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Math142 FA19
Homework 2
Due: Mon, Sep 30, 2019

Orthogonal Transformation Review - Lecture Let ρ and τ be two orthogonal transfomations on a Euclidean space $(V^n, <\ ,\ >)$. Prove that the composition of $\rho \circ \tau$ is again an orthogonal transformation of $(V, <\ ,\ >)$.

Recall that:

A linear Transformation $T : V \to V$ of an inner product space is an orthogonal transformation if for all $u, v \in V$ we have $< T(u), T(v) >=< u, v >$.

Given any $u, v \in V$ we have by definition of the composition

$$(\rho \circ \tau)(v) = \rho(\tau(v)) \quad \text{and} \quad (\rho \circ \tau)(v) = \rho(\tau(v))$$

So for all $u, v \in V$ we have

$$\begin{aligned} < \rho \circ \tau(u), \rho \circ \tau(v) > &=< \rho(\tau(u)), \rho(\tau(v) > && \text{by definition} \\ &=< \tau(u), \tau(v) > && \rho \text{ is orthogonal} \\ &=< u, v > && \tau \text{ is orthogonal} \end{aligned}$$

Therefore $\rho \circ \tau : V \to V$ is orthogonal. ∎

Orthogonal Transformation Review - Lecture Let ρ be an orthogonal transfomations on a Euclidean space $(V^n, <\ ,\ >)$. Prove that the composition of ρ^{-1} is again an orthogonal transformation of $(V, <\ ,\ >)$.

Firt we show that ρ has an inverse. In other words, ρ is one to one.
You should be able to see visually that all full rank (determinant not equal zero) linear transformations are one to one to one and orthogonal transformations are full rank. A more detailed proof is below.

We want to show that for all $u, v \in V^n$, $\rho(u) = \rho(v) \implies u = v$. Let $u, v \in V^n$ such $\rho(u) = \rho(v)$ be given. Now suppose we have $\{a_1, \cdots a_n\}$ as an orthonormal basis for V^n. Then u, v are just linear combinations of those basis vectors. Since the basis vectors is not only linearly indepedent but also orthogonal $\rho(u) = \rho(v)$ actually gives a very nice system of equations. Note that $u = u_1 a_1 + \cdots u_n a_n$ and similarly for $v = v_1 a_1 + \cdots v_n a_n$.

$$u_1 \rho(a_1) = v_1 \rho(a_1) \implies u_1 = v_1$$
$$\vdots$$
$$u_n \rho(a_n) = v_n \rho(a_n) \implies u_n = v_n$$

Therefore, $u = v$.
Now we show that ρ^{-1} is orthogonal. For any $u, v \in V$ we have that

$$\begin{aligned}
< \rho^{-1}(u), \rho^{-1}(v) > &=< \rho(\rho^{-1}(u)), \rho(\rho^{-1}(v)) > && \rho \text{ is orthogonal} \\
&=< (\rho \circ \rho^{-1})(u), (\rho \circ \rho^{-1})(v) > && \text{definition of composition} \\
&=< u, v > && \text{defining property of inverses}
\end{aligned}$$

which shows that ρ^{-1} is orthogonal.

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Problem 6

Some definitions:

- A translation by a vector $v \in \mathbb{R}^3$ is the map $A : \mathbb{R}^3 \to \mathbb{R}^3$ that is given by $A(p) = p + v$, where $p \in \mathbb{R}^3$.
- A linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation when $< \rho(u), \rho(v) >=< u, v >$ for all $u, v \in \mathbb{R}^3$.
- A rigid motion of $\mathbb{R}^3$ is the result of composing a translation with an orthogonal transformation with postive determinant (this last condition is included because we expect rigid motions to preserve orientation).

Problem 6(a) Section 1-5 - Do Carmo Show that both the norm of a vector and the angle θ between two vectors, where $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations.

Norm is invariant:

Suppose that $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transfromation and that $u \in \mathbb{R}^3$. By definition we have that

$$\|u\| = (< u, u >)^2$$

and since ρ is orthogonal we have

$$< \rho(u), \rho(u) >=< u, u >$$

It follows that

$$\|\rho(u)\| =< \rho(u), \rho(u) >=< u, u > \|u\|$$

Demonstrating that an orthogonal transformation of $\mathbb{R}^3$ preserves the norma of a vector $u \in \mathbb{R}^3$, i..e. the norm is invariant under orthogonal transformations.

By definition the orthogonal transformation ρ preserves inner product $< , >$, and we have just shown that ρ preserves the norm $\| \|$.

Angle is invariant:

By definition the angle θ between two vector $u, v \in \mathbb{R}^3$ is defined to be the unique angle θ such that:

$$0 \leq \theta \leq \pi \quad \text{and} \quad cos(\theta) = \frac{\|u\| \|v\|}{< u, v >}$$

Note that by the cauchy-schwartz inequality

$$| < u, v > | \leq \|u\| \|v\|$$

we have that

$$-1 \leq \frac{\|u\| \|v\|}{< u, v >} \leq 1$$

So if we write θ_1 for the angle between u and v, and write θ_2 for the angle between $\rho(u)$ and $\rho(v)$ then we have

$$cos(\theta_2) = \frac{\|\rho(u)\| \|\rho(v)\|}{< \rho(u), \rho(v) >} = \frac{\|u\| \|v\|}{< u, v >} = cos(\theta_1)$$

So since we require that both $-1 \leq \theta_1, \theta_2 \leq 1$ we have that

$$\theta_1 = \theta_2$$

demonstrating that an orthogonal transformation preserves the angle between two vectors, i.e. the angle between two vectors is invariant under orthogonal transformations.

