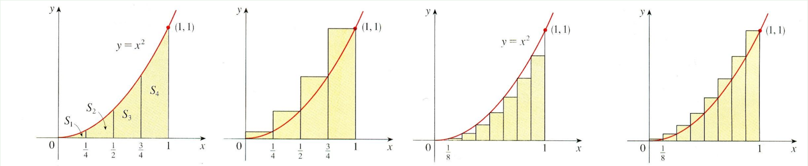
Example

Calculate the area under the graph $f(x) = x^2$ over the interval $0 \leq x \leq 1$.



Velocity

Suppose that a particle travels a distance $d(t)$ at time t. Then

$$\textbf{Average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\Delta d}{\Delta t}$$

$$\textbf{Instantaneous velocity} = \lim_{h \to 0} \frac{\Delta d}{h} = \lim_{h \to 0} \frac{d(t+h) - d(t)}{h}$$

Limits

A number **L** is a limit of a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$ if the numbers a_n approach the number **L** as $n \to \infty$. In this case, we write:

$$\lim_{n\to\infty} a_n = \mathbf{L}.$$

Example

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Example

Let $f(x) = x^2$. Then, since $f(x)$ is continuous,

$$\lim_{x\to 3} f(x) = f(3) = 3^2 = 9.$$

Sum of a series - Zeno's paradox

Consider the sequence:

$$a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Then

$$\lim_{n \to \infty} a_n = 1.$$

Definition of Derivative

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function $f(x)$. Then the **derivative** $f'(x)$ is given by the following formula if the limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example

$f(x) = x^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left[\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{2hx + h^2}{h} = 2x + h \right] = 2x$$

Exponential Rule

For $f(x) = e^x$, $f'(x) = e^x$

Example

If $f(x) = e^x - x^2$, find $f'(x)$.

$$f'(x) = (e^x - x^2)' = (e^x)' - (x^2)' = e^x - 2x$$

Product and quotient rules

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Example

Compute $f'(x)$ for $f(x) = \frac{e^x}{1+x^2}$.

Solution.

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}.$$

$\square$

Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Example

Find $F'(x)$ if $F(x) = e^{x^2+1}$.

Solution:

- Let $f(x) = e^x$ and $g(x) = x^2 + 1$.
- Then $F(x) = f \circ g(x)$.
- So,

$$F'(x) = f'(g(x)) \cdot g'(x) = e^{x^2+1}(2x)$$

$\square$

Derivatives of classical functions

- $\sin'(x) = \cos(x)$
- $\cos'(x) = -\sin(x)$
- $(e^x)' = e^x$
- $\ln'(x) = \frac{1}{x}$, where $\ln(x)$ is the natural log.
- $\log_a'(x) = \frac{1}{x\ln(a)}$, where $\log_a(x)$ is the log in base a.
- $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$

Antiderivatives

Definition

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example

Let $f(x) = x^2$. Then an antiderivative $F(x)$ for x^2 is $F(x) = \frac{x^3}{3}$.

Theorem

If F is an antiderivative of f on an interval $I = (a, b)$, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Table of Anti-differentiation Formulas

Function	**Particular antiderivative**
$c \cdot f(x)$	$c \cdot F(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert$
e^x	e^x
$\cos x$	$\sin x$

Table of Anti-differentiation Formulas

Function	**Particular antiderivative**
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

Example

A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

Solution:

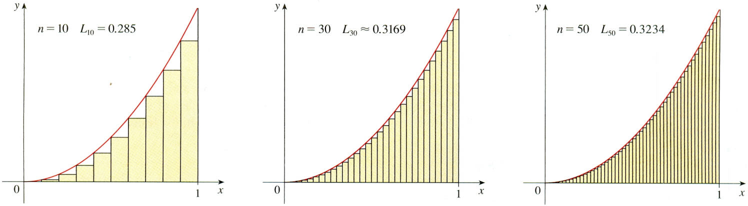
- At time t the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing.
- Therefore, the acceleration must be negative:
$$a(t) = \frac{dv}{dt} = -32$$
- Taking antiderivatives
$$v(t) = -32t + C$$
- To determine C, use that $v(0) = 48$.
- This gives $48 = 0 + C$, so
$$v(t) = -32t + 48$$
- The maximum height is reached when $v(t) = 0$, that is after 1.5 s.
- Since $s'(t) = v(t)$, we anti-differentiate and obtain
$$s(t) = -16t^2 + 48t + D.$$

Continuation of Solution:

- Using that $s(0) = 432$, we have $432 = 0 + D$ and so,
$$s(t) = -16t^2 + 48t + 432$$
- The expression for $s(t)$ is valid until the ball hits the ground.
- This happens when $s(t) = 0$:
$$-16t^2 + 48t + 432 = 0$$
  or, equivalently,
$$t^2 - 3t - 27 = 0$$
$$t = \frac{3 \pm 3\sqrt{13}}{2}$$
- We reject solution with the minus sign since it gives a negative value for t.
- Therefore, the ball hits the ground after $3(1 + \sqrt{13})/2 \approx 6.9$ s.

$\square$

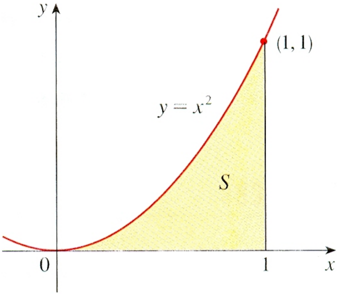
Area under a graph



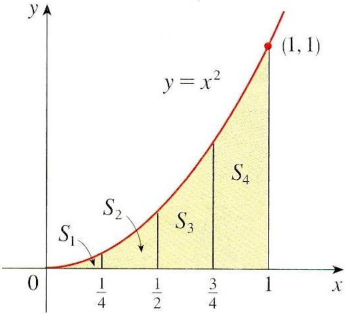
Area under $y = x^2$ from 0 to 1.

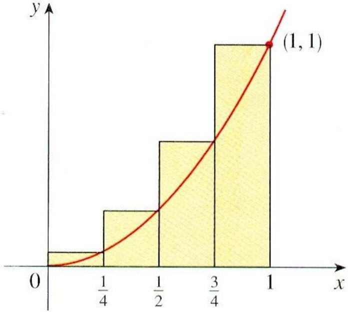
Example

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.



Area estimate using right end points

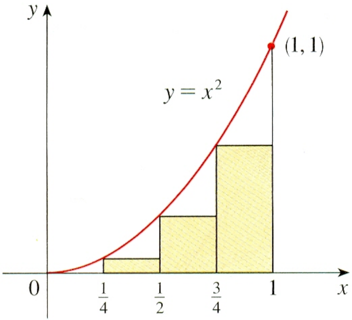




$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 = \frac{15}{32}$$

Note area $A < \frac{15}{32} = .46875$

Area estimate using left end points

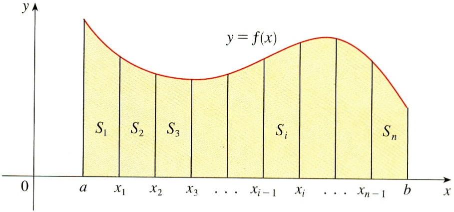


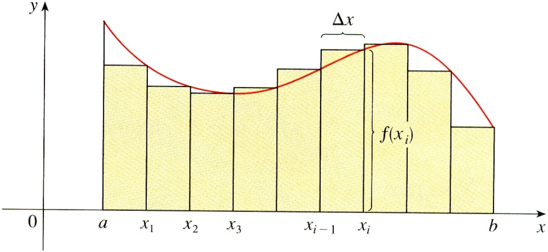
$$L_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{c}{1}4\right) + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} = .21875$$

Note area **A** satisfies

$$.21875 \leq \mathbf{A} \leq .46875$$

General calculation using right end points





Area definition using right end points

Definition

The **area** $\mathbf{A}$ of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$\begin{aligned} \mathbf{A} &= \lim_{n\to\infty} R_n \\ &= \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] \end{aligned}$$

Note in the above definition that if $I = [a, b]$, then

$$\Delta x = \frac{b-a}{n},$$

where n is the number of rectangles or divisions.

Sigma notation and $\mathbf{A}$ = Area

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$\begin{aligned} \mathbf{A} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \quad x_i^* \in [x_{i-1}, x_i], \end{aligned}$$

when f is continuous.

Definition of a Definite Integral

Definition

- If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$.
- We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1,x_i}]$.
- Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1} f(x_i^*) \Delta x$$

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x = \Delta x[f\bar{(}x_1) + \cdots + f(\overline{x_n})]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Properties of the Integral

- $\int_a^b c\,dx = c(b-a)$, where c is any constant
- $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b c \cdot f(x)\,dx = c\,\int_a^b f(x)\,dx$, where c is any constant
- $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

The Fundamental Theorem of Calculus, Part 1

Theorem (Fundamental Theorem of Calculus, Part 1)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = f(x).$$

The Fundamental Theorem of Calculus, Part 2

Theorem (Fundamental Theorem of Calculus, Part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, that is, a function such that

$$F'(x) = f(x).$$

Application of **FTC**

Example

Evaluate $\displaystyle\int_3^6 \frac{1}{x}\,dx.$

Solution:

- An antiderivative for $\frac{1}{x}$ is $F(x) = \ln x$.
- So, by **FTC**,

$$\int_3^6 \frac{1}{x}\,dx = F(6) - F(3) = \ln 6 - \ln 3.$$

$\square$

Example

Evaluate $\int_1^3 e^x\,dx$.

Solution:

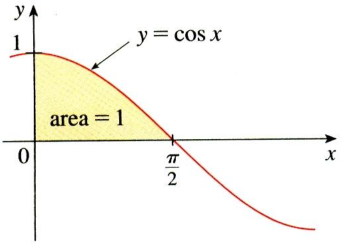
- Note that an antiderivative for e^x is $F(x) = e^x$.
- So, by **FTC**,

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e.$$

$\square$

Example

Find area $\mathbf{A}$ under the cosine curve from 0 to b, where $0 \le b \le \frac{\pi}{2}$.



Solution:

Since an antiderivative of $f(x) = \cos(x)$ is $F(x) = \sin(x)$, we have

$$\mathbf{A} = \int_0^b \cos(x)\,dx = \sin(x)\big|_0^b = \sin(b) - \sin(0) = \sin(b).$$

□

The Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Suppose f is continuous on $[a, b]$.

- If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
- $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$, that is, $F'(x) = f(x)$.

Notation: Indefinite integral

$$\boxed{\int f(x)\,dx = F(x) \quad \textbf{means} \quad F'(x) = f(x).}$$

We use the notation $\int f(x)\,dx$ to denote an antiderivative for $f(x)$ and it is called **an indefinite integral.**

A **definite integral** has the form:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b = F(b) - F(a)$$

Table of Indefinite Integrals

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \tfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \tfrac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

Table of Indefinite Integrals

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int \frac{1}{x^2+1} \, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

Example

Find $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\,dx$ and interpret the result in terms of areas.

Solution:

FTC gives

$$\begin{aligned}
&\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\,dx \\
&= 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1}x\Big|_0^2. \\
&= \frac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\Big|_0^2 \\
&= \frac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1}2 - 0 \\
&= -4 + 3\tan^{-1}2
\end{aligned}$$

□

The Net Change Theorem

Theorem (Net Change)

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second). Find the displacement of the particle during the time period $1 \le t \le 4$.

Solution:

By the Net Change Theorem, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)\, dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}.$$

This means that the particle moved 4.5 m toward the left. □

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second). Find the distance traveled during the time period $1 \leq t \leq 4$.

Solution:

- Note $v(t) = t^2 - t - 6 = (t-3)(t+2)$ and so $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$ on $[3, 4]$.
- From the Net Change Theorem, distance traveled is

$$\begin{aligned}\int_1^4 |v(t)|\,dt &= \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt \\ &= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt \\ &= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\ &= \frac{61}{6} \approx 10.17\mathrm{m}\end{aligned}$$

$\square$

The **Substitution Rule** is one of the main tools used in this class for finding antiderivatives. It comes from the Chain Rule:

$$[F(g(x))]' = F'(g(x))g'(x).$$

So,

$$\int F'(g(x))g'(x)\, dx = F(g(x)).$$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Find $\int x^3 \cos(x^4 + 2)\,dx$.

Solution:

- Make the substitution: $u = x^4 + 2$.
- Get $du = 4x^3\,dx$.

$$\int x^3 \cos(x^4 + 2)\,dx = \int \cos u \cdot \frac{1}{4}\,du = \frac{1}{4}\int \cos u\,du$$

$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4 + 2) + C.$$

Note at the final stage we return to the original variable x. □

Example

Evaluate $\int \sqrt{2x+1}\, dx$.

Proof.

Solution:

- Let $u = 2x + 1$.
- Then $du = 2\, dx$, so $dx = \frac{du}{2}$.
- The Substitution Rule gives

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\, \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}}\ du$$

$$= \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3} u^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

$\square$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Calculate $\int e^{5x}\,dx$.

Solution:

- If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$
- Therefore

$$\int e^{5x}\,dx = \frac{1}{5}\int e^{u}\,du = \frac{1}{5}e^{u} + C = \frac{1}{5}e^{5x} + C.$$

$\square$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Calculate $\int \tan x\,dx$.

Solution:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

This suggests substitution $u = \cos x$, since then $du = -\sin x\,dx$ and so, $\sin x\,dx = -du$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}$$

$$= -\ln|u| + C = -\ln|\cos x| + C.$$

$\square$

The Substitution Rule for Definite Integrals

Theorem (Substitution Rule for Definite Integrals)

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

The **Substitution Rule for Definite Integrals:**
If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example

Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

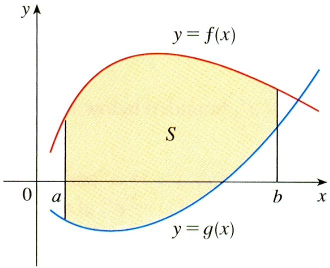
Solution:

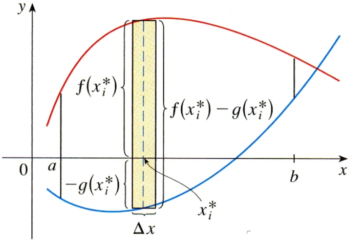
- We let $u = \ln x$ because its differential $du = \frac{dx}{x}$ occurs in the integral.
- When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$.
- Thus

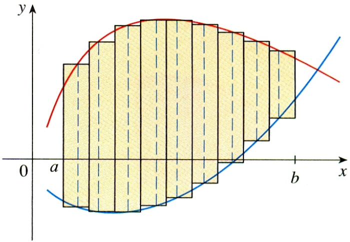
$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}.$$

$\square$

Areas between curves

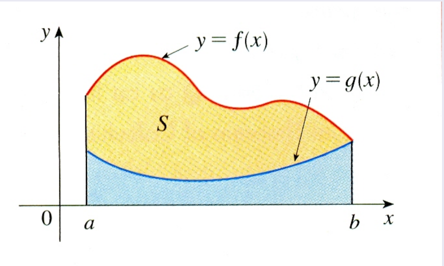






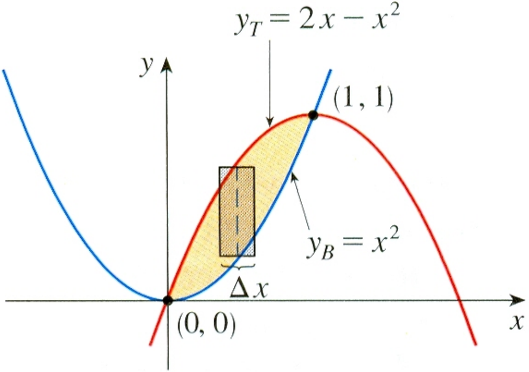
The area **A** of the region **S** bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$\mathbf{A} = \int_a^b [f(x) - g(x)]\, dx$$



Example

Find the area **A** of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.



