Math 551: Scientific Programming

Lecture 7: Newton's method and the secant method

Weiqi Chu

Department of Mathematics and Statistics, UMass Amherst

Last time

- Fixed-point iterations
- Convergence rate

Today: Sections 1.3 & 1.4

- Relation between fixed-point problems and root-finding problems
- Newton's method
- Secant method

Review: Fixed point iterations

Suppose $x = r$ is the solution of $f(x) = 0$ on $[a, b]$. To find the value of r:

1. we construct a fixed-point problem for g, so that $x = r$ is the fixed-point of g
   this means that $g(r) = r$
2. we solve this fixed-point problem using fixed-point iterations
   $$x_{n+1} = g(x_n) \qquad \text{for} \qquad n = 0, 1, \ldots$$

Theorem: The sequence x_n converges to r if g is a contraction:

$$\max_{z \in [a,b]} |g'(z)| = \kappa < 1\,.$$

Fixed-point problems VS root-finding problems

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One idea: Construct $g(x) = f(x) + x$ and apply fixed-point iterations to $g(x)$ using

$$x_{n+1} = g(x_n)\,, \quad n = 0, 1, 2, \dots .$$

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$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots.$$

Does x_n converge to $\sqrt{2}$? NO!!! The above method does not converge to x_* so such a fixed-point construction does not work!

Example

Can you build a fixed-point problem to find the root of $f(x) = x^2 - 2 = 0$?

How about the following choices?

- $g(x) = 2x - \frac{2}{x}$
- $g(x) = \frac{x}{x^2 - 1}$

Example

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To design a fixed-point problem for $f(x_*) = 0$, we need to make sure that

- $g(x_*) = x_*$
- $g'(x_*) < 1$

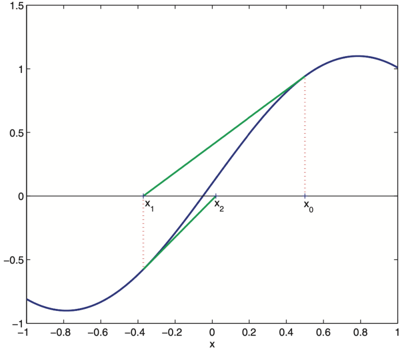
Newton's Method

An iterative method to solve $f(x) = 0$ for x

Iterative formula for Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use a straight line to approximate the function and find the next step point.



Convergence of Newton's method

Theorem

Suppose $f(x) \in C^2[a,b]$ and $f'(x^\star) \neq 0$. There exists $\delta > 0$ such that Newton's method converges quadratically for $x_0 \in [x^\star - \delta, x^\star + \delta]$.

proof sketch (won't be tested).

1. Continuity $+\ f'(x^\star) \neq 0 \Rightarrow$ exists $\delta > 0$ with $f'(x) \neq 0$ all $x \in [x^\star - \delta, x^\star + \delta]$
2. Let $g(x) = x - \dfrac{f(x)}{f'(x)}$.
3. $g(x)$ well-defined on $[x^\star - \delta, x^\star + \delta]$ and $g'(x^\star) = 0$.
4. the fixed point iteration of $g(x)$ converges quadratically.

$\square$

Example

Apply Newton's method to solve $f(x) = x^2 - 2 = 0$. This is the same algorithm as the Babylonian square-root method.

The Secant Method

- **Newton's method:** $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

  Newton's method requires computing $f'(x_n)$.

  Computing f' could be expensive or impossible.

- Improvement idea: Use $f'(x_n) \approx \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$.

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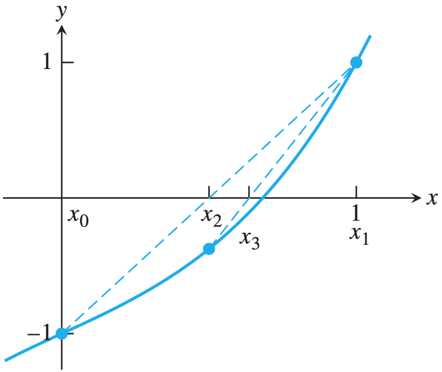
- **Secant Method:** $x_{n+1} = x_n - \dfrac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$.

  Store and reuse $f(x_{n-1})$.

- Converges with rate $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Geometric illustration of secant method



Other root-finding methods

- Method of False Position
- Muller's Method
- Inverse Quadratic Interpolation
- Brent's Method

Next time: Solving systems of linear and nonlinear equation