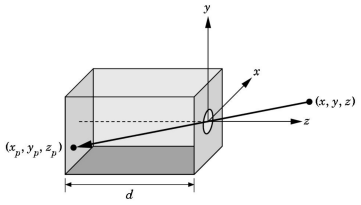
CS3241

INTRO

Image formation

Elements Objects | Viewer | Light sources | Materials

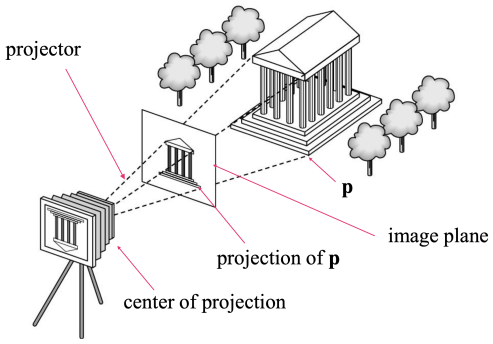
Pinhole camera



Use trigonometry to find **projection** of 3D point at (x, y, z)

$x_p = -dx/z \qquad y_p = -dy/z \qquad z_p = -d$

Synthetic camera model



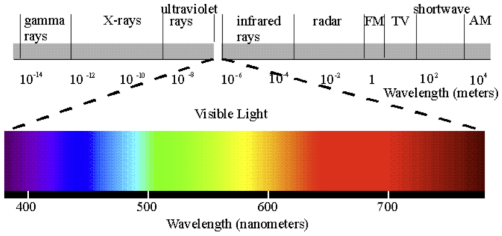
Luminance and color

Monochromatic images Grayscale values stored

Color images Hue | Saturation | Lightness

Light

- Visible spectrum with wavelenght in range 350-750 nanometers
- Long wavelengths are red, short wavelengths are blue



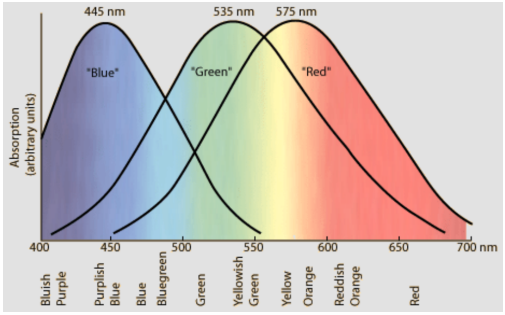
Three-color theory

Human visual system has two types of light sensors

- Rods: good for monochromatic, night vision
- Cones: sensitive to color

Cones

- Three types of cones, sensitive to different wavelengths
- Use RGB to excite the cones
- Humans are more sensitive to green, then red, then blue. If there are only 8 bits total for RGB, allocate 3 : 3 : 2 for R:G:B

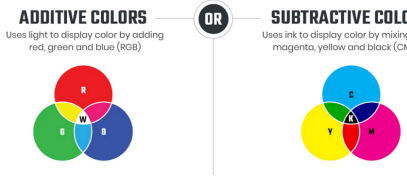


Additive and subtractive color

Additive color Add amounts of RGB

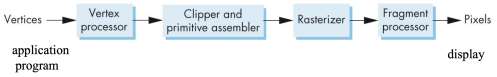
Subtractive color

- Filter white light with cyan, magenta, and yellow filters
- There are 2 ways to think about subtractive color:
  1. What does it filter? Cyan = -Red; Magneta = -Green; Yellow = -Blue
  2. What color is remaining? Refer to image below



Polygon rasterization

- Image is generated by processing a mesh of polygons
- Can be modelled as a pipeline



ELEMENTARY OPENGL

Function format

`glVertex3f(x,y,z)`

- `gl` means it belongs to GL library
- `Vertex` is the function name
- `3` is the number of dimensions
- `f` means the parameters are floats

Coordinate systems

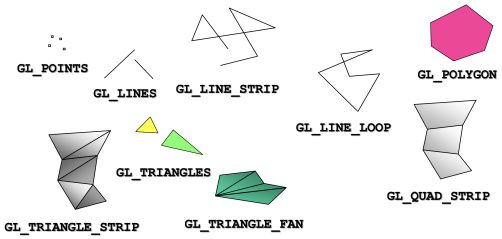
- Units in `glVertex` are determined by application; called object coords
- Camera is positioned in world coords
- Viewing specifications are specified in camera (eye) coords
- Internally, OpenGL converts vertices to camera (eye) coords, then to window coords

Camera

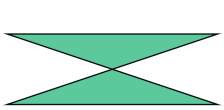
Defaults:

- Placed at origin in world space
- Faces negative z direction
- Viewing volume is a box, centered at origin, with side length 2

OpenGL primitives



- Polygons are displayed correctly if they are
  - Simple: edges do not cross
  - Convex: all points on line segment between any two points in the polygon or on the boundary are also in the polygon
  - Flat: all vertices are in the same plane
- If some condition is not true, OpenGL will still produce output, but it may not be what is desired
- Triangles satisfy all conditions



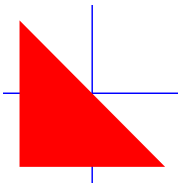
non-simple polygon

non-convex polygon

Usage (right-angled triangle)

- Use `glBegin` and `glEnd` to mark the start/end of declaring vertices for a polygon
- Try to write in the display callback, so it is run once per display callback

```
glBegin(GL_TRIANGLES);
  glColor3f(1, 0, 0);
  glVertex3f(1, -1, 0);
  glVertex3f(-1, 1, 0);
  glVertex3f(-1, -1, 0);
glEnd();
```



Number of entities formed Assume pass in N vertices.

Primitive	Result
GL_POINTS	N points
GL_LINES	$\frac{N}{2}$ lines
GL_LINE_STRIP	$N-1$ lines
GL_LINE_LOOP	N lines
GL_TRIANGLES	$\frac{N}{3}$ triangles
GL_TRIANGLE_STRIP	$N-2$ triangles
GL_TRIANGLE_FAN	$N-2$ triangles
GL_QUADS	$\frac{N}{4}$ quads
GL_QUAD_STRIP	$\frac{N}{2}-1$ quads
GL_POLYGON	1 polygon

Color

- Each color component/channel stored separately in frame buffer
- Usually 8 bits per component
- `glColor3f` expects range from 0.0 to 1.0
- `glColor3ub` expects range from 0 to 255
- Is part of OpenGL state, will be used until changed

Smooth color

- Default is smooth shading
  - Interpolates vertex colors across visible polygons
  - `glShadeModel(GL_SMOOTH)`
- Alternative is flat shading
  - Color of first vertex determines fill color
  - `glShadeModel(GL_FLAT)`

Viewport

- Specifies the drawing area within the window
- `glViewport(x, y, width, height)`
- `x, y` specify lower left corner of viewport rectangle
- Can have multiple, can overlap
- Use `glClear(GL_COLOR_BUFFER_BIT)` to clear **entire window**

