Solving DEs and IVPs with Laplace Transforms

Department of Mathematics

Salt Lake Community College

(Slides by Adam Wilson)

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Derivative Theorem for Laplace Transforms

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$, $f^{(n)}$ is piecewise continuous on $[0, \infty)$, and $f, f', \ldots, f^{(n)}$ are of exponential order α, then for $s > a$, and $n = 1, 2, \ldots$

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$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$
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Strategy to Solve DEs with Laplace Transforms

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1. Using the Laplace transform, transform the IVP with unknown function $y(t)$ into an algebraic problem with unknown function $Y(s)$.
2. Solve the algebraic problem for $Y(s)$.
3. Manipulating $Y(s)$ algebraically if necessary, use the inverse Laplace transform to transform $Y(s)$ into the IVP solution $y(t)$.

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Next, we need to calculate the Laplace transforms of y'' and y'.

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2Y(s) - 2s + 10$$
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$$\begin{aligned} 0 &= \left(s^2 Y(s) - 2s + 10\right) - 2\left(sY(s) - 2\right) - 3Y(s) \\ Y(s) &= \frac{2s - 14}{s^2 - 2s - 3} \end{aligned}$$

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Which means

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Y(s) &= \frac{s^2+2}{(s^2+1)(s^2+4)}
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$$\begin{aligned} s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} &= \mathcal{L}\{\sin(t)\} \\ s^2Y(s) - 1 + 4Y(s) &= \frac{1}{s^2+1} \\ (s^2+1)Y(s) &= \frac{s^2+2}{s^2+1} \\ Y(s) &= \frac{s^2+2}{(s^2+1)(s^2+4)} \\ &= \frac{\frac{1}{3}}{s^2+1} + \frac{\frac{2}{3}}{s^2+4} \end{aligned}$$

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Thus, the solution is

$$y(t) = \frac{1}{3}\sin(t) + \frac{1}{3}\sin(2t)$$

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$$y''' + y' = e^t \quad \text{where} \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 0$$

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$$y(t) = -1 + \frac{1}{2}e^t + \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t)$$

Translation Property for Multiplication by e^{at}

If the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad \text{for } s > a + \alpha$$

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&= e^{-t}\left(3\cos(2t) - 2\sin(2t)\right)
\end{aligned}$$

Multiplication by t^n Rule for the Laplace Transform

If $f(t)$ is a piecewise continuos function on $[0, \infty)$ and is of exponential order α, then for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s) \quad \text{where} \quad n \in \mathbb{N}^+$$

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We will prove the result for $n = 1$.

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This process can be repeated for an arbitrary n.

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