

# Tight Last-Iterate Convergence of the Extragradient and the Optimistic Gradient Descent-Ascent Algorithm for Constrained Monotone Variational Inequalities

Yang Cai<sup>\*†</sup>  
Yale University  
yang.cai@yale.edu

Argyris Oikonomou<sup>\*†</sup>  
Yale University  
argyris.oikonomou@yale.edu

Weiqiang Zheng<sup>†</sup>  
Yale University  
weiqiang.zheng@yale.edu

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## Abstract

The *monotone variational inequality* is a central problem in mathematical programming that unifies and generalizes many important settings such as smooth convex optimization, two-player zero-sum games, convex-concave saddle point problems, etc. The *extragradient algorithm* by Korpelevich [1976] and the *optimistic gradient descent-ascent algorithm* by Popov [1980] are arguably the two most classical and popular methods for solving monotone variational inequalities. Despite its long history and intensive attention from the optimization and machine learning community, the following major problem remains open. *What is the last-iterate convergence rate of the extragradient algorithm or the optimistic gradient descent-ascent algorithm for monotone and Lipschitz variational inequalities with constraints?* We resolve this open problem by showing that both the extragradient algorithm and the optimistic gradient descent-ascent algorithm have a tight  $O\left(\frac{1}{\sqrt{T}}\right)$  last-iterate convergence rate for *arbitrary convex feasible sets*, which matches the lower bound by Golowich et al. [2020a,b]. Our rate is measured in terms of the standard *gap function*. At the core of our results lies a *new performance measure* – the *tangent residual*, which can be viewed as an adaptation of the norm of the operator that takes the *local constraints* into account. We use the tangent residual (or a slight variation of the tangent residual) as the performance measure in our analysis of the extragradient algorithm (or the optimistic gradient descent-ascent algorithm). To establish the monotonicity of these performance measures, we develop a new approach that combines the power of the *sum-of-squares programming* with the *low dimensionality* of the update rule of the extragradient or the optimistic gradient descent-ascent algorithm. We believe our approach has many additional applications in the analysis of iterative methods.

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# 1 Introduction

The *monotone variational inequality* (VI) problem plays a crucial role in mathematical programming, providing a unifying setting for the study of optimization and equilibrium problems. It also serves as a computational framework for numerous important applications in fields such as Economics, Engineering, and Finance [Facchinei and Pang, 2007]. Monotone VIs have been studied since the 1960s [Hartman and Stampacchia, 1966, Browder, 1965, Lions and Stampacchia, 1967, Brezis and Sibony, 1968, Sibony, 1970]. Formally, a monotone VI is specified by a closed convex set  $\mathcal{Z} \subseteq \mathbb{R}^n$  and a **monotone** operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$ ,<sup>1</sup> with the goal of finding a  $z^* \in \mathcal{Z}$  such that

$$\langle F(z^*), z^* - z \rangle \leq 0 \quad \forall z \in \mathcal{Z}. \quad (1)$$

We further assume the operator  $F$  to be **Lipschitz**, which is a natural assumption that is satisfied in most applications and is also made in the majority of algorithmic works concerning monotone VIs. An important special case of the monotone and Lipschitz VI is the convex-concave saddle point problem:

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y), \quad (2)$$

where  $\mathcal{X}$  and  $\mathcal{Y}$  are closed convex sets in  $\mathbb{R}^n$ , and  $f(\cdot, \cdot)$  is smooth, convex in  $x$ , and concave in  $y$ .<sup>2</sup> Besides its central importance in Game Theory, Convex Optimization, and Online Learning, the convex-concave saddle point problem has recently received a lot of attention from the machine learning community due to several novel applications such as the generative adversarial networks (GANS) (e.g., [Goodfellow et al., 2014, Arjovsky et al., 2017]), adversarial examples (e.g., [Madry et al., 2018]), robust optimization (e.g., [Ben-Tal et al., 2009]), and reinforcement learning (e.g., [Du et al., 2017, Dai et al., 2018]).

The extragradient (EG) algorithm by Korpelevich [1976] and the optimistic gradient descent-ascent (OGDA) algorithm by Popov [1980] are arguably the two most classical and popular methods for solving monotone and Lipschitz VIs. Despite the extensive effort from the optimization and machine learning community, the following major problem remains open.

*What is the last-iterate convergence rate of the extragradient algorithm and the optimistic gradient descent-ascent algorithm for monotone and Lipschitz variational inequalities with constraints?*

Indeed, the same problem has not been answered even for two-player zero-sum games, arguably one of the most basic monotone and Lipschitz VIs.<sup>3</sup> Korpelevich [1976] and Facchinei and Pang [2007] prove that the last iterate of EG converges to a solution of the monotone and Lipschitz VI, but do not provide any upper bound on the rate of convergence. Similarly, Popov [1980] and Hsieh et al. [2019] show the asymptotic last-iterate convergence for OGDA. Prior to our work,

<sup>1</sup> $F$  is monotone if  $\langle F(z) - F(z'), z - z' \rangle \geq 0$  for all  $z, z' \in \mathcal{Z}$ .

<sup>2</sup>If we set  $F(x, y) = \begin{pmatrix} \nabla_x f(x, y) \\ -\nabla_y f(x, y) \end{pmatrix}$  and  $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ , then (i)  $F(x, y)$  is a monotone and Lipschitz operator, and (ii) the set of saddle points coincide with the solutions of the monotone VI for operator  $F$  and domain  $\mathcal{Z}$ .

<sup>3</sup>A two-player zero-sum game can be specified by its payoff matrix  $A \in \mathbb{R}^{\ell \times m}$ . It is a special case of the convex-concave saddle point problem, where  $\mathcal{X} = \Delta^\ell$ ,  $\mathcal{Y} = \Delta^m$  ( $\Delta^k$  denotes the  $k$ -dimensional simplex), and the function  $f(x, y) = x^\top A y$ .

the only setting where an upper bound on the rate of convergence exists for either EG or OGDA is when the problem is unconstrained, i.e.,  $\mathcal{Z} = \mathbb{R}^n$ , [Golowich et al., 2020b, Gorbunov et al., 2021, Golowich et al., 2020a]. Determining the last-iterate convergence rate of EG or OGDA in the constrained setting has been posed as an open question in several recent works [Wei et al., 2021b, Gorbunov et al., 2021, Golowich et al., 2020a, Hsieh et al., 2019]. We resolve this open problem by providing the tight last-iterate convergence rate of EG and OGDA under arbitrary convex constraints.

**Main Result:** For any monotone and Lipschitz variational inequality problem with an **arbitrary convex constraint set**  $\mathcal{Z}$ , both EG and OGDA with **constant step size** achieve a **tight** last-iterate convergence of  $O\left(\frac{1}{\sqrt{T}}\right)$  in terms of the standard convergence measures – the gap function (Definition 1) and the tangent residual (Definition 3).<sup>a</sup> See Theorem 3 for the formal statement. We further show that the tangent residual is an upper bound of the natural residual (Lemma 1), so our result also implies a tight last-iterate convergence rate of  $O\left(\frac{1}{\sqrt{T}}\right)$  for the natural residual.

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<sup>a</sup>In the unconstrained setting, the tangent residual is simply the  $\ell_2$ -norm of  $F$ . In the constrained setting, the tangent residual is the  $\ell_2$ -norm of  $F$ 's projection to the tangent cone.

Our upper bounds in terms of the gap function and the natural residual match the lower bounds of Golowich et al. [2020b,a] in a very strong sense, that is, they match in all of the following terms:  $T$ , the Lipschitz constant of  $F$ , and the distance between the starting point  $z_0$  and the solution  $z^*$ .

To the best of our knowledge, our result is the first to provide a last-iterate convergence rate for solving monotone and Lipschitz VIs using any algorithm that belongs to the general class known as *p-stationary canonical linear iterative algorithms* (*p-SCLI*) [Arjevani and Shamir, 2016], which contains the EG, OGDA, and other well-known algorithms. Although often viewed as an approximation to EG, OGDA has an additional feature compared to EG, i.e., it is a *no-regret* learning algorithm (see e.g., [Rakhlin and Sridharan, 2013]). A nice implication of our result for OGDA is that for smooth and monotone games (see Appendix A for the definition), players can each play a no-regret learning algorithm, i.e., OGDA, with constant learning rate, and the overall player behavior exhibits  $O\left(\frac{1}{\sqrt{T}}\right)$  last-iterate convergence rate to a Nash equilibrium in terms of the gap function.

**Why Last-Iterate Convergence?** Both the EG and OGDA algorithms are known to have *average-iterate* convergence. In particular, the average of the iterates of the algorithm converges at a rate of  $O(1/T)$  [Nemirovski, 2004, Auslender and Teboulle, 2005, Tseng, 2008, Monteiro and Svaiter, 2010, Mokhtari et al., 2020, Hsieh et al., 2019]. Nonetheless, there are several important reasons to study last-iterate convergence. First, not only is last-iterate convergence theoretically stronger and more appealing, it is also the only type of convergence that describes the trajectory of an algorithm. As demonstrated by Mertikopoulos et al. [2018], the trajectory of an algorithm may be cycling around in the space perpetually while still converges in the average-iterate sense. In game theory, we often view these algorithms as models of agents' behavior in a system/game. Thus, only last-iterate convergence provides a description of the evolution of the system. Additionally, EG and

OGDA have been successfully applied to improve the training dynamics in GANs, as the training of GANs can be formulated as a saddle point problem [Daskalakis et al., 2018, Yadav et al., 2018, Liang and Stokes, 2019, Gidel et al., 2019a,b, Chavdarova et al., 2019]. On the one hand, in this formulation of GANs, the objective function  $f$  is usually non-convex and non-concave, making existing theoretical guarantees for the average iterate inapplicable. On the other hand, the last iterate typically has good performance in practice. Thus, it is crucial to develop machinery that allows us to analyze the behavior of the last iterate of these algorithms.

## 1.1 A New Performance Measure: the Tangent Residual

A major challenge we face for establishing the last-iterate convergence for EG or OGDA in the constrained setting is the choice of the convergence measure. For simplicity, we focus on our choice of the performance measure for EG, as our performance measure for OGDA is similar and inspired by our performance measure for EG. In the unconstrained case, the central performance measure for EG is the norm of the operator. The key component in both [Golowich et al., 2020b] and [Gorbunov et al., 2021] is to establish that the norm of the operator at the last iterate (also the  $T$ -th iterate) is upper bounded by  $O(\frac{1}{\sqrt{T}})$ , which implies a  $O(\frac{1}{\sqrt{T}})$  last-iterate convergence rate for the gap function.

In the constrained setting, the norm of the operator is a poor choice to measure convergence, as it can be far away from 0 even in the limit, and is hence insufficient to guarantee convergence in terms of the gap function. A standard generalization of the norm of the operator in the constrained setting is the natural residual (Definition 4), which takes the constraints into account and is guaranteed to converge to 0 in the limit. Unfortunately, we observe that the natural residual is *not monotonically decreasing* even in basic bilinear games (see Appendix G), making it difficult to directly analyze. Similar non-monotonicity has been observed for several other natural performance measures such as the norm of the operator mapping introduced in [Diakonikolas, 2020] and the gap function, leaving all these performance measures unsuitable. See more discussion about these performance measures in Section 3 and Appendix G.

The first main contribution of this paper is the introduction of a new performance measure: the **tangent residual**, which can be viewed as the norm of the operator projected to the tangent cone of the current iterate (Definition 3). The tangent residual plays a crucial role in our analyses for both EG and OGDA. Unlike the aforementioned performance measures, we show that the *tangent residual is monotonically decreasing* and has a *last-iterate convergence rate of  $O(\frac{1}{\sqrt{T}})$*  for EG. For OGDA, we prove that a small modification of the tangent residual is monotonically decreasing, which implies that the tangent residual has a last-iterate convergence rate of  $O(\frac{1}{\sqrt{T}})$ . Using the convergence rate of the tangent residual, we can easily derive the last-iterate convergence rate of other classical performance measures such as the natural residual or the gap function. However, we suspect these rates can be challenging to obtain directly without understanding the tangent residual first. Given its central role in our analyses of EG and OGDA, we suspect that the tangent residual will be useful in the analysis of other iterative methods.

## 1.2 Sum-of-Squares based Analysis

To obtain our main result, we propose a novel approach to analyze iterative methods using the *Sum-of-Squares* (SOS) programming [Nesterov, 2000, Parrilo, 2000, 2003, Lasserre, 2001, Laurent, 2009]. In the past two decades, the SOS programming has found numerous applications in fields such as combinatorial optimization, control theory, quantum mechanics, etc. To the best of our knowledge, this is the first application of SOS programming in the analysis of iterative methods for solving VIs.

Very often, the convergence analysis of an iterative method builds on a potential function argument, which boils down to demonstrating the non-negativity of certain function of the iterates. This is usually done by chaining together a sequence of inequalities that are satisfied due to the update rule of the iterative method. Although easy to verify, such a proof could be extremely challenging to discover, and is often found by much trial and error. **We propose to use the SOS programming to significantly automate this process.**

**Sum-of-Squares (SOS) Programming.** Suppose we want to prove that a polynomial  $p(x) \in \mathbb{R}[x_1, \dots, x_n]$  is non-negative over a semialgebraic set  $S = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \forall i \in [m]\}$ , where each  $g_i(x)$  is also a polynomial. One way is to construct a *certificate of non-negativity*, for example, by providing a set of nonnegative coefficients  $\{a_i\}_{i \in [m]} \in \mathbb{R}_{\geq 0}^m$  such that  $p(x) + \sum_{i \in [m]} a_i \cdot g_i(x)$  is a sum-of-squares polynomial, that is, a polynomial that can be expressed as the sum of squares of further polynomials. Surprisingly, if  $p(x)$  is indeed non-negative over  $S$ , a certificate of non-negativity always exists as guaranteed by a foundational result in real algebraic geometry – the Krivine-Stengle Positivstellensatz [Krivine, 1964, Stengle, 1974], a generalization of Artin’s resolution of Hilbert’s 17th problem [Artin, 1927]. Although the certificate in the example above is relatively simple, it is necessary to allow more sophisticated forms in general, e.g., replacing each coefficient  $a_i$  with a SOS polynomial  $s_i(x)$ , etc. The complexity of a certificate is parametrized by the highest degree of the polynomial involved. Such added complexity is useful and indeed needed in our analysis for the EG algorithm. The SOS programming consists of a hierarchy of algorithms, where the  $d$ -th hierarchy is an algorithm that searches for a *certificate of non-negativity* up to degree  $2d$  based on semidefinite programming.

We mainly discuss the analysis of EG here, as the analysis of OGDA is similar and also based on SOS programming. At the core of our analysis of the EG algorithm lies the monotonicity of the tangent residual, which can be formulated as the non-negativity of a **degree-6 polynomial** in the iterates.<sup>4</sup> However, we cannot directly apply the SOS programming to search for such a certificate due to the following subtlety. As the EG algorithm applies to any dimension  $n$ , we need to prove the monotonicity of the corresponding polynomial for every positive integer  $n$ . Thus we must certify the non-negativity for an infinite family of polynomials, one for every  $n$ . We circumvent this difficulty by **exploiting the low-dimensionality of the EG update rule**, i.e., it only involves

<sup>4</sup>The tangent residual is not a polynomial, but the squared tangent residual is a rational function, i.e., quotient of two polynomials. To establish the monotonicity of the tangent residual, we only need to prove that the difference between the squared tangent residual in two consecutive iterates is non-negative. Although, the difference is still only a rational function, proving its non-negativity turns out to be the same as proving the non-negativity of its numerator as the denominator is an SOS polynomial.



a constant number of points and projections per update. In particular, we manage to show that *the number of relevant dimensions in each update* is only a constant regardless of the dimension of the ambient space and the domain  $\mathcal{Z}$ .<sup>5</sup> Taking advantage of this property, we show that it suffices to prove the non-negativity of a single polynomial, which we then use SOS programming to certify. The certificate is rather complex and involves a polynomial identity of a degree-8 polynomial in 27 variables, which we discover by solving a degree-8 SOS program. We think such a proof will be extremely difficult to discover using the traditional approach, if not impossible.

For OGDA, we are not able to show that the tangent residual is monotone. Inspired by the adaptive potential proof in [Golowich et al., 2020a], we suspect that some extra correction term is needed to construct the potential function. Instead of trying to devise such a correction term manually, we manage to directly find one by searching over a family of performance measures using SOS programming. The search we perform is heuristic, but, in our opinion, also a powerful approach to discover potential functions. See Section 6 for details.

### 1.3 Related Work

**Last-Iterate Convergence Rate for EG-like Algorithms in the Unconstrained Setting.** Golowich et al. [2020b,a] show a lower bound of  $\Omega(\frac{1}{\sqrt{T}})$  for solving bilinear games using any p-SCLI algorithms, which include EG and OGDA. For EG, Golowich et al. [2020b] show an matching upper bound under an additional second-order smoothness condition. Gorbunov et al. [2021] improve the result and show that the same upper bound holds without the second-order smoothness condition. For OGDA, Golowich et al. [2020a] provides a matching upper bound under the same second-order smoothness condition. These upper bounds hold for all smooth and Lipschitz VIs. With the additional assumption that the operator  $F$  is cocoercive, Lin et al. [2020] show a  $O(\frac{1}{\sqrt{T}})$  convergence rate for online gradient descent. If we further assume that either  $F$  is strongly monotone in VI or the payoff matrix  $A$  in a bilinear game has all singular values bounded away from 0, linear convergence rate is known for EG, OGDA, and several of their variants [Daskalakis et al., 2018, Gidel et al., 2019a, Liang and Stokes, 2019, Mokhtari et al., 2020, Peng et al., 2020, Zhang and Yu, 2020].

**Last-Iterate Convergence Rate for EG-like Algorithms in the Constrained Setting.** The results for the constrained setting are sparser. If the operator  $F$  is strongly monotone, we know that EG and some of its variants have linear convergence rate [Tseng, 1995, Malitsky, 2015]. Several papers establish the asymptotic convergence, i.e., converge in the limit, of the optimistic multiplicative weight updates in constrained convex-concave saddle point problems [Daskalakis and Panageas, 2019, Lei et al., 2021]. Finally, a recent paper by Wei et al. [2021b] provides a linear rate convergence of OGDA for bilinear games when the domain is a polytope. They show that there is a *problem dependent* constant  $0 < c < 1$  that depends on the payoff matrix of the game as well as the constraint set, so that the error shrinks by a  $1 - c$  factor. However,  $c$  may be arbitrarily close to 0, even if we assume the corresponding operator to be  $L$ -Lipschitz. As a result, their convergence rate is slower

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<sup>5</sup>Note that which constant number of dimensions are relevant change from iterate to iterate, so all  $n$  dimensions might be relevant if we look at the complete trajectory of the algorithm.

than ours when  $T$  is not comparable to  $\frac{1}{c}$ , which may be exponentially large in the dimension  $n$ , though their rate will eventually catch up. Overall, their “instance-specific” bound is incomparable and complements the worst-case view taken in this paper, where we want to derive the worst-case convergence rate for all VIs with monotone and  $L$ -Lipschitz operator  $F$ . Our result is the first last-iterate convergence rate in this worst-case view and matches the lower bound by Golowich et al. [2020b,a].

**Other Algorithms and Performance Measures.** Other than the gap function, one can also measure the convergence using the norm of the operator if the setting is unconstrained, or the natural residual (Definition 4) or similar notions if the setting is constrained. In the unconstrained setting, Kim [2021], Yoon and Ryu [2021], and Lee and Kim [2021] provide algorithms that obtain  $O(\frac{1}{T})$  convergence rate in terms of the norm of the operator, which is shown to be optimal by Yoon and Ryu [2021] for Lipschitz and monotone VIs. In the constrained setting, Diakonikolas [2020] shows the same  $O(\frac{1}{T})$  convergence rate under the extra assumption that the operator is cocoercive and loses an additional logarithmic factor when the operator is only monotone. Our result implies a  $O(\frac{1}{\sqrt{T}})$  last-iterate convergence rate in terms of the natural residual. Although these algorithms achieve faster rate than EG and OGDA,<sup>6</sup> they are based on different methods, i.e., accelerated proximal point or anchoring. As EG and OGDA are arguably the most classical and popular methods for solving variational inequalities, we believe it is crucial to understand their last-iterate convergence rate.

**Computer-Aided Proofs.** A powerful computer-aided proof framework – the *performance estimation problem* (PEP) technique (e.g., [Drori and Teboulle, 2014, Taylor et al., 2017b]) is widely applied to analyze first-order iterative methods. Indeed, the last-iterate convergence rate of EG in the unconstrained setting by Gorbunov et al. [2021] is obtained via the PEP technique. Although the PEP framework can handle projections [Taylor et al., 2017a, Ryu et al., 2020, Goujaud et al., 2022, Dragomir et al., 2021], the main challenge for applying it to the constrained setting is that, the PEP framework requires the performance measures to be polynomials of degree 2 or less (see e.g., [Taylor et al., 2017a]).<sup>7</sup> In the constrained setting, we use the squared tangent residual to measure the algorithm’s progress, which we formulate as a *rational function* with a degree 4 polynomial numerator and a quadratic denominator, making it unsuitable for the PEP framework.<sup>8</sup> Compared to PEP, our SOS approach is more flexible and can accommodate polynomial objectives and constraints of any degree, which allows us to directly apply it to certify the monotonicity of the tangent residual in the constrained setting. We would like to add that there might be a clever reformulation of our performance measure that has a lower degree, which would likely simplify the proof as well.

<sup>6</sup>In terms of the gap function, these algorithms have the same or slower convergence rate as the average-iterate of EG and OGDA, i.e.,  $O(\frac{1}{T})$ .

<sup>7</sup>More specifically, the PEP framework requires the performance measure as well as the constraints to be linear in (i) the function values at the iterates and (ii) the Gram matrix of a set of vectors consisting of the iterates and their gradients.

<sup>8</sup>The tangent residual is the square root of a rational function and can only be even harder to handle.



Lessard et al. [2016] analyze first-order iterative algorithms for convex optimization using a technique inspired by the stability analysis from control theory. They model first-order iterative algorithms using discrete-time dynamical systems and search over quadratic potential functions that satisfy a set of Integral Quadratic Constraints (IQC). Zhang et al. [2021] extend the IQC framework to study smooth and *strongly* monotone VIs in the unconstrained setting. However, since the IQC framework only concerns quadratic potential functions, it is unclear how to encode our potential function and constraints into the IQC framework.

**SOS Programming and Analysis of Iterative Methods.** SOS programming has been employed in the design and analysis of algorithms in convex optimization. To the best of our knowledge, these results only concern minimization of smooth and strongly-convex functions in the unconstrained setting. Fazlyab et al. [2018] propose a framework to search the optimal parameters of the algorithm, e.g., step size. They use SOS programming to search over quadratic potential functions and parameters of the algorithm with the goal of optimizing the exponential decay rate of the potential function. Tan et al. [2021] proposes to apply SOS programming to study the convergence rates of first-order methods. Their motivation is similar to ours, that is, it may be beneficial or even necessary to analyze performance measures that cannot be handled by the PEP framework, e.g., non-quadratic functions. However, their analyses only use degree-2 SOS programs, which can thus be derived within the PEP framework. Moreover, their approach is dimension-dependent for higher degree SOS proofs, that is, for every  $n$ , they need to run a separate SOS program for the dimension- $n$  version of the problem. Compared to the literature, we are the first to analyse a high degree polynomial performance measure, and the first to apply SOS programming to analyze variational inequalities in both unconstrained and constrained settings. Moreover, unlike [Tan et al., 2021], our proof is dimension-free. We hope our result demonstrates the flexibility and power of the SOS programming based approach, and believe that there are many further applications.

**Simultaneous Result on Last-Iterate Convergence of OGDA in the Unconstrained Setting.** Shortly after we obtained the last-iterate convergence rate for OGDA in the constrained setting, we learned in early March, 2022 from private communication that Eduard Gorbunov, Gauthier Gidel, and Adrien Taylor had been working on the same problem. At the time of the communication, they could obtain the same last-iterate convergence rate for OGDA in the unconstrained case using a different method based on PEP.

## 2 Preliminaries

We consider the Euclidean Space  $(\mathbb{R}^n, \|\cdot\|)$ , where  $\|\cdot\|$  is the  $\ell_2$  norm and  $\langle \cdot, \cdot \rangle$  denotes inner product on  $\mathbb{R}^n$ . We use  $z[i]$  to denote the  $i$ -th coordinate of  $z \in \mathbb{R}^n$  and  $e_i$  to denote the unit vector such that  $e_i[j] := \mathbb{1}[i = j]$ , the dimension of  $e_i$  is going to be clear from context.

**Variational Inequality.** Given a closed convex set  $\mathcal{Z} \subseteq \mathbb{R}^n$  and an operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$ , a variational inequality problem is defined as follows: find  $z^* \in \mathcal{Z}$  such that

$$\langle F(z^*), z^* - z \rangle \leq 0 \quad \forall z \in \mathcal{Z}. \quad (3)$$

We say  $F$  is **monotone** if  $\langle F(z) - F(z'), z - z' \rangle \geq 0$ , for all  $z, z' \in \mathcal{Z}$ , and is  $L$ -Lipschitz if,  $\|F(z) - F(z')\| \leq L\|z - z'\|$  for all  $z, z' \in \mathcal{Z}$ .

**Remark 1.** One sufficient condition for such a  $z^*$  to exist is when the set  $\mathcal{Z}$  is bounded, but there are also other sufficient conditions that apply to unbounded  $\mathcal{Z}$ . See [Facchinei and Pang, 2007] for more details. Throughout this paper, we only consider monotone VIs that have a solution.

**Definition 1** (Gap Function). A standard way to measure the performance of  $z \in \mathcal{Z}$  is by its gap function defined as  $\text{GAP}_{\mathcal{Z}, F, D}(z) = \max_{z' \in \mathcal{Z} \cap \mathcal{B}(z, D)} \langle F(z), z - z' \rangle$ , where  $D > 0$  is a fixed parameter and  $\mathcal{B}(z, D)$  is a ball with radius  $D$  centered at  $z$ .<sup>9</sup> When  $\mathcal{Z}, F$  and  $D$  are clear from context, we omit the subscripts and write the gap function at  $z$  as  $\text{GAP}(z)$ .

**The Extragradient Algorithm.** Let  $z_k$  be the  $k$ -th iterate of the Extragradient (EG) Algorithm. The update rule of EG is as follows:

$$z_{k+\frac{1}{2}} = \Pi_{\mathcal{Z}} [z_k - \eta F(z_k)] = \arg \min_{z \in \mathcal{Z}} \|z - (z_k - \eta F(z_k))\|, \quad (4)$$

$$z_{k+1} = \Pi_{\mathcal{Z}} [z_k - \eta F(z_{k+\frac{1}{2}})] = \arg \min_{z \in \mathcal{Z}} \|z - (z_k - \eta F(z_{k+\frac{1}{2}}))\|. \quad (5)$$

**The Optimistic Gradient Descent-Ascent Algorithm.** Let  $z_k$  and  $w_k$  be the  $k$ -th iterate of the Optimistic Gradient Descent Ascent Method (OGDA) method. Let  $z_0, w_0$  be arbitrary starting points in  $\mathcal{Z}$ . The update rule is as follows:

$$w_{k+1} = \Pi_{\mathcal{Z}} [z_k - \eta F(w_k)] = \arg \min_{z \in \mathcal{Z}} \|z - (z_k - \eta F(w_k))\| \quad (6)$$

$$z_{k+1} = \Pi_{\mathcal{Z}} [z_k - \eta F(w_{k+1})] = \arg \min_{z \in \mathcal{Z}} \|z - (z_k - \eta F(w_{k+1}))\| \quad (7)$$

Note that the OGDA method only requires  $T$  queries to the operator at  $\{w_k\}_{0 \leq k \leq T-1}$ , while EG requires  $2T$  queries to the operator. Additionally, OGDA is a more natural algorithm in multi-agent online learning settings [Cesa-Bianchi and Lugosi, 2006, Shalev-Shwartz et al., 2012], as players play according to the strategy profile  $w_k$  and receive gradient feedback  $F(w_k)$  to compute  $z_k$  and  $w_{k+1}$ , while EG requires players to play every half step  $z_k$  and  $z_{k+\frac{1}{2}}$  to get gradient feedback. Finally, as we mentioned before, OGDA is a no-regret algorithm while EG is not.

In Section 4 and 5, we present the analysis of the EG algorithm and provide a detailed description about how to use SOS programming to derive the proof. The analysis of the OGDA algorithm is a simple extension of our analysis to the EG algorithm. We formally state the results of OGDA in Section 6 and postpone the detailed analysis of OGDA in Section H.

<sup>9</sup>Sometimes the gap function is defined to allow  $z'$  to take value in  $\mathcal{Z} \cap \mathcal{B}(z^*, \|z^* - z_0\|)$ , where  $z_0$  is the starting point of the EG algorithm, and  $z^*$  is the solution that the last iterate of the algorithm converges to. Due to Lemma 3,  $\|z_k - z^*\| \leq \|z_0 - z^*\|$  for every  $k$ , so  $\mathcal{B}(z_k, 2\|z^* - z_0\|)$  contains  $\mathcal{B}(z^*, \|z^* - z_0\|)$ .

**Sum-of-Squares (SOS) Polynomials.** Let  $\mathbf{x}$  be a set of variables. We denote the set of real polynomials in  $\mathbf{x}$  as  $\mathbb{R}[\mathbf{x}]$ . We say that polynomial  $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$  is an SOS polynomial if there exist polynomials  $\{q_i(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]\}_{i \in [M]}$  such that  $p(\mathbf{x}) = \sum_{i \in [M]} q_i(\mathbf{x})^2$ . We denote the set of SOS polynomials in  $\mathbf{x}$  as  $\text{SOS}[\mathbf{x}]$ . Note that any SOS polynomial is non-negative.

**SOS Programs.** In Figure 1 we present a generic formulation of a degree- $d$  SOS program. The SOS program takes three kinds of input, a polynomial  $g(\mathbf{x})$ , sets of polynomials  $\{g_i(\mathbf{x})\}_{i \in [M]}$  and  $\{h_i(\mathbf{x})\}_{i \in [N]}$ . Each polynomial in  $\{g(\mathbf{x})\} \cup \{g_i(\mathbf{x})\}_{i \in [M]} \cup \{h_i(\mathbf{x})\}_{i \in [N]}$  has degree of at most  $d$ . The SOS program searches for an SOS polynomial in the set of polynomials  $\Sigma = \{g(\mathbf{x}) + \sum_{i \in [M]} p_i(\mathbf{x}) \cdot g_i(\mathbf{x}) + \sum_{i \in [N]} q_i(\mathbf{x}) \cdot h_i(\mathbf{x})\}$ , where  $\{p_i(\mathbf{x})\}_{i \in [M]}$  and  $\{q_i(\mathbf{x})\}_{i \in [N]}$  are polynomials in  $\mathbf{x}$ . More precisely for each  $i \in [M]$ ,  $p_i(\mathbf{x})$  is an SOS polynomial with degree at most  $d - \deg(g_i(\mathbf{x}))$ . For each  $i \in [N]$ ,  $q_i(\mathbf{x})$  is a (not necessarily SOS) polynomial with degree at most  $d - \deg(h_i(\mathbf{x}))$ . Note that any polynomial in set  $\Sigma$  is at most degree  $d$ . In our applications, we choose  $\{g_i(\mathbf{x})\}_{i \in [M]}$  to be non-positive polynomials and  $\{h_i(\mathbf{x})\}_{i \in [N]}$  to be polynomials that are equal to 0. Any feasible solution to the program certifies the non-negativity of  $g(\mathbf{x})$ .

**Input Fixed Polynomials.**

- Polynomial  $g(\mathbf{x})$
- Polynomial  $g_i(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$  for all  $i \in [M]$ .
- Polynomial  $h_i(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$  for all  $i \in [N]$ .

**Decision Variables of the SOS Program:**

- $p_i(\mathbf{x}) \in \text{SOS}[\mathbf{x}]$  is an SOS polynomial with degree at most  $d - \deg(g_i)$ , for all  $i \in [M]$ .
- $q_i(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$  is a polynomial with degree at most  $d - \deg(h_i)$ , for all  $i \in [N]$ .

**Constraints of the SOS Program:**

$$g(\mathbf{x}) + \sum_{i \in [M]} p_i(\mathbf{x}) \cdot g_i(\mathbf{x}) + \sum_{i \in [N]} q_i(\mathbf{x}) \cdot h_i(\mathbf{x}) \in \text{SOS}[\mathbf{x}]$$

Figure 1: Generic degree  $d$  SOS program.

**Roadmap of the Paper.** In Section 3, we introduce our new performance measure – the tangent residual and prove some of its properties. In Section 4, we show that the EG algorithm enjoys best-iterate convergence. In Section 5, we strengthen the convergence guarantee for the EG algorithm and obtain the tight last-iterate convergence rate by showing that the tangent residual (Definition 3) is non-increasing across the iterations of the EG algorithm. The last-iterate convergence rate for the tangent residual also implies a last-iterate convergence rate for the gap function (Definition 1) and the natural residual (definition 4) as shown in Lemma 2 and Lemma 1. In Section 6, we further prove the tight last-iterate convergence rate for the OGDA algorithm. The analysis of the OGDA algorithm follows the same steps as in the analysis of the EG algorithm, and we

postpone most of the details in Appendix H.