INPUT AND INTERACTION

Input

- Devices contain a **trigger** that sends a signal to the OS
- When triggered,
  - Return info (**measure**) to the system
  - Generate event, put into event queue
  - The event queue starts after `glutMainLoop()` is called in the `main()` function
- e.g. Window (resize, expose, minimize) | Mouse (click, motion) | Keyboard (press, release) | Idle (if no event)

Callbacks

A callback is defined for each event

- `glutDisplayFunc` (required for GLUT programs)
  - `void glutDisplayFunc(void (*func)(void))`
- `glutMouseFunc`
  - `void glutMouseFunc(void (*func)(int button, int state, int x, int y))`
  - `button` is either left, middle, or right
  - `state` is either down or up
- `glutReshapeFunc` (window resizing)
  - `void glutReshapeFunc(void (*func)(int width, int height))`
- `glutKeyboardFunc`
  - `void glutKeyboardFunc(void (*func)(unsigned char key, int x, int y))`
  - Returns ASCII code of depressed key and mouse location
- `glutIdleFunc` (useful for animations)
  - `void glutIdleFunc(void (*func)(void))`
- `glutMotionFunc` (mouse moving while button is pressed)
  - `void glutMotionFunc(void (*func)(int x, int y))`
- `glutPassiveMotionFunc` (mouse moving without button press)
  - `void glutPassiveMotionFunc(void (*func)(int x, int y))`

Display callback

Called when

- Window is first opened | reshaped | uncovered | restored from minimized state
- User program wants to change the display

Posting redisplays

- Many events may invoke display callback function
- Hence, display callback may run multiple times in a single event loop pass
- Use `glutPostRedisplay()` instead to set a flag that tells GLUT to execute the display call back function at the end of a single event loop

Reshape callback

- Good place to put viewing functions, since it is invoked when window is first opened
- Redisplay is posted automatically at the end of execution of the callback

Double buffering

Issue

- Drawing of info in frame buffer is decoupled from displaying the frame buffer
- We may see partially rendered frames on the display, that includes info from different frames (known as flickering)

Fix

- Maintain two buffers. Front buffer is displayed, while back buffer is written to
- Swap buffers at end of display callback (after writing is complete)
- Turn on: `glutInitDisplayMode(... | GLUT_DOUBLE)`
- Swap buffers: `glutSwapBuffers()`

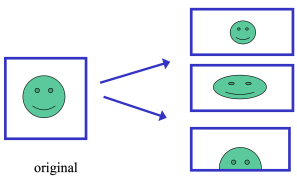
Positioning

- OpenGL has origin at bottom left corner
- But for window system, mouse callback, motion callback, origin at top left corner
- Remember to invert y-coord when necessary

$$y_{\text{opengl}} = h - 1 - y_{\text{win}}$$

Reshaping a window

There are a few ways to redraw the scene.



Obtaining OpenGL state variables

Use `void glGetIntegerv( GLenum pname, GLint *params )`

- `pname` is the state name
- `params` is used to return the state variables

Special keys

- Special keys have constants defined in `glut.h`:
  - Function key 1: `GLUT_KEY_F1`
  - Up arrow key: `GLUT_KEY_UP`
- Check if modifier keys are depressed
  - `if(glutGetModifiers() == GLUT_ACTIVE_CTRL)`
  - Can emulate three-button mouse with one- or two-button mice

GEOMETRY

Elements

Scalar Fully described using magnitude

Vector Has direction and magnitude

Vector space Operations:

- Scalar-vector multiplication
- Vector-vector addition

Point Location in space

- Point-point subtraction yields vector

Affine space Point + vector space. Operations:

- Scalar-vector multiplication
- Vector-vector addition
- Point-vector addition
- Scalar-scalar operations
- For a point P, define $1 \cdot P = P$, $0 \cdot P = \vec{0}$

Lines

For specific P_0, d, a line is the set of all points of the form

$$P(\alpha) = P_0 + \alpha d$$

Representations

- Parametric form (using a point and a direction vector)
  $$P(\alpha) = P_0 + \alpha d$$
- Paramteric form (using 2 points)
  $$P(\alpha) = \alpha P_0 + (1 - \alpha) P_1$$
- Explicit: $y = mx + h$
- Implicit: $ax + by + c = 0$

Ray

- Using the 1 point parametric definition of a line, a ray is a line with $\alpha \geq 0$.
  $$P(\alpha) = P_0 + \alpha d$$

Line segment

- A line segment is bounded by 2 distinct points on a line.
- Using the 2 point parametric definition of a line, the line segment between P_0 and P_1 is a line with $0 \leq \alpha \leq 1$
  $$P(\alpha) = \alpha P_0 + (1 - \alpha) P_1$$

Planes

Defined either by

- A point and two vectors, or
  $$Plane(\alpha, \beta) = R + \alpha \vec{u} + \beta \vec{v}$$
- Three points
  $$Plane(\alpha, \beta) = R + \alpha(Q - R) + \beta(P - Q)$$
- A point and a normal to the plane
  $$(P - R) \cdot \vec{n} = 0$$
  where P is any point on the plane, and $\vec{n} = \vec{u} \times \vec{v}$.

Representation

Coordinate system (vector spaces)

- A basis $\{v_1, v_2, \cdots, v_n\}$ is a set of linearly independent vectors
- A vector can be expressed as $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$
- The list of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is the representation of v with respect to the given basis

- Can write the representation as a row or column vector

$$\vec{a} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}^T = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Frames (affine spaces)

- Coordinate system is only able to specify vectors in 3D space
- Need a reference point to represent a point
- Frame is determined by tuple (P_0, v_1, v_2, v_3)
- Vector is still written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$
$$= \begin{pmatrix} v_1 & v_2 & v_3 & P_0 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{pmatrix}^T$$

- Point can now be written as

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$$
$$= \begin{pmatrix} v_1 & v_2 & v_3 & P_0 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & 1 \end{pmatrix}^T$$

Homogeneous coordinates

Given a coordinate vector $\begin{pmatrix} x & y & z & w \end{pmatrix}^T$,

- If $d = 0$ it is a vector
- Else it represents the point $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$.

