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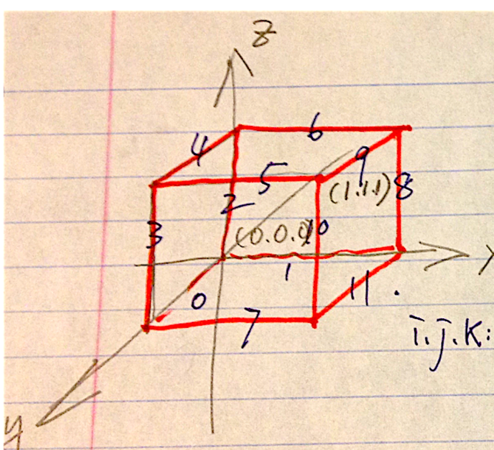
Extra Problem: Antithetic variables

> Study the reliability of a network.
>
> Determine the reliability of a large network with irregular topology is hard to do analytically, so we may explore it by simulation. As a simple example, consider an 8 node cube with vertices labelled {000, 001, 010,….,111}. We assume that the probability that any particular link j is working at some point in time is p_j and that the nodes never fail; for simplicity assume that $p_j = p\ \forall j$. We want to determine the probability that there is a path between a pair of nodes (source to destination). Due to its regular structure of this simple case it is possible to determine the probability that there is a path from node $0=000_B$ to node $8=111_B$ analytically. We can also do this by simulation. The network in question has 12 links. Define a set of Bernoulli Random Variables $\{U_i\}$ to represent whether link i is operational. Consider a path that traverses links i,j, and k and define $Z_{ijk} = U_iU_jU_k$. If the path is operational then $Z_{ijk} = 1$. If we identify all possible paths between the pair of nodes we can rapidly check whether there is a path that is operating. Repeating this for several samples for $\{U_i\}$ we can estimate the probability that there is a path between the pair of nodes 000 and 111.
>
> A. (10 points) Use simulation to estimate the probability that there is a path from 000 to 111 as a function of p. By making several runs estimate the variance of this probability estimator.
>
> B. (10 points) For a particular value of p, repeat the assignment using the method of antithetic variables to see if the variance is reduced. Suppose you made 20 runs in part A, now use only 10 runs. For each run generate a set of Bernoulli Random Variables $\{U_i\}$ and another set using $\{U_i' = 1 - U_i\}$ and use the two sets to generate 2 estimates for the probability that the path exists.

To simulate the network connectivity, I develop the following model with the 8 node cube inside a 3D coordinate system. The source is the node with coordinate (0,0,0) and destination is the node with coordinate(1,1,1). Only the paths with 3 links between them are taken into account so there are 6 paths, namely, 6 combinations of links. All the links are numbered from 1 to 12 although they are numbered from 0 to 11 in the following model illustration.

In the Matlab simulation, probability mean and variance are generated for p = 0:0.01:1. For each p, 20 runs of simulations are done to generate the connectivity probability. The mean and variance of these 20 results are used as probability mean and variance.
For each run, 100 sets of Bernoulli Random Variables {Ui} i = 1 to 12 are generated and checked whether (0,0,0) and (1,1,1) are connected. The total number of connectivities is divided by 100 to give the probability in this run.



(0.0.0) : src → (1.1.1) : dest.
through 3 links i, j, k

with link # in graph. the 3 links combination linking (0.0.0) → (1.1.1). are
i.j.k: 0.7.10; 0.3.5; 2.4.5; 2.6.9; 1.11.10; 1.8.9.

∴ In simulation, each run {Ui} i = 0...11 are generated. and 6 combinations are checked. If any one of them equals 1. There is link between (0,0,0). (1.1.1).

