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Value Function Approximation

Data Intelligence and Social Computing Lab (DISC)

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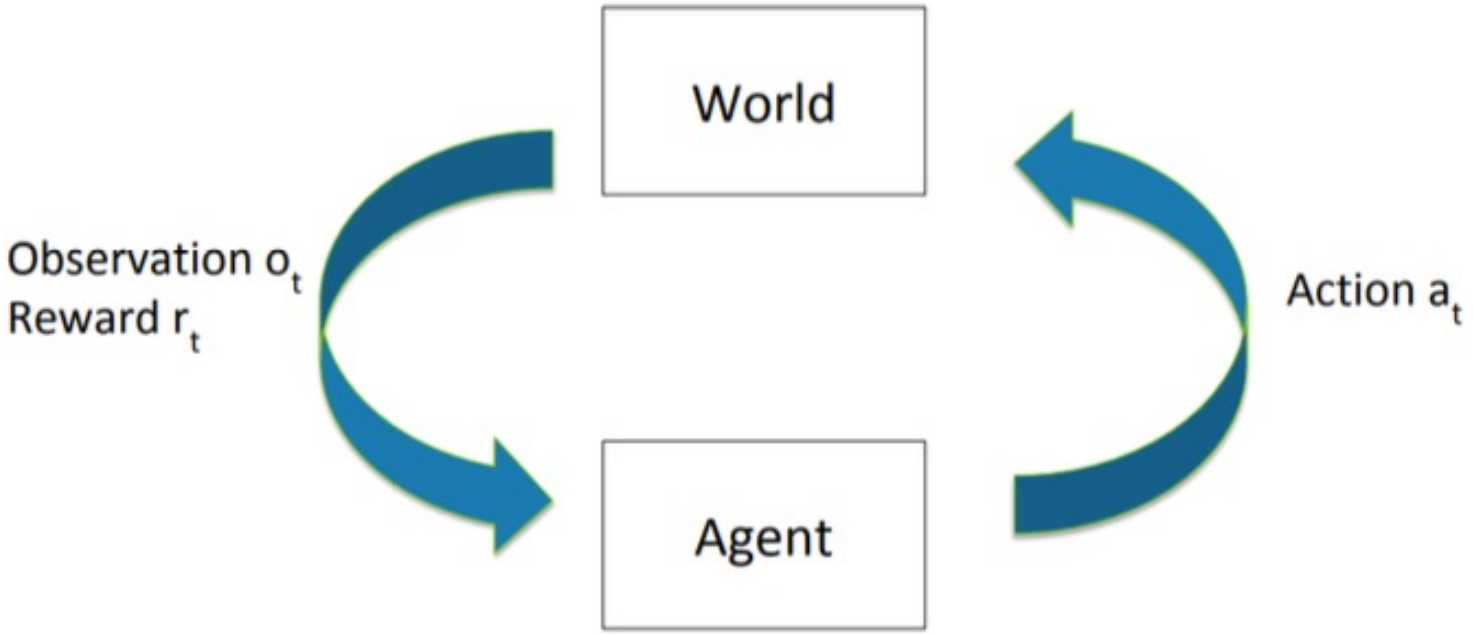
Sequential Decision Making

- An agent makes a sequence of actions: $\{a_t\}$
- Observes a sequence of observations: $\{o_t\}$ i.e., $\{s_t\}$
- Receives a sequence of rewards: $\{r_t\}$
- Trajectory / episodes: $s_1, a_1, r_1, s_2, a_2, r_2, s_3, a_3, r_3$ …



Markov Decision Processes (MDPs)

- Markov Process + Reward + Action
- S is a (finite) set of Markov states s $\in$ S
- A is a (finite) set of actions a $\in$ A
- P is dynamics / transition-model for each action,

$$P(s_{t+1} = s' | s_t = s, a_t = a)$$

- R is a reward function

$$R(s_t = s, a_t = a) = E[r_t | s_t = s, a_t = a]$$

- γ is discount factor $\gamma \in [0,1]$
- MDP is a tuple: (S, A, P, R, γ)

Important Elements for MDPs

- Trials: Interact with the environment to collect trials

$$\tau = s_1, a_1, r_1, s_2, a_2, r_2, \dots, s_T$$

- Return: from time step t

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$$

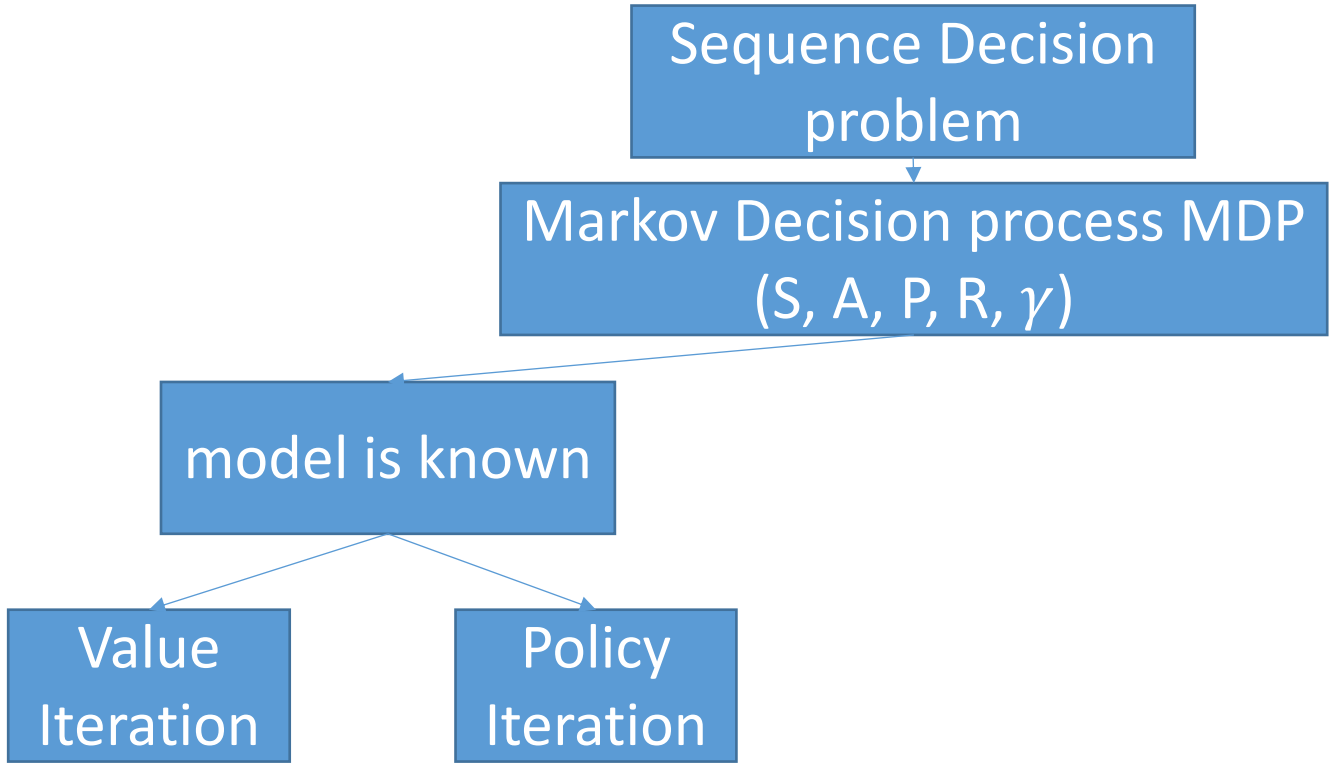
- Value Function: expected return from starting in state s

$$V(s) = E[G_t | s_t = s] = E[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s]$$

- Policy : a distribution over actions / a one-to-one mapping

$$\pi(s) = \pi(a|s) = P(a_t = a | s_t = s)$$

Summary of Tabular RL learning



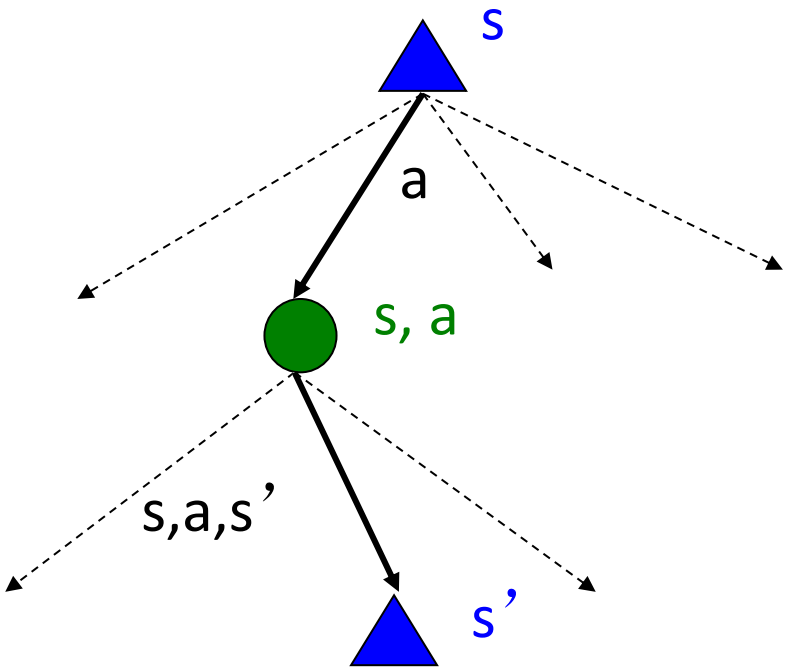
Bellman equations for solving MDPs

- Bellman equation:

$$V^*(s) = \max_{a \in A(s)} Q^*(s,a)$$

$$Q^*(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a) V^*(s')$$

$$V^*(s) = \max_{a \in A(s)} \{R(s,a) + \gamma \sum_{s'} P(s'|s,a) V^*(s')\}$$



Value Iteration (Bellman Update Equation)

- Start with $V_0(s) = 0$
- Given vector of $V_k(s)$ values:

$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \{R(s,a) + \gamma \sum_{s'} P(s'|s,a) V_k(s')\}$$

- Repeat until convergence

Policy Iteration for solving MDPs

- Step 1: Policy evaluation: calculate values for **fixed policy** until convergence.

$$V^{\pi}(s) \leftarrow R(s,a) + \gamma \sum_{s'} P(s'|s,a)V^{\pi}(s')$$

- Step 2: Policy improvement: **update policy** using one-step look-ahead with **current value**

$$\pi^*(s) = \underset{a \in A(s)}{\operatorname{argmax}}\{R(s,a) + \sum_{s'} P(s'|s,a)V^*(s')\}$$

- Repeat steps until policy converges

When models are unknown

- Assume a Markov decision process (MDP):
  - A set of states $s \in S$
  - A set of actions (per state) A
- Looking for a policy $\pi(s)$

