Defining Probability

Colby Community College

Definition

The result of **random process** is called an **outcome**.

Note

A **die**, the singular of **dice**, is a cube with six sides numbered 1 to 6.

Example 1

Assume we roll a die and get a 1.

What is the random process?

Rolling the die.

What is the outcome?

The 1 that was rolled.

What is the chance of rolling a 1 on this die?

If the dice is fair, each side has the same chance of being rolled. So a 1 has a one-in-six chance, equivalently $\frac{1}{6}$.

Definition

A **probability** of an outcome is the proportion of times the outcome would occur if we observed the random process an infinite number of times.

Note

$$P(X) = \frac{\text{Number of outcomes corresponding to } X}{\text{Total number of outcomes}}$$

Example 2

What is the probability of rolling a 1 or 2 on a die?

There are two outcomes, a 1 or a 2, and six faces on a die.

$$P(\text{roll 1 or 2}) = \frac{2}{6} = \frac{1}{3}$$

Note

A standard deck of 52 playing cards consists of four **suits** in two colors: Hearts ♥, Spades ♠, Diamonds ♦, and Clubs ♣

Each suit contains 13 cards, each of a different **rank**:
2 through 10, Jack, Queen, King, and Ace

The Jack, Queen, and King cards are called **face cards.**

The Jack, Queen, King, and Ace cards are called **honour cards.**

The cards numbered 2 to 10 are called **numerals.**

♣2	♣3	♣4	♣5	♣6	♣7	♣8	♣9	♣10	♣J	♣Q	♣K	♣A
♦2	♦3	♦4	♦5	♦6	♦7	♦8	♦9	♦10	♦J	♦Q	♦K	♦A
♥2	♥3	♥4	♥5	♥6	♥7	♥8	♥9	♥10	♥J	♥Q	♥K	♥A
♠2	♠3	♠4	♠5	♠6	♠7	♠8	♠9	♠10	♠J	♠Q	♠K	♠A

Example 3

What is the probability of drawing a single card from a deck and getting an Ace?

There are four aces in a deck of 52 cards. Which gives the probability

$$P\left(\text{Ace}\right) = \frac{4}{52} = \frac{1}{13} = 0.0769 = 7.67\%$$

Example 4

What is the probability of rolling a 1, 2, 3, 4, 5, or 6 on a die?

Every side of the die is listed, so

$$P\left(\text{roll 1 or 2 or 3 or 4 or 5 or 6}\right) = \frac{6}{6} = 1 = 100\%$$

Definition

An outcome with a probability of 1 is called **certain**.

Example 5

If a year is selected at random, what is the probability that Thanksgiving Day (in the United States) will be on a Wednesday?

In the United States, Thanksgiving Day always falls on the fourth Thursday in November.

This means it is impossible for Thanksgiving Day to fall on a Wednesday.

$$P\,(\text{Thanksgiving on a Wednesday}) = 0 = 0\%$$
$$P\,(\text{Thanksgiving on a Thursday}) = 1 = 100\%$$

Definition

An outcome with a probability of 0 is called **impossible**.

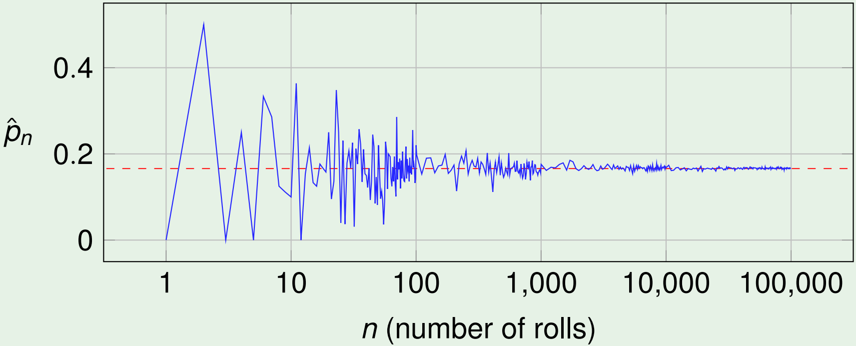
Note

Probabilities are always between 0 and 1.