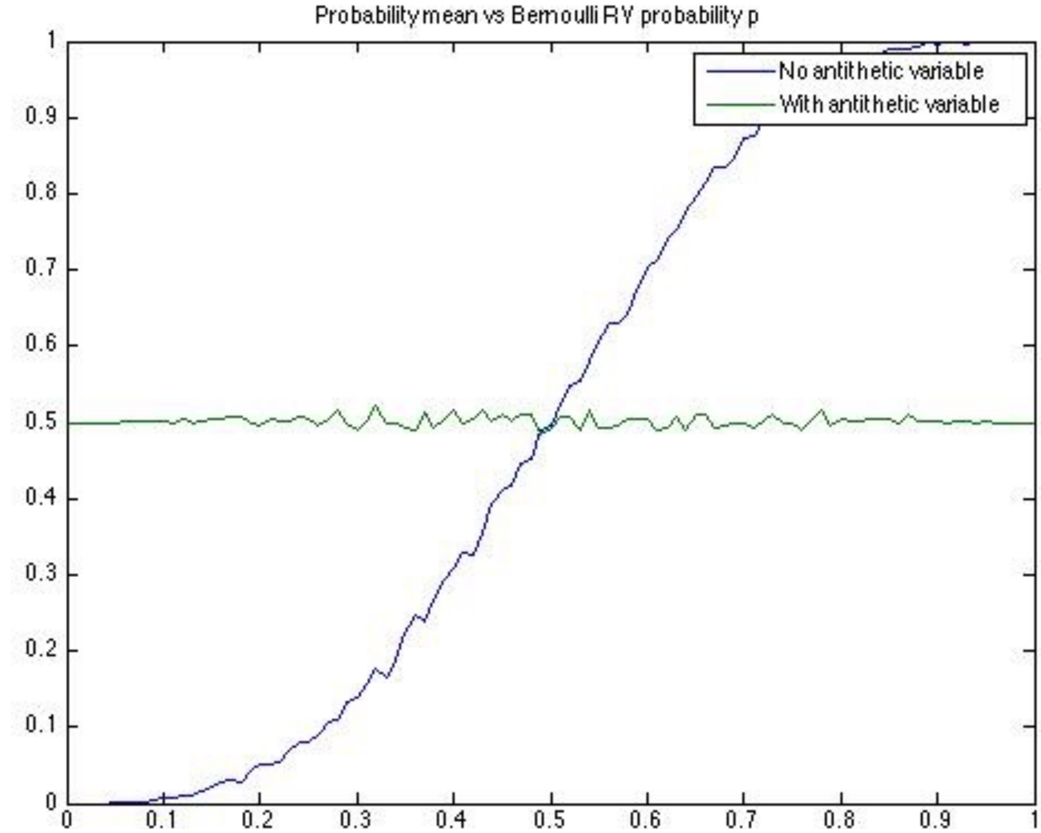
Here are the simulation codes:

