Stat 306. Mathmetical result about $\mathbf{X}^T\mathbf{X}$.

$\mathbf{X}^T\mathbf{X}$ is invertible $\iff$ $\mathbf{X}$ has full column rank (p 3-3) ($\iff$ the columns of $\mathbf{X}$ are linearly independent)

$\mathbf{X}^T\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{y}$ is equation that least squares estimate $\widehat{\boldsymbol{\beta}}$ satisfies.

If $\mathbf{X}^T\mathbf{X}$ is non-singular, solution $\widehat{\boldsymbol{\beta}}$ is unique.
If $\mathbf{X}^T\mathbf{X}$ is singular, solution $\widehat{\boldsymbol{\beta}}$ is non-unique (there exists $\widehat{\boldsymbol{\beta}}$ from the geometry of least squares).

If $n < k =$ ncol($\mathbf{X}$), $\mathbf{X}^T\mathbf{X}$ is singular.
In practice, $n =$ #rows $\geq k$, preferably $n \gg k$.

To understand these results, regress on two variables where $x_{i2} = 3x_{i1}$ or regress on three variables where $x_{i3} = 2x_{i1} + x_{i2}$.

To understand the effect of near multicollinearity (or $\mathbf{X}^T\mathbf{X}$ is nearly singular) on the SEs of the slopes $\widehat{\beta}_j$, regress on two variables where $x_{i2} = 3x_{i1}+$noise, where the variability of the noise is small but positive.