Example

Find the area $\mathbf{A}$ of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution:

- We first find the points of intersection of the parabolas by solving their equations simultaneously.
- This gives $x^2 = 2x - x^2$, or
$$2x^2 - 2x = 0.$$
- Thus $2x(x-1) = 0$, so $x = 0$ or 1. The points of intersection are $(0, 0)$ and $(1, 1)$.
- So the total area is:
$\mathbf{A} = \int_0^1 (2x - 2x^2)\, dx = 2\int_0^1 (x - x^2)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$
$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$

$\square$

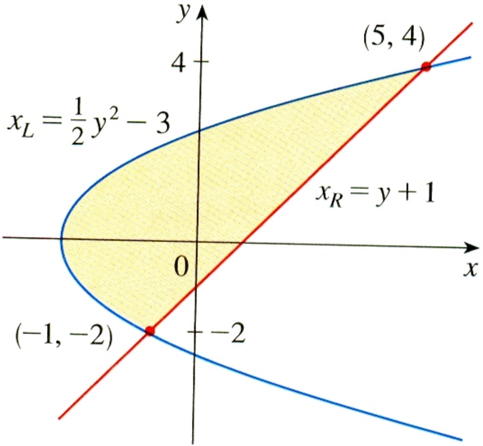
The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$\mathbf{A} = \int_a^b |f(x) - g(x)|\, dx$$

.

Example

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



Example

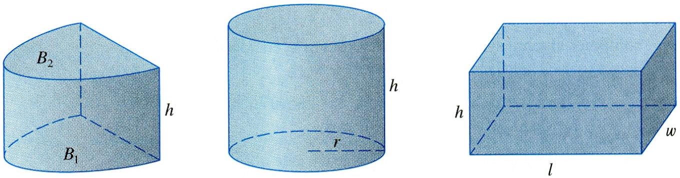
Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution:

- Solving the two equations we find points of intersection $(-1, -2)$ and $(5, 4)$.
- Next solve the equation of the parabola for x:
  $x_L = \frac{1}{2}y^2 - 3 \qquad x_R = y + 1$
- We must integrate between the appropriate y-values, $y = -2$ and $y = 4$.
- Thus $\mathbf{A} = \int_{-2}^{4} (x_R - x_L)\, dy = \int_{-2}^{4} [(y+1) - (\frac{1}{2}y^2 - 3)]\, dy =$
  $\int_{-2}^{4} (-\frac{1}{2}y^2 + y + 4)\, dy$
  $= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\Big]_{-2}^{4} = -\frac{1}{6}(64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18$

$\square$

Volumes



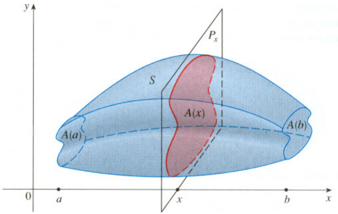
(a) Cylinder
$V = Ah$

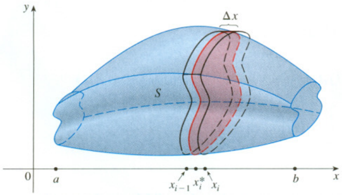
(b) Circular cylinder
$V = \pi r^2 h$

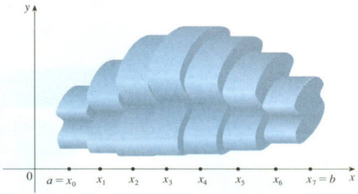
(c) Rectangular box
$V = lwh$

- Cylinder: $\mathbf{V} = \mathbf{A}h$
- Circular Cylinder: $\mathbf{V} = \pi r^2 h$
- Rectangular Box: $\mathbf{V} = lwh$

Volumes







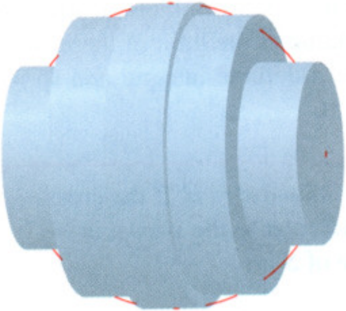
$$\mathbf{V} = \text{volume} \approx \sum_{i=1}^{n} \mathbf{A}(x_i^*)\Delta x$$

Definition of volume

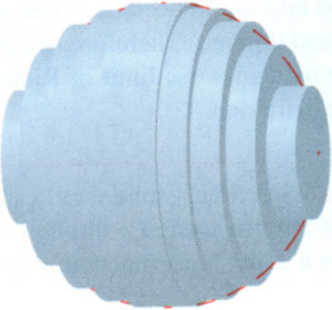
Definition (Definition of Volume)

- Let S be a solid that lies between $x = a$ and $x = b$.
- Suppose the cross-sectional area of S in the plane P_x, through x and perpendicular to the $x-$axis, is $\mathbf{A}(x)$, where $\mathbf{A}$ is a continuous function.
- Then the **volume** of S is

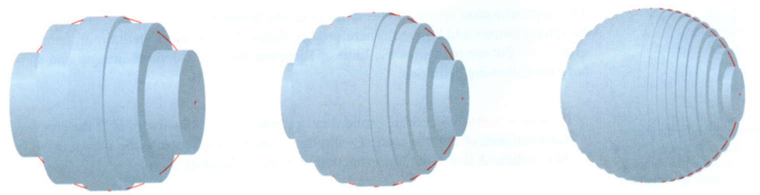
$$\mathbf{V} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{A}(x_i^*)\Delta x = \int_a^b \mathbf{A}(x)\, dx$$



(a) Using 5 disks, $V \approx 4.2726$



(b) Using 10 disks, $V \approx 4.2097$



(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Example

Show the volume of a sphere of radius r is $\mathbf{V} = \frac{4}{3}\pi r^3$.

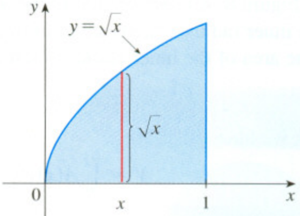
Solution:

- $x^2 + y^2 = r^2 \qquad y = \sqrt{r^2 - x^2}$.
- The cross sectional area is $\mathbf{A}(x) = \pi y^2 = \pi(r^2 - x^2)$. So:
- $\mathbf{V} = \int_{-r}^{r} \mathbf{A}(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = 2\pi \int_0^r (r^2 - x^2)\,dx$

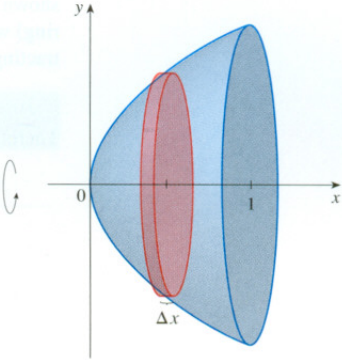
  $= 2\pi \left[r^2 x - \frac{x^3}{3}\right]_0^r = 2\pi \left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3$ $\square$

Example

Find the volume **V** of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



(a)

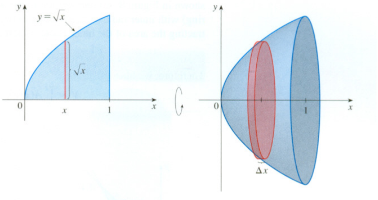


(b)

Computing a volume of revolution

Example

Find the volume $\mathbf{V}$ of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



(a) (b)

Solution:

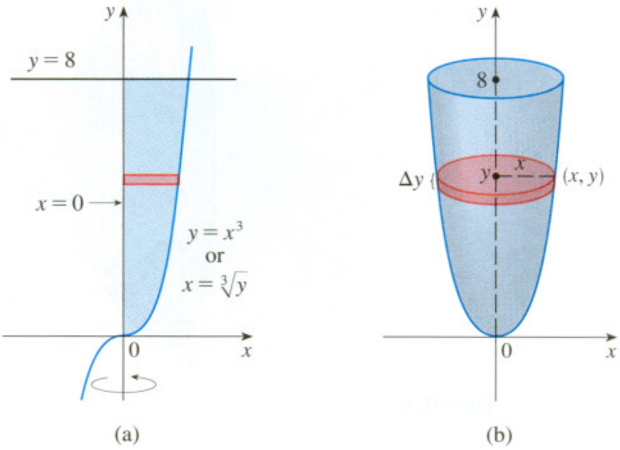
The area of the cross section is: $\mathbf{A}(x) = \pi(\sqrt{x})^2 = \pi x$.

So, $\mathbf{V} = \int_0^1 \mathbf{A}(x)\, dx = \int_0^1 \pi x\, dx = \pi \left. \frac{x^2}{2} \right]_0^1 = \frac{\pi}{2}$

$\square$

Example

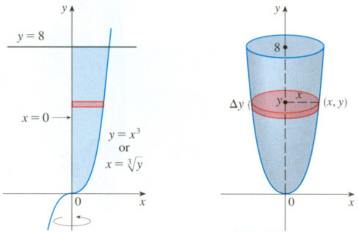
Find the volume **V** of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.



Volume of a solid paraboloid

Example

Find the volume $\mathbf{V}$ of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.



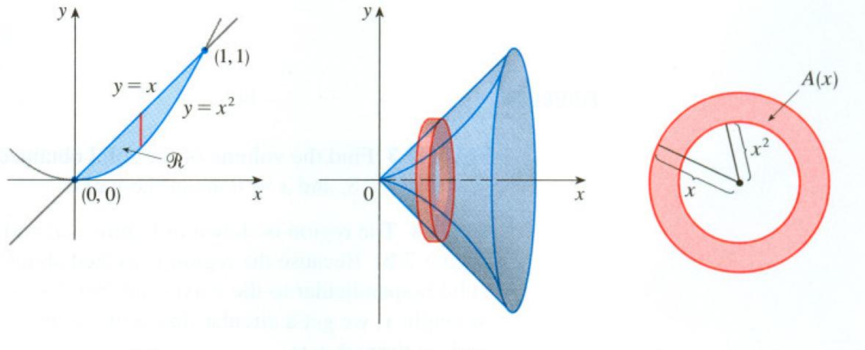
Solution:

- Note that $x = \sqrt[3]{y}$.
- The area of cross section is: $\mathbf{A}(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{\frac{2}{3}}$.
- So, $\mathbf{V} = \int_0^8 \mathbf{A}(y)\, dy = \int_0^8 \pi y^{\frac{2}{3}\, dy} = \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^8 = \frac{96\pi}{5}$.

$\square$

Example

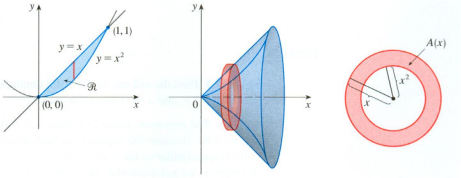
The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume **V** of the solid region.



(a) (b) (c)

Example

The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume $\mathbf{V}$ of the solid region.



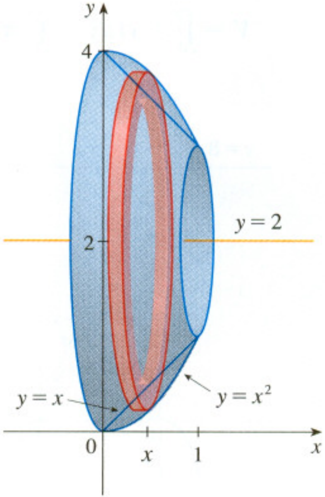
Solution:

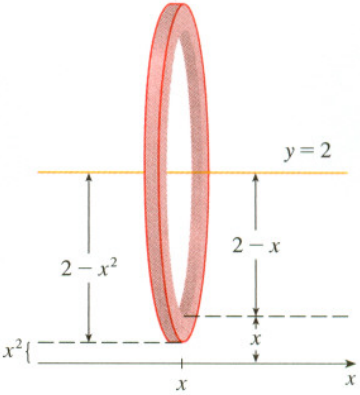
- The curves $y = x$ and $y = x^2$ intersect at the points $(0, 0)$ and $(1, 1)$.
- Cross section of rotated surface has the shape of a **washer** (annular ring).
- So the cross sectional area is: $\mathbf{A}(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$.
- So, $\mathbf{V} = \int_0^1 \mathbf{A}(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right] = \frac{2\pi}{15}$.

$\square$

Example

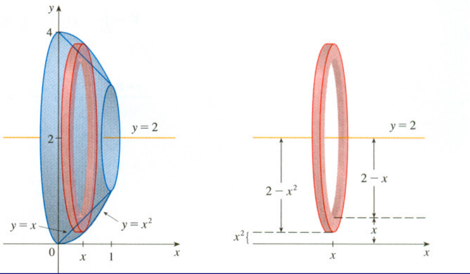
The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume **V**.





Example

The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume $\mathbf{V}$.



Solution:

- The cross section is again a washer.
- The cross sectional area is:
  $\mathbf{A}(x) = \pi(2 - x^2)^2 - \pi(2 - x)^2 = \pi(x^4 - 5x^2 + 4x)$.
- So, $\mathbf{V} = \int_0^1 \mathbf{A}(x)\, dx = \pi \int_0^1 (x^4 - 5x^2 + 4x)\, dx =$
  $\pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2} \right]_0^1 = \frac{8\pi}{15}$.

