Linear Equations

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

Definition

An equation $F(x_1, x_2, \ldots, x_n) = C$ is **linear** if it is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = C$$

where $a_1, a_2, \ldots, a_n$ and C are constants.

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Which of the following are linear equations?

$$\begin{aligned} 4x - 3e^x &= 15 \\ 4x - 2y + 3\sqrt{z} &= 12 \\ 2x - 3y + 4z + 3 &= w \end{aligned}$$

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First and Second Order Notation

It is common to write first-order differential equations as

$$y' + p(t)y = f(t)$$

and second-order differential equations as

$$y'' + p(t)y' + q(t)y = f(t)$$

Example 2

Let us classify the following differential equations.

Differential Equation	Order	Linear?	Homogeneous?	Coefficients
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Notation

We will use a **vector** notation to represent a whole set of variables:

Linear Algebraic Equations:

$$\vec{\boldsymbol{x}} = [x_1, x_2, \ldots, x_n]$$

Linear Differential Equations:

$$\vec{\boldsymbol{y}} = [y^{(n)}, y^{(n-1)}, \ldots, y', y]$$

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A **linear operator** L is an entire operation performed on a set of variables.

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Linear Differential Equations:

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Linear Operator Properties

$$L(k\vec{\boldsymbol{u}}) = kL(\vec{\boldsymbol{u}}), \quad k \in \mathbb{R}$$
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Proof

The properties can be proved directly for algebraic operators.

For differential operators, the proof follows from the derivative properties:

- $(kf)' = kf'$
- $(f + g)' = f' + g'$

Superposition Principle for Linear Homogeneous Equations

Let $\vec{\boldsymbol{u}}_1$ and $\vec{\boldsymbol{u}}_2$ be any solutions of the *homogeneous linear* equation

$$L(\vec{\boldsymbol{u}}) = 0$$

- The sum $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_1 + \vec{\boldsymbol{u}}_2$ is also a solution.
- For any constant k, $\vec{\boldsymbol{u}} = k\vec{\boldsymbol{u}}_1$ is also a solution.

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Proof

The proof of the Superposition Principle follows directly from the properties of linear operators from the previous slides.

$$L(\vec{\boldsymbol{u}}) = L(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}}) = L(\vec{\boldsymbol{u_1}}) + L(\vec{\boldsymbol{u_2}}) = 0 + 0 = 0$$

$$L(\vec{\boldsymbol{u}}) = L(k\vec{\boldsymbol{u_1}}) = kL(\vec{\boldsymbol{u_1}}) = k \cdot 0 = 0$$

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has the solutions $y = e^{2t}$ and $y = e^{-2t}$.

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Nonhomogeneous Principle

Let $\vec{\boldsymbol{u}}_p$ be any solution (called a particular solution) to *linear nonhomogeneous* equation

$$L(\vec{\boldsymbol{u}}) = C \qquad \text{(algebraic)}$$

or

$$L(\vec{\boldsymbol{u}}) = f(t) \qquad \text{(differential)}$$

Then,

$$\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$$

is also a solution, here $\vec{\boldsymbol{u}}_h$ is a solution to the **associated homogeneous** equation

$$L(\vec{\boldsymbol{u}}) = 0$$

Furthermore, *every solution of the nonhomogeneous equation must be of the form* $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$.

Proof

It is easy to show that $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$ is a solution.

$$L(\vec{\boldsymbol{u}}) = L(\vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p) = L(\vec{\boldsymbol{u}}_h) + L(\vec{\boldsymbol{u}}_p) = 0 + f(t) = f(t)$$

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To show that every solution has to be of this form, suppose that $\vec{\boldsymbol{u}}_q$ is any solution. Note that $\vec{\boldsymbol{u}}_q = \vec{\boldsymbol{u}}_p + (\vec{\boldsymbol{u}}_q - \vec{\boldsymbol{u}}_p)$.