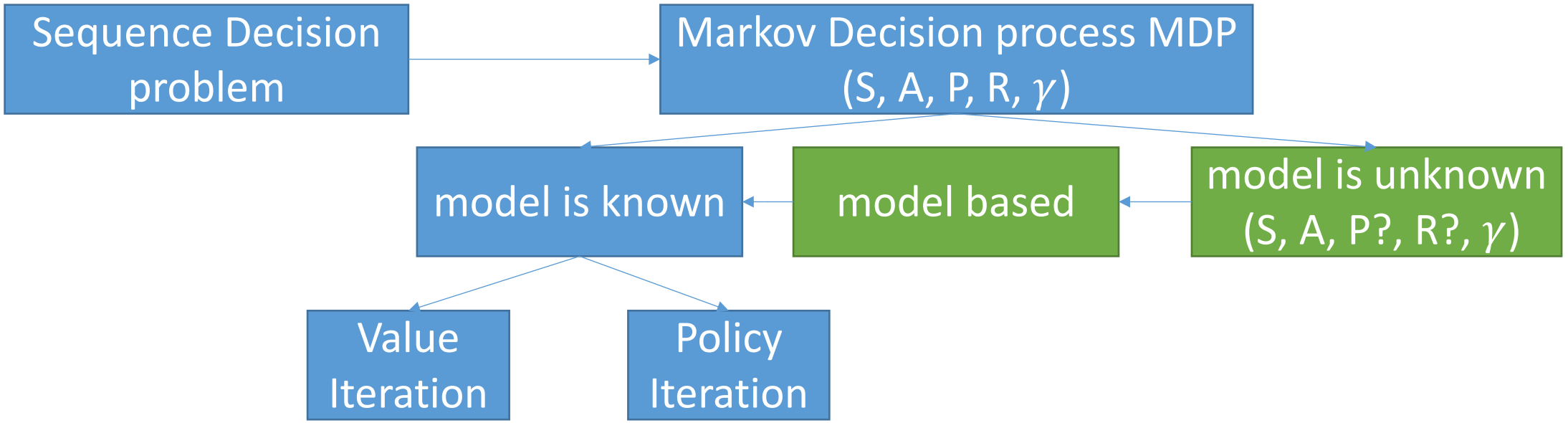
- Try actions and states out to learn, i.e. collect trials

$$\tau = s_1, a_1, r_1, s_2, a_2, r_2, \dots, s_T$$

- The return of a trial. $\gamma \in [0,1]$ is the discount factor

$$G(\tau) = \sum_{t=0}^{T-1} \gamma^t r_{t+1}$$

Summary of Tabular RL learning



Model-Based Learning

- Model-Based Idea:
  - Learn an approximate model based on trials
  - Solve for values as if the learned model were correct

- Step 1: Learn empirical MDP model
  - Count outcomes s’ for each (s, a)
  - Normalize to give an estimate of $P(s_{t+1} = s' | s_t = s, a_t = a)$
  - Discover each $R(s_t = s, a_t = a)$ when we experience (s, a, s’)

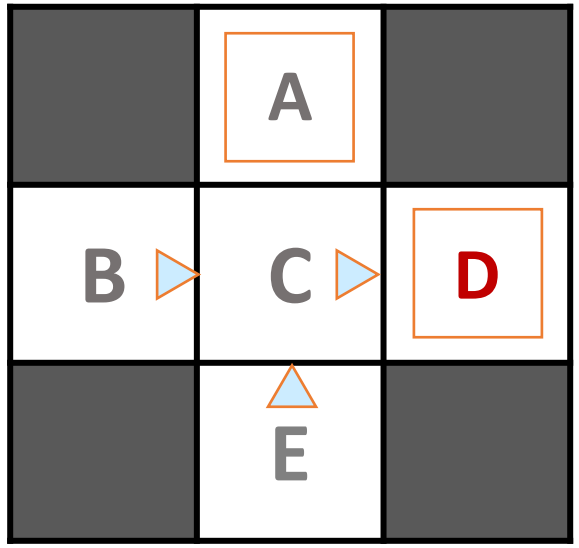
- Step 2: Solve the learned MDP
  - For example, use value iteration

Example: Model-Based Learning

Input Policy π: {(B, east), (C, east), (E, north)}
End states: A and D
Start states: B and E
Assume: $\gamma = 1$

Observed Episodes (Training)

T1: { (B, east, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T2: { (B, east, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T3: { (E, north, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T4: { (E, north, C, -1), (C, east, A, -1), (A, exit, x, -10)}



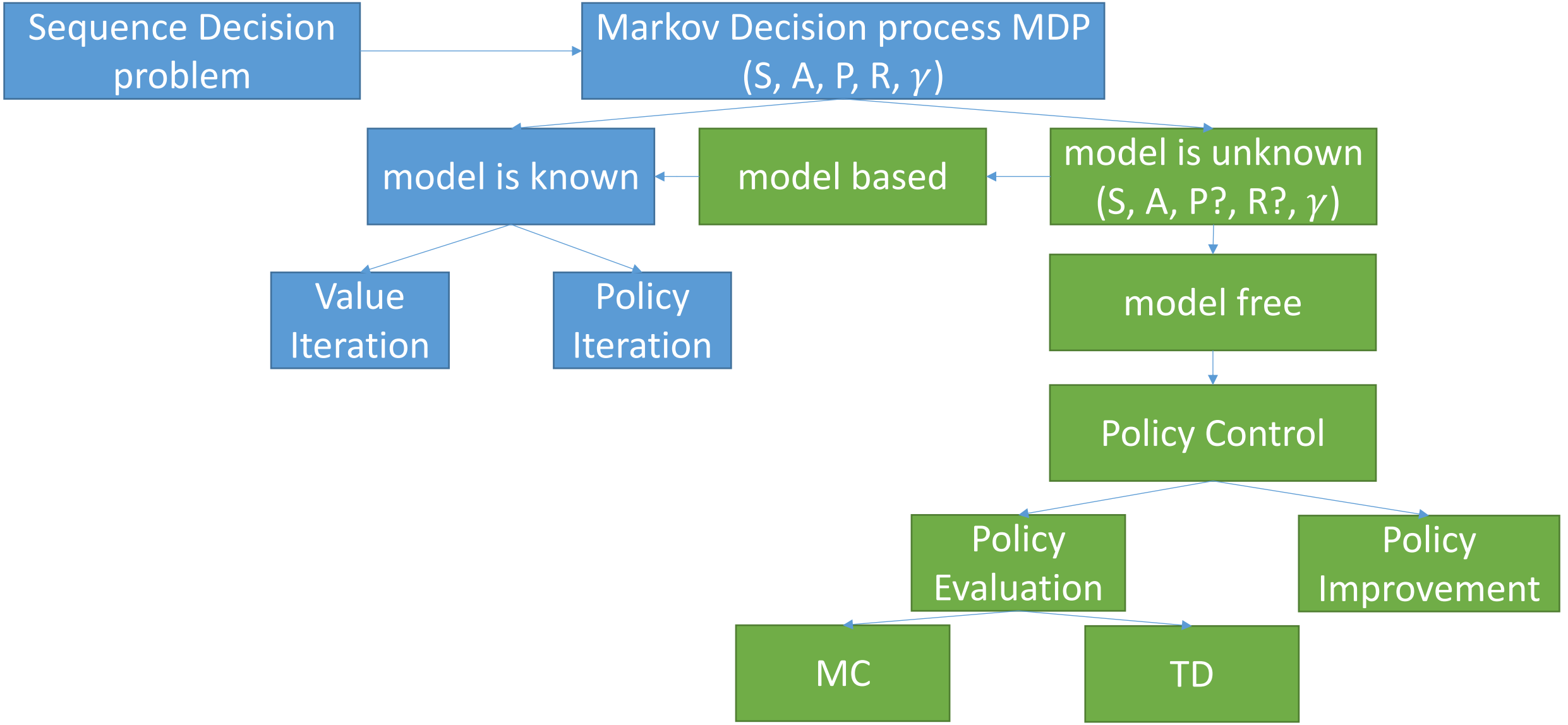
Transition Model

(s, a)	A	B	C	D	E	x	Total #
(A, exit)						1.0	1
(B, east)			1.0				2
(C, east)	.25			.75			4
(D, exit)						1.0	3
(E, north)			1.0				2

Reward Model

(s, a)	reward	Total #	utility
(A, exit)	-10	1	-10
(B, east)	-1, -1	2	-1
(C, east)	-1, -1, -1, -1	4	-1
(D, exit)	10, 10, 10	3	10
(E, north)	-1, -1	2	-1

Summary of Tabular RL learning



Policy Evaluation

- Input: a fixed policy $\pi(s)$
- You don't know the transitions $P(s' | s, a)$
- You don't know the rewards $R(s, a, s')$
- Goal: learn state values $V^{\pi}(S)$

- In this case:
  - No choice about what actions to take
  - Just execute the policy and learn from experience

Monte Carlo (MC), i.e., Direct Evaluation

- Given episodes, $s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_T$

- Goal: Compute values for each state under π

$$V^{\pi}(s) = E_{\pi}[G_t | S_t = s]$$

- Idea: Average together observed sample values

$$V^{\pi}(s) = \frac{1}{N} \left(\sum_{i=1}^{n} v_i^{\pi} \quad (s)\right)$$

$$v_i^{\pi} \quad (s) = G_t | S_t = \text{s}$$

Incremental Monto Carlo (MC)

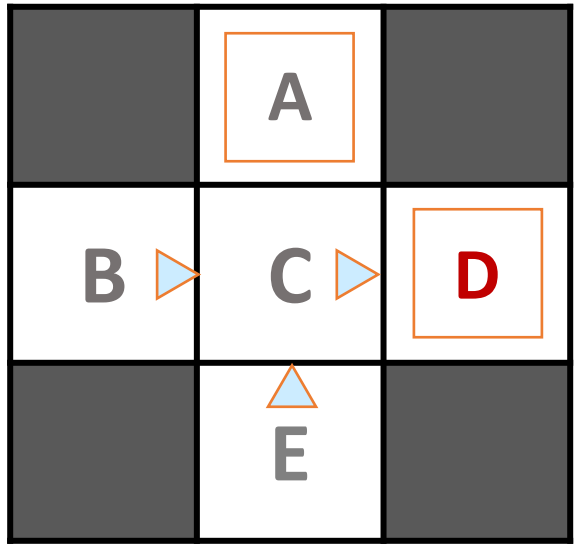
- After each episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots$ as return from time step t onwards in i_{th} episoder
- For state s visited at time step t in episode i
  - $G_i(s)$ is the total return obtained for state s after episode i.
  - Increment counter of total first visits: $N_i(s) = N_{i-1}(s) + 1$
  - Update estimate:

$$V_i^{\pi}(s) = V_{i-1}^{\pi}(s)\left(\frac{N_i(s)-1}{N_i(s)} + \frac{G_{i,t}}{N(s)}\right) = V_{i-1}^{\pi}(s) + \frac{1}{N_i(s)}(G_{i,t} - V_{i-1}^{\pi}(s))$$

Example: Monte Carlo

Input Policy π: {(B, east), (C, east), (E, north)}
End states: A and D
Start states: B and E
Assume: $\gamma = 1$

Observed Episodes
T1: { (B, east, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T2: { (B, east, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T3: { (E, north, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T4: { (E, north, C, -1), (C, east, A, -1), (A, exit, x, -10)}



