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Differential Geometry
Homework 2
Monday, September 21 2019

A.a) Problem 2 on page 14, Section 1-4, Baby Do Carmo.

***2.** A plane P contained in R^3 is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

First note that this plane is parallel to the kernel of (a, b, c) in R^3. Recall that the kernel of a vector in R^3 is always a plane. So the normal vector to this plane is $v = (a, b, c)$. v is perpendicular to the plane. To show We can normalize the equation for any line in order to derive the normal vector and distance to the origin, $ax + by + c = 0 \implies \frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{cz}{\sqrt{a^2+b^2+c^2}} + \frac{d}{\sqrt{a^2+b^2+c^2}} = 0$.

Theorem The distance from the line to the origin is $\frac{d}{\sqrt{a^2+b^2+c^2}}$.

Proof: We want to find the point on the line (x, y, z) that is closest to the origin. Then the distance from the origin to the line is just $||(x, y, z)||$. We can find this by using the lagrange multiplier to minimize $||(x, y, z)||$ under the constraint that $ax + by + cz + d = 0$. We get this system of equations,

$$
\begin{aligned}
ax + by + cz + d &= 0 \\
\frac{x}{\sqrt{x^2 + y^2 + z^2}} &= 0 \\
\frac{y}{\sqrt{x^2 + y^2 + z^2}} &= 0 \\
\frac{z}{\sqrt{x^2 + y^2 + z^2}} &= 0
\end{aligned}
$$

Solving this system of equations gives us the point $(\frac{ad}{(a^2+b^2+c^2)}, \frac{bd}{(a^2+b^2+c^2)}, \frac{cd}{(a^2+b^2+c^2)})$. So the distance to the line is,

$$\sqrt{\frac{a^2d^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2d^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2d^2}{(a^2 + b^2 + c^2)^2}} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

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A.b) Problem 5 on page 14, Section 1-4, Baby Do Carmo.

5. Show that the equation of a plane passing through three noncolinear points $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, $p_3 = (x_3, y_3, z_3)$ is given by

$$(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0,$$

where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$, for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.

$\Longleftarrow$ First I show that if the equation is true then p is in the plane. Let the difference vectors matrix be $D = [(p - p_1)\ (p - p_2)\ (p - p_3)]$ The given formula is equal to $det(D)$. When the determinant is zero the vectors columns of D are not all linearly independent. This means that the columns of D all contained within a 2d subspace. The points p, p_1, p_2, p_3 differ from the columns of D only by a translation so they must lie within a 2d flat (they are coplanar).
$\Longrightarrow$ Now I show that if p is in the plane then the equation is true. If p is in the plane then they are all coplanar. Subtracting, p from every point in the plane gives a plane that passes through the origin. So the columns of D must be linearly dependent. Then $det(D) = 0$. Meaning that equation is true. ∎

A.c) Problem 11 on page 15, Section 1-4, Baby Do Carmo.

11. a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in R^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an *oriented volume* in R^3.

b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$

Proof

Consider the vectors u and v and the parallelogram generated by these two vectors. The area of this parallelogram is given by $||u||\ ||v|| \sin\theta$ where θ is the angle between u and v. We recognize this as the familiar cross-product in $\mathbb{R}^3$, hence the area of this parallelogram is $||u \times v||$. The volume of the parallelepiped is the area of this parallelogram times a distance perpendicular to the plane containing u and v. Notice that since w is linearly independent from u and v, that there is a component of w that is perpendicular to both u and v, and hence either parallel or anti-parallel to $u \times v$. Therefore, the volume of the parallelepiped is given by $|(u \times v) \cdot w|$. Observe that we can interpret the volume as the determinant of a matrix given by

$$V = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Observe that V does not change if we perform an equal number of row exchanges on the above matrix. First exchanging the first and second rows, and subsequently the second and third rows we obtain

$$V = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

If the above matrix is given by A, we know that $V^2 = \det(A)^2 = \det(A)\det(A) = \det(A)\det(A^T) = \det(AA^T)$. We therefore write

$$\begin{aligned} AA^T &= \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 + u_2^2 + u_3^2 & u_1v_1 + u_2v_2 + u_3v_3 & u_1w_1 + u_2w_2 + u_3w_3 \\ v_1u_1 + v_2u_2 + v_3u_3 & v_1^2 + v_2^2 + v_3^2 & v_1w_1 + v_2w_2 + v_3w_3 \\ w_1u_1 + w_2u_2 + w_3u_3 & w_1v_1 + w_2v_2 + w_3v_3 & w_1^2 + w_2^2 + w_3^2 \end{bmatrix} \\ &= \begin{bmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{bmatrix}. \end{aligned}$$

Therefore

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$

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A.d) Problem 13 on page 16, Section 1-4, Baby Do Carmo.

13. Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into R^3. If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions

$$u'(t) = au(t) + bv(t), \qquad v'(t) = cu(t) - av(t),$$

where a, b, and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

To show that cross product is constant we show that it's gradient is always zero,

$$\begin{aligned} \nabla(u(t) \times v(t)) &= u'(t) \times v(t) + u(t) \times v'(t) && \text{(from Do Carmo pg. 14)} \\ &= (au(t) + bv(t)) \times v(t) + u(t) \times (cu(t) - av(t)) \\ &= au(t) \times v(t) + bv(t) \times v(t) + cu(t) \times u(t) - au(t) \times v(t) && \text{(cross product is linear)} \\ &= \vec{0} && (Since\ v(t) \times v(t) = \vec{0}) \end{aligned}$$

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B.a) Find the length of the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x-1)^2 + y^2 = 1$ in R^3.

