Vector Spaces

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Vector Spaces

Definition

A **vector space** $\mathbb{V}$ is a nonempty collection of objects called **vectors** for which the following operations

- Vector addition, denoted $\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}$
- Scalar multiplication, denoted $c\vec{\boldsymbol{x}}$

satisfy the following nine properties. (For all $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \in \mathbb{V}$ and all $c, d \in \mathbb{R}$)

Vector Spaces

Closure

1. $c\vec{x} + d\vec{y} \in \mathbb{V}$

Addition

2. There exists a **zero vector** $\vec{0} \in \mathbb{V}$ such that $\vec{x} + \vec{0} = \vec{x}$
3. For all $\vec{x} \in \mathbb{V}$ there exists $-\vec{x} \in \mathbb{V}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
4. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
5. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

Scalar Multiplication

6. $1\vec{x} = \vec{x}$
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
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All vectors $< x_1, x_2, \ldots, x_n >$ in $\mathbb{R}^n$ satisfy these properties.
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Definition

Let $\mathbb{M}_{mn}$ denote the collection of all $m \times n$ matrices.

Example

Thinking back, we can see that the properties for addition and scalar multiplication of matrices we saw in section 3.1 satisfy all nine requirements to be a vector space.
Which means, for any $m, n \in \mathbb{R}$, $\mathbb{M}_{mn}$ is a vector space.

Vector Function Spaces

Definition

A **function space** is a vector space where the "vectors" are functions defined on an interval I. The addition and scalar multiplication operations are defined in the usual way:

- $(f + g)(t) = f(t) + g(t)$, for all $t \in I$
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Solutions to *linear homogeneous* DEs form a vector space.

Vector Function Spaces

Example

The set of all solutions of the first order linear homogeneous DE

$$y' + p(t)y = 0$$

(where p and y are defined on some interval I) is a vector space.

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For solutions $u(t)$ and $v(t)$, as well as scalars a and b, we need to verify that $a \cdot u(t) + b \cdot v(t)$ is a solution.

$$(au + bv)' + p(t)(au + bv)$$

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&= a \cdot 0 + b \cdot 0 \\
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is **not** a vector space. Why?

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Example

Consider the collection of all polynomials of degree ≤ 3. A vector in this space is given by

$$P(t) = a_3x^3 + a_2x^2 + a_1x + a_0$$

where $a_3, a_2, a_1, a_0 \in \mathbb{R}$.
This collection is a vector space, verified using basic algebra.

Prominent Vector Spaces

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- $\mathbb{R}^3$, the space of all real ordered triples.
- $\mathbb{R}^n$, the space of all real ordered n-tuples.
- $\mathbb{C}^n$, the space of all complex n-tuples.
- $\mathbb{P}$, the space of all polynomials.
- $\mathbb{P}_n$, the space of all polynomials of degree $\leq n$
- $\mathbb{M}_{mn}$, the space of all $m \times n$ matrices.
- $\mathcal{C}(I)$, the space of all continuous functions defined on the interval I.
- $\mathcal{C}^n(I)$, the space of all functions, defined on the interval I, having n continuous derivatives.
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- $\mathbb{P}$, the space of all polynomials.
- $\mathbb{P}_n$, the space of all polynomials of degree $\leq n$
- $\mathbb{M}_{mn}$, the space of all $m \times n$ matrices.
- $\mathcal{C}(I)$, the space of all continuous functions defined on the interval I.
- $\mathcal{C}^n(I)$, the space of all functions, defined on the interval I, having n continuous derivatives.
- $\mathcal{C}^n$, the space of all functions, defined on the interval $(-\infty, +\infty)$, having n continuous derivatives.

Vector Subspaces

Theorem

A nonempty subset, $\mathbb{W}$, of a vector space $\mathbb{V}$ is a **subspace** *of $\mathbb{V}$ if*

- $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} \in \mathbb{W}$ *for all* $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{W}$
- $c\vec{\boldsymbol{u}} \in \mathbb{W}$ *for all* $\vec{\boldsymbol{u}} \in \mathbb{W}$ *and* $c \in \mathbb{R}$

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Proof

The definition of a subspace guarantees closure, everything else is inherited from the parent vector space.

For example, given $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{W}$, consider $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$.

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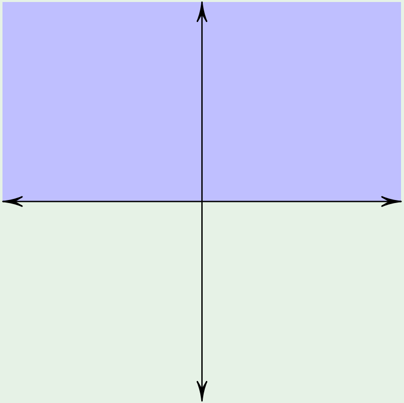
A vector space is a subspace of itself.

Subspaces of $\mathbb{R}^2$

Example

Is the upper half plane a subspace of $\mathbb{R}^2$?