T1:

$$v^{\pi}(B) = G_{1,1} = (-1) + \gamma(-1) + \gamma * \gamma * (+10) = 8$$
$$v^{\pi}(C) = G_{1,2} = (-1) + \gamma * (+10) = 9$$
$$v^{\pi}(D) = G_{1,3} = +10 = 10$$

T2:

$$v^{\pi}(B) = G_{2,1} = 8; v^{\pi}(C) = G_{2,2} = 9; v^{\pi}(D) = G_{2,3} = 10$$

T3:

$$v^{\pi}(E) = 8; v^{\pi}(C) = 9; v^{\pi}(D) = 10$$

T4:

$$v^{\pi}(E) = -12; v^{\pi}(C) = -11; v^{\pi}(A) = -10$$

	$V^{\pi}(B)$	$V^{\pi}(C)$	$V^{\pi}(D)$	$V^{\pi}(E)$	$V^{\pi}(A)$
T1 updating	8 / 1	9 / 1	10 / 1		
T2 updating	(8+8)/2 = 8	(9+9) / 2 = 9	(10+10)/2 = 10		
T3 updating		(9+9+9)/3 = 9	(10*3)/3=10	8 / 1 = 8	
T4 updating		(9*3-11)/4 = 4		(8-12)/2=-2	-10/1 = -10
Total #	2	4	3	2	1

Temporal Difference Learning

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- Simplest TD learning: update value towards estimated value

$$V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha(\underbrace{[r_t + \gamma V^{\pi}(s_{t+1})]}_{\text{TD target}} - V^{\pi}(s_t))$$

- TD error:

$$\delta_t = r_t + \gamma V^{\pi}(s_{t+1}) - V^{\pi}(s_t)$$

- Can immediately update value estimate after (s, a, r, s') tuple

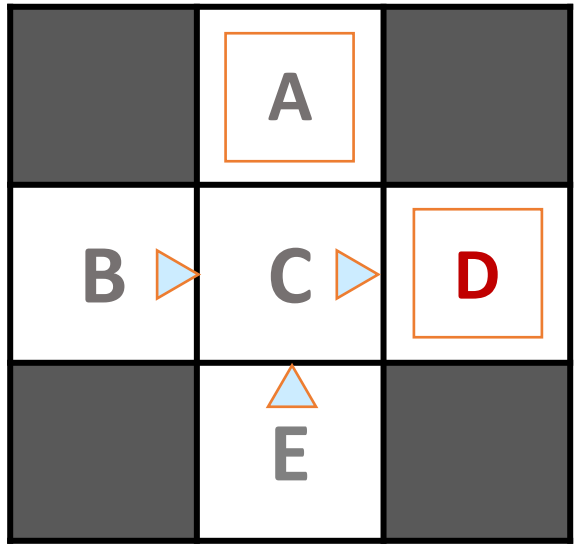
"If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be temporal-difference (TD) learning." – Sutton and Barto 2017

Example: Temporal Difference Policy Evaluation

Input Policy π: {(B, east), (C, east), (E, north)}
End states: A and D
Start states: B and E
Assume: $\gamma = 1$
Learning rate: alpha = 0.5
States value initialized as 0 (including x)

Observed Episodes (Training)

T1: { (B, east, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T2: { (B, east, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T3: { (E, north, C, -1), (C, east, D, -1), (D, exit, x, +10)}
T4: { (E, north, C, -1), (C, east, A, -1), (A, exit, x, -10)}



Update for T1

	$V^{\pi}(B)$	$V^{\pi}(C)$	$V^{\pi}(D)$	$V^{\pi}(E)$	$V^{\pi}(A)$
Initial	0	0	0	0	0
(B, east, C, -1)	-0.5	0	0	0	0
(C, east, D, -1)	-0.5	-0.5	0	0	0
(D, exit, x, +10)	-0.5	-0.5	5	0	0

$$V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha(\underbrace{[r_t + \gamma V^{\pi}(s_{t+1})]}_{\text{TD target}} - V^{\pi}(s_t))$$

Temporal Difference VS Monte Carlo

- With episode $s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_T$
- **Both update the state value by samples**

$$V^{\pi}(s) = V^{\pi}(s) + \alpha(v^{\pi}(s) - V^{\pi}(s))$$

- **Monte Carlo**

$$v^{\pi}(s) = \sum_{k=0}^{T-1} \gamma^k r_{t+k} | S_t = s$$

- **Temporal Difference**

$$v^{\pi}(s) = r_t + \gamma V^{\pi}(S_{t+1}) | S_t = s$$

- In TD, use (s, a, r s') to update V(S)
  - O(1) operation per update; in an episode of length L, O(L)
- In MC have to wait till episode finishes, then also O(L)
- TD exploits Markov structure

Model-free Control

- You don't know the transitions $P(s'|\ s,a)$
- You don't know the rewards $R(s,a)$
- You choose the actions now
- Goal: learn the optimal policy / values：　| S | X | A | values

- In this case:
  - **On-policy** training VS **off-policy** training
  - Fundamental tradeoff: **exploration** vs. **exploitation**

On and Off-Policy Learning

- **Behavior policy π_2** : gather experience from
- **Estimated policy π_1** : the one you want to estimate the value of

- On-policy learning ($\pi_1 = \pi_2$)
  - Learn to estimate a policy from experience obtained from following that policy
- Off-policy learning ($\pi_1 \neq \pi_2$)
  - Learn to estimate a policy using experience gathered from following a different policy
  - ✓ Sometimes trying actions out is costly
  - ✓ Would like to use historical data from old policies

Sample-based Policy Evaluation

- Sample Episodes following a policy $\boldsymbol{\pi_2}$
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_T$

- Compute values for each state under $\boldsymbol{\pi_1}$
  - $Q^{\boldsymbol{\pi_1}}(s, a)$

- **Behavior policy $\boldsymbol{\pi_2}$** : gather experience from
- **Estimated policy $\boldsymbol{\pi_1}$** : the one you want to estimate the value of

Monte Carlo (MC) Off Policy Evaluation

- Aim: estimate value of policy π_1, $V^{\pi_1}(s)$, given episodes generated under behavior policy π_2
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π_2
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^{\pi}(s) = \mathbb{E}_{\pi}[G_t | s_t = s]$
- Have data from a different policy, behavior policy π_2

π_2
Sample generation

π_1
Value update

- Note that, not all G_t can be used in this case.

Exploration with ε-greedy policy

- Let |A| be the number of actions
- Then an **ε-greedy policy** w.r.t. a state-value $Q^{\pi}(s,a)$ is

$$\pi^{\epsilon}(s) = \begin{cases} a, & with\ probabiliy\ of\ \frac{\varepsilon}{|A|} \\ \underset{a}{\operatorname{argmax}}\, Q^{\pi}(s,a), & with\ probabiliy\ of\ 1-\varepsilon \end{cases}$$

Monte Carlo for On-policy Policy Iteration

Initialize $Q(s,a) = 0, N(s,a) = 0\ \forall (s,a)$, Set $\epsilon = 1,\ k = 1$

$\pi_k = \epsilon$-greedy(Q) // Create initial ϵ-greedy policy

loop

  Sample k-th episode $(s_{k,1}, a_{k,1}, r_{k,1}, s_{k,2}, \ldots, s_{k,T})$ given π_k

  $G_{k,t} = r_{k,t} + \gamma r_{k,t+1} + \gamma^2 r_{k,t+2} + \cdots \gamma^{T_i - 1} r_{k,T_i}$

  for $t = 1, \ldots, T$ **do**

    if First visit to (s,a) in episode k **then**

      $N(s,a) = N(s,a) + 1$

      $Q(s_t, a_t) = Q(s_t, a_t) + \frac{1}{N(s,a)}(G_{k,t} - Q(s_t, a_t))$

    end if

  end for

  $k = k + 1,\ \epsilon = 1/k$

  $\pi_k = \epsilon$-greedy(Q) // Policy improvement

end loop

π_k **Sample generation**

π_k **Value update**

π_{k+1} **Policy Improvement**

Policy Evaluation without model using TD

- MC: Sample Episodes
  - $s_1, a_1, r_1, s_2, a_2, r_2, \dots, s_T$
- TD: Sample Transitions
  - $(s_1, a_1, r_1, s_2, a_2), (s_2, a_2, r_2, s_2, a_2), \dots, (s_{T-1}, a_{T-1}, r_{T-1}, s_T, a_T)$
- Compute values for each state
  - $Q^{\boldsymbol{\pi}} \quad (s, a)$
- TD based policy evaluation updates values with transition samples.

Q-Learning: Learning towards Optimal Q(s,a)

- Key idea: Maintain state-action Q estimates and use the value of the best future action for bootstrap update

- SARSA: $(s_1, a_1, r_1, s_2, a_2)$

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(\boxed{(r_t + \gamma Q(s_{t+1}, a_{t+1}))} - Q(s_t, a_t))$$

- Q-learning: $(s_1, a_1, r_1, s_2, argmax(s_2))$

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(\boxed{(r_t + \gamma \max_{a'} Q(s_{t+1}, a'))} - Q(s_t, a_t))$$

Refresh your knowledge

- In tabular MDPs, if using a decision policy that visits all states an infinite number of times, and in each state **randomly selects** an action.

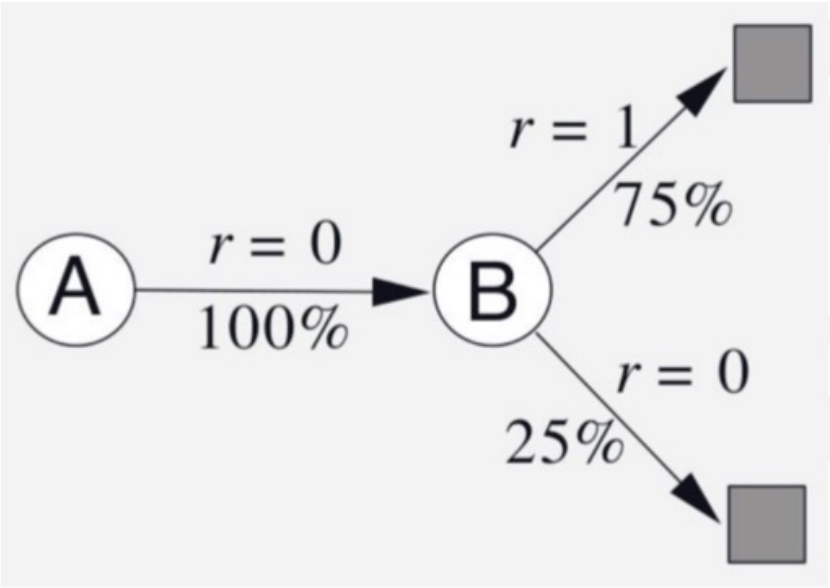
Experience Replay Does this	T or F
Q-learning will converge to the optimal Q-values	
SARSA will converge to the optimal Q-values	
Q-learning is learning off-policy	
SARSA is learning off-policy	

Batch MC and TD

- Batch (Offline) solution for finite dataset
  - Given set of K episodes
  - Repeatedly sample an episode from K
  - Apply MC or TD to the sampled episode

