Variation of Parameters

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Which gives the general solution

$$y_h = c_1 y_1(t) + c_2 y_2(t)$$

where c_1 and c_2 are arbitrary constants.

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$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$

So, we can choose $v_1 y_1' + v_2 y_2' = 0$ as our auxiliary conditions, which reduces y_p' to:

$$y_p' = v_1' y_1 + v_2' y_2$$

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We can then obtain

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$$y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

We then substitute y_p, y_p', and y_p'' into $L(y) = f$.

$$(v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2') + p \cdot (v_1' y_1 + v_2' y_2) + q \cdot (v_1 y_1 + v_2 y_2) = f$$

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$$\begin{aligned}(v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2') + p \cdot (v_1' y_1 + v_2' y_2) + q \cdot (v_1 y_1 + v_2 y_2) &= f \\ v_1(y_1'' + p y_1' + q y_1) + v_2(y_2'' + p y_2' + q y_2) + (v_1' y_1' + v_2' y_2') &= f\end{aligned}$$

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v_1 (y_1'' + p y_1' + q y_1) + v_2 (y_2'' + p y_2' + q y_2) + (v_1' y_1' + v_2' y_2') &= f \\
v_1 \cdot 0 + v_2 \cdot 0 + v_1' y_1' + v_2' y_2' &= f
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Which, using Cramer's Rule, has solution

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

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The denominator is just the Wronskian $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$, because y_1 and y_2 are linearly independent.

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Thus, we can integrate to find v_1 and v_2.

$$v_1 = -\int \frac{y_2 f}{W(y_1, y_2)} \quad \text{and} \quad v_2 = \int \frac{y_1 f}{W(y_1, y_2)}$$

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So,

$$v_1' = -\frac{y_2 f}{1} = -\sin(t)\sec(t) = -\frac{\sin(t)}{\cos(t)} \text{ and } v_2' = \frac{y_1 f}{1} = \cos(t)\sec(t) = 1$$

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Thus, the general solution is

$$y = c_1 \cos(t) + c_2 \sin(t) + \ln(\cos(t))\cos(t) + t\sin(t)$$

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Thus, the particular solution is

$$\begin{aligned} y_p &= (-2t + 2\sin(t)\cos(t))(\cos(t)) + (1 - 2\cos^2(t))(\sin(t)) \\ &= -2t\cos(t) + \sin(t) \end{aligned}$$

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(Cramer's Rule is recommended.)

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This method can be extended to higher orders.

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$$y'' - 2y' + y = \frac{e^t}{t^2 + 1}$$

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The Wronskian is $W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = (t+1)e^{2t} - te^{2t} = e^{2t}$

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So, using the Cramer's Rule formulas from before

$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} = -\frac{t}{t^2+1} \quad \text{and} \quad v_2' = \frac{y_1 f}{W(y_1, y_2)} = \frac{1}{t^2+1}$$

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$$v_1 = -\tfrac{1}{2}\ln\left(t^2+1\right) \quad \text{and} \quad v_2 = \tan^{-1}(t)$$

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$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} = -\frac{\ln(t)}{t} \quad \to v_1 = -\frac{1}{2}\ln^2(t)$$

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The general solution is

$$y = c_1 t + c_2 t^2 - \tfrac{t}{2}\ln^2(t) - t\ln(t) - t$$