$\square$

Integration by parts

- We now return to integration methods.
- Recall our first method was substitution which came from the chain rule.
- **Integration by parts** comes from the **product rule**:
$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$
- So,
$$f(x)g'(x) = (f(x)g(x))' - g(x)f'(x).$$
- Thus, after taking antiderivatives, we get
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

- $$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$
- The above is called the **formula for integration by parts**.
- If we let $u = f(x)$ and $v = g(x)$, then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.
- So the formula becomes:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

- Recall the **integration by parts formula**:

$$\int u\,dv = uv - \int v\,du.$$

- To apply this formula we must choose dv so that we can integrate it!
- Frequently, we choose u so that the derivative of u is simpler than u.
- If both properties hold, then you have made the correct choice.

Examples using strategy: $\int u\,dv = uv - \int v\,du$

- $\displaystyle\int xe^x\,dx$: Choose $\mathbf{u} = \mathbf{x}$ and $\mathbf{dv} = \mathbf{e^x\,dx}$
- $\displaystyle\int t^2e^t\,dt$: Choose $\mathbf{u} = \mathbf{t^2}$ and $\mathbf{dv} = \mathbf{e^t\,dt}$
- $\displaystyle\int \ln x\,dx$: Choose $\mathbf{u} = \ln \mathbf{x}$ and $\mathbf{dv} = \mathbf{dx}$
- $\displaystyle\int x\sin x\,dx$: Choose $\mathbf{u} = \mathbf{x}$ and $\mathbf{dv} = \sin \mathbf{x\,dx}$
- $\displaystyle\int x^2\sin 2x\,dx$: Choose $\mathbf{u} = \mathbf{x^2}$ and $\mathbf{dv} = \sin \mathbf{2x\,dx}$

Example

Find $\int xe^x\,dx$.

Solution:

- Let
$$\mathbf{u = x \quad dv = e^x\,dx.}$$
- Then
$$\mathbf{du = dx \quad v = e^x.}$$
- **Integrating by parts** gives
$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$
$\square$

Example

Evaluate $\int \ln x \, dx$.

Solution:

- Let

$$\mathbf{u} = \ln \mathbf{x} \qquad \mathbf{dv} = \mathbf{dx}.$$

- Then

$$\mathbf{du} = \frac{\mathbf{1}}{\mathbf{x}}\mathbf{dx} \qquad \mathbf{v} = \mathbf{x}.$$

- **Integrating by parts**, we get

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx = x \ln x - x + C.$$

$\square$

Example

Find $\int t^2 e^t \, dt$.

Solution:

- Let $\mathbf{u = t^2} \quad \mathbf{dv = e^t dt}$

  Then $\mathbf{du = 2tdt} \quad \mathbf{v = e^t}$.

- Integration by parts gives $\boxed{\int t^2 e^t \, dt = t^2 e^t - 2 \int t e^t \, dt \quad \mathbf{(1)}}$

- Using integration by parts a second time, this time with

$$\mathbf{u = t} \qquad \mathbf{dv = e^t \, dt}.$$

  Then and

$$\mathbf{du = dt}, \qquad \mathbf{v = e^t},$$

  $\int t e^t \, dt = t e^t - \int e^t \, dt = t e^t - e^t + C.$

- Putting this in Equation **(1)**, we get

$$\int t^2 e^t \, dt = t^2 e^t - 2 \int t e^t \, dt = t^2 e^t - 2(t e^t - e^t + C)$$
$$= t^2 e^t - 2 t e^t + 2 e^t + C_1 \ \text{where} \, C_1 = -2C.$$

□

Example

Evaluate $\int e^x \sin x \, dx$.

Solution:

- Integrate by parts **twice**. Choose $\mathbf{u = e^x}$ and $\mathbf{dv = \sin x \, dx}$. Then $\mathbf{du = e^x \, dx}$ and $\mathbf{v = -\cos x}$, so integration by parts gives

$$\boxed{\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.} \quad \mathbf{(2)}$$

- Next use $\mathbf{u = e^x}$ and $\mathbf{dv = \cos x \, dx}$. Then $\mathbf{du = e^x \, dx}$, $\mathbf{v = \sin x}$, and

$$\boxed{\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.} \quad \mathbf{(3)}$$

- Put Equation **(3)** into Equation **(2)** and we get $\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$. This can be regarded as an equation to be solved for the unknown integral.

- Adding $\int e^x \sin x \, dx$ to both sides, we obtain $2 \int e^x \sin x \, dx = -e \cos x + e^x \sin x$.

- Dividing by 2 and adding the constant of integration, we get $\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$.

.

$\square$

Trigonometric integrals

- Trigonometric integrals are integrals of functions $f(x)$ that can be expressed as a product of functions from trigonometry.
- For example;
  - $f(x) = \cos^3 x$
  - $f(x) = \sin^5 x \cos^2 x$
  - $f(x) = \sin^2 x$.
- Integrating such functions involve several techniques and strategies which we will describe today.

Aside from the most basic relations such as $\tan x = \dfrac{\sin(\theta)}{\cos(\theta)}$ and $\sec x = \dfrac{1}{\cos(\theta)}$, you should know the following trig identities:

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

$$\sec^2(\theta) - \tan^2(\theta) = 1.$$

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$$

Example

Find $\int_0^\pi \sin^2 x\, dx$.

Solution:

- We will use the **double angle formula**:

$$\boxed{\sin^2 x\, dx = \tfrac{1}{2}(1 - \cos 2x).}$$

- $$\int_0^\pi \sin^2 x\, dx = \frac{1}{2}\int_0^\pi (1 - \cos 2x)\, dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^\pi$$

$$= \frac{1}{2}(\pi - \frac{1}{2}\sin 2\pi) - \frac{1}{2}(0 - \frac{1}{2}\sin 0) = \frac{1}{2}\pi.$$

- Here we mentally made the **substitution $\mathbf{u = 2x}$** when integrating $\cos 2x$.

□

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is **odd** ($n = 2k+1$), save one cosine factor and use $\boxed{\cos^2 x = 1 - \sin^2 x}$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

Then **substitute $\mathbf{u} = \sin \mathbf{x}$**.

(b) If the power of sine is **odd** ($m = 2k + 1$), save one sine factor and use $\boxed{\sin^2 x = 1 - \cos^2 x}$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx.$$

Then **substitute $\mathbf{u} = \cos \mathbf{x}$**.

(c) If the powers of both sine and cosine are **even**, use the half-angle identities

$$\boxed{\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)}$$

It is sometimes helpful to use the identity

$$\boxed{\sin x \cos x = \tfrac{1}{2} \sin 2x}$$

(a) If the power of secant is **even** ($n = 2k$, $k \geq 2$), save a factor of $\boxed{\sec^2 x}$ and use $\boxed{\sec^2 = 1 + \tan^2 x}$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then **substitute $\mathbf{u} = \tan \mathbf{x}$**.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(b) If the power of tangent is **odd** $(m = 2k + 1)$, save a factor of $\boxed{\sec x \ \tan x}$ and use $\boxed{\tan^2 x = \sec^2 x - 1}$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then **substitute** $\mathbf{u} = \sec \mathbf{x}$.

Two other useful formulas

- Recall that we proved the following formula in class using integration by parts.

$$\int \tan x \, dx = \ln|\sec x| + C.$$

- The next formula can be checked by differentiating the right hand side.

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

- Also, don't forget that $\frac{d}{dx} \tan x = \sec^2 x$ and $\frac{d}{dx} \sec x = \sec x \tan x$.

Example

Find $\int \tan^3 x \, dx$.

Solution:

- Here only $\tan x$ occurs, so we use $\mathbf{\tan^2 x = \sec^2 x - 1}$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

  $\int \tan^3 x \, dx = \int \tan x \, \tan^2 x \, dx = \int \tan x \, (\sec^2 x - 1) \, dx = \int \tan x \, \sec^2 x \, dx - \int \tan x \, dx = \frac{\tan^2 x}{2} - \ln|\sec x| + C.$
- In the first integral we mentally **substituted** $\mathbf{u = \tan x}$ so that $\mathbf{du = \sec^2 x \, dx}$

$\square$

To evaluate the integrals **(a)** $\int \sin mx \cos nx \; dx$, **(b)** $\int \sin mx \sin n; dx$, or **(c)** $\int \cos mx \cos nx \; dx$, use the corresponding identity:

$$\boxed{\begin{aligned} (\mathbf{a})\ \sin A \cos B &= \tfrac{1}{2}[\sin(A-B)+\sin(A+B)] \\ (\mathbf{b})\ \sin A \sin B &= \tfrac{1}{2}[\cos(A-B)+\cos(A+B)] \\ (\mathbf{c})\ \cos A \cos B &= \tfrac{1}{2}[\cos(A-B)+\sin(A+B)] \end{aligned}}$$

Example

Evaluation $\int \sin 4x \cos 5x \; dx$.

Solution:

$$\int \sin 4x \cos 5x \; dx = \int \frac{1}{2}[\sin(-x)+\sin 9x] \; dx$$

$$= \frac{1}{2}\int(-\sin x + \sin 9x) \; dx = \frac{1}{2}(\cos x - \frac{1}{9}\cos 9x) + C.$$

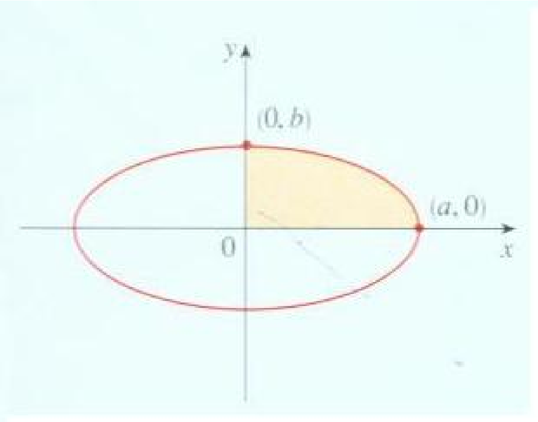
□

Table of Trigonometric Substitution

Expression	**Substitution**	**Identity**
$\sqrt{a^2 - x^2}$	$\mathbf{x} = \mathbf{a}\sin\theta, \quad -\frac{\pi}{\mathbf{2}} \le \theta \le \frac{\pi}{\mathbf{2}}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$\mathbf{x} = \mathbf{a}\tan\theta, \quad -\frac{\pi}{\mathbf{2}} < \theta < \frac{\pi}{\mathbf{2}}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$\mathbf{x} = \mathbf{a}\sec\theta, \quad \mathbf{0} \le \theta < \frac{\pi}{\mathbf{2}}$ or $\pi \le \theta < \frac{\mathbf{3}\pi}{\mathbf{2}}$	$\sec^2\theta - 1 = \tan^2\theta$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$.



Example

Find the area enclosed by the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$.

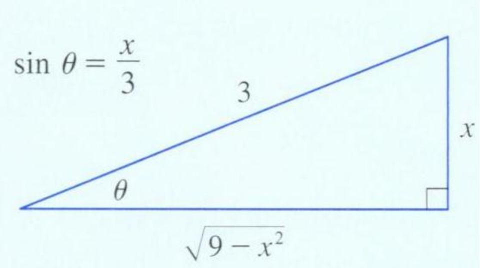
Solution:

- Solving for y gives
  $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and $\mathbf{A} = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$
- **Substitute** $\mathbf{x} = \mathbf{a}\sin\theta$, $\mathbf{dx} = \mathbf{a}\cos\theta\,\mathbf{d}\theta$ and use $\sqrt{a^2 - x^2} = a\cos\theta$.
- $\int \sqrt{a^2 - x^2}\,dx = \int a\cos\theta \cdot a\cos\theta\,d\theta$
  $= a^2\int \cos^2\theta\,d\theta = a^2\int \frac{1}{2}(1 + \cos 2\theta)\,d\theta = \frac{1}{2}a^2(\theta + \frac{1}{2}\sin 2\theta)$.
- $\mathbf{A} = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2} = 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \pi ab$.

$\square$

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\, dx$.



$\sqrt{a^2 - x^2}$, $x = a\sin\theta$, $1 - \sin^2\theta = \cos\theta$

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution:

- Let $\mathbf{x} = \mathbf{3}\sin\theta$, $\mathbf{dx} = \mathbf{3}\cos\theta\ \mathbf{d}\theta$.
  $\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$
-
$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta\ d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\ d\theta \\ &= \int (\csc^2\theta - 1)\ d\theta = -\cot\theta + C = \frac{\sqrt{9-x^2}}{x}\end{aligned}$$

$\square$

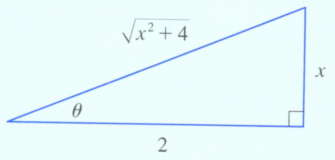


FIGURE 3

$\tan\theta = \frac{x}{2}$

Example

Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

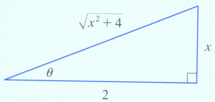


FIGURE 3

$\tan\theta = \frac{x}{2}$

Example

Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\, dx$.

Solution:

- Let $\mathbf{x = 2\tan\theta}$, $\mathbf{dx = 2\sec^2\theta\, d\theta}$ and $\sqrt{x^2+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2\sec\theta$.
- Thus, $\int \frac{dx}{x^2\sqrt{x^4+4}} = \int \frac{2\sec^2\theta\; dt}{4\tan^2\theta\cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$
- Put everything in terms of $\sin\theta$, $\cos\theta$: $\int \frac{dx}{x^2\sqrt{x^4+4}} = \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}(-\frac{1}{u}) + C = -\frac{1}{4\sin\theta} + C = -\frac{\csc\theta}{4} + C = \frac{\sqrt{x^2+4}}{4x} + C$. □

Definition

A function $f(x)$ is called a **rational function** if it can be expressed as the ratio of two polynomials.

Example

$f(x) = \frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$, is a rational function.
So,
$\int \frac{x+5}{x^2-x-2}\, dx = \int \frac{1}{x-1} - \frac{1}{x+2}\, dx = 2\ln|x-1| - \ln|x+2| + C.$

Recall that the **degree** polynomial
$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is n if $a_n \neq 0$. We write this as
$\mathbf{deg(P) = n}$.

A rational function $f(x) = \frac{P(x)}{Q(x)}$ is called **proper** if the degree of $P(x)$ is less than the degree of $Q(x)$; otherwise it is called **improper**.

If $\deg(P) \geq \deg(Q)$, then after **long division**,
$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where $R(x)$ is the **remainder** and
$\deg(R) < \deg(Q)$.

Example

Find $\int \frac{x^3+x}{x-1}\, dx$.