### 3 The Tangent Residual and Its Properties

We formally introduce our performance measure the tangent residual . As discussed in Section 1.1, many standard and natural performance measures, i.e., the natural residual,  $\|z_k - z_{k+1/2}\|$ ,<sup>10</sup>  $\|z_k - z_{k+1}\|$ ,  $\max_{z \in \mathcal{Z}} \langle F(z), z_k - z \rangle$  and  $\max_{z \in \mathcal{Z}} \langle F(z_k), z_k - z \rangle$ , are unfortunately non-decreasing for EG. See Appendix G for numerical examples.

**Definition 2** (Unit Normal Cone). *Given a closed convex set  $\mathcal{Z} \subseteq \mathbb{R}^n$  and a point  $z \in \mathcal{Z}$ , we denote by  $N_{\mathcal{Z}}(z) = \{v \in \mathbb{R}^n : \langle v, z' - z \rangle \leq 0, \forall z' \in \mathcal{Z}\}$  the normal cone of  $\mathcal{Z}$  at point  $z$  and by  $\hat{N}_{\mathcal{Z}}(z) = \{v \in N_{\mathcal{Z}}(z) : \|v\| \leq 1\}$  the intersection of the unit ball with the the normal cone of  $\mathcal{Z}$  at  $z$ . Note that  $\hat{N}_{\mathcal{Z}}(z)$  is nonempty and compact for any  $z \in \mathcal{Z}$ , as  $(0, \dots, 0) \in \hat{N}_{\mathcal{Z}}(z)$ .*

**Definition 3** (Tangent Residual). *Given an operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$  and a closed convex set  $\mathcal{Z}$ , let  $T_{\mathcal{Z}}(z) := \{z' \in \mathbb{R}^n : \langle z', a \rangle \leq 0, \forall a \in C_{\mathcal{Z}}(z)\}$  be the tangent cone of  $z$ , and define  $J_{\mathcal{Z}}(z) := \{z\} + T_{\mathcal{Z}}(z)$ . The tangent residual of  $F$  at  $z \in \mathcal{Z}$  is defined as  $r_{(F, \mathcal{Z})}^{\text{tan}}(z) := \|\Pi_{J_{\mathcal{Z}}(z)}[z - F(z)] - z\|$ . An equivalent definition is  $r_{(F, \mathcal{Z})}^{\text{tan}}(z) := \sqrt{\|F(z)\|^2 - \max_{a \in \hat{N}_{\mathcal{Z}}(z), \langle F(z), a \rangle \leq 0} \langle a, F(z) \rangle^2}$ .*

**Remark 2.** We show the equivalence of the two definitions of tangent residual in Lemma 7. For the rest of the paper, we may use either of the two equivalent definitions depending on which one is more convenient.

When the convex set  $\mathcal{Z}$  and the operator  $F$  are clear from context, we are going to omit the subscript and denote the unit normal cone as  $\hat{N}(z) = \hat{N}_{\mathcal{Z}}(z)$  and the tangent residual as  $r^{\text{tan}}(z) = r_{(F, \mathcal{Z})}^{\text{tan}}(z)$ . Although the definition is slightly technical, one can think of the tangent residual as the norm of another operator  $\hat{F}$ , which is  $F$  projected to all directions that are not “blocked” by the boundary of  $\mathcal{Z}$  if one takes an infinitesimally small step  $\epsilon \cdot F(z)$ , which is the same as projecting  $F$  to  $J_{\mathcal{Z}}(z)$ . Intuitively, if the tangent residual is small, then the next iterate will not be far away from the current one.

Next, we formally define the natural residual associated with the instance formally stated in Definition 4, and show how it is related to the tangent residual.

**Definition 4.** *Consider an instance  $\mathcal{I}$  of the variational inequality problem on convex set  $\mathcal{Z} \subseteq \mathbb{R}^n$  and monotone operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$ . For  $z \in \mathcal{Z}$ , the natural map and natural residual associated with  $\mathcal{I}$  is defined as follows*

$$F_K^{\text{nat}}(z) = z - \Pi_{\mathcal{Z}}(z - F(z)), \quad r_{(F, \mathcal{Z})}^{\text{nat}}(z) = \|F_K^{\text{nat}}(z)\|.$$

Given an instance of the monotone VI constrained on convex set  $\mathcal{Z} \subseteq \mathbb{R}^n$  and operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$ , point  $z^*$  is a solution of the monotone VI iff  $r_{(F, \mathcal{Z})}^{\text{nat}}(z^*) = 0$ . In Lemma 1, we show that the tangent residual upper bounds the the natural residual. See Figure 2 for illustration of how the tangent residual relates to the natural residual.

<sup>10</sup>  $\|z_k - z_{k+1/2}\|$  is proportional to the norm of the operator mapping introduced in [Diakonikolas, 2020].

**Lemma 1.** Consider an instance  $\mathcal{I}$  of the variational inequality problem on convex set  $\mathcal{Z} \subseteq \mathbb{R}^n$  and monotone operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$ . For any  $z \in \mathcal{Z}$ ,  $r_{(F, \mathcal{Z})}^{tan} \geq r_{(F, \mathcal{Z})}^{nat}(z)$ .

*Proof.* Let  $w = \Pi_{\mathcal{Z}}(z - F(z))$  and  $a_1 = z - F(z) - w$ . Observe that

$$\|F(z)\|^2 = \|z - w\|^2 + \|a_1\|^2 - 2\langle z - w, a_1 \rangle.$$

Since  $r_{(F, \mathcal{Z})}^{nat}(z)^2 = \|z - w\|^2$  and  $\langle z - w, a_1 \rangle \leq 0$ , we have  $r_{(F, \mathcal{Z})}^{nat}(z)^2 \leq \|F(z)\|^2 - \|a_1\|^2$ .

According to Lemma 7,  $r^{tan}(z)^2 = \|\Pi_{J_{\mathcal{Z}}(z)}[z - F(z)] - z\|^2$ , where  $J_{\mathcal{Z}}(z) := z + T_{\mathcal{Z}}(z)$  and  $T_{\mathcal{Z}}(z) = \{z' \in \mathbb{R}^n : \langle z', a \rangle \leq 0, \forall a \in \mathcal{C}_{\mathcal{Z}}(z)\}$  is the tangent cone of  $z$ . Since  $J_{\mathcal{Z}}(z)$  is a cone with origin  $z$ , we have  $\langle z - F(z) - \Pi_{J_{\mathcal{Z}}(z)}[z - F(z)], z - \Pi_{J_{\mathcal{Z}}(z)}[z - F(z)] \rangle = 0$ , and  $\|\Pi_{J_{\mathcal{Z}}(z)}[z - F(z)] - z\|^2 = \|F(z)\|^2 - \|z - F(z) - \Pi_{J_{\mathcal{Z}}(z)}[z - F(z)]\|^2$ . As  $\mathcal{Z} \subseteq J_{\mathcal{Z}}(z)$ ,  $\|a_1\|^2 \geq \|z - F(z) - \Pi_{J_{\mathcal{Z}}(z)}[z - F(z)]\|^2$ , which implies that

$$r_{(F, \mathcal{Z})}^{nat}(z)^2 \leq \|\Pi_{J_{\mathcal{Z}}(z)}[z - F(z)] - z\|^2 = r^{tan}(z)^2.$$

□

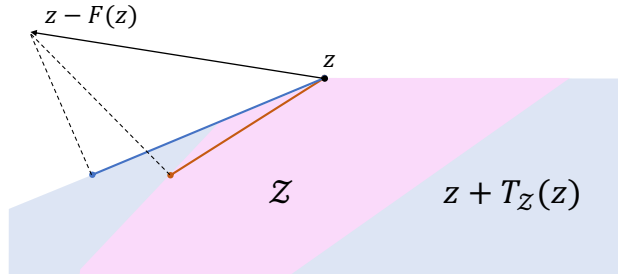


Figure 2: Illustration of the tangent residual and the natural residual. The blue line represents the tangent residual and the red line represents the natural residual. It is clear that the tangent residual upper bounds the natural residual.

Due to the above lemma, an upper bound of the tangent residual is also an upper bound of the natural residual. We show in Theorem 2 the monotonicity of the tangent residual of the EG updates, which is the technical core of our analysis and implies the  $O(\frac{1}{\sqrt{T}})$  convergence rate of the tangent residual. As a result, we also show that the natural residual has a  $O(\frac{1}{\sqrt{T}})$  convergence rate. One may be tempted to directly use the natural residual as the convergence measure. However, from our numerical experiments, the natural residual of the EG updates is not monotone, and we believe that it is very challenging to directly establish the convergence rate for the natural residue without using the tangent residual as a proxy.

In the next lemma, we argue why a small tangent residual implies a small gap function, hence an approximate solution of the variational inequality. The proof is postponed to Appendix A.

**Lemma 2.** [Adapted from the proof of Theorem 10 in [Golowich et al., 2020b].] Given a closed convex set  $\mathcal{Z} \in \mathbb{R}^n$ , an operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$  and  $z \in \mathcal{Z}$ , we have

$$\text{GAP}_{\mathcal{Z}, F, D}(z) := \max_{z' \in \mathcal{Z} \cap \mathcal{B}(z, D)} \langle F(z), z - z' \rangle \leq D \cdot r_{(F, \mathcal{Z})}^{\text{tan}}(z).$$

If we have a convex-concave function  $f(z) : \mathcal{Z} \rightarrow \mathbb{R}$  such that  $z = (x, y)$ ,  $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$  where  $\mathcal{X}$  and  $\mathcal{Y}$  are closed convex sets, let  $F(x, y) = \begin{pmatrix} \nabla_x f(x, y) \\ -\nabla_y f(x, y) \end{pmatrix}$ , then the duality gap at  $z$  with respect to  $\mathcal{X}'$  and  $\mathcal{Y}'$  is  $\text{dg}_f^{\mathcal{X}', \mathcal{Y}'}(z) := \max_{y' \in \mathcal{Y}'} f(x, y') - \min_{x' \in \mathcal{X}'} f(x', y) \leq D\sqrt{2} \cdot r_{(F, \mathcal{Z})}^{\text{tan}}(z)$ , if  $z \in \mathcal{X}' \times \mathcal{Y}'$  and the diameters of  $\mathcal{X}'$  and  $\mathcal{Y}'$  are both upper bounded by  $D$ .<sup>11</sup>

## 4 Best-Iterate Convergence of EG with Constant Step Size

En route to establish the last-iterate convergence of the EG algorithm, we first show a weaker guarantee known as the best-iterate convergence. Lemma 3 implies that after running EG for  $T$  steps, there exists an iteration  $t^* \in [T]$  where  $\|z_{t^*} - z_{t^* + \frac{1}{2}}\|^2 \leq O(\frac{1}{T})$ . The proof can be found in [Korpelevich, 1976] and [Facchinei and Pang, 2007] (included in Appendix B for completeness).

**Lemma 3** ([Korpelevich, 1976, Facchinei and Pang, 2007]). Let  $\mathcal{Z}$  be a closed convex set in  $\mathbb{R}^n$ ,  $F(\cdot)$  be a monotone and  $L$ -Lipschitz operator mapping from  $\mathcal{Z}$  to  $\mathbb{R}^n$ . For any solution  $z^*$  of the monotone VI, that is,  $\langle F(z^*), z^* - z \rangle \leq 0$  for all  $z \in \mathcal{Z}$ . For all  $k$ ,

$$\|z_k - z^*\|^2 \geq \|z_{k+1} - z^*\|^2 + (1 - \eta^2 L^2) \|z_k - z_{k+\frac{1}{2}}\|^2. \quad (8)$$

In Lemma 4, we relate  $\|z_k - z_{k+\frac{1}{2}}\|$  with the tangent residual at  $z_{k+1}$ , and derive the best-iterate convergence guarantee in terms of the tangent residual in Lemma 5. The proofs of Lemma 4 and 5 are postponed to Appendix B.

**Lemma 4.** For all  $k$ ,  $r^{\text{tan}}(z_{k+1}) \leq (1 + \eta L + (\eta L)^2) \frac{\|z_k - z_{k+\frac{1}{2}}\|}{\eta}$ .

In lemma 5, we argue the tangent residual has a best-iterate convergence with rate  $O(\frac{1}{\sqrt{T}})$ .

**Lemma 5.** Let  $\mathcal{Z}$  be a closed convex set in  $\mathbb{R}^n$ ,  $F(\cdot)$  be a monotone and  $L$ -Lipschitz operator mapping from  $\mathcal{Z}$  to  $\mathbb{R}^n$ . Suppose the step size of the EG algorithm  $\eta \in (0, \frac{1}{L})$ , then for any solution  $z^*$  of the monotone VI and any integer  $T > 0$ , there exists  $t^* \in [T]$  such that:

$$\left\| z_{t^*} - z_{t^* + \frac{1}{2}} \right\|^2 \leq \frac{1}{T} \frac{\|z_0 - z^*\|^2}{1 - (\eta L)^2}, \quad \text{AND} \quad r^{\text{tan}}(z_{t^*+1}) \leq \frac{1 + \eta L + (\eta L)^2}{\eta} \frac{1}{\sqrt{T}} \frac{\|z_0 - z^*\|}{\sqrt{1 - (\eta L)^2}}.$$

<sup>11</sup>When  $\mathcal{X}$  and  $\mathcal{Y}$  are bounded, we choose  $\mathcal{X}' = \mathcal{X}$  and  $\mathcal{Y}' = \mathcal{Y}$ , otherwise the convention is to choose  $\mathcal{X}'$  and  $\mathcal{Y}'$  to be  $\mathcal{X} \cap \mathcal{B}(x^*, 2\|x_0 - x^*\|)$  and  $\mathcal{Y} \cap \mathcal{B}(y^*, 2\|y_0 - y^*\|)$  respectively, where  $(x^*, y^*)$  is a saddle point.



## 5 Last-Iterate Convergence of EG with Constant Step Size

In this section, we show that the last-iterate convergence rate is  $O(\frac{1}{\sqrt{T}})$ . In particular, we prove that the tangent residual is non-increasing, which, in combination with Lemma 5, implies the last-iterate convergence rate of EG. To establish the monotonicity of the tangent residual, we combine SOS programming with the low-dimensionality of the EG update rule. To better illustrate our approach, we first prove the result in the unconstrained setting (Section 5.1), then show how to generalize it to the constrained setting (Section 5.2).

### 5.1 Warm Up: Unconstrained Case

As a warm-up, we consider the unconstrained setting where  $\mathcal{Z} = \mathbb{R}^n$ . Although the last-iterate convergence rate is known in the unconstrained setting due to Golowich et al. [2020b] and Gorbunov et al. [2021], we provide a simpler proof, which also permits a larger step size. Our analysis holds for any step size  $\eta \in (0, \frac{1}{L})$ , while the previous analysis requires  $\eta \leq \frac{1}{\sqrt{2}L}$  [Gorbunov et al., 2021].

Let  $z_k$  be the  $k$ -th iterate of the EG method. In Theorem 1, we show that the tangent residual is monotone in the unconstrained setting.<sup>12</sup> Our approach is to apply SOS programming to search for a certificate of non-negativity for  $\|F(z_k)\|^2 - \|F(z_{k+1})\|^2$  for every  $k$ , over the semialgebraic set defined by the following polynomial constraints in variables  $\{z_i[\ell], \eta F(z_i)[\ell]\}_{i \in \{k, k+\frac{1}{2}, k+1\}, \ell \in [n]}$ :

$$\begin{aligned} z_{k+\frac{1}{2}}[\ell] - z_k[\ell] + \eta F(z_k)[\ell] &= 0, \quad z_{k+1}[\ell] - z_k[\ell] + \eta F(z_{k+\frac{1}{2}})[\ell] = 0, \quad \forall \ell \in [n], \quad (\text{EG Update}) \\ \|\eta F(z_i) - \eta F(z_j)\|^2 - (\eta L)^2 \|z_i - z_j\|^2 &\leq 0, \quad \forall i, j \in \{k, k+\frac{1}{2}, k+1\}, \quad (\text{Lipschitzness}) \\ \langle \eta F(z_i) - \eta F(z_j), z_j - z_i \rangle &\leq 0, \quad \forall i, j \in \{k, k+\frac{1}{2}, k+1\}. \quad (\text{Monotonicity}) \end{aligned}$$

We always multiply  $F$  with  $\eta$  in the constraints as it will be convenient later. We use  $K$  to denote the set  $\{k, k+\frac{1}{2}, k+1\}$ . To obtain a certificate of non-negativity, we apply SOS programming to search for a degree-2 SOS proof. More specifically, we want to find non-negative coefficients  $\{\lambda_{i,j}^*, \mu_{i,j}^*\}_{i>j, i,j \in K}$  and degree-1 polynomials  $\gamma_1^{(\ell)}(w)$  and  $\gamma_2^{(\ell)}(w)$  in  $\mathbb{R}[w]$  for each  $\ell \in [n]$ , where  $w := \{z_i[\ell], \eta F(z_i)[\ell]\}_{i \in K, \ell \in [n]}$ , such that the following is an SOS polynomial:

$$\begin{aligned} &\|\eta F(z_k)\|^2 - \|\eta F(z_{k+1})\|^2 + \sum_{i>j \text{ and } i,j \in K} \lambda_{i,j}^* \cdot \left( \|\eta F(z_i) - \eta F(z_j)\|^2 - (\eta L)^2 \|z_i - z_j\|^2 \right) \\ &+ \sum_{i>j \text{ and } i,j \in K} \mu_{i,j}^* \cdot \langle \eta F(z_i) - \eta F(z_j), z_j - z_i \rangle + \sum_{\ell \in [n]} \gamma_1^{(\ell)}(w) (z_{k+\frac{1}{2}}[\ell] - z_k[\ell] + \eta F(z_k)[\ell]) \\ &+ \sum_{\ell \in [n]} \gamma_2^{(\ell)}(w) (z_{k+1}[\ell] - z_k[\ell] + \eta F(z_{k+\frac{1}{2}})[\ell]). \end{aligned} \tag{9}$$

<sup>12</sup>In the unconstrained setting, the tangent residual is simply the norm of the operator  $r_{(F, \mathbb{R}^n)}^{\text{tan}}(z) = \|F(z)\|$ .

Due to constraints satisfied by the EG iterates, the non-negativity of Expression (9) clearly implies that  $\|F(z_k)\|^2 - \|F(z_{k+1})\|^2$  is non-negative. However, Expression (9) is in fact an infinite family of polynomials rather than a single one. Expression (9) corresponds to a different polynomial for every integer  $n$ . To directly search for the solution, we would need to solve an infinitely large SOS program, which is clearly infeasible. By exploring the symmetry in Expression (9), we show that it suffices to solve a constant size SOS program. Let us first expand Expression (9) as follows:

$$\begin{aligned} & \sum_{\ell \in [n]} \left( (\eta F(z_k)[\ell])^2 - (\eta F(z_{k+1})[\ell])^2 + \sum_{i>j \text{ and } i,j \in K} \lambda_{i,j}^* \left( (\eta F(z_i)[\ell] - \eta F(z_j)[\ell])^2 - (\eta L)^2 (z_i[\ell] - z_j[\ell])^2 \right) \right. \\ & + \sum_{i>j \text{ and } i,j \in K} \mu_{i,j}^* (\eta F(z_i)[\ell] - \eta F(z_j)[\ell]) (z_j[\ell] - z_i[\ell]) + \gamma_1^{(\ell)}(\mathbf{w}) (z_{k+\frac{1}{2}}[\ell] - z_k[\ell] + \eta F(z_k)[\ell]) \\ & \left. + \gamma_2^{(\ell)}(\mathbf{w}) (z_{k+1}[\ell] - z_k[\ell] + \eta F(z_{k+\frac{1}{2}})[\ell]) \right). \end{aligned} \quad (10)$$

What we will argue next is that, due to the *symmetry across coordinates*, it suffices to directly search for a single SOS proof that shows that each of the  $n$  summands in Expression (10) is an SOS polynomial. More specifically, we make use of the following two key properties. (i) For any  $\ell, \ell' \in [n]$ , the  $\ell$ -th summand and  $\ell'$ -th summand are identical subject to a change of variable;<sup>13</sup> (ii) the  $\ell$ -th summand only depends on the coordinate  $\ell$ , i.e., variables in  $\{z_i[\ell], \eta F(z_i)[\ell]\}_{i \in K}$  and does not involve any other coordinates.<sup>14</sup> Both properties crucially rely on the assumption that we are in the unconstrained setting. As we will see in Section 5.2, the main source of difficulty in the constrained setting is the lack of these properties. We solve the following SOS program, whose solution can be used to construct  $\{\lambda_{i,j}^*, \mu_{i,j}^*\}_{i>j, i,j \in K}$  and  $\{\gamma_1^{(\ell)}(\mathbf{w}), \gamma_2^{(\ell)}(\mathbf{w})\}_{\ell \in [n]}$  so that each of the summands in Expression (10) is an SOS polynomial.

The proof of the following theorem is based on a feasible solution to the SOS program in Figure 3.

**Theorem 1.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a monotone and  $L$ -Lipschitz operator. Then for any  $k \in \mathbb{N}$ , the EG algorithm with step size  $\eta \in (0, \frac{1}{L})$  satisfies  $\|F(z_k)\|^2 \geq \|F(z_{k+1})\|^2$ .*

*Proof.* Since  $F$  is monotone and  $L$ -Lipschitz, we have

$$\langle F(z_{k+1}) - F(z_k), z_k - z_{k+1} \rangle \leq 0$$

and

$$\left\| F(z_{k+\frac{1}{2}}) - F(z_{k+1}) \right\|^2 - L^2 \left\| z_{k+\frac{1}{2}} - z_{k+1} \right\|^2 \leq 0.$$

We simplify them using the update rule of EG and  $\eta L < 1$ . In particular, we replace  $z_k - z_{k+1}$  with  $\eta F(z_{k+\frac{1}{2}})$  and  $z_{k+\frac{1}{2}} - z_{k+1}$  with  $\eta F(z_{k+\frac{1}{2}}) - \eta F(z_k)$ .

<sup>13</sup>Simply replace  $\{z_i[\ell]\}_{i \in K}$  and  $\{\eta F(z_i)[\ell]\}_{i \in K}$  with  $\{z_i[\ell']\}_{i \in K}$  and  $\{\eta F(z_i)[\ell']\}_{i \in K}$ .

<sup>14</sup>We mainly care about the polynomials arise from the constraints. Although  $\gamma_1^{(\ell)}(\mathbf{w})$  and  $\gamma_2^{(\ell)}(\mathbf{w})$  could depend on other coordinates, we show that it suffices to consider polynomials in  $\{z_i[\ell], \eta F(z_i)[\ell]\}_{i \in K}$ .

**Input Fixed Polynomials.** We use  $\mathbf{x}$  to denote  $(x_0, x_1, x_2)$  and  $\mathbf{y}$  to denote  $(y_0, y_1, y_2)$ . Interpret  $x_i$  as  $z_{k+\frac{i}{2}}[\ell]$  and  $y_i$  as  $\eta F(z_{k+\frac{i}{2}})[\ell]$  for  $0 \leq i \leq 2$ . Observe that  $h_1(\mathbf{x}, \mathbf{y})$  and  $h_2(\mathbf{x}, \mathbf{y})$  come from the EG update rule on coordinate  $\ell$ .  $g_{i,j}^L(\mathbf{x}, \mathbf{y})$  and  $g_{i,j}^m(\mathbf{x}, \mathbf{y})$  come from the  $\ell$ -th coordinate's contribution in the Lipschitzness and monotonicity constraints.

- $h_1(\mathbf{x}, \mathbf{y}) := x_1 - x_0 + y_0$  and  $h_2(\mathbf{x}, \mathbf{y}) := x_2 - x_0 + y_1$ .
- $g_{i,j}^L(\mathbf{x}, \mathbf{y}) := (y_i - y_j)^2 - C \cdot (x_i - x_j)^2$  for any  $0 \leq j < i \leq 2$ .<sup>a</sup>
- $g_{i,j}^m(\mathbf{x}, \mathbf{y}) := (y_i - y_j)(x_j - x_i)$  for any  $0 \leq j < i \leq 2$ .

**Decision Variables of the SOS Program:**

- $p_{i,j}^L \geq 0$ , and  $p_{i,j}^m \geq 0$ , for all  $0 \leq j < i \leq 2$ .
- $q_1(\mathbf{x}, \mathbf{y})$  and  $q_2(\mathbf{x}, \mathbf{y})$  are two degree 1 polynomials in  $\mathbb{R}[\mathbf{x}, \mathbf{y}]$ .

**Constraints of the SOS Program:**

$$\begin{aligned} \text{s.t. } y_0^2 - y_2^2 + \sum_{2 \geq i > j \geq 0} p_{i,j}^L \cdot g_{i,j}^L(\mathbf{x}, \mathbf{y}) + \sum_{2 \geq i > j \geq 0} p_{i,j}^m \cdot g_{i,j}^m(\mathbf{x}, \mathbf{y}) \\ + q_1(\mathbf{x}, \mathbf{y}) \cdot h_1(\mathbf{x}, \mathbf{y}) + q_2(\mathbf{x}, \mathbf{y}) \cdot h_2(\mathbf{x}, \mathbf{y}) \in \text{SOS}[\mathbf{x}, \mathbf{y}]. \end{aligned} \quad (11)$$

<sup>a</sup>C represents  $(\eta L)^2$ . Larger  $C$  corresponds to a larger step size and makes the SOS program harder to satisfy. Through binary search, we find that the largest possible value of  $C$  is 1 while maintaining the feasibility of the SOS program.

Figure 3: Our SOS program in the unconstrained setting.

$$\left\langle F(z_{k+1}) - F(z_k), F(z_{k+\frac{1}{2}}) \right\rangle \leq 0, \quad (12)$$

$$\left\| F(z_{k+\frac{1}{2}}) - F(z_{k+1}) \right\|^2 - \left\| F(z_{k+\frac{1}{2}}) - F(z_k) \right\|^2 \leq 0. \quad (13)$$

In Proposition 1 at Appendix C we verify the following identity.

$$\|F(z_k)\|^2 - \|F(z_{k+1})\|^2 + 2 \cdot \text{LHS of Inequality (12)} + \text{LHS of Inequality (13)} = 0.$$

Thus,  $\|F(z_k)\|^2 - \|F(z_{k+1})\|^2 \geq 0$ . □

Corollary 1 is implied by combining Lemma 2, Lemma 5, Theorem 1 and the fact that  $\eta \in (0, \frac{1}{L})$ .

**Corollary 1.** Let  $F(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a monotone and  $L$ -Lipshitz operator and  $z^* \in \mathbb{R}^n$  be a solution to the variational inequality. For any  $T \geq 1$ , let  $z_T$  be  $T$ -th iterate of the EG algorithm with constant step size  $\eta \in (0, \frac{1}{L})$ , then  $\text{GAP}_{\mathbb{R}^n, F, D}(z_T) \leq \frac{1}{\sqrt{T}} \frac{3D\|z_0 - z^*\|}{\eta\sqrt{1-(\eta L)^2}}$ .

## 5.2 Last-Iterate Convergence of EG with Arbitrary Convex Constraints

We establish the last-iterate convergence rate of the EG algorithm in the constrained setting in this section. The plan is similar to the one in Section 5.1. First, we use the assistance of SOS programming to prove the monotonicity of the tangent residual (Theorem 2), then combine it with the best-iterate convergence guarantee from Lemma 5 to derive the last-iterate convergence rate (Theorem 3).

The crux is proving the monotonicity of the tangent residual. Due to the constraints introduced by the domain  $\mathcal{Z}$ , our approach in the unconstrained case no longer applies. More specifically, the coordinates are now entangled due to projection, ruining the two key properties – Property (i) and (ii) that we rely on in the unconstrained setting. We show how to reduce *both the number of constraints and the dimension by exploiting the low dimensionality of the EG update rule*. Equipped with the reduction, we convert the problem of proving the monotonicity of the tangent residual to solving a small constant size SOS program.

**Reducing the Number of Constraints.** Suppose we are not given the description of  $\mathcal{Z}$ , and we only observe one iteration of the EG algorithm. In other words, we know  $z_k$ ,  $z_{k+\frac{1}{2}}$ , and  $z_{k+1}$ , as well as  $F(z_k)$ ,  $F(z_{k+\frac{1}{2}})$ , and  $F(z_{k+1})$ . To compute the squared tangent residual at  $z_k$ , let us also assume that the unit vector  $-a_k \in \hat{N}(z_k)$  satisfies  $r_{(F, \mathcal{Z})}^{\tan}(z_k)^2 = \|F(z_k)\|^2 - \langle F(z_k), -a_k \rangle^2$ . From this limited information, what can we learn about  $\mathcal{Z}$ ? We can conclude that  $\mathcal{Z}$  must lie in the intersection of the following halfspaces: (a)  $\langle a_k, z \rangle \geq b_k$ , where  $b_k = \langle a_k, z_k \rangle$ . This is true because  $-a_k \in \hat{N}(z_k)$ . (b)  $\langle a_{k+\frac{1}{2}}, z \rangle \geq b_{k+\frac{1}{2}}$ , where  $a_{k+\frac{1}{2}} = \frac{z_{k+\frac{1}{2}} - z_k + \eta F(z_k)}{\|z_{k+\frac{1}{2}} - z_k + \eta F(z_k)\|}$  and  $b_{k+\frac{1}{2}} = \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle$ . This is true because  $z_{k+\frac{1}{2}} = \Pi_{\mathcal{Z}}(z_k - \eta F(z_k))$ , so  $\langle z_{k+\frac{1}{2}} - z_k + \eta F(z_k), z - z_{k+\frac{1}{2}} \rangle \geq 0$  for all  $z \in \mathcal{Z}$ . (c)  $\langle a_{k+1}, z \rangle \geq b_{k+1}$ , where  $a_{k+1} = \frac{z_{k+1} - z_k + \eta F(z_{k+\frac{1}{2}})}{\|z_{k+1} - z_k + \eta F(z_{k+\frac{1}{2}})\|}$  and  $b_{k+1} = \langle a_{k+1}, z_{k+1} \rangle$ . This is true because  $z_{k+1} = \Pi_{\mathcal{Z}}(z_k - \eta F(z_{k+\frac{1}{2}}))$ , so  $\langle z_{k+1} - z_k + \eta F(z_{k+\frac{1}{2}}), z - z_{k+1} \rangle \geq 0$  for all  $z \in \mathcal{Z}$ . See Figure 4 for illustration.

The “hardest instance” of  $\mathcal{Z}$  that is consistent with our knowledge of  $z_k$ ,  $z_{k+\frac{1}{2}}$ , and  $z_{k+1}$  is when  $\mathcal{Z}$  is exactly the intersection of these three halfspaces. In such case, the squared tangent residual of  $z_{k+1}$  is  $\|F(z_{k+1})\|^2 - \langle F(z_{k+1}), a_{k+1} \rangle^2 \cdot \mathbb{1}[\langle F(z_{k+1}), a_{k+1} \rangle \geq 0]$ , and it is an upper bound of  $r_{(F, \mathcal{Z})}^{\tan}(z_{k+1})^2$  for any other consistent  $\mathcal{Z}$ . Our goal is to prove the tangent residual is non-increasing even in the “hardest case”, that is, to prove the non-negativity of

$$\|F(z_k)\|^2 - \langle F(z_k), a_k \rangle^2 - \left( \|F(z_{k+1})\|^2 - \langle F(z_{k+1}), a_{k+1} \rangle^2 \cdot \mathbb{1}[\langle F(z_{k+1}), a_{k+1} \rangle \geq 0] \right) \quad (14)$$

**Low-dimensionality of an EG Update.** As there are only three hyperplanes  $\langle a_i, z \rangle \geq b_i$  for  $i \in \{k, k+\frac{1}{2}, k+1\}$  involved, we can choose a new basis, so that  $a_{k+1} = (1, 0, \dots, 0)$ ,  $a_{k+\frac{1}{2}} = (\theta_1, \theta_2, 0, \dots, 0)$ , and  $a_k = (\sigma_1, \sigma_2, \sigma_3, 0, \dots, 0)$ . As  $a_{k+\frac{1}{2}}$  and  $z_{k+\frac{1}{2}} - z_k + \eta F(z_k)$  are co-directed, and  $a_{k+1}$  and  $z_{k+1} - z_k + \eta F(z_{k+\frac{1}{2}})$  are co-directed, *an important property of this change of basis is that the*

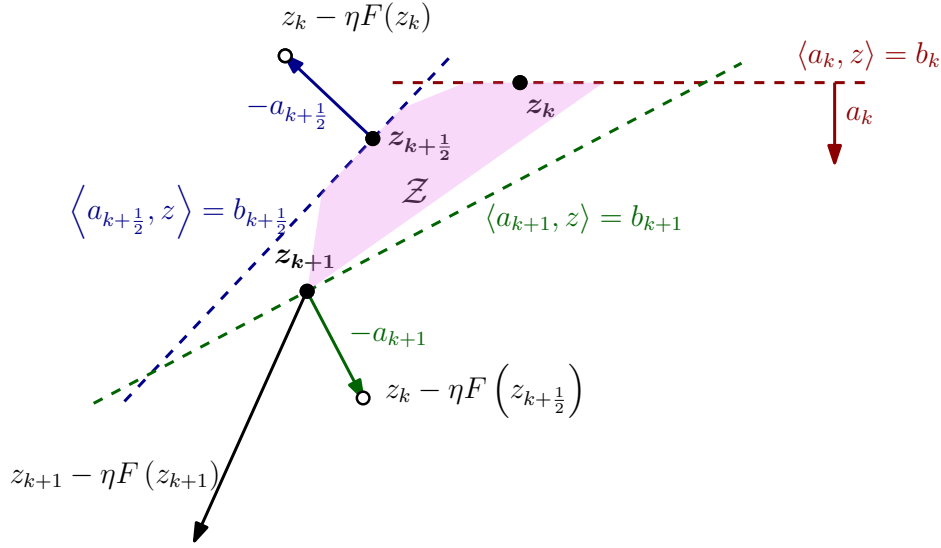


Figure 4: We illustrate how to reduce the number of constraints. Let  $\mathcal{Z}' = \{z : \langle a_{k+\frac{i}{2}}, z \rangle \geq b_{k+\frac{i}{2}}, \text{ for } i \in \{0, 1, 2\}\}$ . Pictorially  $-F(z_{k+1}) \in N_{\mathcal{Z}}(z_{k+1})$ , while  $-F(z_{k+1}) \notin N_{\mathcal{Z}'}(z_{k+1})$  which further implies  $r_{(F, \mathcal{Z})}^{\tan}(z_{k+1}) = 0$  and  $r_{(F, \mathcal{Z}')}^{\tan}(z_{k+1}) > 0$ .

