Basis and Dimension

Colby Community College

Definition

For a vector space $\mathbb{V}$, a **linear combination** of vectors is:

$$c_1 \vec{\boldsymbol{v_1}} + c_2 \vec{\boldsymbol{v_2}} + \cdots + c_n \vec{\boldsymbol{v_k}}$$

where $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{v_i}} \in \mathbb{V}$

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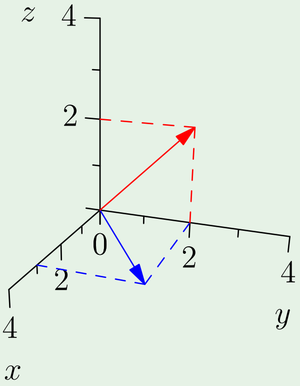
Note

If the **span** $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\} = \mathbb{V}$ we say the set spans the vector space.

Example 1

Consider **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.



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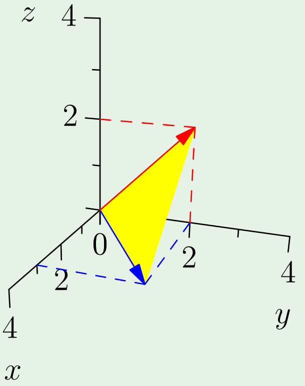

This spanning set is the plane defined by these two vectors.

Example 2

Let up look closer at this spanning set. Where we give names to the two vectors:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

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$$\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

We can then write a general vector in the spanning set as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{v}}$$

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Example 2

We can write the vector equation as the system:

$$
\begin{aligned}
x &= 3a && \Rightarrow a = \frac{x}{3} \\
y &= 2a + 2b \\
z &= 2b && \Rightarrow b = \frac{z}{2}
\end{aligned}
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Which is equivalent to $2x - 3y + 3z = 0$, the equation of the yellow plane.

Theorem 3

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Example 4

Consider adding $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ to **span** $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$.

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Since we can write

$$-1 \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

we see that this doesn't change to the spanning set.

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To show this, let us try to find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

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Which is equivalent to solving the inconsistent system

$$\begin{aligned} 1 &= 3c_1 \\ 1 &= 2c_1 + 2c_2 \\ 1 &= 2c_2 \end{aligned}$$

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What is **span** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$?

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To show this, we then need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Which has a unique solution for any $x, y, z \in \mathbb{R}$.

Theorem 7

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{R}^n$, a vector $\vec{b} \in \mathbb{R}^n$ is in **span** $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ *if and only if there is at least one solution to the matrix equation $A\vec{x} = \vec{b}$. Where A is formed from the column vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$.*

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Note

We can write spanning sets using set builder notation.

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$$\textbf{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\} = \left\{c\begin{bmatrix}2\\1\end{bmatrix} \;\middle|\; c \in \mathbb{R}\right\}$$

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$$\textbf{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \left\{c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix}\;\middle|\; c_1, c_2, c_3 \in \mathbb{R}\right\}$$

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Theorem 10

For $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{V}$, $\textbf{span}\left\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\right\}$ *is a subspace of* $\mathbb{V}$.

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Let $\vec{u}$ and $\vec{w}$ be two vectors in the spanning set, which means there are scalars r_i and s_i such that

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So, for any $a, b \in \mathbb{R}$:

$$a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{w}}$$

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So, for any $a, b \in \mathbb{R}$:

$$a\vec{u} + b\vec{w} = a(r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_n\vec{v_k}) + b(s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_k})$$

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Which means $a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{w}}$ is in the spanning set and we have closure.

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Example 11

Consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

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$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 2 & 4 & 1 & 1 & 5 \end{bmatrix}$$

The column space of $\boldsymbol{B}$ is a subspace of $\mathbb{R}^2$ and defined:

$$\textbf{Col } \boldsymbol{B} = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 5 \end{bmatrix} \;\middle|\; c_1, \ldots, c_5 \in \mathbb{R} \right\}$$

Definition

A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of vectors in a vector space $\mathbb{V}$ is **linearly independent** if no vector of the set can be written as a linear combination of the others. Otherwise it is **linearly dependent**.