Example 6

The probability of rolling a 1 on a die is $p = 1/6 \approx 0.167$, but if we roll six dice, we may get no 1's or multiple 1's.

Let $\hat{p}_n$ be the proportion of number of 1's rolled after n rolls.



Note

It is not a coincidence that $\hat{p}_n$ get closer to p as n increases.

Law of Large Numbers

As more observations are collected, the proportion $\hat{p}_n$ of occurrences with a particular outcome converges to the probability p of that outcome.

Cautions

- The law of large numbers applies to behavior over a large number of trails, and it does not apply to any one individual outcome.
  - Gamblers sometimes foolishly loose large sums of money by incorrectly thinking that a string of losses increases the chances of a win on the next bet, or that a string of wins is likely to continue.
- If we know nothing about the likelihood of different possible outcomes, we should not assume that they are equally likely.
  - You should not think that the probability of passing the next exam is $\frac{1}{2}$, or 0.5. The actual probability depends on factors such as the amount of preparation and the difficulty of the exam.

Definition

Outcomes A and B are **disjoint** (or **mutually exclusive**) if they cannot occur at the same time.

Example 7

Are the outcomes "roll a 1" and "roll a 2" disjoint?

Yes, it is impossible to roll two different numbers at the same time.

Example 8

Are the outcomes "roll a 1" and "roll an odd number" disjoint?

No, 1 is an odd number.

Example 9

Are the outcomes "draw an ace" and "draw a diamond" disjoint?

No, the ♦A is both an ace and a diamond.

Addition Rule of Disjoint Outcomes

If A_1 and A_2 represent two disjoint outcomes, then the probability that one of them occurs is given by:

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$

Example 10

The probability of rolling a 1 or rolling a 2 on a die can be calculated two ways:

Directly: There are two outcomes that are either a 1 or a 2, so the probability is:

$$P(\text{roll a 1 or roll a 2}) = \frac{2}{6} = \frac{1}{3}$$

Addition Rule: Rolling a 1 and rolling a 2 are disjoint, so:

$$\begin{aligned} P(\text{roll a 1 or roll a 2}) &= P(\text{roll a 1}) + P(\text{roll a 2}) \\ &= \frac{1}{6} + \frac{1}{6} = \frac{1+1}{6} = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Example 11

Let's calculate the probability of rolling a 1, 2, or 3.

$$\begin{aligned} P\left(\text{roll a 1 or roll a 2 or roll a 3}\right) &= P\left(\text{roll a 1}\right) + P\left(\text{roll a 2}\right) + P\left(\text{roll a 3}\right) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{3}{6} = \frac{1}{2} \end{aligned}$$

Example 12

Let's calculate the probability of rolling a 1, 2, 3 or 4.

$$\begin{aligned} P\left(\text{roll a 1 or 2 or 3 or 4}\right) &= P\left(1\right) + P\left(2\right) + P\left(3\right) + P\left(4\right) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{4}{6} = \frac{2}{3} \end{aligned}$$

Definition

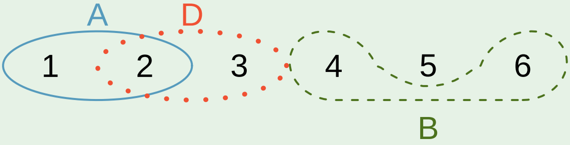
A **event** is a collection of outcomes.

Note

If two events have no elements in common, they are disjoint.

Example 13

Consider the following events.



Are A and B disjoint? Yes.

Are A and D disjoint? No, 2 is in both.

Are B and D disjoint? Yes.

Example 14

Suppose we flip a fair coin and roll a fair die.

The list of all possible outcomes is:

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

We want to calculate the probability of getting a head or a six.

The relevant outcomes are: H1, H2, H3, H4, H5, H6, T6.

Meaning that $P(\text{H or } 6) = \frac{7}{12}$.

Notice that $\frac{6}{12} = \frac{1}{2}$ of the outcomes have heads and $\frac{2}{12} = \frac{1}{6}$ have a six.

But, $\frac{1}{2} + \frac{1}{6} = \frac{8}{12}$, is wrong because we have double counted H6.