EG update from  $z_k$  to  $z_{k+1}$  is unconstrained in all coordinates  $\ell \geq 4$ . More specifically,

$$z_{k+\frac{1}{2}}[\ell] - z_k[\ell] + \eta F(z_k)[\ell] = 0, \quad z_{k+1}[\ell] - z_k[\ell] + \eta F(z_{k+\frac{1}{2}})[\ell] = 0, \quad \forall \ell \geq 4.$$

Hence, we can represent all of the coordinates  $\ell \geq 4$  with one coordinate in the SOS program similar to the unconstrained case. We still need to keep the first three dimensions, but now we only face a problem in dimension 4 rather than in dimension  $n$ , and we can form a constant size SOS program to search for a certificate of non-negativity for Expression (14).

In Lemma 6, we further simplify the instance that we need to consider. In particular, we argue that it is w.l.o.g. to assume that (1)  $a_{k+1}$ ,  $a_{k+\frac{1}{2}}$ , and  $a_k$  are linear independent and (2) the intersection of the three halfspaces forms a cone, i.e.,  $b_k = b_{k+\frac{1}{2}} = b_{k+1} = 0$ . Both assumption (1) and (2) reduce the number of variables we need to consider in the SOS program, so a low degree SOS proof is more likely to exist. To maximally reduce the number of variables, we only included the minimal number of constraints that suffice to derive an SOS proof.

The proof of Lemma 6 is postponed to Appendix E.

**Lemma 6** (Simplification Procedure). *Let  $\mathcal{I}$  be a variational inequality problem for a closed convex set  $\mathcal{Z} \subseteq \mathbb{R}^n$  and a monotone and  $L$ -Lipschitz operator  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$ . Suppose the EG algorithm has a constant step size  $\eta$ . Let  $z_k$  be the  $k$ -th iteration of the EG algorithm,  $z_{k+\frac{1}{2}}$  be the  $(k + \frac{1}{2})$ -th iteration as defined in (4), and  $z_{k+1}$  be the  $(k + 1)$ -th iteration as defined in (5).*

*Then either  $r_{(F, \mathcal{Z})}^{\tan}(z_k) \geq r_{(F, \mathcal{Z})}^{\tan}(z_{k+1})$ , or there exist vectors  $\bar{a}_k, \bar{a}_{k+\frac{1}{2}}, \bar{a}_{k+1}, \bar{z}_k, \bar{z}_{k+\frac{1}{2}}, \bar{z}_{k+1}$ ,  $\bar{F}(\bar{z}_k), \bar{F}(\bar{z}_{k+\frac{1}{2}}), \bar{F}(\bar{z}_{k+1}) \in \mathbb{R}^N$  with  $N \leq n + 5$  that satisfy the following conditions.*

1.  $\bar{a}_k = (\beta_1, \beta_2, 1, 0, \dots, 0)$ ,  $\bar{a}_{k+\frac{1}{2}} = (\alpha, 1, 0, \dots, 0)$ , and  $\bar{a}_{k+1} = (1, 0, \dots, 0)$  for some  $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ .
2.  $\left\| \bar{F}(\bar{z}_k) - \frac{\langle F(\bar{z}_k), \bar{a}_k \rangle \cdot \bar{a}_k}{\|\bar{a}_k\|^2} \right\|^2 - \left\| \bar{F}(\bar{z}_{k+1}) - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \cdot \bar{a}_{k+1}}{\|\bar{a}_{k+1}\|^2} \mathbb{1}[\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \geq 0] \right\|^2 < 0$ .
3. Additionally,  $\langle \bar{a}_i, \bar{z}_j \rangle \geq 0$  and  $\langle \bar{a}_i, \bar{z}_i \rangle = 0$  for all  $i, j \in \{k, k + \frac{1}{2}, k + 1\}$ .  $\bar{a}_{k+\frac{1}{2}}$  and  $\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k)$  are co-directed, i.e., they are colinear and have the same direction, and  $\bar{a}_{k+1}$  and  $\bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})$  are co-directed.
- 4.

$$\left\| \bar{F}(\bar{z}_{k+1}) - \bar{F}(\bar{z}_{k+\frac{1}{2}}) \right\|^2 \leq L^2 \left\| \bar{z}_{k+1} - \bar{z}_{k+\frac{1}{2}} \right\|^2 \quad (15)$$

$$\langle \bar{F}(\bar{z}_{k+1}) - \bar{F}(\bar{z}_k), \bar{z}_{k+1} - \bar{z}_k \rangle \geq 0 \quad (16)$$

$$\langle \bar{a}_k, \bar{F}(\bar{z}_k) \rangle \geq 0. \quad (17)$$

In Theorem 2, we establish the monotonicity of the tangent residual. Our proof is based on the solution to the degree-8 SOS program concerning polynomials in 27 variables (Figure 5). We include a proof sketch of Theorem 2 in the main body, and the full proof in Appendix E.

**Theorem 2.** Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$  be a monotone and  $L$ -Lipschitz operator. For any step size  $\eta \in (0, \frac{1}{L})$  and any  $z_k \in \mathcal{Z}$ , the EG method update satisfies  $r_{(F, \mathcal{Z})}^{\text{tan}}(z_k) \geq r_{(F, \mathcal{Z})}^{\text{tan}}(z_{k+1})$ .

**Proof Sketch** Assume towards contradiction that  $r^{\text{tan}}(z_k) < r^{\text{tan}}(z_{k+1})$ , using Lemma 6 there exist numbers  $\alpha, \beta_1, \beta_2 \in \mathbb{R}$  and vectors  $\bar{a}_k, \bar{a}_{k+\frac{1}{2}}, \bar{a}_{k+1}, \bar{z}_k, \bar{z}_{k+\frac{1}{2}}, \bar{z}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{z}_{k+\frac{1}{2}}), \bar{F}(\bar{z}_{k+1}) \in \mathbb{R}^N$  where  $N = n + 5$  that satisfy the properties in the statement of Lemma 6 and

$$0 > \left\| \bar{F}(\bar{z}_k) - \frac{\langle F(\bar{z}_k), \bar{a}_k \rangle}{\|\bar{a}_k\|^2} \bar{a}_k \right\|^2 - \left\| \bar{F}(\bar{z}_{k+1}) - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \bar{a}_{k+1}}{\|\bar{a}_{k+1}\|^2} \mathbb{1}[\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \geq 0] \right\|^2.$$

We use TARGET to denote the RHS above. We first present 8 inequalities/equations, which directly follow from Lemma 6. We verify their correctness in the complete proof in Appendix E.



$$\langle \eta \bar{F}(\bar{z}_{k+1}) - \eta \bar{F}(\bar{z}_k), \bar{z}_k - \bar{z}_{k+1} \rangle \leq 0, \quad (18)$$

$$\left\| \eta \bar{F}(\bar{z}_{k+1}) - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) \right\|^2 - \left\| \bar{z}_{k+1} - \bar{z}_{k+\frac{1}{2}} \right\|^2 \leq 0, \quad (19)$$

$$\bar{z}_{k+\frac{1}{2}}[1] \left( \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right)[1] - \alpha \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right)[2] \right) = 0, \quad (20)$$

$$\bar{z}_{k+1}[2] \left( \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right)[1] - \alpha \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right)[2] \right) = 0, \quad (21)$$

$$(\alpha(\bar{z}_k - \eta \bar{F}(\bar{z}_k))[1] + (\bar{z}_k - \eta \bar{F}(\bar{z}_k))[2])(\alpha \bar{z}_{k+1}[1] + \bar{z}_{k+1}[2]) \leq 0, \quad (22)$$

$$-\eta(\beta_1 \bar{F}(\bar{z}_k)[1] + \beta_2 \bar{F}(\bar{z}_k)[2] + \bar{F}(\bar{z}_k)[3]) \left( \beta_1 \bar{z}_{k+\frac{1}{2}}[1] + \beta_2 \bar{z}_{k+\frac{1}{2}}[2] + \bar{z}_{k+\frac{1}{2}}[3] \right) \leq 0, \quad (23)$$

$$\bar{z}_k[1] \left( \bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1] \right) \leq 0, \quad (24)$$

$$-\eta \bar{F}(\bar{z}_{k+1})[1] \mathbb{1} \left[ \bar{F}(\bar{z}_{k+1})[1] \leq 0 \right] \left( \bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1] \right) \leq 0. \quad (25)$$

We first add several non-positive terms to TARGET to derive Expression 26.

$$\begin{aligned} & \eta^2 \cdot \text{TARGET} + 2 \cdot \text{LHS of Inequality (18)} + \text{LHS of Inequality (19)} + 2 \cdot \text{LHS of Equation (20)} \\ & + \frac{2\alpha}{1+\alpha^2} \cdot \text{LHS of Equation (21)} + \frac{2}{1+\alpha^2} \cdot \text{LHS of Inequality (22)} + \frac{2}{1+\beta_1^2+\beta_2^2} \cdot \text{LHS of Inequality (23)} \\ & + 2 \cdot \text{LHS of Inequality (24)} + 2 \cdot \text{LHS of Inequality (25)} \end{aligned} \quad (26)$$

After substituting the following six variables  $\bar{z}_k[3], \bar{z}_{k+\frac{1}{2}}[2], \bar{z}_{k+\frac{1}{2}}[3], \bar{z}_{k+1}[1], \bar{z}_{k+1}[2], \bar{z}_{k+1}[3]$  using Equation (67) to (72), Expression (26) equals to the following polynomial, which is clearly non-negative. We verify the validity of the substitutions (Equation (67) to (72)) in the full proof.

$$\begin{aligned} & \left( \bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1] + \eta \bar{F}(\bar{z}_{k+1})[1] \cdot \mathbb{1}[\bar{F}(\bar{z}_{k+1})[1] \geq 0] \right)^2 \\ & + \frac{\left( \bar{z}_k[1] - \eta \bar{F}(\bar{z}_k)[1] - \bar{z}_{k+\frac{1}{2}}[1] \right)^2}{1 + \beta_1^2 + \beta_2^2} \\ & + \frac{\left( \eta \bar{F}(\bar{z}_k)[3] + \beta_1 \bar{z}_k[1] + \beta_2 \bar{z}_k[2] + (\alpha \beta_2 - \beta_1) \bar{z}_{k+\frac{1}{2}}[1] \right)^2}{1 + \beta_1^2 + \beta_2^2} \\ & + \frac{\left( \bar{z}_k[2] - \eta \bar{F}(\bar{z}_k)[2] + \alpha \bar{z}_{k+\frac{1}{2}}[1] \right)^2}{1 + \beta_1^2 + \beta_2^2} \\ & + \frac{\left( \beta_1 (\bar{z}_k[2] - \eta \bar{F}(\bar{z}_k)[2] + \alpha \bar{z}_{k+\frac{1}{2}}[1]) - \beta_2 (\bar{z}_k[1] - \eta \bar{F}(\bar{z}_k)[1] - \bar{z}_{k+\frac{1}{2}}[1]) \right)^2}{1 + \beta_1^2 + \beta_2^2} \end{aligned}$$

■

**Theorem 3.** Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set,  $F(\cdot) : \mathcal{Z} \rightarrow \mathbb{R}^n$  be a monotone and  $L$ -Lipschitz operator and  $z^* \in \mathcal{Z}$  be the solution to the variational inequality. Then for any  $T \geq 1$ ,  $z_T$  produced by EG with any constant step size  $\eta \in (0, \frac{1}{L})$  satisfies

- $\text{GAP}(z_T) \leq \frac{1}{\sqrt{T}} \frac{3D\|z_0 - z^*\|}{\eta\sqrt{1-(\eta L)^2}},$
- $r^{\text{nat}}(z_T) \leq r^{\text{tan}}(z_T) \leq \frac{1}{\sqrt{T}} \frac{3\|z_0 - z^*\|}{\eta\sqrt{1-(\eta L)^2}}.$

Theorem 3 is implied by combining Lemma 1, Lemma 2, Lemma 5, Theorem 2 and the fact that  $\eta \in (0, \frac{1}{L})$ . Choosing  $\eta$  to be  $\frac{1}{2L}$  and  $D = O(\|z_0 - z^*\|)$ , then  $\text{GAP}(z_T) = O(\frac{D^2L}{\sqrt{T}})$  matching the  $\Omega(\frac{D^2L}{\sqrt{T}})$  lower bound for EG, OGDA, and more generally all p-SCLI algorithms [Golowich et al., 2020b,a] in terms of the dependence on  $D$ ,  $L$ , and  $T$ . Additionally,  $r_{\mathcal{Z},F,D}^{\text{nat}}(z_T) = O(\frac{DL}{\sqrt{T}})$ , and  $r_{\mathcal{Z},F,D}^{\text{tan}}(z_T) = O(\frac{DL}{\sqrt{T}})$ , so our upper bounds for both the natural residual and tangent residual also match the  $\Omega(\frac{DL}{\sqrt{T}})$  lower bounds with respect to natural residual and tangent residual for EG [Golowich et al., 2020b]. This is because both the natural residual and tangent residual are equivalent to the norm of the operator, and [Golowich et al., 2020b] shows that in the unconstrained setting  $\|F(z_T)\| = \Omega(\frac{DL}{\sqrt{T}})$ .

## 6 Last-Iterate Convergence of OGDA with Constant Step Size

In this section, we show that the OGDA algorithm with constant step size  $\eta \in (0, \frac{1}{2L})$  has  $O(\frac{1}{\sqrt{T}})$  last-iterate convergence rate with respect to the tangent residual or the gap function. The analysis of the OGDA algorithm follows the same steps as in the analysis of the EG algorithm. Compared to the EG algorithm, the last iterate convergence of OGDA follows by builds on the monotonicity and best-iterate convergence of the following potential function

$$\Phi_k = \|F(z_k) - F(w_k)\|^2 + r^{\text{tan}}(z_k)^2. \quad (27)$$

The potential function can be thought of as the tangent residual ( $r^{\text{tan}}(z_k)^2$ ) and an extra correction term  $\|F(z_k) - F(w_k)\|^2$ . The potential function is discovered directly through SOS programming. The SOS program was formulated by searching over linear combinations of  $\|F(z_k)\|^2 - \|F(z_{k+1})\|^2, \|F(w_k)\|^2 - \|F(w_{k+1})\|^2, \langle F(z_k), F(w_k) \rangle - \langle F(z_{k+1}), F(w_{k+1}) \rangle$  and  $r^{\text{tan}}(z_k) - r^{\text{tan}}(z_{k+1})$ , under (i) the constraint that the linear combination is non-increasing,<sup>15</sup> and (ii) the constraints induced by properties of the operator  $F(\cdot)$ , the update rule of OGDA and the set  $\mathcal{Z}$  (See Figure 5 for a demonstration of the induced constraints of EG algorithm for solving a monotone VI over convex constraints). We then use the linear combination output by the SOS program as the potential function in our analysis. We believe our heuristic for finding a potential function could be useful

<sup>15</sup>To avoid finding the trivial linear combination, i.e., all coefficients equal to 0, we also use the objective function in the SOS program to encourage a non-trivial solution if one exists by, for example, maximizing the sum of the coefficients of the linear combination.

in other settings. In general, one can first choose a collection of basis functions that may be part of a potential function, then use SOS programming to search over all linear combinations of the basis functions subject to the constraint that the linear combination is non-negative to discover the potential function. We postpone all technical details to Appendix H.

## 6.1 Warm Up: Unconstrained Case

In this section, we show that in the unconstrained setting the potential function  $\Phi_k$  is non-increasing, which implies the  $O(\frac{1}{\sqrt{T}})$  last-iterate convergence rate for OGDA with respect to the tangent residual, the natural residual, and the gap function.

**Theorem 4.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a monotone and  $L$ -Lipschitz operator. Then for any  $k \in \mathbb{N}$ , the OGDA algorithm with step size  $\eta \in (0, \frac{1}{2L})$  satisfies  $\|F(z_k) - F(w_k)\|^2 + \|F(z_k)\|^2 \geq \|F(z_{k+1}) - F(w_{k+1})\|^2 + \|F(z_{k+1})\|^2$ .*

*Proof.* Since  $F$  is monotone and  $L$ -Lipschitz, we have  $\langle F(z_{k+1}) - F(z_k), z_k - z_{k+1} \rangle \leq 0$  and  $\|F(w_{k+1}) - F(z_{k+1})\|^2 - L^2\|w_{k+1} - z_{k+1}\|^2 \leq 0$ . We simplify them using the update rule of OGDA and  $\eta^2 L^2 < \frac{1}{4}$ . In particular, we replace  $z_k - z_{k+1}$  by  $\eta F(w_{k+1})$  and  $w_{k+1} - z_{k+1}$  with  $\eta F(w_{k+1}) - \eta F(w_k)$ .

$$\langle F(z_{k+1}) - F(z_k), F(w_{k+1}) \rangle \leq 0, \quad (28)$$

$$\|F(w_{k+1}) - F(z_{k+1})\|^2 - \frac{1}{4}\|F(w_{k+1}) - F(w_k)\|^2 \leq 0. \quad (29)$$

MATLAB code for the verification of the following identity can be found at this [link](#).

$$\begin{aligned} & \|F(z_k) - F(w_k)\|^2 + \|F(z_k)\|^2 - \|F(z_{k+1}) - F(w_{k+1})\|^2 - \|F(z_{k+1})\|^2 \\ & + 2 \cdot \text{LHS of Inequality (28)} + 2 \cdot \text{LHS of Inequality (29)} \\ & = \frac{1}{2}\|F(w_k) + F(w_{k+1}) - 2F(z_k)\|^2. \end{aligned} \quad (30)$$

Thus,  $\|F(z_k) - F(w_k)\|^2 + \|F(z_k)\|^2 \geq \|F(z_{k+1}) - F(w_{k+1})\|^2 + \|F(z_{k+1})\|^2$ .  $\square$

The following theorem is a combination of Corollary 2, Theorem 4, Lemma 13, Lemma 1 and Lemma 2.

**Theorem 5.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a monotone and  $L$ -Lipschitz operator. Let  $z_0 = w_0 \in \mathbb{R}^n$  be arbitrary starting point and  $\{z_k, w_k\}_{k \geq 0}$  be the iterates of the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$ . Denote  $D_0 := \sqrt{(4 + 6\eta^4 L^4)\|z_0 - z^*\|^2 + (16\eta^2 L^2 + 6\eta^4 L^4)\|w_0 - z_0\|^2} = O(\max\{\|z_0 - z^*\|, \|w_0 - z_0\|\})$ . Then for any  $T \geq 1$ ,*

- $\text{GAP}_{Z,F,D}(z_T) \leq \frac{1}{\sqrt{T}} \cdot \frac{DD_0}{\eta\sqrt{1-4(\eta L)^2}}.$

- $r_{\mathcal{Z},F,D}^{\text{nat}}(z_T) \leq r_{\mathcal{Z},F,D}^{\text{tan}}(z_T) \leq \frac{1}{\sqrt{T}} \cdot \frac{D_0}{\eta\sqrt{1-4(\eta L)^2}}.$
- $\text{GAP}_{\mathcal{Z},F,D}(w_{T+1}) \leq \frac{1}{\sqrt{T}} \cdot \frac{\sqrt{2}(2+\eta L) \cdot DD_0}{\eta\sqrt{1-4(\eta L)^2}}.$
- $r_{\mathcal{Z},F,D}^{\text{nat}}(w_{T+1}) \leq r_{\mathcal{Z},F,D}^{\text{tan}}(w_{T+1}) \leq \frac{1}{\sqrt{T}} \cdot \frac{\sqrt{2}(2+\eta L) \cdot D_0}{\eta\sqrt{1-4(\eta L)^2}}.$

## 6.2 Last-Iterate Convergence of OGDA with Arbitrary Convex Constraints

In this section, we formally state the last-iterate convergence of OGDA algorithm with respect to the gap function, the natural residual and the tangent residual in the constrained setting. All the details are postponed to Appendix H.

**Theorem 6.** *Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be a monotone and  $L$ -Lipschitz operator. Let  $z_0, w_0 \in \mathcal{Z}$  be arbitrary starting point and  $\{z_k, w_k\}_{k \geq 0}$  be the iterates of the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$ . Let  $D_0 := \sqrt{(4 + 6\eta^4 L^4) \|z_0 - z^*\|^2 + (16\eta^2 L^2 + 6\eta^4 L^4) \|w_0 - z_0\|^2} = O(\max\{\|z_0 - z^*\|, \|w_0 - z_0\|\})$ . Then for any  $T \geq 1$ ,*

- $\text{GAP}_{\mathcal{Z},F,D}(z_T) \leq \frac{DD_0}{\eta \cdot \sqrt{T \cdot (1-4(\eta L)^2)}}.$
- $r_{\mathcal{Z},F,D}^{\text{nat}}(z_T) \leq r_{\mathcal{Z},F,D}^{\text{tan}}(z_T) \leq \frac{D_0}{\eta \cdot \sqrt{T \cdot (1-4(\eta L)^2)}}.$
- $\text{GAP}_{\mathcal{Z},F,D}(w_{T+1}) \leq \frac{\sqrt{2}(2+\eta L) \cdot D \cdot D_0}{\eta \cdot \sqrt{T \cdot (1-4(\eta L)^2)}}.$
- $r_{\mathcal{Z},F,D}^{\text{nat}}(w_{T+1}) \leq r_{\mathcal{Z},F,D}^{\text{tan}}(w_{T+1}) \leq \frac{\sqrt{2}(2+\eta L) D_0}{\eta \cdot \sqrt{T \cdot (1-4(\eta L)^2)}}.$

Setting  $D = \max\{\|z_0 - z^*\|, \|w_0 - z_0\|\}$ , and  $\eta = \frac{1}{2\sqrt{2}L}$ , we have  $\text{GAP}_{\mathcal{Z},F,D}(z_T)$  (or  $\text{GAP}_{\mathcal{Z},F,D}(w_T)$ )  $= O(\frac{D^2 L}{\sqrt{T}})$ , which matches the lower bound of  $\Omega(\frac{D^2 L}{\sqrt{T}})$  by Golowich et al. [2020a].

## 7 Conclusion

We propose a new method based on SOS programming to analyze iterative algorithms. Combining with the low-dimensionality of the EG and OGDA update rules, we provide the first and tight last-iterate convergence rate of the EG and ODGA algorithm for constrained monotone and Lipschitzs VIs, which, to the best of our knowledge, is also the first last-iterate convergence rate for any p-SCLI algorithm in the constrained setting. We believe many other iterative methods also have the low-dimensionality that make them amenable to our SOS programming based analysis.

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## A Additional Preliminaries

For  $z \in \mathbb{R}^n$  and  $D > 0$ , we use  $\mathcal{B}(z, D) = \{z' \in \mathbb{R}^n : \|z' - z\| \leq D\}$  to denote the ball of radius  $D$ , centered at  $z$ .

**Min-Max Saddle Points.** A special case of the variational inequality problem is the constrained min-max problem  $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y)$ , where  $\mathcal{X}$  and  $\mathcal{Y}$  are closed convex sets in  $\mathbb{R}^n$ , and  $f(\cdot, \cdot)$  is smooth, convex in  $x$ , and concave in  $y$ . It is well known that if one set  $F(x, y) = \begin{pmatrix} \nabla_x f(x, y) \\ -\nabla_y f(x, y) \end{pmatrix}$ , then  $F(x, y)$  is a monotone and Lipschitz operator [Facchinei and Pang, 2007].

**Equilibria of Monotone Games.** Monotone games are a large class of multi-player games that include many common and well-studied class of games such as bilinear games,  $\lambda$ -cocoercive games [Lin et al., 2020], zero-sum polymatrix games [Cai and Daskalakis, 2011, Cai et al., 2016], and zero-sum socially-concave games [Even-Dar et al., 2009]. Besides, the min-max saddle point problem is a special case of two-player monotone games. We include the definition of monotone games here and remind readers that finding a Nash Equilibrium of a monotone game is exactly the same as finding a solution to a monotone variational inequality.

A continuous game  $\mathcal{G}$  is denoted as  $(\mathcal{N}, (\mathcal{X}_i)_{i \in [N]}, (f_i)_{i \in [N]})$  where there are  $N$  players  $\mathcal{N} = \{1, \dots, N\}$ . Player  $i \in \mathcal{N}$  chooses action from a closed convex set  $\mathcal{X}_i \in \mathbb{R}^{n_i}$  such that  $\mathcal{X} := \prod_{i \in \mathcal{N}} \mathcal{X}_i \in \mathbb{R}^n$  and wants to minimize its cost function  $f_i : \mathcal{X} \rightarrow \mathbb{R}$ . For each player  $i$ , we denote  $x_{-i}$  the vector of actions of all the other players. A *Nash Equilibrium* of game  $\mathcal{G}$  is an action profile  $x^* \in \mathcal{X}$  such that  $f_i(x^*) \leq f_i(x'_i, x^*_{-i})$  for any  $x'_i \in \mathcal{X}_i$ . Let  $F(x) = (\nabla_{x_i} f_i(x), \dots, \nabla_{x_N} f_N(x)) \in \mathbb{R}^n$ . We say  $\mathcal{G}$  is *monotone* if  $\langle F(x) - F(x'), x - x' \rangle \geq 0$  for any  $x, x' \in \mathcal{X}$ .

In Lemma 7 we present several equivalent formulations of the tangent residual.

**Lemma 7.** Let  $\mathcal{Z}$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$  be an operator. Denote  $N_{\mathcal{Z}}(z)$  the normal cone of  $z$  and  $J_{\mathcal{Z}}(z) := \{z\} + T_{\mathcal{Z}}(z)$ , where  $T_{\mathcal{Z}}(z) = \{z' \in \mathbb{R}^n : \langle z', a \rangle \leq 0, \forall a \in N_{\mathcal{Z}}(z)\}$  is the tangent cone of  $z$ . Then all of the following quantities are equivalent:

1.  $\sqrt{\|F(z)\|^2 - \max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \langle F(z), a \rangle^2}$
2.  $\min_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \|F(z) - \langle F(z), a \rangle \cdot a\|$
3.  $\|\Pi_{T_{\mathcal{Z}}(z)}[-F(z)]\|$
4.  $\|\Pi_{J_{\mathcal{Z}}(z)}[z - F(z)] - z\|$
5.  $\|-F(z) - \Pi_{N_{\mathcal{Z}}(z)}[-F(z)]\|$
6.  $\min_{a \in N_{\mathcal{Z}}(z)} \|F(z) + a\|$

*Proof.* **(quantity 1 = quantity 2).** Observe that

$$\min_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \|F(z) - \langle F(z), a \rangle \cdot a\|^2 = \|F(z)\|^2 - \max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \langle F(z), a \rangle^2 \cdot (2 - \|a\|^2).$$

Therefore, it is enough to show that  $\max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \langle F(z), a \rangle^2 \cdot (2 - \|a\|^2) = \max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \langle F(z), a \rangle^2$ . If  $\hat{N}_{\mathcal{Z}}(z) = \{(0, \dots, 0)\}$ , then the equality holds trivially. Now we assume that  $\{(0, \dots, 0)\} \subsetneq \hat{N}_{\mathcal{Z}}(z)$  and consider any  $a \in \hat{N}_{\mathcal{Z}}(z) \setminus \{(0, \dots, 0)\}$ . Let  $c \in [1, \frac{1}{\|a\|}]$ . By Definition 2,  $\|a\| \leq 1$ , which implies that  $c \cdot a \in \hat{N}_{\mathcal{Z}}(z)$ . We try to maximize the following objective

$$\langle F(z), c \cdot a \rangle^2 \cdot (2 - c^2 \|a\|^2) = \frac{\langle F(z), a \rangle^2}{\|a\|^2} \cdot c^2 \|a\|^2 \cdot (2 - c^2 \|a\|^2).$$

One can easily verify that function  $c^2 \|a\|^2 \cdot (2 - c^2 \|a\|^2)$  is maximized when  $c^2 \|a\|^2 = 1 \Leftrightarrow c = \frac{1}{\|a\|}$ . Thus when  $\{(0, \dots, 0)\} \subsetneq \hat{N}_{\mathcal{Z}}(z)$ ,

$$\begin{aligned} \max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \langle F(z), a \rangle^2 \cdot (2 - \|a\|^2) &= \max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0, \\ \|a\|=1}} \langle F(z), a \rangle^2 \cdot (2 - \|a\|^2) \\ &= \max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0, \\ \|a\|=1}} \langle F(z), a \rangle^2 \\ &= \max_{\substack{a \in \hat{N}_{\mathcal{Z}}(z), \\ \langle F(z), a \rangle \leq 0}} \langle F(z), a \rangle^2, \end{aligned}$$

which concludes the proof.

**(quantity 3 = quantity 4).** By definition,  $J_{\mathcal{Z}}(z) = \{z\} + T_{\mathcal{Z}}(z)$ . Thus we have

$$\left\| \Pi_{J_{\mathcal{Z}}(z)} [z - F(z)] - z \right\| = \left\| \Pi_{T_{\mathcal{Z}}(z)} [-F(z)] \right\|.$$

**(quantity 4 = quantity 5).** By definition, the tangent cone  $T_{\mathcal{Z}}(z)$  is the polar cone of the normal cone  $N_{\mathcal{Z}}(z)$ . Since  $N_{\mathcal{Z}}(z)$  is a closed convex cone, by Moreau's decomposition theorem, we have for any vector  $x \in \mathbb{R}^n$ ,

$$x = \Pi_{N_{\mathcal{Z}}(z)}(x) + \Pi_{T_{\mathcal{Z}}(z)}(x), \quad \langle \Pi_{N_{\mathcal{Z}}(z)}(x), \Pi_{T_{\mathcal{Z}}(z)}(x) \rangle = 0.$$

Thus it is clear that we have

$$\begin{aligned} \left\| \Pi_{J_{\mathcal{Z}}(z)} [z - F(z)] - z \right\| &= \left\| \Pi_{T_{\mathcal{Z}}(z)} [-F(z)] \right\| \\ &= \left\| -F(z) - \Pi_{N_{\mathcal{Z}}(z)} [-F(z)] \right\|. \end{aligned}$$

**(quantity 5 = quantity 6).** Denote  $a^* := \Pi_{N_{\mathcal{Z}}(z)}[-F(z)]$ . By definition of projection, we have

$$a^* = \operatorname{argmin}_{a \in N_{\mathcal{Z}}(z)} \|F(z) + a\|^2.$$

Thus

$$\left\| -F(z) - \Pi_{N_{\mathcal{Z}}(z)}[-F(z)] \right\|^2 = \|F(z) + a^*\|^2 = \min_{a \in N_{\mathcal{Z}}(z)} \|F(z) + a\|^2.$$

**(quantity 6 = quantity 2).** For any fix non-zero  $a \in N_{\mathcal{Z}}(z)$ , (i) if  $\langle F(z), a \rangle > 0$ , then  $\|F(z) + a\|^2 \geq \|F(z)\|^2$ , and (ii) if  $\langle F(z), a \rangle \leq 0$ ,  $\|F(z) + a\|^2 \geq \|F(z) - \langle F(z), \frac{a}{\|a\|} \rangle \cdot \frac{a}{\|a\|}\|^2$ , as

$$\min_{r \geq 0} \|F(z) + r \cdot a\|^2 = \left\| F(z) - \left\langle F(z), \frac{a}{\|a\|} \right\rangle \cdot \frac{a}{\|a\|} \right\|^2.$$

Hence,

$$\min_{a \in N_{\mathcal{Z}}(z)} \|F(z) + a\|^2 = \min_{\substack{a' \in \widehat{N}_{\mathcal{Z}}(z) \\ \langle F(z), a' \rangle \leq 0}} \|F(z) - \langle F(z), a' \rangle \cdot a'\|^2$$

The first equality is because for any  $a \in N_{\mathcal{Z}}(z)$ , there exists  $a' \in \widehat{N}_{\mathcal{Z}}(z)$  so that  $\|F(z) + a\|^2 \geq \|F(z) - \langle F(z), a' \rangle \cdot a'\|^2$ .  $\square$

In the following Lemma, we show a useful property of the tangent residual that we use repeatedly.

