Direction Fields and Euler's Method

Colby Community College

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Analytic Definition of a Solution

Analytically, $y(t)$ is a **solution** of a differential equation if substituting $y(t)$ for y reduced the equation to an identity:

$$y'(t) = f(t, y(t))$$

on an appropriate domain for t.

Example 1

Verify that $y(t)$ is a solution to the DE.

$$y'(t) = 2y, \quad y(t) = e^{2t}$$

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Similarly, we could show that

$$y(t) = 2e^{2t} \quad \text{and} \quad y(t) = \frac{-3}{2}e^{2t}$$

are also solutions. In fact, any constant multiple of e^{2t} is a solution.

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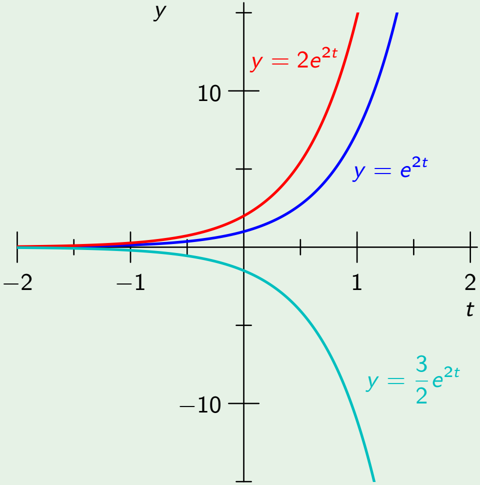

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Other solutions are of the form $y(t) = \sqrt{k-t^2}$.

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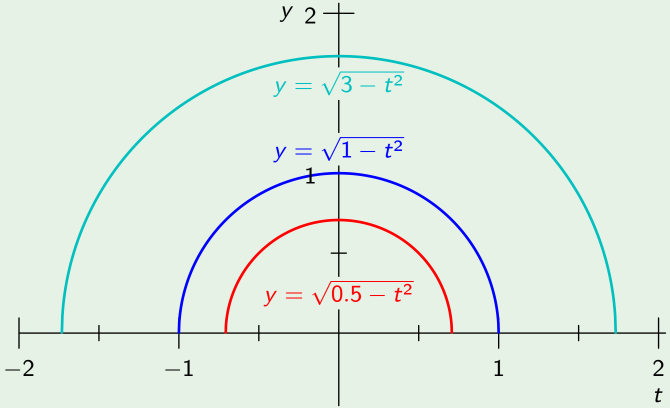

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$$y = \int f(t)dt + c$$

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Family of Solutions

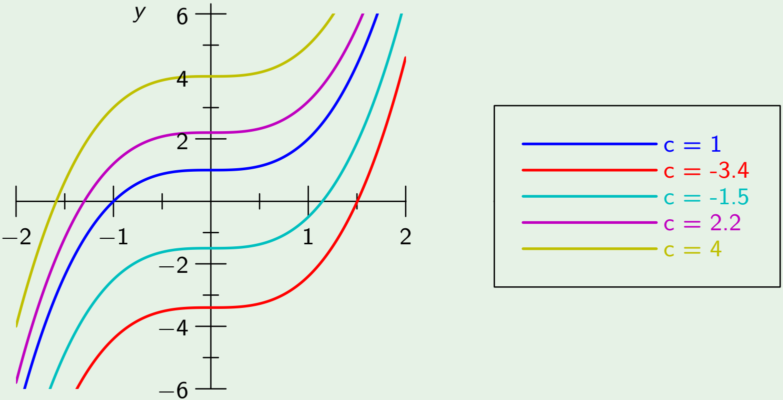
In general, all solutions of a first-order DE form a **family** of solutions expressed with a single parameter c. Such a family is called the **general solution**. A member of the family that results from a specific value of c is called a **particular solution**.

Example 4

The general solution of $y' = 3t^2$ is

$$y = t^3 + c$$

where c may be any real value.



Initial-Value Problem

The combination of a first-order differential equation and an **initial condition**

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

is called an **initial-value problem**. It's solution will pass through the point (t_0, y_0).

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Note

While a family of solutions for a DE contains multiple solutions, an IVP usually has only one solution. That is, the solution to an IVP is a particular solution to the DE.