Solution:

- After **long division**, we find

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}.$$

- So,

$$\int \frac{x^3+x}{x-1}\, dx = \int x^2 + x + 2 + \frac{2}{x-1}\, dx$$
$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C.$$

□

Partial Fraction decomposition

- First step, **factor** $Q(x)$ and express $\frac{R(x)}{Q(x)}$ using partial fractions.
- Any polynomial $Q(x)$ can be factored into a product of linear factors $(\mathbf{ax} + \mathbf{b})$ and quadratic factors $(\mathbf{ax^2} + \mathbf{bx} + \mathbf{c})$, where $b^2 - 4ac < 0$.
- For example, suppose $Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$.
- Next express $\frac{R(x)}{Q(x)}$ as a sum of partial fractions of the form

$$\frac{\mathbf{A}}{(\mathbf{ax} + \mathbf{b})^{\mathbf{i}}} \quad \textbf{or} \quad \frac{\mathbf{Ax} + \mathbf{B}}{(\mathbf{ax^2} + \mathbf{bx} + \mathbf{c})^{\mathbf{j}}}$$

.

Case $Q(x)$ is a product of linear factors

Example

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\, dx$.

Solution:

- **First factor Q(x)**.
  $Q(x) = 2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x-1)(x+2)$.
- $$\text{So,} \quad \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$
- **Now multiply both sides by Q(x)**:
  $$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1).$$
- **Next simplify**:
  $x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B + C)x - 2A$.

Continuation:

- **Next solve the linear equations:**

  $2A + B + 2C = 1$

  $3A + 2B - C = 2$

  $-2A = -1$

- **Solving** we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, $C = -\frac{1}{10}$.

- Thus,

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$$

$$= \int \left(\frac{1}{2} \cdot \frac{1}{x} + \frac{1}{5} \cdot \frac{1}{2x - 1} - \frac{1}{10} \cdot \frac{1}{x + 2} \right) dx$$

$$= \frac{1}{2} \ln |x| + \frac{1}{10} \ln |2x - 1| - \frac{1}{10} \ln |x + 2| + C.$$

□

$Q(x)$ is a product of repeated linear factors

Example

Find $\int \frac{4x}{x^3 - x^2 - x + 1}\, dx$.

Solution:

- The first step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$.
- Since $Q(1) = 0$, we know that $x - 1$ is a factor.
- So, $x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$.
- Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

- Now solve for A, B, and C and integrate. $\square$

Q(x) has irreducible repeated quadratic factors

Example

Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}.$$

Solution:

The form of the partial fraction decomposition is

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3} =$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}.$$

□

Q(x) contains irreducible quadratic factors

Example

Evaluate $\int \frac{2x^2-x+4}{x^3+4x}; dx$.

Solution:

- Since $x^3+4x = x(x^2+4)$ can't be factored further, we write
$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$
- Multiplying by $x(x^2+4)$, we have
$2x^2-x+4 = A(x^2+4) + (Bx+C)x = (A+B)x^2 + Cx + 4A$
- Equating coefficients, we obtain $A+B=2 \quad C=-1 \quad 4A=4$
- Thus $A=1$, $B=1$, and $C=-1$ and so
$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)\,dx$$
- In order to integrate the second term we split it into two parts:

Continuation:

- $$\int \frac{x-1}{x^2+4}\, dx = \int \frac{x}{x^2+4}\, dx - \int \frac{1}{x^2+4}\, dx$$

- **We make the substitution** $\mathbf{u = x^2 + 4}$ in the first of these integrals so that $\mathbf{du = 2x\, dx}$.

- **Integrating** we obtain:

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\, dx = \int \frac{1}{x}\, dx + \int \frac{x}{x^2+4}\, dx - \int \frac{1}{x^2+4}\, dx$$
$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(\frac{x}{2}) + K$$

□

Example

Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{4}\, dx$.

Solution:

Let $\mathbf{u = \sqrt{x+4}}$, $\mathbf{u^2 = x+4}$, $\mathbf{x = u^2 - 4}$, $\mathbf{dx = 2u\, du}$.
Therefore,

$$\int \frac{\sqrt{x+4}}{x}\, dx = \int \frac{u}{u^2-4}\, 2u\, du = 2\int \frac{u^2}{u^2+4}\, du$$
$$= 2\int (1 + \frac{4}{u^2-4})\, du = 2\int \left(1 + \frac{4}{(u-2)(u+3)}\right)\, du$$
$$= 2\int du + 8\int \frac{1}{4}\left(\frac{1}{u-2} - \frac{1}{u+2}\right) = 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C$$

$\square$

Strategies: Simplify the integrand if possible

$$\int \frac{\tan\theta}{\sec^2\theta}\, d\theta = \int \frac{\sin\theta}{\cos\theta} \cos^2\theta\, d\theta = \int \sin\theta \cos\theta\ d\theta$$

$$= \frac{1}{2}\int \sin 2\theta\ d\theta.$$

$$\int (\sin x + \cos x)^2\ dx = \int \sin^2 x + 2\sin x \cos x + \cos^2 x\ dx$$

$$= \int 1 + 2\sin x \cos x\ dx.$$

Strategies: Look for an obvious substitution

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1|.$$

Here, $\mathbf{u = x^2 - 1}$, $\mathbf{du = 2x\,dx}$.

Strategies: Classify integrand according to form

- **Trigonometric functions**. Use recommended substitutions.
- **Rational functions.** Use partial fractions.
- **Integration by parts.** Use if $f(x)$ is a product of x^n and a transcendental function.
- **Radicals**. Particular substitutions are recommended.
  (a) If $\sqrt{\pm x^2 \pm a^2}$ occurs, use **trig substitutions.**
  (b) If $\sqrt{ax+b}$ occurs, use the **rationalizing substitution** $\mathbf{u = \sqrt{ax+b}}$.

Strategies: Try again

- **Try substitution.**
- **Try parts.**
- **Manipulate the integrand.** For example,
  $\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}\, dx = \int \frac{1+\cos^2 x}{\sin^2 x}\, dx$
- **Relate the problem to previous problems.** For example,
  $\int \tan^2 x\ \sec x\, dx = \int \sec^3 x\, dx - \int \sec x\, dx$ and you know $\int \sec^3 x\, dx$ by previous work.
- **Use several methods.**

We have already considered several methods for estimating $\int_a^b f(x)\,dx$.

- **left endpoint approximation**
- **right endpoint approximation**
- **midpoint approximation**

Example

Use the **Midpoint Rule** with $n = 4$ to approximate the integral $\int_0^8 e^{x^2}\,dx$.

Solution:

Since $a = 0$, $b = 8$, and $n = 4$, the **Midpoint Rule** gives
$\int_0^8 e^{x^2}\,dx \approx \Delta x[f(1) + f(3) + f(5) + f(7)]$
$= 2\left[e^1 + e^9 + e^{25} + e^{49}\right].$

$\square$

Trapezoidal Rule

$\int_a^b f(x)\,dx \approx T_n =$
$\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$
where $\Delta x = \frac{(b-a)}{n}$ and $x_i = a + i\Delta x$.

Example

Use the **Trapezoidal Rule** to approximate the integral $\int_1^2 \frac{1}{x}\,dx$.

Solution:

With $n = 5$, $a = 1$, and $b = 2$, we have $\Delta x = \frac{(2-1)}{5} = 0.2$, and so the **Trapezoidal Rule** gives

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{0.2}{2}[f(1)+2f(1.2)+2f(1.4)+2f(1.6)+2f(1.8)+f(2)] =$$
$$0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right).$$

$\square$

Error Bounds

Theorem (Error Bounds)

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

and

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Example

Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\, dx$.

Solution:

Since $a = 0$, $b = 1$, and $n = 10$, the Midpoint Rule gives

$$\int_0^1 e^{x^2}\, dx \approx \Delta \times [f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)]$$

$$= 0.1\left[e^{0.0025} + e^{0.0225} + e^{0.0625} + \ldots + e^{0.7225} + e^{0.9025}\right] \approx 1.460393$$

$\square$

- Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$.
- If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then
  $|E_T| \leq \frac{K(b-a)^3}{12n^2}$ and $|E_M| \leq \frac{K(b-a)^3}{24n^2}$

Example

Give an upper bound for the error involved in this approximation.

Solution:

- Since $f(x) = e^{x^2}$, we have $f'(x) = 2xe^{x^2}$ and $f''(x) = (2 + 4x^2)e^{x^2}$.
- Also, since $0 \leq x \leq 1$, we have $x^2 \leq 1$ and so
  $0 \leq f''(x) = (2 + 4x^2)e^{x^2} \leq 6e$
- Taking $K = 6e$, $a = 0$, $b = 1$, and $n = 10$ in the error estimate, we see that an upper bound for the error is
  $$\frac{6e(1)^3}{24(10)^2} = \frac{e}{400} \approx 0.007$$

□

- **Simpson's Rule** generalizes the method in the Trapezoidal Rule by using **parabola approximations**.
- It gives a much better approximation.
- **Simpson's Rule:** $\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$, where n is even and $\Delta x = \frac{(b-a)}{n}$.
- **Error Bound for Simpson's Rule** Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then $|E_S| \leq \frac{K(b-a)^5}{180n^4}$.

Example

How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (\frac{1}{x})\, dx$ is accurate to within 0.0001?

Solution:

- If $f(x) = \frac{1}{x}$, then $f^{(4)}(x) = \frac{24}{x^5}$.
- Since $x \geq 1$, we have $\frac{1}{x} \leq 1$ and so $|f^{(4)}(x)| = \frac{24}{x^5} \leq 24$.
- Therefore, we can take $K = 24$.
- Thus, for an error less than 0.0001 we should choose n so that $\frac{24(1)^5}{180n^4} < 0.0001$
- This gives $n^4 > \frac{24}{180(0.0001)}$ or $n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04$
- Therefore, $n = 8$ (n must be even) gives the desired accuracy.

$\square$

Definition of an Improper Integral of Type 1

Definition (Improper Integral of Type 1)

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$\int_{-\infty}^b f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$

In part (c) any real number a can be used.

Example

Find the area **A** under the graph of $f(x) = \frac{1}{x^2}$ from $x = 1$ to $x = \infty$.

Solution:

$$\mathbf{A} = \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\bigg|_0^t = 1.$$

This means that the area is equal to 1 also.

$\square$

Example

Determine whether the integral $\int_1^\infty \frac{1}{x}\, dx$ is convergent or divergent.

Solution:

- According to part (a) of Definition above, we have

$$\begin{aligned} \int_1^\infty \frac{1}{x}\, dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x}\, dx \\ &= \lim_{t\to\infty} \ln|x| \Big|_1^t = \lim_{t\to\infty} (\ln t - \ln 1) \\ &= \lim_{t\to\infty} \ln t = \infty. \end{aligned}$$

- The limit does not exist as a finite number and so the improper integral $\int_1^\infty (\frac{1}{x})\, dx$ is **divergent**.

□

Example

Evaluate $\int_{-\infty}^{0} e^x \, dx$.

Solution:

Using part (b) of Definition above,

$$\int_{-\infty}^{0} e^x \, dx = \lim_{t \to -\infty} \int_{t}^{0} e^x \, dx$$

$$= \lim_{t \to -\infty} e^x \Big|_t^0 = 1 - \lim_{t \to \infty} \frac{1}{e^t} = 1.$$

$\square$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$.

Solution:

- $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{-\infty} \frac{1}{1+x^2}\, dx$
- Since $f(x) = \frac{1}{1+x^2}$, then $f(x) = f(-x)$ and so $\int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{-\infty} \frac{1}{1+x^2}\, dx = 2\int_{0}^{\infty} \frac{1}{1+x^2}\, dx$.
- Then $\int_{0}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{t\to\infty} \int_{0}^{t} \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x\Big|_{0}^{t} = \lim_{t\to\infty}(\tan^{-1} t - \tan^{-1} 0) = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$.
- So, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 2 \cdot \frac{\pi}{2} = \pi$.

$\square$

Example

Evaluate $\int_{-\infty}^{0} xe^{x}\, dx$.

Solution:

- Using part (b) of Definition above, we have
  $\int_{-\infty}^{0} xe^{x}\, dx = \lim_{t\to\infty} \int_{t}^{0} xe^{x}\, dx$
- **We integrate by parts** with $\mathbf{u} = \mathbf{x}$, $dv = e^{x}\, dx$ so that $du = dx$, $v = e^{x}$:
  $\int_{t}^{0} xe^{x}\, dx = \left. xe^{x}\right]_{t}^{0} - \int_{t}^{0} e^{x}\, dx = -te^{t} - 1 + e^{t}$.
- We know that $e^{t} \to 0$ as $t \to -\infty$, and by 1' Hospital's Rule we have
  $\lim_{t\to-\infty} te^{t} = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}} = \lim_{t\to-\infty}(-e^{t}) = 0$
- Therefore
  $\int_{-\infty}^{0} xe^{x}\, dx = \lim_{t\to-\infty}(-te^{t} - 1 + e^{t}) = -0 - 1 + 0 = -1$

□

Definition of an Improper Integral of Type 2

Definition (Improper Integral of Type 2)

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$
if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then
$\displaystyle\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$ if this limit exists (as a finite number).

Definition (Improper Integral of Type 2)

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example

$$\int_{-1}^1 \frac{1}{x^2} = \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx.$$

Example

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$

Solution:

- We note first that the given integral is improper because $f(x) = \frac{1}{\sqrt{x-2}}$ has the vertical asymptote $x = 2$.
- Since the infinite discontinuity occurs at the left endpoint of $[2, 5]$, we use (b):

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}$$

$$= \lim_{t\to 2^+} 2\sqrt{x-2}\Big]_t^5 = \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2}) = 2\sqrt{3}$$

$\square$

Example

Evaluate $\int_0^1 \ln x \, dx = \lim_{t\to 0^+} \int_t^1 \ln x \, dx$.

Solution:

- We know that the function $f(x) = \ln x$ has a vertical asymptote at 0, since $\lim_{x\to 0^+} \ln x = -\infty$.
- Thus, the given integral is improper and we have
  $\int_0^1 \ln x \, dx = \lim_{t\to 0^+} \int_t^1 \ln x \, dx$
- Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = \frac{dx}{x}$, and $v = x$:
  $\int_t^1 \ln x \, dx = \; x \ln x \,]_t^1 - \int_t^1 \; dx = 1 \ln 1 - t \ln t - (1-t) = -t \ln t - 1 + t$
- To find the limit of the first term we use 1'Hospital's Rule:
  $\lim_{t\to 0^+} t \ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+}(-t) = 0$
- Therefore
  $\int_0^1 \ln x \, dx = \lim_{t\to 0^+}(\lim_{t\to 0^+}(-t \ln t - 1 + t) = -0 - 1 + 0 = -1.$ □

Example

For what values of p is the integral
$\int_1^\infty \frac{1}{x^p}\, dx$
convergent?