Batch MC and TD: AB Example

- Two states A,B with $\gamma = 1$
- Given 8 episodes of experience:
  - A, 0, B, 0
  - B, 1 (observed 6 times)
  - B, 0
- Run the 8 episodes once：
  - $V^{MC}(A) =$ $;V^{MC}(B) =$
  - $V^{TD}(A) =$ $;V^{TD}(B) =$
- Run the 8 episodes infinite number of times：
  - $V^{TD}(A) =$ $;V^{TD}(B) =$
  - $V^{MC}(A) =$ $;V^{MC}(B) =$



Outline

- States Generalization

Generalizing Across States

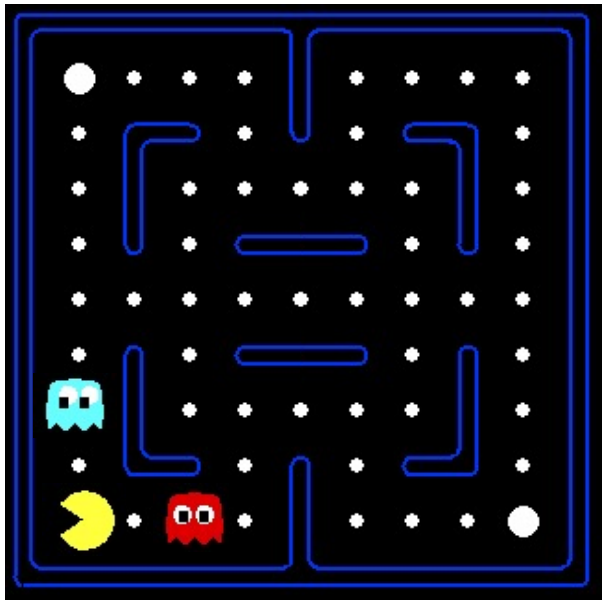
- Tabular RL keeps a table of all state and q values

- In realistic situations, we cannot possibly learn about every single state.
  - Storage: too many states to hold the tables in **memory**
  - Experience: too many **trajectories** to visit every single state
  - Computation: too much **iterations** to take

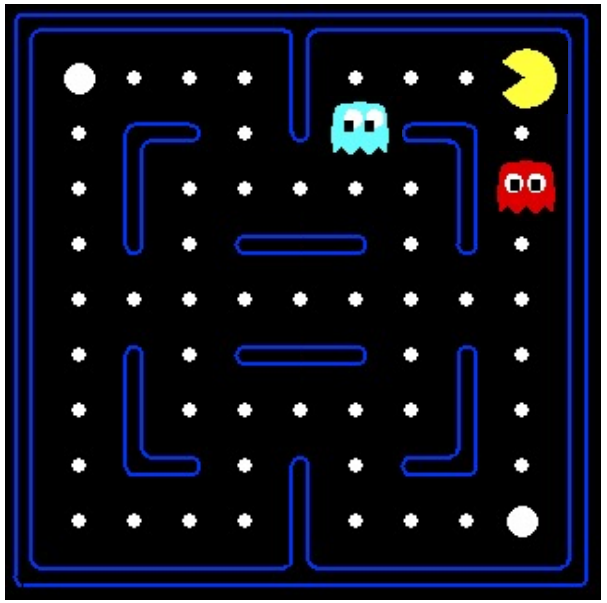
- We want more compact representation that generalizes across states or states and actions

Example: Pacman

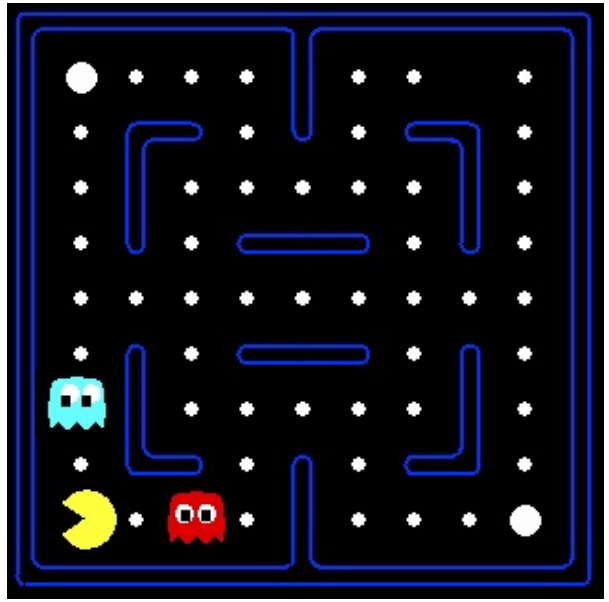
Let's say we discover through experience that this state is bad:



In naïve q-learning, we know nothing about this state:

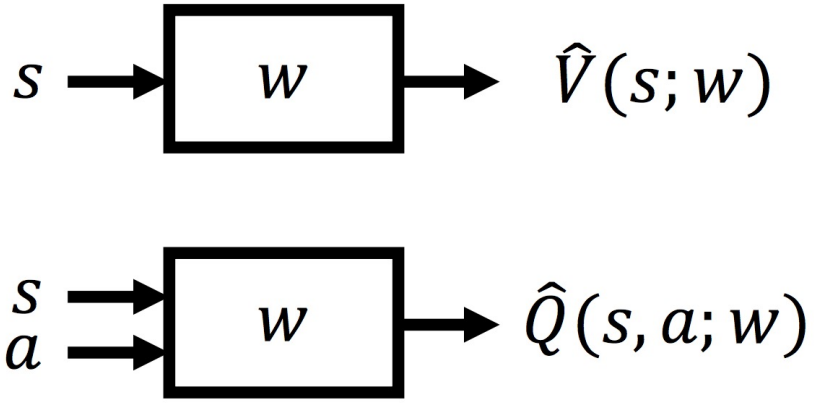


Or even this one!



Approximate Value Functions

- Represent a (state-action/state) value function with a parameterized function instated of a table



- Map a larger space (original state space) into a smaller one (parameter space). And similar states can be mapped closer for generalization.
- Advantages of Generalization
  - Reduce memory needed to store (P, R)/V/Q/π
  - Reduce experience needed to find a good P, R/V/Q/ π
  - Reduce computation needed to compute (P, R)/V/Q/ π

Outline

- States Generalization
- Function Approximation Preliminaries

Function Approximators

- Many possible function approximators including
  - Linear combinations of features
  - Neural networks
  - Decision trees
- linear combinations of features
  - Feature extractor

$$\boldsymbol{f}(s) = (f_1(s), f_2(s), \ldots, f_n(s))$$

  - Weight vector

$$\mathbf{w} = (w_1, w_2, \ldots, w_n)$$

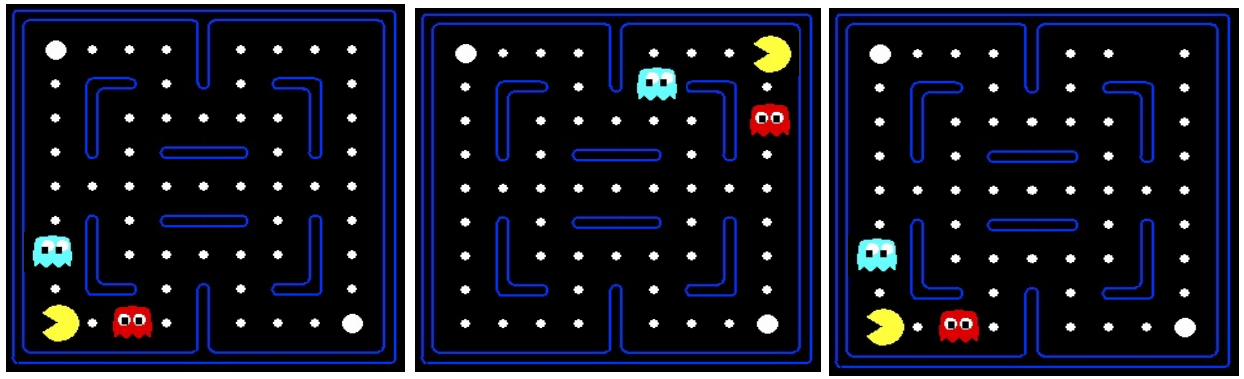
  - Approximated values

$$\hat{V}(s; \boldsymbol{w}) = w_1 f_1(s) + w_2 f_2(s) + \cdots + w_n f_n(s)$$

$$\hat{Q}(s, a; \boldsymbol{w}) = w_1 f_1(s, a) + w_2 f_2(s, a) + \cdots + w_n f_n(s, a)$$

Linear Combination of Features

- Features are functions from states to real numbers that capture important properties of the state
  - f_1: Distance to closest ghost
  - f_2: Distance to closest dot
  - f_3: Number of ghosts
  - f_4: Is Pacman in a tunnel? (0/1)



s_1 s_2 s_3

- $f(s_1) = f(s_2) = f(s_3) = (2,1,2,1)$
- w: [1,1,1,1]
- $\hat{V}(s;w) = w_1 * f_1(s) + w_2 * f_2(s) + w_3 * f_3(s) + w_4 * f_4(s)$

Recall Gradient Descent

- Consider an **objective function J(w)** is the loss between true value and the approximated value. It is a differentiable function of a parameter vector **w**

$$J(\boldsymbol{w}) = E \quad [(y - \hat{f}(y; \boldsymbol{w}))^2]$$

- Goal is to find parameter **w** that minimizes J.
- The gradient of **J(w)** is :

$$\nabla_w J(\boldsymbol{w}) = [\frac{\partial J(\boldsymbol{w})}{\partial w_1}, \frac{\partial J(\boldsymbol{w})}{\partial w_2}, \frac{\partial J(\boldsymbol{w})}{\partial w_3}, \ldots, \frac{\partial J(\boldsymbol{w})}{\partial w_n}]$$

- Compute the gradient and update parameters with a learning rate

$$\mathbf{w} \leftarrow \boldsymbol{w} - \alpha \nabla_w J(\boldsymbol{w})$$

- Until converge

Outline

- States Generalization
- Function Approximation Preliminaries
- Value Function Approximation (VFA) for Policy Evaluation

Model free policy evaluation

- Recall model-free policy evaluation
  - Following a fixed policy π
  - Goal is to estimate V^{π}

$$V_t{}^{\pi}(s) = V_{t-1}{}^{\pi}(s) + \alpha((v^{\pi}(s) - V_{t-1}{}^{\pi}(s))$$

- Maintained a look up table to store V^{π}
- Updated these estimates after each episode (Monte Carlo methods) or after each step (TD methods)
- In value function approximation, change the estimate update step to include fitting the function approximator

VFA for Policy Evaluation

- **True Value:** Assume we could query any state s and an oracle would return the true value for $V^{\pi}(S)$.

- **Approximation:** Find the **approximate representation** for V^{π} given a particular parameterized function $\hat{V}^{\pi}(S; \boldsymbol{w})$.

- **Goal:** Find the parameter vector **w** that minimizes the loss between a true value function $V^{\pi}(s)$ and its approximation $\hat{V}^{\pi}(S; \boldsymbol{w})$.

- How to learn these parameters?