**Lemma 8.** Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be an operator. Let  $\eta > 0$  and  $z_1, z_2, z_3 \in \mathcal{Z}$  be three points such that  $z_1 = \Pi_{\mathcal{Z}}[z_2 - \eta F(z_3)]$ , then we have

$$r^{tan}(z_1) \leq \left\| \frac{z_2 - z_1}{\eta} + F(z_1) - F(z_3) \right\|.$$

*Proof.* If  $z_1 = z_2 - \eta F(z_3)$ , then the lemma holds since

$$r^{tan}(z_1) \leq \|F(z_1)\| = \|F(z_3) + F(z_1) - F(z_3)\| = \left\| \frac{z_2 - z_1}{\eta} + F(z_1) - F(z_3) \right\|.$$

For the rest of the proof, we assume that  $z_1 \neq z_2 - \eta F(z_3)$ . Since  $z_2 - \eta F(z_3) - z_1 \in N(z_1)$ , we have

$$\langle z_2 - \eta F(z_3) - z_1, z - z_1 \rangle \leq 0, \quad \forall z \in \mathcal{Z}.$$

Define  $a := \frac{z_2 - \eta F(z_3) - z_1}{\|z_2 - \eta F(z_3) - z_1\|}$ . We thus know  $a \in \widehat{N}(z_1)$ . Let

$$a_{\perp} := \begin{cases} \frac{F(z_1) - \langle a, F(z_1) \rangle \cdot a}{\|F(z_1) - \langle a, F(z_1) \rangle \cdot a\|} & \text{if } \|F(z_1) - \langle a, F(z_1) \rangle \cdot a\| \neq 0, \\ (0, \dots, 0) & \text{otherwise.} \end{cases}$$



Observe that  $\langle a, a \rangle = 1$ ,  $\langle a_\perp, a \rangle = 0$  and  $F(z_1) = \langle a, F(z_1) \rangle a + \langle a_\perp, F(z_1) \rangle a_\perp$ . Thus

$$\begin{aligned} 0 &= \langle a_\perp, a \rangle = \langle a_\perp, z_2 - \eta F(z_3) - z_1 \rangle \\ &\Leftrightarrow \langle a_\perp, F(z_3) \rangle = \frac{\langle a_\perp, z_2 - z_1 \rangle}{\eta}. \end{aligned} \quad (31)$$

Moreover, the fact that  $\langle a, z_1 - z_2 + \eta F(z_3) \rangle \leq 0$  implies  $\langle a, F(z_3) \rangle \leq \frac{\langle a, z_2 - z_1 \rangle}{\eta}$ , which further implies that

$$\langle a, F(z_1) \rangle = \langle a, F(z_3) + F(z_1) - F(z_3) \rangle \leq \left\langle a, \frac{z_2 - z_1}{\eta} + F(z_1) - F(z_3) \right\rangle. \quad (32)$$

Combining the definition of  $r^{tan}(z_1)$ , Equation (31), and Equation (32) we have

$$\begin{aligned} r^{tan}(z_1)^2 &\leq \|F(z_1)\|^2 - \langle a, F(z_1) \rangle^2 \cdot \mathbb{1}[\langle a, F(z_1) \rangle \leq 0] \\ &= \langle a_\perp, F(z_1) \rangle^2 + \langle a, F(z_1) \rangle^2 \cdot \mathbb{1}[\langle a, F(z_1) \rangle > 0] \\ &= \langle a_\perp, F(z_3) + F(z_1) - F(z_3) \rangle^2 + \langle a, F(z_1) \rangle^2 \cdot \mathbb{1}[\langle a, F(z_1) \rangle > 0] \\ &\leq \left\langle a_\perp, \frac{z_2 - z_1}{\eta} + F(z_1) - F(z_3) \right\rangle^2 + \left\langle a, \frac{z_2 - z_1}{\eta} + F(z_1) - F(z_3) \right\rangle^2 \\ &\leq \left\| \frac{z_2 - z_1}{\eta} + F(z_1) - F(z_3) \right\|^2. \end{aligned} \quad \square$$

**Proof of Lemma 2:** If  $\langle a, F(z) \rangle \geq 0$  for all  $a \in \hat{N}(z)$ , then we have  $r^{tan}(z) = \|F(z)\|$ . Thus for any  $z' \in \mathcal{Z}$ , by Cauchy-Schwarz inequality, we have

$$\langle F(z), z - z' \rangle \leq \|F(z)\| \|z - z'\| \leq D \cdot r^{tan}(z).$$

Otherwise there exists  $a \in \hat{N}(z)$  such that  $\|a\| = 1$ ,  $\langle a, F(z) \rangle < 0$  and  $r^{tan}(z) = \sqrt{\|F(z)\|^2 - \langle a, F(z) \rangle^2} = \|F(z) - \langle a, F(z) \rangle a\|$ . Then for any  $z' \in \mathcal{Z}$ , we have

$$\begin{aligned} \langle F(z), z - z' \rangle &= \langle F(z) - \langle a, F(z) \rangle a, z - z' \rangle + \langle a, F(z) \rangle \cdot \langle a, z - z' \rangle \\ &\leq \langle F(z) - \langle a, F(z) \rangle a, z - z' \rangle \\ &\leq \|F(z) - \langle a, F(z) \rangle a\| \|z - z'\| \\ &\leq D \cdot r^{tan}(z), \end{aligned}$$

where we use  $\langle a, F(z) \rangle < 0$  and  $\langle a, z - z' \rangle \geq 0$  in the first inequality and Cauchy-Schwarz inequality in the second inequality.

If  $\mathcal{Z}' = \mathcal{X}' \times \mathcal{Y}'$  and  $F(x, y) = \begin{pmatrix} \nabla_x f(x, y) \\ -\nabla_y f(x, y) \end{pmatrix}$  for a convex-concave function  $f$  then

$$\begin{aligned}
\text{dg}_f^{\mathcal{X}', \mathcal{Y}'}(z) &= \max_{y' \in \mathcal{Y}'} f(x, y') - \min_{x' \in \mathcal{X}'} f(x', y) \\
&= \max_{y' \in \mathcal{Y}'} (f(x, y') - f(x, y)) - \min_{x' \in \mathcal{X}'} (f(x', y) - f(x, y)), \\
&\leq \max_{y' \in \mathcal{Y}'} \langle \nabla_y f(x, y), y' - y \rangle + \max_{x' \in \mathcal{X}'} \langle \nabla_x f(x, y), x - x' \rangle, \\
&= \max_{z' \in \mathcal{Z}'} \langle F(z), z - z' \rangle, \\
&\leq D\sqrt{2} \cdot r^{\tan}(z),
\end{aligned}$$

where we use the fact that  $f$  is a convex-concave function in the first inequality and  $\|z - z'\| = \sqrt{\|x - x'\|^2 + \|y - y'\|^2} \leq \sqrt{2}D$  in the second inequality.  $\blacksquare$

## B Missing Proofs from Section 4

**Proof of Lemma 3:** By Pythagorean inequality,

$$\begin{aligned}
\|z_{k+1} - z^*\|^2 &\leq \|z_k - \eta F(z_{k+\frac{1}{2}}) - z^*\|^2 - \|z_k - \eta F(z_{k+\frac{1}{2}}) - z_{k+1}\|^2 \\
&= \|z_k - z^*\|^2 - \|z_k - z_{k+1}\|^2 + 2\eta \langle F(z_{k+\frac{1}{2}}), z^* - z_{k+1} \rangle \\
&= \|z_k - z^*\|^2 - \|z_k - z_{k+1}\|^2 + 2\eta \langle F(z_{k+\frac{1}{2}}), z^* - z_{k+\frac{1}{2}} \rangle + 2\eta \langle F(z_{k+\frac{1}{2}}), z_{k+\frac{1}{2}} - z_{k+1} \rangle.
\end{aligned} \tag{33}$$

We first use monotonicity of  $F(\cdot)$  to argue that  $\langle F(z_{k+\frac{1}{2}}), z^* - z_{k+\frac{1}{2}} \rangle \leq 0$ .

**Fact 1.** For all  $z \in \mathcal{Z}$ ,  $\langle F(z), z^* - z \rangle \leq 0$ .

*Proof.*

$$\begin{aligned}
0 &\leq \langle F(z^*) - F(z), z^* - z \rangle && \text{(monotonicity of } F(\cdot) \text{)} \\
&= \langle F(z^*), z^* - z \rangle - \langle F(z), z^* - z \rangle \\
&\leq -\langle F(z), z^* - z \rangle && \text{(optimality of } z^* \text{ and } z \in \mathcal{Z} \text{)}
\end{aligned}$$

$\square$

We can simplify Equation (33) using Fact 1:

$$\begin{aligned}
\|z_{k+1} - z^*\|^2 &\leq \|z_k - z^*\|^2 - \|z_k - z_{k+1}\|^2 + 2\eta \langle F(z_{k+\frac{1}{2}}), z_{k+\frac{1}{2}} - z_{k+1} \rangle \\
&= \|z_k - z^*\|^2 - \|z_k - z_{k+\frac{1}{2}}\|^2 - \|z_{k+\frac{1}{2}} - z_{k+1}\|^2 - 2\langle z_k - \eta F(z_{k+\frac{1}{2}}) - z_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} - z_{k+1} \rangle \\
&= \|z_k - z^*\|^2 - \|z_k - z_{k+\frac{1}{2}}\|^2 - \|z_{k+\frac{1}{2}} - z_{k+1}\|^2 \\
&\quad - 2\langle z_k - \eta F(z_k) - z_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} - z_{k+1} \rangle - 2\langle \eta F(z_k) - \eta F(z_{k+\frac{1}{2}}), z_{k+\frac{1}{2}} - z_{k+1} \rangle \\
&\leq \|z_k - z^*\|^2 - \|z_k - z_{k+\frac{1}{2}}\|^2 - \|z_{k+\frac{1}{2}} - z_{k+1}\|^2 - 2\eta \langle F(z_k) - F(z_{k+\frac{1}{2}}), z_{k+\frac{1}{2}} - z_{k+1} \rangle
\end{aligned}$$

The last inequality is because  $\langle z_k - \eta F(z_k) - z_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} - z_{k+1} \rangle \geq 0$ , which follows from the fact that  $z_{k+\frac{1}{2}} = \Pi_{\mathcal{Z}}[z_k - \eta F(z_k)]$  and  $z_{k+1} \in \mathcal{Z}$ .

Finally, since  $F(\cdot)$  is  $L$ -Lipschitz, we know that

$$-\langle F(z_k) - F(z_{k+\frac{1}{2}}), z_{k+\frac{1}{2}} - z_{k+1} \rangle \leq \|F(z_k) - F(z_{k+\frac{1}{2}})\| \cdot \|z_{k+\frac{1}{2}} - z_{k+1}\| \leq L\|z_k - z_{k+\frac{1}{2}}\| \cdot \|z_{k+\frac{1}{2}} - z_{k+1}\|.$$

So we can further simplify the inequality as follows:

$$\begin{aligned} \|z_{k+1} - z^*\|^2 &\leq \|z_k - z^*\|^2 - \|z_k - z_{k+\frac{1}{2}}\|^2 - \|z_{k+\frac{1}{2}} - z_{k+1}\|^2 - 2\eta \langle F(z_k) - F(z_{k+\frac{1}{2}}), z_{k+\frac{1}{2}} - z_{k+1} \rangle \\ &\leq \|z_k - z^*\|^2 - \|z_k - z_{k+\frac{1}{2}}\|^2 - \|z_{k+\frac{1}{2}} - z_{k+1}\|^2 + 2\eta L\|z_k - z_{k+\frac{1}{2}}\| \cdot \|z_{k+\frac{1}{2}} - z_{k+1}\| \\ &\leq \|z_k - z^*\|^2 - (1 - \eta^2 L^2)\|z_k - z_{k+\frac{1}{2}}\|^2 \end{aligned}$$

Hence,

$$\|z_k - z^*\|^2 \geq \|z_{k+1} - z^*\|^2 + (1 - \eta^2 L^2)\|z_k - z_{k+\frac{1}{2}}\|^2.$$

■

**Proof of Lemma 4:** We need the following fact for our proof.

**Fact 2.**  $\|z_{k+\frac{1}{2}} - z_{k+1}\| \leq \eta L\|z_k - z_{k+\frac{1}{2}}\|$ . Moreover, when  $\eta L < 1$ ,  $\|z_{k+\frac{1}{2}} - z_{k+1}\| \leq \frac{\|z_k - z_{k+1}\|}{1 - \eta L}$ .

*Proof.* Recall that  $z_{k+\frac{1}{2}} = \Pi_{\mathcal{Z}}[z_k - \eta F(z_k)]$  and  $z_{k+1} = \Pi_{\mathcal{Z}}[z_k - \eta F(z_{k+\frac{1}{2}})]$ . By the non-expansiveness of the projection operator and the  $L$ -Lipschitzness of operator  $F$ , we have that  $\|z_{k+\frac{1}{2}} - z_{k+1}\| \leq \|\eta(F(z_{k+\frac{1}{2}}) - F(z_k))\| \leq \eta L\|z_k - z_{k+\frac{1}{2}}\|$ .

Finally, by the triangle inequality

$$\|z_k - z_{k+1}\| \geq \|z_k - z_{k+\frac{1}{2}}\| - \|z_{k+\frac{1}{2}} - z_{k+1}\| \geq (1 - \eta L)\|z_k - z_{k+\frac{1}{2}}\|.$$

□

Now we prove Lemma 4. By the  $L$ -Lipschitzness of operator  $F$  we have

$$\|F(z_{k+1}) - F(z_{k+\frac{1}{2}})\| \leq L\|z_{k+1} - z_{k+\frac{1}{2}}\| \leq \eta L^2\|z_k - z_{k+\frac{1}{2}}\|. \quad (34)$$

Recall that  $z_{k+1} = \Pi_{\mathcal{Z}} \left[ z_k - \eta F(z_{k+\frac{1}{2}}) \right]$ . Using Lemma 8, we have

$$\begin{aligned}
r^{tan}(z_{k+1}) &\leq \left\| \frac{z_k - z_{k+1}}{\eta} + F(z_{k+1}) - F(z_{k+\frac{1}{2}}) \right\| \\
&\leq \frac{\|z_k - z_{k+1}\|}{\eta} + \|F(z_{k+1}) - F(z_{k+\frac{1}{2}})\| \\
&\leq \frac{\|z_k - z_{k+1}\| + (\eta L)^2 \|z_k - z_{k+\frac{1}{2}}\|}{\eta} \\
&\leq \frac{\|z_k - z_{k+\frac{1}{2}}\| + \|z_{k+\frac{1}{2}} - z_{k+1}\| + (\eta L)^2 \|z_k - z_{k+\frac{1}{2}}\|}{\eta} \\
&\leq (1 + \eta L + (\eta L)^2) \frac{\|z_k - z_{k+\frac{1}{2}}\|}{\eta}.
\end{aligned}$$

The second and the fourth inequality follow from the triangle inequality. The third inequality follows from Equation (34). In the final inequality we use  $\|z_{k+\frac{1}{2}} - z_{k+1}\| \leq \eta L \|z_k - z_{k+\frac{1}{2}}\|$  by Fact 2. ■

**Proof of Lemma 5:** By Lemma 3 we have

$$\|z_0 - z^*\|^2 \geq \|z_{T+1} - z^*\|^2 + (1 - \eta^2 L^2) \sum_{k=0}^T \|z_k - z_{k+\frac{1}{2}}\|^2 \geq (1 - \eta^2 L^2) \sum_{k=0}^T \|z_k - z_{k+\frac{1}{2}}\|^2$$

Thus there exists a  $t^* \in [T]$  such that  $\|z_{t^*} - z_{t^*+\frac{1}{2}}\|^2 \leq \frac{\|z_0 - z^*\|^2}{T(1-\eta^2 L^2)}$ . We conclude the proof by applying Lemma 4. ■

## C Missing Proofs from Section 5.1

**Proposition 1.**

$$\begin{aligned}
&\|F(z_k)\|^2 - \|F(z_{k+1})\|^2 + 2 \cdot \langle F(z_{k+1}) - F(z_k), F(z_{k+\frac{1}{2}}) \rangle \\
&\quad + \left( \|F(z_{k+\frac{1}{2}}) - F(z_{k+1})\|^2 - \|F(z_{k+\frac{1}{2}}) - F(z_k)\|^2 \right) = 0.
\end{aligned}$$

*Proof.* Expanding the LHS of the equation in the statement we can verify that

$$\begin{aligned}
&\|F(z_k)\|^2 - \|F(z_{k+1})\|^2 + 2 \cdot \langle F(z_{k+1}), F(z_{k+\frac{1}{2}}) \rangle - 2 \cdot \langle F(z_k), F(z_{k+\frac{1}{2}}) \rangle \\
&\quad + \|F(z_{k+\frac{1}{2}})\|^2 - 2 \cdot \langle F(z_{k+1}), F(z_{k+\frac{1}{2}}) \rangle + \|F(z_{k+1})\|^2 \\
&\quad - \|F(z_{k+\frac{1}{2}})\|^2 + 2 \cdot \langle F(z_k), F(z_{k+\frac{1}{2}}) \rangle - \|F(z_k)\|^2 = 0.
\end{aligned}$$

□

## **D The SOS Program for the Constrained Setting**

### Interpretation and Notation.

- Interpret  $\bar{\alpha}$  as  $\alpha$ ,  $\bar{\beta}_1$  as  $\beta_1$ , and  $\bar{\beta}_2$  as  $\beta_2$ . We use  $\bar{a}_0$  to denote  $(\bar{\beta}_1, \bar{\beta}_2, 1)$ ,  $\bar{a}_1$  to denote  $(\bar{\alpha}, 1, 0)$ , and  $\bar{a}_2$  to denote  $(1, 0, 0)$ .
- Interpret  $\bar{z}_i[\ell]$  as  $\bar{z}_{k+\frac{i}{2}}[\ell]$  and  $\bar{F}_i[\ell]$  as  $\eta \bar{F}(\bar{z}_{k+\frac{i}{2}})[\ell]$  for  $0 \leq i \leq 2$  and  $\ell \leq 3$ . We use  $\bar{z}_i$  to denote  $(\bar{z}_i[1], \bar{z}_i[2], \bar{z}_i[3])$  and  $\bar{F}_i$  to denote  $(\bar{F}_i[1], \bar{F}_i[2], \bar{F}_i[3])$ .
- Interpret  $x_i$  and  $y_i$  as the representatives for  $\bar{z}_{k+\frac{i}{2}}[\ell]$  and  $\eta \bar{F}(\bar{z}_{k+\frac{i}{2}})[\ell]$  for all  $\ell \geq 4$ , for  $0 \leq i \leq 2$ .
- We use  $\mathbf{x}$  to denote  $(\bar{z}_0, \bar{z}_1, \bar{z}_2, x_0, x_1, x_2, \bar{\alpha}, \bar{\beta}_1, \bar{\beta}_2)$  and  $\mathbf{y}$  to denote  $(F_0, F_1, F_2, y_0, y_1, y_2)$ .

### Input Fixed Polynomials.

- Polynomial  $g(\mathbf{x}, \mathbf{y}) = \|\bar{a}_0\|^2 \cdot (\|\bar{F}_0 - \frac{\langle \bar{F}_0, \bar{a}_0 \rangle \cdot \bar{a}_0}{\|\bar{a}_0\|^2}\|^2 + y_0^2 - \|\bar{F}_2 - \langle \bar{F}_2, \bar{a}_2 \rangle \cdot \bar{a}_2\|^2 - y_2^2)$
- Set of polynomials

$$\begin{aligned}
S^{(=0)} &= \{ \langle \bar{a}_0, \bar{z}_0 \rangle, \langle \bar{a}_1, \bar{z}_1 \rangle, \langle \bar{a}_2, \bar{z}_2 \rangle \} \\
&\cup \{ \langle (1, -\bar{\alpha}, 0), \bar{z}_1 - \bar{z}_0 + \bar{F}_0 \rangle \} \\
&\cup \{ \langle e_3, \bar{z}_1 - \bar{z}_0 + \bar{F}_0 \rangle, \langle e_2, \bar{z}_2 - \bar{z}_0 + \bar{F}_1 \rangle, \langle e_3, \bar{z}_2 - \bar{z}_0 + \bar{F}_1 \rangle \} \\
&\cup \{ x_1 - x_0 + y_0, x_2 - x_0 + y_1 \}, \\
S^{(\geq 0)} &= \{ \langle \bar{a}_i, \bar{z}_j \rangle \}_{0 \leq i \neq j \leq 2} \\
&\cup \{ \langle \bar{a}_0, \bar{F}_0 \rangle, \langle \bar{a}_1, \bar{z}_1 - \bar{z}_0 + \bar{F}_0 \rangle, \langle \bar{a}_2, \bar{z}_2 - \bar{z}_0 + \bar{F}_1 \rangle \}, \\
S^{(G, \leq 0)} &= \{ \langle \bar{F}_i - \bar{F}_j, \bar{z}_j - \bar{z}_i \rangle + (y_i - y_j) \cdot (x_j - x_i) \}_{i,j \in \{0,1,2\}} \\
&\cup \left\{ \|\bar{F}_i - \bar{F}_j\|^2 - \|\bar{z}_i - \bar{z}_j\|^2 + (y_i - y_j)^2 - (x_i - x_j)^2 \right\}_{i,j \in \{0,1,2\}}.
\end{aligned}$$

### Inferred “Negative” Polynomials (Not part of Input).

- Let  $S^{(\leq 0)} = \left\{ -h(\mathbf{x}, \mathbf{y}) \cdot h'(\mathbf{x}, \mathbf{y}) : h(\mathbf{x}, \mathbf{y}), h'(\mathbf{x}, \mathbf{y}) \in S^{(\geq 0)} \right\} \cup S^{(G, \leq 0)} \cup \{g(\mathbf{x}, \mathbf{y})\}$ .

### Decision Variables of the SOS Program:

- $p_{(h(\mathbf{x}, \mathbf{y}))}(\mathbf{x}, \mathbf{y}) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$  is a polynomial with degree at most  $d - \deg(h(\mathbf{x}, \mathbf{y}))$ , for all  $h(\mathbf{x}, \mathbf{y}) \in S^{(=0)}$ .
- $q_{(h(\mathbf{x}, \mathbf{y}))}(\mathbf{x}, \mathbf{y}) \in \text{SOS}[\mathbf{x}, \mathbf{y}]$  is an SOS polynomial with degree at most  $d - \deg(h(\mathbf{x}, \mathbf{y}))$ , for all  $h(\mathbf{x}, \mathbf{y}) \in S^{(\leq 0)}$ .

### Constraints of the SOS Program:

$$g(\mathbf{x}, \mathbf{y}) + \sum_{h(\mathbf{x}, \mathbf{y}) \in S^{(=0)}} p_{(h(\mathbf{x}, \mathbf{y}))}(\mathbf{x}, \mathbf{y}) \cdot h(\mathbf{x}, \mathbf{y}) + \sum_{h(\mathbf{x}, \mathbf{y}) \in S^{(\leq 0)}} q_{(h(\mathbf{x}, \mathbf{y}))}(\mathbf{x}, \mathbf{y}) \cdot h(\mathbf{x}, \mathbf{y}) \in \text{SOS}[\mathbf{x}, \mathbf{y}]$$

Figure 5: SOS program for the constrained case of degree  $d$ . **We need  $d$  to be 8 to discover our proof.**  $S^{(=0)}$  contains all polynomials that are equal to 0 due to the constraints.  $S^{(\geq 0)}$  contains all polynomials that only involve the first 3 coordinates and are non-negative due to the constraints.  $S^{(G, \leq 0)}$  contains all polynomials that involve all coordinates and are non-negative. In our program, we need to include the inferred non-positive polynomials  $S^{(\leq 0)}$ .

## E Missing Proofs from Section 5.2

**Proof of Lemma 6:** For our proof, it will be more convenient to work with terms  $r^{tan}(z_k)^2, r^{tan}(z_{k+1})^2$  rather than  $r^{tan}(z_k), r^{tan}(z_{k+1})$ . Since  $r^{tan}(z_k), r^{tan}(z_{k+1}) \geq 0$ , then  $r^{tan}(z_k) - r^{tan}(z_{k+1}) \geq 0$  iff  $r^{tan}(z_k)^2 - r^{tan}(z_{k+1})^2 \geq 0$ . For the rest of the proof, we refer to the property that  $\bar{a}_k = (\beta_1, \beta_2, 1, 0, \dots, 0), \bar{a}_{k+\frac{1}{2}} = (\alpha, 1, 0, \dots, 0), \bar{a}_{k+1} = (1, 0, \dots, 0)$  as the form property.

Recall that the  $k$ -th update of EG is as follows  $z_{k+\frac{1}{2}} = \Pi_{\mathcal{Z}}[z_k - \eta F(z_k)]$  and  $z_{k+1} = \Pi_{\mathcal{Z}}[z_k - \eta F(z_{k+\frac{1}{2}})]$ . We define the following vectors:

$$-a_k \in \underset{\substack{a \in \hat{N}_{\mathcal{Z}}(z_k), \\ \langle F(z_k), a \rangle \leq 0}}{\text{argmin}} \|F(z_k) - \langle F(z_k), a \rangle \cdot a\|^2, \quad (35)$$

$$a_{k+\frac{1}{2}} = z_{k+\frac{1}{2}} - z_k + \eta F(z_k), \quad (36)$$

$$a_{k+1} = z_{k+1} - z_k + \eta F(z_{k+\frac{1}{2}}). \quad (37)$$

For now, let us assume that  $a_k, a_{k+\frac{1}{2}}$ , and  $a_{k+1}$  satisfy (i) the form property, and (ii)  $\langle a_k, z_k \rangle = \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle = \langle a_{k+1}, z_{k+1} \rangle = 0$ . We use this simple case as the basis of our construction, and will remove these assumptions later.

We set  $\bar{a}_k = a_k, \bar{a}_{k+\frac{1}{2}} = a_{k+\frac{1}{2}}, \bar{a}_{k+1} = a_{k+1}, \bar{z}_k = z_k, \bar{z}_{k+\frac{1}{2}} = z_{k+\frac{1}{2}}, \bar{z}_{k+1} = z_{k+1}, \bar{F}(\bar{z}_k) = F(z_k), \bar{F}(\bar{z}_{k+\frac{1}{2}}) = F(z_{k+\frac{1}{2}})$  and  $\bar{F}(\bar{z}_{k+1}) = F(z_{k+1})$ . We first argue that Property 4 holds. Since  $F(\cdot)$  is monotone and  $L$ -Lipschitz, Inequality (15) and (16) are satisfied. In addition, Inequality (17) is satisfied due to the definition of  $a_k$ .

Next, we show that property 3 holds. By the definition of  $\bar{a}_{k+\frac{1}{2}}$  (or  $\bar{a}_{k+1}$ ), it is clear that  $\bar{a}_{k+\frac{1}{2}}$  (or  $\bar{a}_{k+1}$ ) and  $\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k)$  (or  $\bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})$ ) are co-directed. As  $-a_k \in \hat{N}_{\mathcal{Z}}(z_k)$ ,

$$\langle \bar{a}_k, \bar{z} \rangle \geq \langle \bar{a}_k, \bar{z}_k \rangle = 0, \quad \bar{z} \in \{\bar{z}_k, \bar{z}_{k+\frac{1}{2}}, \bar{z}_{k+1}\}. \quad (38)$$

According to the update rule of the EG algorithm (Equation (4) and (5)), Equation (36), and Equation (37), we know that for all  $z \in \mathcal{Z}$ ,

$$\langle a_{k+\frac{1}{2}}, z - z_{k+\frac{1}{2}} \rangle = \langle z_{k+\frac{1}{2}} - z_k + \eta F(z_k), z - z_{k+\frac{1}{2}} \rangle \geq 0, \quad (39)$$

$$\langle a_{k+1}, z - z_{k+1} \rangle = \langle z_{k+1} - z_k + \eta F(z_{k+\frac{1}{2}}), z - z_{k+1} \rangle \geq 0, \quad (40)$$

which implies that for any  $i \in \{k + \frac{1}{2}, k + 1\}, j \in \{k, k + \frac{1}{2}, k + 1\}, \langle \bar{a}_i, \bar{z}_j \rangle \geq \langle \bar{a}_i, \bar{z}_i \rangle = 0$ .

Finally, we verify Property 2. By Equation (40),  $-\frac{a_{k+1}}{\|a_{k+1}\|} \in \hat{N}(z_{k+1})$ , which in combination with

Lemma 7 implies

$$\begin{aligned}
& \left\| \bar{F}(\bar{z}_{k+1}) - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \cdot \bar{a}_{k+1}}{\|\bar{a}_{k+1}\|^2} \mathbb{1}[\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \geq 0] \right\|^2 \\
& \geq \min_{\substack{a \in \hat{N}_{\mathcal{Z}}(z_{k+1}), \\ \langle F(z_{k+1}), a \rangle \leq 0}} \|F(z_{k+1}) - \langle F(z_{k+1}), a \rangle \cdot a\|^2 \\
& = r^{\tan}(z_{k+1})^2.
\end{aligned}$$

According to Lemma 7 and Equation (35), we know that  $r^{\tan}(z_k)^2 = \|F(z_k) - \langle F(z_k), a_k \rangle \cdot a_k\|^2 = \|\bar{F}(\bar{z}_k) - \frac{\langle \bar{F}(\bar{z}_k), \bar{a}_k \rangle \cdot \bar{a}_k}{\|\bar{a}_k\|^2}\|^2$ . If  $r^{\tan}(z_k)^2 - r^{\tan}(z_{k+1})^2 < 0$ , then

$$\begin{aligned}
0 & > r^{\tan}(z_k)^2 - r^{\tan}(z_{k+1})^2 \\
& \geq \left\| \bar{F}(\bar{z}_k) - \frac{\langle \bar{F}(\bar{z}_k), \bar{a}_k \rangle \cdot \bar{a}_k}{\|\bar{a}_k\|^2} \right\|^2 - \left\| \bar{F}(\bar{z}_{k+1}) - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \cdot \bar{a}_{k+1}}{\|\bar{a}_{k+1}\|^2} \mathbb{1}[\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \geq 0] \right\|^2.
\end{aligned} \tag{41}$$

Observe that when vectors  $a_k$  and  $a_{k+1}$  satisfy the form property, then  $\|\bar{a}_k\| = 1$ , and  $\|\bar{a}_{k+1}\| \geq 1$  and Equation (41) is well-defined.

This completes the proof for the case, where (i) vectors  $a_k, a_{k+\frac{1}{2}}$ , and  $a_{k+1}$  satisfy the form property, and (ii)  $\langle a_k, z_k \rangle = \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle = \langle a_{k+1}, z_{k+1} \rangle = 0$ . Our next step is to remove the assumptions. We first show how to modify the construction so that for any  $a_k, a_{k+\frac{1}{2}}, a_{k+1}$ , we can construct vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}, \hat{a}_{k+1}, \hat{z}_k, \hat{z}_{k+\frac{1}{2}}, \hat{z}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{z}_{k+\frac{1}{2}}), \hat{F}(\hat{z}_{k+1}) \in \mathbb{R}^{n+5}$ , that satisfy, among other properties, (a)  $\langle \hat{a}_k, \hat{z}_k \rangle = \langle \hat{a}_{k+\frac{1}{2}}, \hat{z}_{k+\frac{1}{2}} \rangle = \langle \hat{a}_{k+1}, \hat{z}_{k+1} \rangle = 0$ , and (b) vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}, \hat{a}_{k+1}$  are linear independent. In our final step, we choose a proper basis to construct vectors  $\bar{a}_k, \bar{a}_{k+\frac{1}{2}}, \bar{a}_{k+1}, \bar{z}_k, \bar{z}_{k+\frac{1}{2}}, \bar{z}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{z}_{k+\frac{1}{2}}), \bar{F}(\bar{z}_{k+1}) \in \mathbb{R}^{n+5}$  that satisfy all four properties in the statement of Lemma 6.

