# The Harmonic Oscillator

Department of Mathematics

Salt Lake Community College

## Newton's Dot Notation

Scientists and Engineers who work with many variables where the independent variables is always $t$ commonly use dot notation:

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{dx^2}{d^2t} \quad \text{and} \quad \dddot{x} = \frac{dx^3}{d^3t} \quad \text{and} \quad \ddddot{x} = \frac{dx^4}{d^4t}$$

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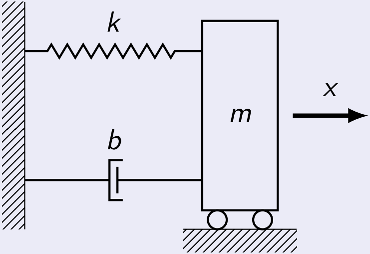
## Definition 1

A very important DE is the second-order homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

where $m > 0$, $b$, and $k$ are constants.
This models a class of phenomena called **damped harmonic oscillation**.

## The Mass-Spring System

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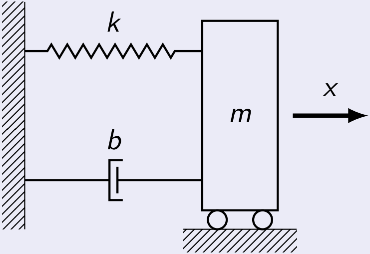


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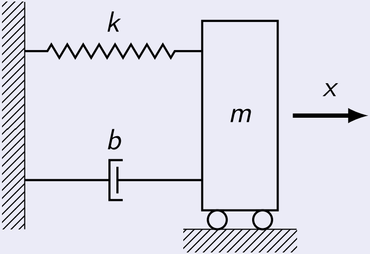


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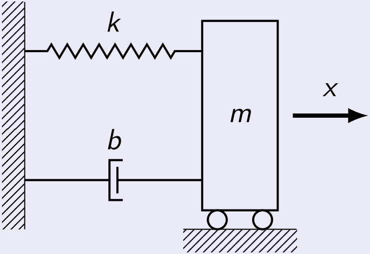


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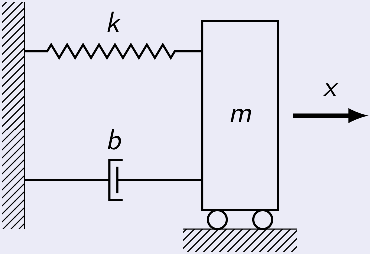


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Summing these forces gives:

$$\begin{array}{rlll} \text{mass} \times \text{acceleration} & = F_{\text{restoring}} & + F_{\text{damping}} & + F_{\text{external}} \\ m\ddot{x} & = -kx & \text{-}\ b\dot{x} & + f(t) \end{array}$$

## Simple Harmonic Oscillator

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- When $b = 0$, the motion is called **undamped**; otherwise it is **damped**.
- If $f(t) = 0$ for all $t$, then the equation is homogeneous:

  $$m\ddot{x} + b\dot{x} + kx = 0$$

  and the motion is called **unforced**, **undriven**, or **free**; otherwise it is called **forced** or **driven**.

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We also measure the damping force of the object sliding on the table to be 0.5 newtons when the velocity is 0.25 meters per second.

$$b = \frac{0.5 \text{ newton}}{0.25 \; \frac{\text{meter}}{second}} = 2 \; \frac{\text{newton second}}{\text{meter}}$$

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Notice that a second-order DE requires **two** initial conditions.

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| Gravity (Earth) | $9.8\ \frac{m}{s^2}$ | $980.665\ \frac{cm}{s^2}$ | $32\ \frac{ft}{s^2}$ |

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$$(\sin(\omega_0 t))'' = -\omega_0^2 \sin(\omega_0 t) \quad \text{and} \quad \ddot{x} = -\frac{k}{m}x$$

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## Note

Another solutions is $x(t) = \cos(\omega_0 t)$.

## Solution of the Undamped Unforced Oscillator

For the undamped unforced oscillator

$$m\ddot{x} + kx = 0$$

we know two solutions:

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## Note

We will see next section that these cover all solutions.