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Problem 6(b) Section 1-5 - Do Carmo Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion true if we drop the condition on the determinant?

Trick question! The vector product is not invariant under orthogonal transformations with positive determinant. But the magnitude of the vector product is. Reacll that $\forall u, v \in V^n$, $||u \times v|| = ||u|| \cdot ||v|| \cdot sin(\theta)$ where θ is the angle between u and v. We showed in the last problem that the angle and magnitudes arre invariant under orthogonal transformation. So it follows that the magnitude of the vector product is invariant under orthogonal transformations. This will still be true if the determinant is not positive.

Problem 6(c) Section 1-5 - Do Carmo Show that the arc length, the curvature, and the torsion of aparametrized curve are (whenever defined) invariant under rigid motions.

c)

Arc Length

To prove that arc length is invariant under a rigid motion R, we must show that

$$\int_a^b \left|\frac{d\alpha}{dt}\right| dt = \int_a^b \left|\frac{d(R \circ \alpha)}{dt}\right| dt.$$

We note that the rigid motion $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$R(\mathbf{x}) = M\mathbf{x} + \mathbf{c},$$

where M is a constant matrix in $SO(3)$ and c is a real vector. Expanding our initial equation, we can then write

$$\begin{aligned}
\int_a^b \left|\frac{d\alpha}{dt}\right| dt &= \int_a^b \left|\frac{d}{dt}(M\alpha(t) + c)\right| dt \\
&= \int_a^b \left|M\alpha'(t)\right| dt && \text{(Multivariable product rule.)} \\
&= \int_a^b \sqrt{(M\alpha'(t))^T M\alpha'(t)}\, dt && \text{(Definition of magnitude.)} \\
&= \int_a^b \sqrt{(\alpha'(t)^T M^T M\alpha'(t)}\, dt \\
&= \int_a^b \sqrt{(\alpha'(t)^T \alpha'(t)}\, dt && \text{(Property of } SO(3).) \\
&= \int_a^b \left|\alpha'(t)\right| dt
\end{aligned}$$

which proves the desired claim.

Note to grader: We could have worked directly from our earlier lemmas regarding norm and angle, but I found that the matrix manipulation – which is equivalent – was easier and more compact. To use these matrices, should I have explained more how $SO(3)$ matrices are isomorphic to orthogonal transformations?

Curvature

Curvature, k, is defined by $\alpha''(t) = k\,n(t)$, where $n(t)$ is a unit vector. Thus, $k(t) = |\alpha''(t)|$ and we proceed to show that

$$\begin{aligned}
\left|\alpha''(t)\right| &= \left|\frac{d^2}{dt^2}(R \circ \alpha(t))\right| \\
&= \left|\frac{d^2}{dt^2}(M\alpha(t) + c)\right| \\
&= \left|\frac{d}{dt}(M\alpha'(t))\right| && \text{(Multivariable product rule.)} \\
&= \left|M\alpha''(t))\right| \\
&= \sqrt{(\alpha''(t)^T M^T M\alpha''(t)} \\
&= \sqrt{(\alpha''(t)^T \alpha''(t)} && \text{(Property of } SO(3).) \\
&= \left|\alpha''(t)\right|
\end{aligned}$$

as desired.

Torsion

Recall that torsion is defined by

$$b'(s) = \tau(s)n(s).$$

We can write that

$$\tau(s) = \frac{|b'(s)|}{|n(s)|},$$

since we know that $b'(s)$ is parallel to the normal vector. Substituting do Carmo's derivation of $b'(s)$, we get

$$\begin{aligned}\tau(s) &= \frac{|t(s) \wedge n'(s)|}{1} \\ &= |t(s)|\,|n'(s)|\,\sin\theta,\end{aligned}$$

where θ is the angle between the vectors $t(s)$ and $n'(s)$. We note here that, for plane curves, $\theta = 0$.

As above, we may apply the linearity of a rigid transformation R. We see that

$$\begin{aligned}\alpha(s) &\mapsto M\alpha(s) + c \\ \implies \alpha'(s) &\mapsto M\,\alpha'(s) \\ \implies \alpha''(s) &\mapsto M\,\alpha''(s)\end{aligned}$$

Above, we proved that curvature is invariant under rigid transformations; recall now that $\alpha''(s) = k(s)n(s)$ by definition. Therefore,

$$\begin{aligned}n(s) = \frac{1}{k(s)}\alpha''(s) &\mapsto \frac{1}{k(s)}M\,\alpha''(s) = Mn(s) \\ \implies n'(s) &\mapsto Mn'(s)\end{aligned}$$

