Topic 2: Regular Curves

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Math 142:
Differential Geometry

Local Theory: Key Ideas

1. How to define a curve so that we can study it by using differential calculus?

Roughly speaking, we want a "smooth curve." In fact,

parametrized curve $\rightarrow$ regular curve of order 0 $\rightarrow$
parametrized by arc length $\rightarrow$ regular curve of order 1

2. Frenet Frame and Frenet Formulas

The Frenet Frame is a coordinate system which is best adapted to a problem involving a curve. To apply calculus and liear algebra, we use the Frenet Formulas!

Fundamental Theorem of the Local Theory of Curves

Roughly speaking, this theorem says that the curvature and torsion completely describe the local behavior of a curve.

Parametrized and Regular Curves

Definition

A *parametrized differentiable curve* is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

Definition

A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

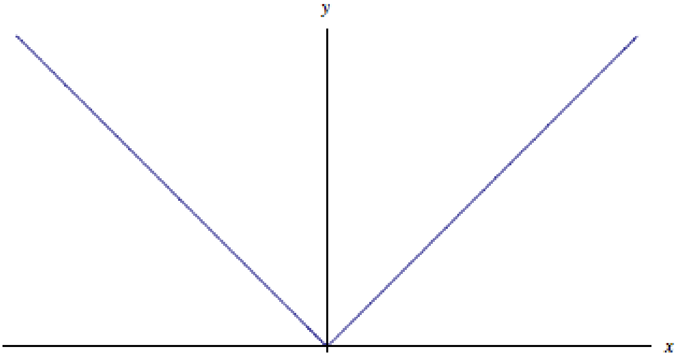
Definition

We say that $s \in I$ is a *singular point of order 1* if $\alpha''(s) = 0$ (in this context, the points where $\alpha'(s) = 0$ are called singular points of order 0).

Examples

Example 1

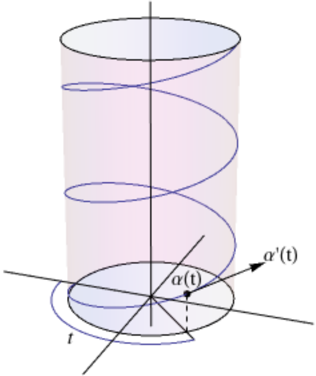
The map $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t, |t|), t \in \mathbb{R}$ (not differentiable).



Examples

Example 2

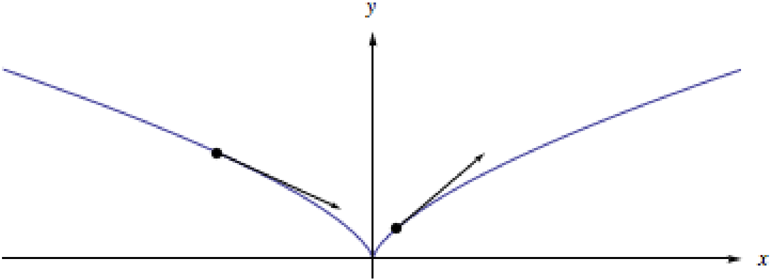
A helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$.



Examples

Example 3

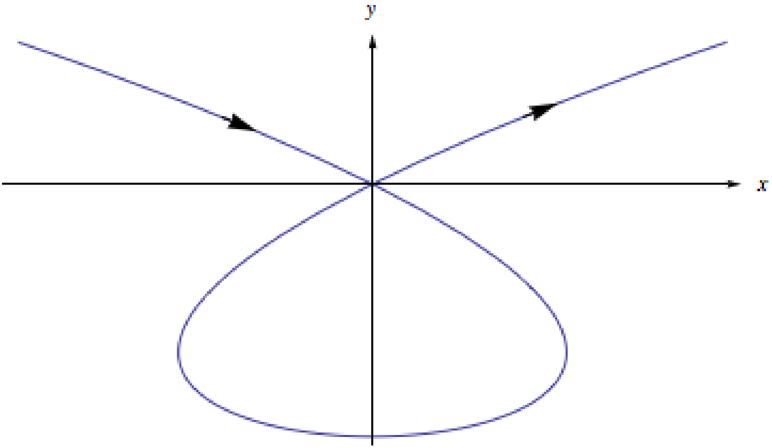
The map $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^3, t^2), t \in \mathbb{R}$.



Examples

Example 4

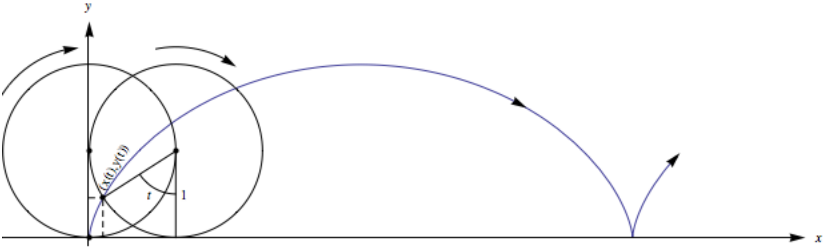
The map $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^3 - 4t, t^2 - 4), t \in \mathbb{R}$.