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The function $y(t) = t^3 + 1$ is a solution to the IVP

$$y' = 3t^2, \quad y(0) = 1$$

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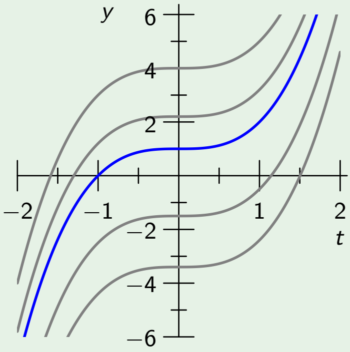
Differentiating $y(t)$ confirms that

$$y'(t) = (t^3 + 1)' = 3t^2, \quad \text{and} \quad y(0) = 0^3 + 1 = 1$$



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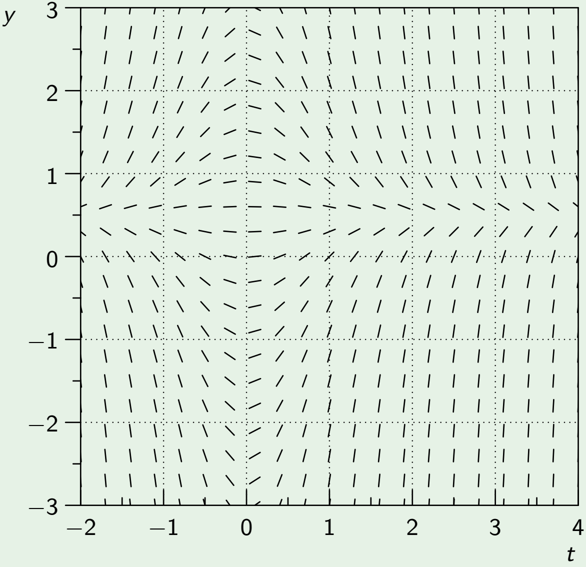
Direction Fields

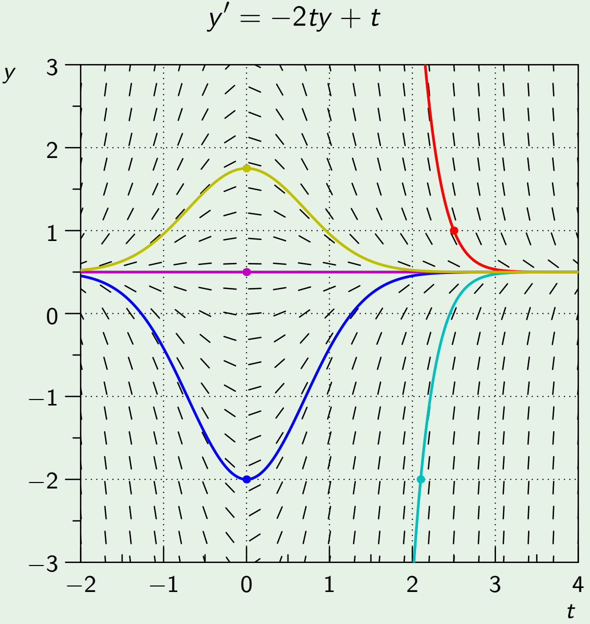
We can see what solution curves look like by, on regular intervals, drawing short line segments with slope determined by the DE for that point. The collection of these segments are called **direction field** (or a **slope field**).

Example 6

$$y' = -2ty + t$$



Example 6



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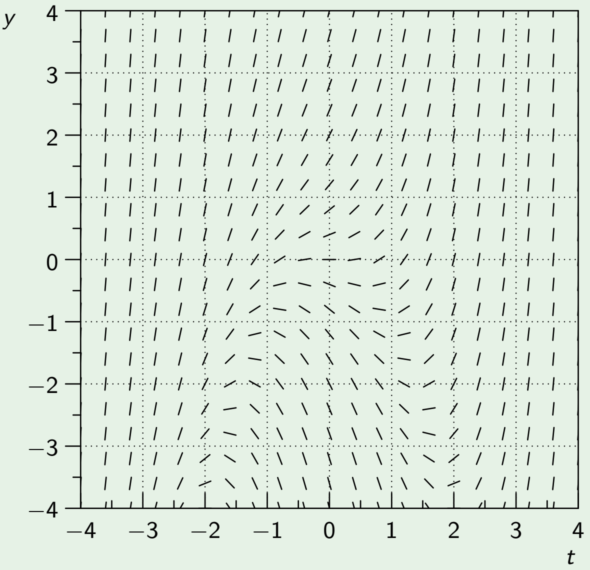

