Systems of Linear Equations

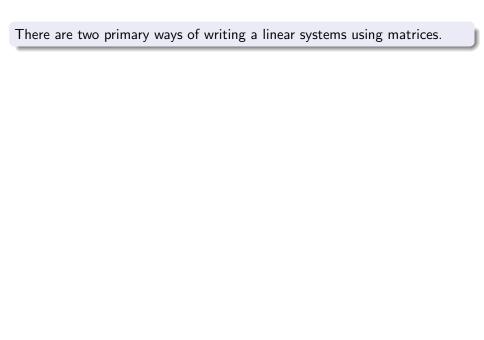
Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

System of Linear Equations

A $m \times n$ system of linear equations is a set of m equations in n variables x_1, x_2, \dots, x_n of the form



There are two primary ways of writing a linear systems using matrices.

An Augmented Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

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A Matrix Equation (We will look at these in section 3.3)

As the matrix equation $A\vec{x} = \vec{b}$, where:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

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Elementary Row Operations

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 (or $R_i = r_j$, $R_j = r_i$)

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Add row j to row i (leaving row j unchanged):

$$R_i = r_i + r_i$$

Gaussian Elimination

Use row operations until the augmented matrix is in Row Echelon Form:

$$\begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & d_2 \\ 0 & 0 & 1 & \cdots & c_{3n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_m \end{bmatrix}$$

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Then back solve the system:

$$x_1 + c_{12}x_2 + c_{13}x_3 + \dots + c_{1n}x_n = d_1$$

 $x_2 + c_{23}x_3 + \dots + c_{2n}x_n = d_2$
 \vdots
 $x_n = d_m$

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We can write this as the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{bmatrix}$$

We now want to use row operations to transform this augmented matrix into Row Echelon Form.

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$$\left[\begin{array}{ccc|ccc}
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\right]$$

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$$\begin{array}{rclcrcr}
x & + & y & + & z & = & 3 \\
y & + & 3z & = & -2 \\
z & = & -2
\end{array}$$

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Plug both into the first equation and solve for x:

$$x + (4) + (-2) = 3 \implies x = 1$$

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 - A system with more than one solution is called dependent.

Reduced Row Echelon Form

An augmented matrix is said to be in Reduced Row Echelon Form if:

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Rank

The **rank** r of a matrix is equal to how many 1's are in the diagonal of it's Reduced Row Echelon Form.

- If r equals the number of variables, there is a unique solution.
- If *r* is less than the number of variables, the solutions are not unique.