Solution:

- If $p = 1$, then we know integral is divergent.
- Assume now $p \neq 1$.
- $\int_1^\infty \frac{1}{x^p}\, dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big]_{x=1}^{x=t} = \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}-1}\right]$
- If $p > 1$, then as $t \to \infty$, $t^{p-1} \to \infty$ and $\frac{1}{t^{p-1}} \to 0$.
- So,
  $\int_1^0 \frac{1}{x^p}\, dx = \frac{1}{p-1}$ if $p > 1$.
- But if $p < 1$, $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$ and so the integral is divergent.

$\square$

Example

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\, dx$.

Solution:

- We note first that the given integral is improper because $f(x) = \frac{1}{\sqrt{x-2}}$ has the vertical asymptote $x = 2$.
- Since the infinite discontinuity occurs at the left endpoint of $[2, 5]$, we use :

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}$$

$$= \lim_{t\to 2^+} 2\sqrt{x-2}\Big]_t^5 = \lim_{t\to 2^+} 2(\sqrt{3} - \sqrt{t-2}) = 2\sqrt{3}$$

$\square$

Comparison Theorem

Theorem (Comparison Theorem)

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\displaystyle\int_a^\infty f(x)\,dx$ is convergent, then $\displaystyle\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\displaystyle\int_a^\infty g(x)\,dx$ is divergent, then $\displaystyle\int_a^\infty f(x)\,dx$ is divergent.

Example

Show that $\int_0^\infty e^{-x^2}\, dx$ is convergent.

Proof.

- First write

$$\int_0^\infty e^{-x^2} = \int_0^1 e^{-x^2}\, dx + \int_1^\infty e^{-x^2}\, dx.$$

- Since on $[1,\infty)$, $e^{-x^2} \le e^{-x}$ and

$$\int_1^\infty e^{-x}\, dx = \lim_{t\to\infty} \left(-e^{-x}\Big|_1^t = e^{-1} - e^{-t} \right) = e^{-1},$$

the integral is convergent by the Comparison Theorem.

□

Example

The integral

$$\int_1^\infty \frac{1 + e^{-x}}{x} \, dx$$

is divergent by the Comparison Theorem because

$$\frac{1 + e^{-x}}{x} > \frac{1}{x}$$

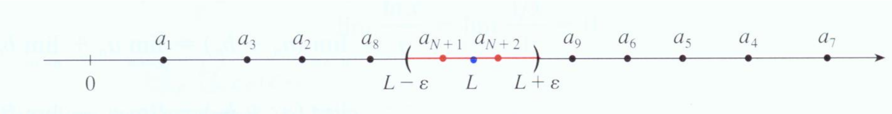
and $\displaystyle\int_1^\infty \frac{1}{x} \, dx$ is divergent.

Definition

A sequence $\{a_n\}$ has the **limit L** and we write

$$\lim_{n\to\infty} a_n = \mathbf{L} \qquad \text{or} \qquad a_n \to \mathbf{L} \quad \text{as} \quad n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that if $n > N$, then $|a_n - L| < \varepsilon$.



Definition

- A sequence a_n has the **limit** $\mathbf{L}$ and we write

$$\lim_{n\to\infty} a_n = \mathbf{L} \quad \text{or} \quad a_n \to \mathbf{L} \ \text{as} \ n \to \infty$$

  if we can make the terms a_n as close to $\mathbf{L}$ as we like by taking n sufficiently large.
- If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**).
- Otherwise, we say the sequence **diverges** (or is **divergent**).

Example

Let $f(x) = \frac{1}{x}$. Consider the sequence $a_n = f(n) = \frac{1}{n}$ for n an integer. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Definition

- A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$.
- It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
- It is called **monotonic** if it is either increasing or decreasing.

Example

The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because $\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$ and so $a_n > a_{n+1}$ for all $n \geq 1$.

Definition

- A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
- It is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$.
- If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Theorem (Monotonic Sequence Theorem)

Every bounded, monotonic sequence is convergent.

If $\{a_n\}$ and $\{b\}_n$ are convergent sequences and c if a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \ \text{ if } \ \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p \ \text{ if } \ p > 0 \ \text{ and } \ a_n > 0$$

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example

Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution:

- Notice that both numerator and denominator approach infinity as $n \to \infty$.
- We can apply 1' Hospital's Rule to the related function $f(x) = \frac{\ln x}{x}$ and obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

$\square$

Theorem

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum c \cdot a_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$:

(i) $$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$$

(ii) $$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(iii) $$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Theorem

If $\lim_{n\to\infty} a_n = \mathbf{L}$ and the function f is continuous at $\mathbf{L}$, then $\lim_{n\to\infty} f(a_n) = f(\mathbf{L})$

Example

Discuss the convergence of the sequence $a_n = \frac{n!}{n^n}$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

Solution:

- Write out a few terms:

  $a_1 = 1 \quad a_2 = \frac{1\cdot 2}{2\cdot 2} \quad a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$

  $a_n = \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot n\cdot\cdots\cdot} n.$

- Observe from above equation that $a_n = \frac{1}{n}\left(\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n}\right)$.
- The expression in parentheses is at most 1 because the numerator is less than the denominator.
- So $0 < a_n \leq \frac{1}{n}$. We know that $\frac{1}{n} \to 0$ as $n \to \infty$.
- Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

$\square$

Definition

- Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its **n-th partial sum**

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

- If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent**.
- When it is convergent, we write

  $a_1 + a_2 + \cdots + a_n + \cdots = s$ or $\sum_{n=1}^{\infty} a_n = s$.
- The number s is called the **sum** of the series.
- Otherwise, the series is called **divergent.**

- An important example of an infinite series is the **geometric series**

$$\mathbf{a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}}.$$

- Each term is obtained from the preceding one by multiplying it by the common ratio r.
- If $r \neq 1$, we have the **n-th partial sum**

$$\mathbf{s_n = a + ar + ar^2 + \cdots + \cdots + ar^{n-1}}$$

and

$$\mathbf{rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n}.$$

- Subtracting these equations, we get $\mathbf{s_n - rs_n = a - ar^n}$.
- Thus,

$$\mathbf{s_n = \frac{a(1 - r^n)}{1 - r}}.$$

- The **geometric series**

$$\mathbf{a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}},$$

  has the **n-th partial sum**

$$\mathbf{s_n = \frac{a(1 - r^n)}{1 - r}}.$$

- If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{\mathbf{n}\to\infty} \mathbf{s_n} = \lim_{\mathbf{n}\to\infty} \left(\mathbf{\frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \cdot r^n} \right)$$

$$= \mathbf{\frac{a}{1 - r} - \frac{a}{1 - r}} \cdot \lim_{\mathbf{n}\to\infty} \mathbf{r^n = \frac{a}{1 - r}}.$$

Theorem

- The geometric series $\sum_{i=1}^{n} ar^{n-1} = a + ar + ar^2 + \cdots$ is convergent if $|r| < 1$ and its **sum** is
$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1.$$
- If $|r| \geq 1$, the geometric series is divergent.

Example

Find the sum of the geometric series
$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = 5 \cdot 1 + 5 \cdot -\frac{2}{3} + 5 \cdot (-\frac{2}{3})^2 \cdots$

Solution:

- The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$.
- Since $|r| = \frac{2}{3} < 1$, the series is convergent and its sum is
$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3.$

Example

Find the exact value of the repeating decimal $\mathbf{0.\overline{156} = 0.156156156\ldots}$ as a rational number.

Solution:

Note that

$$\mathbf{0.\overline{156} = \sum_{n=0}^{\infty} \frac{156}{1000}\left(\frac{1}{1000}\right)^{n}}.$$

This is a **geometric series** with $\mathbf{a = \frac{156}{1000}}$ and $\mathbf{r = \frac{1}{1000}}$ and has the value:

$$\mathbf{\frac{\frac{156}{1000}}{\left(1 - \frac{1}{1000}\right)} = \frac{\frac{156}{1000}}{\frac{999}{1000}} = \frac{156}{999}}.$$

$\square$

Example

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Solution:

- Let's rewrite the n-th term of the series in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)}$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

.

- We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$.
- Since $r > 1$, the series diverge.

$\square$

Example

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

Solution:

- Compute the partial sums
  $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdots 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$.
- Simplify expression using the partial fractions

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

- Thus, $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) =$
  $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.
- So $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$.
- Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

□

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

- **The converse of Theorem is not true.**
- If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum a_n$ is convergent.
- For the harmonic series $\sum \frac{1}{n}$, we have $a_n = \frac{1}{n} \to 0$ as $n \to \infty$, but $\sum \frac{1}{n}$ is **divergent!**

Theorem (Test For Divergence)

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution:

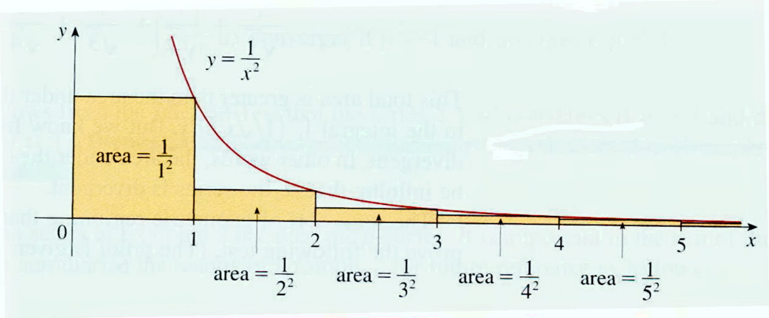
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0.$$

So the series diverges by the **Test of Divergence.** □

Example

Consider the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$



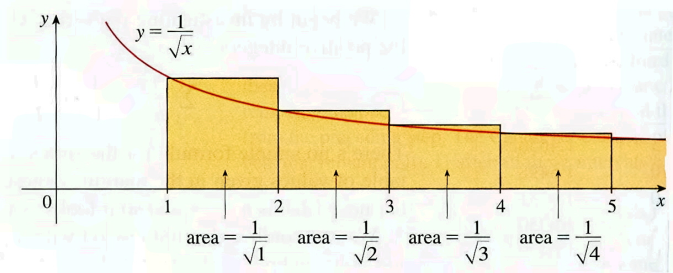
By comparing the areas, we see that $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is less than the area $\int_1^{\infty} \frac{1}{x^2}\, dx = 1$. So,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + 1 = 2.$$

Example

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots .$$



By comparing the areas, we see that
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = \infty$, so the series fails to converge.

Theorem (Integral Test)

- Suppose f is a **continuous, positive, decreasing** function on $[1, \infty)$ and let $a_n = f(n)$.
- Then the series $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is convergent if and only if the improper integral $\int_{\mathbf{1}}^{\infty} \mathbf{f(x)\,dx}$ is convergent.
- In other words:

  (i) If $\displaystyle\int_{\mathbf{1}}^{\infty} \mathbf{f(x)\,dx}$ is convergent, then $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is convergent.

  (ii) If $\displaystyle\int_{\mathbf{1}}^{\infty} \mathbf{f(x)\,dx}$ is divergent, then $\displaystyle\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is divergent.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution:

- The function $f(x) = \frac{1}{(x^2+1)}$ is continuous, positive and decreasing on $[1, \infty)$.
- By the **Integral Test:**
$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1} x \Big|_1^t$$
$$= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$
- Thus $\int_1^\infty \frac{1}{(x^2+1)}\,dx$ is a convergent integral and the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

$\square$

Theorem (p-Series Test)

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is **convergent if $p > 1$** and **divergent if $p \leq 1$.**

Example

(a) The series

$$\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^3}} = \frac{\mathbf{1}}{\mathbf{1^3}} + \frac{\mathbf{1}}{\mathbf{2^3}} + \frac{\mathbf{1}}{\mathbf{3^3}} + \frac{\mathbf{1}}{\mathbf{4^3}} + \cdots$$

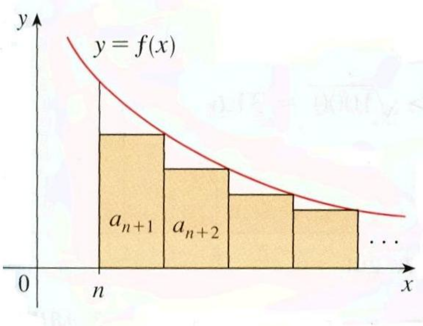
is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^{\frac{1}{3}}}} = \sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\sqrt[3]{n}} = \mathbf{1} + \frac{\mathbf{1}}{\sqrt[3]{2}} + \frac{\mathbf{1}}{\sqrt[3]{3}} + \frac{\mathbf{1}}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

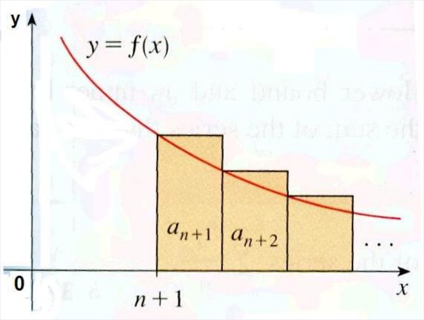
Suppose $f(x)$ is a positive decreasing function of x.



Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in figure above, we see that

$$\mathbf{R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)\, dx.}$$

Similarly, we see from figure below



$$\mathbf{R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\, dx}$$

So we have proved the following error estimate.

Theorem (Remainder Estimate For The Integral Test)

- Suppose $f(k) = a_k$, where f is a **continuous, positive, decreasing** function for $x \geq n$ and $\sum \mathbf{a_n}$ is convergent.
- If $R_n = s - s_n$, then

$$\int_{\mathbf{n+1}}^{\infty} \mathbf{f(x)\,dx} \leq \mathbf{R_n} \leq \int_{\mathbf{n}}^{\infty} \mathbf{f(x)\,dx}.$$

Example

It can be shown that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

How close is $R_k = \sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}$ to $\frac{\pi^2}{6}$?

Solution:

$$R_n \leq \int_k^{\infty} \frac{1}{x^2}\, dx = \lim_{t \to 0} \frac{-1}{x}\Big|_k^t = \frac{1}{k}.$$

$\square$

Theorem (Comparison Test)

Suppose that $\sum \mathbf{a_n}$ and $\sum \mathbf{b_n}$ are series with positive terms.

(i) If $\sum \mathbf{b_n}$ is convergent and $a_n \leq b_n$ for all n, then $\sum \mathbf{a_n}$ is also convergent.

(ii) If $\sum \mathbf{b_n}$ is divergent and $a_n \geq b_n$ for all n, then $\sum \mathbf{a_n}$ is also divergent.

