Real insight of lienar transformation

Let V^n be a vector space with basis $\{v_1, v_2, \cdots, v_n\}$
Let W^n be another wector space with basis $\{w_1, w_2, \cdots, w_n\}$
<u>Definition</u> A function $L : V^n \to W^m$ is linear if and only if, for all $x \in V^n$,

$$L(x + y) = L(x) + L(y)$$
$$L(\gamma x) = \gamma L(x), \forall \gamma \in \mathbb{R}$$

All linear transformations have a matrix reprresentation. The matrix for L is $(L(v_1), L(v_2), \cdots, L(v_n))$ The matrix just tells where the basis vectors will go and the other vectors follow in a linear way. if you don't see this visually please see the animation at https://youtu.be/kYB8IZa5AuE?t=72.

Steps to visualize diagonalization

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if it can be written in the form $A = PDP^{-1}$ where P is a invertable n by n matrix, and D is a diagonal matrix.

1. If you can't already visualize the eigenvectors watch at least the first 5 minutes and 15 seconds of https://youtu.be/PFDu9oVAE-g.
2. P always has the eigenvectors of A as the columns. So it is a change of basis into the eigenbasis. That is, it moves the original basis vectors (e.g. the x and y axes) to the eigenvectors.
3. D is a diagonal matrix with the eigenvalues along it's diagonal. That means it has the affect of stretching alongst the eigenvectors.
4. P^{-1} just moves us back into our original basis. That is, it moves the eigenvectors to the original basis vectors (e.g. the x and y axes).

Symetric matrices are diagonalizable

Let $A \in \mathbb{R}^n$ be a symmetric matrix. Symmetric matrices are orthogonally diagonalizable so A has n distinct real eigenvectors which are all orthogonal to one another. So let the eigenvalues of A be $\lambda_1, \lambda_2, \cdots, \lambda_n$. The proof of this involves first showing algebraicly that it works for 2 by 2 symetric matrices and then using induction to show it works for larger symetric matrices.

Eigenvalues of A^{-1}

Given an eigenvalue and eigenvector for A we can find a corresponding eigenvalue and eigenvector for A^{-1}. Suppose that λ, x are an eigenvalue and eigenvector of A. Then

$$\begin{aligned} Ax &= \lambda x \\ \implies A^{-1}Ax &= \lambda A^{-1}x \\ \implies A^{-1}x &= \frac{1}{\lambda}x \end{aligned}$$

So x is also an eigenvector of A^{-1} and the corresponding eigenvalue is $1/\lambda$. Repeat this process for each of the n eigenvectors of A to find that the n eigenvalues of A^{-1} are $1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_n$. This fact should also be pretty intuitive since eigenvalues are used to multiply and division is the inverse of mulitplication.

the determinant of A^{-1}

The determinant of a matrix is equal to the product of the eigenvalues. To gain an intuition for why this is true you must recall that the determinant is the volume of the parallelotype sppaned by the columns of the matrix. You also need a visual understanding of the eigenvalues. By using this fact in combination with the last slide we derive this useful formula,

$$det(A^{-1}) = \frac{1}{\lambda_1}\frac{1}{\lambda_2}\cdots\frac{1}{\lambda_n} = \frac{1}{\lambda_1\lambda_2\cdots\lambda_n} = \frac{1}{det(A)}$$

Eigenvalues of A^2

This is very similar to what we did for A^{-1}

$$AAx = A\lambda x = \lambda\lambda x = \lambda^2 x$$

Therefore, the corresponding eigenvalues of A^2 is $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$. More generally, $A^k x = \lambda^k x$. In fact, we can generalize this to any polynomial of A. Let P be a function that takes some polynomial. So $P(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$. Then the eigenvalues of $P(A)$ are $P(\lambda_1), P(\lambda_2), \cdots, P(\lambda_n)$

Matrix exponent Suppose A is diagonalized with $A = PDP^{-1}$. Then $A^n = (PD\cancel{P^{-1})(P}DP^{-1}) \cdots (PDP^{-1}) = PD^nP^{-1}$

Similar matrices Matrices A, B are similar if there exists P such that $B = P^{-1}AP$. Then $det(B) = det(P^{-1}AP) = det(P^{-1})det(A)det(P) = \frac{1}{det(P)}det(A)det(P) = det(A)$. Then the determinant of A is the product of the eigenvalues of A.

inner product once you define the inner product on a vector space, the angles and norm follow. $||v|| = v \cdot v$. Angle between vectors u, v is $cos^{-1}(\frac{u \cdot v}{||u|| \cdot ||v||})$

A tensor can take multiple vectors as input and is multilinear.
For example $T : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a tensor. The inner product is also a tensor.
We call a finite dimensional vector space with an inner product a euclidean space.