Linear Value Function Approximation

- Represent a value function with a weighted linear combination of features. And f(s) is the feature vector.

$$\hat{V}(s;\boldsymbol{w}) = \sum_{j=1}^{n} f_j(s) * w_j = \boldsymbol{f}(s)^T \boldsymbol{w}$$

- Given $V^{\pi}(s)$ is the true state value of s, Objective function can be:

$$J(\boldsymbol{w}) = E_{\pi}[(V^{\pi}(s) - \hat{V}(s;\boldsymbol{w}))^2]$$

- Weight update for gradient descent is:

$$\Delta \boldsymbol{w} = -\frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{\alpha} \nabla_w J(\boldsymbol{w}) \qquad \nabla_w J(\boldsymbol{w}) = E_{\pi}[-2 * (V^{\pi}(s) - \hat{V}(s;\boldsymbol{w})) * \nabla_w \hat{V}(s;\boldsymbol{w})]$$

- Weight update using stochastic gradient descent, using linear function

$$\Delta w = \alpha * ((V^{\pi}(s) - \boldsymbol{f}(s)^T \boldsymbol{w}) * \boldsymbol{f}(s))$$

- Update equals = step-size * prediction error * feature value

Outline

- States Generalization
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- MC and TD for VFA in Policy Evaluation

Difference of MC and TD in VFA in Policy Evaluation

$$\Delta w = \alpha * \left(V^{\pi}(s) - \hat{V}^{\pi}(s; \boldsymbol{w}) \right) * \nabla_w \hat{V}^{\pi}(s; \boldsymbol{w})$$

- They can share the same approximation function.
- They can use the same way to search for the optimized parameters.

- TD and MC differs in the way of obtaining true value for each sample.

- MC obtain a sample value after each episode, while TD can update the parameter after each transition

Monte Carlo Value Function Approximation

- Use the accumulated rewards as the ground true value for updating.

$$G_t = \sum_{t=1}^{T} \gamma^{t-1} r_t$$

- Do supervised learning on a set of (state, return) pairs
  - $(s_1, G_1), (s_2, G_2), (s_3, G_3), \ldots, (s_T, G_T)$
  - Substitute G_t for the true $V^{\pi}(s_t)$ when fit function approximator
- Find weights to minimize mean squared error

$$J(\boldsymbol{w}) = E_{\pi}[G_t - \hat{V}(s; \boldsymbol{w}))^2]$$

- Using linear VFA for policy evaluation, gradient computing

$$\begin{aligned}\Delta \boldsymbol{w} &= \alpha * \left(\left(V^{\pi}(\boldsymbol{s}) - \hat{V}^{\pi}(s_t; \boldsymbol{w})\right) * \nabla_w \hat{V}^{\pi}(s_t; \boldsymbol{w})\right) \\ &= \alpha * \left(\left(G_t - \hat{V}^{\pi}(s_t; \boldsymbol{w})\right) * \nabla_w \hat{V}^{\pi}(s_t; \boldsymbol{w})\right) \\ &= \alpha * ((G_t - \boldsymbol{f}(s_t)^T \boldsymbol{w}) * \boldsymbol{f}(s_t))\end{aligned}$$

MC Linear VFA for Policy Evaluation

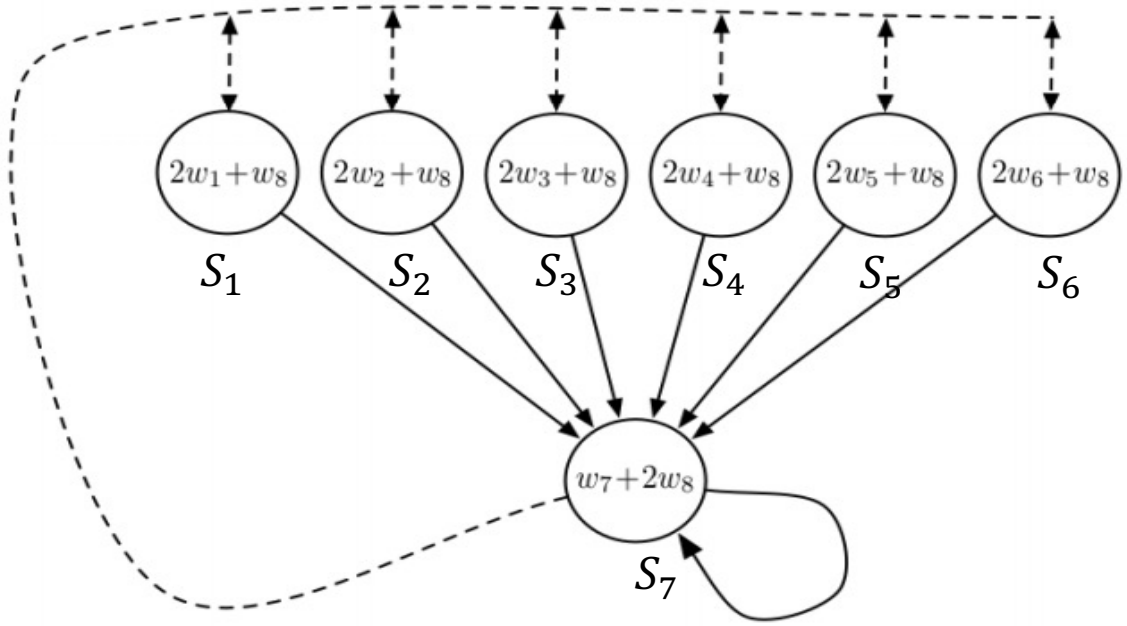
Initialize $\mathbf{w} = \mathbf{0}$, $k = 1$
loop
  Sample k-th episode $(s_{k,1}, a_{k,1}, r_{k,1}, s_{k,2}, \ldots, s_{k,L_k})$ given π
  for $t = 1, \ldots, L_k$ **do**
    if First visit to (s) in episode k **then**
      $G_t(s) = \sum_{j=t}^{L_k} r_{k,j}$
      Update weights:

$$\boldsymbol{w} = \boldsymbol{w} + \alpha * ((G_t - \boldsymbol{f}(s)^T \boldsymbol{w}) * \boldsymbol{f}(s))$$

    end if
  end for
  $k = k + 1$
end loop

Example for MC Policy Evaluation

- 7 states, 8 features
- 2 actions
  - a_1 ->: deterministic
  - a_2 -->: random walk to other states (1/7)
  - Terminate in S_7 for some probability
- no reward
- Trajectory:
  - τ: $S_1, a_1, 0, S_7, a_1, 0, S_7, a_1, 0, terminate$
- Suppose $w_0 = [1,1,1,1,1,1,1,1], \alpha = 0.5, \gamma = 0.9$
- Demonstrate the update of $V^{\pi}(S)$, following τ



$f(s_1)$: [2,0,0,0,0,0,0,1]
$f(s_2)$: [0,2,0,0,0,0,0,1]
$f(s_3)$: [0,0,2,0,0,0,0,1]
$f(s_4)$: [0,0,0,2,0,0,0,1]
$f(s_5)$: [0,0,0,0,2,0,0,1]
$f(s_6)$: [0,0,0,0,0,2,0,1]
$f(s_7)$: [0,0,0,0,0,0,1,2]

Feature Vectors

$$MC\ updates: \Delta w = \alpha\left(G_t(s_t) - \boldsymbol{f}(s_t\ )^T w\right) * \boldsymbol{f}(s_t)$$

TD Value Function Approximation

- In value function approximation update is based on (s_t, a_t, r_t, s_{t+1})

$$r + \gamma \hat{V}^{\pi}(s', w)$$

- Do supervised learning on a set of (state, next state value) pairs

$$\left(s_1, r_1 + \gamma \hat{V}^{\pi}(s_2, \boldsymbol{w})\right), \left(s_2, r_2 + \gamma \hat{V}^{\pi}(s_3, \boldsymbol{w})\right), \left(s_3, r_3 + \gamma \hat{V}^{\pi}(s_4, \boldsymbol{w})\right), \ldots, \left(s_4, r_4 + \gamma \hat{V}^{\pi}(s_5, \boldsymbol{w})\right)$$

- Find weights to minimize mean squared error

$$J(\boldsymbol{w}) = E_{\pi}[(r + \gamma \hat{V}^{\pi}(s_{t+1}, \boldsymbol{w}) - \hat{V}^{\pi}(s_t; \boldsymbol{w}))^2]$$

- Gradient:

$$\Delta \boldsymbol{w} = \alpha * \left(\left(V^{\pi}(\boldsymbol{s}) - \hat{V}^{\pi}(s_t; \boldsymbol{w})\right) * \nabla_w \hat{V}^{\pi}(s_t; \boldsymbol{w})\right)$$

$$= \alpha * \left(\left(r_{\mathrm{t}} + \gamma \hat{V}^{\pi}(s_{t+1}, \boldsymbol{w}) - \hat{V}^{\pi}(s_t; \boldsymbol{w})\right) * \nabla_w \hat{V}^{\pi}(s_t; \boldsymbol{w})\right)$$

$$= \alpha * \left(\left(r_{\mathrm{t}} + \gamma \hat{V}^{\pi}(s_{t+1}, \boldsymbol{w}) - \boldsymbol{f}(s_t)^T \boldsymbol{w}\right) * \boldsymbol{f}(s_t)\right)$$

TD Value Function Approximation

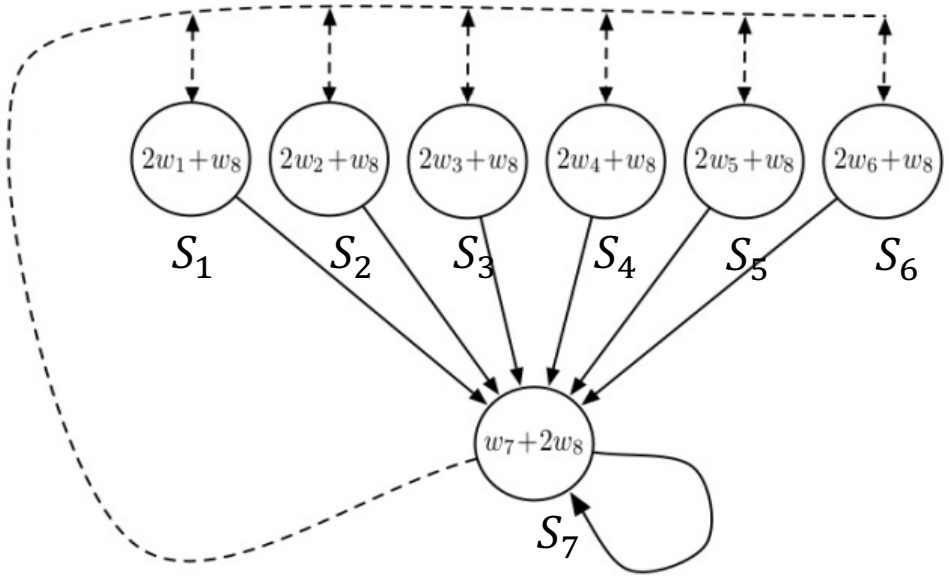
1: Initialize $\mathbf{w} = \mathbf{0},\ k = 1$
2: **loop**
3: Sample tuple (s_k, a_k, r_k, s_{k+1}) given π
4: Update weights:

$$\boldsymbol{w} = \boldsymbol{w} + \alpha * \left(\left(r + \gamma \widehat{V}^{\pi}(s_{t+1}, \boldsymbol{w}) - \boldsymbol{f}(s_t)^T \boldsymbol{w}\right) * \boldsymbol{f}(s_t)\right)$$