Solution

We can first parametrize the cylinder by taking $x(t) = 1 + \cos t$ and $y(t) = \sin t$ for $t \in [0, 2\pi]$. Observe this satisfies the identity $\cos^2 t + \sin^2 t = 1$. Next we obtain an equation of a parametrized curve in $\mathbb{R}^3$ by substituting these values of $x(t)$ and $y(t)$ into the equation of the sphere and solving for $z(t)$. We see that

$$\begin{aligned}
(1+\cos t)^2 + \sin^2 t + z^2(t) &= 4 \\
\cos^2 t + \sin^2 t + 2\cos t + 1 + z^2(t) &= 4 \\
2 + 2\cos t + z^2(t) &= 4 \\
z^2(t) &= 2(1-\cos t).
\end{aligned}$$

We now find the arclength of the curve in the first octant. Observe that at $t = 0$ we have $y(0) = 0$, and hence this is the smallest value of t such that the curve lies in the first octant. At $t = \pi$ we have $z^2(\pi) = 4$ and hence t obtains its largest value in the first octant at $t = \pi$. The equation of the curve is then

$$\alpha(t) = \left(1+\cos t, \sin t, \sqrt{2(1-\cos t)}\right) \qquad t \in [0, \pi].$$

To find the arclength we first find the derivative $\alpha'(t)$.

$$\alpha'(t) = \left(-\sin t, \cos t, \frac{\sin t}{\sqrt{2(1-\cos t)}}\right).$$

Now find the norm of $\alpha'(t)$.

$$\begin{aligned}
||\alpha'(t)||^2 &= \sin^2 t + \cos^2 t + \frac{\sin^2 t}{2(1-\cos t)} \\
&= 1 + \frac{1-\cos^2 t}{2(1-\cos t)} \\
&= 1 + \frac{(1-\cos t)(1+\cos t)}{2(1-\cos t)} \\
&= \frac{1}{2}(3+\cos t).
\end{aligned}$$

Therefore the arclength of this curve in the first octant is

$$\begin{aligned}
\int_0^\pi ||\alpha'(t)||\, dt &= \int_0^\pi \sqrt{\frac{3+\cos t}{2}}\, dt \\
&\approx 3.8202.
\end{aligned}$$

Integration was done numerically on Wolfram Alpha. $\square$

C.a) Problem 1 on page 5, Section 1-2, Baby Do Carmo.

1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

$\alpha(t) = (sin(t), cos(t))$ $\blacksquare$

C.b) Problem 3 on page 5, Section 1-2, Baby Do Carmo.

3. A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

It is like a point with that undergoes zero acceleration. So it has constant velocity. So it goes in a straight line. $\blacksquare$

C.c) Problem 4 on page 5, Section 1-2, Baby Do Carmo.

4. Let $\alpha : I \longrightarrow R^3$ be a parametrized curve and let $v \in R^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Proof

Let $g(t)$ be the function given by $\alpha(t) \cdot v = g(t)$. Taking the derivative of both sides we obtain $\alpha'(t) \cdot v = g'(t)$. We know however that since $\alpha'(t)$ is orthogonal to v for all $t \in I$ that $g'(t) = 0$. Therefore $g(t) = c$ where c is a constant. Hence $\alpha(t) \cdot v = c$ is constant for all $t \in I$. Notice however that we are given that $\alpha(0) \cdot v = 0$. This initial condition implies that $c = 0$. Therefore $\alpha(t) \cdot v = 0$ for all $t \in I$, hence $\alpha(t)$ is orthogonal to v for all $t \in I$. $\square$

C.d) Problem 5 on page 5, Section 1-2, Baby Do Carmo.

5. Let $\alpha : I \longrightarrow R^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof

$\Longrightarrow$

Suppose $||\alpha(t)|| = c$ for all $t \in I$ where c is a nonzero constant. We therefore have that $\alpha(t) \cdot \alpha(t) = c^2$. Taking the derivative of both sides we obtain

$$\alpha'(t) \cdot \alpha(t) + \alpha(t) \cdot \alpha'(t) \quad = \quad 2\alpha'(t) \cdot \alpha(t) \quad = \quad 0$$

implying that $\alpha'(t) \cdot \alpha(t) = 0$ for all $t \in I$.

$\Longleftarrow$

Suppose $\alpha(t) \cdot \alpha'(t) = 0$ for all $t \in I$. Consider the norm $||\alpha(t)||$. Taking the derivative of this we obtain

$$\frac{d}{dt}||\alpha(t)||^2 \quad = \quad \alpha(t) \cdot \alpha'(t) + \alpha'(t) \cdot \alpha(t) \quad = \quad 2\alpha(t) \cdot \alpha'(t) \quad = \quad 0.$$

This implies that $||\alpha(t)|| = c$ where c is a constant. We know that $c \neq 0$, since this would imply that the curve $\alpha(t) \equiv 0$, which is a contradiction since it violates the definition of a parametrized curve. $\square$