The torsion of the new curve, computed using the transformed values of $t(s)$ and $n'(s)$, yields

$$\tau_{\text{transformed}}(s) = |M\,\alpha'(s)|\,|Mn'(s)|\,\sin\theta_{\text{transformed}}.$$

We have seen that orthogonal matrices such as M have a determinant of 1 and thus do not affect the norm of vectors; we have also seen that angles such as θ are unaffected by rigid transformations. Thus, we can conclude that

$$\begin{aligned}\tau_{\text{transformed}}(s) &= |M\,\alpha'(s)|\,|Mn'(s)|\,\sin\theta_{\text{transformed}} \\ &= |t(s)|\,|n'(s)|\,\sin\theta \\ &= \tau(s)\end{aligned}$$

which proves that torsion is invariant under rigid transformations. ∎

B.a) Show SO(n) is a group with respect to the usual matrix multiplication. (Later, we will see that SO(n) is in fact a Lie group.)

Recall that a set is a group if it is equipped with a binary operation that satisfies the axioms of closure, associativity, identity, and invertibility.

As described in class, $SO(n)$ is the set of all $n \times n$ orthogonal matrices with unit determinant. That is,

$$SO(n) = \left\{ M_n \mid M^T M = I, \ \det M = 1 \right\}.$$

■

Closure

To prove that $SO(n)$ is closed under matrix multiplication, take matrices $A, B \in SO(n)$. Then, the n matrix $AB \in SO(n)$ since the transpose of the product is the inverse of the product,

$$\begin{aligned} (AB)^T AB &= I \\ B^T A^T AB &= I, \end{aligned}$$

which is true since the inverses of A and B are their respective transposes. We must also show that $\det(AB) = 1$, which is true because $\det(AB) = \det A \det B = 1$. Thus, $AB \in SO(n)$, which proves closure.

Associativity

$SO(n)$ is associative under matrix multiplication because all matrices are associative under matrix multiplication, and all elements of $SO(n)$ can be represented as matrices.

Identity

There is an identity element in $SO(n)$. It is I_n. Observe that $I_n \in SO(n)$ since $I^T I = I$ and $\det I = 1$. Observe further that for all $A \in SO(n)$,

$$IA = A.$$

Invertibility

Suppose a matrix $A \in SO(n)$. We say it is invertible if there is an element $A^{-1} \in SO(n)$ such that $A^{-1}A = I$. From the definition of $SO(n)$, that element exists and, furthermore, is A^T.

Since $SO(n)$ satisfies closure, associativity, identity, and invertibility, $SO(n)$ is a group.

B.b) Show that the mirror reflection τ (as defined in the lecture) is an orthogonal transformation and $\tau^2 = id$, where id is the identity transformation.

In class, we defined $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\tau(x) = Mx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x,$$

which reflects the vector x across the xy-plane.

τ is an orthogonal transformation since $M \in O(3)$, that is, $M^T M = I$ and $\det M = \pm 1$. In this case, $\det M = -1$. It follows that τ is an orthogonal transformation since $O(3)$ is the set of all orthogonal transformations in $\mathbb{R}^3$.

$$M^T M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = I_3$$

$$\det M = (1)(1)(-1) = -1$$

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C Choose 2 problems out of following problems:

- a) Problem 3 on page 7, Section 1-3, Baby Do Carmo.
- b) Problem 5 on page 8, Section 1-3, Baby Do Carmo.
- c) Problem 6 on page 8, Section 1-3, Baby Do Carmo.

Choose 3 problems out of following problems:

- a) Problem 1 on page 22, Section 1-5, Baby Do Carmo.
- b) Problem 2 on page 22, Section 1-5, Baby Do Carmo.
- c) Problem 5 on page 23, Section 1-5, Baby Do Carmo.
- d) Problem 12 on page 25, Section 1-5, Baby Do Carmo.

The problems have been reproduced below. Please don't accidentally do them all! If this format is confusing, refer to the original problem set at https://weiqinggu.github.io/Math142/resources.html.

C, Part 1, a) Problem 3 on page 7, Section 1-3, Baby Do Carmo.

3. Let $0A = 2a$ be the diameter of a circle S^1 and $0Y$ and AV be the tangents to S^1 at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle S^1 at C and the line AV at B. On $0B$ mark off the segment $0p = CB$. If we rotate r about 0, the point p will describe a curve called the *cissoid of Diocles*. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that

a. The trace of

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \qquad t \in R,$$

is the cissoid of Diocles ($t = \tan\theta$; see Fig. 1-8).

b. The origin (0, 0) is a singular point of the cissoid.

c. As $t \longrightarrow \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \longrightarrow (2a, 0)$. Thus, as $t \longrightarrow \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an *asymptote* to the cissoid.

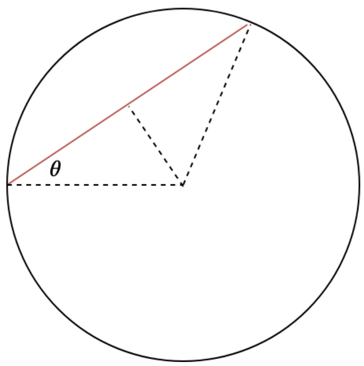

Figure 1: Proof that $OB = 2a\cos\theta$.