5: $k = k + 1$
6: **end loop**

Example for TD Policy Evaluation

- 7 states, 8 features
- 2 actions
  - a_1 ->: deterministic
  - a_2 -->: random walk to other states (1/7)
- no reward
- Transition: $(S_1, a_1, 0, S_7)$
- Suppose $w_0 = [1,1,1,1,1,1,1,1], \alpha = 0.5, \gamma = 0.9$
- Demonstrate the update of $V^{\pi}(S)$



$f(s_1)$: [2,0,0,0,0,0,0,1]
$f(s_2)$: [0,2,0,0,0,0,0,1]
$f(s_3)$: [0,0,2,0,0,0,0,1]
$f(s_4)$: [0,0,0,2,0,0,0,1]
$f(s_5)$: [0,0,0,0,2,0,0,1]
$f(s_6)$: [0,0,0,0,0,2,0,1]
$f(s_7)$: [0,0,0,0,0,0,1,2]

Feature Vectors

$$TD\ updates: \Delta w = \alpha\left(r + \gamma * f(s_{t+1})^T w - f(s_t\ )^T w\right) * f(s_t)$$

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Policy Control using Value Function Approximation

- Use value function approximation to represent state-action values

$$\widehat{\mathrm{Q}}^{\pi}(s,a;\boldsymbol{w}) \approx Q^{\pi}(s,a)$$

- Policy Iteration
  - Policy evaluation： using value function approximation (MC or TD)
  - Policy Improvement: $\varepsilon - greedy$, same as non-VFA setting

Linear State Action Value Function Approximation

- Use features to represent both the state and action

$$\boldsymbol{f}\,(s,a) = (f_1(s,a), f_2(s,a), f_n(s,a))$$

- Represent state-action value function with a weighted linear combination of features

$$\widehat{\mathrm{Q}}^{\pi}(s,a;w) = \boldsymbol{f}(s,a)^T\boldsymbol{w} = \sum_{j=1}^{n} f_j(s,a)w_j$$

- Objective function:

$$J(w) = E_{\pi}[(Q^{\pi}(s,a) - \widehat{\mathrm{Q}}^{\pi}(s,a;\boldsymbol{w}))^2]$$

- Stochastic gradient descent update

$$\Delta\boldsymbol{w} = \alpha * ((\mathrm{Q}^{\boldsymbol{\pi}}(\boldsymbol{s},\boldsymbol{a}) - \boldsymbol{f}(s,a)^T\boldsymbol{w}) * \boldsymbol{f}(s,a))$$

Incremental Model-Free Control Approaches

- Similar to policy evaluation, true state-action value function is unknown and need to substitute a target value
- In Monte Carlo methods, use a return G_t as substitute target

$$\Delta \boldsymbol{w} = \alpha * \left(\left(G_t - \hat{Q}^{\pi}(s_t, a_t; \boldsymbol{w}) \right) * \nabla_w \hat{Q}^{\pi}(s_t, a_t; \boldsymbol{w}) \right)$$

- For SARSA, use a TD target $r + \gamma \hat{Q}(s', a'; \boldsymbol{w})$ which leverages the current function approximation value

$$\Delta \boldsymbol{w} = \alpha * \left(\left(r + \gamma \hat{Q}^{\pi}(s_{t+1}, a_{t+1}; \boldsymbol{w}) - \hat{Q}^{\pi}(s_t, a_t; \boldsymbol{w}) \right) * \nabla_w \hat{Q}^{\pi}(s_t, a_t; \boldsymbol{w}) \right)$$

- For Q-learning, use a TD target $r + \gamma \max_a Q^{\pi}(s_{t+1}, a_{t+1}; \boldsymbol{w})$ which leverages the max of the current function approximation value

$$\Delta \boldsymbol{w} = \alpha * \left(\left(r + \gamma \max_a Q^{\pi}(s_{t+1}, a_{t+1}; \boldsymbol{w}) - \hat{Q}^{\pi}(s_t, a_t; \boldsymbol{w}) \right) * \nabla_w \hat{Q}^{\pi}(s_t, a_t; \boldsymbol{w}) \right)$$

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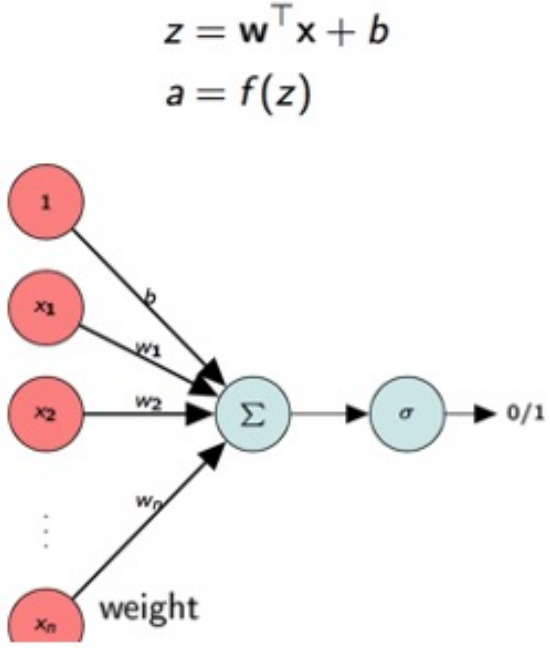
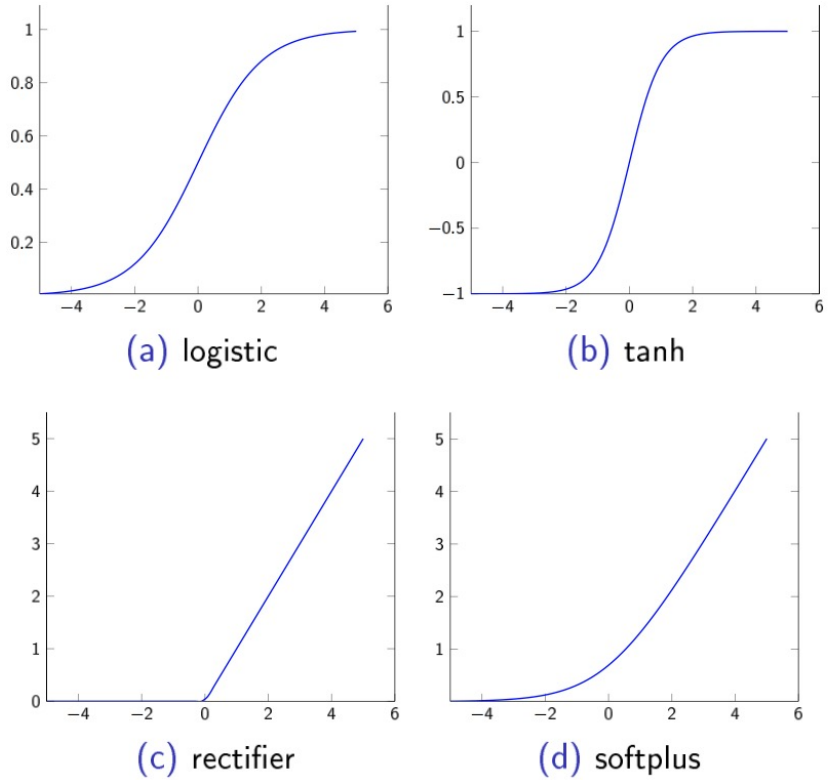
Artificial Neuron

- An artificial neuron is a mathematical function to simulate biological neurons. Input: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$

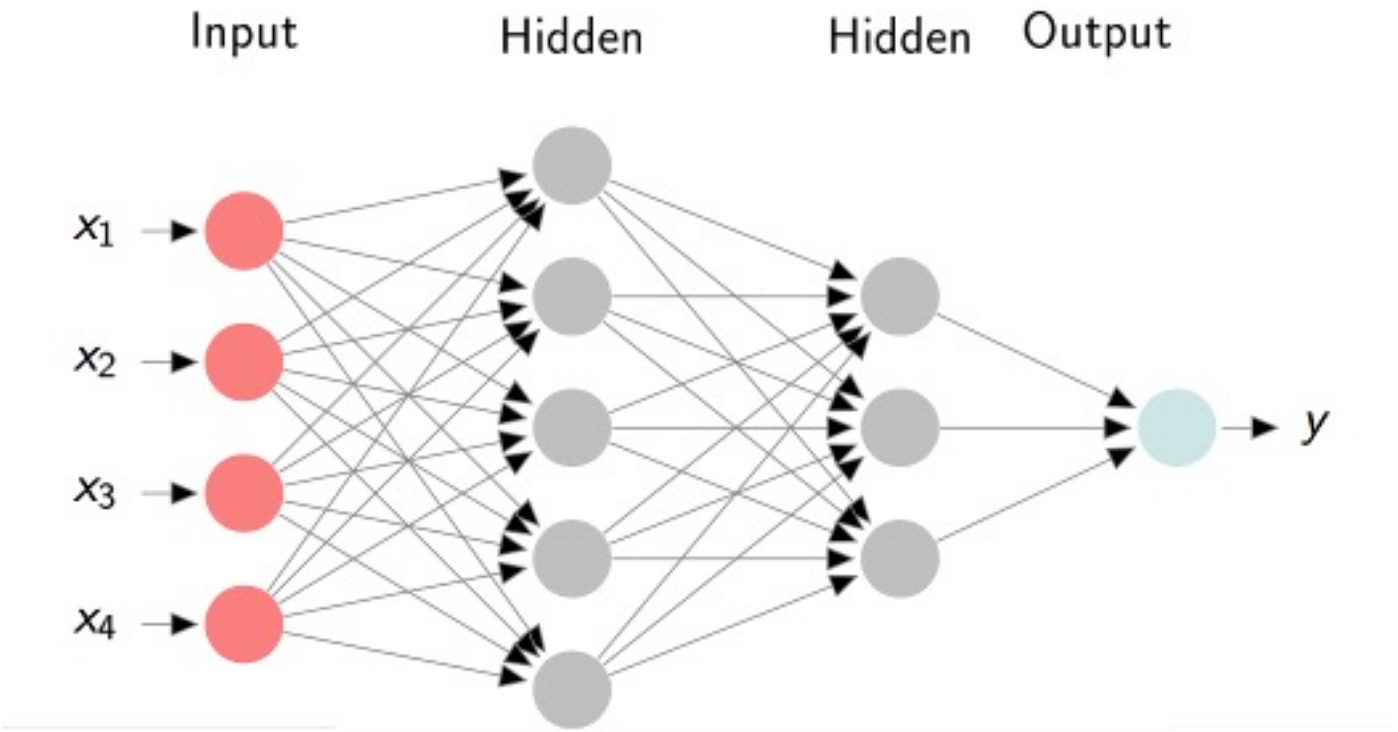
- State: **z**
- Output: **a**
- Activation Function: **f**





Feedforward Neural Network

- In feedforward neural network, the information moves in only one direction forward: from the input nodes data goes through the hidden nodes and to the output nodes.