Solution to part (i):

- Let
$$\mathbf{s_n} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{a_i} \qquad \mathbf{t_n} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{b_i} \qquad \mathbf{t} = \sum_{\mathbf{n=1}}^{\infty} \mathbf{b_n}.$$
- The sequences $\{s_n\}$ and $\{t_n\}$ are increasing $(s_{n+1} = s_n + a_{n+1} \geq s_n)$.
- Also $t_n \to t$, so $t_n \leq t$ for all n.
- Since $a_i \leq b_i$, we have $s_n \leq t_n$. Thus $s_n \leq t$ for all n.
- So $\{s_n\}$ is increasing and bounded above and converges by the **Monotonic Sequence Theorem.**

$\square$

Example

Determine whether the series $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{5}}{\mathbf{2n^2+4n+3}}$ converges or diverges.

Solution:

- $$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$
  because left side has a bigger denominator.
- $$\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{5}}{\mathbf{2n^2}} = \frac{\mathbf{5}}{\mathbf{2}} \sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^2}}$$
  is convergent because it's a constant times a $p-$series with $p = 2 > 1$.
- Therefore
  $$\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{5}}{\mathbf{2n^2+4n+3}}$$
  is convergent by part **(i)** of the **Comparison Test.**

$\square$

Example

Test the series $\sum_{\mathbf{n=1}}^{\infty} \frac{\ln \mathbf{n}}{\mathbf{n}}$ for convergence or divergence.

Solution:

- Observe that $\ln n > 1$ for $n \geq 3$ and so

$$\frac{\ln \mathbf{n}}{\mathbf{n}} > \frac{\mathbf{1}}{\mathbf{n}} \qquad \mathbf{n} \geq \mathbf{3}.$$

- We know that $\sum \frac{\mathbf{1}}{\mathbf{n}}$ is divergent ($p-$series with $p = 1$).
- Thus the given series is divergent by the **Comparison Test**.

$\square$

Theorem (Limit Comparison Test)

- Suppose that $\sum \mathbf{a_n}$ and $\sum \mathbf{b_n}$ are series with positive terms.
- If

$$\lim_{\mathbf{n}\to\infty} \frac{\mathbf{a_n}}{\mathbf{b_n}} = \mathbf{c}$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.

Solution:

- We use the **Limit Comparison Test** with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

- We obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 > 0$$

- Since this limit exists and $\sum \frac{1}{2^n}$ is a convergent geometric series, the given series converges by the **Limit Comparison Test.** □

Theorem (**The Alternating Series Test.**)

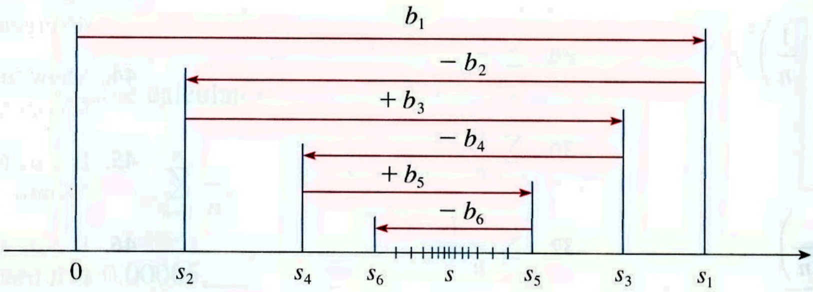
If the alternating series
$\sum_{\mathbf{n=1}}^{\infty}(-\mathbf{1})^{\mathbf{n-1}}\mathbf{b_n} = \mathbf{b_1} - \mathbf{b_2} + \mathbf{b_3} - \mathbf{b_4} + \mathbf{b_5} - \mathbf{b_6} + \cdots$ for $b_n > 0$
satisfies:

(i) $b_{n+1} \leq b_n$ for all n,

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

The proof is contained in the next figure.



Example

The **alternating harmonic series**

$$\mathbf{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}$$

satisfies

(i) $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

So the series is convergent by the **Alternating Series Test.**

Theorem (Alternating Series Estimation Theorem)

If $\mathbf{s} = \sum(-\mathbf{1})^{\mathbf{n}-\mathbf{1}}\mathbf{b_n}$ is the sum of an alternating series that satisfies

$$\text{(i) } 0 \leq b_{n+1} \leq b_n \quad \text{and} \quad \text{(ii) } \lim_{n\to\infty} b_n = 0,$$

then $|\mathbf{R_n}| = |\mathbf{s} - \mathbf{s_n}| \leq \mathbf{b_{n+1}}$.

Proof.

- We know from the proof of the **Alternating Series Test** that s lies between any two consecutive partial sums s_n and s_{n+1}.
- It follows that
$$|s_n - s_n| \leq |s_{n+1} - s_n| = b_{n+1}.$$

□

Definition

A series $\sum \mathbf{a_n}$ is called **absolutely convergent** if the series of absolute values $\sum |\mathbf{a_n}|$ is convergent.

Definition

A series $\sum \mathbf{a_n}$ is called **conditionally convergent** if it is convergent but **not** absolutely convergent.

Example

The series $\sum_{\mathbf{n=1}}^{\infty} \frac{(-\mathbf{1})^{\mathbf{n-1}}}{\mathbf{n}^2} = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}^2} + \frac{\mathbf{1}}{\mathbf{3}^2} - \frac{\mathbf{1}}{\mathbf{4}^2} + \cdots$ is absolutely convergent because
$\sum_{\mathbf{n=1}}^{\infty} \left| \frac{(-\mathbf{1})^{\mathbf{n-1}}}{\mathbf{n}^2} \right| = \sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n}^2} = \mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}^2} + \frac{\mathbf{1}}{\mathbf{3}^2} + \frac{\mathbf{1}}{\mathbf{4}^2} + \cdots$ is a convergent **p-series** $(p = 2)$.

Theorem

If a series $\sum \mathbf{a_n}$ is **absolutely convergent**, then it is **convergent**.

Proof.

- Observe the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

- If $\sum \mathbf{a_n}$ is absolutely convergent, then $\sum |\mathbf{a_n}|$ is convergent, so $\sum \mathbf{2}|\mathbf{a_n}|$ is convergent.
- By the **Comparison Test**, $\sum(\mathbf{a_n} + |\mathbf{a_n}|)$ is convergent, then

$$\sum \mathbf{a_n} = \sum(\mathbf{a_n} + |\mathbf{a_n}|) - \sum |\mathbf{a_n}|$$

is the difference of two convergent series and is convergent. $\square$

Theorem (Ratio Test)

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is **absolutely convergent** (and therefore **convergent**).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is **divergent**.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the **Ratio Test** is **inconclusive**; no conclusion can be drawn about the convergence or divergence of $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$.

Proof of the Ratio Test.

- Assume the series $\sum_{i=1}^{n} a_n$ satisfies $\left|\frac{a_{j+1}}{a_j}\right| \to L < r < 1$ as $j \to \infty$.
- Then for a large integer i and $n \geq i$, $\left|\frac{a_{n+1}}{a_n}\right| < r$ or $|a_{n+1}| < |a_n|r$.
- Thus, for any $k > 0$,

$$|a_{i+k}| < |a_{i+k-1}|r < |a_{i+k-2}|r^2 \ldots < |a_i|r^k.$$

- Hence, for $k > 0$, $|a_{i+k}| < |a_i|r^k$.
- Since $\sum_{k=0}^{\infty} |a_i|r^k$ is a **convergent geometric series**, the **Comparison Test** implies $\sum_{k=0}^{\infty} |a_{i+k}|$ converges and so $\sum_{i=1}^{n} a_n$ **converges absolutely**.
- Thus, $\sum_{n=0}^{\infty} a_n$ converges absolutely.

$\square$

Example

Test the series $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{n^3}}{\mathbf{3^n}}$ for absolute convergence.

Solution:

- We use the **Ratio Test** with $a_n = \frac{n^3}{3^n}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1 \quad \text{as} \quad n \to \infty.$$

- Thus, by the **Ratio Test**, the given series is absolutely convergent and therefore convergent.

$\square$

Theorem (Root Test)

(i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is **absolutely convergent** (and therefore **convergent**).

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is **divergent**.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the **Root Test** is **inconclusive**.

Example

Test the convergence of the series $\sum_{\mathbf{n=1}}^{\infty} \left(\frac{\mathbf{2n+3}}{\mathbf{3n+2}}\right)^{\mathbf{n}}$.

Solution:

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1 \quad \text{as} \quad n \to \infty.$$

Thus the given series converges by the **Root Test.** □

Example

$\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{n-1}}{\mathbf{2n+1}}$

Solution:

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, it fails to converge by the **Test for Divergence**. □

Example

$\sum_{\mathbf{n=1}}^{\infty} \frac{\sqrt{\mathbf{n^3+1}}}{\mathbf{3n^3+4n^2+2}}$

Solution:

- Since a_n is an algebraic function of n, we compare the given series with a **p-series**.
- The comparison series for the **Limit Comparison Test** is $\sum \mathbf{b_n}$, where

  $$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{\frac{3}{2}}}{3n^3} = \frac{1}{3n^{\frac{3}{2}}}.$$

□

Example

$\sum_{n=1}^{\infty} \mathbf{n}e^{-n^2}$

Solution:

- Since the integral $\int_1^\infty xe^{-x^2}\,dx$ is easily evaluated, we use the **Integral Test**.
- The **Ratio Test** also works.

□

Example

$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Solution:

Since the series is alternating, we use the **Alternating Series Test.** □

Example

$\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Solution:

Since the series involves $k!$, we use the **Ratio Test.** □

Example

$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Solution:

Since the series is closely related to the **geometric series** $\sum \frac{1}{3^n}$, we use the **Comparison Test.** □

- A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \qquad (1)$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

- For each fixed x, the series **(1)** is a series of constants that we can test for convergence or divergence.

- A power series may converge for some values of x and diverge for other values of x.

- The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges.

Definition

A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called any of the following:

- a **power series in $(x-a)$**
- a **power series centered at a**
- a **power series about a.**

Example

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Solution:

- Let $a_n = \frac{(x-3)^n}{n}$.
- Then
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \frac{n}{n+1}|x-3|$$
$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \quad \text{as} \quad n \to \infty.$$
- By the **RatioTest**, the given series is absolutely convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$.
- So the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.
- For $x = 4$, the series is the **harmonic series** $\sum \frac{1}{n}$, which is divergent by **p-Test**.
- For $x = 2$, the series is $\sum \frac{(-1)^n}{n}$, which converges by the **Alternating Series Test**.
- Thus the given power series converges for $\mathbf{2 \leq x < 4}$. □

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

- The number R in case **(iii)** is called the **radius of convergence** of the power series.
- By convention, the radius of convergence is $R = 0$ in case **(i)** and $R = \infty$ in case **(ii)**.
- The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

Example

Find the radius of convergence and the interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Solution:

- Let $a_n = \frac{x^n}{n}$.
- Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \frac{n}{n+1}|x| \to |x| \text{ as } n \to \infty.$$

- By the **Ratio Test**, this series converges for $|x| < 1$ and diverges for $|x| > 1$.
- So the **radius of convergence** is $\mathbf{R} = \mathbf{1}$ and the series converges on the interval $(-1, 1)$.
- We now check **end points**.

  For $x = 1$, we have the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by the **Integral Test** or by the **p-Test**.
- For $x = -1$, we have the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the **Alternating Series Test**.
- So the interval of convergence is $[-1, 1)$.

$\square$

Example

Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution:

- Recall $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$.
- Thus,

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}$$
$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots .$$

- Because this is a **geometric series**, it converges precisely when $|-x^2| < 1$, that is $x^2 < 1$, or $|x| < 1$.
- Therefore the **interval of convergence** is $(-1, 1)$. $\square$

Theorem

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

The radii of convergence of the power series in Equations **(i)** and **(ii)** are both R.

Example

Express $\frac{1}{(1-x)^2}$ as a power series by differentiating $\frac{1}{(1-\mathbf{x})} = \sum_{\mathbf{n}=0}^{\infty} \mathbf{x}^{\mathbf{n}}$.
What is the radius of convergence?

Solution:

- Differentiate each side of the equation
  $\frac{1}{(1-\mathbf{x})} = 1 + \mathbf{x} + \mathbf{x}^2 + \mathbf{x}^3 + \cdots = \sum_{\mathbf{n}=0}^{\infty} \mathbf{x}^{\mathbf{n}}$.
- We get
$$\frac{1}{(1-\mathbf{x})^2} = 1 + 2\mathbf{x} + 3\mathbf{x}^2 + \cdots = \sum_{\mathbf{n}=1}^{\infty} \mathbf{n}\mathbf{x}^{\mathbf{n}-1}.$$
- Replace n by $n+1$ and write the answer as
$$\frac{1}{(1-\mathbf{x})^2} = \sum_{\mathbf{n}=0}^{\infty} (\mathbf{n}+1)\mathbf{x}^{\mathbf{n}}.$$
- By the theorem the radius of convergence of the differentiated series is the same, so, $\mathbf{R} = 1$.

$\square$

Example

Find a power series representation for $\ln(1-x)$ and its radius of convergence.

Solution:

- By **FTC** and the **geometric series**,
  $-\ln(\mathbf{1}-\mathbf{x}) = \int \frac{\mathbf{1}}{\mathbf{1}-\mathbf{x}}\,\mathbf{dx} = \int(\mathbf{1}+\mathbf{x}+\mathbf{x}^2+\cdots)\,\mathbf{dx}$
  $= \mathbf{x}+\frac{\mathbf{x}^2}{\mathbf{2}}+\frac{\mathbf{x}^3}{\mathbf{3}}+\cdots+\mathbf{C} = \sum_{\mathbf{n}=\mathbf{1}}^{\infty}\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}}+\mathbf{C} \qquad |\mathbf{x}|<\mathbf{1}.$
- To determine the value of C we put $x=0$ is this equation and obtain $-\ln(1-0)=C$.
- Thus $C=0$ and
  $\ln(\mathbf{1}-\mathbf{x}) = -\mathbf{x}-\frac{\mathbf{x}^2}{\mathbf{2}}-\frac{\mathbf{x}^3}{\mathbf{3}}-\cdots = -\sum_{\mathbf{n}=\mathbf{1}}^{\infty}\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}} \quad |\mathbf{x}|<\mathbf{1}.$
- The radius of convergence is the same as for the original series: $\mathbf{R}=\mathbf{1}$.

$\square$

Example

Find a power series representation for $f(x) = \tan^{-1} x$.

