

# Basis and Dimension

Department of Mathematics

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## Definition 1

For a vector space  $\mathbb{V}$ , a **linear combination** of vectors is:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_k$$

where  $c_i \in \mathbb{R}$  and  $\vec{v}_i \in \mathbb{V}$

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The **span** of a set  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  of vectors in a vector space  $\mathbb{V}$  is the set of all linear combinations of these vectors. Denoted **span**  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$

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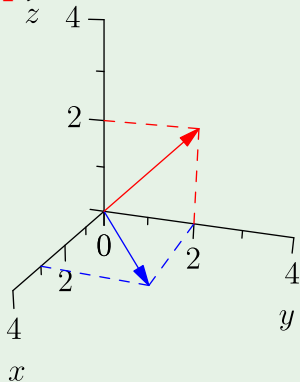
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## Note

If the **span**  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \mathbb{V}$  we say the set spans the vector space.

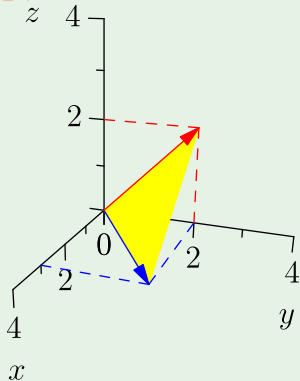
### Example 3

Consider **span**  $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$ .



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This spanning set is the plane defined by these two vectors.

### Example 4

Let us look closer at this spanning set. Where we give names to the two vectors:

$$\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$



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We can then write a general vector in the spanning set as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{u} + b\vec{v}$$

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### Example 4

We can write the vector equation as the system:

$$x = 3a \quad \Rightarrow \quad a = \frac{x}{3}$$

$$y = 2a + 2b$$

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Which is equivalent to  $2x - 3y + 3z = 0$ , the equation of the yellow plane.



## Theorem 5

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## Example 6

Consider adding  $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$  to  $\text{span} \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$ .

## Theorem 5

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Since we can write

$$-1 \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

we see that this doesn't change to the spanning set.

## Example 7

Consider adding  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  to **span**  $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$ .

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Consider adding  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  to **span**  $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$ .

This would expand the spanning set.

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This would expand the spanning set.

To show this, let us try to find  $c_1, c_2 \in \mathbb{R}$  such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

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Which is equivalent to solving the inconsistent system

$$1 = 3c_1$$

$$1 = 2c_1 + 2c_2$$

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## Example 8

What is **span**  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$ ?



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It turns out the spanning set is  $\mathbb{R}^3$ .

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To show this, we then need to find  $c_1, c_2, c_3 \in \mathbb{R}$  such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Which has a unique solution for any  $x, y, z \in \mathbb{R}$ .

## Theorem 9

For  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$ , a vector  $\vec{b} \in \mathbb{R}^n$  is in  $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  if and only if there is at least one solution to the matrix equation  $\mathbf{A}\vec{x} = \vec{b}$ . Where  $\mathbf{A}$  is formed from the column vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ .

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## Note

We can write spanning sets using set builder notation.

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$$\text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} = \left\{ c \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid c \in \mathbb{R} \right\}$$

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$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid c_1, c_2, c_3 \in \mathbb{R} \right\}$$



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## Theorem 12

For  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{V}$ ,  $\text{span} \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \}$  is a subspace of  $\mathbb{V}$ .

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Let  $\vec{u}$  and  $\vec{w}$  be two vectors in the spanning set, which means there are scalars  $r_i$  and  $s_j$  such that

$$\vec{u} = r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_n \vec{v}_k \quad \text{and} \quad \vec{w} = s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_n \vec{v}_k$$

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So, for any  $a, b \in \mathbb{R}$ :

$$a\vec{u} + b\vec{w}$$

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So, for any  $a, b \in \mathbb{R}$ :

$$a\vec{u} + b\vec{w} = a(r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_n \vec{v}_k) + b(s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_n \vec{v}_k)$$

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Which means  $a\vec{u} + b\vec{w}$  is in the spanning set and we have closure.



### Definition 13

For any  $m \times n$  matrix  $\mathbf{A}$ , the **column space**, denoted **Col**  $\mathbf{A}$ , is the span of the column vectors of  $\mathbf{A}$ , and is a subspace of  $\mathbb{R}^m$ .