- Composition of multiple functions
- Can use a chain rule to backpropagate the gradient

Convolutional Neural Network

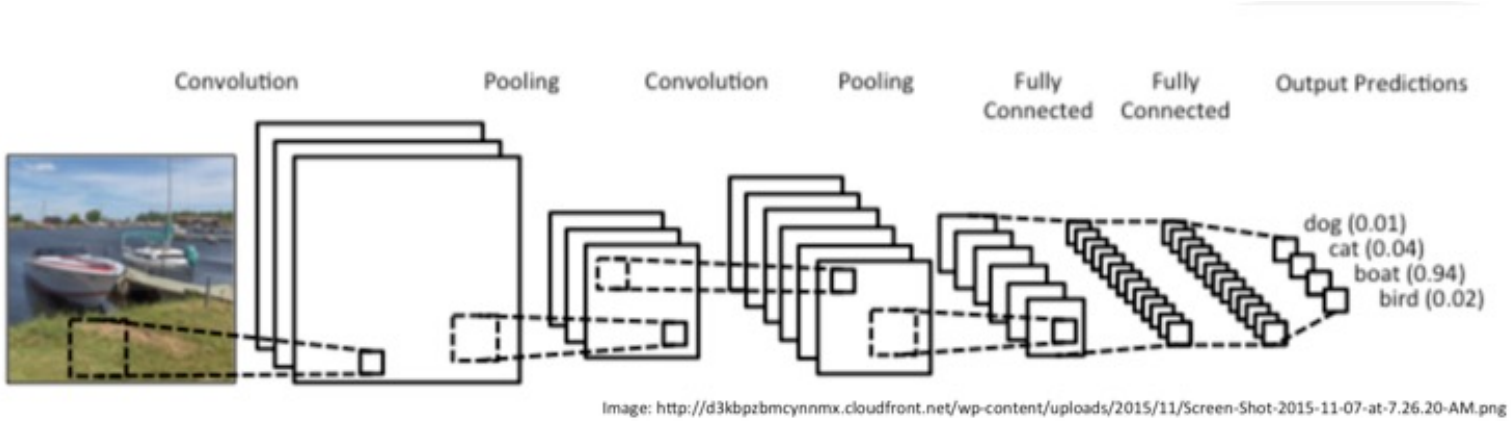


Image: http://d3kbpzbmcynnmx.cloudfront.net/wp-content/uploads/2015/11/Screen-Shot-2015-11-07-at-7.26.20-AM.png

- Consider local structure and common extraction of features
- Not fully connected
- Locality of processing
- Weight sharing for parameter reduction
- Learn the parameters of multiple convolutional filter banks
- Compress to extract salient features & favor generalization

The Benefit of Deep Neural Network Approximators

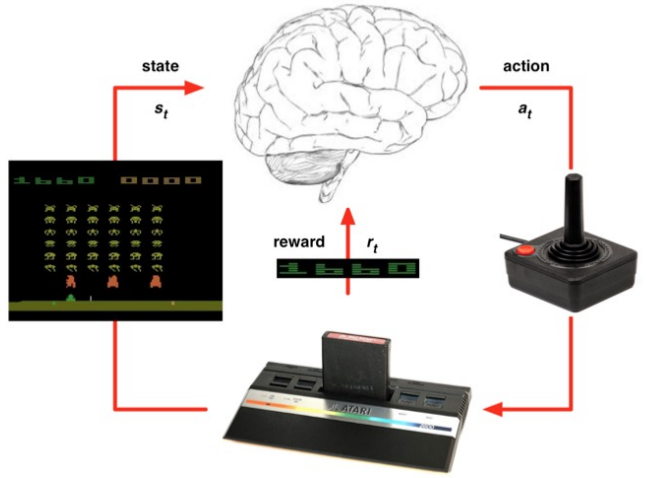
- Linear value function approximators assume value function is a weighted combination of a set of features, where each feature is a function of the state.

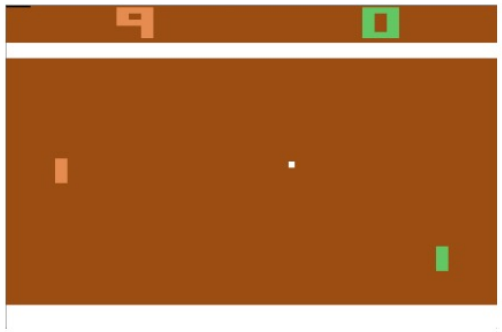
- Linear VFA often work well given the right set of features, but can require carefully hand designing.

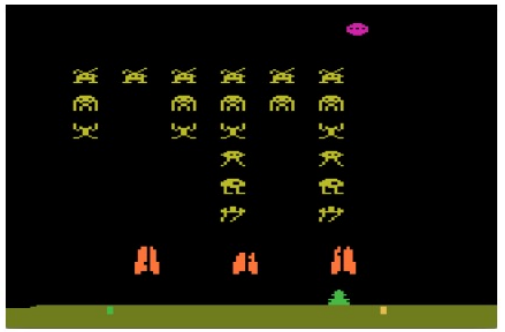
- Deep neural networks
  - Uses distributed representations instead of local representations
  - Need exponentially less nodes/parameters, (compared to a shallow net) to represent the same approximated function
  - Learn the parameters using stochastic gradient descent

Using DQN to do RL in Atari

- **Atari is one the most recognized and celebrated brands in the world.**
- Founded in 1972, Atari played an integral role in the development of the arcade game, game console and personal computer industries. Atari's iconic games, including Pong®, Asteroids®, Centipede® Missile Command®.







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- Issues in Value Function Approximation

Issues in Value Function Approximation

- Q-learning converges to the optimal $Q*(s,a)$ using table lookup representation
- In value function approximation Q-learning we can minimize MSE loss by stochastic gradient descent using a target Q estimate instead of true Q (as we saw with linear VFA)

- But Q-learning with VFA can diverge and two issues causing problems:
  - Correlations between samples
  - Non-stationary targets

- Deep Q-learning (DQN)
  - Experience replay
  - Fixed Q-targets

DQNs: Experience Replay

- To help remove correlations, store dataset D from prior experience

$s_1, a_1, r_2, s_2, a_2, r_3, s_3, \ldots, s_t, a_t, r_{t+1}, s_{t+1}$ →

s_1, a_1, r_2, s_2
s_2, a_2, r_3, s_3
s_3, a_3, r_4, s_4
…
$s_t, a_t, r_{t+1}, s_{t+1}$

→ s, a, r, s'

- To perform experience replay, repeat the following:
  - (s,a,r,s') : sample an experience tuple from the dataset D
  - Compute the sampled target value s : $r + \gamma \max_{a'} \hat{Q}(s', a'; \boldsymbol{w})$
  - Use stochastic gradient descent to update the network weights

$$\Delta \boldsymbol{w} = \alpha * \left(\left(r + \gamma \max_{a'} \hat{Q}^{\pi}(s', a'; \boldsymbol{w}) - \hat{Q}^{\pi}(s, a; \boldsymbol{w}) \right) * \nabla_w \hat{Q}^{\pi}(s, a; \boldsymbol{w}) \right)$$

- Can treat the target as a scalar, but the weights will get updated on the next round, changing the target value

DQNs: Fixed Q-Targets

- To help improve stability, fix the target weights used in the target calculation for multiple updates
- Target network uses a different set of weights than the weights being updated
- Let parameters $\boldsymbol{w}^{-}$ be the set of weights used in the target, and $\boldsymbol{w}$ be the weights that are being updated
- Slight change to computation of target value:
  - (s,a,r,s′) : sample an experience tuple from the dataset
  - Compute the sampled target value s : $r + \gamma \max_{a'} \hat{Q}(s', a'; \boldsymbol{w}^{-})$
  - Use stochastic gradient descent to update the network weights

$$\Delta \boldsymbol{w} = \alpha * \left(\left(r + \gamma \max_{a'} \hat{Q} \quad (s', a'; \boldsymbol{w}^{-}) - \hat{Q}^{\pi}(s, a; \boldsymbol{w}) \right) * \nabla_{w} \hat{Q}^{\pi}(s, a; \boldsymbol{w}) \right)$$

DQNs Pseudocode

Input C, α, $D = \{\}$, Initialize w, $w^- = w$, $t = 0$
Get initial state s_0
loop
    Sample action a_t given ϵ-greedy policy for current $\hat{Q}(s_t, a; w)$
    Observe reward r_t and next state s_{t+1}
    Store transition (s_t, a_t, r_t, s_{t+1}) in replay buffer D
    Sample random minibatch of tuples (s_i, a_i, r_i, s_{i+1}) from D
    for j in minibatch **do**
        if episode terminated at step $i + 1$ **then**
            $y_i = r_i$
        else
            $y_i = r_i + \gamma \max_{a'} \hat{Q}(s_{i+1}, a'; w^-)$
        end if
        Do gradient descent step on $(y_i - \hat{Q}(s_i, a_i; w))^2$ for parameters w: $\Delta w = \alpha(y_i - \hat{Q}(s_i, a_i; w))\nabla_w \hat{Q}(s_i, a_i; w)$
    end for
    $t = t + 1$
    if mod(t,C) == 0 **then**
        $w^- \leftarrow w$
    end if
end loop

- One needs to choose the **neural network architecture**, the **learning rate**, and **how often to update the target network**.
- Often a fixed **size replay buffer** is used for experience replay, which introduces a parameter to control the size, and the need to decide how to populate it.

DQNs in Atari

- End to end learning of values Q(s, a) from pixels s
- Input state s is stack of raw pixels from last 4 frames
- Output is Q(s, a) for 18 joystick/button positions
- Reward is change in score for that step
- Network architecture and hyperparameters fixed across all games

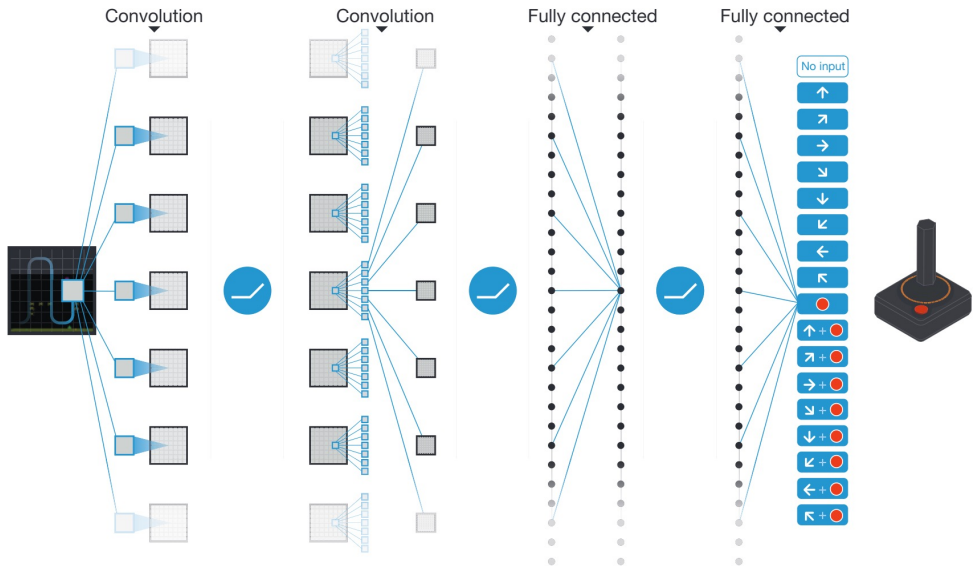


Figure 1: Human-level control through deep reinforcement learning, Mnih et al, 2015