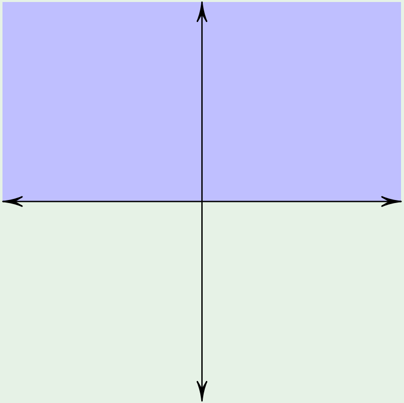


Subspaces of $\mathbb{R}^2$

Example

Is the upper half plane a subspace of $\mathbb{R}^2$?

No, points in the upper half plane are not closed under scalar multiplication.



Subspaces of $\mathbb{R}^2$

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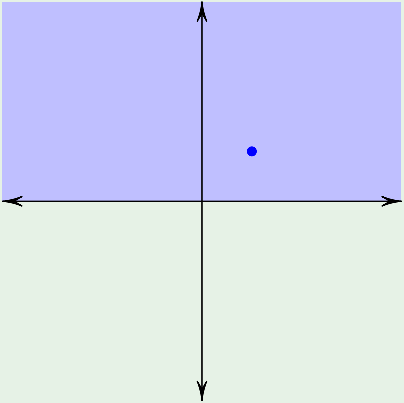
Consider $(1, 1)$.



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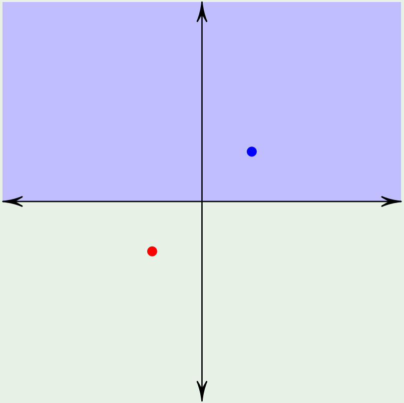
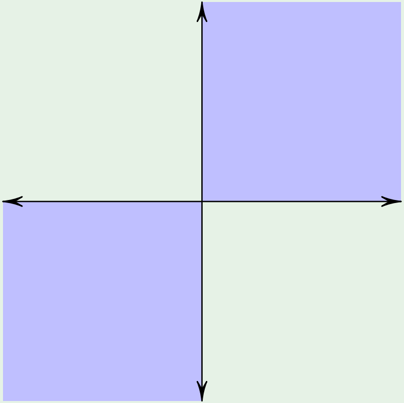
Multiplying by the scalar -1 gives $(-1 \cdot 1, -1 \cdot 1) = (-1, -1)$, a point in Q3.



Example

Is the set containing Q1 and Q3 a subspace of $\mathbb{R}^2$?

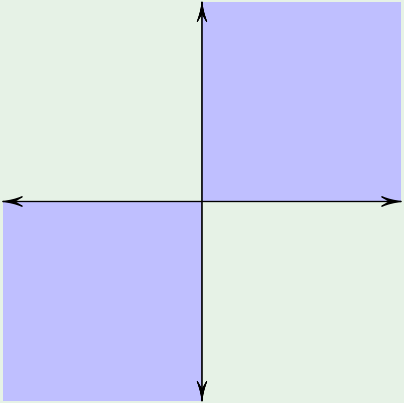


Subspaces of $\mathbb{R}^2$

Example

Is the set containing Q1 and Q3 a subspace of $\mathbb{R}^2$?

No, points in the set containing Q1 and Q3 are not closed under addition.



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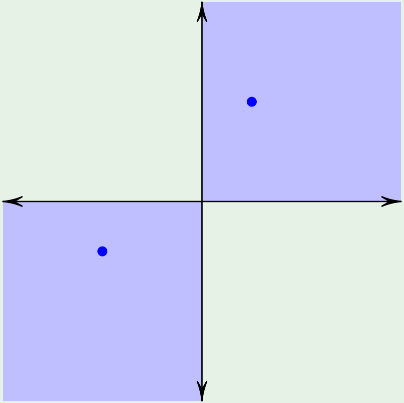
Consider $(1, 2)$ and $(-2, -1)$.



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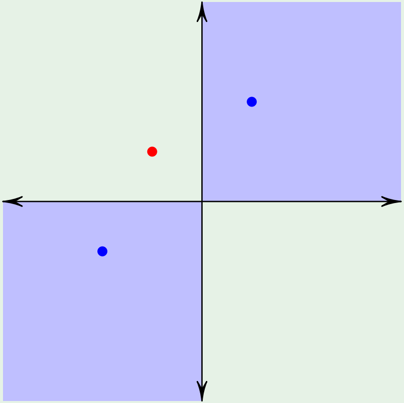
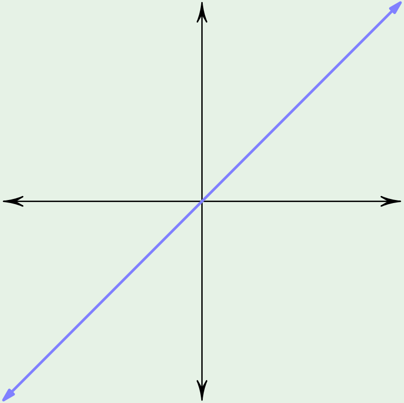
Adding these points gives $(1 + (-2), 2 + (-1)) = (-1, 1)$, a point in Q2.