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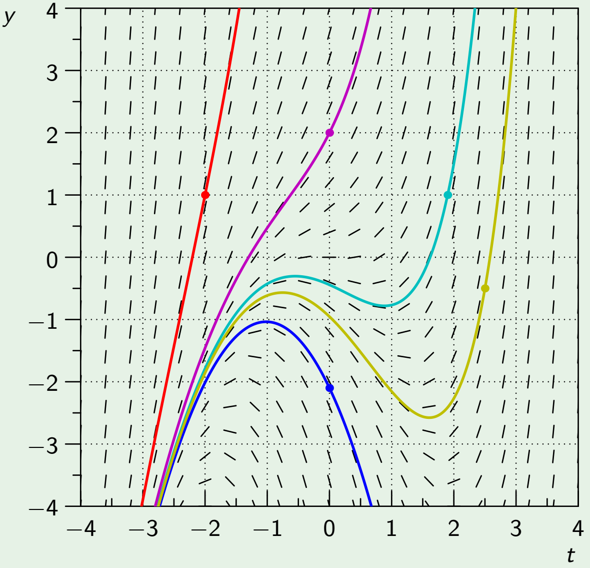
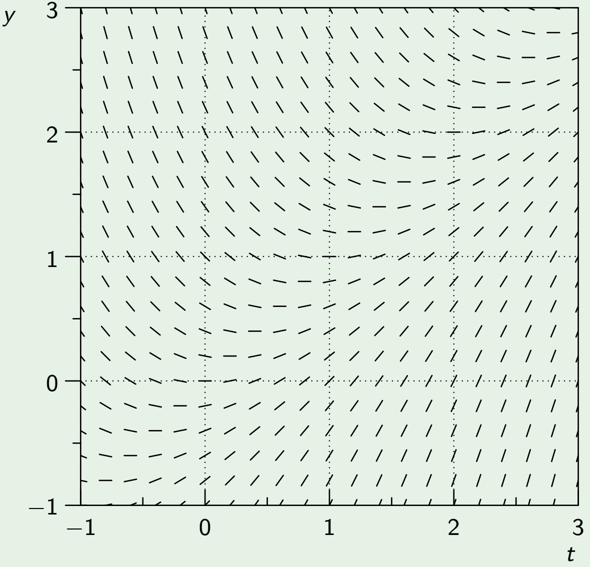


Example 8

$$y' = t - y$$



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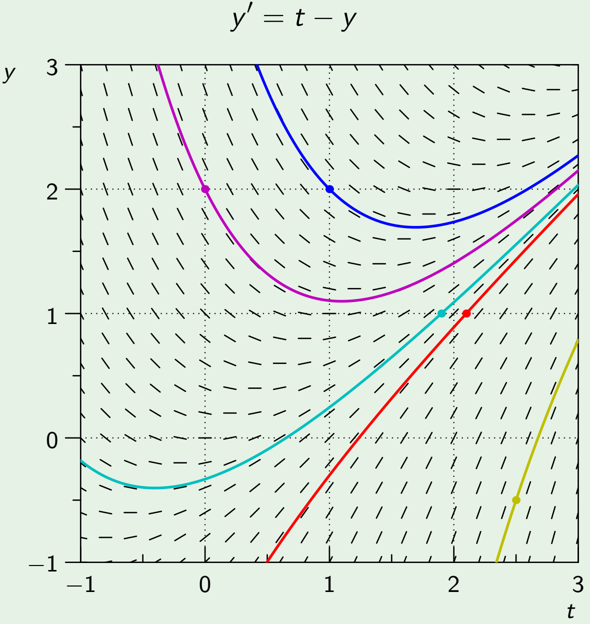
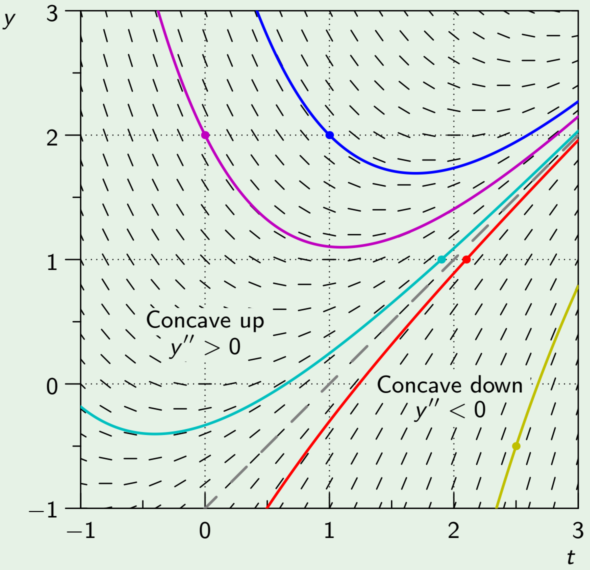