Observe that OC is a chord of a circle, and that $\overline{CB} = \overline{OB} - \overline{OC}$. We see in Fig. 1 that OC divides part of a sector of the circle into two right triangles; we may thus use elementary trig to show that $\overline{OB} = 2a\cos\theta$.

Similarly, we may observe that the entire length $\overline{OB}$ is given by $\frac{2a}{\cos\theta} = 2a\sec\theta$ by relating the sides of the right triangle OAB. Then,

$$\begin{aligned}\overline{CB} &= \overline{OB} - \overline{OC} \\ &= 2a\sec\theta - 2a\cos\theta \\ &= 2a(\sec\theta - \cos\theta)\end{aligned}$$

Recall from trigonometry that

$$\begin{aligned}\sec - \cos &= \frac{1}{\cos} - \cos \\ &= \frac{1}{\cos} - \frac{\cos^2}{\cos} \\ &= \frac{1 - \cos^2}{\cos} \\ &= \frac{\sin^2}{\cos} \\ &= \tan\theta\sin\theta\end{aligned}$$

where we alternate willy-nilly from typesetting θ. Thus,

$$\overline{CB} = 2a \tan\theta \sin\theta.$$

The point αt corresponds to a given angle θ, and in fact $\alpha(t)$ lies along the line

$$y = \tan\theta\, x$$

and the circle

$$x^2 + y^2 = (2a \tan\theta \sin\theta)^2$$

We solve the system by substitution. For x,

$$\begin{aligned} x^2 + y^2 &= (2a \tan\theta \sin\theta)^2 \\ x^2 \sec^2\theta &= 4a^2 \tan^2\theta \sin^2\theta \\ x^2 &= 4a^2 \tan^2\theta \sin^2\theta \cos^2\theta \end{aligned}$$

Recall now that

$$\sin^2 = \tan^2 \cos^2.$$

Then

$$\begin{aligned} x^2 &= 4a^2 \sin^4\theta \\ x &= \pm 2a \sin^2\theta \end{aligned}$$

We observe from the problem definition that we are only interested in those value $x > 0$, so we drop the $\pm$. Then, by being very clever, we see that

$$\sin^2 = \frac{\tan^2}{\csc^2} = \frac{\tan^2}{1 + \tan^2}$$

so that we may write

$$x = \frac{2a \tan^2\theta}{1 + \tan^2\theta}$$

or, taking $t = \tan\theta$,

$$x(t) = \frac{2at^2}{1 + t^2}$$

.

We now solve for y, noting that $x = \cot y$.

$$
\begin{aligned}
(\cot\theta y)^2 + y^2 &= 4a^2\tan^2\theta\sin^2\theta \\
y^2(1+\cot^2\theta) &= 4a^2\tan^2\theta\sin^2\theta \\
y^2 &= 4a^2\tan^2\theta\sin^2\theta\sin^2\theta \\
y &= \pm 2a\tan\theta\sin^2\theta
\end{aligned}
$$

Here, we note that either the positive or negative function could yield the desired curve as $\tan\theta$ varies through $(-\frac{\pi}{2}, \frac{\pi}{2}$, so we arbitrarily pick the positive function and drop the $\pm$. Thus,

$$
\begin{aligned}
y &= \frac{2a\tan^3\theta}{\csc^2\theta} \\
&= \frac{2a\tan^3\theta}{1+\tan^2\theta}
\end{aligned}
$$

Again, we let $t = \tan\theta$, so

$$y(t) = \frac{2at^3}{1+t^2}.$$

This yields the desired curve and

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right).$$

b)

Differentiating $\alpha(t)$ with Wolfram Alpha yields

$$\alpha'(t) = \left(\frac{4at}{(1+t^2)^2}, \frac{2at^2(t^2+3)}{(1+t^2)^2}\right).$$

Then,

$$\alpha'(t=0) = (0,0).$$

Thus, $\alpha(t=0) = (0,0)$ is a singular point.

c)

We confirm that

$$
\begin{aligned}
\lim_{t\to\infty}\alpha(t) &= \lim_{t\to\infty}\left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right) \\
&= (2a\,,\,\infty)
\end{aligned}
$$

and that

$$
\begin{aligned}
\lim_{t\to\infty}\alpha'(t) &= \lim_{t\to\infty}\left(\frac{4at}{(1+t^2)^2}, \frac{2at^2(t^2+3)}{(1+t^2)^2}\right) \\
&= (0, 2a).
\end{aligned}
$$

These two facts confirm that the function reaches an asymptote of the line $x = 2a$ at t tends toward infinity.

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C, Part 1, b) Problem 5 on page 8, Section 1-3, Baby Do Carmo.