We now present the construction of vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}, \hat{a}_{k+1}, \hat{z}_k, \hat{z}_{k+\frac{1}{2}}, \hat{z}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{z}_{k+\frac{1}{2}}), \hat{F}(\hat{z}_{k+1})$ . High-levelly speaking, we introduce five dummy dimensions. The purpose of the first dummy dimension is to ensure property (a). We use the remaining four dummy dimensions to ensure that the newly created vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}$  and  $\hat{a}_{k+1}$  are linearly independent satisfying property (b). More specifically for parameters  $\ell, \epsilon > 0$  that we determine later, we define  $\hat{z}_k, \hat{z}_{k+\frac{1}{2}}, \hat{z}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{z}_{k+\frac{1}{2}}),$



$\widehat{F}(\widehat{z}_{k+1}), \widehat{a}_k, \widehat{a}_{k+\frac{1}{2}}, \widehat{a}_{k+1}$  as follows

$$\widehat{z}_i := (-\epsilon^{-1}, 0, 0, 0, z_i) \quad \forall i \in \{k, k + \frac{1}{2}, k + 1\} \quad (42)$$

$$\widehat{F}(\widehat{z}_k) := (\frac{\epsilon}{\eta} \cdot \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle, 0, \frac{\epsilon}{\eta}, 0, \frac{\ell\epsilon}{\eta}, F(z_k)) \quad (43)$$

$$\widehat{F}(\widehat{z}_{k+\frac{1}{2}}) := (\frac{\epsilon}{\eta} \cdot \langle a_{k+1}, z_{k+1} \rangle, 0, 0, \frac{\epsilon}{\eta}, \frac{\ell\epsilon}{\eta}, F(z_{k+\frac{1}{2}})) \quad (44)$$

$$\widehat{F}(\widehat{z}_{k+1}) := (\frac{\epsilon}{\eta} \cdot \langle a_{k+1}, z_{k+1} \rangle, 0, 0, \frac{\epsilon}{\eta}, \frac{\ell\epsilon}{\eta}, F(z_{k+1})) \quad (45)$$

$$\widehat{a}_k := (\epsilon \cdot \langle a_k, z_k \rangle, \epsilon, 0, 0, \ell\epsilon, a_k) \quad (46)$$

$$\widehat{a}_{k+\frac{1}{2}} := (\epsilon \cdot \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle, 0, \epsilon, 0, \ell\epsilon, a_{k+\frac{1}{2}}) \quad (47)$$

$$\widehat{a}_{k+1} := (\epsilon \cdot \langle a_{k+1}, z_{k+1} \rangle, 0, 0, \epsilon, \ell\epsilon, a_{k+1}) \quad (48)$$

Clearly,  $\widehat{a}_k, \widehat{a}_{k+\frac{1}{2}}, \widehat{a}_{k+1}$  are linear independent,  $\widehat{a}_{k+1}$  and  $\widehat{z}_{k+1} - \widehat{z}_k + \eta\widehat{F}(\widehat{z}_{k+\frac{1}{2}})$  are co-directed, and  $\widehat{a}_{k+\frac{1}{2}}$  and  $\widehat{z}_{k+\frac{1}{2}} - \widehat{z}_k + \eta\widehat{F}(\widehat{z}_k)$  are co-directed. Note that the following inequalities hold. By Equation (35)-(37), it is clear that  $-a_k \in \widehat{N}(z_k) \subseteq N(z_k)$ ,  $-a_{k+\frac{1}{2}} \in N(z_{k+\frac{1}{2}})$  and  $-a_{k+1} \in N(z_{k+1})$ , which further implies

$$\langle \widehat{a}_i, \widehat{z}_j \rangle = \langle a_i, z_j \rangle - \langle a_i, z_i \rangle \geq 0, \quad \forall i, j \in \{k, k + \frac{1}{2}, k + 1\} \quad (49)$$

$$\langle \widehat{a}_i, \widehat{z}_i \rangle = 0, \quad \forall i \in \{k, k + \frac{1}{2}, k + 1\} \quad (50)$$

$$\|\widehat{F}(\widehat{z}_{k+1}) - \widehat{F}(\widehat{z}_{k+\frac{1}{2}})\|^2 = \|F(z_{k+1}) - F(z_{k+\frac{1}{2}})\|^2 \leq L^2 \|z_{k+1} - z_{k+\frac{1}{2}}\|^2 = \|\widehat{z}_{k+1} - \widehat{z}_{k+\frac{1}{2}}\|^2 \quad (51)$$

$$\langle \widehat{F}(\widehat{z}_{k+1}) - \widehat{F}(\widehat{z}_k), \widehat{z}_{k+1} - \widehat{z}_k \rangle = \langle F(z_{k+1}) - F(z_k), z_{k+1} - z_k \rangle \geq 0 \quad (52)$$

Moreover,  $\langle \widehat{a}_k, \widehat{F}(\widehat{z}_k) \rangle = \frac{\epsilon^2}{\eta} \cdot (\langle a_k, z_k \rangle \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle + \ell^2) + \langle a_k, F(z_k) \rangle$ . We choose  $\ell$  to be sufficiently large so that  $\langle a_k, z_k \rangle \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle + \ell^2 \geq 0$ . Hence, for our choice of  $\ell$ ,

$$\langle \widehat{a}_k, \widehat{F}(\widehat{z}_k) \rangle \geq \langle a_k, F(z_k) \rangle \geq 0. \quad (53)$$

We define function

$$\begin{aligned} \widehat{H}(\widehat{z}_k) &:= \left\| \widehat{F}(\widehat{z}_k) - \frac{\langle \widehat{F}(\widehat{z}_k), \widehat{a}_k \rangle}{\|\widehat{a}_k\|^2} \cdot \widehat{a}_k \right\|^2 \\ &= \left\| \widehat{F}(\widehat{z}_k) - \frac{\frac{\epsilon^2}{\eta} \cdot (\langle a_k, z_k \rangle \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle + \ell^2) + \langle a_k, F(z_k) \rangle}{\epsilon^2 (\langle a_k, z_k \rangle^2 + 1 + \ell^2) + \|a_k\|^2} \cdot \widehat{a}_k \right\|^2. \end{aligned}$$

Define  $f(\epsilon) := \frac{\frac{\epsilon^2}{\eta} \cdot (\langle a_k, z_k \rangle \langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle + \ell^2)}{\epsilon^2 (\langle a_k, z_k \rangle^2 + 1 + \ell^2) + \|a_k\|^2}$ . We can simplify  $\hat{H}(\hat{z}_k)$  to be

$$\epsilon^2 \cdot \left\| \left( \frac{\langle a_{k+\frac{1}{2}}, z_{k+\frac{1}{2}} \rangle}{\eta} - f(\epsilon) \langle a_k, z_k \rangle, -f(\epsilon), \frac{1}{\eta}, 0, \frac{\ell}{\eta} - f(\epsilon)\ell \right) \right\|^2 + \|F(z_k) - f(\epsilon) \cdot a_k\|^2.$$

When  $a_k = (0, \dots, 0)$ ,  $r^{tan}(z_k)^2 = \|F(z_k)\|^2$  (Equation (35)) and  $f(\epsilon) = \frac{\ell^2}{\eta(1+\ell^2)}$  for any  $\epsilon > 0$ . Therefore,  $\lim_{\epsilon \rightarrow 0^+} \hat{H}(\hat{z}_k) = \|F(z_k)\|^2 = r^{tan}(z_k)^2$ , when  $a_k = (0, \dots, 0)$ . When  $a_k \neq (0, \dots, 0)$ ,  $r^{tan}(z_k)^2 = \|F(z_k) - \frac{\langle a_k, F(z_k) \rangle}{\|a_k\|^2} \cdot a_k\|^2$  (Equation (35)) and  $\lim_{\epsilon \rightarrow 0^+} f(\epsilon) = \frac{\langle a_k, F(z_k) \rangle}{\|a_k\|^2}$ , so  $\lim_{\epsilon \rightarrow 0^+} \hat{H}(\hat{z}_k) = \|F(z_k) - \frac{\langle a_k, F(z_k) \rangle}{\|a_k\|^2} \cdot a_k\|^2 = r^{tan}(z_k)^2$ .

Similarly, we define function

$$\begin{aligned} \hat{H}(\hat{z}_{k+1}) &:= \left\| \hat{F}(\hat{z}_{k+1}) - \frac{\langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle \cdot \hat{a}_{k+1}}{\|\hat{a}_{k+1}\|^2} \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle \geq 0 \right] \right\|^2 \\ &= \left\| \hat{F}(\hat{z}_{k+1}) - \frac{\frac{\epsilon^2}{\eta} \cdot (\langle a_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \langle F(z_{k+1}), a_{k+1} \rangle}{\epsilon^2 (\langle a_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \|a_{k+1}\|^2} \cdot \hat{a}_{k+1} \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle \geq 0 \right] \right\|^2. \end{aligned}$$

Define  $g(\epsilon) := \frac{\frac{\epsilon^2}{\eta} \cdot (\langle a_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \langle F(z_{k+1}), a_{k+1} \rangle}{\epsilon^2 (\langle a_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \|a_{k+1}\|^2}$ , and we can simplify  $\hat{H}(\hat{z}_{k+1})$  to be

$$\begin{aligned} \epsilon^2 \left( \frac{1}{\eta} - g(\epsilon) \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle \geq 0 \right] \right)^2 \cdot \|(\langle a_{k+1}, z_{k+1} \rangle, 0, 0, 1, \ell)\|^2 \\ + \left\| F(z_{k+1}) - g(\epsilon) \cdot a_{k+1} \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle \geq 0 \right] \right\|^2. \end{aligned}$$

When  $a_{k+1} = (0, \dots, 0)$ , we have  $g(\epsilon) = \frac{1}{\eta}$  and  $\langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle \geq 0$ , so  $\hat{H}(\hat{z}_{k+1}) = \|F(z_{k+1})\|^2 \geq r^{tan}(z_{k+1})^2$ . When  $a_{k+1} \neq (0, \dots, 0)$ , we have  $\lim_{\epsilon \rightarrow 0^+} g(\epsilon) = \frac{\langle F(z_{k+1}), a_{k+1} \rangle}{\|a_{k+1}\|^2}$  and  $\lim_{\epsilon \rightarrow 0^+} \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle \geq 0 \right] = \mathbb{1} \left[ \langle F(z_{k+1}), a_{k+1} \rangle \geq 0 \right]$ ,<sup>16</sup> hence

$$\lim_{\epsilon \rightarrow 0^+} \hat{H}(\hat{z}_{k+1}) = \left\| F(z_{k+1}) - \frac{\langle F(z_{k+1}), a_{k+1} \rangle}{\|a_{k+1}\|^2} \cdot a_{k+1} \mathbb{1} \left[ \langle F(z_{k+1}), a_{k+1} \rangle \geq 0 \right] \right\|^2 \geq r^{tan}(z_{k+1})^2.$$

The last inequality is because  $\frac{-a_{k+1}}{\|a_{k+1}\|} \in \hat{N}_{\mathcal{Z}}(z_{k+1})$  and Lemma 7.

If  $r^{tan}(z_k)^2 - r^{tan}(z_{k+1})^2 < 0$ , then

$$\lim_{\epsilon \rightarrow 0^+} \left( \hat{H}(\hat{z}_k) - \hat{H}(\hat{z}_{k+1}) \right) \leq r^{tan}(z_k)^2 - r^{tan}(z_{k+1})^2 < 0.$$

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<sup>16</sup>This is because  $\langle \hat{F}(\hat{z}_{k+1}), \hat{a}_{k+1} \rangle$  is never smaller than  $\langle F(z_{k+1}), a_{k+1} \rangle$ , and the function  $\mathbb{1}[x \geq 0]$  is right continuous.

Thus, we can choose a sufficiently small  $\epsilon$  so that  $\hat{H}(\hat{z}_k) - \hat{H}(\hat{z}_{k+1}) < 0$ . Together with Inequalities (49)-(53), we can satisfy all properties excluding the form property using  $\hat{z}_k, \hat{z}_{k+\frac{1}{2}}, \hat{z}_{k+1}, \hat{a}_k, \hat{a}_{k+\frac{1}{2}}, \hat{a}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{z}_{k+\frac{1}{2}}), \hat{F}(\hat{z}_{k+1})$ .

Now we show how to make the vectors also satisfy the form property. We perform a change of basis so that vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}$  and  $\hat{a}_{k+1}$  only depend on the first three coordinates. We use the Gram-Schmidt process to generate a basis, where vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}, \hat{a}_{k+1}$  all lie in the span of the first three vector of the new basis. More formally, let  $N = n + 5$  and  $\{b_i\}_{i \in [N]}$  be a sequence of orthonormal vectors produced by the Gram-Schmidt process on ordered input  $\hat{a}_{k+1}, \hat{a}_{k+\frac{1}{2}}, \hat{a}_k$  and  $\{e_i\}_{i \in [N]}$ . Let  $Q$  be the  $N \times N$  matrix, where the  $i$ -th row of  $Q$  is vector  $b_i$ . Observe that any vector  $z \in \mathbb{R}^N$  written in the basis  $\{e_i\}_{i \in [N]}$  can be represented by the basis  $\{b_i\}_{i \in [N]}$  with coefficients  $Q \cdot z$ .

Let  $\bar{a}_k = Q \cdot \hat{a}_k, \bar{a}_{k+\frac{1}{2}} = Q \cdot \hat{a}_{k+\frac{1}{2}}, \bar{a}_{k+1} = Q \cdot \hat{a}_{k+1}, \bar{z}_k = Q \cdot \hat{z}_k, \bar{z}_{k+\frac{1}{2}} = Q \cdot \hat{z}_{k+\frac{1}{2}}, \bar{z}_{k+1} = Q \cdot \hat{z}_{k+1}, \bar{F}(\bar{z}_k) = Q \cdot \hat{F}(\hat{z}_k), \bar{F}(\bar{z}_{k+\frac{1}{2}}) = Q \cdot \hat{F}(\hat{z}_{k+\frac{1}{2}})$  and  $\bar{F}(\bar{z}_{k+1}) = Q \cdot \hat{F}(\hat{z}_{k+1})$ . Note that  $\bar{a}_k$  is the coefficients of  $\hat{a}_k$  written in the basis  $\{b_i\}_{i \in [N]}$  and similar reasoning holds for the rest of the defined vectors. Clearly, for any  $\hat{z}, \hat{z}' \in \mathbb{R}^N$ , if we define  $\bar{z} = Q \cdot \hat{z}$  and  $\bar{z}' = Q \cdot \hat{z}'$ , then we have

$$\langle \bar{z}, \bar{z}' \rangle = \langle \hat{z}, \hat{z}' \rangle. \quad (54)$$

Combining Equation (54) and the fact that all properties but the form property hold for vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}, \hat{a}_{k+1}, \hat{z}_k, \hat{z}_{k+\frac{1}{2}}, \hat{z}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{z}_{k+\frac{1}{2}})$  and  $\hat{F}(\hat{z}_{k+1})$ , we conclude that the same properties also hold for vectors  $\bar{a}_k, \bar{a}_{k+\frac{1}{2}}, \bar{a}_{k+1}, \bar{z}_k, \bar{z}_{k+\frac{1}{2}}, \bar{z}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{z}_{k+\frac{1}{2}})$  and  $\bar{F}(\bar{z}_{k+1})$ .

Finally, by properties of the Gram-Schmidt process, the order of the vector in its input, and the fact that vectors  $\hat{a}_k, \hat{a}_{k+\frac{1}{2}}$  and  $\hat{a}_{k+1}$  are linearly independent, we have  $\hat{a}_{k+1} \in \text{Span}(b_1)$ ,  $\hat{a}_{k+\frac{1}{2}} \in \text{Span}(b_1, b_2)$ ,  $\hat{a}_k \in \text{Span}(b_1, b_2, b_3)$ ,  $\langle \hat{a}_{k+1}, b_1 \rangle > 0$ ,  $\langle \hat{a}_{k+\frac{1}{2}}, b_2 \rangle > 0$  and  $\langle \hat{a}_k, b_3 \rangle > 0$ . Thus  $\bar{a}_k = (\beta_1, \beta_2, b, 0, \dots, 0)$ ,  $\bar{a}_{k+\frac{1}{2}} = (\alpha, c, 0, \dots, 0)$  and  $\bar{a}_{k+1} = (d, 0, \dots, 0)$ , where  $\beta_1, \beta_2, \alpha \in \mathbb{R}$  and  $b, c, d > 0$ . By properly scaling  $\bar{a}_k, \bar{a}_{k+\frac{1}{2}}$  and  $\bar{a}_{k+1}$ , we can make them satisfy the form property. This completes the proof.  $\blacksquare$

**Proof of Theorem 2:** Assume towards contradiction that  $r^{\tan}(z_{k+1}) > r^{\tan}(z_k)$ , using Lemma 6 there exists numbers  $\alpha, \beta_1, \beta_2 \in \mathbb{R}$  and vectors  $\bar{a}_k, \bar{a}_{k+\frac{1}{2}}, \bar{a}_{k+1}, \bar{z}_k, \bar{z}_{k+\frac{1}{2}}, \bar{z}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{z}_{k+\frac{1}{2}}), \bar{F}(\bar{z}_{k+1}) \in \mathbb{R}^N$  where  $N = n + 5$  that satisfy the properties in the statement of Lemma 6 and

$$\begin{aligned} 0 &> \left\| \bar{F}(\bar{z}_k) - \frac{\langle \bar{F}(\bar{z}_k), \bar{a}_k \rangle}{\|\bar{a}_k\|^2} \bar{a}_k \right\|^2 - \left\| \bar{F}(\bar{z}_{k+1}) - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \bar{a}_{k+1}}{\|\bar{a}_{k+1}\|^2} \mathbb{1}[\langle \bar{F}(\bar{z}_{k+1}), \bar{a}_{k+1} \rangle \geq 0] \right\|^2 \\ &= \|\bar{F}(\bar{z}_k)\|^2 - \|\bar{F}(\bar{z}_{k+1})\|^2 - \frac{(\beta_1 \bar{F}(\bar{z}_k)[1] + \beta_2 \bar{F}(\bar{z}_k)[2] + \bar{F}(\bar{z}_k)[3])^2}{\beta_1^2 + \beta_2^2 + 1} + \bar{F}(\bar{z}_{k+1})[1]^2 \mathbb{1}[\bar{F}(\bar{z}_{k+1})[1] \geq 0], \end{aligned}$$

where we use the fact that  $\bar{a}_k = (\beta_1, \beta_2, 1, 0, \dots, 0)$  and  $\bar{a}_{k+1} = (1, 0, \dots, 0)$ .

We use TARGET to denote

$$\|\bar{F}(\bar{z}_k)\|^2 - \|\bar{F}(\bar{z}_{k+1})\|^2 - \frac{(\beta_1 \bar{F}(\bar{z}_k)[1] + \beta_2 \bar{F}(\bar{z}_k)[2] + \bar{F}(\bar{z}_k)[3])^2}{\beta_1^2 + \beta_2^2 + 1} + \bar{F}(\bar{z}_{k+1})[1]^2 \mathbb{1}[\bar{F}(\bar{z}_{k+1})[1] \geq 0],$$

and our goal is to show that TARGET is non-negative, and thus reach a contradiction.

Our plan is to show that we can obtain a sum of quotients of SOS polynomials by adding non-positive terms to TARGET, which implies the non-negativity of TARGET. We add the non-positive terms in a few steps.

Combining Lemma 6 and the fact that  $\eta > 0$  and  $(\eta L)^2 \leq 1$ , we derive the following two inequalities:

$$\langle \eta \bar{F}(\bar{z}_{k+1}) - \eta \bar{F}(\bar{z}_k), \bar{z}_k - \bar{z}_{k+1} \rangle \leq 0, \quad (55)$$

$$\left\| \eta \bar{F}(\bar{z}_{k+1}) - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) \right\|^2 - \left\| \bar{z}_{k+1} - \bar{z}_{k+\frac{1}{2}} \right\|^2 \leq 0. \quad (56)$$

Equipped with these two inequalities, it is clear that

$$\eta^2 \cdot \text{TARGET} \geq \eta^2 \cdot \text{TARGET} + 2 \cdot \text{LHS of Inequality (55)} + \text{LHS of Inequality (56)}. \quad (57)$$

Therefore, it is sufficient to show that the RHS of Inequality (57) is non-negative.

We take advantage of the sparsity of vectors  $\bar{a}_k, \bar{a}_{k+\frac{1}{2}}, \bar{a}_{k+1}$  by considering the following partition of  $[N]$ . We define  $P_1 = \{1, 2, 3\}$  and  $P_2 = \{4, 5, \dots, N\}$ . For any vector  $z \in \mathbb{R}^N$  and  $j \in \{1, 2\}$ , we define  $p_j(z) \in \mathbb{R}^n$  to be the vector such that  $p_j(z)[i] = z[i]$  for  $i \in P_j$  and  $p_j(z)[i] = 0$  otherwise. We divide the RHS of Inequality (57) using the partition to Expression (58) and Expression (59):

$$\begin{aligned} & \eta^2 \|p_2(\bar{F}(\bar{z}_k))\|^2 - \eta^2 \|p_2(\bar{F}(\bar{z}_{k+1}))\|^2 + 2\eta \langle p_2(\bar{F}(\bar{z}_{k+1})) - p_2(\bar{F}(\bar{z}_k)), p_2(\bar{z}_k) - p_2(\bar{z}_{k+1}) \rangle \\ & + \eta^2 \|p_2(\bar{F}(\bar{z}_{k+1})) - p_2(\bar{F}(\bar{z}_{k+\frac{1}{2}}))\|^2 - \|p_2(\bar{z}_{k+1}) - p_2(\bar{z}_{k+\frac{1}{2}})\|^2, \end{aligned} \quad (58)$$

and

$$\begin{aligned} & \eta^2 \|p_1(\bar{F}(\bar{z}_k))\|^2 - \eta^2 \|p_1(\bar{F}(\bar{z}_{k+1}))\|^2 + 2\eta \langle p_1(\bar{F}(\bar{z}_{k+1})) - p_1(\bar{F}(\bar{z}_k)), p_1(\bar{z}_k) - p_1(\bar{z}_{k+1}) \rangle \\ & + \eta^2 \|p_1(\bar{F}(\bar{z}_{k+1})) - p_1(\bar{F}(\bar{z}_{k+\frac{1}{2}}))\|^2 - \|p_1(\bar{z}_{k+1}) - p_1(\bar{z}_{k+\frac{1}{2}})\|^2 \\ & - \frac{\eta^2}{\beta_1^2 + \beta_2^2 + 1} (\beta_1 \bar{F}(\bar{z}_k)[1] + \beta_2 \bar{F}(\bar{z}_k)[2] + \bar{F}(\bar{z}_k)[3])^2 + \eta^2 \bar{F}(\bar{z}_{k+1})[1]^2 \mathbb{1}[\bar{F}(\bar{z}_{k+1})[1] \geq 0]. \end{aligned} \quad (59)$$

First we show that the Expression (58) is non-negative. According to Lemma 6,  $\bar{a}_{k+\frac{1}{2}}$  and  $\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k)$  are co-directed, and  $\bar{a}_{k+1}$  and  $\bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})$  are co-directed. For any  $i \in P_2$ , we know  $\bar{a}_{k+\frac{1}{2}}[i] = \bar{a}_{k+1}[i] = 0$  and thus we have

$$\begin{aligned} 0 &= \langle e_i, \bar{a}_{k+\frac{1}{2}} \rangle = \langle e_i, \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_{k+1}) \rangle \Leftrightarrow \bar{z}_{k+\frac{1}{2}}[i] = \bar{z}_k[i] - \eta \bar{F}(\bar{z}_{k+1})[i], \\ 0 &= \langle e_i, \bar{a}_{k+1} \rangle = \langle e_i, \bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) \rangle \Leftrightarrow \bar{z}_{k+1}[i] = \bar{z}_k[i] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[i], \end{aligned}$$

which implies that

$$\begin{aligned} p_2(\bar{z}_{k+\frac{1}{2}}) &= p_2(\bar{z}_k) - \eta \cdot p_2(\bar{F}(\bar{z}_k)), \\ p_2(\bar{z}_{k+1}) &= p_2(\bar{z}_k) - \eta \cdot p_2(\bar{F}(\bar{z}_{k+\frac{1}{2}})). \end{aligned}$$

Intuitively, one can think of the coordinates in  $P_2$  as the ones where the EG update is unconstrained. With the two new equalities, it is easy to verify that Expression (58) is always 0.

$$\begin{aligned}
& \eta^2 \|p_2(\bar{F}(\bar{z}_k))\|^2 - \eta^2 \|p_2(\bar{F}(\bar{z}_{k+1}))\|^2 + 2\eta \langle p_2(\bar{F}(\bar{z}_{k+1})) - p_2(\bar{F}(\bar{z}_k)), p_2(\bar{z}_k) - p_2(\bar{z}_{k+1}) \rangle \\
& + \eta^2 \|p_2(\bar{F}(\bar{z}_{k+1})) - p_2(\bar{F}(\bar{z}_{k+\frac{1}{2}}))\|^2 - \|p_2(\bar{z}_{k+1}) - p_2(\bar{z}_{k+\frac{1}{2}})\|^2 \\
& = \eta^2 \|p_2(\bar{F}(\bar{z}_k))\|^2 - \eta^2 \|p_2(\bar{F}(\bar{z}_{k+1}))\|^2 + 2\eta^2 \langle p_2(\bar{F}(\bar{z}_{k+1})) - p_2(\bar{F}(\bar{z}_k)), p_2(\bar{F}(\bar{z}_{k+\frac{1}{2}})) \rangle \\
& + \eta^2 \|p_2(\bar{F}(\bar{z}_{k+1})) - p_2(\bar{F}(\bar{z}_{k+\frac{1}{2}}))\|^2 - \eta^2 \|p_2(\bar{F}(\bar{z}_k) - p_2(\bar{F}(\bar{z}_{k+\frac{1}{2}}))\|^2 \\
& = 0.
\end{aligned}$$

We now turn our attention to Expression (59) and show that it is non-negative. The analysis is more challenging here. We introduce the following six non-positive expressions, multiply each of them with a carefully chosen coefficient, then add them together with Expression (59). We finally verify that the sum is a sum of quotients of SOS polynomials implying the non-negativity of Expression (59). We believe it will be extremely challenging if not impossible for human beings to discover these non-positive expressions and their associated coefficients manually to complete this proof. We instead harness the power of the SOS programming to overcome the difficulty and make the discovery.

We first present the six non-positive expressions.

$$\bar{z}_{k+\frac{1}{2}}[1] \left( \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right) [1] - \alpha \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right) [2] \right) = 0, \quad (60)$$

$$\bar{z}_{k+1}[2] \left( \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right) [1] - \alpha \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right) [2] \right) = 0, \quad (61)$$

$$(\alpha(\bar{z}_k - \eta \bar{F}(\bar{z}_k)) [1] + (\bar{z}_k - \eta \bar{F}(\bar{z}_k)) [2]) (\alpha \bar{z}_{k+1} [1] + \bar{z}_{k+1} [2]) \leq 0, \quad (62)$$

$$-\eta (\beta_1 \bar{F}(\bar{z}_k) [1] + \beta_2 \bar{F}(\bar{z}_k) [2] + \bar{F}(\bar{z}_k) [3]) (\beta_1 \bar{z}_{k+\frac{1}{2}} [1] + \beta_2 \bar{z}_{k+\frac{1}{2}} [2] + \bar{z}_{k+\frac{1}{2}} [3]) \leq 0, \quad (63)$$

$$\bar{z}_k [1] (\bar{z}_k [1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) [1]) \leq 0, \quad (64)$$

$$-\eta \bar{F}(\bar{z}_{k+1}) [1] \mathbb{1} [\bar{F}(\bar{z}_{k+1}) [1] \leq 0] (\bar{z}_k [1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) [1]) \leq 0. \quad (65)$$

Equation (60) and Equation (61) follow from the combination of the fact that  $\langle (1, -\alpha, 0, \dots, 0), \bar{a}_{k+\frac{1}{2}} \rangle = 0$  and that  $\bar{a}_{k+\frac{1}{2}}$  and  $\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k)$  are co-directed:

$$\begin{aligned}
& \langle (1, -\alpha, 0, \dots, 0), \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \rangle = 0 \\
& \Leftrightarrow \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right) [1] - \alpha \left( \bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k) \right) [2] = 0.
\end{aligned}$$

Note that the LHS of Inequality (62) is equal to  $\langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_k - \eta \bar{F}(\bar{z}_k) \rangle \cdot \langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_{k+1} \rangle$ . Lemma 6 guarantees that  $\langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_{k+1} \rangle \geq 0$ . Since  $\bar{a}_{k+\frac{1}{2}}$  and  $\bar{z}_k - \eta \bar{F}(\bar{z}_k) - \bar{z}_{k+\frac{1}{2}}$  are oppositely directed, and  $\langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_{k+\frac{1}{2}} \rangle = 0$ , we have that

$$0 \geq \langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_k - \eta \bar{F}(\bar{z}_k) - \bar{z}_{k+\frac{1}{2}} \rangle = \langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_k - \eta \bar{F}(\bar{z}_k) \rangle.$$

Hence,  $\langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_k - \eta \bar{F}(\bar{z}_k) \rangle \cdot \langle \bar{a}_{k+\frac{1}{2}}, \bar{z}_{k+1} \rangle \leq 0$ .

Observe that the LHS of Inequality (63) is equal to  $-\eta \langle \bar{a}_k, \bar{F}(\bar{z}_k) \rangle \cdot \langle \bar{a}_k, \bar{z}_{k+\frac{1}{2}} \rangle$ . Lemma 6 guarantees that  $\langle \bar{a}_k, \bar{F}(\bar{z}_k) \rangle \geq 0$  and  $\langle \bar{a}_k, \bar{z}_{k+\frac{1}{2}} \rangle \geq 0$ .

Finally, we argue Inequality (64) and Inequality (65). Note that the LHS of Inequality (64) is equal to  $\langle \bar{a}_{k+1}, \bar{z}_k \rangle \cdot \langle \bar{a}_{k+1}, \bar{z}_k - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) \rangle$ . Since  $\bar{a}_{k+1}$  and  $\bar{z}_k - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) - \bar{z}_{k+1}$  are oppositely directed, and  $\langle \bar{a}_{k+1}, \bar{z}_{k+1} \rangle = 0$ , we have that

$$0 \geq \langle \bar{a}_{k+1}, \bar{z}_k - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) - \bar{z}_{k+1} \rangle = \langle \bar{a}_{k+1}, \bar{z}_k - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}}) \rangle.$$

Clearly,  $-\bar{F}(\bar{z}_{k+1}) \mathbb{1}[\bar{F}(\bar{z}_{k+1})[1] \leq 0]$  is non-negative and  $\bar{z}_k[1] = \langle \bar{a}_{k+1}, \bar{z}_k \rangle$  is also non-negative due to Lemma 6. Thus Inequality (64) and Inequality (65) hold.

Our next step is to show that the following is non-negative.

$$\begin{aligned} & \text{Expression (59)} + 2 \times (\text{LHS of Equation (60)} + \text{LHS of Inequality (64)} + \text{LHS of Inequality (65)}) \\ & + \frac{2\alpha}{1+\alpha^2} \times \text{LHS of Equation (61)} + \frac{2}{1+\alpha^2} \times \text{LHS of Inequality (62)} + \frac{2}{1+\beta_1^2+\beta_2^2} \times \text{LHS of Inequality (63)} \end{aligned} \quad (66)$$

We first simplify Expression (66), using the following relationship between the variables.

$$\bar{z}_k[3] = -\beta_1 \bar{z}_k[1] - \beta_2 \bar{z}_k[2], \quad (67)$$

$$\bar{z}_{k+\frac{1}{2}}[2] = -\alpha \bar{z}_{k+\frac{1}{2}}[1], \quad (68)$$

$$\bar{z}_{k+\frac{1}{2}}[3] = \bar{z}_k[3] - \eta \bar{F}(\bar{z}_k)[3], \quad (69)$$

$$\bar{z}_{k+1}[1] = 0, \quad (70)$$

$$\bar{z}_{k+1}[2] = \bar{z}_k[2] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[2], \quad (71)$$

$$\bar{z}_{k+1}[3] = \bar{z}_k[3] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[3]. \quad (72)$$

Equation (67), Equation (68), and Equation (70) follows by  $\langle \bar{a}_i, \bar{z}_i \rangle = 0$  for  $i \in \{k, k+\frac{1}{2}, k+1\}$  due to Lemma 6. We know that (i)  $\langle \bar{a}_{k+\frac{1}{2}}, e_3 \rangle = \langle \bar{a}_{k+1}, e_2 \rangle = \langle \bar{a}_{k+1}, e_3 \rangle = 0$  by the definition of  $\bar{a}_{k+\frac{1}{2}}$  and  $\bar{a}_{k+1}$ , and (ii)  $\bar{a}_{k+\frac{1}{2}}$  and  $\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k)$  are co-directed,  $\bar{a}_{k+1}$  and  $\bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})$  are co-directed by Lemma 6, thus Equation (69), Equation (71), and Equation (72) follow from the combination of (i) and (ii).

We simplify Expression (66) by substituting  $\bar{z}_k[3]$ ,  $\bar{z}_{k+\frac{1}{2}}[2]$ ,  $\bar{z}_{k+\frac{1}{2}}[3]$ ,  $\bar{z}_{k+1}[1]$ ,  $\bar{z}_{k+1}[2]$ , and  $\bar{z}_{k+1}[3]$  using Equations (67)-(72).

