Picard's Theorem: Theoretical Analysis

Department of Mathematics

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(Slides by Adam Wilson)

Why Study Theory?

When considering a mathematical model two important questions are often considered:

- Does a solution actually exist? (Existence Theorems.)
- Is that solution unique? (Uniqueness Theorems.)

The theorems we will study can answer both of these questions, without actually needing to find any solutions.

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Note

Questions of existence and uniqueness are not limited to the study of differential equations. All branches of mathematics ask these questions, as well as many other fields such as philosophy and physics.

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We know, from the Intermediate Value Theorem, that on the interval $[0, 1]$ at least one root of $f(x)$ exists. Thus, the IVT gives us existence. Further, given that

$$f'(x) = 5x^4 + 1$$

we know, from Rolle's theorem, that there can't exist more than one root in $[0, 1]$. Thus, Rolle gives us uniqueness.

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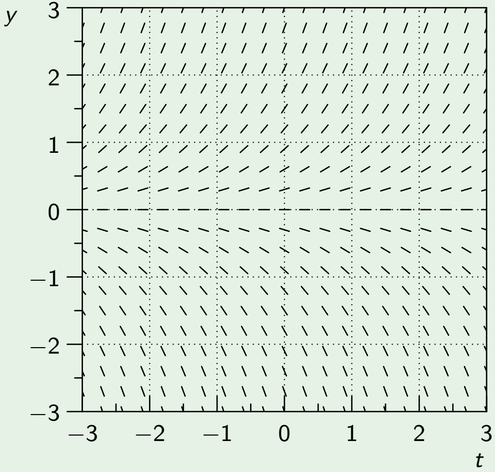
It will often be useful to look at the direction field first, as some pretty obvious information may be gleamed.

Example 2

What can we determine from the direction field for $y' = y$?



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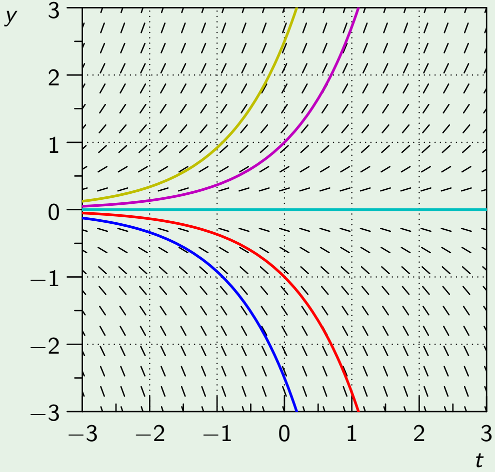
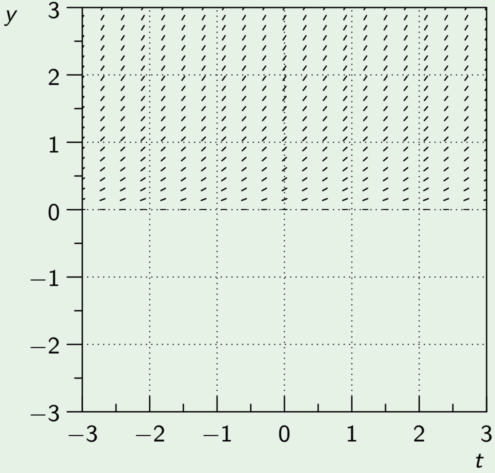


The direction field suggests that a unique solution exists for every point, and these solutions are defined for all t-values.

Example 3

What can we determine from the direction field for $y' = \sqrt{y}$?