DQNs Summary

- DQN uses experience replay and fixed Q-targets
- Store transition $(s_t, a_t, r_{t+1}, s_{t+1})$ in replay memory D
- Sample random mini-batch of transitions (s, a, r, s') from D
- Compute Q-learning targets w.r.t. old, fixed parameters w
- Optimizes MSE between Q-network and Q-learning targets
- Uses stochastic gradient descent

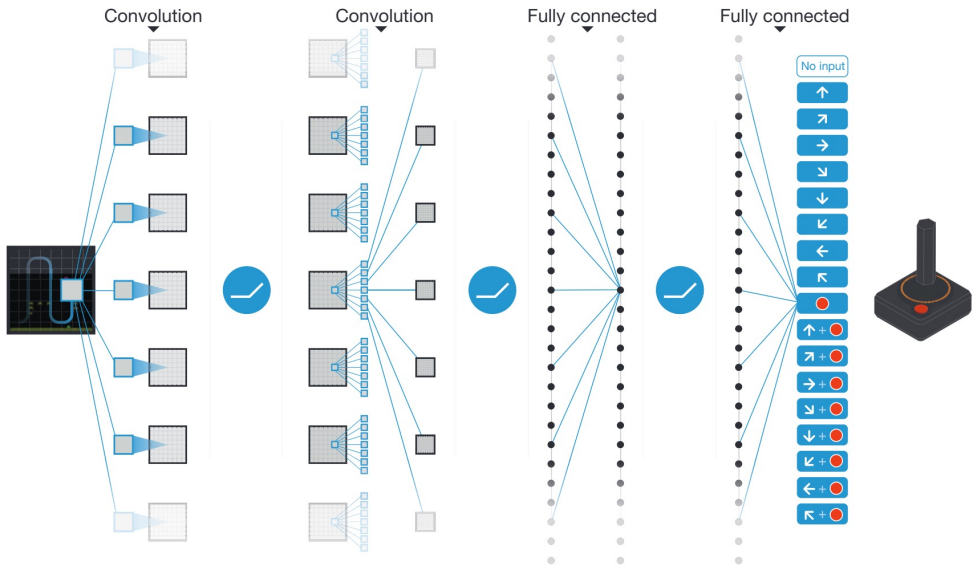
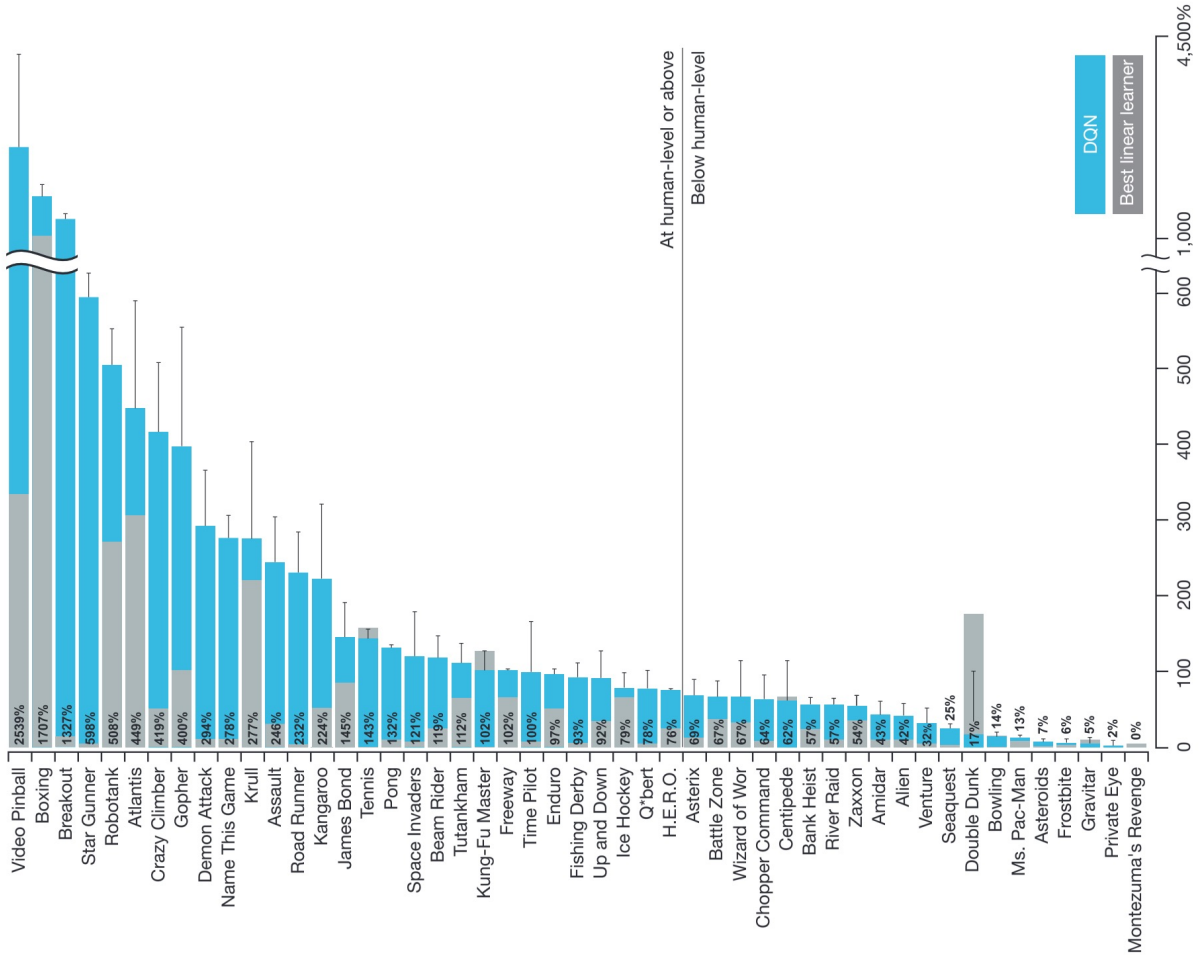


Figure 1: Human-level control through deep reinforcement learning, Mnih et al, 2015

DQN Results in Atari



- Figure 2: Human-level control through deep reinforcement learning, Mnih et al, 2015

Which Aspects of DQN were Important for Success?

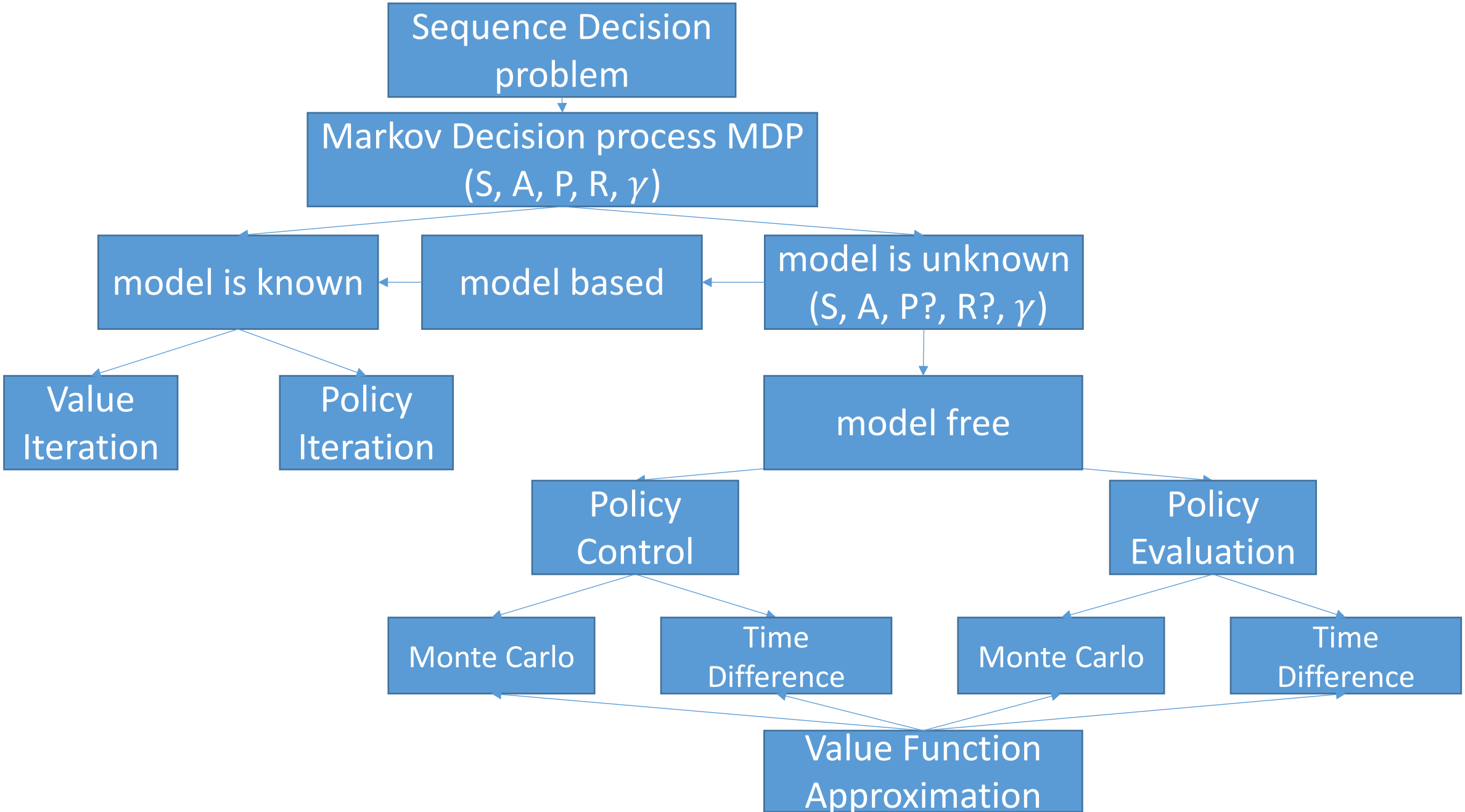
Game	Linear	Deep Network	DQN w/fixed Q	DQN w/reply	DQN w/ both
Breakout	3	3	10	241	317
Enduro	62	29	141	831	1,006
River Raid	2,345	1,453	2,868	4,102	7,447
Seaquest	656	275	1,003	823	2,894
Space Invaders	301	302	373	826	1,089

- Just using a deep NN hurts performance sometimes
- Experience replay is important

More about experience replay

Experience Replay Does this	T or F
Involves using a bank of prior (s,a,r,s) tuples and doing Q-learning updates on the tuples in the bank.	
Always uses the most recent history of tuples	
Reduces the data efficiency of DQN	
Increases the computational cost	

Summary of Reinforcement Learning



More Readings

- Reinforcement Learning: An introduction, Richard S. Sutton and Andrew G. Barto
- CS234, Reinforcement Learning
- 《神经网络与深度学习》，邱锡鹏，Chapter 14