Subspaces of $\mathbb{R}^2$

Example

Is the line $y = x$ a subspace of $\mathbb{R}^2$?

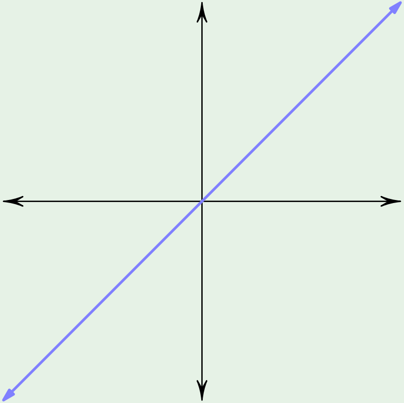


Example

Is the line $y = x$ a subspace of $\mathbb{R}^2$?

Yes. Given (s, s) and (t, t), two points on the line, then

$a \cdot (s, s) + b \cdot (t, t) =$



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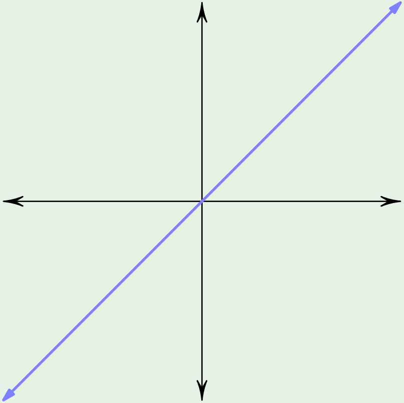

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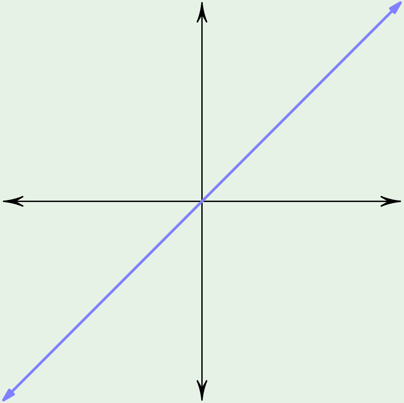

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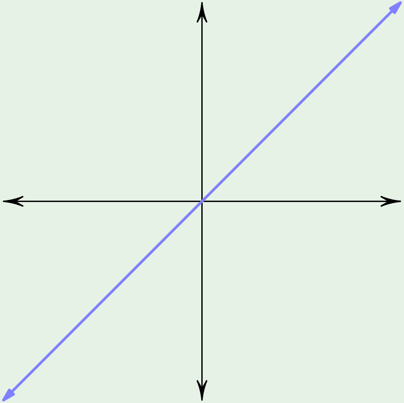
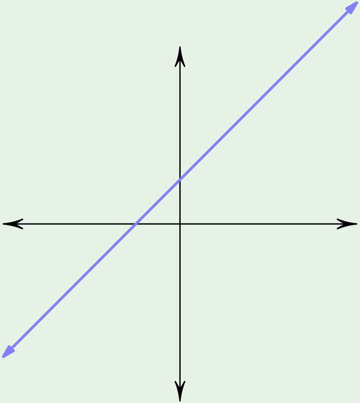
which is a point on the line.



Subspaces of $\mathbb{R}^2$

Example

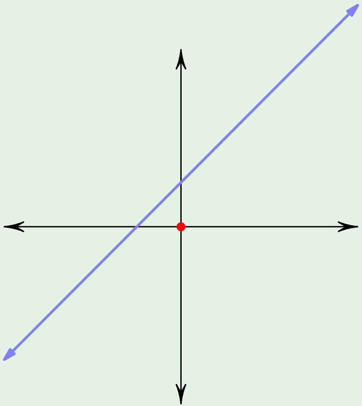
Is the line $y = x + 1$ a subspace of $\mathbb{R}^2$?



Example

Is the line $y = x + 1$ a subspace of $\mathbb{R}^2$?

No, the zero vector, $(0, 0)$ is not on the line.



Subspaces of $\mathbb{R}^2$

Corollary

The only subspaces of $\mathbb{R}^2$ *are*

- *The zero subspace* $(0, 0)$
- *Any line passing through the origin*
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We call a subspace of $\mathbb{V}$ **trivial** if it is the subspace containing just the zero vector, or $\mathbb{V}$ itself. All other subspaces are called **nontrivial**.

Theorem

The set of solutions of the linear system $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ is a subspace of $\mathbb{R}^m$, where A is a $m \times n$ matrix and $\vec{\boldsymbol{x}} \in \mathbb{R}^m$, is a subspace of $\mathbb{R}^m$.

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Proof

Closure is given by the Superposition Principle from section 2.1.

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Proof

Closure is given by the Superposition Principle from section 2.1.
Since solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ are vectors in $\mathbb{R}^m$, the remaining properties are inherited from $\mathbb{R}^m$