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Testing for Linear Independence

To test for linear independence of a set of k vectors $\vec{v_i} \in \mathbb{R}^n$, we consider the system:

$$\begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_k} \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$$

The column vectors of A are linearly independent if and only if the solution $c_1 = c_2 = \cdots = c_k = 0$ is unique.

Example 12

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\3\\2\end{bmatrix}$ linearly independent?

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To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 3\\1 & -1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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Since $|\boldsymbol{A}| = 5$, we know that $\boldsymbol{A}$ is invertible and hence a unique solution exists. This means that these vectors are linearly independent.

Example 13

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\3\\2\end{bmatrix}$, and $\begin{bmatrix}5\\-1\\0\end{bmatrix}$ linearly independent?

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$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 5\\1 & 1 & 3 & -1\\1 & -1 & 2 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\3\\2\end{bmatrix}$, and $\begin{bmatrix}5\\-1\\0\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 5\\1 & 1 & 3 & -1\\1 & -1 & 2 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

We have more columns than rows, which means there will be at least one free variable. Thus, the solution (if one exists) won't be unique, so these vectors are not linearly independent.

Example 14

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ linearly independent?

Example 14

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\boldsymbol{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & -1 & -2\\1 & 0 & 1\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent?

To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

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To determine if they are, we need to look at the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And thus, these vectors are not linearly independent.

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$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And thus, these vectors are not linearly independent.

Moreover, since

$$\begin{bmatrix}1\\1\\1\end{bmatrix} + 3\begin{bmatrix}-1\\0\\1\end{bmatrix} - \begin{bmatrix}-2\\1\\4\end{bmatrix} = 0$$

we can see that any vector can be written as a combination of the others.

Definition

A set of vector functions $\{\vec{v_1}(t), \vec{v_2}(t), \ldots, \vec{v_k}\}$ in a vector space $\mathbb{V}$ is **linearly independent** on an interval I if, for *all* $t \in I$, the equation

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + \cdots + c_k\vec{v_k}(t) = \vec{0} \quad (\text{where } c_i \in \mathbb{R})$$

has the only solution: $c_1 = c_2 = \cdots = c_k = 0$.

If for any value $t_0 \in I$ there is any solution with $c_i \neq 0$, the vector functions $\vec{v_1}(t), \vec{v_2}(t), \ldots, \vec{v_k}(t)$ are **linearly dependent**.

Example 15

Are the vectors

$$\vec{v_1}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \quad \vec{v_2}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \quad \vec{v_3}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

linearly independent on $(-\infty, \infty)$?

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We need to see what the solution, for $c_1, c_2, c_3 \in \mathbb{R}$, is:

$$c_1 \vec{\boldsymbol{v_1}}(t) + c_2 \vec{\boldsymbol{v_2}}(t) + c_3 \vec{\boldsymbol{v_3}}(t) = \vec{\boldsymbol{0}}$$

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Since this equation has to hold for all t, it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} e^{(0)} \\ 0 \\ 2e^{(0)} \end{bmatrix} + c_2 \begin{bmatrix} e^{-(0)} \\ 3e^{-(0)} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{2(0)} \\ e^{2(0)} \\ e^{2(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Since this equation has to hold for all t, it has to hold for $t = 0$:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Since the unique solution is $c_1 = c_2 = c_3 = 0$, these vectors are linearly independent.

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Are the following functions linearly independent?

$$\vec{v_1}(t) = e^t, \quad \vec{v_2}(t) = 5e^{-t}, \quad \vec{v_3}(t) = e^{3t}$$

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We can think of each of these as a one-dimensional vector.