Usefulness

- Standardized coordinate system that can represent both vectors and points simultaneously
- All standard transformation (rotaion, translation, scaling) can be implemented with 4 by 4 matrix multiplications
- Allows perspective projection (and perspective division) using matrix multiplication

Planar test for polygon in 3D space

Method 1:

- Suppose polygon is defined by vertices $v_1, v_2, \cdots, v_n$
- Assume three consecutive vertices are not collinear
- Find normal vectors of triangle v_1, v_2, v_3, triangle v_3, v_4, v_5, and so on until triangle v_{n-2}, v_{n-1}, v_n
- Polygon is planar iff all the normal vectors are parallel

Method 2:

- Suppose polygon is defined by vertices $v_1, v_2, \cdots, v_n$
- Assume three consecutive vertices are not collinear
- Form equation of plane using v_1, v_2, v_3
- Substitute in the rest of the points. Polygon is planar iff all the points agree with the equation

Convex test for polygon in xy plane

- Suppose polygon is defined by vertices $v_1, v_2, \cdots, v_n$
- Assume three consecutive vertices are not collinear
- Compute normal vectors of triangle v_1, v_2, v_3, triangle v_2, v_3, v_4, and so on until triangle v_n, v_1, v_2
- Polygon is convex iff all normal vectors are in the same direction (not just parallel)

TRANSFORMATIONS

Affine transformations

- Preserves lines, so we can just transform endpoints of line segments and let implementation draw line segments
- Rigid body transformations: rotation, translation
- Scaling, shearing

Translation

- Move/translate/displace point to new location
- Displacement determined by vector d

Coordinate representation

Define

$$p = \begin{pmatrix} x & y & z & 1 \end{pmatrix}^T$$
$$p' = \begin{pmatrix} x' & y' & z' & 1 \end{pmatrix}^T$$
$$d = \begin{pmatrix} d_x & d_y & d_z & 0 \end{pmatrix}^T$$

Here, $p' = p + d$, so

$$x' = x + d_x$$
$$y' = y + d_y$$
$$z' = z + d_z$$

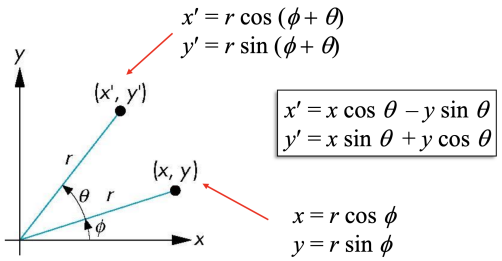
Matrix representation

We can also get $p' = Tp$ with the following matrix T:

$$T = T(d_x, d_y, d_z) = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation (2D / about z-axis)

Consider anti-clockwise rotation about origin by θ degrees.



Note that this is equivalent to rotation about z-axis, since z is kept unchanged.

Coordinate representation

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

Matrix representation

We have $p' = R_z(\theta)p$, where

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation about x or y-axis

Similarly, for rotation about x or y-axis, x or y is kept unchanged. Again, θ is measured anti-clockwise.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that rotations do not commute.

Scaling

Coordinate representation

$$x' = s_x x$$
$$y' = s_y y$$
$$z' = s_z z$$

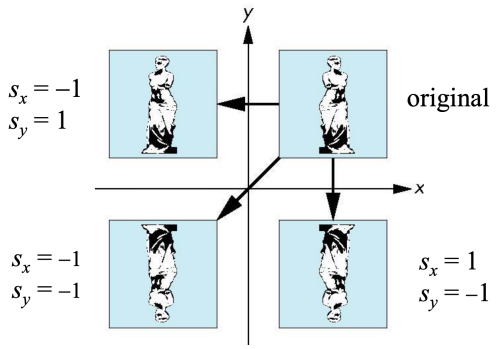
Matrix representation

We have $p' = S(s_x, s_y, s_z)p$, where

$$S(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Reflection

Corresponds to negative scale factors



Concatenation

- We can form arbitrary affine transformation matrices by taking the product of the rotation, translation, scaling matrices
- $p' = DCBA(p)$ applies transformation A first, then B, C, D
- Note:

$$(ABC)^T = C^T B^T A^T$$

Inverses

Use geometric observations to get inverses:

- Translation:

$$T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z)$$

- Rotation: $R^{-1}(\theta) = R(-\theta) = R^T$
- Scaling:

$$S^{-1}(s_x, s_y, s_z) = S\left(\frac{1}{s_x}, \frac{1}{s_y}, \frac{1}{s_z}\right)$$

- Concatenation:

$$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$$

General rotation in 3D

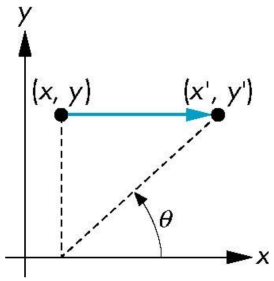
1. Translate so that rotation axis passes through origin
2. Rotate about the x-axis, so that the rotation axis lies in the $x - z$ plane
3. Rotate about the y-axis, so that the rotation axis lies along the z-axis
4. Perform the desired rotation by θ, about the z-axis
5. Apply inverse of step 3
6. Apply inverse of step 2
7. Apply inverse of step 1

Instance transformation

- Draw object centered at origin, oriented with the axis, and at a standard size
- Scale, rotate, then translate

Shearing

Consider the following shear along the x-axis.



Coordinate representation

$$x' = x + y\cot\theta$$
$$y' = y$$
$$z' = z$$

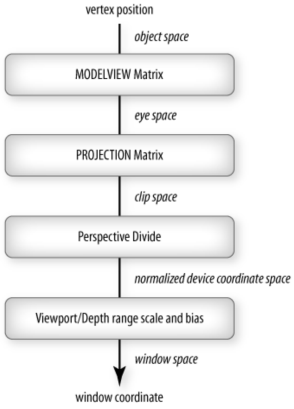
Matrix representation

We have $p' = H(\theta)p$, where

$$H(\theta) = \begin{pmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

OpenGL transformations

- Model-View: `glMatrixMode(GL_MODELVIEW)`
- Projection: `glMatrixMode(GL_PROJECTION)`



Current Transformation Matrix (CTM)

- Part of OpenGL state
- Can load/post-multiply matrices
- Since only post-multiply allowed, the last operation specified is actually the first transformation applied to vertices
  - When drawing a vertex, you pre-multiply the CTM to the point
  - Hence the last operation is first applied
- OpenGL stores has a different matrix for Model-View and Projection, which is loaded using `glMatrixMode`

Model-View matrix

- (Modelling transformation) Transforms objects into the world space
- (View transformation) Positions the camera, aligning camera and world coordinate frames
- Projection matrix is used to defined view volume

Operations

- Load identity matrix: `glLoadIdentity()`
- Rotation: `glRotatef(theta, vx, vy, vz)`
  - `theta` is in degrees
  - `(vx, vy, vz)` is axis of rotation
  - Convert radian to degrees by multiplying by $\frac{180}{\pi}$
- Translation: `glTranslatef(dx, dy, dz)`
- Scale: `glScalef(sx, sy, sz)`
- Each has `(f)loat` and `(d)ouble` format

Arbitrary matrices

- Load arbitrary matrix: `glLoadMatrixf(m)`
- Post-multiply by arbitrary matrix: `glMultMatrixf(m)`

Matrix stacks

- Might want to save transformation matrices for later use
- OpenGL maintains a stack for each matrix mode
- `glPushMatrix()` pushes a copy of the top of the stack onto it
- `glPopMatrix()` pops the stack

CAMERA & VIEWING

Planar geometric projections

- Project onto a plane
- Preserves lines, so we can just transform endpoints of line segments and let implementation draw line segments
- May not preserve angles
- Non-planar projection surfaces needed for applications such as map construction

Examples Perspective projection, parallel projection

Projectors are lines that either

- Converge at center of projection
- Are parallel

Orthographic projection

- Special case of parallel projection, where projectors are orthogonal to projection surface
- In the default orthographic view, points are projected along the z-axis, onto the plane $z = 0$