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We can then show that $\vec{\boldsymbol{u}}_q - \vec{\boldsymbol{u}}_p$ is also a solution to $L(\vec{\boldsymbol{u}}) = 0$:

$$\begin{aligned} L(\vec{\boldsymbol{u}}_q - \vec{\boldsymbol{u}}_p) &= L(\vec{\boldsymbol{u}}_q) + L(-\vec{\boldsymbol{u}}_p) \\ &= L(\vec{\boldsymbol{u}}_q) - L(\vec{\boldsymbol{u}}_p) \end{aligned}$$

Proof

It is easy to show that $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_h + \vec{\boldsymbol{u}}_p$ is a solution.

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Process for Solving Nonhomogeneous Linear Equations

Step 1: Find all solutions $\vec{u}_h$ of $L(\vec{u}) = 0$.

Step 2: Find any solution $\vec{u}_p$ of $L(\vec{u}) = f$.

Step 3: Add $\vec{u}_h + \vec{u}_p = \vec{u}$ to find all solutions of $L(\vec{u}) = f$.

Example 6

Consider

$$y' - y = t$$

To solve using superposition we need to complete three steps.

Step 1:

Step 2:

Step 3:

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To solve using superposition we need to complete three steps.

Step 1: Solve the associated homogeneous equation $y' - y = 0$, or $y' = y$. (Note: first-order homogeneous linear differential equations are always separable.)

$$y_h = ce^t, \quad \text{for any } c \in \mathbb{R}$$

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Step 2: We can verify by differentiation and substitution that $y_p = -t - 1$ is a particular solution.

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Step 3: Superposition tells us that

$$y = y_h + y_p = ce^t - t - 1$$

is a solution for any $c \in \mathbb{R}$.

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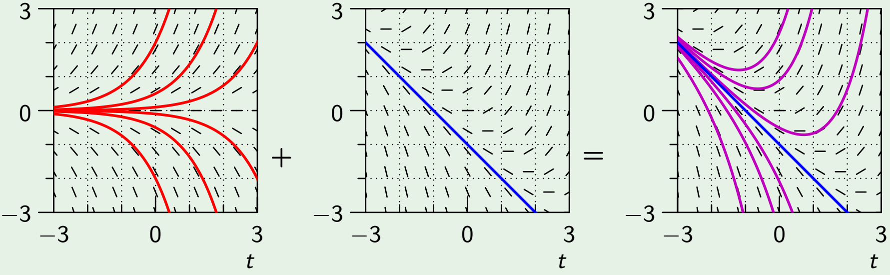
Consider

$$y' - y = t$$



$\{y_h\}$ $\qquad$ y_p $\qquad$ $\{y_h\} + y_p$

Euler-Lagrange Two-Stage Method

We have seen that the general solution for the linear first-order differential equation

$$y' + p(t)y = f(t)$$

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where $c \in \mathbb{R}$.

The second step is to find a particular solution, which we will accomplish using **variation of parameters**, which was developed by French mathematician Joseph Louis Lagrange.

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The idea of variation of parameters is to start with

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The idea of variation of parameters is to start with

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where the unknown function $v(t)$ is called the **varying parameter**.

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Our goal is to find $v(t)$, to do so we need to substitute y_p into the DE.

$$\underbrace{\left(v'(t)e^{-\int p(t)dt} - p(t)v(t)e^{-\int p(t)dt}\right)}_{y_p'} + \underbrace{p(t)v(t)e^{-\int p(t)dt}}_{p(t)y_p} = f(t)$$

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Our goal is to find $v(t)$, to do so we need to substitute y_p into the DE.

$$v(t) = \int f(t)e^{\int p(t)dt}dt$$

Now that we have $v(t)$, we have determined a particular solution.

$$y_p(t) = v(t)e^{-\int p(t)dt} = e^{-\int p(t)dt}\int f(t)e^{\int p(t)dt}dt$$

Example 7

Consider the IVP

$$y' + \left(\frac{1}{t+1}\right) y = 2, \quad y(0) = 0, \quad t \geq 0$$

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Let us assume for the moment that $y \neq 0$ and use separation of variables.