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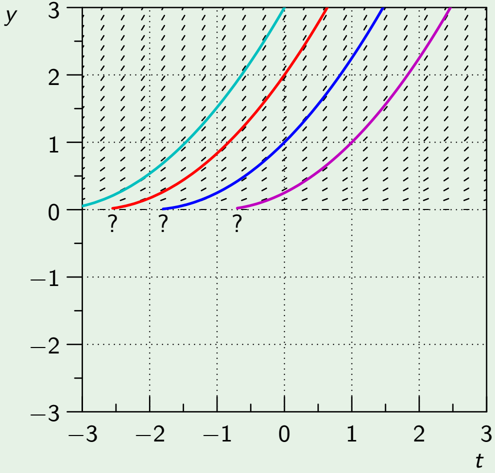
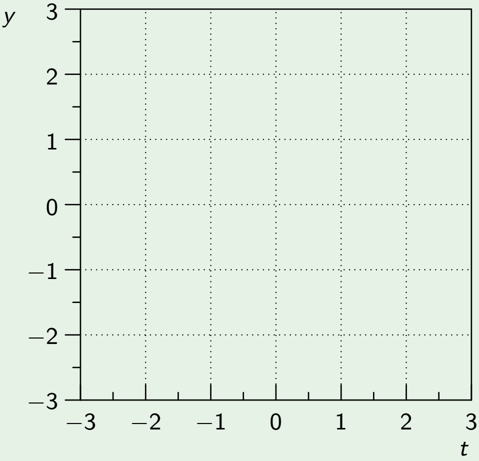


The direction field suggests that a unique solution exists for every point in the upper half place, and that initial points on the t-axis need a closer look.

Example 4

Let us look at the direction field for $y' = \sqrt{y} + \sqrt{-y} + \dfrac{1}{y}$



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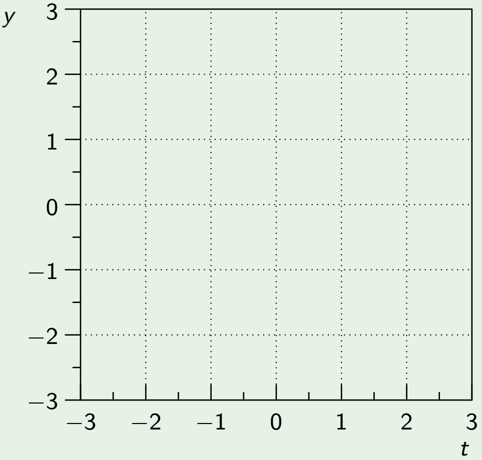

There is no y-value for which this DE is defined, hence no solutions exist.

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Separation of Variables gives us:

$$y(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{1}{4}t^2 & \text{if } t \geq 0 \end{cases}$$

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In fact, the situation is even worse. For any t-value $c > 0$, we have the following solution.

$$y(t) = \begin{cases} 0 & \text{if } t \leq c \\ \frac{1}{4}(t-c)^2 & \text{if } t \geq c \end{cases}$$

We can now introduce the theorem we will mainly use, thanks to the French mathematician Charles Émile Picard (1856 - 1941).

Picard's Existence and Uniqueness Theorem

Suppose that the function $f(t, y)$ is continuous on the region

$$R = \{(t, y) \mid a < t < b, c < y < d\}$$

and $(t_0, y_0) \in \mathbb{R}$.

Then there exists a positive number h such that the initial-value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a solution for t in the interval $(t_0 - h, t_0 + h)$.

More over, if $f_y(t, y)$ is also continuous in R, that solution is unique.

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Both of these are polynomials, which are continuous on any open rectangle.

This means that Picard's theorem guarantees a unique solution exists for the initial point $(t_0, y_0) = (0, 0)$ on *some* interval (though we don't know how large) around $t = 0$.

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We could, if so desired, use Separation of Variables to solve this IVP. Doing so, we find that the solution is $y(t) = \tan(t)$, which is defined on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

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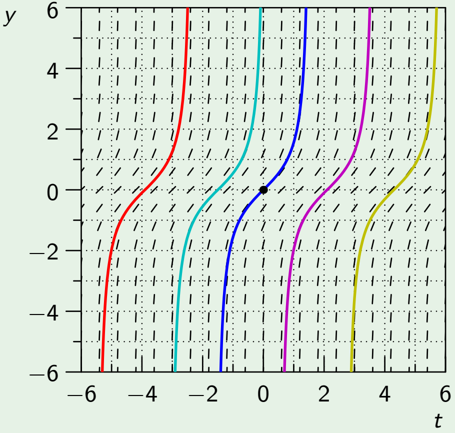

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