Perspective projection

- Projectors converge at center of projection
- Objects further from viewer are projected smaller (dimunition)
- Equal distances along a line are not projected into equal distances (non-uniform foreshortening)
- Angles preserved only in planes which are parallel to the projection plane

Projection matrix

- Basically selects a lens
- Sets view volume/clipping volume, defining the region of the 3D world that appears in the viewport

Spaces

- Refer to slides.
- Note that world space is after the model transformation, but since OpenGL combines the model and view transformations into a single matrix, this space is not explicitly in the pipeline

Model transformation

- Prior to this, objects are typically at the origin, possibly at unit size
- This transformation uses rotation, scaling, translation to put the object in its intended position in the world

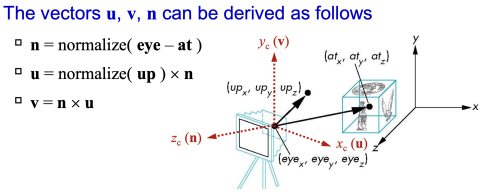
`gluLookAt()` / View transformation

Usage

- `gluLookAt(eyeX, eyeY, eyeZ, atX, atY, atZ, upX, upY, upZ)`
- `(eyeX, eyeY, eyeZ)` specifies the position of the camera
- `(atX, atY, atZ)` specifies the point that the camera should look at
- up-vector is the vector from the eye position to `(upX, upY, upZ)`.
- Note that camera should face negative z direction

up-vector

- Lies anywhere on the original $y - z$ plane, but should not lie on z-axis (view direction)
- Does not need to be perpendicular to view direction (negative z direction)
- Cross product of view direction and up-vector gives unique x-axis
- Cross product of x-axis and view direction gives unique y-axis
- Cross product of x-axis and y-axis gives unique z-axis



Concept

- Positions camera at required location and orientation wrt world frame
- Internally generates transformation matrix that can be used to express all points in world frame wrt camera frame (i.e. view transformation)

$$M_{view} = RT$$
$$= \begin{pmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Translation T moves camera to origin
- Rotation R aligns camera frame axes to world frame axes
- u, v, n defined in picture above
- e is the eye position

Transformation of normal vectors

Let a plane equation be given by $n \cdot v = 0$. Let M_t be the upper-left 3 by 3 submatrix of the model-view transformation matrix M.

$$n \cdot v = 0$$

$$n^T v = 0$$

$$n^T M_t^{-1} M_t v = 0$$

But notice that $M_t v$ produces a vector in eye coordinates. Now we have

$$n^T M_t^{-1} \cdot v_{eye} = 0$$

which is the equation of a plane, which was transformed by M_t. Hence, the new normal is $n^T M_t^{-1}$. This is a row vector pre-multiplied to a matrix, which gives a row vector. To get a column vector, transpose this, getting $(M_t^{-1})^T n$.

Result

- Normal vectors are pre-multiplied by $(M_t^{-1})^T$, where M_t is the upper-left 3 by 3 submatrix of M.
- If M_t is a rotation, then $M_n = M_t$, since $(M_t)^{-1} = (M_t)^T$.

Projection matrix

Projections

- Specifies view volume for projection wrt camera frame
- 3D region of scene to appear in the rendered image

Matrix

- Maps points in view volume to canonical view volume
- Canonical view volume is the 2 by 2 by 2 cube bounded by the planes $x = \pm 1, y = \pm 1, z = \pm 1$, also called Normalized Device Coordinates
- Preserves depth order (in z-coordinate)
- Preserves lines
- This intermediate canonical view volume is then mapped to the viewport

Orthographic projection

Example code:

```
// Set the matrix mode
glMatrixMode(GL_PROJECTION);
// Reset the matrix and apply default
// view volume
glLoadIdentity();
glOrtho(-1.0, 1.0, -1.0, 1.0, -1.0, 1.0);
```

`glOrtho`

- `glOrtho(left, right, bottom, top, near, far)`
- `left, right, bottom, top` specify coords of clipping planes
- `near` and `far` specify displacements from $z = 0$ to near/far clipping planes in the -ve z-direction. Since the camera faces the -ve z direction, a -ve number means that the plane is behind the camera.
- `gluOrtho2D(left, right, bottom, top)` is equivalent to `glOrtho(left, right, bottom, top, -1, 1)`
- Transformation matrix

$$\begin{pmatrix} \frac{2}{right-left} & 0 & 0 & \frac{-(right+left)}{right-left} \\ 0 & \frac{2}{top-btm} & 0 & \frac{-(top+btm)}{top-btm} \\ 0 & 0 & \frac{-2}{far-near} & \frac{-(far+near)}{far-near} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= S\left(\frac{2}{right-left}, \frac{2}{top-btm}, \frac{2}{near-far}\right) \cdot T\left(\frac{-(right+left)}{2}, \frac{-(top+btm)}{2}, \frac{far+near}{2}\right)$$

- Note that $z = -near$ is mapped to $z = -1$, and $z = -far$ to $z = 1$

Viewport transformation

- Canonical view volume is mapped to the viewport (from NDC to window coords). Consider viewport of size (w, h) and bottom left corner (x_{vp}, y_{vp}).

$$\frac{x_{NDC} - (-1)}{2} = \frac{x_{win} - x_{vp}}{w} \Rightarrow x_{win} = x_{vp} + \frac{w(x_{NDC}+1)}{2}$$

$$\frac{y_{NDC} - (-1)}{2} = \frac{y_{win} - y_{vp}}{h} \Rightarrow y_{win} = y_{vp} + \frac{h(y_{NDC}+1)}{2}$$

$$z_{win} = \frac{z_{NDC}+1}{2}$$

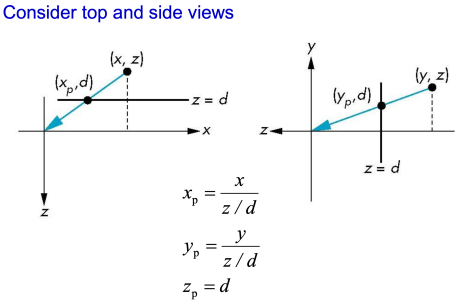
- By default, $0 \le z_{win} \le 1$, for z-buffer hidden surface removal

Perspective projection

Example

- Center of projection at origin
- Projection plane is $z = d$ where $d < 0$

Coordinates



Matrix We have $p = Mq$, where

$$p = \begin{pmatrix} x \\ y \\ z \\ z/d \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{pmatrix} \quad q = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Perspective division

- Notice that in the transformed point p, $z/d \neq 1$, so we divide by z/d to return from homogeneous coordinates
- This is called perspective division

$$p = \begin{pmatrix} x \\ y \\ z \\ z/d \end{pmatrix} \xrightarrow{\substack{perspective \\ division}} p' = \begin{pmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ d \\ 1 \end{pmatrix}$$