Expression (59) is equal to the sum of the following three parts.

The first part is

$$\begin{aligned}
& \eta^2 \|p_1(\bar{F}(\bar{z}_k))\|^2 - \eta^2 \|p_1(\bar{F}(\bar{z}_{k+1}))\|^2 - \frac{\eta^2}{\beta_1^2 + \beta_2^2 + 1} (\beta_1 \bar{F}(\bar{z}_k)[1] + \beta_2 \bar{F}(\bar{z}_k)[2] + \bar{F}(\bar{z}_k)[3])^2 \\
& + \eta^2 \bar{F}(\bar{z}_{k+1})[1]^2 \mathbb{1}[\bar{F}(\bar{z}_{k+1})[1] \geq 0] \\
& = \sum_{i=1}^3 (\eta^2 \bar{F}(\bar{z}_k)[i]^2 - \eta^2 \bar{F}(\bar{z}_{k+1})[i]^2) - \frac{\eta^2}{\beta_1^2 + \beta_2^2 + 1} (\beta_1 \bar{F}(\bar{z}_k)[1] + \beta_2 \bar{F}(\bar{z}_k)[2] + \bar{F}(\bar{z}_k)[3])^2 \\
& + \eta^2 \bar{F}(\bar{z}_{k+1})[1]^2 \mathbb{1}[\bar{F}(\bar{z}_{k+1})[1] \geq 0].
\end{aligned} \tag{73}$$

The second part is

$$\begin{aligned}
& 2\eta \langle p_1(\bar{F}(\bar{z}_{k+1})) - p_1(\bar{F}(\bar{z}_k)), p_1(\bar{z}_k) - p_1(\bar{z}_{k+1}) \rangle \\
& = 2\eta \bar{z}_k[1] (\bar{F}(\bar{z}_{k+1})[1] - \bar{F}(\bar{z}_k)[1]) + 2\eta^2 \bar{F}(\bar{z}_{k+\frac{1}{2}})[2] (\bar{F}(\bar{z}_{k+1})[2] - \bar{F}(\bar{z}_k)[2]) \\
& + 2\eta^2 \bar{F}(\bar{z}_{k+\frac{1}{2}})[3] (\bar{F}(\bar{z}_{k+1})[3] - \bar{F}(\bar{z}_k)[3]).
\end{aligned} \tag{74}$$

The third part is

$$\begin{aligned}
& \eta^2 \left\| p_1(\bar{F}(\bar{z}_{k+1})) - p_1(\bar{F}(\bar{z}_{k+\frac{1}{2}})) \right\|^2 - \left\| p_1(\bar{z}_{k+1}) - p_1(\bar{z}_{k+\frac{1}{2}}) \right\|^2 \\
& = \sum_{i=1}^3 \eta^2 \left( \bar{F}(\bar{z}_{k+1})[i] - \bar{F}(\bar{z}_{k+\frac{1}{2}})[i] \right)^2 - \bar{z}_{k+\frac{1}{2}}[1]^2 - \left( \bar{z}_k[2] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[2] + \alpha \bar{z}_{k+\frac{1}{2}}[1] \right)^2 \\
& - \left( \eta \bar{F}(\bar{z}_k)[3] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[3] \right)^2.
\end{aligned} \tag{75}$$

$2 \times$  LHS of Equation (60) is equal to

$$\begin{aligned}
& 2\bar{z}_{k+\frac{1}{2}}[1] \left( (\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k))[1] - \alpha(\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k))[2] \right) \\
& = 2\bar{z}_{k+\frac{1}{2}}[1] \left( (\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k))[1] - \alpha(-\alpha \bar{z}_{k+\frac{1}{2}}[1] - \bar{z}_k[2] + \eta \bar{F}(\bar{z}_k)[2]) \right).
\end{aligned} \tag{76}$$

$\frac{2\alpha}{1+\alpha^2} \times$  LHS of Equation (61) is equal to

$$\begin{aligned}
& \frac{2\alpha}{1+\alpha^2} \bar{z}_{k+1}[2] \left( (\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k))[1] - \alpha(\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k))[2] \right) \\
& = \frac{2\alpha}{1+\alpha^2} \left( \bar{z}_k[2] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[2] \right) \left( (\bar{z}_{k+\frac{1}{2}} - \bar{z}_k + \eta \bar{F}(\bar{z}_k))[1] - \alpha(-\alpha \bar{z}_{k+\frac{1}{2}}[1] - \bar{z}_k[2] + \eta \bar{F}(\bar{z}_k)[2]) \right).
\end{aligned} \tag{77}$$

$\frac{2}{1+\alpha^2} \times$  LHS of Inequality (62) is equal to

$$\begin{aligned}
& \frac{2}{1+\alpha^2} (\alpha(\bar{z}_k - \eta \bar{F}(\bar{z}_k))[1] + (\bar{z}_k - \eta \bar{F}(\bar{z}_k))[2]) (\alpha \bar{z}_{k+1}[1] + \bar{z}_{k+1}[2]) \\
& = \frac{2}{1+\alpha^2} (\alpha(\bar{z}_k - \eta \bar{F}(\bar{z}_k))[1] + (\bar{z}_k - \eta \bar{F}(\bar{z}_k))[2]) \left( \bar{z}_k[2] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[2] \right).
\end{aligned} \tag{78}$$

$\frac{2}{1+\beta_1^2+\beta_2^2} \times \text{LHS of Inequality (63)}$  is equal to

$$\begin{aligned}
& -\frac{2}{1+\beta_1^2+\beta_2^2} (\beta_1 \eta \bar{F}(\bar{z}_k)[1] + \beta_2 \eta \bar{F}(\bar{z}_k)[2] + \eta \bar{F}(\bar{z}_k)[3]) (\beta_1 \bar{z}_{k+\frac{1}{2}}[1] + \beta_2 \bar{z}_{k+\frac{1}{2}}[2] + \bar{z}_{k+\frac{1}{2}}[3]) \\
& = -\frac{2}{1+\beta_1^2+\beta_2^2} (\beta_1 \eta \bar{F}(\bar{z}_k)[1] + \beta_2 \eta \bar{F}(\bar{z}_k)[2] + \eta \bar{F}(\bar{z}_k)[3]) ((\beta_1 - \alpha \beta_2) \bar{z}_{k+\frac{1}{2}}[1] + \bar{z}_k[3] - \eta \bar{F}(\bar{z}_k)[3]) \\
& = -\frac{2}{1+\beta_1^2+\beta_2^2} (\eta \beta_1 \bar{F}(\bar{z}_k)[1] + \eta \beta_2 \bar{F}(\bar{z}_k)[2] + \eta \bar{F}(\bar{z}_k)[3]) \\
& \quad \cdot ((\beta_1 - \alpha \beta_2) \bar{z}_{k+\frac{1}{2}}[1] - \beta_1 \bar{z}_k[1] - \beta_2 \bar{z}_k[2] - \eta \bar{F}(\bar{z}_k)[3]). \tag{79}
\end{aligned}$$

$2 \times \text{LHS of Inequality (64)}$  is equal to

$$2 \bar{z}_k[1] (\bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1]). \tag{80}$$

$2 \times \text{LHS of Inequality (65)}$  is equal to

$$-2 \eta \bar{F}(\bar{z}_{k+1})[1] \mathbb{1} [\bar{F}(\bar{z}_{k+1})[1] \leq 0] (\bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1]). \tag{81}$$

After the substitution, we need to argue that the sum of Expression (73) to (81) is a sum of quotients of SOS polynomials, which we prove by establishing the following identity.

$$\begin{aligned}
& \text{Expression (73)} + \text{Expression (74)} + \text{Expression (75)} + \text{Expression (76)} + \text{Expression (77)} \\
& \quad + \text{Expression (78)} + \text{Expression (79)} + \text{Expression (80)} + \text{Expression (81)} \\
& = (\bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1] + \eta \bar{F}(\bar{z}_{k+1})[1] \cdot \mathbb{1} [\bar{F}(\bar{z}_{k+1})[1] \geq 0])^2 \tag{82}
\end{aligned}$$

$$+ \frac{(\bar{z}_k[1] - \eta \bar{F}(\bar{z}_k)[1] - \bar{z}_{k+\frac{1}{2}}[1])^2}{1 + \beta_1^2 + \beta_2^2} \tag{83}$$

$$+ \frac{(\eta \bar{F}(\bar{z}_k)[3] + \beta_1 \bar{z}_k[1] + \beta_2 \bar{z}_k[2] + (\alpha \beta_2 - \beta_1) \bar{z}_{k+\frac{1}{2}}[1])^2}{1 + \beta_1^2 + \beta_2^2} \tag{84}$$

$$+ \frac{(\bar{z}_k[2] - \eta \bar{F}(\bar{z}_k)[2] + \alpha \bar{z}_{k+\frac{1}{2}}[1])^2}{1 + \beta_1^2 + \beta_2^2} \tag{85}$$

$$+ \frac{(\beta_1 (\bar{z}_k[2] - \eta \bar{F}(\bar{z}_k)[2] + \alpha \bar{z}_{k+\frac{1}{2}}[1]) - \beta_2 (\bar{z}_k[1] - \eta \bar{F}(\bar{z}_k)[1] - \bar{z}_{k+\frac{1}{2}}[1]))^2}{1 + \beta_1^2 + \beta_2^2} \tag{86}$$

$\geq 0$ .

We verify this identity by expanding both the LHS and RHS in Section F only for the case where  $\bar{F}(\bar{z}_{k+1})[1] \geq 0$ . Observe that only Expression (73), Expression (81) and Term (82) depend



on the sign of  $\bar{F}(\bar{z}_{k+1})[1]$ . It is sufficient for us to verify the case where  $\bar{F}(\bar{z}_{k+1})[1] \geq 0$ , as when  $\bar{F}(\bar{z}_{k+1})[1] < 0$ , we only need to subtract  $\eta^2 \bar{F}(\bar{z}_{k+1})[1]^2 + 2\eta \bar{F}(\bar{z}_{k+1})[1](\bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1])$  from both the LHS and the RHS,<sup>17</sup> and the identity still holds.

Hence, Expression (59) is non-negative. Combining with the non-negativity of Expression (58), we conclude that TARGET is non-negative. This completes the proof. ■

## F Table Verification

In this section, we expand Expression (73)-(86) and verify the identity in the proof of Theorem 2. We expand Expression (73)-(86) in the tables below, In each table, rows that correspond to an expression contain the coefficients of that expression; rows that are marked by "Sum" contain the sum of the coefficients of the expressions above it.

	$\eta \bar{F}(\bar{z}_k)[1] \bar{z}_k[1]$	$\eta \bar{F}(\bar{z}_k)[1] \bar{z}_{k+\frac{1}{2}}[1]$	$\eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1] \bar{z}_k[1]$	$\eta \bar{F}(\bar{z}_{k+1})[1] \bar{z}_k[1]$	$\eta \bar{F}(\bar{z}_k)[2] \bar{z}_k[1]$
Expression(73)	0	0	0	0	0
Expression(74)	-2	0	0	2	0
Expression(75)	0	0	0	0	0
Expression(76)	0	2	0	0	0
Expression(77)	0	0	0	0	0
Expression(78)	0	0	0	0	0
Expression(79)	$\frac{2\beta_1^2}{\beta_1^2+\beta_2^2+1}$	$-\frac{\beta_1(2\beta_1-2\alpha\beta_2)}{\beta_1^2+\beta_2^2+1}$	0	0	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$
Expression(80)	0	0	-2	0	0
Sum	$-\frac{2\beta_2^2+2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_2^2+2\alpha\beta_1\beta_2+2}{\beta_1^2+\beta_2^2+1}$	-2	2	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$
Expression(82)	0	0	-2	2	0
Expression(83)	$-\frac{2}{\beta_1^2+\beta_2^2+1}$	$\frac{2}{\beta_1^2+\beta_2^2+1}$	0	0	0
Expression(84)	0	0	0	0	0
Expression(85)	0	0	0	0	0
Expression(86)	$-\frac{2\beta_2^2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_2^2+2\alpha\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	0	0	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$
Sum	$-\frac{2\beta_2^2+2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_2^2+2\alpha\beta_1\beta_2+2}{\beta_1^2+\beta_2^2+1}$	-2	2	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$

<sup>17</sup>Notice that  $(\bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1] + \eta \bar{F}(\bar{z}_{k+1})[1])^2 = (\bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1])^2 + \eta^2 \bar{F}(\bar{z}_{k+1})[1]^2 + 2\eta \bar{F}(\bar{z}_{k+1})[1](\bar{z}_k[1] - \eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1])$ .

	$\eta \bar{F}(\bar{z}_k)[2] \bar{z}_{k+\frac{1}{2}}[1]$	$\eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[2] \bar{z}_k[1]$	$\eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[2] \bar{z}_{k+\frac{1}{2}}[1]$	$\eta \bar{F}(\bar{z}_k)[3] \bar{z}_k[1]$
Expression(73)	0	0	0	0
Expression(74)	0	0	0	0
Expression(75)	0	0	$2\alpha$	0
Expression(76)	$-2\alpha$	0	0	0
Expression(77)	0	$\frac{2\alpha}{\alpha^2+1}$	$-2\alpha$	0
Expression(78)	0	$-\frac{2\alpha}{\alpha^2+1}$	0	0
Expression(79)	$-\frac{\beta_2(2\beta_1-2\alpha\beta_2)}{\beta_1^2+\beta_2^2+1}$	0	0	$\frac{2\beta_1}{\beta_1^2+\beta_2^2+1}$
Expression(80)	0	0	0	0
Sum	$-\frac{2\alpha(\beta_1^2+1)}{\beta_1^2+\beta_2^2+1} - \frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	0	0	$\frac{2\beta_1}{\beta_1^2+\beta_2^2+1}$

Expression(82)	0	0	0	0
Expression(83)	0	0	0	0
Expression(84)	0	0	0	$\frac{2\beta_1}{\beta_1^2+\beta_2^2+1}$
Expression(85)	$-\frac{2\alpha}{\beta_1^2+\beta_2^2+1}$	0	0	0
Expression(86)	$-\frac{2\beta_1(\alpha\beta_1+\beta_2)}{\beta_1^2+\beta_2^2+1}$	0	0	0
Sum	$-\frac{2\alpha(\beta_1^2+1)}{\beta_1^2+\beta_2^2+1} - \frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	0	0	$\frac{2\beta_1}{\beta_1^2+\beta_2^2+1}$

	$\eta \bar{F}(\bar{z}_k)[3] \bar{z}_{k+\frac{1}{2}}[1]$	$\eta \bar{F}(\bar{z}_k)[1] \bar{z}_k[2]$	$\eta \bar{F}(\bar{z}_k)[2] \bar{z}_k[2]$	$\eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[2] \bar{z}_k[2]$
Expression(73)	0	0	0	0
Expression(74)	0	0	0	0
Expression(75)	0	0	0	2
Expression(76)	0	0	0	0
Expression(77)	0	$\frac{2\alpha}{\alpha^2+1}$	$-\frac{2\alpha^2}{\alpha^2+1}$	$-\frac{2\alpha^2}{\alpha^2+1}$
Expression(78)	0	$-\frac{2\alpha}{\alpha^2+1}$	$-\frac{2}{\alpha^2+1}$	$-\frac{2}{\alpha^2+1}$
Expression(79)	$-\frac{2\beta_1-2\alpha\beta_2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_2^2}{\beta_1^2+\beta_2^2+1}$	0
Expression(80)	0	0	0	0
Sum	$-\frac{2\beta_1-2\alpha\beta_2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	$-\frac{2\beta_1^2+2}{\beta_1^2+\beta_2^2+1}$	0

Expression(82)	0	0	0	0
Expression(83)	0	0	0	0
Expression(84)	$-\frac{2\beta_1-2\alpha\beta_2}{\beta_1^2+\beta_2^2+1}$	0	0	0
Expression(85)	0	0	$-\frac{2}{\beta_1^2+\beta_2^2+1}$	0
Expression(86)	0	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	$-\frac{2\beta_1^2}{\beta_1^2+\beta_2^2+1}$	0
Sum	$-\frac{2\beta_1-2\alpha\beta_2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	$-\frac{2\beta_1^2+2}{\beta_1^2+\beta_2^2+1}$	0

	$\eta \bar{F}(\bar{z}_k)[3] \bar{z}_k[2]$	$\bar{z}_k[1] \bar{z}_{k+\frac{1}{2}}[1]$	$\bar{z}_k[1] \bar{z}_k[2]$	$\bar{z}_{k+\frac{1}{2}}[1] \bar{z}_k[2]$
Expression(73)	0	0	0	0
Expression(74)	0	0	0	0
Expression(75)	0	0	0	$-2\alpha$
Expression(76)	0	$-2$	0	$2\alpha$
Expression(77)	0	0	$-\frac{2\alpha}{\alpha^2+1}$	$2\alpha$
Expression(78)	0	0	$\frac{2\alpha}{\alpha^2+1}$	0
Expression(79)	$\frac{2\beta_2}{\beta_1^2+\beta_2^2+1}$	0	0	0
Expression(80)	0	0	0	0
Sum	$\frac{2\beta_2}{\beta_1^2+\beta_2^2+1}$	$-2$	0	$2\alpha$

Expression(82)	0	0	0	0
Expression(83)	0	$-\frac{2}{\beta_1^2+\beta_2^2+1}$	0	0
Expression(84)	$\frac{2\beta_2}{\beta_1^2+\beta_2^2+1}$	$-\frac{2\beta_1(\beta_1-\alpha\beta_2)}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	$-\frac{2\beta_2(\beta_1-\alpha\beta_2)}{\beta_1^2+\beta_2^2+1}$
Expression(85)	0	0	0	$\frac{2\alpha}{\beta_1^2+\beta_2^2+1}$
Expression(86)	0	$-\frac{2\beta_2^2+2\alpha\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	$-\frac{2\beta_1\beta_2}{\beta_1^2+\beta_2^2+1}$	$\frac{2\beta_1(\alpha\beta_1+\beta_2)}{\beta_1^2+\beta_2^2+1}$
Sum	$\frac{2\beta_2}{\beta_1^2+\beta_2^2+1}$	$-2$	0	$2\alpha$

	$(\eta \bar{F}(\bar{z}_k)[1])^2$	$(\eta \bar{F}(\bar{z}_{k+\frac{1}{2}})[1])^2$	$(\eta \bar{F}(\bar{z}_{k+1})[1])^2$	$(\eta \bar{F}(\bar{z}_k)[2])^2$	$(\eta \bar{F}(\bar{z}_{k+1})[2])^2$
Expression(73)	$1 - \frac{\beta_1^2}{\beta_1^2+\beta_2^2+1}$	0	0	$1 - \frac{\beta_2^2}{\beta_1^2+\beta_2^2+1}$	$-1$
Expression(74)	0	0	0	0	0
Expression(75)	0	1	1	0	1
Expression(76)	0	0	0	0	0
Expression(77)	0	0	0	0	0
Expression(78)	0	0	0	0	0
Expression(79)	0	0	0	0	0
Expression(80)	0	0	0	0	0
Sum	$\frac{\beta_2^2+1}{\beta_1^2+\beta_2^2+1}$	1	1	$\frac{\beta_1^2+1}{\beta_1^2+\beta_2^2+1}$	0

Expression(82)	0	1	1	0	0
Expression(83)	$\frac{1}{\beta_1^2+\beta_2^2+1}$	0	0	0	0
Expression(84)	0	0	0	0	0
Expression(85)	0	0	0	$\frac{1}{\beta_1^2+\beta_2^2+1}$	0
Expression(86)	$\frac{\beta_2^2}{\beta_1^2+\beta_2^2+1}$	0	0	$\frac{\beta_1^2}{\beta_1^2+\beta_2^2+1}$	0
Sum	$\frac{\beta_2^2+1}{\beta_1^2+\beta_2^2+1}$	1	1	$\frac{\beta_1^2+1}{\beta_1^2+\beta_2^2+1}$	0

	$(\eta\bar{F}(\bar{z}_k)[3])^2$	$(\eta\bar{F}(\bar{z}_{k+1})[3])^2$	$\bar{z}_k[1]^2$	$\bar{z}_{k+\frac{1}{2}}[1]^2$
Expression(73)	$1 - \frac{1}{\beta_1^2 + \beta_2^2 + 1}$	-1	0	0
Expression(74)	0	0	0	0
Expression(75)	-1	1	0	$-\alpha^2 - 1$
Expression(76)	0	0	0	$2\alpha^2 + 2$
Expression(77)	0	0	0	0
Expression(78)	0	0	0	0
Expression(79)	$\frac{2}{\beta_1^2 + \beta_2^2 + 1}$	0	0	0
Expression(80)	0	0	2	0
Sum	$\frac{1}{\beta_1^2 + \beta_2^2 + 1}$	0	2	$\alpha^2 + 1$

Expression(82)	0	0	1	0
Expression(83)	0	0	$\frac{1}{\beta_1^2 + \beta_2^2 + 1}$	$\frac{1}{\beta_1^2 + \beta_2^2 + 1}$
Expression(84)	$\frac{1}{\beta_1^2 + \beta_2^2 + 1}$	0	$\frac{\beta_1^2}{\beta_1^2 + \beta_2^2 + 1}$	$\frac{(\beta_1 - \alpha\beta_2)^2}{\beta_1^2 + \beta_2^2 + 1}$
Expression(85)	0	0	0	$\frac{\alpha^2}{\beta_1^2 + \beta_2^2 + 1}$
Expression(86)	0	0	$\frac{\beta_2^2}{\beta_1^2 + \beta_2^2 + 1}$	$\frac{(\beta_2 + \alpha\beta_1)^2}{\beta_1^2 + \beta_2^2 + 1}$
Sum	$\frac{1}{\beta_1^2 + \beta_2^2 + 1}$	0	2	$\alpha^2 + 1$

	$\bar{z}_k[2]^2$	$\eta\bar{F}(\bar{z}_{k+\frac{1}{2}})[1] \eta\bar{F}(\bar{z}_{k+1})[1]$	$\eta\bar{F}(\bar{z}_k)[1] \eta\bar{F}(\bar{z}_k)[2]$	$\eta\bar{F}(\bar{z}_k)[1] \eta\bar{F}(\bar{z}_{k+\frac{1}{2}})[2]$
Expression(73)	0	0	$-\frac{2\beta_1\beta_2}{\beta_1^2 + \beta_2^2 + 1}$	0
Expression(74)	0	0	0	0
Expression(75)	-1	-2	0	0
Expression(76)	0	0	0	0
Expression(77)	$\frac{2\alpha^2}{\alpha^2 + 1}$	0	0	$-\frac{2\alpha}{\alpha^2 + 1}$
Expression(78)	$\frac{2}{\alpha^2 + 1}$	0	0	$\frac{2\alpha}{\alpha^2 + 1}$
Expression(79)	0	0	0	0
Expression(80)	0	0	0	0
Sum	1	-2	$-\frac{2\beta_1\beta_2}{\beta_1^2 + \beta_2^2 + 1}$	0

Expression(82)	0	-2	0	0
Expression(83)	0	0	0	0
Expression(84)	$\frac{\beta_2^2}{\beta_1^2 + \beta_2^2 + 1}$	0	0	0
Expression(85)	$\frac{1}{\beta_1^2 + \beta_2^2 + 1}$	0	0	0
Expression(86)	$\frac{\beta_1^2}{\beta_1^2 + \beta_2^2 + 1}$	0	$-\frac{2\beta_1\beta_2}{\beta_1^2 + \beta_2^2 + 1}$	0
Sum	1	-2	$-\frac{2\beta_1\beta_2}{\beta_1^2 + \beta_2^2 + 1}$	0

	$\eta\bar{F}(\bar{z}_k)[1] \eta\bar{F}(\bar{z}_k)[3]$	$\eta\bar{F}(\bar{z}_k)[2] \eta\bar{F}(\bar{z}_{k+\frac{1}{2}})[2]$	$\eta\bar{F}(\bar{z}_{k+\frac{1}{2}})[2] \eta\bar{F}(\bar{z}_{k+1})[2]$	$\eta\bar{F}(\bar{z}_k)[2] \eta\bar{F}(\bar{z}_k)[3]$
Expression(73)	$-\frac{2\beta_1}{\beta_1^2+\beta_2^2+1}$	0	0	$-\frac{2\beta_2}{\beta_1^2+\beta_2^2+1}$
Expression(74)	0	-2	2	0
Expression(75)	0	0	-2	0
Expression(76)	0	0	0	0
Expression(77)	0	$\frac{2\alpha^2}{\alpha^2+1}$	0	0
Expression(78)	0	$\frac{2}{\alpha^2+1}$	0	0
Expression(79)	$\frac{2\beta_1}{\beta_1^2+\beta_2^2+1}$	0	0	$\frac{2\beta_2}{\beta_1^2+\beta_2^2+1}$
Expression(80)	0	0	0	0
Sum	0	0	0	0
Expression(82)	0	0	0	0
Expression(83)	0	0	0	0
Expression(84)	0	0	0	0
Expression(85)	0	0	0	0
Expression(86)	0	0	0	0
Sum	0	0	0	0
	$\eta\bar{F}(\bar{z}_k)[3] \eta\bar{F}(\bar{z}_{k+\frac{1}{2}})[3]$	$\eta\bar{F}(\bar{z}_{k+\frac{1}{2}})[3] \eta\bar{F}(\bar{z}_{k+1})[3]$		
Expression(73)	0	0		
Expression(74)	-2	2		
Expression(75)	2	-2		
Expression(76)	0	0		
Expression(77)	0	0		
Expression(78)	0	0		
Expression(79)	0	0		
Expression(80)	0	0		
Sum	0	0		
Expression(82)	0	0		
Expression(83)	0	0		
Expression(84)	0	0		
Expression(85)	0	0		
Expression(86)	0	0		
Sum	0	0		

## G Non-Monotonicity of Several Standard Performance Measures

We conduct numerical experiments by trying to find saddle points in constrained bilinear games using EG, and verified that the following performance measures are not monotone: the (squared) natural residual,  $\|z_k - z_{k+\frac{1}{2}}\|^2$ ,  $\|z_k - z_{k+1}\|^2$ ,  $\max_{z \in \mathcal{Z}} \langle F(z), z_k - z \rangle$ ,  $\max_{z \in \mathcal{Z}} \langle F(z_k), z_k - z \rangle$ .

All of our counterexamples are constructed by trying to find a saddle point in bilinear games of the following form:

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} x^\top A y - b^\top x - c^\top y \quad (87)$$

where  $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^2$ ,  $A$  is a  $2 \times 2$  matrix and  $b, c$  are 2-dimensional column vectors. All of the instances of the bilinear game considered in this section have  $\mathcal{X}, \mathcal{Y} = [0, 10]^2$ . We denote by  $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$  and by  $F(x, y) = \begin{pmatrix} A y - b \\ -A^\top x + c \end{pmatrix} : \mathcal{Z} \rightarrow \mathbb{R}^n$ . We remind readers that finding a saddle point of bilinear game (87), is equivalent to solving the monotone VI with operator  $F(z)$  on set  $\mathcal{Z}$ .

### G.1 Non-Monotonicity of the Natural Residual and its Variants

**Performance Measure: Natural Residual.** Let  $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ ,  $b = c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Running the EG method on the corresponding VI problem with step-size  $\eta = 0.1$  starting at  $z_0 = (0.3108455, 0.4825575, 0.4621875, 0.5768655)^T$  has the following trajectory:

$$\begin{aligned} z_1 &= (0.24923465, 0.47967569, 0.43497808, 0.57458145)^T, \\ z_2 &= (0.19396855, 0.48164918, 0.40193211, 0.56061753)^T. \end{aligned}$$

Thus we have

$$\begin{aligned} r^{nat}(z_0)^2 &= 0.15170013184049996, \\ r^{nat}(z_1)^2 &= 0.13617654362050116, \\ r^{nat}(z_2)^2 &= 0.16125792556139756. \end{aligned}$$

It is clear that the natural residual is not monotone.

**Performance Measure:**  $\|z_k - z_{k+\frac{1}{2}}\|^2$ . Note that the norm of the operator mapping defined in [Dikonikolas, 2020] is exactly  $\frac{1}{\eta} \cdot \|z_k - z_{k+\frac{1}{2}}\|$ . Let  $A = \begin{bmatrix} 0.50676631 & 0.15042569 \\ 0.46897595 & 0.96748026 \end{bmatrix}$ ,  $b = c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Running the EG method on the corresponding VI problem with step-size  $\eta = 0.1$  starting at  $z_0 = (2.35037432, 0.00333996, 1.70547279, 0.71065999)^T$  has the following trajectory:

$$\begin{aligned} z_{\frac{1}{2}} &= (2.35325656, 0, 1.72473848, 0.64633879)^T, \\ z_1 &= (2.35324779, 0, 1.72472791, 0.64605901)^T, \\ z_{1+\frac{1}{2}} &= (2.35612601, 0, 1.74398258, 0.58145791)^T, \\ z_2 &= (2.35612201, 0, 1.74412844, 0.5815012)^T, \\ z_{2+\frac{1}{2}} &= (2.35898819, 0, 1.76352876, 0.51694333)^T. \end{aligned}$$

Thus we have

$$\begin{aligned}\|z_0 - z_{\frac{1}{2}}\|^2 &= 0.00452784581555656, \\ \|z_1 - z_{1+\frac{1}{2}}\|^2 &= 0.004552329544896258, \\ \|z_2 - z_{2+\frac{1}{2}}\|^2 &= 0.004552306444552208.\end{aligned}$$

It is clear that the  $\|z_k - z_{k+\frac{1}{2}}\|^2$  is not monotone.

**Performance Measure:**  $\|z_k - z_{k+1}\|^2$ . Let  $A = \begin{bmatrix} 0.50676631 & 0.15042569 \\ 0.46897595 & 0.96748026 \end{bmatrix}$ ,  $b = c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Running the EG method on the corresponding VI problem with step-size  $\eta = 0.1$  starting at  $z_0 = (2.37003485, 0, 1.84327237, 0.25934775)^T$  has the following trajectory:

$$\begin{aligned}z_1 &= (2.37267186, 0, 1.86351397, 0.1950396)^T, \\ z_2 &= (2.37524308, 0, 1.88388624, 0.13077023)^T, \\ z_3 &= (2.37774149, 0.00426125, 1.90438549, 0.06653856)^T.\end{aligned}$$

Thus we have

$$\begin{aligned}\|z_0 - z_1\|^2 &= 0.004552214685275266, \\ \|z_1 - z_2\|^2 &= 0.004552191904998012, \\ \|z_2 - z_3\|^2 &= 0.004570327450598002.\end{aligned}$$

It is clear that the  $\|z_k - z_{k+1}\|^2$  is not monotone.

## G.2 Non-Monotonicity of the Gap Functions and its Variant

**Performance Measure:** **Gap Function** and  $\max_{z \in \mathcal{Z}} \langle F(z), z_k - z \rangle$ . Let  $A = \begin{bmatrix} -0.21025101 & 0.22360196 \\ 0.40667685 & -0.2922158 \end{bmatrix}$ ,  $b = c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . One can easily verify that  $\langle F(z), z_k - z \rangle = \langle F(z_k), z_k - z \rangle$ , which further implies that  $\max_{z \in \mathcal{Z}} \langle F(z), z_k - z \rangle = \max_{z \in \mathcal{Z}} \langle F(z_k), z_k - z \rangle = \text{GAP}(z_k)$ , which implies that non-monotonicity of the gap function implies non-monotonicity of  $\max_{z \in \mathcal{Z}} \langle F(z), z_k - z \rangle$ . Running the EG method on the corresponding VI problem with step-size  $\eta = 0.1$  starting at  $z_0 = (0.53095379, 0.29084076, 0.62132986, 0.49440498)^T$  has the following trajectory:

$$\begin{aligned}z_1 &= (0.53290086, 0.28009156, 0.62151204, 0.4981395)^T, \\ z_2 &= (0.5347502, 0.26947398, 0.62122195, 0.50222691)^T.\end{aligned}$$

One can easily verify that

$$\begin{aligned}\text{GAP}(z_0) &= 0.6046398415472187, \\ \text{GAP}(z_1) &= 0.58462873354003214, \\ \text{GAP}(z_2) &= 0.5914026255469654.\end{aligned}$$

It is clear that the duality gap is not monotone.