5. Let $\alpha: (-1, +\infty) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

a. For $t = 0$, α is tangent to the x axis.

b. As $t \longrightarrow +\infty$, $\alpha(t) \longrightarrow (0, 0)$ and $\alpha'(t) \longrightarrow (0, 0)$.

c. Take the curve with the opposite orientation. Now, as $t \longrightarrow -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative to the line $y = x$ is called the *folium of Descartes* (see Fig. 1-10).

a)

Differentiation of $\alpha(t)$ yields

$$\alpha'(t) = \left(3a\frac{1-2t^3}{(1+t^3)^2}, 3a\frac{t(2-t^3)}{(1+t^3)^2}\right).$$

Then, since $\alpha'(t) = (1, 0)$, and $\alpha(0) = (0, 0)$, the curve α lies tangent to the x axis for $t = 0$.

b)

$$\begin{aligned}
\lim_{t\to\infty} \alpha(t) &= \lim_{t\to\infty} \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right) \\
&= (0, 0) \\
\lim_{t\to\infty} \alpha'(t) &= \lim_{t\to\infty} \left(3a\frac{1-2t^3}{(1+t^3)^2}, 3a\frac{t(2-t^3)}{(1+t^3)^2}\right) \\
&= (0, 0)
\end{aligned}$$

c)

To show that the curve approaches the line $x + y + a = 0$, we will show that the distance between the trace and the line approaches zero. We define the distance between the trace and the line l as

$$d = \left\|\alpha - \text{proj}_l \alpha\right\|.$$

Without loss of generality, we move our origin to the point $p = (-a, 0)$. As we showed above, distances are preserved under rigid body motion so this will not affect the distance function. Then,

$$d = \left\|(\alpha - p) - \text{proj}_l(\alpha - p)\right\|.$$

Since the line l now passes through the origin, we may take $\text{proj}_l(\alpha - p)$ to be the projection onto the vector $v = (-1, 1)^T$ which gives the slope of l, so

$$\begin{aligned}
d &= \left\|(\alpha - p) - \text{proj}_v(\alpha - p)\right\| \\
&= \left\|(\alpha - p) - \frac{v \cdot (\alpha - p)}{v \cdot v} v\right\| \\
&= \left\|(\alpha - p) - \frac{vv^T}{v^T v}(\alpha - p)\right\|
\end{aligned}$$

where we switch to using the projection matrix rather than the dot product form. Take a moment to reassure yourself that, indeed,

$$\frac{v \cdot x}{v \cdot v}v = \frac{vv^T}{v^T v}x = \left(\frac{1}{v^T v}\right) vv^T x = \left(\frac{1}{v^T v}\right) v(v^T x) = \left(\frac{1}{v \cdot v}\right) v(v \cdot x) = \frac{v \cdot x}{v \cdot v}v.$$

Continuing with d, we observe that

$$\begin{aligned} d &= \left\| \left(I - \frac{vv^T}{v^T v}\right)(\alpha - p) \right\| \\ &= \sqrt{(\alpha - p)^T \left(I - \frac{vv^T}{v^T v}\right)^T \left(I - \frac{vv^T}{v^T v}\right)(\alpha - p)} \\ &= \sqrt{(\alpha - p)^T \left(I^T I - I^T \frac{vv^T}{v^T v} - \left(\frac{vv^T}{v^T v}\right)^T I + \left(\frac{vv^T}{v^T v}\right)^T \frac{vv^T}{v^T v}\right)(\alpha - p)}. \end{aligned}$$

We note here that

$$\left(\frac{vv^T}{v^T v}\right)^T = \frac{vv^T}{v^T v}$$

as the denominator is a scalar and in the numerator $(vv^T)^T = vv^T$. Therefore,

$$\left(\frac{vv^T}{v^T v}\right)^T \frac{vv^T}{v^T v} = \left(\frac{vv^T}{v^T v}\right) \frac{vv^T}{v^T v} = \frac{vv^T vv^T}{(v^T v)(v^T v)} = \frac{v(v^T v)v^T}{(v^T v)(v^T v)} = \frac{vv^T}{v^T v}$$

and

$$\begin{aligned} d &= \sqrt{(\alpha - p)^T \left(I - 2\frac{vv^T}{v^T v} + \frac{vv^T}{v^T v}\right)(\alpha - p)} \\ &= \sqrt{(\alpha - p)^T \left(I - \frac{vv^T}{v^T v}\right)(\alpha - p)} \end{aligned}$$

.

Here, we substitute $p = (-a, 0)^T$, $\alpha = (x, y)^T$, and $v = (-1, 1)^T$.

$$
\begin{aligned}
d &= \sqrt{(\alpha - p)^T \left(I - \frac{vv^T}{v^T v} \right) (\alpha - p)} \\
&= \sqrt{\begin{pmatrix} x+a & y \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix}}{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}} \right) \begin{pmatrix} x+a \\ y \end{pmatrix}} \\
&= \sqrt{\begin{pmatrix} x+a & y \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) \begin{pmatrix} x+a \\ y \end{pmatrix}} \\
&= \sqrt{\frac{1}{2} \begin{pmatrix} x+a & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x+a \\ y \end{pmatrix}} \\
&= \sqrt{\frac{1}{2} \begin{pmatrix} x+a & y \end{pmatrix} \begin{pmatrix} x+a+y \\ x+a+y \end{pmatrix}} \\
&= \sqrt{\frac{1}{2} \left[(x+a)(x+a+y) + y(x+a+y) \right]} \\
&= \sqrt{\frac{1}{2} \left[x^2 + 2ax + yxy + 2ay + a^2 + y^2 \right]} \\
&= \sqrt{\frac{1}{2} (x+a+y)^2} \\
&= \frac{|x+a+y|}{\sqrt{2}}
\end{aligned}
$$

We see in spectacular hindsight that we could have identified the quadratic form

$$\begin{pmatrix} x+a & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x+a \\ y \end{pmatrix} = (x+a+y)^2.$$

We see in spectacularer hindsight that this is simply a case of the distance-from-a-line formula, given on Wikipedia by

$$d = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$

which would have yielded the same result in a single step.