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### Example 14

Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

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$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

The column space of  $\mathbf{B}$  is a subspace of  $\mathbb{R}^2$  and defined:

$$\text{Col } \mathbf{B} = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 5 \end{bmatrix} \mid c_1, \dots, c_5 \in \mathbb{R} \right\}$$

## Definition 15

A set  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  of vectors in a vector space  $\mathbb{V}$  is **linearly independent** if no vector of the set can be written as a linear combination of the others. Otherwise it is **linearly dependent**.

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## Testing for Linear Independence

To test for linear independence of a set of  $k$  vectors  $\vec{v}_i \in \mathbb{R}^n$ , we consider the system:

$$\begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$$

The column vectors of  $A$  are linearly independent if and only if the solution  $c_1 = c_2 = \cdots = c_k = 0$  is unique.

### Example 16

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$  linearly independent?

### Example 16

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$  linearly independent?

To determine if they are, we need to look at the system

$$\mathbf{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Since  $|\mathbf{A}| = 5$ , we know that  $\mathbf{A}$  is invertible and hence a unique solution exists. This means that these vectors are linearly independent.



### Example 17

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$  linearly independent?

### Example 17

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$  linearly independent?

To determine if they are, we need to look at the system

$$\mathbf{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 1 & 3 & -1 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

### Example 17

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$  linearly independent?

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$$\mathbf{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 1 & 3 & -1 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have more columns than rows, which means there will be at least one free variable. Thus, the solution (if one exists) won't be unique, so these vectors are not linearly independent.

### Example 18

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$  linearly independent?

## Example 18

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$  linearly independent?

To determine if they are, we need to look at the system

$$\mathbf{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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To determine if they are, we need to look at the system

$$\left[ \begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array} \right]$$

## Example 18

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$  linearly independent?

To determine if they are, we need to look at the system

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

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Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$  linearly independent?

To determine if they are, we need to look at the system

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

And thus, these vectors are not linearly independent.



### Example 18

Are the vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$  linearly independent?

To determine if they are, we need to look at the system

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

And thus, these vectors are not linearly independent.

Moreover, since

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = 0$$

we can see that any vector can be written as a combination of the others.

## Definition 19

A set of vector functions  $\{\vec{v}_1(t), \vec{v}_2(t), \dots, \vec{v}_k\}$  in a vector space  $\mathbb{V}$  is **linearly independent** on an interval  $I$  if, for *all*  $t \in I$ , the equation

$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + \cdots + c_k \vec{v}_k(t) = \vec{0} \quad (\text{where } c_i \in \mathbb{R})$$

has the only solution:  $c_1 = c_2 = \cdots = c_k = 0$ .

If for any value  $t_0 \in I$  there is any solution with  $c_i \neq 0$ , the vector functions  $\vec{v}_1(t), \vec{v}_2(t), \dots, \vec{v}_k(t)$  are **linearly dependent**.

## Example 20

Are the vectors

$$\vec{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on  $(-\infty, \infty)$ ?

## Example 20

Are the vectors

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linearly independent on  $(-\infty, \infty)$ ?

We need to see what the solution, for  $c_1, c_2, c_3 \in \mathbb{R}$ , is:

$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0}$$

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$$\vec{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

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$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0}$$

Since this equation has to hold for all  $t$ , it has to hold for  $t = 0$ :

$$c_1 \begin{bmatrix} e^{(0)} \\ 0 \\ 2e^{(0)} \end{bmatrix} + c_2 \begin{bmatrix} e^{-(0)} \\ 3e^{-(0)} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{2(0)} \\ e^{2(0)} \\ e^{2(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## Example 20

Are the vectors

$$\vec{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

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$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0}$$

Since this equation has to hold for all  $t$ , it has to hold for  $t = 0$ :

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Since this equation has to hold for all  $t$ , it has to hold for  $t = 0$ :

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array} \right]$$



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Since this equation has to hold for all  $t$ , it has to hold for  $t = 0$ :

Since the unique solution is  $c_1 = c_2 = c_3 = 0$ , these vectors are linearly independent.