Example 8

$$y' = t - y \quad \text{and} \quad y'' = 1 - y' = 1 - t + y$$



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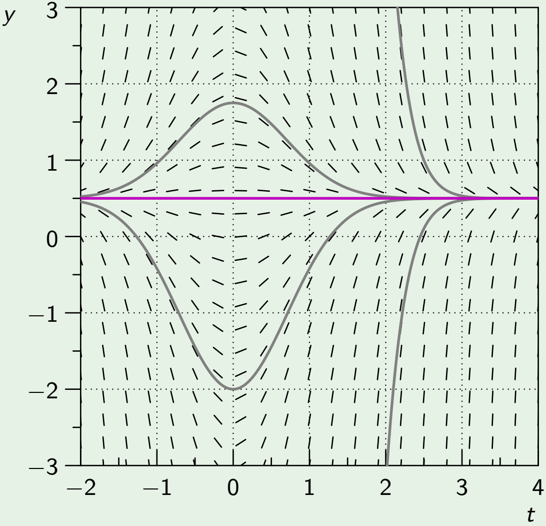
Note

A equilibrium solution is often called **semistable** if it is stable on one side and unstable on the other.

Example 9

The DE $y' = -2ty + t$ has the constant solution $y(t) = \frac{1}{2}$.



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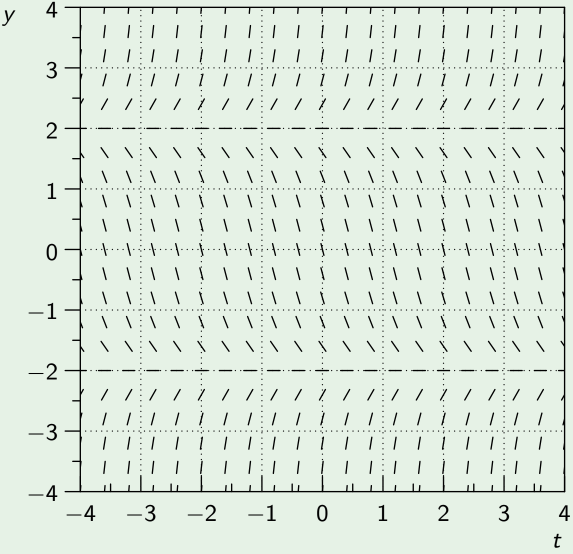
$$y' = y^2 - 4$$