Regardless, we now have a convenient closed-form solution for the distance of an arbitrary point from the line. We thus confirm that

$$
\begin{aligned}
\lim_{t \to -1} d(t) &= \lim_{t \to -1} \frac{|x(t) + a + y(t)|}{\sqrt{2}} \\
&= \lim_{t \to -1} x(t) + a + y(t) \\
&= \lim_{t \to -1} \frac{3at}{1+t^3} + a + \frac{3at^2}{1+t^3} \\
&= \lim_{t \to -1} \frac{3a(t+t^2)}{1+t^3} + a \\
&= \lim_{t \to -1} \frac{3a(1+t)t}{1+t^3} + a \\
&= \lim_{t \to -1} \frac{3at}{t^2 - t + 1} + a \\
&= \lim_{t \to -1} -a + a \\
&= 0
\end{aligned}
$$

which shows that the curve approaches the line. Similarly, we show that its tangent approaches the line since the tangent's angle from the x-axis goes to

$$
\begin{aligned}
\lim_{t \to -1} \arctan \alpha'(t) &= \lim_{t \to -1} \arctan \left(\frac{3a \frac{t(2-t^3)}{(1+t^3)^2}}{3a \frac{1-2t^3}{(1+t^3)^2}} \right) \\
&= \lim_{t \to -1} \arctan \left(\frac{t(2-t^3)}{1-2t^3} \right) \\
&= \arctan -1
\end{aligned}
$$

which places the tangent parallel with any line of slope -1.

■

C, Part 1, c) Problem 6 on page 8, Section 1-3, Baby Do Carmo.

6. Let $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$, $t \in R$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

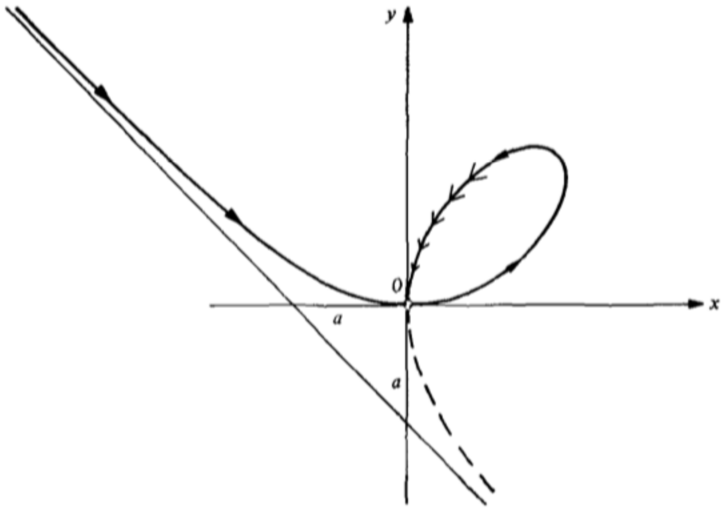

Figure 1-10. Folium of Descartes.

a. Show that as $t \longrightarrow +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the *logarithmic spiral*; see Fig. 1-11).

b. Show that $\alpha'(t) \longrightarrow (0, 0)$ as $t \longrightarrow +\infty$ and that

$$\lim_{t \to +\infty} \int_{t_0}^{t} |\alpha'(t)|\, dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

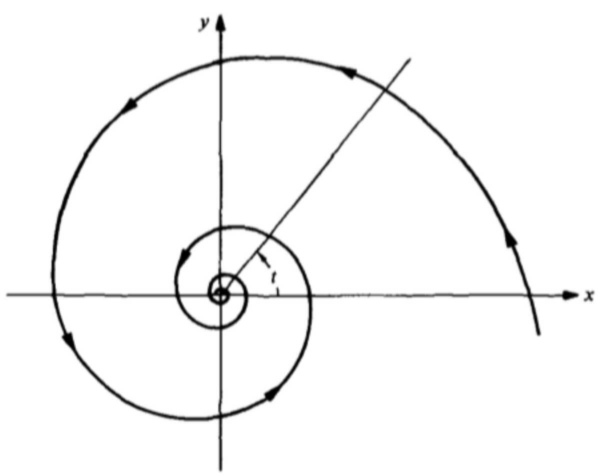

Figure 1-11. Logarithmic spiral.

I didn't do this one.

C, Part 2, a) Problem 1 on page 22, Section 1-5, Baby Do Carmo.

Unless explicity stated, $\alpha: I \longrightarrow R^3$ *is a curve parametrized by arc length* s, *with curvature* $k(s) \neq 0$, *for all* $s \in I$.