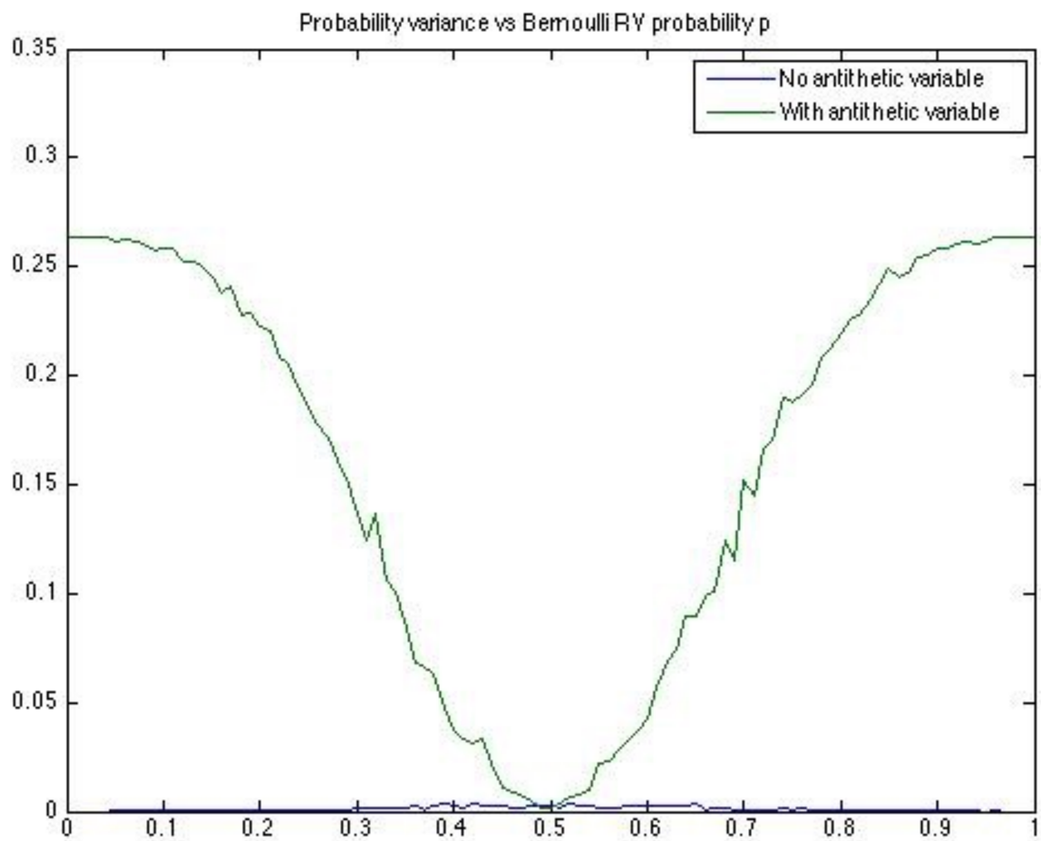
```
line = 12; % 12 lines
p = 0:0.01:1; % probability list to be test
run = 20; % run per p
r = 100; % repeat per run
l = length(p);
list = [1,8,11;
    1,4,6;
    3,5,6;
    3,7,10;
    2,12,11;
    2,9,10];

m = zeros(1, l); % resulting list of mean
v = zeros(1, l); % resulting list of variance

for i = 1:l
  result = zeros(1, run); % result list of simulated p
  for j = 1:run
    connect = 0; % times of connectivity
    for k = 1:r
      b = binornd(1, p(i), 1, line);
      for n = 1:length(list)
        if (b(list(n,1)) * b(list(n,2)) * b(list(n,3)) == 1)
          connect = connect + 1;
          break
        end
      end
    end
    result(j) = connect / r;
  end
  m(i) = mean(result);
  v(i) = var(result);
end

ma = zeros(1, l); % resulting list of mean with antithetic variables
va = zeros(1, l); % resulting list of variance with antithetic variables

for i = 1:l
  result = zeros(1, run); % result list of simulated p
  for j = 1:(run/2)
    connect = 0; % times of connectivity
    aconnect = 0; % times of connectivity with antithetic variable
    for k = 1:r
      b = binornd(1, p(i), 1, line);
      for n = 1:length(list)
        if (b(list(n,1)) * b(list(n,2)) * b(list(n,3)) == 1)
```

```
                connect = connect + 1;
                break
            end
        end
        ba = ones(size(b)) - b; % generate antithetic variable
        for n = 1:length(list)
            if (ba(list(n,1)) * ba(list(n,2)) * ba(list(n,3)) == 1)
                aconnect = aconnect + 1;
                break
            end
        end
    end
    result(j) = connect / r;
    result(j+run/2) = aconnect / r;
  end

  ma(i) = mean(result);
  va(i) = var(result);
end

figure;
plot(p, m); hold all;
plot(p, ma); hold off;
title('Probability mean vs Bernoulli RV probability p');
legend('No antithetic variable', 'With antithetic variable');
figure;
plot(p, v); hold all;
plot(p, va); hold off;
title('Probability variance vs Bernoulli RV probability p');
legend('No antithetic variable', 'With antithetic variable');
```

The results are in next page.
It is reasonable to have a horizontal line at 0.5 for mean with antithetic variable. For a set of {Ui}, corresponding set is {1-Ui}. So if p = 0, all Ui = 0 and 1-Ui = 1. This means half of the connectivity probabilities are 1 and the other half is 0. So the mean is 1/2. This applies to all p similarly. So it is also reasonable to have variance like below with antithetic variable. With larger p, the difference between half, half of the probabilities from the 20 runs will be larger thus larger variance. It is symmetric to generate Ui and 1-Ui along 0 to 1, so the plots of mean and variance are also symmetric along the line p = 0.5.

Normally the variance with antithetic variable is larger than the one without because the explanation above. However when p = 0.5, the mean are the same with or without antithetic variable so it is comparable for two variances. In this case, variance with antithetic variable is smaller than the one without.


Probability mean vs Bernoulli RV probability p
No antithetic variable
With antithetic variable


Probability variance vs Bernoulli RV probability p
No antithetic variable
With antithetic variable