Eigenvalues and Eigenvectors

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Example

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{\boldsymbol{u}}) = \boldsymbol{A}\vec{\boldsymbol{u}}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

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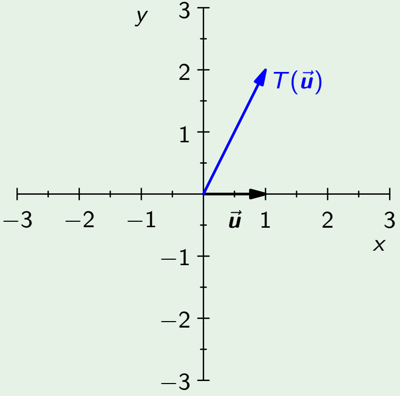
We can see how T maps a few vectors:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \longrightarrow$$



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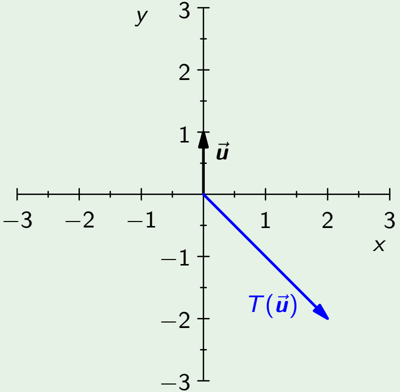

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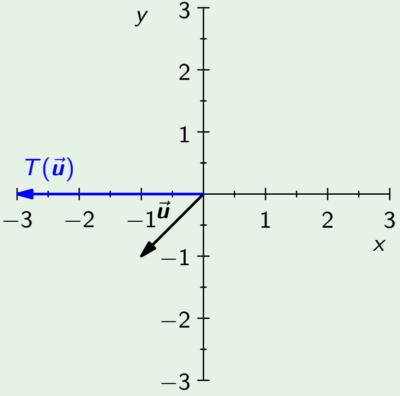
$$T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \qquad \longrightarrow$$



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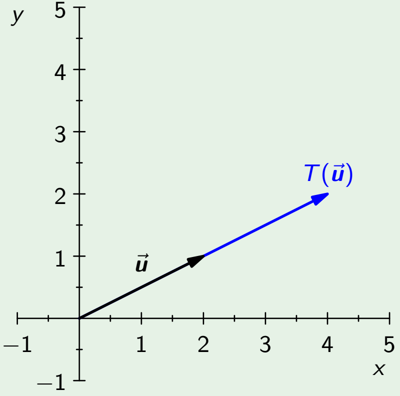
But something interesting happens for some special vectors.

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \longrightarrow$$



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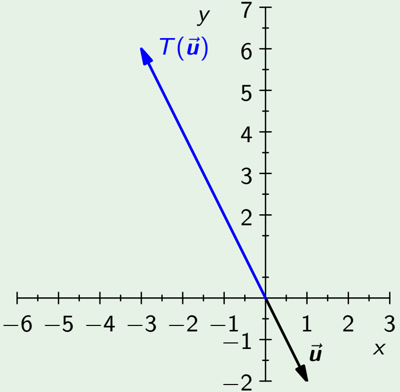

Eigenvalues and Eigenvectors

Let $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation from vector space $\mathbb{V}$ into itself. A scalar λ is a **eigenvalue** of T if there is a *nonzero* vector $\vec{\boldsymbol{v}} \in \mathbb{V}$ such that

$$T(\vec{\boldsymbol{v}}) = \lambda\vec{\boldsymbol{v}}$$

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If the linear transformation T is represented by an $n \times n$ matrix $\boldsymbol{A}$, where $\mathbb{V} = \mathbb{R}^n$ and $T(\vec{\boldsymbol{v}}) = \boldsymbol{A}\vec{\boldsymbol{v}}$, then λ and $\vec{\boldsymbol{v}}$ are characterized by the equation

$$\boldsymbol{A}\vec{\boldsymbol{v}} = \lambda \vec{\boldsymbol{v}}$$

Computing Eigenvalues and Eigenvectors

If $\boldsymbol{A}$ is a $n \times n$ matrix, and $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, then

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While this equation always has the trivial solution $\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$, we are looking for any non-zero solutions. Therefore, we are looking for when

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The polynomial in λ denoted by

$$p(\lambda) = |\boldsymbol{A} - \lambda\boldsymbol{I}_n|$$

is called the **characteristic polynomial** of $\boldsymbol{A}$.