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \qquad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

a)

Differentiating yields

$$t(s) = \alpha'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$$

with unit norm

$$|\alpha'(s)| = \sqrt{\frac{a^2}{c^2}\sin^2\left(\frac{s}{c}\right) + \frac{a^2}{c^2}\cos^2\left(\frac{s}{c}\right) + \frac{b^2}{c^2}} \qquad = \sqrt{\frac{a^2}{c^2} + \frac{b^2}{c^2}} = \sqrt{\frac{a^2+b^2}{c^2}} \qquad = 1$$

so $\alpha(s)$ is parameterized by arc length.

b)

To find the curvature, we find

$$\alpha''(s) = \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right)$$

and identify the curvature

$$k(s) = \frac{\alpha''(s)}{n(s)} = \|\alpha''(s)\| = \frac{a}{c^2}.$$

Then, the normal vector is

$$n(s) = \frac{\alpha''(s)}{k(s)} = \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right).$$

We compute the binormal vector

$$\begin{aligned}
b(s) &= t(s) \wedge n(s) \\
&= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right) \wedge \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right) \\
&= \begin{vmatrix} i & j & k \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\cos\left(\frac{s}{c}\right) & -\sin\left(\frac{s}{c}\right) & 0 \end{vmatrix} \\
&= \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\sin^2\left(\frac{s}{c}\right) + \frac{a}{c}\cos^2\left(\frac{s}{c}\right)\right) \\
&= \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right)
\end{aligned}$$

which has derivative

$$\begin{aligned}
b'(s) &= \left(\frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right) \\
&= -\frac{b}{c^2}n(s)
\end{aligned}$$

which tells gives the curvature

$$\tau(s) = \frac{b'(s)}{n(s)} = -\frac{b}{c^2}.$$

c)

The osculating plane is

$$P(s) = \left\{x \in \mathbb{R}^3 \mid (\alpha(s) - x) \cdot b(s) = 0\right\},$$

where $\alpha(s)$ and $b(s)$ are given above. Performing the arithmetic gives

$$\frac{ab}{c^2}s = \frac{b}{c}\sin\left(\frac{s}{c}\right)x - \frac{b}{c}\cos\left(\frac{s}{c}\right)y + \frac{a}{c}z$$

which is easier to interpret in terms of component functions x, y, and z.

d) First, we show that a line containing $n(s)$ and $\alpha(s)$ meets the z-axis. This is the case if

$$m(s)n(s) + \alpha(s) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t(s)$$

for some real scalars m and t. This can be written as a linear system parameterized by s,

$$\begin{pmatrix} n_x(s) & 0 \\ n_y(s) & 0 \\ n_z(s) & -1 \end{pmatrix} \begin{pmatrix} m(s) \\ t(s) \end{pmatrix} = -\begin{pmatrix} \alpha_x(s) \\ \alpha_y(s) \\ \alpha_z(s) \end{pmatrix}$$
$$\begin{pmatrix} -\cos\left(\frac{s}{c}\right) & 0 \\ -\sin\left(\frac{s}{c}\right) & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} m(s) \\ t(s) \end{pmatrix} = -\begin{pmatrix} a\cos\left(\frac{s}{c}\right) \\ a\sin\left(\frac{s}{c}\right) \\ b\frac{s}{c} \end{pmatrix}$$

It is apparent from the equation above that the x and y components of $n(s)$ and $\alpha(s)$ are always linearly dependent by the same scalar multiple, and thus that a solution $(m(s), t(s))$ will exist for any real value of s. This implies that the normal line will intersect the z-axis.

Now, to show that the angle between them is $\frac{\pi}{2}$ we show that n is orthogonal to the z axis.

$$n \cdot \hat{z} = \begin{pmatrix} -\cos\left(\frac{s}{c}\right) \\ -\sin\left(\frac{s}{c}\right) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

At any value of s, we have that a line through n meets the z axis at an angle of $\frac{\pi}{2}$, as desired.

e)

The tangent lines to α have direction

$$t(s) = \alpha'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$$

. Because the z-component is constant, the projection of the unit tangent vector t onto the z-axis is constant, which means that the tangent line makes a constant angle with the z-axis. ∎

C, Part 2, b) Problem 2 on page 22, Section 1-5, Baby Do Carmo.

***2.** Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

Recall that

$$\begin{aligned}
\alpha' &= t \\
\alpha'' &= kn \\
\alpha''' &= kn' + k'n \\
&= -k^t - k\tau b + k'n.
\end{aligned}$$

Now we will show (rather than derive) that the given equation is true.

$$\begin{aligned}
\tau &= -\frac{\alpha' \wedge \alpha'' \cdot \alpha'''}{|k|^2} \\
&= -\frac{t \wedge kn \cdot (-k^2 t - k\tau b + k'n)}{|k|^2} \\
&= -\frac{kb \cdot (-k^2 t - k\tau b + k'n)}{|k|^2}.
\end{aligned}$$

We choose a natural basis in which to evaluate the dot product: the Frenet frame t, n, b. Then,

$$\begin{aligned}
\tau &= -\frac{k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -k^2 t \\ k' \\ -k\tau \end{pmatrix}}{|k|^2} \\
&= \frac{k^2 \tau}{|k|^2} \\
&= \tau
\end{aligned}$$

as desired. ■

C, Part 2, c) Problem 5 on page 23, Section 1-5, Baby Do Carmo.