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Are the following functions linearly independent?

$$\vec{v_1}(t) = \left[e^{t}\right], \quad \vec{v_2}(t) = \left[5e^{-t}\right], \quad \vec{v_3}(t) = \left[e^{3t}\right]$$

We can think of each of these as a one-dimensional vector.
Which means we have to see if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad (\text{for all } t \in \mathbb{R})$$

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For $t = \quad 0$: $\qquad c_1 \cdot 5e^{(0)} + \quad c_2 \cdot e^{-(0)} + \quad c_3 \cdot e^{3(0)} = 0$

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$$\begin{array}{llll} \text{For } t = \ 0: & c_1 \cdot 5e^{(0)} + & c_2 \cdot e^{-(0)} + & c_3 \cdot e^{3(0)} = 0 \\ \text{For } t = \ 1: & c_1 \cdot 5e^{(1)} + & c_2 \cdot e^{-(1)} + & c_3 \cdot e^{3(1)} = 0 \end{array}$$

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$$\begin{aligned}
\text{For } t = 0\text{:} \quad & c_1 \cdot 5e^{(0)} + c_2 \cdot e^{-(0)} + c_3 \cdot e^{3(0)} = 0 \\
\text{For } t = 1\text{:} \quad & c_1 \cdot 5e^{(1)} + c_2 \cdot e^{-(1)} + c_3 \cdot e^{3(1)} = 0 \\
\text{For } t = -1\text{:} \quad & c_1 \cdot 5e^{(-1)} + c_2 \cdot e^{-(-1)} + c_3 \cdot e^{3(-1)} = 0
\end{aligned}$$

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$$\begin{array}{lrrr}
\text{For } t = 0: & c_1 + & 5c_2 + & c_3 = 0 \\
\text{For } t = 1: & ec_1 + & \frac{5}{e}c_2 + & e^3c_3 = 0 \\
\text{For } t = -1: & \frac{1}{e}c_1 + & ec_2 + & \frac{1}{e^3}c_3 = 0
\end{array}$$

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$$c_1\vec{v_1}(t) + c_2\vec{v_2}(t) + c_3\vec{v_3}(t) = \vec{0} \quad \text{(for all } t \in \mathbb{R})$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ e & \frac{5}{e} & e^3 & 0 \\ \frac{1}{e} & e & \frac{1}{e^3} & 0 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since we have a unique solution, $c_1 = c_2 = c_3 = 0$, these functions are linearly independent.

Definition

The **Wronskian** of functions $f_1, f_2, \ldots, f_k$ on interval I is the determinant:

$$W[f_1, f_2, \ldots, f_k](t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{vmatrix}$$

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Theorem 17

If $W[f_1, f_2, \ldots, f_k](t) \neq 0$ for all $t \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is a linearly independent set of functions on I.

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Theorem 17

If $W[f_1, f_2, \ldots, f_k](t) \neq 0$ for all $t \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is a linearly independent set of functions on I.

Note

If $\{f_1, f_2, \ldots, f_k\}$ are linearly dependent, then $W[f_1, f_2, \ldots, f_k](t) = 0$ for all $t \in I$. Thus, to show independence we only need to find a single t that makes the Wronskian nonzero.

Example 18

Use the Wronskian to check that

$$\left\{t^2+1, t^2-1, 2t+5\right\}$$

are linearly independent on $\mathbb{P}_2$.

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$$W(t) = \begin{vmatrix} t^2 + 1 & t^2 - 1 & 2t + 5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

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$$\begin{aligned} W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \end{aligned}$$

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&= (t^2+1)(0-4) - (t^2-1)(0-4) + (2t+5)(4t-4t) \\
&= -4t^2-4+4t^2-4
\end{aligned}$$

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Use the Wronskian to check that

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are linearly independent on $\mathbb{P}_2$.

$$\begin{aligned} W(t) &= \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} \\ &= (t^2+1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} - (t^2-1)\begin{vmatrix} 2t & 2 \\ 2 & 0 \end{vmatrix} + (2t+5)\begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix} \\ &= (t^2+1)(0-4) - (t^2-1)(0-4) + (2t+5)(4t-4t) \\ &= -4t^2 - 4 + 4t^2 - 4 = -8 \end{aligned}$$

Since $W(t) = -8 \neq 0$, this is a set of linearly independent functions.

Example 19

Let us consider the converse:

Does the Wronskian being zero imply dependence?

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In general, the answer is no.

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In general, the answer is no.

Consider the linearly independent functions:

$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

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Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

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Does the Wronskian being zero imply dependence?