## H Optimistic Gradient Descent Ascent Algorithm

Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be an operator. Let  $z_k$  and  $w_k$  be the  $k$ -th iterate of the Optimistic Gradient Descent Ascent algorithm (OGDA) algorithm. Let  $z_0, w_0$  be arbitrary point in  $\mathcal{Z}$  and  $\{z_k, w_k\}_{k \geq 0}$  be the iterated of the OGDA algorithm. The update rule for any  $k \geq 0$  is as follows:

$$\begin{aligned}w_{k+1} &= \Pi_{\mathcal{Z}} [z_k - \eta F(w_k)] = \arg \min_{z \in \mathcal{Z}} \|z - (z_k - \eta F(w_k))\| \\ z_{k+1} &= \Pi_{\mathcal{Z}} [z_k - \eta F(w_{k+1})] = \arg \min_{z \in \mathcal{Z}} \|z - (z_k - \eta F(w_{k+1}))\|\end{aligned}\tag{88}$$

We prove last-iterate convergence for OGDA with respect to the gap function, natural residual and tangent residual in Theorem 8 at Section H.5. The last-iterate convergence proof for OGDA is a simple extension of the proof for EG. The last-iterate convergence for the performance measures we mentioned follow from the last-iterate convergence of the following monotonically decreasing potential function:

$$\Phi_k = \|F(z_k) - F(w_k)\|^2 + r^{\tan}(z_k)^2\tag{89}$$

In Section H.1 we show that OGDA enjoys last-iterate convergence with respect to the quantity  $\|z_k - w_{k+1}\|$  [Wei et al., 2021a, Hsieh et al., 2019] and in Section H.2 we show how to upper bound the potential function  $\Phi_k$  by the best-iterate. In Section H.4 we show that the potential function  $\Phi_k$  is monotonically decreasing across iterates and finally in Section H.5 we show how to translate the last-iterate convergence with respect to the potential function  $\Phi_k$  to last-iterate convergence of the performance measures of interest.

### H.1 Best-Iterate Convergence of OGDA with Constant Step Size

Best-iterate convergence guarantees for OGDA are known [Wei et al., 2021a] and can easily be derived by Hsieh et al. [2019]. We include the proof here for completeness.

**Lemma 9.** *Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set,  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be a monotone and  $L$ -Lipschitz operator, and  $z^*$  be a saddle point. Let  $z_0, w_0 \in \mathcal{Z}$  be arbitrary starting points and  $\{z_k, w_k\}_{k \geq 0}$  be the iterates of the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$ . Then for all  $T \geq 1$ ,*

$$\sum_{k=0}^T \|z_k - w_{k+1}\|^2 \leq \frac{1 - 2\eta^2 L^2}{1 - 4\eta^2 L^2} \|z_0 - z^*\|^2 + \frac{2\eta^2 L^2}{1 - 4\eta^2 L^2} \|w_0 - z_0\|^2.\tag{90}$$



**Proof of Lemma 9:** In order to upper bound  $\sum_{k=0}^T \|w_k - w_{k+1}\|^2$ , we first relate the quantity  $\|w_k - w_{k+1}\|^2$  to the weighted sum of  $\{\|z_t - w_{t+1}\|^2\}_{0 \leq t \leq k}$ .

**Lemma 10.** For all  $k \geq 0$ ,

$$\|w_k - w_{k+1}\|^2 \leq 2(2\eta^2 L^2)^k \|w_0 - z_0\|^2 + \sum_{t=0}^k 2(2\eta^2 L^2)^t \|z_{k-t} - w_{k+1-t}\|^2. \quad (91)$$

Moreover, for all  $T \geq 0$ ,

$$\sum_{k=0}^T \|w_k - w_{k+1}\|^2 \leq \frac{2}{1 - 2\eta^2 L^2} \left( \|w_0 - z_0\|^2 + \sum_{k=0}^T \|z_k - w_{k+1}\|^2 \right). \quad (92)$$

*Proof.* We first prove Equation (91) by induction. Note that for all  $k \geq 0$ , we have

$$\begin{aligned} \|w_k - w_{k+1}\|^2 &= \|w_k - z_k + z_k - w_{k+1}\|^2 \\ &\leq 2\|w_k - z_k\|^2 + 2\|z_k - w_{k+1}\|^2. \end{aligned} \quad (93)$$

The inequality follows from the fact that  $(a + b)^2 \leq 2a^2 + 2b^2$ . Thus Equation (91) holds for the base case  $k = 0$ . For the sake of induction, we assume that Equation (91) holds for some  $k - 1 \geq 0$ . Using the update rule of OGD, the non-expansiveness of the projection operator, and the  $L$ -Lipschitzness of  $F$ , for all  $k \geq 1$  we have

$$\|w_k - z_k\|^2 \leq \eta^2 \|F(w_{k-1}) - F(w_k)\|^2 \leq \eta^2 L^2 \|w_{k-1} - w_k\|^2. \quad (94)$$

Combining Equation (93), Equation (94), and the induction assumption, we have

$$\begin{aligned} \|w_k - w_{k+1}\|^2 &\leq 2\|w_k - z_k\|^2 + 2\|z_k - w_{k+1}\|^2 \\ &\leq 2\eta^2 L^2 \|w_{k-1} - w_k\|^2 + 2\|z_k - w_{k+1}\|^2 \\ &\leq 2\eta^2 L^2 \left( 2(2\eta^2 L^2)^{k-1} \|w_0 - z_0\|^2 + \sum_{t=0}^{k-1} 2(2\eta^2 L^2)^t \|z_{k-1-t} - w_{k-t}\|^2 \right) + 2\|z_k - w_{k+1}\|^2 \\ &= 2(2\eta^2 L^2)^k \|w_0 - z_0\|^2 + \sum_{t=1}^k 2(2\eta^2 L^2)^t \|z_{k-t} - w_{k+1-t}\|^2 + 2\|z_k - w_{k+1}\|^2 \\ &= 2(2\eta^2 L^2)^k \|w_0 - z_0\|^2 + \sum_{t=0}^k 2(2\eta^2 L^2)^t \|z_{k-t} - w_{k+1-t}\|^2. \end{aligned}$$

This completes the proof of Equation (91).

Summing Equation (91) with  $k = 0, 1, \dots, T$ , we have

$$\begin{aligned}
\sum_{k=0}^T \|w_k - w_{k+1}\|^2 &\leq \sum_{k=0}^T 2(2\eta^2 L^2)^k \|w_0 - z_0\|^2 + \sum_{k=0}^T \sum_{t=0}^k 2(2\eta^2 L^2)^t \|z_{k-t} - w_{k+1-t}\|^2 \\
&= \sum_{k=0}^T 2(2\eta^2 L^2)^k \|w_0 - z_0\|^2 + \sum_{k=0}^T \left( \sum_{t=0}^{T-k} 2(2\eta^2 L^2)^t \right) \cdot \|z_k - w_{k+1}\|^2 \\
&\leq \frac{2}{1 - 2\eta^2 L^2} \left( \|w_0 - z_0\|^2 + \sum_{k=0}^T \|z_k - w_{k+1}\|^2 \right).
\end{aligned}$$

This completes the proof of Equation (92).  $\square$

Back to the proof of Lemma 9. For all  $k \geq 0$ , we have

$$\begin{aligned}
\|z_{k+1} - z^*\|^2 &= \|z_{k+1} - z_k + z_k - z^*\|^2 \\
&= \|z_k - z^*\|^2 + \|z_{k+1} - z_k\|^2 + 2\langle z_{k+1} - z_k, z_k - z^* \rangle \\
&= \|z_k - z^*\|^2 - \|z_{k+1} - z_k\|^2 + 2\langle z_{k+1} - z_k, z_{k+1} - z^* \rangle \\
&\leq \|z_k - z^*\|^2 - \|z_{k+1} - z_k\|^2 - 2\eta \langle F(w_{k+1}), z_{k+1} - z^* \rangle.
\end{aligned} \tag{95}$$

The last inequality follows from  $\langle z_{k+1} - z_k + \eta F(w_{k+1}), z_{k+1} - z^* \rangle \leq 0$  as  $z_{k+1} = \Pi_{\mathcal{Z}} [z_k - \eta F(w_{k+1})]$ .

Similarly, for all  $k \geq 0$ , we have

$$\begin{aligned}
\|z_{k+1} - w_{k+1}\|^2 &= \|z_{k+1} - z_k + z_k - w_{k+1}\|^2 \\
&= \|z_{k+1} - z_k\|^2 + \|z_k - w_{k+1}\|^2 + 2\langle z_k - w_{k+1}, z_{k+1} - z_k \rangle \\
&= \|z_{k+1} - z_k\|^2 - \|z_k - w_{k+1}\|^2 + 2\langle z_k - w_{k+1}, z_{k+1} - w_{k+1} \rangle \\
&\leq \|z_{k+1} - z_k\|^2 - \|z_k - w_{k+1}\|^2 + 2\eta \langle F(w_k), z_{k+1} - w_{k+1} \rangle.
\end{aligned} \tag{96}$$

The last inequality follows from  $\langle z_k - \eta F(w_k) - w_{k+1}, z_{k+1} - w_{k+1} \rangle \leq 0$  as  $w_{k+1} = \Pi_{\mathcal{Z}} [z_k - \eta F(w_k)]$ .

We can further simplify Equation (95) using Fact 1:

$$\begin{aligned}
\|z_{k+1} - z^*\|^2 &\leq \|z_k - z^*\|^2 - \|z_{k+1} - z_k\|^2 - 2\eta \langle F(w_{k+1}), z_{k+1} - z^* \rangle \\
&= \|z_k - z^*\|^2 - \|z_{k+1} - z_k\|^2 - 2\eta \langle F(w_{k+1}), z_{k+1} - w_{k+1} \rangle + 2\eta \langle F(w_{k+1}), z^* - w_{k+1} \rangle \\
&\leq \|z_k - z^*\|^2 - \|z_{k+1} - z_k\|^2 - 2\eta \langle F(w_{k+1}), z_{k+1} - w_{k+1} \rangle.
\end{aligned} \tag{97}$$

Summing Equation (96) and Equation (97), we get

$$\begin{aligned}
\|z_{k+1} - z^*\|^2 &\leq \|z_k - z^*\|^2 - \|z_k - w_{k+1}\|^2 - \|z_{k+1} - w_{k+1}\|^2 + 2\eta \langle F(w_k) - F(w_{k+1}), z_{k+1} - w_{k+1} \rangle \\
&\leq \|z_k - z^*\|^2 - \|z_k - w_{k+1}\|^2 - \|z_{k+1} - w_{k+1}\|^2 + 2\eta \|F(w_k) - F(w_{k+1})\| \|z_{k+1} - w_{k+1}\| \\
&\leq \|z_k - z^*\|^2 - \|z_k - w_{k+1}\|^2 - \|z_{k+1} - w_{k+1}\|^2 + 2\eta L \|w_k - w_{k+1}\| \|z_{k+1} - w_{k+1}\| \\
&\leq \|z_k - z^*\|^2 - \|z_k - w_{k+1}\|^2 + \eta^2 L^2 \|w_k - w_{k+1}\|^2,
\end{aligned} \tag{98}$$

where we use Cauchy-Schwarz inequality in the second inequality and  $L$ -Lipschitzness of  $F(\cdot)$  in the third inequality. In the last inequality, we optimize the quadratic function in  $\|z_{k+1} - w_{k+1}\|$ .

Summing Equation (98) for  $k = 0, 1, \dots, T$  and using Lemma 10, we get

$$\begin{aligned}
\|z_{T+1} - z^*\|^2 &\leq \|z_0 - z^*\|^2 - \sum_{k=0}^T \|z_k - w_{k+1}\|^2 + \eta^2 L^2 \sum_{k=0}^T \|w_k - w_{k+1}\|^2 \\
&\leq \|z_0 - z^*\|^2 - \sum_{k=0}^T \|z_k - w_{k+1}\|^2 + \frac{2\eta^2 L^2}{1 - 2\eta^2 L^2} \left( \|w_0 - z_0\|^2 + \sum_{k=0}^T \|z_k - w_{k+1}\|^2 \right) \\
&\quad \text{(Lemma 10)} \\
&= \|z_0 - z^*\|^2 - \frac{1 - 4\eta^2 L^2}{1 - 2\eta^2 L^2} \sum_{k=0}^T \|z_k - w_{k+1}\|^2 + \frac{2\eta^2 L^2}{1 - 2\eta^2 L^2} \|w_0 - z_0\|^2.
\end{aligned}$$

Since  $\eta^2 L^2 < \frac{1}{4}$ , we complete the proof by rearranging the above inequality.  $\blacksquare$

## H.2 Best-Iterate of $\Phi_k$

In this section, we use Lemma 9 to show that there exists  $t^* \in [T]$  such that  $\Phi_{t^*} = O(\frac{1}{T})$ .

**Lemma 11.** *Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set,  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be a monotone and  $L$ -Lipschitz operator, and  $z^*$  be a saddle point. Let  $z_0, w_0 \in \mathcal{Z}$  be arbitrary starting point and  $\{z_k, w_k\}_{k \geq 0}$  be the iterates of the OGD algorithm with any step size  $\eta \in (0, \frac{1}{2L})$ . Then for all  $T \geq 1$ ,*

$$\sum_{k=1}^T \left( \|\eta F(z_k) - \eta F(w_k)\|^2 + \eta^2 r^{\tan}(z_k)^2 \right) \leq \frac{4 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|z_0 - z^*\|^2 + \frac{16\eta^2 L^2 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|w_0 - z_0\|^2.$$

Moreover, when  $w_0 = z_0$

$$\sum_{k=1}^T \left( \|\eta F(z_k) - \eta F(w_k)\|^2 + \eta^2 r^{\tan}(z_k)^2 \right) \leq \frac{4 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|z_0 - z^*\|^2.$$

**Proof of Lemma 11:** For all  $k \geq 1$ , we have

$$\begin{aligned}
\|\eta F(z_k) - \eta F(w_k)\|^2 &\leq \eta^2 L^2 \|z_k - w_k\|^2 && (L\text{-Lipschitzness of } F) \\
&\leq \eta^4 L^4 \|w_{k-1} - w_k\|^2. && (\text{Equation (94)})
\end{aligned}$$

Using Lemma 8 with the fact that  $z_k = \Pi_{\mathcal{Z}}[z_{k-1} - \eta F(w_k)]$ , we have for all  $k \geq 1$ ,

$$\begin{aligned}
\eta^2 r^{\tan}(z_k)^2 &\leq \|z_{k-1} - z_k + \eta F(z_k) - \eta F(w_k)\|^2 \\
&\leq 2\|z_{k-1} - z_k\|^2 + 2\eta^2 \|F(z_k) - F(w_k)\|^2 \\
&\leq 2\|z_{k-1} - w_k + w_k - z_k\|^2 + 2\eta^2 L^2 \|w_k - z_k\|^2 && (L\text{-Lipschitzness of } F) \\
&\leq 4\|z_{k-1} - w_k\|^2 + (4 + 2\eta^2 L^2) \|w_k - z_k\|^2 \\
&\leq 4\|z_{k-1} - w_k\|^2 + (4 + 2\eta^2 L^2) \eta^2 L^2 \|w_{k-1} - w_k\|^2. && (\text{Equation (94)})
\end{aligned}$$

Summing the above inequalities with  $k = 1, \dots, T$  and using Lemma 9 and Lemma 10, we have

$$\begin{aligned}
& \sum_{k=1}^T \left( \|\eta F(z_k) - \eta F(w_k)\|^2 + \eta^2 r^{\tan}(z_k)^2 \right) \\
& \leq 4 \sum_{k=0}^{T-1} \|z_k - w_{k+1}\|^2 + (4 + 3\eta^2 L^2) \eta^2 L^2 \sum_{k=0}^{T-1} \|w_k - w_{k+1}\|^2 \\
& \leq \frac{2(4 + 3\eta^2 L^2) \eta^2 L^2}{1 - 2\eta^2 L^2} \|w_0 - z_0\|^2 + \left( 4 + \frac{2(4 + 3\eta^2 L^2) \eta^2 L^2}{1 - 2\eta^2 L^2} \right) \sum_{k=0}^{T-1} \|z_k - w_{k+1}\|^2 \\
& \leq \frac{2(4 + 3\eta^2 L^2) \eta^2 L^2}{1 - 2\eta^2 L^2} \|w_0 - z_0\|^2 + \left( \frac{8\eta^2 L^2}{1 - 4\eta^2 L^2} + \frac{4(4 + 3\eta^2 L^2) \eta^4 L^4}{(1 - 2\eta^2 L^2) \cdot (1 - 4\eta^2 L^2)} \right) \|w_0 - z_0\|^2 \\
& \quad + \left( \frac{4 - 8\eta^2 L^2}{1 - 4\eta^2 L^2} + \frac{2(4 + 3\eta^2 L^2) \eta^2 L^2}{1 - 4\eta^2 L^2} \right) \|z_0 - z^*\|^2 \\
& = \frac{16\eta^2 L^2 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|w_0 - z_0\|^2 + \frac{4 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|z_0 - z^*\|^2,
\end{aligned}$$

which concludes the proof. ■

**Corollary 2.** Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set,  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be a monotone and  $L$ -Lipschitz operator, and  $z^*$  be a saddle point. Let  $z_0, w_0 \in \mathcal{Z}$  be arbitrary starting point and  $\{z_k, w_k\}_{k \geq 0}$  be the iterates of the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$ . Then for all  $T \geq 1$ , there exists  $t^* \in [T]$  such that

$$\|\eta F(z_{t^*}) - \eta F(w_{t^*})\|^2 + \eta^2 r^{\tan}(z_{t^*})^2 \leq \frac{1}{T} \frac{4 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|z_0 - z^*\|^2 + \frac{1}{T} \frac{16\eta^2 L^2 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|w_0 - z_0\|^2.$$

Moreover, when  $w_0 = z_0$

$$\|\eta F(z_{t^*}) - \eta F(w_{t^*})\|^2 + \eta^2 r^{\tan}(z_{t^*})^2 \leq \frac{1}{T} \frac{4 + 6\eta^4 L^4}{1 - 4\eta^2 L^2} \|z_0 - z^*\|^2.$$

### H.3 Simplification Lemma

In this section we show how to “reduce” the number of constraints and the dimension of the problem similar to Lemma 6.

**Lemma 12 (Dimension and Constraint Reduction).** Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}^n$  be a monotone and  $L$ -Lipschitz operator. Let  $z_k, w_k \in \mathcal{Z}$  and the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$  produces  $w_{k+1}, z_{k+1}$ . Then either  $\|F(z_k) - F(w_k)\|^2 + r^{\tan}(z_k)^2 \geq \|F(z_{k+1}) - F(w_{k+1})\|^2 + r^{\tan}(z_{k+1})^2$  holds, or we can construct vectors  $\bar{b}_k, \bar{a}_{k+1}, \bar{b}_{k+1}, \bar{z}_k, \bar{w}_k, \bar{z}_{k+1}, \bar{w}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{w}_k), \bar{F}(\bar{z}_{k+1})$ , and  $\bar{F}(\bar{w}_{k+1})$  in  $\mathbb{R}^N$  with  $N = n + 5$  that satisfy the following conditions.

1. **Negativity of TARGET:** We use TARGET to denote the LHS of the inequality below.

$$\begin{aligned} & \|\bar{F}(\bar{z}_k) - \bar{F}(\bar{w}_k)\|^2 + \left\| \bar{F}(\bar{z}_k) - \frac{\langle \bar{F}(\bar{z}_k), \bar{b}_k \rangle \cdot \bar{b}_k}{\|\bar{b}_k\|^2} \right\|^2 \\ & - \left( \|\bar{F}(\bar{z}_{k+1}) - \bar{F}(\bar{w}_{k+1})\|^2 + \left\| \bar{F}(\bar{z}_{k+1}) - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{b}_{k+1} \rangle \cdot \bar{b}_{k+1}}{\|\bar{b}_{k+1}\|^2} \mathbb{1}[\langle \bar{F}(\bar{z}_{k+1}), \bar{b}_{k+1} \rangle \geq 0] \right\|^2 \right) < 0. \end{aligned}$$

2. **Form Property:**  $\bar{b}_k = (\beta_1, \beta_2, 1, 0, \dots, 0)$ ,  $\bar{a}_{k+1} = (\alpha, 1, 0, \dots, 0)$ , and  $\bar{b}_{k+1} = (1, 0, \dots, 0)$  for some  $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ .

3. **Simplified Conic Constraints:** All of the inner products between  $\{\bar{b}_k, \bar{a}_{k+1}, \bar{b}_{k+1}\}$  and  $\{\bar{w}_k, \bar{z}_k, \bar{w}_{k+1}, \bar{z}_{k+1}\}$  are nonnegative. Moreover,  $\langle \bar{b}_k, \bar{z}_k \rangle = \langle \bar{a}_{k+1}, \bar{w}_{k+1} \rangle = \langle \bar{b}_{k+1}, \bar{z}_{k+1} \rangle = 0$ .

We have  $\langle \bar{b}_k, \bar{F}(\bar{z}_k) \rangle \geq 0$ . The two vectors  $\bar{a}_{k+1}$  and  $\bar{w}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_k)$  are co-directed, i.e., they are colinear and have the same direction. The two vectors  $\bar{b}_{k+1}$  and  $\bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_{k+1})$  are co-directed.

4. **Monotonicity and Lipshitzness**

$$\begin{aligned} & \langle \bar{F}(\bar{z}_{k+1}) - \bar{F}(\bar{z}_k), \bar{z}_k - \bar{z}_{k+1} \rangle \leq 0, \\ & \|\bar{F}(\bar{z}_{k+1}) - \bar{F}(\bar{w}_{k+1})\|^2 \leq L^2 \|\bar{z}_{k+1} - \bar{w}_{k+1}\|^2. \end{aligned}$$

**Proof of Lemma 12:** The Lemma holds when  $\|F(z_k) - F(w_k)\|^2 + r^{\tan}(z_k)^2 \geq \|F(z_{k+1}) - F(w_{k+1})\|^2 + r^{\tan}(z_{k+1})^2$ . In the rest of the proof, we assume that  $\|F(z_k) - F(w_k)\|^2 + r^{\tan}(z_k)^2 < \|F(z_{k+1}) - F(w_{k+1})\|^2 + r^{\tan}(z_{k+1})^2$ .

High levelly speaking, the proof consists of two steps. In Step 1, we introduce five dummy dimensions to construct vectors  $\hat{b}_k, \hat{a}_{k+1}, \hat{b}_{k+1}, \hat{z}_k, \hat{w}_k, \hat{z}_{k+1}, \hat{w}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{w}_k), \hat{F}(\hat{z}_{k+1})$ , and  $\hat{F}(\hat{w}_{k+1})$  in  $\mathbb{R}^{n+5}$  that satisfy all conditions except for the form property. We also ensure that  $\hat{b}_k, \hat{a}_{k+1}$ , and  $\hat{b}_{k+1}$  are linear independent. In Step 2, we perform a change of basis to construct  $\bar{b}_k, \bar{a}_{k+1}, \bar{b}_{k+1}, \bar{z}_k, \bar{w}_k, \bar{z}_{k+1}, \bar{w}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{w}_k), \bar{F}(\bar{z}_{k+1})$ , and  $\bar{F}(\bar{w}_{k+1})$  that also satisfy, among all conditions, the form property.

**Step 1: Introducing Dummy Dimensions.** We first define the following three vectors:

$$-b_k \in \underset{\substack{a \in N_{\mathcal{Z}}(z_k), \\ \langle F(z_k), a \rangle \leq 0}}{\operatorname{argmin}} \|F(z_k) - \langle F(z_k), a \rangle \cdot a\|^2, \quad (99)$$

$$a_{k+1} = w_{k+1} - z_k + \eta F(w_k), \quad (100)$$

$$b_{k+1} = z_{k+1} - z_k + \eta F(w_{k+1}). \quad (101)$$

By the above definition, we know

$$\langle b_k, F(z_k) \rangle \geq 0. \quad (102)$$

Moreover, according to the fact that  $-b_k \in N_{\mathcal{Z}}(z_k)$  and the OGD update rule (6), we know for any  $z \in \mathcal{Z}$ ,

$$\begin{aligned}\langle b_k, z \rangle &\geq \langle b_k, z_k \rangle, \\ \langle a_{k+1}, z \rangle &\geq \langle a_{k+1}, w_{k+1} \rangle, \\ \langle b_{k+1}, z \rangle &\geq \langle b_{k+1}, z_{k+1} \rangle.\end{aligned}\tag{103}$$

Now we present the construction of  $\hat{b}_k, \hat{a}_{k+1}, \hat{b}_{k+1}, \hat{z}_k, \hat{w}_k, \hat{z}_{k+1}, \hat{w}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{w}_k), \hat{F}(\hat{z}_{k+1})$ , and  $\hat{F}(\hat{w}_{k+1})$ . For parameters  $\ell, \epsilon > 0$  that we determine later, we define:

$$\begin{aligned}\hat{w}_i &:= (-\epsilon^{-1}, 0, 0, 0, 0, w_i) & \forall i \in \{k, k+1\} \\ \hat{z}_i &:= (-\epsilon^{-1}, 0, 0, 0, 0, z_i) & \forall i \in \{k, k+1\} \\ \hat{F}(\hat{w}_k) &:= \left( \frac{\epsilon}{\eta} \cdot \langle a_{k+1}, w_{k+1} \rangle, 0, \frac{\epsilon}{\eta}, 0, \frac{\ell\epsilon}{\eta}, F(w_k) \right) \\ \hat{F}(\hat{z}_k) &:= \left( \frac{\epsilon}{\eta} \cdot \langle a_{k+1}, w_{k+1} \rangle, 0, \frac{\epsilon}{\eta}, 0, \frac{\ell\epsilon}{\eta}, F(z_k) \right) \\ \hat{F}(\hat{w}_{k+1}) &:= \left( \frac{\epsilon}{\eta} \cdot \langle b_{k+1}, z_{k+1} \rangle, 0, 0, \frac{\epsilon}{\eta}, \frac{\ell\epsilon}{\eta}, F(w_{k+1}) \right) \\ \hat{F}(\hat{z}_{k+1}) &:= \left( \frac{\epsilon}{\eta} \cdot \langle b_{k+1}, z_{k+1} \rangle, 0, 0, \frac{\epsilon}{\eta}, \frac{\ell\epsilon}{\eta}, F(z_{k+1}) \right) \\ \hat{b}_k &:= (\epsilon \cdot \langle b_k, z_k \rangle, \epsilon, 0, 0, \ell\epsilon, b_k) \\ \hat{a}_{k+1} &:= (\epsilon \cdot \langle a_{k+1}, w_{k+1} \rangle, 0, \epsilon, 0, \ell\epsilon, a_{k+1}) \\ \hat{b}_{k+1} &:= (\epsilon \cdot \langle b_{k+1}, z_{k+1} \rangle, 0, 0, \epsilon, \ell\epsilon, b_{k+1})\end{aligned}$$

Clearly,  $\hat{b}_k, \hat{a}_{k+1}$ , and  $\hat{b}_{k+1}$  are linear independent. It is also clear that  $\hat{a}_{k+1}$  equals to  $\hat{w}_{k+1} - \hat{z}_k + \eta \hat{F}(\hat{w}_k)$ , thus they are co-directed. Similarly,  $\hat{b}_{k+1}$  and  $\hat{z}_{k+1} - \hat{z}_k + \eta \hat{F}(\hat{w}_{k+1})$  are co-directed. Besides, the following inequalities/equations hold. The first three inequalities hold by Equation (103). The last two inequalities hold since  $F$  is a monotone and  $L$ -Lipschitz operator.

$$\langle \hat{b}_k, \hat{z} \rangle = \langle b_k, z \rangle - \langle b_k, z_k \rangle \geq 0, \quad \forall z \in \{w_k, z_k, w_{k+1}, z_{k+1}\} \tag{104}$$

$$\langle \hat{a}_{k+1}, \hat{z} \rangle = \langle a_{k+1}, z \rangle - \langle a_{k+1}, w_{k+1} \rangle \geq 0, \quad \forall z \in \{w_k, z_k, w_{k+1}, z_{k+1}\} \tag{105}$$

$$\langle \hat{b}_{k+1}, \hat{z} \rangle = \langle b_{k+1}, z \rangle - \langle b_{k+1}, z_{k+1} \rangle \geq 0, \quad \forall z \in \{w_k, z_k, w_{k+1}, z_{k+1}\} \tag{106}$$

$$\langle \hat{b}_k, \hat{z}_k \rangle = \langle \hat{a}_{k+1}, \hat{w}_{k+1} \rangle = \langle \hat{b}_{k+1}, \hat{z}_{k+1} \rangle = 0 \tag{107}$$

$$\left\| \hat{F}(\hat{z}_{k+1}) - \hat{F}(\hat{w}_{k+1}) \right\|^2 = \|F(z_{k+1}) - F(w_{k+1})\|^2 \leq L^2 \|z_{k+1} - w_{k+1}\|^2 = L^2 \|\hat{z}_{k+1} - \hat{w}_{k+1}\|^2 \tag{108}$$

$$\langle \hat{F}(\hat{z}_{k+1}) - \hat{F}(\hat{z}_k), \hat{z}_k - \hat{z}_{k+1} \rangle = \langle F(z_{k+1}) - F(z_k), z_k - z_{k+1} \rangle \leq 0 \tag{109}$$

Moreover,  $\langle \hat{b}_k, \hat{F}(\hat{z}_k) \rangle = \frac{\epsilon^2}{\eta} (\langle b_k, z_k \rangle \langle a_{k+1}, w_{k+1} \rangle + \ell^2) + \langle b_k, F(z_k) \rangle$ . We choose  $\ell$  to be sufficiently

large (for example,  $\sqrt{|\langle b_k, z_k \rangle \langle a_{k+1}, w_{k+1} \rangle|}$ ) so that  $(\langle b_k, z_k \rangle \langle a_{k+1}, w_{k+1} \rangle + \ell^2) \geq 0$ . Then by Equation (102), we have

$$\langle \hat{b}_k, \hat{F}(\hat{z}_k) \rangle \geq \langle b_k, F(z_k) \rangle \geq 0. \quad (110)$$

Now we focus on Condition 1. We define the following function

$$\begin{aligned} \hat{H}(\hat{z}_k) &:= \left\| \hat{F}(\hat{z}_k) - \frac{\langle \hat{F}(\hat{z}_k), \hat{b}_k \rangle}{\|\hat{b}_k\|^2} \cdot \hat{b}_k \right\|^2 \\ &= \left\| \hat{F}(\hat{z}_k) - \frac{\frac{\epsilon^2}{\eta} \cdot (\langle b_k, z_k \rangle \langle a_{k+1}, w_{k+1} \rangle + \ell^2) + \langle b_k, F(z_k) \rangle}{\epsilon^2 (\langle b_k, z_k \rangle^2 + 1 + \ell^2) + \|b_k\|^2} \cdot \hat{b}_k \right\|^2. \end{aligned}$$

Define  $f(\epsilon) := \frac{\frac{\epsilon^2}{\eta} \cdot (\langle b_k, z_k \rangle \langle a_{k+1}, w_{k+1} \rangle + \ell^2) + \langle b_k, F(z_k) \rangle}{\epsilon^2 (\langle b_k, z_k \rangle^2 + 1 + \ell^2) + \|b_k\|^2}$ . We can simplify  $\hat{H}(\hat{z}_k)$  to be

$$\epsilon^2 \cdot \left\| \left( \frac{\langle a_{k+1}, w_{k+1} \rangle}{\eta} - f(\epsilon) \langle b_k, z_k \rangle, -f(\epsilon), \frac{1}{\eta}, 0, \frac{\ell}{\eta} - f(\epsilon) \ell \right) \right\|^2 + \|F(z_k) - f(\epsilon) \cdot b_k\|^2.$$

When  $b_k = (0, \dots, 0)$ ,  $r^{\tan}(z_k)^2 = \|F(z_k)\|^2$  (Equation (99)) and  $f(\epsilon) = \frac{\ell^2}{\eta(1+\ell^2)}$  for any  $\epsilon > 0$ . Therefore,  $\lim_{\epsilon \rightarrow 0^+} \hat{H}(\hat{z}_k) = \|F(z_k)\|^2 = r^{\tan}(z_k)^2$ , when  $b_k = (0, \dots, 0)$ . When  $b_k \neq (0, \dots, 0)$ ,  $r^{\tan}(z_k)^2 = \|F(z_k) - \frac{\langle b_k, F(z_k) \rangle}{\|b_k\|^2} \cdot b_k\|^2$  (Equation (99)) and  $\lim_{\epsilon \rightarrow 0^+} f(\epsilon) = \frac{\langle b_k, F(z_k) \rangle}{\|b_k\|^2}$ , so  $\lim_{\epsilon \rightarrow 0^+} \hat{H}(\hat{z}_k) = \|F(z_k) - \frac{\langle b_k, F(z_k) \rangle}{\|b_k\|^2} \cdot b_k\|^2 = r^{\tan}(z_k)^2$ .

