Electrical Circuits

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Series Circuit Equation (Charge)

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)$$

If there is no voltage source $(V(t) = 0)$, then

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Modeling Electrical Circuits

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$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}(t)$$

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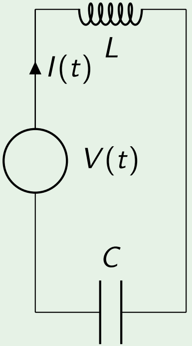
Modeling Electrical Circuits

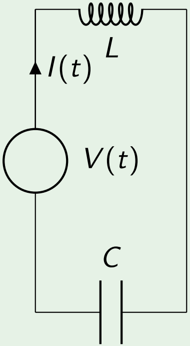
Example



Consider a circuit composed of a capacitor and inductor hooked up in series. Suppose that at $t = 0$ a charge Q_0 is put on the capacitor.

Modeling Electrical Circuits

Example



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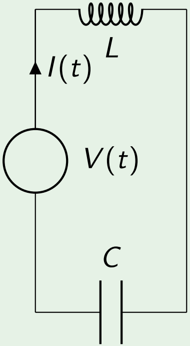
The IVP is

$$L\ddot{Q} + \frac{1}{C} = 0, \quad Q(0) = Q_0, \quad \dot{Q}(0) = 0$$

Modeling Electrical Circuits

Example



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Thus, the solution is

$$Q(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

where

$$\omega_0 = \sqrt{\frac{1}{LC}}$$