Summary of Steps for Finding Eigenvalues and Eigenvectors

1. Write the characteristic equation $|\boldsymbol{A} - \lambda \boldsymbol{I}_n| = 0$.
2. Solve the characteristic equation for λ.
3. For each eigenvalue λ_i, find the corresponding eigenvector $\vec{\boldsymbol{v}_i}$ by solving the system of equations

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For large matrices they steps become cumbersome, so computer algebra systems are often employed.

Example

In our first example we saw two eigenvectors, let us verify these using the characteristic equation.

$$\left| \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

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$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

The characteristic equation is

$$|\boldsymbol{A} - \lambda \boldsymbol{I}_2| = 0 \quad \rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \quad \rightarrow \quad (1-\lambda)^2 - 4 = 0$$

Which has solutions $\lambda_1 = 3$ and $\lambda_2 = -1$.

To find the eigenvector for λ_1 we need to solve

$$\begin{bmatrix} 1-(3) & 1 \\ 4 & 1-(3) \end{bmatrix} \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}} \rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{\boldsymbol{v_1}} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find the eigenvector for λ_2 we need to solve

$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

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$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}} \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right]$$

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Which simplifies to:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

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Which simplifies to:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$
$$(\lambda - 2)(\lambda - 1)(\lambda + 1) = 0$$

So, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$.

Example

Let us find the eigenvalues and eigenvectors for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

To find the eigenvector for $\lambda_1 = 2$ we need to solve the system:

$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_1 = v_3$ and $v_2 = 3v_3$. Replacing v_3 with parameter s gives

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Example

Let us find the eigenvalues and eigenvectors for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

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$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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To find the eigenvector for $\lambda_2 = 1$ we need to solve the system:

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$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_1 = v_3$ and $v_2 = 0$. Replacing v_3 with parameter s gives

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Special Cases

Triangular Matrices: The eigenvalues of an upper (or lower) triangular matrix appear on the main diagonal.

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Trace

The **trace** of a matrix, $\textbf{tr}\,\boldsymbol{A}$, is the sum of all elements in the diagonal.

Eigenspace Theorem for Linear Transformations

For each eigenvalue λ of a linear transformations $T : \mathbb{V} \to \mathbb{V}$, the **eigenspace**, defined by

$$\mathbb{E}_\lambda = \{\vec{\boldsymbol{v}} \in \mathbb{V} \mid T(\vec{\boldsymbol{v}}) = \lambda \vec{\boldsymbol{v}}\}$$

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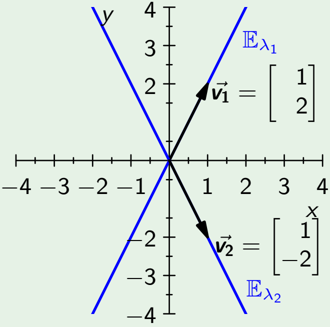

Example

For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

we had the following eigenvectors:

$$\lambda_1 = 2 \qquad \vec{\boldsymbol{v_1}} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \qquad \vec{\boldsymbol{v_2}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -1 \qquad \vec{\boldsymbol{v_3}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

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$$\lambda_2 = 1 \qquad \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \mathbb{E}_{\lambda_2} = \textbf{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_3 = -1 \quad \vec{\boldsymbol{v}}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbb{E}_{\lambda_3} = \textbf{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Distinct Eigenvalue Theorem

Let $\boldsymbol{A}$ be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p}$, then $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p}\}$ is a set of linearly independent vectors.