Examples

Example 5

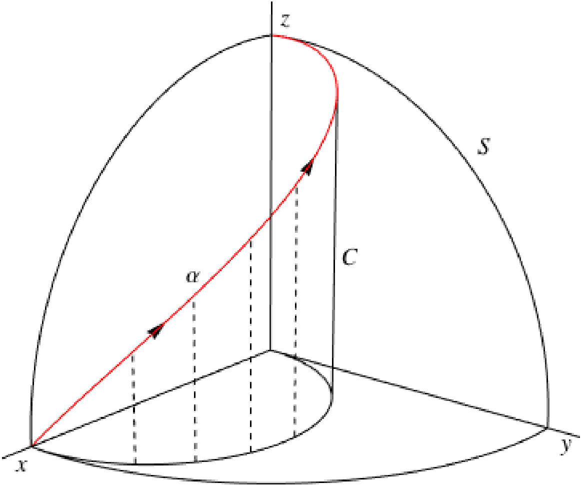
A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid*.



Examples

Example 6

The intersection of a sphere and a cylinder.



Arc Length of a Curve

Definition

Given $t \in I$, the *arc length* of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$, from the point t_0, is by definition

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| \, dt,$$

where

$$\|\alpha'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

is the length of the vector $\alpha'(t)$.

Definition

A parametrized curve $\alpha : I \to \mathbb{R}^3$ is said to be parametrized by arc length if $\|\alpha'(t)\| = 1$ (that is, if α has unit speed) for all $t \in I$.

Parametrization by Arc Length

Proposition (Geometric meaning of above definition)

A curve $\alpha : I \to \mathbb{R}^3$ is parametrized by arc length if and only if the parameter t is the arc length of α measured from some point.

Proof.

$\square$

Proposition (Advantages of $\|\alpha'(s)\| = 1$)

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length. Then $\alpha''(s)$ is orthogonal to $\alpha'(s)$ for all $s \in I$.

Proof.

$\square$

Reparametrization by Arc Length

Theorem

If α is a regular curve in $\mathbb{R}^3$, then there exists a reparametrization β of α such that β has unit speed.

Proof.

$\square$

Reparametrization by Arc Length

Example

Consider the helix $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by $\alpha(t) = (\cos t, \sin t, t)$.

- From now on, we are going to assume curves are parametrized by arc length.

Curvature

Geometric Meaning

Let $\alpha : I = (a, b) \to \mathbb{R}^3$ be a curve parametrized by arc length s. Since the tangent vector $\alpha'(s)$ has unit length, the norm $\|\alpha''(s)\|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s. $\|\alpha''(s)\|$ gives, therefore, a measure of how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s.

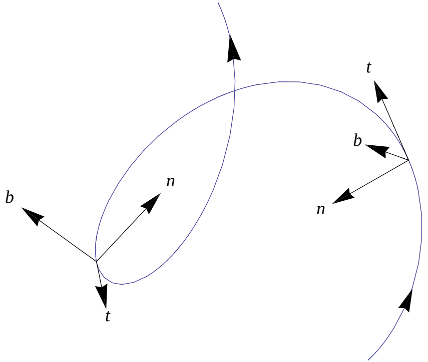
Definition

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $\|\alpha''(s)\| = k(s)$ is called the *curvature* of α at s.

Torsion

Geometric Meaning

Since $b(s)$ is a unit vector, the length $\|b'(s)\|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is $b'(s)$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s.



Special Cases

1.
$k(s) = \|\alpha''(s)\| = 0$ for any $s \in \mathbb{R}$ if and only if α is a straight line.

Proof.

□

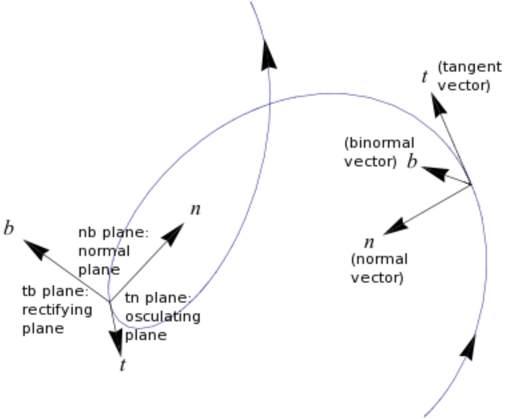
2.
If α is a plane curve, the $\tau = 0$. However, the converse does not necessarily hold.

Proof.

□

Frenet Frame

$$\alpha'(s) \stackrel{\triangle}{=} t(s)$$
$$\alpha''(s) = k(s)n(s)$$
$$t(s) \wedge n(s) = b(s)$$



Frenet Formulas

$$\begin{cases} t' = kn, \\ n' = -kt - \tau b, \\ b' = \tau n \end{cases}$$

Proof. $\square$

Fundamental Theorem of the Local Theory of Curves

Theorem

Given differentiable functions $k(s) > 0$ and $\tau(s), s \in I$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α Moreover, any other curve $\overline{\alpha}$ satisfying the same conditions differs from α by a rigid motion; that is, there exists an orthogonal map ρ of $\mathbb{R}^3$, with positive determinant, and a vector c such that $\overline{\alpha} = \rho \circ \alpha + c$.

Proof of uniqueness.

Claim: arc length, curvature, and torsion are invariant under the rigid motion. $\square$

Examples (helpful for HW)

Example 1

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Example 2

Show that the set of rigid motions forms a group.

Helpful for HW

- Idea on obtaining a parametrized equation for a cycloid.
- Hint on proving $SO(3)$ is a group.
- Techniques on parametrization by arc length, Frenet Formula, etc.

Example

Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.