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$$f_1(t) = \begin{cases} t^3, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^3, & t < 0 \end{cases}$$

Then

$$f_1'(t) = \begin{cases} 3t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{and} \quad f_2'(t) = \begin{cases} 0, & t \geq 0 \\ 3t^2, & t < 0 \end{cases}$$

So,

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

Definition

The set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a **basis** for vector space $\mathbb{V}$, provided that

- $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a linearly independent set
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Example 20

The vectors

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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We saw earlier that these vectors span $\mathbb{R}^3$.

It's easy to see that $c_1\vec{i} + c_2\vec{j} + c_3\vec{k} = \vec{0}$ has the unique solution $c_1 = c_2 = c_3 = 0$.

Definition

The **standard basis** for $\mathbb{R}^n$ is $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$ where

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix},\ \cdots,\ \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are the column vectors of the identity matrix $\boldsymbol{I}_n$.

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$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 2a + 3b - 4c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

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Since $a, b, c \in \mathbb{R}$ were arbitrary, we see these three vectors span the hyperplane.

Example 21

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Now, we need to show that the vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

Example 21

Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Which means, for $c_1, c_2, c_3 \in \mathbb{R}$, solving the equation:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The (unique) solution is $c_1 = c_2 = c_3 = 0$, thus these vectors are linearly independent.

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Let us find a basis for the hyperplane in $\mathbb{R}^4$ that is the solution to

$$2x_1 + 3x_2 - 4x_3 - x_4 = 0$$

So, we see that the hyperplane has a basis of three vectors.

It looks like this hyperplane is a three-dimensional subspace of a four-dimensional space.

Example 22

It is possible for a vector space to have multiple bases.

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For $\mathbb{R}^2$, one is the standard basis

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For $\mathbb{R}^2$, one is the standard basis

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but, another basis is given by

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Theorem 23

The number of vectors in a basis is always the same for a particular vector space

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Proof

The proof is in Appendix LT of your textbook, on page 602.

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The **dimension** of a vector space $\mathbb{V}$ is the number of vectors in any basis of $\mathbb{V}$.

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The proof is in Appendix LT of your textbook, on page 602.

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The **dimension** of a vector space $\mathbb{V}$ is the number of vectors in any basis of $\mathbb{V}$.

Definition

If a vector space is so large that cannot be spanned by a finite set of vectors, it is called **infinite-dimensional**.

Example 24

The solution to the system

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \end{array}$$

is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

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Writing this system in RREF gives

$$\begin{array}{rcrcrcrcl} x_1 & & & - & 5x_3 & - & x_4 & = & 0 \\ & & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}$$

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The two free variables tell us that the solution to this system will be a two-parameter family.

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5a+b \\ -2a-b \\ a \\ b \end{bmatrix}$$

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is a subspace of $\mathbb{R}^4$. (The intersection of two 3D hyperplanes.)

What is its dimension?

The two vectors

$$\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

span the subspace.

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They are also linearly independent, thus this a basis of the subspace.

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Which means the dimension is 2.

Properties of the Column Space of a Matrix

1. The pivot columns of a matrix $\boldsymbol{A}$ form a basis for Col $\boldsymbol{A}$.
   - A pivot column is a column of $\boldsymbol{A}$ that corresponds to a column in the RREF of $\boldsymbol{A}$ with a leading 1.

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Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 & 8 \end{bmatrix}$$

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The pivot columns are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which means $\textbf{rank}(\boldsymbol{A}) = 2$ and thus the dimension of the column space is 2.

Invertible Matrix Characterization

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent:

- $\boldsymbol{A}$ is invertible.
- The column vectors of $\boldsymbol{A}$ are linearly independent.
- Every column of $\boldsymbol{A}$ is a pivot column.
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$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

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So, since we have the unique solution $c_1 = c_2 = c_3 = 0$, these functions are linearly independent and thus form a basis of $\mathbb{P}_2$.

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Which means $\mathbf{dim}\, \mathbb{P}_2 = 3$.

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Note

There are many infinite-dimensional spaces.

We have seen $\mathbb{P}$, $\mathcal{C}(I)$, and $\mathcal{C}^n(I)$.