`glFrustum()`

- `glFrustum(left, right, bottom, top, near, far)`
- Maps the view frustum to canonical view volume, which will then be mapped to viewport
- Matrix

$$\begin{pmatrix} \frac{2 \cdot near}{right-left} & 0 & \frac{right+left}{right-left} & 0 \\ 0 & \frac{2 \cdot near}{top-btm} & \frac{top+btm}{top-btm} & 0 \\ 0 & 0 & \frac{-(far+near)}{far-near} & \frac{-2 \cdot far \cdot near}{far-near} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

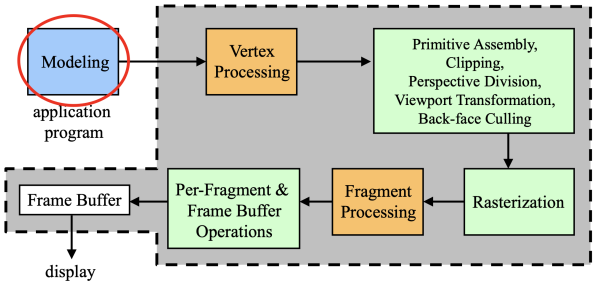
`gluPerspective()`

- `glFrustum` allows for off-center, non-symmetric view volume
- But we often want a symmetric view volume. We can use `gluPerspective(fovy, aspect, near, far)` instead
  - `fovy` specifies the fov angle in degrees in the y-direction
  - `aspect` is $\frac{width}{height}$
  - `near` and `far` specify distances to near/far planes
- Let $f = \cot\left(\frac{fovy}{2}\right)$. The matrix is

$$\begin{pmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{far+near}{near-far} & \frac{2 \cdot far \cdot near}{near-far} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

OPENGL RENDERING PIPELINE

- To render a primitive using OpenGL, the primitive goes through the following main stages
  - Turning each primitive into pixels / fragments



Modelling

- Performed in application
- Provides a set of vertices that specifies geometric objects
- May perform scene processing to reduce amount of geometric data passed to rendering pipeline
  - View-frustum culling: do not pass objects that are outside the frustum
  - Occlusion culling: do not pass objects that are completely blocked

Vertex processing

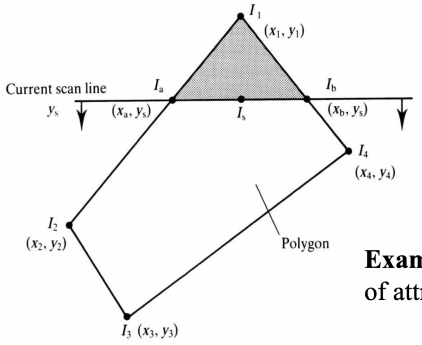
- In this stage, model-view transformation is applied (now in camera/-eye space), affecting vertices and vertex normals
- Assigns color to vertices using lighting computation
- Compute texture coordinates at vertices
- Projection matrix is applied (now in clip space)

Primitive assembly

- Primitive assembly
  - Vertex data was processed separately in previous vertex processing stage
  - Vertex data is collected into complete primitives
  - Necessary for clipping, rasterization, back face culling
- Clipping
- Perspecive division (to NDC space)
- Viewport transformation (to window space, include depth-range scaling)
  - For rasterization to color specific pixels in the window
- Back-face culling

Rasterization

- If geometric primitive not clipped out, appropriate pixels in frame buffer must be assigned colors
- Rasterizer produces a set of fragments for each primitive
  - Fragments are potential pixels, which have a pixel location, color, depth attributes
- Vertex attributes are interpolated over the primitive
- In bilinear interpolation, take weighted average of endpoints on a line segment
  - Attribute values at fragments are computed by interpolating attribute values assigned to vertices
    - Interpolation is performed in window space (2D)



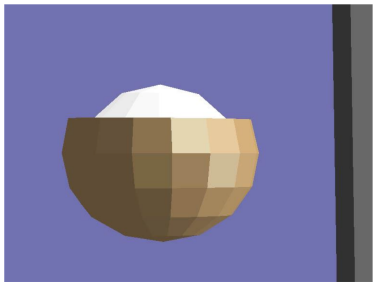
$$I_a = \frac{1}{y_1 - y_2}\left[I_1(y_s - y_2) + I_2(y_1 - y_s)\right]$$

$$I_b = \frac{1}{y_1 - y_4}\left[I_1(y_s - y_4) + I_4(y_1 - y_s)\right]$$

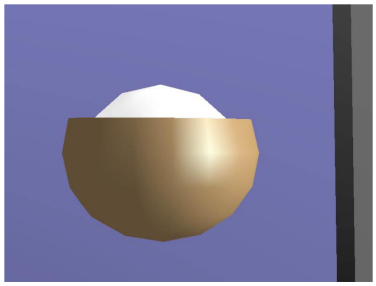
$$I_s = \frac{1}{x_b - x_a}\left[I_a(x_b - x_s) + I_b(x_s - x_a)\right]$$

Example: Bilinear interpolation of attribute values

- Color interpolation



Flat Shading
No interpolation of vertex colors



Smooth (Gouraud) Shading
Interpolation of vertex colors

- Can also interpolate z-value (depth), or texture coordinates

Fragment processing

- Each generated fragment is processed to determine the color of the corresponding pixel in the frame buffer
- Fragment color can be modified by texture mapping
  - Texture access (use the texture directly)
  - Texture application (combine texture with fragment color)

Per-fragment operations

Z-buffer hidden surface removal Discard fragment if it is blocked (occluded) by corresponding pixel in frame buffer

Blending Blend fragment with corresponding pixel in frame buffer

CLIPPING

- Easy for line segments and polygons
- Hard for curves and text - convert to lines and polygons first

Clipping 2D line segments

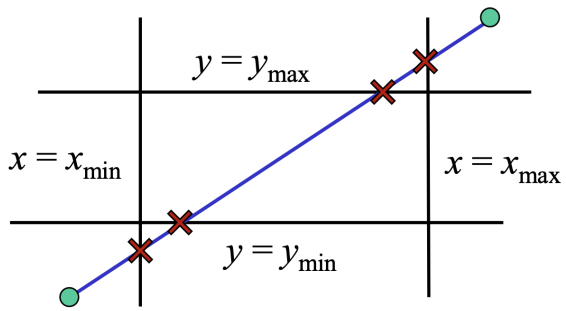
Brute force

- Compute intersections with all side of clipping window
- Inefficient: one division per intersection
- Alternatives: Cohen-Sutherland, Liang-Barsky line clipping algorithms

Cohen-Sutherland

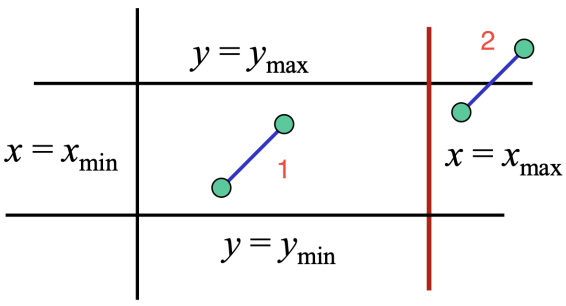
Split window into distinct regions

Extend sides of clipping window to get 9 distinct regions

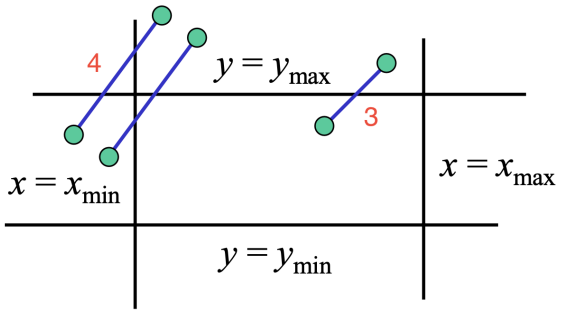


4 cases

1. Both endpoints inside all four lines - Accept
2. Both endpoints outside same line - Reject



3. One endpoint inside all four lines, one outside - Perform intersection
4. Both outside ⇒ may have some part inside. Do one intersection, reducing to Case 2 or Case 3.