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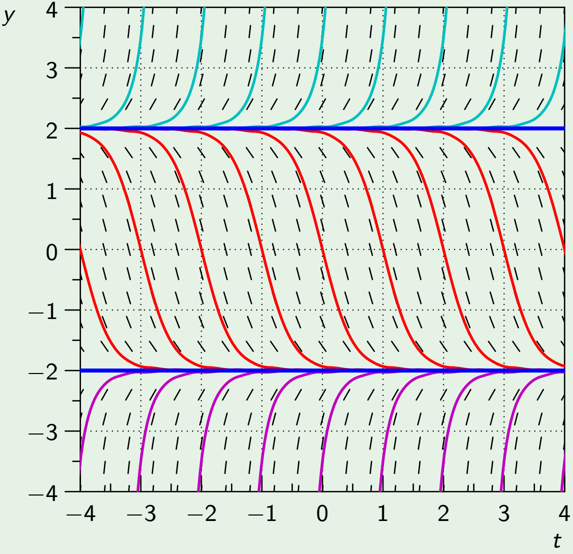
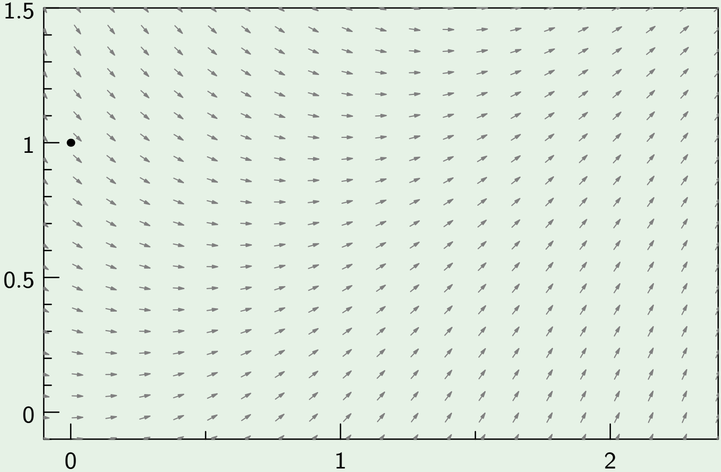


Example 11

We can approximate the solution of the IVP

$$y' = t - y, \; y(0) = 1$$



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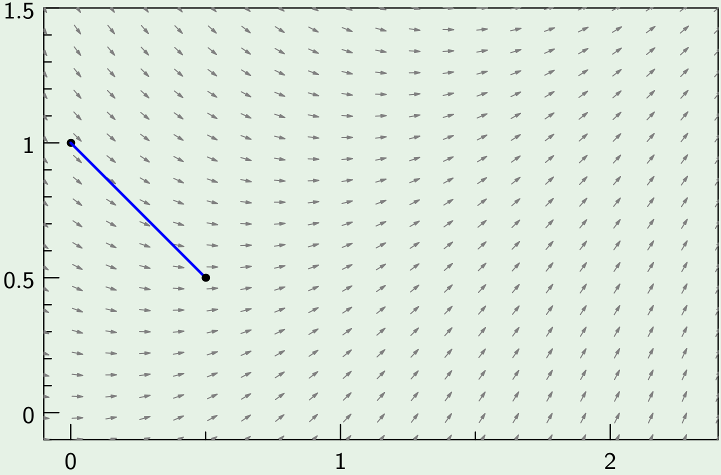

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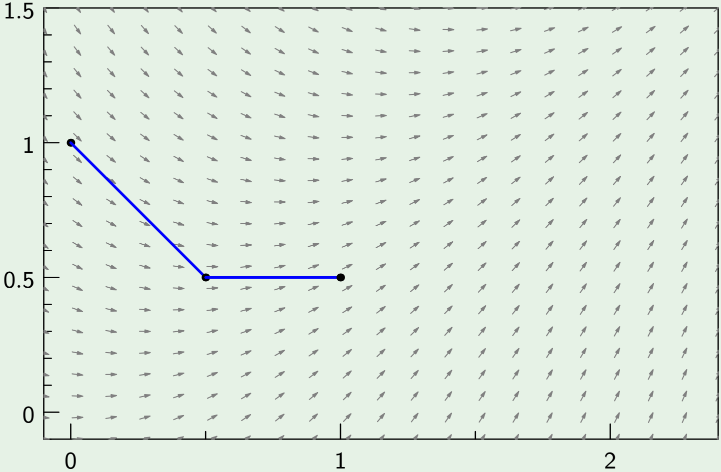