$$\frac{dy}{y} = -\frac{dt}{t+1}$$

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Let us assume for the moment that $y \neq 0$ and use separation of variables.

$$y_h = \frac{k}{t+1}$$

where $k = \pm e^c$

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Consider the IVP

$$y' + \left(\frac{1}{t+1}\right) y = 2, \quad y(0) = 0, \quad t \geq 0$$

Step 2. Next, using variation of parameters, we try

$$y_p = \frac{v(t)}{t+1}$$

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Step 2. Next, using variation of parameters, we try

$$y_p = \frac{v(t)}{t+1}$$

which gives:

$$v(t) = t^2 + 2t + c$$

But, we only need a single $v(t)$, so we can let $c = 0$, giving

$$y_p = \frac{t^2 + 2t}{t+1}$$

Example 7

Consider the IVP

$$y' + \left(\frac{1}{t+1}\right) y = 2, \quad y(0) = 0, \quad t \geq 0$$

Step 3. Thus, the general solution is:

$$y(t) = y_h + y_p = \frac{k}{t+1} + \frac{t^2 + 2t}{t+1}$$

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$$0 = y(0) = \frac{k}{(0)+1} + \frac{(0)^2 + 2(0)}{(0)+1}$$

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Which means that the solution to the IVP is

$$y(t) = \frac{t^2 + 2t}{t+1}$$

Euler-Lagrange Method for Solving Linear First-Order DEs

To solve a linear differential equation

$$y' + p(t)y = f(t)$$

where p and f are continuous on a domain I, use the following steps.

Step 1. Solve the corresponding homogenous equation $y' + p(t)y = 0$ to obtain the one-parameter family.

$$y_h = ce^{-\int p(t)dt}$$

Step 2. Solve

$$v'(t)e^{-\int p(t)dt} = f(t)$$

for $v(t)$ to obtain a particular solution $y_p = v(t)e^{-\int p(t)dt}$.

Step 3. Combine the results of Step 1 and Step 2 to form the general solution

$$y(t) = y_h + y_p$$

Step 4. If you are solving an IVP, only after Step 3 can you plug in the initial condition.

Note

Variation of Parameters is a very powerful method, and you will see it again in a proper differential equations course. But, for first-order (and *only* first-order) equations we have a second method, called the **Integrating Factor Method** which may also be used.

Note

Variation of Parameters is a very powerful method, and you will see it again in a proper differential equations course. But, for first-order (and *only* first-order) equations we have a second method, called the **Integrating Factor Method** which may also be used.

For the differential equation

$$y' + p(t)y = f(t)$$

we will break this new method down into two cases:

- $p(t)$ is constant.
- $p(t)$ is variable.

Integrating Factor Method (Constant Coefficient)

Let us look at the first-order linear differential equation

$$y' + ay = f(t), \quad a \in \mathbb{R}$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

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Let us start with the differential equation.

$$y' + ay = f(t)$$

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This method uses a simple observation made by Euler:

$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

We first multiply both sides of the equation by e^{at}.

$$e^{at}\left(y' + ay\right) = e^{at}f(t)$$

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$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

We then apply Euler's observation to the left-hand side.

$$\frac{d}{dt}\left(e^{at}y\right) = e^{at}f(t)$$

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$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

Next we integrate both sides.

$$e^{at}y = \int e^{at}f(t)dt + c$$

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$$e^{at}\left(y' + ay\right) = \frac{d}{dt}\left(e^{at}y\right)$$

Solving for y gives:

$$y(t) = e^{-at}\int e^{at}f(t)dt + ce^{-at}$$

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Solving for y gives:

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Note

This is the same answer we got from Variation of Parameters, though achieved through a different route. We have obtained both y_h and y_p at the same time.