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Proof (sketch)

If we have two eigenvalues with $\lambda_1 \neq \lambda_2$, then if the associated eigenvectors $\vec{\boldsymbol{v_1}}$ and $\vec{\boldsymbol{v_2}}$ were linearly dependent, we have

$$\vec{\boldsymbol{v_2}} = c\vec{\boldsymbol{v_1}} \quad \text{where } c \neq 0$$

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But, we could also have multiplied by $\boldsymbol{A}$

$$\boldsymbol{A}\vec{\boldsymbol{v_2}} = c\boldsymbol{A}\vec{\boldsymbol{v_1}}$$

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Let $\boldsymbol{A}$ be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_p}}$, then $\{\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_p}}\}$ is a set of linearly independent vectors.

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$$\boldsymbol{A}\vec{\boldsymbol{v_2}} = c\boldsymbol{A}\vec{\boldsymbol{v_1}}$$
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Which would imply that $\lambda_1 = \lambda_2$,

Example

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

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$$\lambda(\lambda + 3)^2 = 0$$

So, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -3$. (Note that -3 is a repeated root.)

Example

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

To find the eigenvector for $\lambda_1 = 0$ we need to solve the system:

$$\begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Example

Consider the matrix

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To find the eigenvector for $\lambda_1 = 0$ we need to solve the system:

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So, we have $v_1 = v_3$ and $v_2 = v_3$. Replacing v_3 with parameter s gives

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To find the eigenvector for $\lambda_2 = -3$ we need to solve the system:

$$\begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_1 = -v_2 - v_3$ This means we need two parameters, $v_2 = r$ and $v_3 = s$. Which means we have two linearly independent eigenvectors.

$$\vec{\boldsymbol{v_2}} = \begin{bmatrix} -r-s \\ 1 \\ 1 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Example

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

This means the eigenspace is

$$\mathbb{E}_{\lambda_2} = \textbf{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Any linear combination of these two vectors is also an eigenvector, which means that the eigenspace is a plane.

Example

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Since this is an upper diagonal matrix, we know that the eigenvalue is $\lambda = 1$, with multiplicity of 3.

Example

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$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the eigenvector for $\lambda = 1$ we need to solve the system:

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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So, we have $v_2 + v_3 = 0$ and $v_3 = 0$. Replacing v_1 with parameter s gives

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Which means the eigenspace has dimension 1.

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$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

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We can continue in the same way and find that the eigenvectors are

$$\vec{\boldsymbol{v}}_{\boldsymbol{1}} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\boldsymbol{2}} = \begin{bmatrix} -1 \\ -i \end{bmatrix}$$

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$$\boldsymbol{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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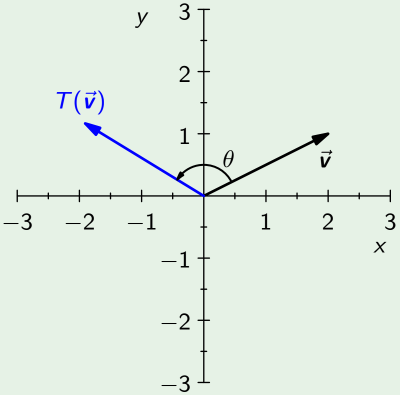
Which means these eigenvalues rotate a vector, instead of scaling it.

Example



Eigenvalue Properties

Let $\boldsymbol{A}$ be an $n \times n$ matrix

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- $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathsf{T}}$ have the same characteristic polynomials and the same eigenvalues.
- If λ is an eigenvalue of an invertible matrix $\boldsymbol{A}$, then $\dfrac{1}{\lambda}$ is an eigenvalue of $\boldsymbol{A}^{-1}$.

Characteristic roots of a linear homogeneous DE are eigenvalues.

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Properties of Linear Homogeneous Differential Equations with Distinct Eigenvalues

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We will explore the connection between eigenvalues and solutions to differential equations in chapter 6