## Alternate Form of the Undamped Unforced Oscillator Solution

Solutions to the undamped unforced oscillator may also be expressed as

$$x(t) = A\cos(\omega_0 t - \delta)$$

- $A$ is the **amplitude**.
- $\delta$ is the **phase angle**, measured in radians.
- The motion has **circular frequency** $\omega_o = \sqrt{\frac{k}{m}}$, measured in $\frac{\text{radians}}{\text{second}}$.
- The **natural frequency** is $f_0 = \frac{\omega_0}{2\pi}$.
- The **period** is $\frac{1}{f_0} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$, measured in seconds.
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## Converting Between the Two Forms

The translation is given by

$$A = \sqrt{c_1^2 + c_2^2}, \qquad \tan(\delta) = \frac{c_2}{c_1}$$

and

$$c_1 = A\cos(\delta), \qquad c_2 = A\sin(\delta)$$

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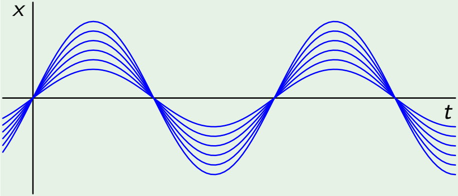
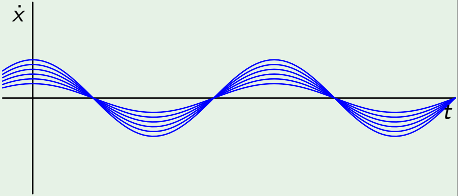
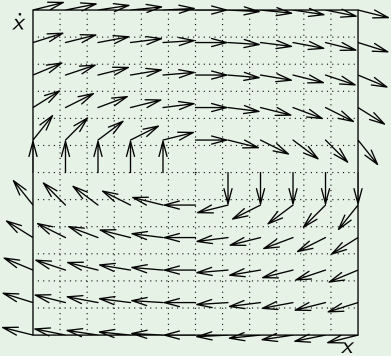
Substituting $t = 0$, $x(0) = 0$, and $\dot{x}(0) = 1$ into this system gives the solution $c_1 = 0$ and $c_2 = 1$.

## Example 5

Let us look at some plots concerning $\ddot{x} + 0.25x = 0$:



## Phase Portraits

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the **phase plane** is the two-dimensional graph with $x$ and $\dot{x}$ axes.

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The phase plane has a **vector field** specified by the DE, which at any point in the phase plane gives a direction vector with

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## Note

Phase portraits can be graphed *without* solving the DE.

## Definition 6

The second-order differential equation

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is equivalent to the system of first-order equations:

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## Example 7

To draw the vector field for $\ddot{x} + 0.25x = 0$ we can build the system:

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$$m\ddot{x} + b\dot{x} + kx = f(t)$$

is equivalent to the system of first-order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \ddot{x} = \frac{f(t)}{m} - \frac{k}{m}x - \frac{b}{m}y \end{aligned}$$

## Example 7

To draw the vector field for $\ddot{x} + 0.25x = 0$ we can build the system:

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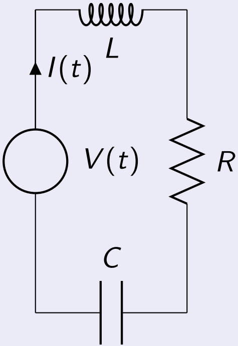
Then, pplane may be used to plot the phase portrait.

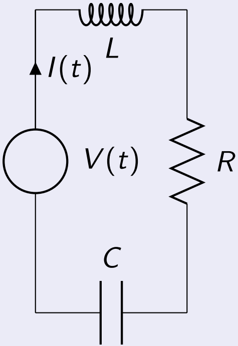
## Electrical Circuits




The current $I$ in a wire, measured in *amperes*, is the flow of charge $Q$. That is, the current is the rate of change of the charge

$$I(t) = \dot{Q}(t)$$

## Electrical Circuits

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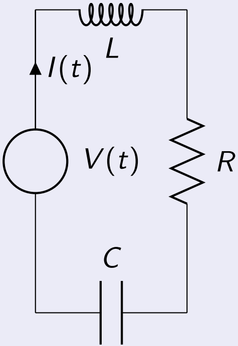
**Kirchoff's Voltage Law** tell us that the input voltage $V(t)$ is the sum of voltage drops around the circuit. In our circuit, we have three such voltage drops.