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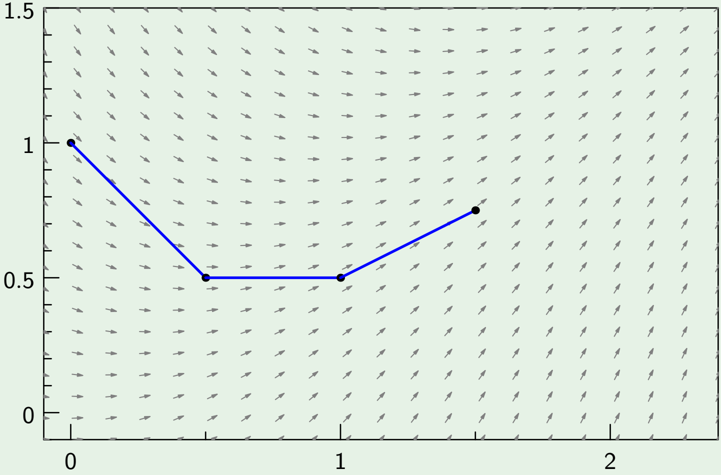

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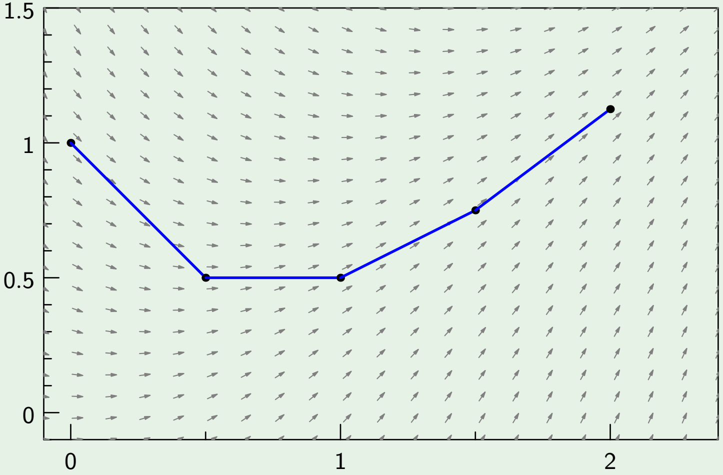

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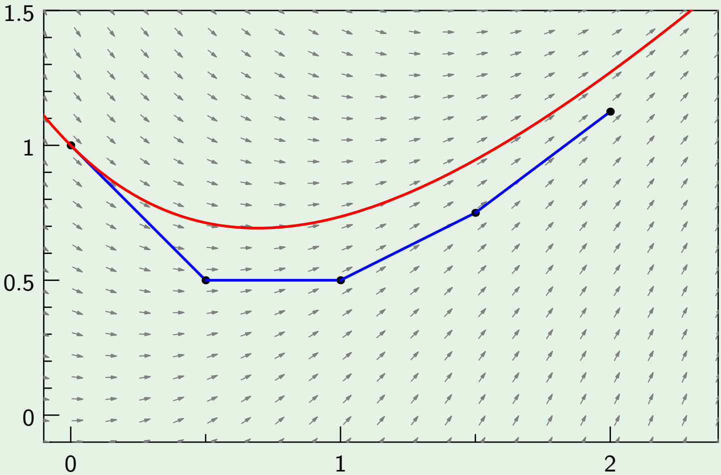

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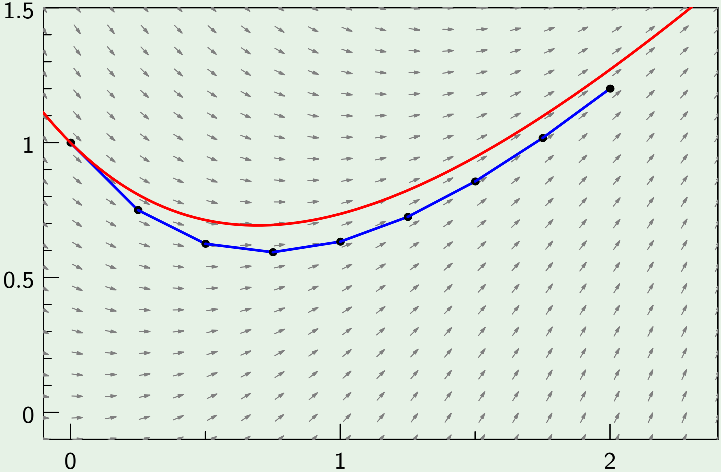

