Joseph Gardi
Differential Geometry
Homework 6
Monday, November 4 2019

A: Read:

- Baby Do Carmo, Differential Geometry of Curves and Surfaces: Sections 2-4, 2-5, 2-6 and Section 5-10 on Abstract surfaces (starting on page 425)
- Handouts 8 and 9
- Lecture Notes

B: Problems from Lectures

a) Let S be a subset of R^3. Show that S is a regular surface if and only if S is locally diffeomorphic to R^2.

b) Find five examples of regular surfaces such that each of them can be represented as a surface of revolution. Write down specifically for each example the generating curve, the rotation axis, and the parameterization (as a map) for the surface (including the domain of the map).

C: Other Problems

a) Problem 10 on page 81, Section 2-3, Baby Do Carmo.

10. Let C be a plane regular curve which lies in one side of a straight line r of the plane and meets r at the points p, q (Fig. 2-21). What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution?

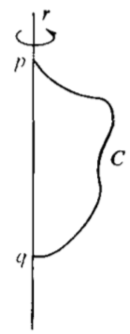

Figure 2-21

b) Problem 9 on page 89, Section 2-4, Baby Do Carmo.

9. Show that the parametrized surface

$$\mathbf{x}(u, v) = (v \cos u, v \sin u, au), \qquad a \neq 0,$$

is regular. Compute its normal vector $N(u, v)$ and show that along the coordinate line $u = u_0$ the tangent plane of $\mathbf{x}$ rotates about this line in such a way that the tangent of its angle with the z axis is proportional to the distance $v\,(= \sqrt{x^2 + y^2})$ of the point $\mathbf{x}(u_0, v)$ to the z axis.

c) Problem 15 on page 90, Section 2-4, Baby Do Carmo.

15. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

d) Problem 18 on page 90, Section 2-4, Baby Do Carmo.

18. Prove that if a regular surface S meets a plane P in a single point p, then this plane coincides with the tangent plane of S at p.

e) Problem 1 on page 99, Section 2-5, Baby Do Carmo.

1. Compute the first fundamental forms of the following parametrized surfaces where they are regular:

a. $\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$; ellipsoid.

b. $\mathbf{x}(u, v) = (au \cos v, bu \sin v, u^2)$; elliptic paraboloid.

c. $\mathbf{x}(u, v) = (au \cosh v, bu \sinh v, u^2)$; hyperbolic paraboloid.

d. $\mathbf{x}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$; hyperboloid of two sheets.

We will find the $\mathbf{x}_u, \mathbf{x}_v$ for each problem. From there we find $E =< \mathbf{x}_u, \mathbf{x}_u >, F- < \mathbf{x}_u, \mathbf{x}_v >, G =< \mathbf{u}_v, \mathbf{u}_v >$. Then the first fundamental form is,

$$I_p((u', v')) = E(u')^2 + 2Fu'v' + G(v')^2$$

where p is a point on the surface.

(a) $\mathbf{x}_u = (a \cos u \cos v, b \cos u \sin v, -c \sin u), \mathbf{x}_v = (-a \sin u \sin v, \sin u \cos v, 0)$.

(b) $\mathbf{x}_u = (a \cos v, b \sin v, 2u), \mathbf{x}_v = (-au \sin v, bu \cos v, 0)$

■

f) Problem 3 on page 99, Section 2-5, Baby Do Carmo.

3. **Obtain the first fundamental form of the sphere in the parametrization given by stereographic projection (cf. Exercise 16, Sec. 2-2).**

The parameterization for the surface is,

$$\pi^{-1}(u,v) = \begin{bmatrix} \frac{4u}{u^2+v^2+4} \\ \frac{4v}{u^2+v^2+4} \\ \frac{2(u^2+v^2)}{u^2+v^2+4} \end{bmatrix}$$

Then,

$$\boldsymbol{\pi^{-1}}_u = \begin{bmatrix} \frac{4}{u^2+v^2+4} - \frac{4u}{(u^2+v^2+4)^2} \cdot 2u \\ -\frac{4v}{(u^2+v^2+4)^2} \cdot 2u \\ \frac{4u}{u^2+v^2+4} - \frac{2(u^2+v^2)}{(u^2+v^2+4)^2} \cdot 2u \end{bmatrix}$$

$$\boldsymbol{\pi^{-1}}_v = \begin{bmatrix} -\frac{4u}{(u^2+v^2+4)^2} \cdot 2v \\ \frac{4}{u^2+v^2+4} - \frac{4v}{(u^2+v^2+4)^2} \cdot 2v \\ \frac{4v}{u^2+v^2+4} - \frac{2(u^2+v^2)}{(u^2+v^2+4)^2} \cdot 2v \end{bmatrix}$$

Then calcualate the first fundamental form as described in the last problem.

g) Problem 9 on page 100, Section 2-5, Baby Do Carmo.

***9.** Show that a surface of revolution can always be parametrized so that

$$E = E(v), \qquad F = 0, \qquad G = 1.$$

D: Extra Credit Problems

a) Let $T \subset R^3$ be a torus of revolution with center in $(0,0,0) \in R^3$ and let $A(x,y,z) = (-x,-y,-z)$. Let K be the quotient space of the torus T by the equivalence relation $p \sim A(p)$. Can you tell what surface K is?

b) Show that K is a differentiable 2-dimensional manifold.

c) Show that K is non orientable in two different ways.