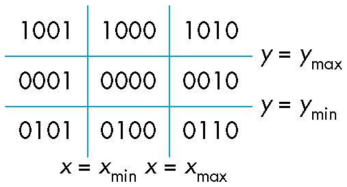
Outcodes

- Define outcodes for easier computation
  - Outcodes divide space into 9 regions
  - Computation of outcode requires at most 4 subtractions

▪ For each endpoint, define an outcode

$b_0 b_1 b_2 b_3$

- $b_0 = 1$ if $y > y_{max}$, 0 otherwise
- $b_1 = 1$ if $y < y_{min}$, 0 otherwise
- $b_2 = 1$ if $x > x_{max}$, 0 otherwise
- $b_3 = 1$ if $x < x_{min}$, 0 otherwise



Consider line segment AB

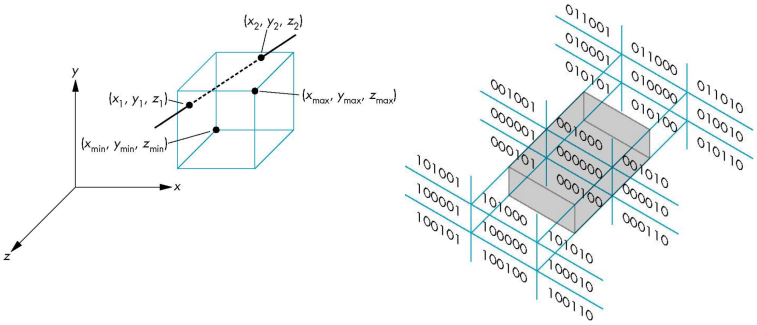
- Case 1: outcodes are both 0
  - Accepted
- Case 2: outcode A AND outcode B (bitwise) is not 0
  - Both outcodes have 1 in same bit location (i.e. both outside the same line)
  - Rejected
- Case 3: outcode A is 0, outcode B not 0
  - Compute intersection, by using the location of 1 in outcode B
  - If outcode B has two 1s, then do two intersections
- Case 4: Neither 0, logical AND yields 0
  - Shorten line segment by intersecting with one side of window
  - Compute outcode of new line segment
  - Repeat algorithm. Yields case 2 or case 3

Efficiency

- Clipping window is usually small, relative to the entire database of line segments
- Most line segments are outside one or more sides of the window, and can be eliminated early
- Inefficient if case 4 occurs often

Application to 3D

- Use 6 bit outcodes
- Clip line segment against planes



Misc

Clipping in clip space

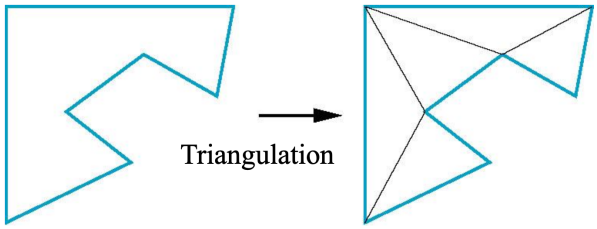
- A vertex is in clip space after multiplication by projection matrix, and before perspective division
- Has coordinates (x, y, z, w)
- If vertex is in canonical view volume, then

$$-1 \leq x/w \leq 1 \quad \Rightarrow \quad -w \leq x \leq w$$
$$-1 \leq y/w \leq 1 \quad \Rightarrow \quad -w \leq y \leq w$$
$$-1 \leq z/w \leq 1 \quad \Rightarrow \quad -w \leq z \leq w$$

- The 6 inequalities are used to decide whether a vertex should be clipped away, in clip space

Polygon clipping

- If convex, clipping yields at most one other polygon
- If concave, use tessellation to split the polygon into a set of smaller convex polygons
  - Usually use triangles
  - Also makes rasterization easier

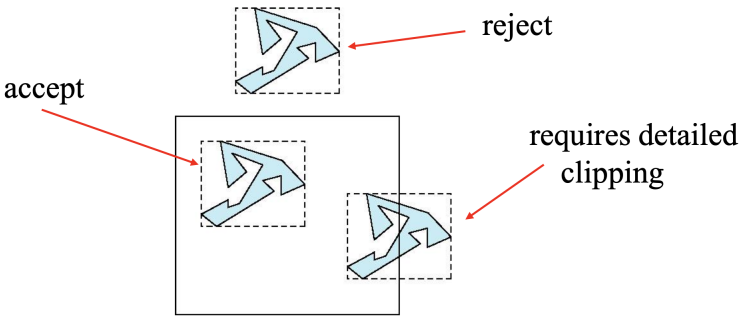


Pipeline clipping of line segments

- Clipping against each side of window is independent of other sides
- Can use four independent clippers in pipeline

Simple early acceptance & rejection

- Use an axis-aligned bounding box (AABB) instead of clipping a complex polygon
- AABB: smallest axis-aligned rectangle that encloses the polygon
- Simple to compute: max and min of x and y



RASTERIZATION

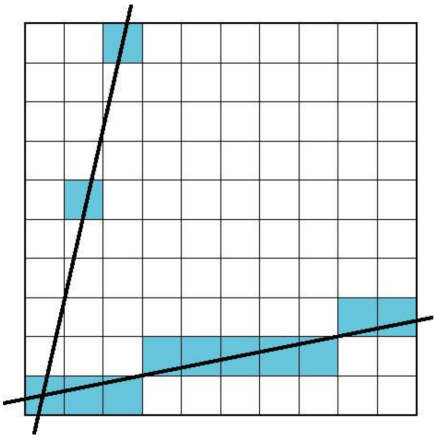
Digital Differential Analyzer - DDA

- For every x, plot pixel at closest y

```
for (x = x0, y = y0; x <= xe; x++) {
    write_pixel(x, round(y), line_color);
    y += m;
}
```

Steep lines

- Problem with steep lines, so use only when $0 \le |m| \le 1$.
- For $|m| > 1$, swap roles of x and y, i.e. for every y, plot pixel at closest x