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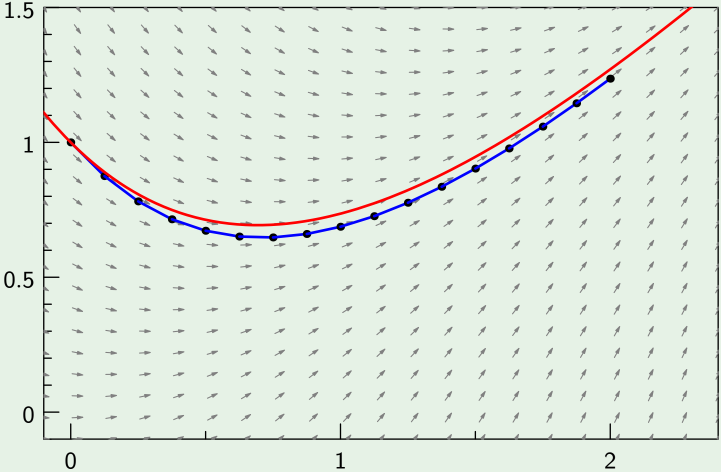

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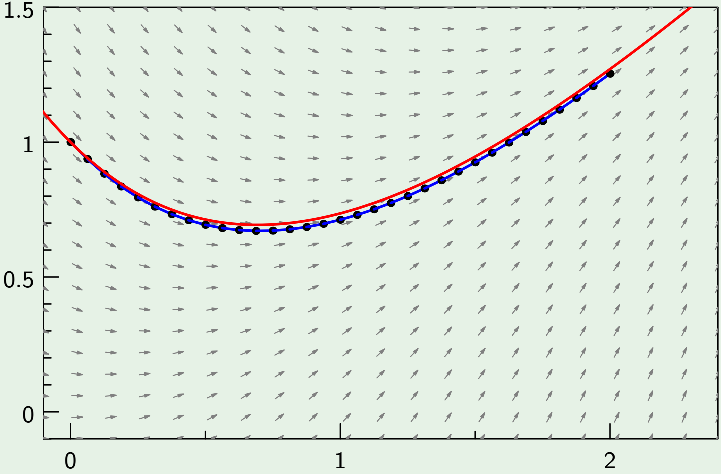

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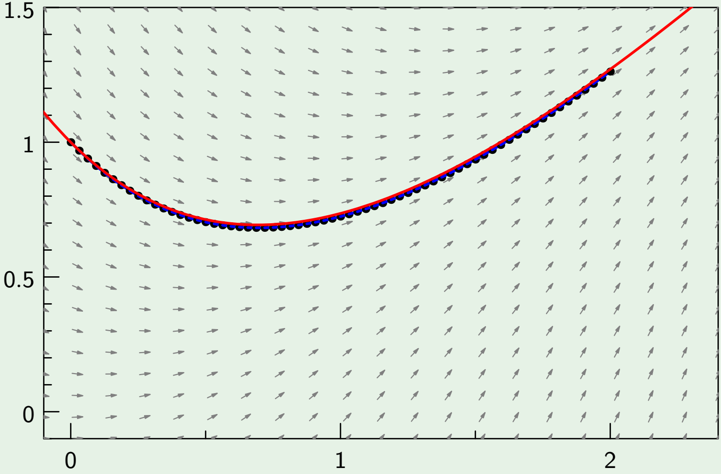

Consider

Given the IVP

$$y' = f(t, y),\ \ y(t_0) = y_0$$

We want to compute approximate values for $y(t_n)$ at the (finite) set of points $t_1, t_2, t_3, \ldots, t_k$.

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to find the approximate solution $(t_1, y(t_1))$:

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\end{aligned}$$

The resulting piecewise-linear function (i.e. play connect-the-dots) is called the **Euler-approximate** solution.

Euler's Method

For the Initial-value problem

$$y' = f(t, y), \ y(t_0) = y_0$$

use the formulas

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + h \cdot f(t_n, y_n) \end{aligned}$$

to iteratively compute the points, using step size h,

$$(t_1, y_1), (t_2, y_2), \ldots, (t_k, y_k).$$

The piecewise-linear function connecting these points is the Euler approximtion to the solution $y(t)$ of the IVP for $t_0 \leq t \leq t_k$.