### Example 21

Are the following functions linearly independent?

$$\vec{v}_1(t) = e^t, \quad \vec{v}_2(t) = 5e^{-t}, \quad \vec{v}_3(t) = e^{3t}$$

## Example 21

Are the following functions linearly independent?

$$\vec{v}_1(t) = [e^t], \quad \vec{v}_2(t) = [5e^{-t}], \quad \vec{v}_3(t) = [e^{3t}]$$

We can think of each of these as a one-dimensional vector.

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Which means we have to see if there exists  $c_1, c_2, c_3 \in \mathbb{R}$  such that

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$$\text{For } t = 0: \quad c_1 \cdot 5e^{(0)} + c_2 \cdot e^{-(0)} + c_3 \cdot e^{3(0)} = 0$$

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$$\text{For } t = 0: \quad c_1 \cdot 5e^{(0)} + c_2 \cdot e^{-(0)} + c_3 \cdot e^{3(0)} = 0$$

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$$\text{For } t = 0: \quad c_1 \cdot 5e^{(0)} + c_2 \cdot e^{-(0)} + c_3 \cdot e^{3(0)} = 0$$

$$\text{For } t = 1: \quad c_1 \cdot 5e^{(1)} + c_2 \cdot e^{-(1)} + c_3 \cdot e^{3(1)} = 0$$

$$\text{For } t = -1: \quad c_1 \cdot 5e^{(-1)} + c_2 \cdot e^{-(-1)} + c_3 \cdot e^{3(-1)} = 0$$



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$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

$$\text{For } t = 0: \quad c_1 + 5c_2 + c_3 = 0$$

$$\text{For } t = 1: \quad ec_1 + \frac{5}{e}c_2 + e^3c_3 = 0$$

$$\text{For } t = -1: \quad \frac{1}{e}c_1 + ec_2 + \frac{1}{e^3}c_3 = 0$$

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$$\left[ \begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ e & \frac{5}{e} & e^3 & 0 \\ \frac{1}{e} & e & \frac{1}{e^3} & 0 \end{array} \right]$$

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$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

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$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + c_3 \vec{v}_3(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Since we have a unique solution,  $c_1 = c_2 = c_3 = 0$ , these functions are linearly independent.

## Definition 22

The **Wronskian** of functions  $f_1, f_2, \dots, f_k$  on interval  $I$  is the determinant:

$$W[f_1, f_2, \dots, f_k](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{vmatrix}$$

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## Theorem 23

*If  $W[f_1, f_2, \dots, f_k](t) \neq 0$  for all  $t \in I$ , then  $\{f_1, f_2, \dots, f_k\}$  is a linearly independent set of functions on  $I$ .*

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## Theorem 23

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## Note

If  $\{f_1, f_2, \dots, f_k\}$  are linearly dependent, then  $W[f_1, f_2, \dots, f_k](t) = 0$  for all  $t \in I$ . Thus, to show independence we only need to find a single  $t$  that makes the Wronskian nonzero.

## Example 24

Use the Wronskian to check that

$$\{t^2 + 1, t^2 - 1, 2t + 5\}$$

are linearly independent on  $\mathbb{P}_2$ .



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$$W(t) = \begin{vmatrix} t^2 + 1 & t^2 - 1 & 2t + 5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

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are linearly independent on  $\mathbb{P}_2$ .

$$\begin{aligned} W(t) &= \begin{vmatrix} t^2 + 1 & t^2 - 1 & 2t + 5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2 + 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2 - 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t + 5) \begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \end{aligned}$$

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## Example 24

Use the Wronskian to check that

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$$\begin{aligned} W(t) &= \begin{vmatrix} t^2 + 1 & t^2 - 1 & 2t + 5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2 + 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2 - 1) \begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t + 5) \begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \\ &= (t^2 + 1)(0 - 4) - (t^2 - 1)(0 - 4) + (2t + 5)(4t - 4t) \\ &= -4t^2 - 4 + 4t^2 - 4 = -8 \end{aligned}$$

Since  $W(t) = -8 \neq 0$ , this is a set of linearly independent functions.

## Example 25

Let us consider the converse:

Does the Wronskian being zero imply dependence?

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Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$



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Let us consider the converse:

Does the Wronskian being zero imply dependence?

In general, the answer is no.