Similarly, we define function

$$\begin{aligned} \hat{H}(\hat{z}_{k+1}) &:= \left\| \hat{F}(\hat{z}_{k+1}) - \frac{\langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle}{\|\hat{b}_{k+1}\|^2} \cdot \hat{b}_{k+1} \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle \geq 0 \right] \right\|^2 \\ &= \left\| \hat{F}(\hat{z}_{k+1}) - \frac{\frac{\epsilon^2}{\eta} \cdot (\langle b_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \langle F(z_{k+1}), b_{k+1} \rangle}{\epsilon^2 (\langle b_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \|b_{k+1}\|^2} \cdot \hat{b}_{k+1} \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle \geq 0 \right] \right\|^2. \end{aligned}$$

Define  $g(\epsilon) := \frac{\frac{\epsilon^2}{\eta} \cdot (\langle b_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \langle F(z_{k+1}), b_{k+1} \rangle}{\epsilon^2 (\langle b_{k+1}, z_{k+1} \rangle^2 + 1 + \ell^2) + \|b_{k+1}\|^2}$ , and we can simplify  $\hat{H}(\hat{z}_{k+1})$  to be

$$\begin{aligned} \epsilon^2 \left( \frac{1}{\eta} - g(\epsilon) \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle \geq 0 \right] \right)^2 \cdot \|\langle b_{k+1}, z_{k+1} \rangle, 0, 0, 1, \ell\|^2 \\ + \left\| F(z_{k+1}) - g(\epsilon) \cdot b_{k+1} \mathbb{1} \left[ \langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle \geq 0 \right] \right\|^2. \end{aligned}$$

When  $b_{k+1} = (0, \dots, 0)$ , we have  $g(\epsilon) = \frac{1}{\eta}$  and  $\langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle \geq 0$ , so  $\hat{H}(\hat{z}_{k+1}) = \|F(z_{k+1})\|^2 \geq r^{\tan}(z_{k+1})^2$ . When  $b_{k+1} \neq (0, \dots, 0)$ , we have  $\lim_{\epsilon \rightarrow 0^+} g(\epsilon) = \frac{\langle F(z_{k+1}), b_{k+1} \rangle}{\|b_{k+1}\|^2}$  and  $\lim_{\epsilon \rightarrow 0^+} \mathbb{1}[\langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle \geq 0] = \mathbb{1}[\langle F(z_{k+1}), b_{k+1} \rangle \geq 0]$ ,<sup>18</sup> hence

$$\lim_{\epsilon \rightarrow 0^+} \hat{H}(\hat{z}_{k+1}) = \left\| F(z_{k+1}) - \frac{\langle F(z_{k+1}), b_{k+1} \rangle}{\|b_{k+1}\|^2} \cdot b_{k+1} \mathbb{1}[\langle F(z_{k+1}), b_{k+1} \rangle \geq 0] \right\|^2 \geq r^{\tan}(z_{k+1})^2.$$

The last inequality is because  $\frac{-b_{k+1}}{\|b_{k+1}\|} \in N_{\mathcal{Z}}(z_{k+1})$  (Equation (103)).

Note that  $\|\hat{F}(\hat{z}_k) - \hat{F}(\hat{w}_k)\|^2 = \|F(z_k) - F(w_k)\|^2$  and  $\|\hat{F}(\hat{z}_{k+1}) - \hat{F}(\hat{w}_{k+1})\|^2 = \|F(z_{k+1}) - F(w_{k+1})\|^2$  hold. Notice that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \left( \|\hat{F}(\hat{z}_k) - \hat{F}(\hat{w}_k)\|^2 + \hat{H}(\hat{z}_k) - \|\hat{F}(\hat{z}_{k+1}) - \hat{F}(\hat{w}_{k+1})\|^2 - \hat{H}(\hat{z}_{k+1}) \right) \\ &= \|F(z_k) - F(w_k)\|^2 - \|F(z_{k+1}) - F(w_{k+1})\|^2 + \lim_{\epsilon \rightarrow 0^+} \left( \hat{H}(\hat{z}_k) - \hat{H}(\hat{z}_{k+1}) \right) \\ &\leq \|F(z_k) - F(w_k)\|^2 - \|F(z_{k+1}) - F(w_{k+1})\|^2 + r^{\tan}(z_k)^2 - r^{\tan}(z_{k+1})^2 \\ &< 0. \end{aligned}$$

Thus we can choose sufficiently small  $\epsilon$  so that  $\|\hat{F}(\hat{z}_k) - \hat{F}(\hat{w}_k)\|^2 + \hat{H}(\hat{z}_k) - \|\hat{F}(\hat{z}_{k+1}) - \hat{F}(\hat{w}_{k+1})\|^2 - \hat{H}(\hat{z}_{k+1}) < 0$ . Together with Inequalities (104)-(110), the constructed vectors  $\hat{b}_k, \hat{a}_{k+1}, \hat{b}_{k+1}, \hat{z}_k, \hat{w}_k, \hat{z}_{k+1}, \hat{w}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{w}_k), \hat{F}(\hat{z}_{k+1})$ , and  $\hat{F}(\hat{w}_{k+1})$  satisfy all the conditions excluding the form property.

**Step 2: Change of basis.** Let  $N = n + 5$ . We denote the current basis as  $E = \{e_i\}_{i \in [N]}$ . According to Step 1, we know  $\hat{b}_k, \hat{a}_{k+1}$ , and  $\hat{b}_{k+1}$  are linear independent. Let  $V$  be an arbitrary sequence of vectors in  $\mathbb{R}^N$  such that  $V$  together with  $\hat{b}_k, \hat{a}_{k+1}$ , and  $\hat{b}_{k+1}$  spans  $\mathbb{R}^N$ .

We then conduct Gram-Schmidt process on the ordered sequence of  $N$  linear independent vectors  $(\hat{b}_{k+1}, \hat{a}_{k+1}, \hat{b}_k, V)$  to get a new basis  $\bar{E} = \{\bar{e}_i\}_{i \in [N]}$  of  $\mathbb{R}^N$ . Note that there exists an orthogonal matrix  $Q$  such that any vector  $z \in \mathbb{R}^N$  with respect to the basis  $E$  can be represented as  $Q \cdot z$  with respect to the new basis  $\bar{E}$ .

We define  $\bar{z} = Q \cdot z$  for  $z \in \{\hat{b}_k, \hat{a}_{k+1}, \hat{b}_{k+1}, \hat{z}_k, \hat{w}_k, \hat{z}_{k+1}, \hat{w}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{w}_k), \hat{F}(\hat{z}_{k+1}), \hat{F}(\hat{w}_{k+1})\}$ . Since  $\langle Q \cdot z, Q \cdot z' \rangle = \langle z, z' \rangle$  holds for any  $z, z' \in \mathbb{R}^N$ , it is clear that all the conditions proved in Step 1 for  $\{\hat{b}_k, \hat{a}_{k+1}, \hat{b}_{k+1}, \hat{z}_k, \hat{w}_k, \hat{z}_{k+1}, \hat{w}_{k+1}, \hat{F}(\hat{z}_k), \hat{F}(\hat{w}_k), \hat{F}(\hat{z}_{k+1}), \hat{F}(\hat{w}_{k+1})\}$  also hold for  $\{\bar{b}_k, \bar{a}_{k+1}, \bar{b}_{k+1}, \bar{z}_k, \bar{w}_k, \bar{z}_{k+1}, \bar{w}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{w}_k), \bar{F}(\bar{z}_{k+1}), \bar{F}(\bar{w}_{k+1})\}$ .

Finally, we show how to make  $\bar{b}_k, \bar{a}_{k+1}$  and  $\bar{b}_{k+1}$  satisfy form property. Since  $\hat{b}_{k+1}, \hat{a}_{k+1}$  and  $\hat{b}_k$  are the first three vectors in Gram-Schmidt process, we have  $\bar{b}_{k+1} \in \text{Span}(\bar{e}_1)$ ,  $\bar{a}_{k+1} \in \text{Span}(\bar{e}_1, \bar{e}_2)$ ,  $\bar{b}_k \in \text{Span}(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ ,  $\langle \bar{b}_{k+1}, \bar{e}_1 \rangle > 0$ ,  $\langle \bar{a}_{k+1}, \bar{e}_2 \rangle > 0$  and  $\langle \bar{b}_k, \bar{e}_3 \rangle > 0$ . Thus  $\bar{b}_k = (\beta_1, \beta_2, b, 0, \dots, 0)$ ,  $\bar{a}_{k+1} = (\alpha, c, 0, \dots, 0)$  and  $\bar{b}_{k+1} = (d, 0, \dots, 0)$ , where  $\beta_1, \beta_2, \alpha \in \mathbb{R}$  and  $b, c, d > 0$ . By properly scaling  $\bar{b}_k, \bar{a}_{k+1}$  and  $\bar{b}_{k+1}$ , we can make them satisfy the form property without affecting other conditions. This completes the proof.  $\blacksquare$

<sup>18</sup>This is because  $\langle \hat{F}(\hat{z}_{k+1}), \hat{b}_{k+1} \rangle$  is never smaller than  $\langle F(z_{k+1}), b_{k+1} \rangle$ , and the function  $\mathbb{1}[x \geq 0]$  is right continuous.



## H.4 Monotonicity of the Potential

In this section we show that the potential function  $\Phi_k$  is monotonically decreasing across iterates of OGDA.

**Theorem 7.** *Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be a monotone and  $L$ -Lipschitz operator. Then for any  $z_k, w_k \in \mathcal{Z}$ , the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$  produces  $w_{k+1}, z_{k+1} \in \mathcal{Z}$  that satisfy  $\|F(z_k) - F(w_k)\|^2 + r^{tan}(z_k)^2 \geq \|F(z_{k+1}) - F(w_{k+1})\|^2 + r^{tan}(z_{k+1})^2$ .*

*Proof.* We prove by contradiction. Assume that  $\|F(z_k) - F(w_k)\|^2 + r^{tan}(z_k)^2 < \|F(z_{k+1}) - F(w_{k+1})\|^2 + r^{tan}(z_{k+1})^2$ . Then by Lemma 12 there exist vectors  $\bar{b}_k, \bar{a}_{k+1}, \bar{b}_{k+1}, \bar{z}_k, \bar{w}_k, \bar{z}_{k+1}, \bar{w}_{k+1}, \bar{F}(\bar{z}_k), \bar{F}(\bar{w}_k), \bar{F}(\bar{z}_{k+1}), \bar{F}(\bar{w}_{k+1}) \in \mathbb{R}^N$ , where  $N \geq 6$  and numbers  $\alpha, \beta_1, \beta_2 \in \mathbb{R}$  that satisfy the inequalities of Lemma 12 such that

$$\begin{aligned} & \|\bar{F}(\bar{z}_k) - \bar{F}(\bar{w}_k)\|^2 + \left\| \bar{F}(\bar{z}_k) - \frac{\langle \bar{F}(\bar{z}_k), \bar{b}_k \rangle \cdot \bar{b}_k}{\|\bar{b}_k\|^2} \right\|^2 \\ & - \left( \|\bar{F}(\bar{z}_{k+1}) - \bar{F}(\bar{w}_{k+1})\|^2 + \left\| \bar{F}(\bar{z}_{k+1}) - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{b}_{k+1} \rangle \cdot \bar{b}_{k+1}}{\|\bar{b}_{k+1}\|^2} \mathbb{1} \left[ \langle \bar{F}(\bar{z}_{k+1}), \bar{b}_{k+1} \rangle \geq 0 \right] \right\|^2 \right) < 0. \end{aligned}$$

We use TARGET to denote the LHS of the above inequality. Similar to the proof of Theorem 2, we show that the sum of TARGET and some non-positive expressions is a sum of squares, which implies that TARGET is non-negative leading to contradiction.

**Deriving Inequalities:** By Item 4 of Lemma 12 and the fact that  $\eta \cdot L \leq \frac{1}{2}$  we have that

$$2 \cdot \langle \eta \bar{F}(\bar{z}_{k+1}) - \eta \bar{F}(\bar{z}_k), \bar{z}_k - \bar{z}_{k+1} \rangle \leq 0, \quad (111)$$

$$2 \cdot \left( \|\eta \bar{F}(\bar{z}_{k+1}) - \eta \bar{F}(\bar{w}_{k+1})\|^2 - \frac{1}{4} \|\bar{z}_{k+1} - \bar{w}_{k+1}\|^2 \right) \leq 0, \quad (112)$$

By co-directness of vectors  $\bar{b}_{k+1}, \bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_{k+1})$  and the fact that  $\langle \bar{b}_{k+1}, \bar{z}_{k+1} \rangle = 0, \langle \bar{b}_{k+1}, \bar{z}_k \rangle \geq 0$  (by Item 3 of Lemma 12) we can show that

$$2 \cdot \langle \bar{b}_{k+1}, \bar{z}_k \rangle \langle \bar{b}_{k+1}, -\bar{z}_{k+1} + \bar{z}_k - \eta \bar{F}(\bar{w}_{k+1}) \rangle = 2 \cdot \langle \bar{b}_{k+1}, \bar{z}_k \rangle \langle \bar{b}_{k+1}, \bar{z}_k - \eta \bar{F}(\bar{w}_{k+1}) \rangle \leq 0, \quad (113)$$

$$2 \cdot \left( -\langle \bar{b}_{k+1}, \eta \bar{F}(\bar{z}_{k+1}) \rangle \mathbb{1} \left[ \langle \bar{b}_{k+1}, \bar{F}(\bar{z}_{k+1}) \rangle \leq 0 \right] \right) \langle \bar{b}_{k+1}, \bar{z}_k - \eta \bar{F}(\bar{w}_{k+1}) \rangle \leq 0, \quad (114)$$

Similarly by co-directness of vectors  $\bar{a}_{k+1}, \bar{w}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_k)$  and the fact that  $\langle \bar{a}_{k+1}, \bar{w}_{k+1} \rangle = 0, \langle \bar{a}_{k+1}, \bar{z}_{k+1} \rangle \geq 0$  we can show that

$$\frac{1}{\|\bar{a}_{k+1}\|^2} \langle \bar{a}_{k+1}, \bar{z}_{k+1} \rangle \langle \bar{a}_{k+1}, \bar{z}_k - \eta \bar{F}(\bar{w}_k) \rangle \leq 0, \quad (115)$$

Combining the fact that  $\langle \bar{b}_k, \bar{F}(\bar{z}_k) \rangle \geq 0$ ,  $\langle \bar{b}_k, \bar{w}_{k+1} \rangle \geq 0$  and  $\langle \bar{b}_k, \bar{z}_{k+1} \rangle \geq 0$  (by Item 3 of Lemma 12) we get that

$$-\frac{1}{\|\bar{b}_k\|^2} \langle \bar{b}_k, \eta \bar{F}(\bar{z}_k) \rangle \langle \bar{b}_k, \bar{w}_{k+1} + \bar{z}_{k+1} \rangle \leq 0, \quad (116)$$

Define vector  $\bar{a}_{k+1}^\perp = (-1, \alpha, 0, \dots, 0)$ . Since  $\langle \bar{a}_{k+1}^\perp, \bar{a}_{k+1} \rangle = 0$ , by co-directness of vectors  $\bar{a}_{k+1}$ ,  $\bar{w}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_k)$  we have that

$$\langle \bar{a}_{k+1}^\perp, \bar{w}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_k) \rangle = 0,$$

which further implies that

$$\bar{w}_{k+1}[1] \langle \bar{a}_{k+1}^\perp, \bar{z}_k - \eta \bar{F}(\bar{w}_k) - \bar{w}_{k+1} \rangle = 0, \quad (117)$$

$$\frac{\alpha}{\|\bar{a}_{k+1}\|^2} \cdot \bar{z}_{k+1}[2] \langle \bar{a}_{k+1}^\perp, \bar{z}_k - \eta \bar{F}(\bar{w}_k) - \bar{w}_{k+1} \rangle = 0. \quad (118)$$

**Showing that TARGET is non-negative:** Since Expression (111)-(118) are non-positive, then showing that the LHS of Expressions (119) is non-negative implies that TARGET is also non-negative.

$$\begin{aligned} \eta^2 \cdot \text{TARGET} + \text{Expression (111)} + \text{Expression (112)} + \text{Expression (113)} + \text{Expression (114)} \\ + \text{Expression (115)} + \text{Expression (116)} + \text{Expression (117)} + \text{Expression (118)} \geq 0 \end{aligned} \quad (119)$$

We denote by TARGET<sub>1</sub> the following expression

$$\begin{aligned} \sum_{i \in [1,3]} \left( \left( \bar{F}(\bar{z}_k)[i] - \bar{F}(\bar{w}_k)[i] \right)^2 + \left( \bar{F}(\bar{z}_k)[i] - \frac{\langle \bar{F}(\bar{z}_k), \bar{b}_k \rangle \cdot \bar{b}_k[i]}{\|\bar{b}_k\|^2} \right)^2 \right. \\ \left. - \left( \left( \bar{F}(\bar{z}_{k+1})[i] - \bar{F}(\bar{w}_{k+1})[i] \right)^2 + \left( \bar{F}(\bar{z}_{k+1})[i] - \frac{\langle \bar{F}(\bar{z}_{k+1}), \bar{b}_{k+1} \rangle \cdot \bar{b}_{k+1}[i]}{\|\bar{b}_{k+1}\|^2} \mathbb{1} \left[ \langle \bar{F}(\bar{z}_{k+1}), \bar{b}_{k+1} \rangle \geq 0 \right] \right)^2 \right) \right), \end{aligned}$$

by TARGET<sub>2</sub> the following expression

$$\sum_{i \in [4,N]} \left( \left( \bar{F}(\bar{z}_k)[i] - \bar{F}(\bar{w}_k)[i] \right)^2 + \left( \bar{F}(\bar{z}_k)[i] \right)^2 - \left( \left( \bar{F}(\bar{z}_{k+1})[i] - \bar{F}(\bar{w}_{k+1})[i] \right)^2 + \left( \bar{F}(\bar{z}_{k+1})[i] \right)^2 \right) \right),$$

and we consider the following expressions

$$2 \cdot \sum_{i \in [1,3]} (\eta \bar{F}(\bar{z}_{k+1})[i] - \eta \bar{F}(\bar{z}_k)[i]) \cdot (\bar{z}_k[i] - \bar{z}_{k+1}[i]), \quad (120)$$

$$2 \cdot \sum_{i \in [4,N]} (\eta \bar{F}(\bar{z}_{k+1})[i] - \eta \bar{F}(\bar{z}_k)[i]) \cdot (\bar{z}_k[i] - \bar{z}_{k+1}[i]), \quad (121)$$

$$2 \cdot \sum_{i \in [1,3]} \left( (\eta \bar{F}(\bar{z}_{k+1})[i] - \eta \bar{F}(\bar{w}_{k+1})[i])^2 - \frac{1}{4} (\bar{z}_{k+1}[i] - \bar{w}_{k+1}[i])^2 \right), \quad (122)$$

$$2 \cdot \sum_{i \in [4,N]} \left( (\eta \bar{F}(\bar{z}_{k+1})[i] - \eta \bar{F}(\bar{w}_{k+1})[i])^2 - \frac{1}{4} (\bar{z}_{k+1}[i] - \bar{w}_{k+1}[i])^2 \right). \quad (123)$$

Observe that  $\text{TARGET} = \text{TARGET}_1 + \text{TARGET}_2$ , Expression (112) = Expression (122) + Expression (123), Expression (111) = Expression (120) + Expression (121). Thus in order to prove Inequality (119), it is enough to show the following two inequalities

$$\eta^2 \cdot \text{TARGET}_1 + \text{Expression (120)} + \text{Expression (122)} + \text{Expression (113)} + \text{Expression (114)} + \text{Expression (115)} + \text{Expression (116)} + \text{Expression (117)} + \text{Expression (118)} \geq 0, \quad (124)$$

$$\eta^2 \cdot \text{TARGET}_2 + \text{Expression (121)} + \text{Expression (123)} \geq 0. \quad (125)$$

**Proving Inequality (125):** Since for  $i \geq 4$ ,  $\langle e_i, \bar{a}_{k+1} \rangle = \langle e_i, \bar{b}_{k+1} \rangle = 0$ , vector  $\bar{a}_{k+1}$  is co-directed with vector  $\bar{w}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_k)$  and vector  $\bar{b}_{k+1}$  is co-directed with vector  $\bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_{k+1})$  we have that

$$\langle e_i, \bar{a}_{k+1} \rangle = 0 = \langle e_i, \bar{w}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_k) \rangle \Leftrightarrow \bar{w}_{k+1}[i] = \bar{z}_k[i] - \eta \bar{F}(\bar{w}_k)[i], \quad (126)$$

$$\langle e_i, \bar{b}_{k+1} \rangle = 0 = \langle e_i, \bar{z}_{k+1} - \bar{z}_k + \eta \bar{F}(\bar{w}_{k+1}) \rangle \Leftrightarrow \bar{z}_{k+1}[i] = \bar{z}_k[i] - \eta \bar{F}(\bar{w}_{k+1})[i]. \quad (127)$$

Substituting  $\bar{w}_{k+1}[i]$  with  $\bar{z}_k[i] - \eta \bar{F}(\bar{w}_k)[i]$  and  $\bar{z}_{k+1}[i]$  with  $\bar{z}_k[i] - \eta \bar{F}(\bar{w}_{k+1})[i]$  in the LHS of Inequality (125), we can prove the following identity for any  $i \geq 4$

$$\begin{aligned} & (\eta \bar{F}(\bar{z}_k)[i] - \eta \bar{F}(\bar{w}_k)[i])^2 + (\eta \bar{F}(\bar{z}_k)[i])^2 - \left( (\eta \bar{F}(\bar{z}_{k+1})[i] - \eta \bar{F}(\bar{w}_{k+1})[i])^2 + (\eta \bar{F}(\bar{z}_{k+1})[i])^2 \right) \\ & + 2 \cdot (\eta \bar{F}(\bar{z}_{k+1})[i] - \eta \bar{F}(\bar{z}_k)[i]) \cdot \eta \bar{F}(\bar{w}_{k+1})[i] \\ & + 2 \cdot \left( (\eta \bar{F}(\bar{z}_{k+1})[i] - \eta \bar{F}(\bar{w}_{k+1})[i])^2 - \frac{1}{4} (\eta \bar{F}(\bar{w}_{k+1})[i] - \eta \bar{F}(\bar{w}_k)[i])^2 \right) \\ & = \frac{1}{2} (\eta \bar{F}(\bar{w}_k)[i] + \eta \bar{F}(\bar{w}_{k+1})[i] - 2\eta \bar{F}(\bar{z}_k)[i])^2, \end{aligned} \quad (128)$$

which implies that  $\text{TARGET}_2$  is non-negative. The proof of Identity (128) follows in the same way as the proof of Identity (30). MATLAB code for the verification of the above identity can be found at this [link](#).

**Proving Inequality (124):** By Item 2 and Item 3 of Lemma 12 we have that

$$\langle \bar{b}_k, \bar{z}_k \rangle = 0 \Leftrightarrow \bar{z}_k[3] = -\beta_1 \bar{z}_k[1] - \beta_2 \bar{z}_k[2], \quad (129)$$

$$\langle \bar{a}_{k+1}, \bar{w}_{k+1} \rangle = 0 \Leftrightarrow \bar{w}_{k+1}[2] = -\alpha \bar{w}_{k+1}[1], \quad (130)$$

$$\langle \bar{b}_{k+1}, \bar{z}_{k+1} \rangle = 0 \Leftrightarrow \bar{z}_{k+1}[1] = 0. \quad (131)$$

Since  $\langle e_2, \bar{b}_{k+1} \rangle = \langle e_3, \bar{b}_{k+1} \rangle = \langle e_3, \bar{a}_{k+1} \rangle = 0$  and  $\langle \bar{z}_{k+1}, \bar{b}_{k+1} \rangle = \langle \bar{w}_{k+1}, \bar{a}_{k+1} \rangle = 0$  we can prove in a similar manner to Inequalities (126)-(127) that

$$\bar{z}_{k+1}[2] = \bar{z}_k[2] - \eta \bar{F}(\bar{w}_{k+1})[2], \quad (132)$$

$$\bar{z}_{k+1}[3] = \bar{z}_k[3] - \eta \bar{F}(\bar{w}_{k+1})[3] = -\beta_1 \bar{z}_k[1] - \beta_2 \bar{z}_k[2] - \eta \bar{F}(\bar{w}_{k+1})[3], \quad (133)$$

$$\bar{w}_{k+1}[3] = \bar{z}_k[3] - \eta \bar{F}(\bar{w}_k)[3] = -\beta_1 \bar{z}_k[1] - \beta_2 \bar{z}_k[2] - \eta \bar{F}(\bar{w}_k)[3]. \quad (134)$$

By substituting  $\{\bar{z}_k[3], \bar{w}_{k+1}[2], \bar{w}_{k+1}[3], \bar{z}_{k+1}[1], \bar{z}_{k+1}[2], \bar{z}_{k+1}[3]\}$  in Inequality (124) with their corresponding RHS using Equalities (129)-(134) we can prove the following identity

$$\begin{aligned}
& \text{LHS of Inequality (124)} = \\
& + \left( \eta \bar{F}(\bar{w}_k)[1] - \eta \bar{F}(\bar{z}_k)[1] + \frac{\bar{w}_{k+1}[1]}{2} \right)^2 \\
& + \left( \bar{z}_k[1] - \eta \bar{F}(\bar{w}_{k+1})[1] + \eta \bar{F}(\bar{z}_{k+1})[1] \mathbb{1} \left[ \bar{F}(\bar{z}_{k+1})[1] \geq 0 \right] \right)^2 \\
& + \left( \frac{\bar{z}_k[2]}{2} - \frac{\bar{w}_{k+1}[2]}{2} - \frac{\eta \bar{F}(\bar{w}_{k+1})[2]}{2} + \eta \bar{F}(\bar{z}_k)[2] - \eta \bar{F}(\bar{w}_k)[2] \right)^2 \\
& + \frac{1}{2} \left( \eta \bar{F}(\bar{w}_k)[3] + \eta \bar{F}(\bar{w}_{k+1})[3] + \frac{\beta_1}{1 + \beta_1^2 + \beta_2^2} \eta \bar{F}(\bar{z}_k)[1] + \frac{\beta_2}{1 + \beta_1^2 + \beta_2^2} \eta \bar{F}(\bar{z}_k)[2] - \frac{2 \cdot \beta_1^2 + 2 \cdot \beta_2^2 + 1}{1 + \beta_1^2 + \beta_2^2} \eta \bar{F}(\bar{z}_k)[3] \right)^2 \\
& + \left( \frac{\bar{z}_k[2]}{2} - \frac{\bar{w}_{k+1}[2]}{2} + \frac{\eta \bar{F}(\bar{w}_{k+1})[2]}{2} + \frac{\beta_1 \beta_2}{1 + \beta_1^2 + \beta_2^2} \eta \bar{F}(\bar{z}_k)[1] - \frac{1 + \beta_1^2}{1 + \beta_1^2 + \beta_2^2} \eta \bar{F}(\bar{z}_k)[2] + \frac{\beta_2}{1 + \beta_1^2 + \beta_2^2} \eta \bar{F}(\bar{z}_k)[3] \right)^2 \\
& + \frac{\left( \frac{1 + \frac{\beta_1^2}{2} + 2\beta_2^2 + \beta_2^4}{1 + \beta_1^2 + \beta_2^2} \eta \bar{F}(\bar{z}_k)[1] - \frac{\beta_1 \beta_2 (1 + 2\beta_2^2)}{2(1 + \beta_1^2 + \beta_2^2)} \eta \bar{F}(\bar{z}_k)[2] - \frac{\beta_1 (1 + 2\beta_2^2)}{2(1 + \beta_1^2 + \beta_2^2)} \eta \bar{F}(\bar{z}_k)[3] + \frac{1 + \beta_2^2}{2} \bar{w}_{k+1}[1] - (1 + \beta_2^2) \bar{z}_k[1] \right)^2}{1 + \frac{\beta_1^2}{2} + 2\beta_2^2 + \beta_2^4} \\
& + \frac{(\beta_2 \eta \bar{F}(\bar{z}_k)[2] + \eta \bar{F}(\bar{z}_k)[3] - \frac{\beta_1}{2} \bar{w}_{k+1}[1] + \beta_1 \bar{z}_k[1])^2}{\beta_1^2 + 2(1 + \beta_2^2)^2} \\
& \geq 0,
\end{aligned}$$

which implies that  $\text{TARGET}_1$  is non-negative and concludes the proof. MATLAB code for the verification of the above identity can be found at this [link](#).  $\square$

## H.5 Combining Everything

In this section, we combine the results of the previous sections and show that  $\Phi_T = O\left(\frac{1}{T}\right)$  and we show the last-iterate convergence rate for performance measures of interest.

**Lemma 13.** *Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be a monotone and  $L$ -Lipschitz operator. Let  $z_0, w_0 \in \mathcal{Z}$  be arbitrary starting point and  $\{z_k, w_k\}_{k \geq 0}$  be the iterates of the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$ . Then for any  $k \geq 0$ ,*

$$r_{(F, \mathcal{Z})}^{\text{tan}}(w_{k+1}) \leq \sqrt{2}(2 + \eta L) \sqrt{r_{(F, \mathcal{Z})}^{\text{tan}}(z_k)^2 + \|F(w_k) - F(z_k)\|^2}.$$

*Proof.* Since  $w_{k+1} = \Pi_{\mathcal{Z}}[z_k - F(w_k)]$ , by using Lemma 8 we have

$$\begin{aligned}
r_{(F,\mathcal{Z})}^{\tan}(w_{k+1}) &\leq \left\| \frac{z_k - w_{k+1}}{\eta} + F(w_{k+1}) - F(w_k) \right\| \\
&\leq \left\| \frac{z_k - w_{k+1}}{\eta} \right\| + \|F(w_k) - F(z_k)\| + \|F(z_k) - F(w_{k+1})\| \\
&\leq \frac{1 + \eta L}{\eta} \|z_k - w_{k+1}\| + \|F(w_k) - F(z_k)\|. \quad (L\text{-Lipschitzness of } F)
\end{aligned}$$

Using Lemma 1 and the non-expansiveness of the projection mapping, we have

$$\begin{aligned}
\|z_k - w_{k+1}\| &\leq \|z_k - \Pi_{\mathcal{Z}}[z_k - \eta F(z_k)]\| + \|\Pi_{\mathcal{Z}}[z_k - \eta F(z_k)] - w_{k+1}\| \\
&= r_{(\eta F, \mathcal{Z})}^{\text{nat}}(z_k) + \|\Pi_{\mathcal{Z}}[z_k - \eta F(z_k)] - \Pi_{\mathcal{Z}}[z_k - \eta F(w_k)]\| \\
&\leq r_{(\eta F, \mathcal{Z})}^{\tan}(z_k) + \eta \|F(z_k) - F(w_k)\| \\
&= \eta r_{(F, \mathcal{Z})}^{\tan}(z_k) + \eta \|F(z_k) - F(w_k)\|.
\end{aligned}$$

Combing the above two inequalities, we have

$$\begin{aligned}
r_{(F, \mathcal{Z})}^{\tan}(w_{k+1}) &\leq (1 + \eta L) r_{(F, \mathcal{Z})}^{\tan}(z_k) + (2 + \eta L) \|F(w_k) - F(z_k)\| \\
&\leq \sqrt{2}(2 + \eta L) \sqrt{r_{(F, \mathcal{Z})}^{\tan}(z_k)^2 + \|F(w_k) - F(z_k)\|^2}. \quad (a + b \leq \sqrt{2}\sqrt{a^2 + b^2})
\end{aligned}$$

□

Combining Corollary 2, Theorem 7, Lemma 13, Lemma 1 and Lemma 2 we get  $\mathcal{O}(\frac{1}{\sqrt{T}})$  last-iterate convergence in terms of the tangent residual, natural residual and gap function for both  $z_T$  and  $w_{T+1}$ . The result is formally stated in Theorem 8.

**Theorem 8.** Let  $\mathcal{Z} \subseteq \mathbb{R}^n$  be a closed convex set and  $F : \mathcal{Z} \rightarrow \mathbb{R}$  be a monotone and  $L$ -Lipschitz operator. Let  $z_0, w_0 \in \mathcal{Z}$  be arbitrary starting point and  $\{z_k, w_k\}_{k \geq 0}$  be the iterates of the OGDA algorithm with any step size  $\eta \in (0, \frac{1}{2L})$ . Let  $D_0 := \sqrt{(4 + 6\eta^4 L^4) \|z_0 - z^*\|^2 + (16\eta^2 L^2 + 6\eta^4 L^4) \|w_0 - z_0\|^2} = \mathcal{O}(\max\{\|z_0 - z^*\|, \|w_0 - z_0\|\})$ . Then for any  $T \geq 1$ ,

- $\text{GAP}_{\mathcal{Z}, F, D}(z_T) \leq \frac{1}{\sqrt{T}} \cdot \frac{DD_0}{\eta \sqrt{1 - 4 \cdot (\eta L)^2}}.$
- $r_{\mathcal{Z}, F, D}^{\text{nat}}(z_T) \leq r_{\mathcal{Z}, F, D}^{\tan}(z_T) \leq \frac{1}{\sqrt{T}} \cdot \frac{D_0}{\eta \sqrt{1 - 4 \cdot (\eta L)^2}}.$
- $\text{GAP}_{\mathcal{Z}, F, D}(w_{T+1}) \leq \frac{1}{\sqrt{T}} \cdot \frac{\sqrt{2}(2 + \eta L) \cdot D \cdot D_0}{\eta \sqrt{1 - 4 \cdot (\eta L)^2}}.$
- $r_{\mathcal{Z}, F, D}^{\text{nat}}(w_{T+1}) \leq r_{\mathcal{Z}, F, D}^{\tan}(w_{T+1}) \leq \frac{1}{\sqrt{T}} \cdot \frac{\sqrt{2}(2 + \eta L) D_0}{\eta \sqrt{1 - 4 \cdot (\eta L)^2}}.$