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Let us obtain the Euler-approximate solution of the IVP

$$y' = -2ty + t,\ \ y(0) = -1$$

with step size 0.1 on $[0, 0.4]$.

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In other words:

$$\begin{aligned} f(t, y) &= -2ty + t = t(1 - 2y) \\ t_0 &= 0 \\ y_0 &= -1 \\ h &= 0.1 \\ k &= 1, 2, 3, 4 \end{aligned}$$

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$$y_1 = y_0 + h \cdot f(t_0, y_0) = -1 + (0.1)(0)(1 - 2(-1)) = -1$$

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$$t_4 = t_3 + h = 0.3 + 0.1 = 0.4$$
$$\begin{aligned} y_4 &= y_3 + h \cdot f(t_3, y_3) \\ &= -0.9112 + (0.1)(0.3)(1 - 2(-0.9112)) = -0.82652 \end{aligned}$$

Example 12

How does this compare to the exact solution $y(t) = 0.5 - 1.5e^{-t^2}$?

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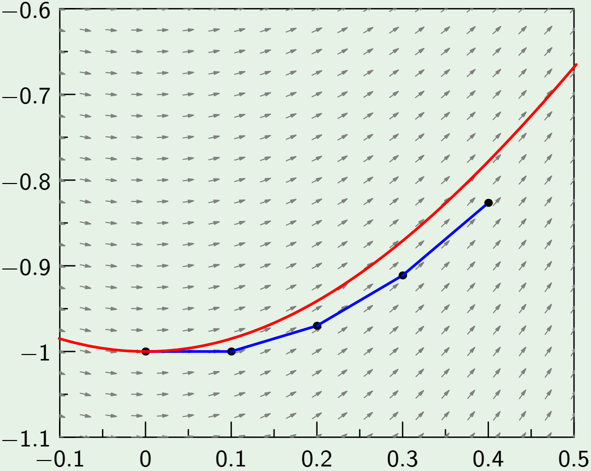
n	t_n	y_n	$y(t_n)$	**Error**
0	0.0	-1.000000	-1.000000	**0.000000**
1	0.1	-1.000000	-0.985075	**-0.014925**
2	0.2	-0.970000	-0.941184	**-0.028815**
3	0.3	-0.911200	-0.870897	**-0.040303**
4	0.4	-0.826528	-0.778216	**-0.048312**

Notice how the error grows rapidly.

Example 12



Example 13

Find the Euler-approximation of

$$y' = -2ty, \; y(0) = 1$$

using a step size of 0.2 over the range of $[0, 2]$.

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9	1.8	0.0197378	0.0391639	**0.019426**
10	2.0	0.0055265	0.0183156	**0.012789**

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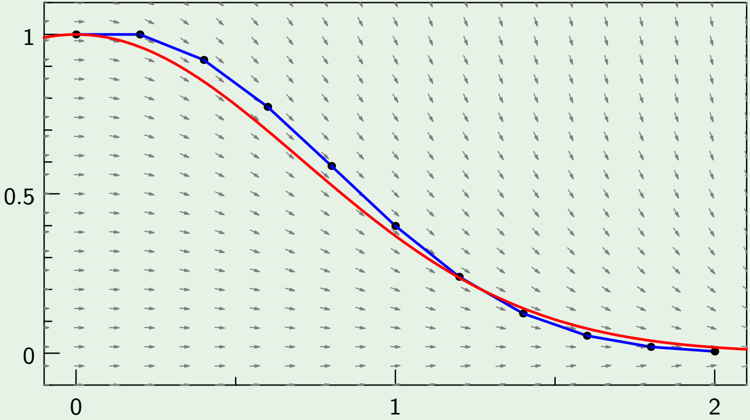

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It can be shown, using Taylor series expansions, that the error is proportional to the square of the step size. (We will talk about Taylor series in chapter 11.)

$$|y_i - y(t_i)| \leq C \cdot h^2$$

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We call this error the **local discretization error** because it estimates the error for a single step only. After n steps, we have n times the error. Which we call the **global discretization error**.