Matrix representation of dot product with any basis

Let V^2 be a vector space with basis vectors v_1, v_2. For all, $v, w \in V^2$ where $v = a_1 v_1 + a_2 v_2, w = b_1 v_1 + b_2 v_2$,

$$\begin{aligned} < v, w > &=< a_1 v_1 + a_2 v_2, w > \\ &= a_1 < v_1, w > + a_2 < v_2, w > \\ &= (a_1, a_2) \boxed{\begin{pmatrix} < v_1, v_1 > & < v_1, V_2 > \\ < v_2, v_1 > & < v_2, v_2 > \end{pmatrix}} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{aligned}$$

The boxed matrix is the matrix representation of the dot product with respect to the basis vectors $\{v_1, v_2\}$. Let's call this matrix C

Theorem The eigenvalues of C are all greater than 0:
Proof: First we show that C is positive definite. Suppose $v = w$ so that $a_1 = b_1, a_2 = b_2$. Then,

$$< v, w >=< v, v >= v^T Cv$$

Suppose $v > 0$. Then $< v, v >= |v| > 0$. Then for all $v > 0$,

$$v^T Cv > 0$$

So C is positive definite. Therefore, C has all positive eigenvalues. Recall that all positive definite matrices have only positive eigenvalues. You should be able to see why visually.

Advantage of orthonormal basis

The orthonormal basis is more convenient because the matrix representation of the dot product becomes,

$$\begin{pmatrix} < v_1, v_1 > & < v_1, V_2 > \\ < v_2, v_1 > & < v_2, v_2 > \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So then the dot product between v and w becomes just what you'd expect: $a_1 b_1 + a_2 b_2$.

Projective geometry

Projective geometry is the study of geometric properties that are invariant with respect to projective transformations. For computer visions this means properties of an image that don't depend on the viewpoint of the camera.

In projective space, parallel lines meet at infinity.

Definition A projective space of dimension n is the set of the vector lines (that is, vector subspaces of dimension one) in a vector space V of dimension $n+1$. Equilvalently, it is the quotient set of $V \setminus \{0\}$ by the equivalence relation of being on the same vector line. As a vector line intersects the unit sphere of V in two antipodal points, projective spaces can be equivalently defined as spheres in which antipodal points are identified. A projective space of dimension 1 is a projective line. A projective space of dimension 2 is a projective plane.

Q: How many lines pass through the origin $\mathbb{R}^n$?. The number of lines is equal to $|\mathbb{R}^1|$.

In homogeneous coordinates, vectors that diiffer only by scale are considered to be equivalent. In homogeneous coordinates you can always use rotations instead of translations. To convert a homegenous vector to an inhomogenous vector just divide by the last element of the vector. So given a homogenous vectors $\widetilde{x} = (x_1, x_2, w)$, the inhmoegnous coordinate is $(x_1/w, x_2/w)$.
Definition A projective (or homogenous) transformation is any invertable matrix $\widetilde{H}$ applied to a homogenous coordinate.
Homogeneous coordinates are ubiquitous in 3d graphics. Here's a link to an alternate explanation of homogenous coordinates along with an explanation of it's use in 3d computer graphics. There is a related set of interactive demos for homogenous coordinates

closest point on the line derived geometriclly and then with calculus

We can normalize the equation for any line in order to derive the normal vector and distance to the origin, $ax + by + c = 0 \implies \frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y + \frac{c}{\sqrt{a^2+b^2}} = 0$. Let $v = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})$. I claim that v is the normal vector to the plain. This is true because the plane is parallel to the kernel space of v. Recall, the kernel space is orthogonal. Theorem The distance from the line to the origin is $\frac{c}{\sqrt{a^2+b^2}}$.

Proof: We want to find the point on the line (x, y) that is closest to the origin. Then the distance from the origin to the line is just $||(x, y)||$. We can find this by using the lagrange multiplier to minimize $||(x, y)||$ under the constraint that $ax + by + c = 0$. Then the closest point is $(\frac{ac}{(a^2+b^2)}, \frac{bc}{(a^2+b^2)})$. So the distance to the line is,

$$\sqrt{\frac{a^2c^2}{(a^2+b^2)^2} + \frac{b^2c^2}{(a^2+b^2)^2}} = \frac{c}{\sqrt{a^2+b^2}}$$

Make sure to notice how this proof is totally consistent with what we found geometrically on the previous slide. This can be generlized to hyperplanes in higher dimension.