5. A regular parametrized curve α has the property that all its tangent lines pass through a fixed point.

a. Prove that the trace of α is a (segment of a) straight line.

b. Does the conclusion in part a still hold if α is not regular?

Assume without loss of generality that α is parameterized by arc length. Let p be the fixed point. A tangent line l at $\alpha(s)$ is the line with direction $\alpha'(s)$ and passing through the point $\alpha(s)$,

$$l(x) = \alpha(s) + x\alpha'(s).$$

By hypothesis, for each choice of s there exists some $x(s)$ such that

$$\alpha'(s)x(s) + \alpha(s) = p.$$

Differentiating with respect to s yields

$$\begin{aligned} \alpha'(s) + \alpha''(s)x(s) + \alpha'(s)x'(s) &= 0 \\ (1 + x'(s))\alpha'(s) + x(s)\alpha''(s) &= 0 \end{aligned}$$

Since $\alpha(s)$ is parameterized by arc length, $\alpha'(s) \perp \alpha''(s)$. Thus, the equation above describes a linear independent system. Since $\alpha'(s) \neq 0$, the solutions of the homogeneous system apply whenever

$$\begin{cases} 1 + x'(s) = 0 \\ x(s) = 0 \end{cases} \quad \text{or} \quad \begin{cases} 1 + x'(s) = 0 \\ \alpha''(s) = 0. \end{cases}$$

The first case can not apply to all s, since it would form a contradiction (the latter equation is impossible if the initial equation is true). The second case is therefore the more general solution.

Since $\alpha''(s) = 0$, the curve has a constant tangent line and therefore forms a segment of a straight line.

The $1 + x'(s) = 0$ condition simply describes that the curve approaches and departs the point p at unit speed, which makes sense given that we have chosen to parameterize the curve by arc length.

∎

C, Part 2, d) Problem 12 on page 25, Section 1-5, Baby Do Carmo.

12. Let $\alpha: I \longrightarrow R^3$ be a regular parametrized curve (not necessarily by arc length) and let $\beta: J \longrightarrow R^3$ be a reparametrization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$ (see Remark 2). Let $t = t(s)$ be the inverse function of s and set $d\alpha/dt = \alpha'$, $d^2\alpha/dt^2 = \alpha''$, etc. Prove that

a. $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha''/|\alpha'|^4)$.

b. The curvature of α at $t \in I$ is

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}.$$

c. The torsion of α at $t \in I$ is

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

d. If $\alpha: I \longrightarrow R^2$ is a plane curve $\alpha(t) = (x(t), y(t))$, the signed curvature (see Remark 1) of α at t is

$$k(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}.$$

I didn't do this problem.

D. EXTRA CREDIT!

- Problems 7, 8 on page 22-23, Section 1-5, Baby Do Carmo.

***7.** Let $\alpha: I \to R^2$ be a regular parametrized plane curve (arbitrary parameter), and define $n = n(t)$ and $k = k(t)$ as in Remark 1. Assume that $k(t) \neq 0$, $t \in I$. In this situation, the curve

$$\beta(t) = \alpha(t) + \frac{1}{k(t)} n(t), \qquad t \in I,$$

is called the *evolute* of α (Fig. 1-17).

a. Show that the tangent at t of the evolute of α is the normal to α at t.

b. Consider the normal lines of α at two neighboring points $t_1, t_2, t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the trace of the evolute of α.

8. The trace of the parametrized curve (arbitrary parameter)

$$\alpha(t) = (t, \cosh t), \qquad t \in R,$$

is called the *catenary*.

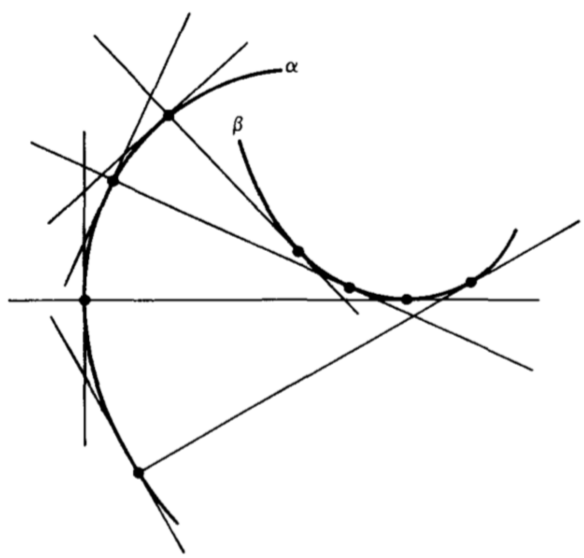

Figure 1-17

a. Show that the signed curvature (cf. Remark 1) of the catenary is

$$k(t) = \frac{1}{\cosh^2 t}.$$

b. Show that the evolute (cf. Exercise 7) of the catenary is

$$\beta(t) = (t - \sinh t \cosh t,\ 2 \cosh t).$$