Bresenham's Algorithm

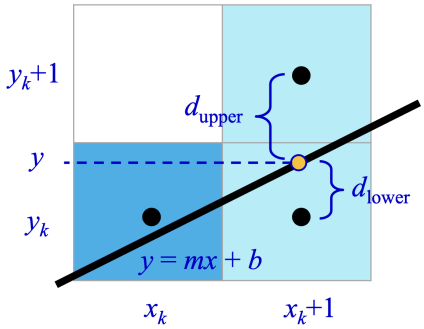
- DDA requires one floating point addition per step
- Bresenham's algorithm can avoid floating point operations
- Consider only $0 \le m \le 1$, the rest can be derived by symmetry
  - Next pixel is either to the right, or right and top
  - Turns into binary decision problem

$$y = m(x_k + 1) + b$$

$$d_{\text{lower}} = y - y_k$$
$$d_{\text{upper}} = (y_k + 1) - y$$

$$p_k = \Delta x\,(d_{\text{lower}} - d_{\text{upper}})$$
$$= 2x_k\,\Delta y - 2y_k\,\Delta x + c$$

where $c = 2\Delta y + \Delta x\,(2b - 1)$ is a **integer constant**.



p_k is an **integer**.

Recall that $\Delta x > 0$, therefore
- if $p_k < 0$, plot lower pixel
- if $p_k > 0$, plot upper pixel

- We can incrementally compute p_{k+1} from p_k

$$p_{k+1} = 2x_{k+1}\Delta y - 2y_{k+1}\,\Delta x + c$$

$$p_{k+1} - p_k = 2(x_{k+1} - x_k)\Delta y - 2(y_{k+1} - y_k)\Delta x$$
$$\Rightarrow p_{k+1} = p_k + 2\Delta y - 2(y_{k+1} - y_k)\Delta x$$

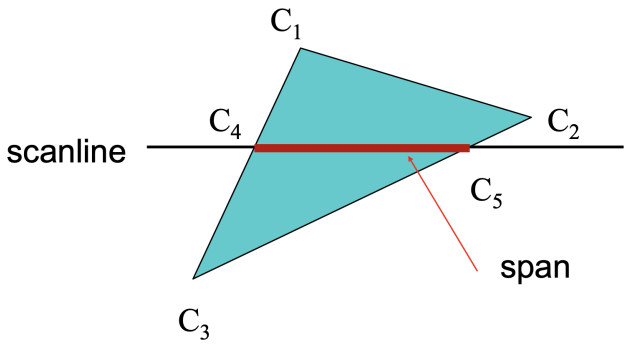
Therefore,
- if $p_k < 0$, $p_{k+1} = p_k + 2\Delta y$
- if $p_k > 0$, $p_{k+1} = p_k + 2\Delta y - 2\Delta x$

where $p_0 = 2\Delta y - \Delta x$

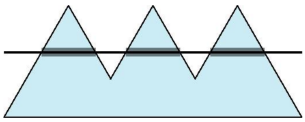
- For each x_k, we need only an integer addition and a test
  - Made into a single instruction on many graphics chips

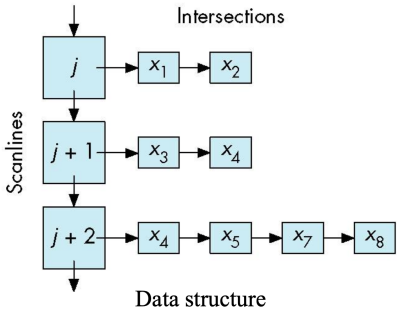
Scan-line fill

- Applies to convex polygons (usually triangles)
  - Non-convex polygons assumed tessellated
- Shades (colors) have been computed for vertices
- Combine with z-buffer algorithm
  - Vertices also have z-values (depth-values)
- March across horizontal scan lines, interpolating shades and z-values
  - $C_1\,C_2\,C_3$ specified by `glColor` or by vertex shading (lighting computation)
  - C_4 determined by interpolating between C_1 and C_3
  - C_5 determined by interpolating between C_2 and C_3
  - Interpolate between C_4 and C_5 along span



- Can also fill by maintaining a data structure of all intersections of polygons with scanlines
  - Sort by scan line
  - Fill each span


Polygon with multiple spans



Data structure

Flood-fill

- Fill can be done recursively if we know a seed point located inside
- But first scan-convert edges using an edge color

HIDDEN-SURFACE REMOVAL

- In 3D graphics, some object should appear in front of another
- Without hidden-surface removal, objects "on top" are the ones that are drawn last
- Hidden-surface removal fixes this problem using the z-buffer algorithm

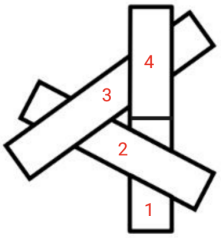
Painter's algorithm (Depth sorting)

- Render polygons in back-to-front order so that polygons behind others are simply painted over
- Depth-sorting can be done in $O(n \log n)$ time
- Cannot handle cyclic overlap of polygons
  - Solution: split original polygons

Not always possible to be able to sort a set of polygons in a back-to-front order

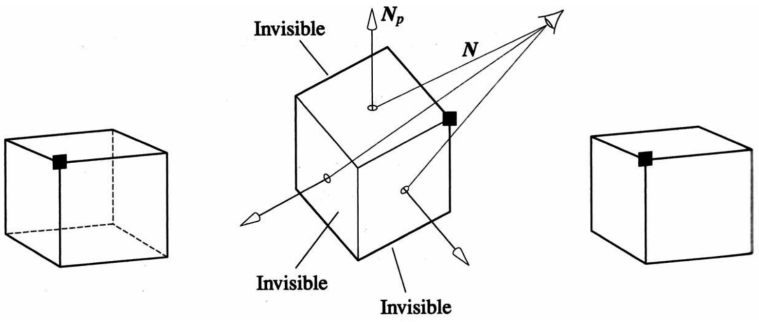
E.g. cyclic occlusion

Solution: Split original polygons



Back-face culling

- Eliminate polygon if it is back-facing and invisible (e.g. back face of a 3d building)
- It is back facing if $N_p \cdot N < 0$



- In OpenGL, simply enable culling
  - Polygon vertices must be provided in counter-clockwise order
  - May not work correctly for non-convex polygon

Image space approach

- For each projector (nm projectors for a n by m frame buffer), find the closest polygon among k polygons
- $O(nmk)$
- e.g. Ray tracing, z-buffer

z-buffer algorithm

- Use a z-buffer or depth buffer to store depth of closest object at each pixel so far
- As we render each polygon, compare depth of each fragment, to depth in z-buffer
- If less than, update, placing fragment in color buffer and its depth in z-buffer
- If not, then discard

```
// In main()
glutInitDisplayMode(... | GLUT_DEPTH);
// In init()
glEnable(GL_DEPTH_TEST);
// Cleared in display callback
glClear(... | GL_DEPTH_BUFFER_BIT);
```

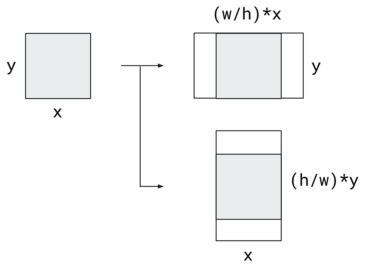
TUTORIALS

T2 Q7

Question

In many old CRT monitors, the pixels are not square. Let's assume the pixel width-to-height aspect ratio is 4:3. Suppose in the camera coordinate frame, there is a disc in the $z = 0$ plane, centered at (100, 200, 0), and has a radius of 10. You want to draw the entire disc as big as possible inside the window, and it should appear circular and not oval. **(A)** If the window size is 600 × 300 (width × height), how would you set up the viewport and the orthographic projection using OpenGL? **(B)** What if the window size is now 300 × 600? **(C)** What if the window size is now 300 × 320?