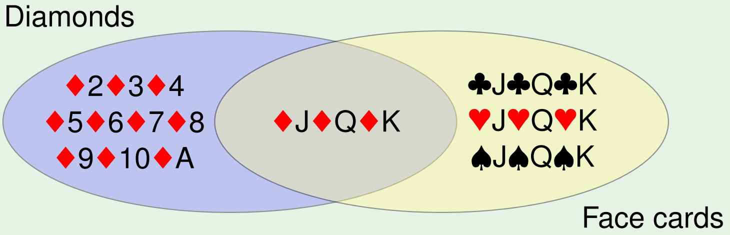
The correct probability is:

$$P(\text{H or } 6) = P(\text{H}) + P(6) - P(\text{H and } 6) = \frac{1}{2} + \frac{1}{6} - \frac{1}{12} = \frac{7}{12}$$

Example 15

Let us consider the events “draw a diamond” and “draw a face card”.

These outcomes are not disjoint, since three cards are both:



The Addition Rule for Disjoint Outcomes would count ♦J♦Q♦K twice!

$$\begin{aligned} P(\blacklozenge \text{ and face}) &= P(\blacklozenge) + P(\text{face}) - P(\blacklozenge \text{ and face}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} \end{aligned}$$

General Addition Rule

If A and B are any two events, disjoint or not, then the probability that at least one of them will occur is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Note

In statistics, when we write “or” what we mean is “and/or”, unless we explicitly say otherwise.

In other words, “A or B” occurring means A, B, or both A and B occur.

Note

If A and B are disjoint this means $P(A \text{ and } B) = 0$, and so we get the Addition Rule for Disjoint Outcomes:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= P(A) + P(B) - 0 \\ &= P(A) + P(B) \end{aligned}$$

Example 16

Suppose we draw one card from a standard deck. Let us find the probability that we get a Queen or a King.

There are 4 Queens and 4 Kings in the deck, hence eight outcomes out of 52 possible outcomes. So, the probability is

$$P\left(\text{Q or K}\right) = \frac{8}{52}$$

Since there are no cards that are both Kings and Queens, we have

$$P\left(\text{Q or K}\right) = P\left(\text{Q}\right) + P\left(K\right) - P\left(\text{Q and K}\right) = \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}$$

Definition

A **probability distribution** is a table of all disjoint outcomes and their associated probabilities.

Example 17

The probability distribution for the sum of two dies:

Dice sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Rules for Probability Distributions

All probability distributions must satisfy the following rules:

1. The outcomes listed must be disjoint.
2. Each probability must be between 0 and 1.
3. The probabilities must sum to 1.

Example 18

The follow table contain three possible distributions for household income in the United States.

Income Range	\$0-\$25k	\$25k-\$50k	\$50k-\$100k	\$100k+
a	0.18	0.39	0.33	0.16
b	0.38	-0.27	0.52	0.37
c	0.28	0.27	0.29	0.16

Only one of the three is actually a probability distribution. Which one?

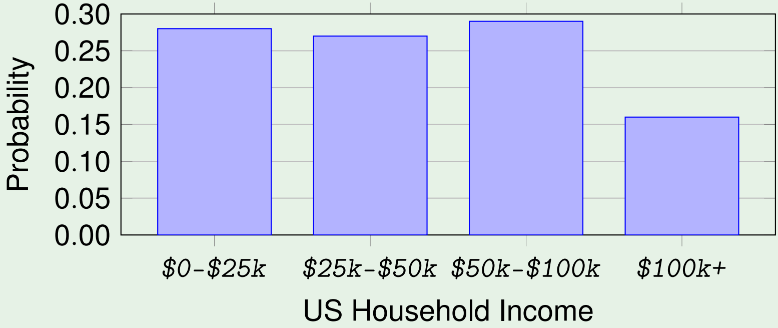
- a sums to 2.5.
- b contains a negative number.
- c is the real probability distribution.

Note

A probability distribution can be represented with a bar plot, where each outcome is represented by a bar, the height of the bar being the probability of the outcome.