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

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$$y' + p(t)y = f(t)$$

We seek a function $\mu(t)$ that satisfies Euler's observation, i.e.

$$\mu(t) \cdot \left(y' + p(t)y\right) = \frac{d}{dt}\left(\mu(t) \cdot y\right)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

Let us carry out the differentiation on the right-hand side

$$\mu(t)y' + p(t)\mu(t)y = \mu'(t)y + \mu(t)y'$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

If we assume $y(t) \neq 0$, this simplifies to

$$\mu'(t) = p(t)\mu(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

Integrating Factor Method (Variable Coefficient)

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$$\ln|\mu(t)| = \int p(t)dt$$

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$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

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Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

We now know the integrating factor, and perform the same steps as before.

$$y' + p(t)y = f(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Multiply both sides by the integrating factor.

$$\mu(t) \cdot \big(y' + p(t)y\big) = \mu(t) \cdot f(t)$$

Integrating Factor Method (Variable Coefficient)

Now let us look at the more general first-order differential equation

$$y' + p(t)y = f(t)$$

We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Apply the property $\mu(t) \cdot (y' + p(t)y) = (\mu(t) \cdot y)'$ to the left-hand side.

$$(\mu(t)y)' = \mu(t)f(t)$$

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Now let us look at the more general first-order differential equation

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We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Integrate both sides.

$$\mu(t)y(t) = \int \mu(t)f(t)dt + c$$

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We can find a solution $\mu(t) > 0$ by Separation of Variables.

$$\mu(t) = e^{\int p(t)dt}$$

Assuming $\mu(t) \neq 0$, we can solve for y.

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t)dt + \frac{c}{\mu(t)}$$

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Assuming $\mu(t) \neq 0$, we can solve for y.

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) f(t) dt + \frac{c}{\mu(t)}$$

Note

We have again found y_h and y_p at the same time.

Integrating Factor Method for First-Order Linear DEs

To solve the linear first-order DE, where p and f are continuous on a domain I.

$$y' + p(t)y = f(t)$$

Step 1. Find the integrating factor $\mu(t) = e^{\int p(t)dt}$, where $\int p(t)dt$ represents *any* anti-derivative of $p(t)$.

Step 2. Multiply both sides of the DE by $mu(t)$, which always simplifies to:

$$\left(e^{\int p(t)dt} y(t)\right)' = e^{\int p(t)dt} f(t)$$

Step 3. Find the anti-derivative to get:

$$e^{\int p(t)dt} y(t) = \int e^{\int p(t)dt} f(t)dt + c$$

Step 4. Solve algebraically for y.

$$y = e^{-\int p(t)dt} \int e^{\int p(t)dt} f(t)dt + ce^{-\int p(t)dt}$$

Step 5. For IVPs, substitute the initial conditions in to find c.

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

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Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int p(t) dt}$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (-1) dt}$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (-1) dt} = e^{-t}$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 1. Find the integrating factor:

$$\mu(t) = e^{\int (-1) dt} = e^{-t}$$

Step 2. Multiply both sides of the DE by $\mu(t)$:

$$e^{-t}\left(y' - y\right) = e^{-t}$$

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Let us solve this DE using the Integrating Factor method.

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Which reduces to:

$$\left(e^{-t}y\right)' = te^{-t}$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t}$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

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$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

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Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^0$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

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$$y(t) = e^t \left(e^{-t}\right)(-t-1) + ce^t = -t - 1 + ce^t$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^0 \Rightarrow c = 2$$

Example 8

Consider the IVP

$$y' - y = t, \quad y(0) = 1$$

Let us solve this DE using the Integrating Factor method.

Step 3. Find the antiderivative:

$$e^{-t}y = \int te^{-t}dt = e^{-t}(-t-1) + c$$

Step 4. Solve for y:

$$y(t) = e^{t}\left(e^{-t}\right)(-t-1) + ce^{t} = -t - 1 + ce^{t}$$

Step 5. Plug in the initial conditions to find the solution to the IVP:

$$1 = y(0) = -0 - 1 + ce^{0} \Rightarrow c = 2$$

Thus, the solution to the IVP is $y(t) = -t - 1 + 2e^{t}$