Consider the linearly independent functions:

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Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

## Example 25

Let us consider the converse:

Does the Wronskian being zero imply dependence?

In general, the answer is no.

Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

So,

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

## Definition 26

The set  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  is a **basis** for vector space  $\mathbb{V}$ , provided that

- $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  is a linearly independent set
- $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \mathbb{V}$

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## Example 27

The vectors

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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We saw earlier that these vectors span  $\mathbb{R}^3$ .

It's easy to see that  $c_1\vec{i} + c_2\vec{j} + c_3\vec{k} = \vec{0}$  has the unique solution  $c_1 = c_2 = c_3 = 0$ .

## Definition 28

The **standard basis** for  $\mathbb{R}^n$  is  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are the column vectors of the identity matrix  $I_n$ .

### Example 29

Let us find a basis for the hyperplane in  $\mathbb{R}^4$  that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$



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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 2a + 3b - 4c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

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Since  $a, b, c \in \mathbb{R}$  were arbitrary, we see these three vectors span the hyperplane.

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Let us find a basis for the hyperplane in  $\mathbb{R}^4$  that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Now, we need to show that the vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

### Example 29

Let us find a basis for the hyperplane in  $\mathbb{R}^4$  that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for  $c_1, c_2, c_3 \in \mathbb{R}$ , solving the equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The (unique) solution is  $c_1 = c_2 = c_3 = 0$ , thus these vectors are linearly independent.

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Let us find a basis for the hyperplane in  $\mathbb{R}^4$  that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

So, we see that the hyperplane has a basis of three vectors.

It looks like this hyperplane is a three-dimensional subspace of a four-dimensional space.

### Example 30

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For  $\mathbb{R}^2$ , one is the standard basis

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but, another basis is given by

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

### Theorem 31

*The number of vectors in a basis is always the same for a particular vector space*

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### Proof

The proof is in Appendix LT of your textbook, on page 602.



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### Definition 32

The **dimension** of a vector space  $\mathbb{V}$  is the number of vectors in any basis of  $\mathbb{V}$ .

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The **dimension** of a vector space  $\mathbb{V}$  is the number of vectors in any basis of  $\mathbb{V}$ .

### Definition 33

If a vector space is so large that cannot be spanned by a finite set of vectors, it is called **infinite-dimensional**.

### Example 34

The solution to the system

$$x_1 + 2x_2 - x_3 + x_4 = 0$$

$$x_1 + 3x_2 + x_3 + 2x_4 = 0$$

is a subspace of  $\mathbb{R}^4$ . (The intersection of two 3D hyperplanes.)

What is its dimension?

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Writing this system in RREF gives

$$x_1 - 5x_3 - x_4 = 0$$

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The two free variables tell us that the solution to this system will be a two-parameter family.

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The two vectors

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span the subspace.

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Which means the dimension is 2.

## Properties of the Column Space of a Matrix

- ① The pivot columns of a matrix  $\mathbf{A}$  form a basis for  $\text{Col } \mathbf{A}$ .
  - A pivot column is a column of  $\mathbf{A}$  that corresponds to a column in the RREF of  $\mathbf{A}$  with a leading 1.

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### Example 35

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 & 8 \end{bmatrix}$$

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### Example 35

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

The pivot columns are  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , which means  $\text{rank}(\mathbf{A}) = 2$  and thus the dimension of the column space is 2.



## Invertible Matrix Characterization

Let  $\mathbf{A}$  be a  $n \times n$  matrix. The following statements are equivalent:

- $\mathbf{A}$  is invertible.
- The column vectors of  $\mathbf{A}$  are linearly independent.
- Every column of  $\mathbf{A}$  is a pivot column.
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$$\text{For } x = -1: \quad c_3 - c_2 + c_1 = 0$$

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So, since we have the unique solution  $c_1 = c_2 = c_3 = 0$ , these functions are linearly independent and thus form a basis of  $\mathbb{P}_2$ .

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Which means **dim**  $\mathbb{P}_2 = 3$ .

### Example 37

What is the dimension of  $\mathbb{P}$ , the space of all polynomials?



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### Note

There are many infinite-dimensional spaces.

We have seen  $\mathbb{P}$ ,  $\mathcal{C}(I)$ , and  $\mathcal{C}^n(I)$ .