Linear Equations: The Nature of Their Solutions

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Definition

An equation $F(x_1, x_2, \ldots, x_n) = C$ is **linear** if it is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = C$$

where $a_1, a_2, \ldots, a_n$ and C are constants.

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Which of the following are linear equations?

$$\begin{aligned} 4x - 3e^x &= 15 \\ 4x - 2y + 3\sqrt{z} &= 12 \\ 2x - 3y + 4z + 3 &= w \end{aligned}$$

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First and Second Order Notation

It is common to write first-order differential equations as

$$y' + p(t)y = f(t)$$

and second-order differential equations as

$$y'' + p(t)y' + q(t)y = f(t)$$

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Let us classify the following differential equations.

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Notation

We will use a **vector** notation to represent a whole set of variables:

Linear Algebraic Equations:

$$\vec{\boldsymbol{x}} = [x_1, x_2, \ldots, x_n]$$

Linear Differential Equations:

$$\vec{\boldsymbol{y}} = [y^{(n)}, y^{(n-1)}, \ldots, y', y]$$

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A **linear operator** L is an entire operation performed on a set of variables.

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Linear Differential Equations:

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Linear Operator Properties

$$L(k\vec{\boldsymbol{u}}) = kL(\vec{\boldsymbol{u}}), \quad k \in \mathbb{R}$$
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Proof

The properties can be proved directly for algebraic operators.

For differential operators, the proof follows from the derivative properties:

- $(kf)' = kf'$
- $(f + g)' = f' + g'$

Superposition Principle for Linear Homogeneous Equations

Let $\vec{\boldsymbol{u}}_1$ and $\vec{\boldsymbol{u}}_2$ be any solutions of the *homogeneous linear* equation

$$L(\vec{\boldsymbol{u}}) = 0$$

- The sum $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_1 + \vec{\boldsymbol{u}}_2$ is also a solution.
- For any constant k, $\vec{\boldsymbol{u}} = k\vec{\boldsymbol{u}}_1$ is also a solution.

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Proof

The proof of the Superposition Principle follows directly from the properties of linear operators from the previous slides.

$$L(\vec{\boldsymbol{u}}) = L(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}}) = L(\vec{\boldsymbol{u_1}}) + L(\vec{\boldsymbol{u_2}}) = 0 + 0 = 0$$

$$L(\vec{\boldsymbol{u}}) = L(k\vec{\boldsymbol{u_1}}) = kL(\vec{\boldsymbol{u_1}}) = k \cdot 0 = 0$$

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This is easily verified:

$$y' = 4e^{2t} - 6e^{-2t}$$

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Nonhomogeneous Principle

Let $\vec{\boldsymbol{u}}_p$ be any solution (called a particular solution) to *linear nonhomogeneous* equation

$$L(\vec{\boldsymbol{u}}) = C \qquad \text{(algebraic)}$$

or

$$L(\vec{\boldsymbol{u}}) = f(t) \qquad \text{(differential)}$$

Then,

$$\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$$

is also a solution, here $\vec{\boldsymbol{u}}_h$ is a solution to the **associated homogeneous** equation

$$L(\vec{\boldsymbol{u}}) = 0$$

Furthermore, *every solution of the nonhomogeneous equation must be of the form* $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$.

Proof

It is easy to show that $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$ is a solution.

$$L(\vec{\boldsymbol{u}}) = L(\vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p) = L(\vec{\boldsymbol{u}}_h) + L(\vec{\boldsymbol{u}}_p) = 0 + f(t) = f(t)$$

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To show that every solution has to be of this form, suppose that $\vec{\boldsymbol{u}}_q$ is any solution. Note that $\vec{\boldsymbol{u}}_q = \vec{\boldsymbol{u}}_p + (\vec{\boldsymbol{u}}_q - \vec{\boldsymbol{u}}_p)$.

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We can then show that $\vec{u}_q - \vec{u}_p$ is also a solution to $L(\vec{u}) = 0$:

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$$L(\vec{\boldsymbol{u}}_q - \vec{\boldsymbol{u}}_p) = L(\vec{\boldsymbol{u}}_q) + L(-\vec{\boldsymbol{u}}_p)$$

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We can then show that $\vec{\boldsymbol{u}}_q - \vec{\boldsymbol{u}}_p$ is also a solution to $L(\vec{\boldsymbol{u}}) = 0$:

$$\begin{aligned} L(\vec{\boldsymbol{u}}_q - \vec{\boldsymbol{u}}_p) &= L(\vec{\boldsymbol{u}}_q) + L(-\vec{\boldsymbol{u}}_p) \\ &= L(\vec{\boldsymbol{u}}_q) - L(\vec{\boldsymbol{u}}_p) \end{aligned}$$

Proof

It is easy to show that $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$ is a solution.

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Process for Solving Nonhomogeneous Linear Equations

Step 1: Find all solutions $\vec{\boldsymbol{u}}_h$ of $L(\vec{\boldsymbol{u}}) = 0$.

Step 2: Find any solution $\vec{\boldsymbol{u}}_p$ of $L(\vec{\boldsymbol{u}}) = f$.

Step 3: Add $\vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p = \vec{\boldsymbol{u}}$ to find all solutions of $L(\vec{\boldsymbol{u}}) = f$.

Example 6

Consider

$$y' - y = t$$

To solve using superposition we need to complete three steps.

Step 1:

Step 2:

Step 3:

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To solve using superposition we need to complete three steps.

Step 1: Solve the associated homogeneous equation $y' - y = 0$, or $y' = y$. (Note: first-order homogeneous linear differential equations are always separable.)

$$y_h = ce^t, \quad \text{for any } c \in \mathbb{R}$$

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Step 3: Superposition tells us that

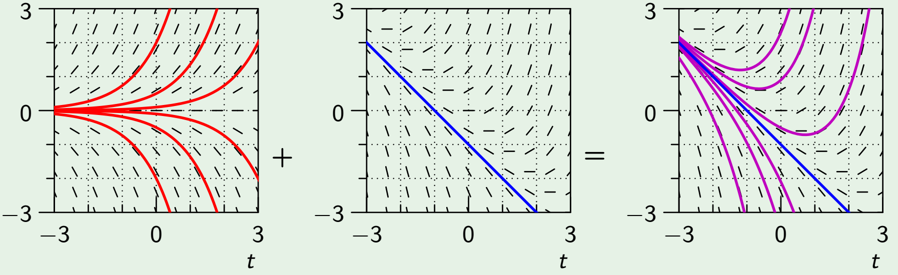
$$y = y_h + y_p = ce^t - t - 1$$

is a solution for any $c \in \mathbb{R}$.

Example 6

Consider

$$y' - y = t$$



$\{y_h\}$ $\quad$ y_p $\quad$ $\{y_h\} + y_p$

Note

Many of you may be wondering how exactly we go about finding particular solutions?

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But, sometimes a particular solution may be staring us in the face.

Example 7

Let us solve

$$y' + ay = b$$

where a and b are constants.

Step 1:

Step 2:

Step 3:

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where a and b are constants.

Step 1: The associated homogeneous equation $y' + ay = 0$ will soon become an old friend.
It has the solution $y_h = ce^{-at}$, where $c \in \mathbb{R}$.

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is a solution for any $c \in \mathbb{R}$.

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where a and b are constants.

Alternatively, we can look for a horizontal line in the direction field.