Answer



Case 1: win width > win height
- clipping rect width greater than object width
- clipping rect height equals object height

Case 2: win width < win height
- clipping rect width equals object width
- clipping rect height greater than object height

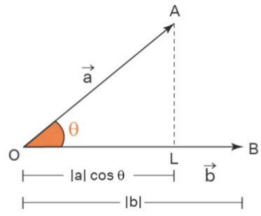
```
double pixelAspectRatio = 4.0 / 3.0;
int winWidth = 600; // or 300 for (B) and (C)
int winHeight = 300; // or 600 for (B) and 320 for (C)
double apparentWinHeight = winHeight / pixelAspectRatio;
double center[2] = {100.0, 200.0};
double radius = 10.0;

glViewport( 0, 0, winWidth, winHeight );
glMatrixMode( GL_PROJECTION );
glLoadIdentity();

if ( winWidth >= apparentWinHeight )
     gluOrtho2D(
          center[0] - radius * ( winWidth / apparentWinHeight ),
          center[0] + radius * ( winWidth / apparentWinHeight ),
          center[1] - radius, center[1] + radius );
else
     gluOrtho2D(
          center[0] - radius, center[0] + radius,
          center[1] - radius * ( apparentWinHeight / winWidth ),
          center[1] + radius * ( apparentWinHeight / winWidth ));
```

T3 Q1c

Projection of a onto b



$a \cdot b = |a|.|b| \cos\theta$

Projection of Vector a on Vector b = $\frac{a.b}{|b|}$

Perpendicular distance of point Q to a plane

Let plane be defined as $ax+by+cz+d=0$. The perpendicular distance from Q to the plane is given by

$$\frac{|ax_q + by_q + cz_q + d|}{\sqrt{a^2+b^2+c^2}}$$

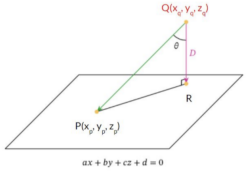
Proof:

What is the **perpendicular distance** of the point Q from the plane $ax + by + cz + d = 0$?

$QP = (x_q - x_p,\ y_q - y_p,\ z_q - z_p)$

$n = (a, b, c)$

$$|QR| = |(x_q - x_p,\ y_q - y_p,\ z_q - z_p) \cdot (a, b, c)| \ / \ \sqrt{(a^2 + b^2 + c^2)}$$
$$= [a(x_q - x_p) + b(y_q - y_p) + c(z_q - z_p)]| \ / \ \sqrt{(a^2 + b^2 + c^2)}$$
$$= |(ax_q + by_q + cz_q) - (ax_p + by_p + cz_p)| \ / \ \sqrt{(a^2 + b^2 + c^2)}$$



For any point in the plane (x, y, z), $ax + by + cz + d = 0$

Thus, $ax_p + by_p + cz_p + d = 0 \Rightarrow d = -(ax_p + by_p + cz_p)$

Therefore, the distance is: $|QR| = \frac{|\ ax_q + by_q + cz_q + d\ |}{\sqrt{(a^2 + b^2 + c^2)}}$

T3 Q4

Given the following 4×4 transformation matrix **M**, where r_{ij} are the elements of a rotation matrix, s_1, s_2, s_3 are non-zero scale factors in the x, y, z directions, and t_1, t_2, t_3 are the translations in the x, y, z directions. Find the *inverse matrix* $\mathbf{M}^{-1}$. Write your answer as a single matrix.

$$\mathbf{M} = \begin{bmatrix} s_1r_{11} & s_1r_{12} & s_1r_{13} & t_1 \\ s_2r_{21} & s_2r_{22} & s_2r_{23} & t_2 \\ s_3r_{31} & s_3r_{32} & s_3r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

M = T·S·R

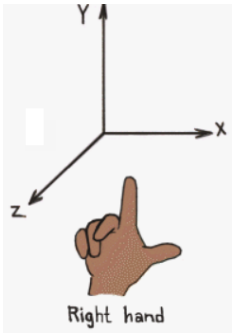
$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

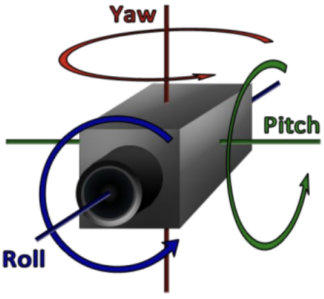
- Clearly T is applied last, since you can reverse the translation and get a clean matrix
- Then, S is applied after R because s_1 is distributed along the row

T4 Q3 - right handed coordinate system

Coordinate system

- By convention we use a right-handed coordinate system
- The right thumb points along the z axis in the positive direction and the curl of the fingers represents a motion from the first or x axis to the second or y axis
- Alternatively, thumb points at x, pointer points at y, middle finger points at z (left image below)





Pitch: rotation about x-axis | Yaw: rotation about y-axis | Roll: rotation about z-axis

Rotation

When rotating around a vector, let thumb be direction of vector. Use grip rule, direction that fingers wrap in is the direction of rotation

T4 Q5 - camera to NDC

- To convert from camera coordinates to NDC, notice that the ratio of the vertex along each axis stays the same after the transformation
- For the z-axis, the near plane maps to -1, while the far plane maps to 1.

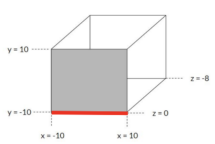
x-axis example

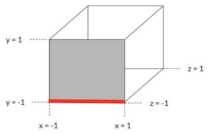
Question 5

glOrtho(left, right, bottom, top, near, far)

A vertex, whose **camera coordinates** are (4, 6, −6)

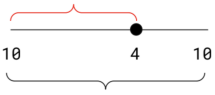
Consider the x axis

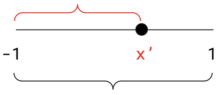




$$\frac{4 - (-10)}{10 - (-10)} = \frac{x' - (-1)}{1 - (-1)}$$

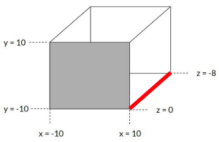
x' = 0.4

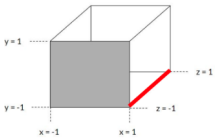




z-axis example

Consider the z axis





$$\frac{-6 - 0}{-8 - 0} = \frac{z' - (-1)}{1 - (-1)}$$

z' = 0.5