Solution:

- Since $f'(x) = \frac{1}{(1+x^2)}$, find the required series by integrating the **geometric series**
  $\frac{1}{(1+x^2)} = \frac{1}{(1-(-x^2))} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.

$$\tan^{-1} x = \int \frac{1}{1+x^2}\, dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\, dx$$
$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

- To find C we put $x = 0$ and obtain $C = \tan^{-1} 0 = 0$.
- Therefore
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

□

Theorem

If **f** has a power series representation (expansion) at **a**, that is, if

$$\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{n}=0}^{\infty} \mathbf{c}_{\mathbf{n}}(\mathbf{x}-\mathbf{a})^{\mathbf{n}} \qquad |\mathbf{x}-\mathbf{a}| < \mathbf{R},$$

then its coefficients are given by the formula

$$\mathbf{c}_{\mathbf{n}} = \frac{\mathbf{f}^{(\mathbf{n})}(\mathbf{a})}{\mathbf{n}!}.$$

Theorem

If $\mathbf{f}$ has a power series expansion at $\mathbf{a}$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .$$

- This power series is called the **Taylor series** of the function $\mathbf{f(x)}$ at $\mathbf{a}$ (or centered at $\mathbf{a}$).
- If $\mathbf{a=0}$, then the Taylor series of $\mathbf{f(x)}$ is called the **Maclaurin series** of $\mathbf{f(x)}$.

Example

Find the **Maclaurin series** of the function $f(x) = e^x$ and its radius of convergence.

Solution:

- If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n.
- Therefore, the **Taylor series** for f at 0 (that is, the **Maclaurin series**) is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

- To find the radius of convergence we let $a_n = \frac{x^n}{n!}$.
- Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1,$$

- So, by the **Ratio Test**, the series converges for all x and the radius of convergence is $\mathbf{R} = \infty$. $\square$

Definition

The partial sum

$$\mathbf{T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)(a)}}{i!}(x-a)^i}$$

$$\mathbf{= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

is called the **n-th-degree Taylor polynomial of f**.

Theorem

If $f(x) = T_n(x) + R_n(x)$, where T_n is the n-th-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < \mathbf{R}$, then f is equal to the sum of its **Taylor series** on the interval $|x-a| < \mathbf{R}$. R_n is called the **reminder.**

Theorem (Taylor's Inequality)

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the reminder $R_n(x)$ of the **Taylor series** satisfies the inequality

$$|\mathbf{R_n(x)}| \leq \frac{\mathbf{M}}{\mathbf{(n+1)!}} |\mathbf{x}-\mathbf{a}|^{\mathbf{n+1}} \quad \text{for} \quad |\mathbf{x}-\mathbf{a}| \leq \mathbf{d}.$$

In applying Taylor's inequality, the following fact is useful:

$$\lim_{\mathbf{n}\to\infty} \frac{\mathbf{x^n}}{\mathbf{n!}} = \mathbf{0} \quad \textbf{for every real number x}$$

Example

Prove that e^x is equal to the sum of its **Maclaurin series**.

Solution:

- If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$, then $f^{(n+1)}(x) = e^x$.
- If d is any positive number and $|x| \le d$, then $|f^{(n+1)}(x)| = e^x \le e^d$.
- So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that
$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \le d.$$
- The same constant $M = e^d$ works for every value of n.
- So we have
$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$
- It follows that $\lim_{n\to\infty} |R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$ for all values of x.
- Hence,
$$\mathbf{e^x} = \sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{x^n}}{\mathbf{n!}} \qquad \text{for all} \quad \mathbf{x}.$$

$\square$

Example

Find the **Taylor series** for $f(x) = e^x$ at $a = 2$.

Solution:

- We have $f^{(n)}(2) = e^2$ and so, putting $a = 2$ in the definition of a **Taylor series**, we get

$$\sum_{\mathbf{n}=0}^{\infty} \frac{\mathbf{f}^{(\mathbf{n})}(\mathbf{2})}{\mathbf{n}!}(\mathbf{x}-\mathbf{2})^{\mathbf{n}} = \sum_{\mathbf{n}=0}^{\infty} \frac{\mathbf{e}^{\mathbf{2}}}{\mathbf{n}!}(\mathbf{x}-\mathbf{2})^{\mathbf{n}}.$$

- Again it can be verified, that the radius of convergence is $\mathbf{R} = \infty$.
- As in the previous example we can verify that $\lim_{n\to\infty} R_n(x) = 0$, so

$$\mathbf{e}^{\mathbf{x}} = \sum_{\mathbf{n}=0}^{\infty} \frac{\mathbf{e}^{\mathbf{2}}}{\mathbf{n}!}(\mathbf{x}-\mathbf{2})^{\mathbf{n}} \quad \text{for all } \mathbf{x}$$

$\square$

Example

Find the **Maclaurin series** for $\sin x$ and prove that it represents $\sin x$ for all x.

Solution:

- Calculating we get: $f(x) = \sin x \quad f'(x) = \cos x$
  $f''(x) = -\sin x \quad f'''(x) = -\cos x \quad f^{(4)}(x) = \sin x;$
  $f(0) = 0 \;\; f'(0) = 1 \;\; f''(0) = 0 \;\; f'''(0) = -1 \quad f^{(4)}(0) = 0.$
- Since the derivatives repeat in a cycle of four, we can write the **Maclaurin series** as:
$$\mathbf{f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots}$$
$$\mathbf{= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$
- Since $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$, we know that $|f^{(n+1)}(x)| \leq 1$ for all x.
- So we can take $M = 1$ in Taylor's Inequality:
$$|R_n(x)| \leq \frac{M}{(n+1)}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!} \to 0 \qquad \textbf{as} \quad \mathbf{n \to \infty}.$$

Example

Find the **Maclaurin series** for $\cos x$.

Solution:

- We could proceed directly as in the previous example but it's easier to differentiate the **Maclaurin series** for $\sin x$:

$$\cos \mathbf{x} = \frac{\mathbf{d}}{\mathbf{dx}}(\sin \mathbf{x}) = \frac{\mathbf{d}}{\mathbf{dx}}\left(\mathbf{x} - \frac{\mathbf{x}^3}{\mathbf{3}!} + \frac{\mathbf{x}^5}{\mathbf{5}!} - \frac{\mathbf{x}^7}{\mathbf{7}!} + \cdots\right)$$

$$= \mathbf{1} - \frac{\mathbf{3x}^2}{\mathbf{3}!} + \frac{\mathbf{5x}^4}{\mathbf{5}!} - \frac{\mathbf{7x}^6}{\mathbf{7}!} + \cdots = \mathbf{1} - \frac{\mathbf{x}^2}{\mathbf{2}!} + \frac{\mathbf{x}^4}{\mathbf{4}!} - \frac{\mathbf{x}^6}{\mathbf{6}!} + \cdots$$

- Since the **Maclaurin series** for $\sin x$ converges for all x, the differentiated series for $\cos x$ also converges for all x.

□

Example

Find the **Maclaurin series** for the function $f(x) = x \cos x$.

Solution:

Instead of computing derivatives, it's easier to multiply the **Maclaurin series** for $\cos x$ by x:

$$\mathbf{x} \cos \mathbf{x} = \mathbf{x} \sum_{\mathbf{n}=0}^{\infty} (-\mathbf{1})^{\mathbf{n}} \frac{\mathbf{x}^{2\mathbf{n}}}{(2\mathbf{n})!} = \sum_{\mathbf{n}=0}^{\infty} (-\mathbf{1})^{\mathbf{n}} \frac{\mathbf{x}^{2\mathbf{n}+1}}{(2\mathbf{n})!}$$

□

Table of Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (-\infty, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [-1, 1]$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \qquad (0, 2)$$

Example

Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

Solution:

- Find the **Maclaurin series** for $f(x) = e^{-x^2}$.
- Replace x with $-x^2$ in the series for e^x given in the table of **Maclaurin series**.
- Thus, for all values of x,

$$\mathbf{e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots}$$

- Now integrate term by term:

$$\mathbf{\int e^{-x^2}\,dx = \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx}$$

$$\mathbf{= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots}$$

- This series converges for all x because the original series for e^{-x^2} converges for all x. $\square$

Example

Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Solution:

- Apply the MacLauren series for E^x:

$$\int_0^1 \mathbf{e}^{-\mathbf{x}^2}\,\mathbf{dx} = \int_0^1 1 - \frac{\mathbf{x}^2}{1!} + \frac{\mathbf{x}^4}{2!} - \frac{\mathbf{x}^6}{3!} + \cdots\ \mathbf{dx}$$

$$= \left[\mathbf{x} - \frac{\mathbf{x}^3}{3\cdot 1!} + \frac{\mathbf{x}^5}{5\cdot 2!} - \frac{\mathbf{x}^7}{7\cdot 3!} + \frac{\mathbf{x}^9}{9\cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots \approx 0.7475$$

- The **Alternating Series Estimation Theorem** shows that the error involved in this approximation is less than

$$\frac{1}{11\cdot 5!} = \frac{1}{1320} < 0.001$$

$\square$

Example

Approximate the function $$f(x) = \sqrt[3]{x}$$

by a Taylor polynomial of degree 2 at $a = 8$.

Solution:

- $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ $\quad f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ $\quad f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$ $\quad f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}$

  $f(8) = 2 \qquad f'(8) = \frac{1}{12} \qquad f''(8) = -\frac{1}{144}$

- Thus, the second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

- The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

□

Example

What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when -0.3 $\leq x \leq$ 0.3?

Solution:

- Notice that the **Maclaurin series**

  $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

  is alternating for all nonzero values of x, and are decreasing in size because $|x| < 1$, so we can use the **Alternating Series Estimation Theorem**.

- The error in approximating $\sin x$ by the first three terms of its **Maclaurin series** is at most

  $$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040}.$$

- If -0.3 $\leq x \leq$ 0.3, then $|x| \leq 0.3$, so the error is smaller than

  $$\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}.$$

$\square$

Example

For what values of x is this approximation accurate to within 0.00005?

Solution:

- The error will be smaller than 0.00005 if

$$\frac{|x|^7}{5040} < 0.00005$$

item Solving this inequality for x, we get

$$|x|^7 < 0.252 \quad \text{or} \quad |x| < (0.252)^{\frac{1}{7}} \approx 0.821$$

- So the given approximation is accurate to within 0.00005 when $|x| < 0.82$.

$\square$

Polar Coordinates $(\mathbf{r}, \theta)$ in the plane are described by

$$\mathbf{r} = \quad \textbf{distance} \quad \textbf{from} \quad \textbf{the} \quad \textbf{origin}$$

and $\quad \boldsymbol{\theta \in [0, 2\pi)}$ **is the counter-clockwise angle**.

We make the convention

$$(-\mathbf{r}, \theta) = (\mathbf{r}, \theta + \pi).$$

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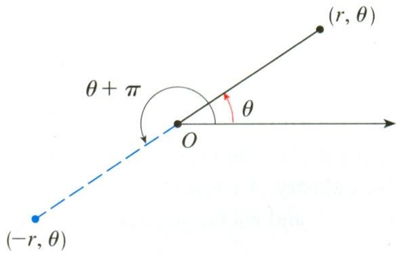


Example

Plot the points whose **polar** coordinates are given.

$$\text{(a)}\ \left(1, 5\frac{\pi}{4}\right) \quad \text{(b)}\ (2, 3\pi) \quad \text{(c)}\ \left(2, -2\frac{\pi}{3}\right) \quad \text{(d)}\ \left(-3, 3\frac{\pi}{4}\right).$$

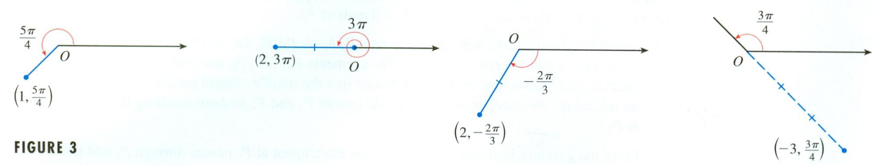
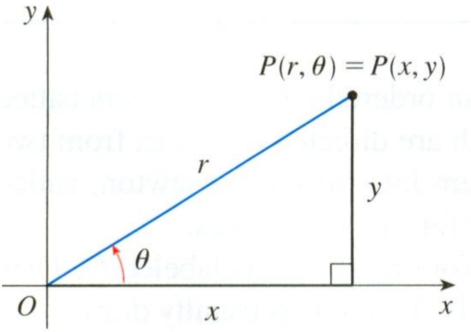


FIGURE 3

Solution:

In part (d) the point $\left(-3, 3\frac{\pi}{4}\right)$ is located three units from the pole in the fourth quadrant because the angle $3\frac{\pi}{4}$ is in the second quadrant and $r = -3$ is negative. $\square$



$$\boxed{\mathbf{x} = \mathbf{r}\cos\theta \qquad \mathbf{y} = \mathbf{r}\sin\theta}$$

$$\boxed{\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 \qquad \tan\theta = \frac{\mathbf{y}}{\mathbf{x}}}$$

Example

Convert the point $\left(2, \frac{\pi}{3}\right)$ from **polar** to **Cartesian** coordinates.

Solution:

Since $r = 2$ and $\theta = \frac{\pi}{3}$,

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in **Cartesian** coordinates. $\square$

Example

Represent the point with **Cartesian** coordinates $(1, -1)$ in terms of **polar** coordinates.

Solution:

- If we choose r to be positive, then

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\tan\theta = \frac{y}{x} = -1.$$

- Since the point $(1, -1)$ lies in the fourth quadrant, we choose $\theta = -\frac{\pi}{4}$ or $\theta = 7\,\frac{\pi}{4}$.
- Thus, one possible answer is $\left(\sqrt{2}, -\frac{\pi}{4}\right)$; another is $\left(\sqrt{2}, 7\,\frac{\pi}{4}\right)$.

□

Polar Coordinates

- The coordinates of a point $(x, y) \in \mathbf{R^3}$ can be described by the equations:
$$\mathbf{x} = \mathbf{r}\cos(\theta) \qquad \mathbf{y} = \mathbf{r}\sin(\theta), \tag{1}$$
where $r = \sqrt{x^2 + y^2}$ is the distance from the origin and $(\frac{x}{r}, \frac{y}{r})$ is $(\cos(\theta), \sin(\theta))$ on the unit circle.

- Note that $r \geq 0$ and θ can be taken to lie in the interval $[0, 2\pi)$.

