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Differential Geometry
Homework 8
Monday, November 18 2019

C.a) Problem 5 on page 168, Section 3-3, Baby Do Carmo.

Consider the parametrized surface (Enneper's surface)

$$x(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

Show that

(a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0$$

(b) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0$$

(c) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = \frac{-2}{(1 + u^2 + v^2)^2}$$

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C.b) Problem 1 on page 185, Section 3-4, Baby Do Carmo.

Prove that the differentiability of a vector field does not depend on the choice of a coordinate system.

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C.c) Problem 2 on page 185, Section 3-4, Baby Do Carmo.

Prove that the vector field obtained on the torus by parametrizing all its meridians by arc length and taking their tangent vectors (Example 1) is differentiable.

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C.d) Problem 5 on page 186, Section 3-4, Baby Do Carmo.

Let S be a surface and let $x : U \to S$ be a parametrization of S. If $ac - b^2 < 0$, show that

$$a(u,v)(u')^2 + 2b(u,v)u'v' + c(u,v)(v')^2 = 0$$

can be factored into two distinct equations, each of which determines a field of directions on $X(U) \subset S$. Prove that these two fields of directions are orthogonal if and only if

$$Ec - 2Fb + Ga = 0$$

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C.e) Problem 8 on page 187, Section 3-4, Baby Do Carmo.

Show that if w is a differentiable vector field on a surface S and $w(p) \neq 0$ for some $p \in S$, then it is possible to parametrize a neighborhood of p by $x(u,v)$ in such a way that $x_u = w$.

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Extra Credit Problems

D.a) Problem 10 on page 187, Section 3-4, Baby Do Carmo.

Let T be the Torus of Example 6 of Sec. 2-2 and define a map $\phi : \mathbb{R}^2 \to T$ by

$$\phi(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u),$$

where u and v are the Cartesian coordinates of $\mathbb{R}^2$. Let $u = at, v = bt$ be a straight line in $\mathbb{R}^2$, passing by $(0, 0) \in \mathbb{R}^2$, and consider the curve in T $\alpha(t) = \phi(at, bt)$. Prove that

(a) ϕ is a local diffeomorphism.

(b) The curve $\alpha(t)$ is a regular curve; $\alpha(t)$ is a closed curve if and only if b/a is a rational number.

(c) If b/a is irrational, the curve $\alpha(t)$ is dense in T; that is, in each neighborhood of a point $p \in T$ there exists a point of $\alpha(t)$

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D.b) Problem 11 on page 187, Section 3-4, Baby Do Carmo.

Use the local uniqueness of trajectories of a vector field w in $U \subset S$ to prove the following result. Given $p \in U$, there exists a unique trajectory $\alpha : I \to U$ of w, with $\alpha(0) = p$, which is maximal in the following sense: Any other trajectory $\beta : J \to U$, with $\beta(0) = p$, is the restriction of α to J (i.e $J \subset I$ and $\alpha|_J = \beta$)

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D.c) Problem 12 on page 187, Section 3-4, Baby Do Carmo.

Prove that if w is a differentiable vector field on a compact surface S and $\alpha(t)$ is the maximal trajectory of w with $\alpha(0) = p \in S$, then $\alpha(t)$ is defined for all $t \in \mathbb{R}$.

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