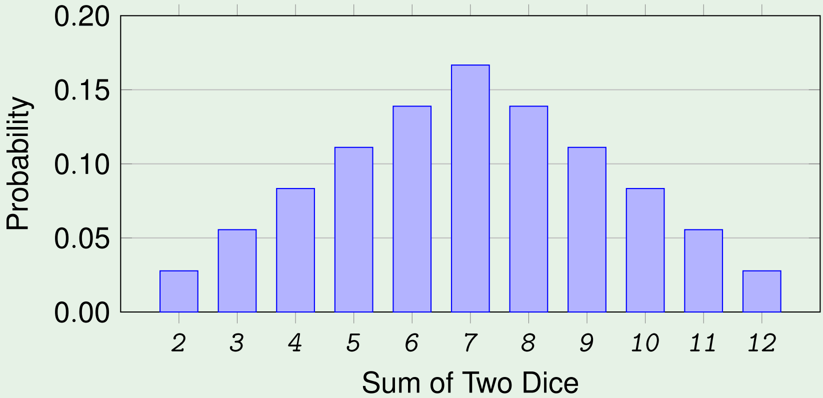
Example 19

Here is the bar plot for the probability distribution in Example 18:



Example 20

Here is the bar plot for the dice sum distribution in Example 17



Definition

The set of all possible outcomes is called the **sample space**, S.

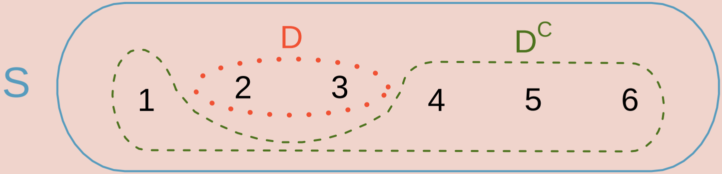
Example 21

The sample space of rolling a single die is:

$$\{\text{roll a 1}, \text{roll a 2}, \text{roll a 3}, \text{roll a 4}, \text{roll a 5}, \text{roll a 6}\}$$

Definition

The **complement** of an event D is all the outcomes in the sample space that are not in D. Denoted D^C.



Note

A and A^C are disjoint events.

Note

Since every outcome in the sample space is either in A or A^C:

$$P\left(A \text{ or } A^C\right) = 1$$

Note

We can also use the Addition Rule $\left(P\left(A \text{ and } A^C\right) = 0\right)$:

$$P\left(A \text{ or } A^C\right) = P(A) + P\left(A^C\right)$$

Note

Finally, $P\left(A^C\right) = 1 - P(A)$ and $P(A) = 1 - P\left(A^C\right)$.

Example 22

Consider rolling two dice and summing the numbers.

What is the complement of the event "the total is less than 12?"

The complement is "the total is 12."

Finding $P\,(\text{total} < 12)$ can be calculated directly, but it is complicated. Complements give an easier way:

$$\begin{aligned} P\,(\text{total} < 12) &= 1 - P\,(\text{total} = 12) \\ &= 1 - \frac{1}{36} \\ &= \frac{36}{36} - \frac{1}{36} \\ &= \frac{36-1}{36} \\ &= \frac{35}{36} \approx 97.2\% \end{aligned}$$

Example 23

Consider rolling two dice and summing the numbers.

What is the complement of the event "the total is at least 4?"

The complement is "the total is 2 or the total is 3."

$$\begin{aligned}
P(\text{total} \geq 4) &= 1 - P(\text{total} = 2 \text{ or total} = 3) \\
&= 1 - (P(\text{total} = 2) + P(\text{total} = 3)) \\
&= 1 - \left(\frac{1}{36} + \frac{2}{36}\right) \\
&= 1 - \frac{3}{36} \\
&= \frac{36}{36} - \frac{3}{36} \\
&= \frac{33}{36} \approx 91.6\%
\end{aligned}$$

Definition

Events A and B are **independent events** if the probability of B occurring is the same, whether or not A occurs.

Example 24

1. A fair coin is tossed two times. The two events are:
   - The first toss is a head.
   - The second toss is a head.

   Are these events independent? Yes
2. You draw a card from a deck, then without replacing the first, draw a second card.

   Are these events independent? No

Note

If two events are not independent, they are called **dependent**.