## Electrical Circuits

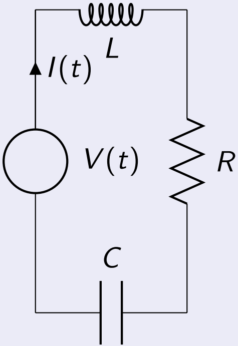



**Drop across a Resistor:** By **Ohm's Law**, the voltage drop across a resistor is proportional to the current passing through it.

$$V_R(t) = R \cdot I(t)$$

Where $R$ is the **resistance** of the resistor and is measured in *ohms*.

## Electrical Circuits

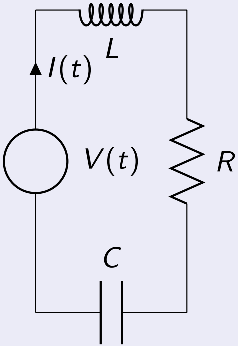



**Drop across an Inductor:** By Faraday's Law, the voltage drop across an inductor is proportional to the time rate of change of the current passing through it.

$$V_L(t) = L \cdot \dot{I}(t)$$

where $L$ is the **inductance** and is measured in *henries*.

## Electrical Circuits

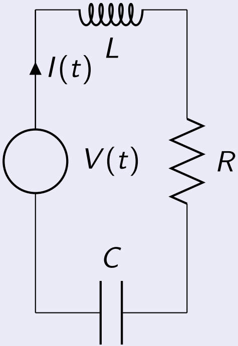



**Drop across a Capacitor:** The voltage drop across a capacitor is proportional to the charge $Q(t)$ on the capacitor.

$$V_C(t) = \frac{1}{C} Q(t) = \frac{1}{C} \int I(t) dt$$

where $C$ is the **capacitance** of the capacitor and is measured in *farads*.

## Electrical Circuits




Thus, the voltage drop across the circuit is

$$V(t) = R \cdot I + L \cdot \dot{I} + \frac{1}{C}\int I(t)dt$$

This is called an **integro-differential equation** because it contains both a derivative and an integral.

Using the fact that $I(t) = \dot{Q}(t)$ we can build the following equations.

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## Series Circuit Equation (Charge)

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)$$

If there is no voltage source ($V(t) = 0$), then

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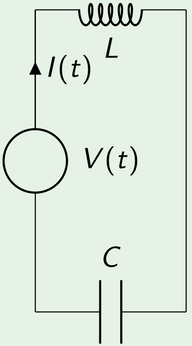
## Series Circuit Equation (Current)

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}(t)$$

If there is no voltage source $(V(t) = 0)$, then

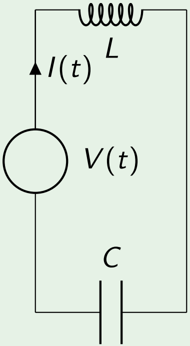
$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0$$

## Example 8




Consider a circuit composed of a capacitor and inductor hooked up in series. Suppose that at $t = 0$ a charge $Q_0$ is put on the capacitor.

## Example 8

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The IVP is

$$L\ddot{Q} + \frac{1}{C} = 0, \quad Q(0) = Q_0, \quad \dot{Q}(0) = 0$$

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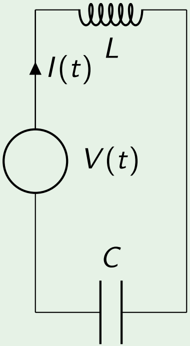


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The IVP is

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Thus, the solution is

$$Q(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

where

$$\omega_0 = \sqrt{\frac{1}{LC}}$$