- To find r and θ when x and y are known, we use the equations:
$$\mathbf{r^2} = \mathbf{x^2} + \mathbf{y^2} \qquad \tan(\theta) = \frac{\mathbf{y}}{\mathbf{x}}. \tag{2}$$

Definition

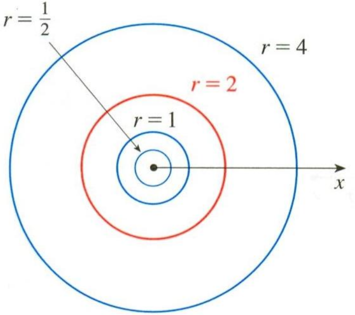
The **graph of a polar equation** $r = f(\theta)$, or more generally $\mathbf{F}(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example

What curve is represented by the **polar equation $\mathbf{r = 2}$**?

Solution:

- Curve consists of points $(\mathbf{r}, \theta)$ with $r = 2$.
- Since r represents the distance from the point to the origin O, the curve $r = 2$ represents the circle with center O and radius 2.
- The equation $r = a$ represents a circle with center O and radius $|a|$.

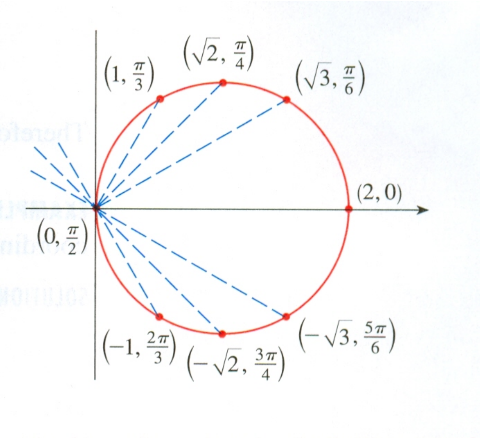


Example

Sketch the curve with **polar equation** $\mathbf{r} = \mathbf{2}\cos\theta$.

Solution:

Plotting points we find what seems to be a circle:



Example

Find the **Cartesian** coordinates for $r = 2\cos\theta$.

Proof.

Solution Since $x = r\cos\theta$, the equation $r = 2\cos\theta$ becomes $r = \frac{2x}{r}$ or

$$2x = r^2 = x^2 + y^2$$

or

$$x^2 - 2x + y^2 = 0$$

or

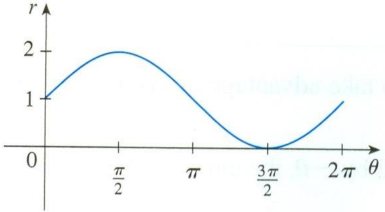
$$(x-1)^2 + y^2 = 1.$$

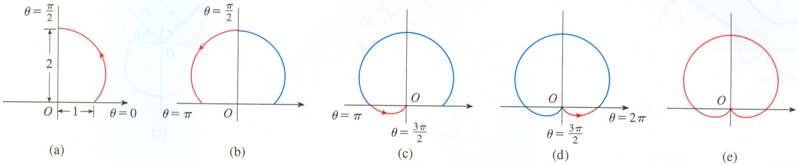
This is the equation of a **circle** of radius 1 centered at $(1, 0)$. $\square$

Example

Sketch the curve $r = 1 + \sin\theta$. This curve is called a **cardioid**.

Solution

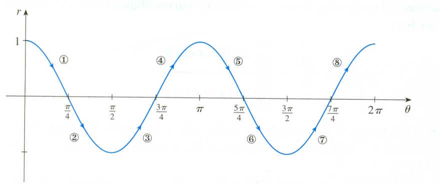


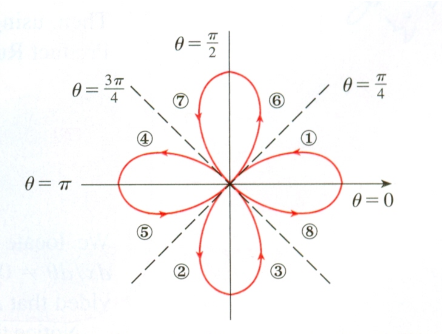


Example

Sketch the curve $r = \cos 2\theta$. This curve is called a **four-leaved rose**.

Solution





Tangents to **Polar Curves**

- To find a **tangent line** to a polar curve $\mathbf{r} = \mathbf{f}(\theta)$ we regard θ as a parameter and write its **parametric equations** as

$$\mathbf{x} = \mathbf{r}\cos\theta = \mathbf{f}(\theta)\cos\theta \qquad \mathbf{y} = \mathbf{r}\sin\theta = \mathbf{f}(\theta)\sin\theta$$

- Then, using the method for finding slopes of parametric curves and the Product Rule, we have

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\frac{\mathbf{dy}}{\mathbf{d}\theta}}{\frac{\mathbf{dx}}{\mathbf{d}\theta}} = \frac{\frac{\mathbf{dr}}{\mathbf{d}\theta}\sin\theta + \mathbf{r}\cos\theta}{\frac{\mathbf{dr}}{\mathbf{d}\theta}\cos\theta - \mathbf{r}\sin\theta}$$

- We locate **horizontal tangents** by finding the points where $\frac{\mathbf{dy}}{\mathbf{d}\theta} = \mathbf{0}$ (provided that $\frac{dx}{d\theta} \neq 0$.)
- Likewise, we locate **vertical tangents** at the points where $\frac{\mathbf{dx}}{\mathbf{d}\theta} = \mathbf{0}$ (provided that $\frac{dy}{d\theta} \neq 0$).

Example

For the **cardioid** $\mathbf{r = 1 + \sin\theta}$ find the **slope** of the tangent line when $\theta = \frac{\pi}{3}$.

Solution:

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\frac{\mathbf{dr}}{\mathbf{d}\theta}\sin\theta + \mathbf{r}\cos\theta}{\frac{\mathbf{dr}}{\mathbf{d}\theta}\cos\theta - \mathbf{r}\sin\theta} = \frac{\cos\theta\sin\theta + (\mathbf{1}+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (\mathbf{1}+\sin\theta)\sin\theta}$$

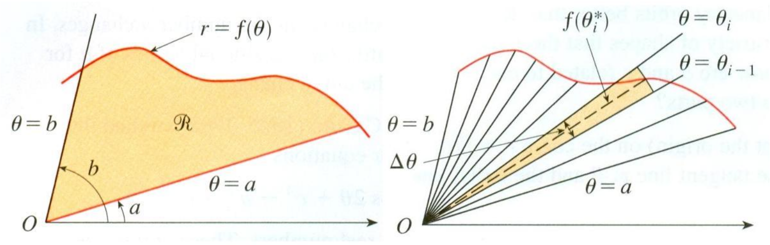
$$= \frac{\cos\theta(\mathbf{1}+\mathbf{2}\sin\theta)}{\mathbf{1}-\mathbf{2}\sin^{\mathbf{2}}\theta - \sin\theta} = \frac{\cos\theta(\mathbf{1}+\mathbf{2}\sin\theta)}{(\mathbf{1}+\sin\theta)(\mathbf{1}-\mathbf{2}\sin\theta)}$$

- The **slope of the tangent** at the point where $\theta = \frac{\pi}{\mathbf{3}}$ is

$$\left.\frac{\mathbf{dy}}{\mathbf{dx}}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos\left(\frac{\pi}{\mathbf{3}}\right)\left(\mathbf{1}+\mathbf{2}\sin\left(\frac{\pi}{\mathbf{3}}\right)\right)}{\left(\mathbf{1}+\sin\left(\frac{\pi}{\mathbf{3}}\right)\right)\left(\mathbf{1}-\mathbf{2}\sin\left(\frac{\pi}{\mathbf{3}}\right)\right)}$$

$$= \frac{\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\sqrt{\mathbf{3}})}{(\mathbf{1}+\frac{\sqrt{\mathbf{3}}}{\mathbf{2}})(\mathbf{1}-\sqrt{\mathbf{3}})} = \frac{\mathbf{1}+\sqrt{\mathbf{3}}}{(\mathbf{2}+\sqrt{\mathbf{3}})(\mathbf{1}-\sqrt{\mathbf{3}})} = \frac{\mathbf{1}+\sqrt{\mathbf{3}}}{-\mathbf{1}-\sqrt{\mathbf{3}}} = -\mathbf{1}$$

$\square$



- The **area** of a region "under" a polar function $\mathbf{r} = \mathbf{f}(\theta)$ is described by either of the following formulas.
- These formulas arise from the fact that the area of a $\theta_1 \leq \theta \leq \theta_2$ portion of a circle of radius r is given by $\frac{1}{2}(\theta_2 - \theta_1)r^2$.

$$\boxed{\mathbf{A} = \int_{\mathbf{a}}^{\mathbf{b}} \frac{\mathbf{1}}{\mathbf{2}}[\mathbf{f}(\theta)]^{\mathbf{2}}\, \mathbf{d}\theta,}$$

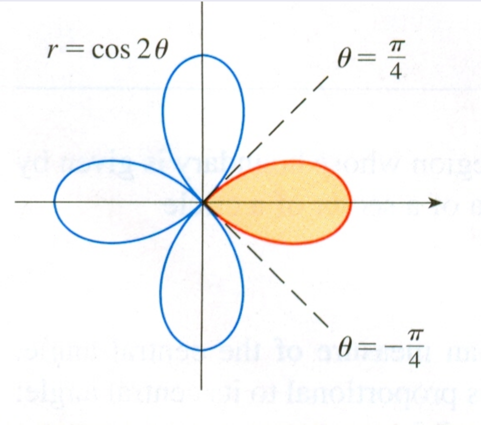
$$\boxed{\mathbf{A} = \int_{\mathbf{a}}^{\mathbf{b}} \frac{\mathbf{1}}{\mathbf{2}}\mathbf{r}^{\mathbf{2}}\, \mathbf{d}\theta,}$$

Example

Find the **area** enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution:

First recall the picture of this curve:



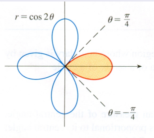
$\square$

Example

Find the **area** enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution:

- First recall the picture of this curve:



- By our **area formulas**,

$$\textbf{Area} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 4\theta) \, d\theta = \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}.$$

$\square$

Speed and length

Definition

- The **velocity vector** of a curve
$$\mathbf{C}(\mathbf{t}) = (\mathbf{x}(\mathbf{t}), \mathbf{y}(\mathbf{t}))$$
is $\mathbf{C}'(\mathbf{t}) = (\mathbf{x}'(\mathbf{t}), \mathbf{y}'(\mathbf{t}))$.
- The **speed** of $\mathbf{C}(\mathbf{t})$ is
$$\mathbf{s}(\mathbf{t}) = |\mathbf{C}'(\mathbf{t})| = \sqrt{(\mathbf{x}'(\mathbf{t}))^2 + (\mathbf{y}'(\mathbf{t}))^2}.$$
- Since the integral of the speed is the distance traveled or **length** for $\mathbf{C} \colon [\mathbf{a}, \mathbf{b}] \to \mathbb{R}^2$,
$$\textbf{Length of a curve}(\mathbf{C}) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{s}(\mathbf{t})\, \mathbf{dt}$$
$$= \int_{\mathbf{a}}^{\mathbf{b}} \sqrt{(\mathbf{x}'(\mathbf{t}))^2 + (\mathbf{y}'(\mathbf{t}))^2}\, \mathbf{dt}.$$

Example

As the parameter t increases forever, starting at $t = 0$, the curve with parametric equations $x = e^{-t}\cos t$, $y = e^{-t}\sin t$
spirals inward toward the origin, getting ever closer to the origin (but never actually reaching) as $t \to \infty$. Find the length of this spiral curve.

Solution:

- The tangent vector to the curve $\mathbf{c(t) = (x(t), y(t)) = (e^{-t}\cos t, e^{-t}\sin t)}$ is $\mathbf{c'(t) = (x'(t), y'(t))}$.
- So, $\mathbf{c'(t) = (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos t) = -e^{-t}(\cos t + \sin t, \sin t - \cos t)}$.
- The speed $\mathbf{s(t)}$ of $\mathbf{c(t)}$ is $|\mathbf{c'(t)}|$ which is equal to $\sqrt{\mathbf{(x'(t))^2 + (y'(t))^2}}$.
- The **length** is the integral of the speed:

$$\int_0^\infty \sqrt{\mathbf{e^{-2t}(\cos^2 t + \sin^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t\sin t)dt}}$$

$$= \int_0^\infty \mathbf{e^{-t}\sqrt{2}\,dt} = \lim_{\mathbf{t\to\infty}} -\sqrt{\mathbf{2}}\mathbf{e^{-t}}\Big|_{\mathbf{0}}^{\mathbf{t}} = \sqrt{\mathbf{2}}. \qquad \square$$

Length formula

- In **polar** coordinates $\mathbf{x} = \mathbf{r}\cos\theta$, $\mathbf{y} = \mathbf{r}\sin\theta$. Then

$$\frac{\mathbf{dx}}{\mathbf{d}\theta} = \frac{\mathbf{dr}}{\mathbf{d}\theta}\cos\theta - \mathbf{r}\sin\theta \qquad \frac{\mathbf{dy}}{\mathbf{d}\theta} = \frac{\mathbf{dr}}{\mathbf{d}\theta}\sin\theta + \mathbf{r}\cos\theta.$$

- Using $\cos^2\theta + \sin^2\theta = 1$, we get

$$\begin{aligned}\left(\frac{\mathbf{dx}}{\mathbf{d}\theta}\right)^{\mathbf{2}} + \left(\frac{\mathbf{dy}}{\mathbf{d}\theta}\right)^{\mathbf{2}} &= \left(\frac{\mathbf{dr}}{\mathbf{d}\theta}\right)^{\mathbf{2}}\cos^{\mathbf{2}}\theta - \mathbf{2r}\frac{\mathbf{dr}}{\mathbf{d}\theta}\cos\theta\sin\theta + \mathbf{r}^{\mathbf{2}}\sin^{\mathbf{2}}\theta \\ &\quad + \left(\frac{\mathbf{dr}}{\mathbf{d}\theta}\right)^{\mathbf{2}}\sin^{\mathbf{2}}\theta + \mathbf{2r}\frac{\mathbf{dr}}{\mathbf{d}\theta}\sin\theta\cos\theta + \mathbf{r}^{\mathbf{2}}\sin^{\mathbf{2}}\theta \\ &= \left(\frac{\mathbf{dr}}{\mathbf{d}\theta}\right)^{\mathbf{2}} + \mathbf{r}^{\mathbf{2}}.\end{aligned}$$

- Thus the **length L** of a polar curve $\mathbf{r} = \mathbf{f}(\theta)$, $a \le \theta \le b$, is:

$$\mathbf{L} = \int_{\mathbf{a}}^{\mathbf{b}} \sqrt{\mathbf{r}^{\mathbf{2}} + \left(\frac{\mathbf{dr}}{\mathbf{d}\theta}\right)^{\mathbf{2}}}\; \mathbf{d}\theta.$$