Multiplication Rule For Independent Events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 25

Suppose we flip a fair coin and roll a fair die.

The sample space is $\{\text{H1}, \text{H2}, \text{H3}, \text{H4}, \text{H5}, \text{H6}, \text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}$.

The probability of getting tails on the coin and three on the die is

$$P(\text{T3}) = \frac{1}{12}$$

Since a coin and dice are independent, we could have used the Multiplication Rule:

$$P(\text{T and 3}) = P(\text{T}) \cdot P(3) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Example 26

Modern aircraft are now highly reliable, and one design feature contributing to that reliability is the use of **redundancy**, whereby critical components are duplicated so that if one fails, the other will still work.

The Airbus 310 airliner has three independent hydraulic systems, so if one system fails, full flight control is maintained. Let us assume the probability of a hydraulic system failing is 0.002.

- *If the Airbus 310 had only a single hydraulic system, what is the probability that the flight control system would work for an entire flight?* $P\,(\text{safe flight}) = 1 - P\,(\text{system fails}) = 1 - 0.002 = 0.998$
- *Given the Airbus 310 has three independent hydraulic systems, what is the probability that the flight control system would work for an entire flight?*

$$\begin{aligned} P\,(\text{safe flight}) &= 1 - P\,(\text{system}_1 \text{ fails and system}_2 \text{ fails and system}_3 \text{ fails}) \\ &= 1 - P\,(\text{system}_1 \text{ fails}) \cdot P\,(\text{system}_2 \text{ fails}) \cdot P\,(\text{system}_3 \text{ fails}) \\ &= 1 - 0.002 \cdot 0.002 \cdot 0.002 \\ &= 1 - 0.000000008 = .999999992 \end{aligned}$$

Sampling

Sampling methods are critically important, and the following relationships hold:

- Sampling *with replacement*: Selections are *independent* events.
- Sampling *without replacement*: Selections are *dependent* events.

Treating Dependent Events as Independent

When sampling without replacement and the sample size is very small compared to the size of the population, treat the selections as being independent (even though they are actually dependent).

Note

A good rule of thumb is that the sample size is small if it is less than 10% of the total population size.

Example 27

Assume that three adults are randomly selected without replacement from the $333,373,690$ adults in the United States. If we assume that 10% of adults use drugs, lets calculate is the probability that the three selected adults all use drugs.

Because the three adults are randomly selected without replacement, the three events are dependent. This means the exact probability is rather cumbersome to calculate:

$$\begin{aligned} P\,(\text{all use drugs}) &= P\left(1^{\text{st}} \text{ uses drugs and } 2^{\text{nd}} \text{ uses drugs and } 3^{\text{rd}} \text{ uses drugs}\right) \\ &= \left(\frac{33,337,369}{333,373,690}\right)\cdot\left(\frac{33,337,368}{333,373,689}\right)\cdot\left(\frac{33,337,367}{333,373,688}\right) \\ &= 0.000999999919 \quad \text{(Imagine selecting 10,000 adults!)} \end{aligned}$$

Since 3 adults is a very small part of the total population, we can simplify the calculations considerably:

$$\begin{aligned} P\,(\text{all use drugs}) &= P\left(1^{\text{st}} \text{ uses drugs and } 2^{\text{nd}} \text{ uses drugs and } 3^{\text{rd}} \text{ uses drugs}\right) \\ &= 0.1\cdot 0.1\cdot 0.1 = 0.00100 \end{aligned}$$

Note

We can also use the Multiplication Rule For Independent Events to check if two events are independent.

Example 28

If we shuffle a deck of playing cards and draw one card, let us check that the events “draw a heart” and “draw an ace” are independent.

We know that

$$P(\heartsuit\text{A}) = \frac{1}{52}$$
$$P(\heartsuit) = \frac{13}{52} = \frac{1}{4}$$
$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$$

So,

$$P(\heartsuit) \cdot P(\text{Ace}) = \frac{1}{4} \cdot \frac{1}{13} = \frac{1}{4 \cdot 13} = \frac{1}{52} = P(\heartsuit \text{ and Ace})$$

Since the equation holds, the two events must be independent.