Predictive Modeling

Forecasting

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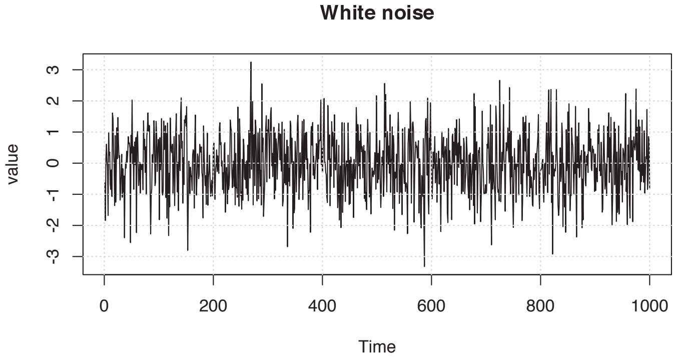
HSLU T&A

1. Repetition : Models of Stochastic Processes

2. Forecasting

Repetition : White Noise

- A white noise process consists of independent and identically distributed random variables $\{W_1, W_2, \dots\}$ where each W_i has mean 0 and variance σ^2

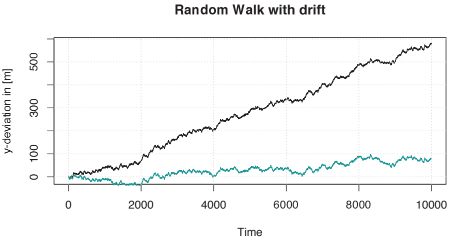


Repetition : Random Walk

- Choose n independent Bernoulli random variables $D_1, \ldots, D_n$ that take on the values 1 and -1 with equal probability, i.e. $p = 0.5$

- Define the random variables $X_i = D_1 + \cdots + D_i$ for each $1 \leq i \leq n$. Then $\{X_1, X_2, \ldots\}$ is a discrete stochastic process modeling the random walk

- Random walk with a drift : $X_i = X_{i-1} + \delta + W_i$



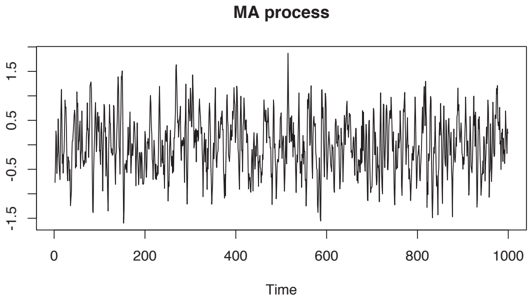
Repetition : Moving Average Process

- We apply a sliding window filter to the white noise process

$$\{W_1, W_2, \dots\}$$

- We obtain a **moving average** process
- If we choose the window length to be 3, we obtain

$$V_i = \frac{1}{3}(W_{i-1} + W_i + W_{i+1}).$$



Repetition : Autoregressive Model

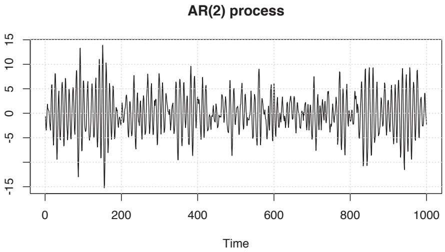
- We consider again the white noise process

$$\{W_1, W_2, \dots, W_n\}$$

- We recursively define the following sequence:

$$X_i = 1.5X_{i-1} - 0.9X_{i-2} + W_i$$

- In other words, the value of the process at time instance i is modeled as a linear combination of the past two values plus some random component : **autoregressive** process



Measures of Dependence

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- We start with the first order moments of the process, the **mean sequence**:

Mean sequence

The mean sequence $\{\mu(1), \mu(2), \dots\}$ (or mean function) of a discrete stochastic process $\{X_1, X_2, \dots\}$ is defined as the sequence of the means:

$$\mu(i) = \mathrm{E}[X_i].$$

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- As second order moments, we consider the covariance within a *single* process

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Autocovariance and autocorrelation

Let $\{X_1, X_2, \dots\}$ be a discrete stochastic process.

1. The *autocovariance* γ_X is defined as

$$\gamma_X(i,j) = \mathrm{Cov}(X_i, X_j) = \mathrm{E}[(X_i - \mu(i))(X_j - \mu(j))].$$

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2. The *autocorrelation* ρ_X is defined as

$$\rho_X(i,j) = \frac{\gamma_X(i,j)}{\sqrt{\gamma_X(i,i)\gamma_X(j,j)}}.$$

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- The first and simplest way to **test** whether a time series is weakly stationary consists of looking for evidence of trend in mean sequence or in the autocorrelation function
- If any such patterns are present then these are **signs of non-stationarity**

Repetition : Estimation of the autocovariance

Due to the stationarity of a process we know that the mean sequence $\mu(k) = \mu$ is constant. A canonical estimator : $\widehat{\mu} = \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

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Sample Autocovariance

1. The *sample autocovariance* is defined by

$$\widehat{\gamma}(h) = \frac{1}{n}\sum_{i=1}^{n-h}(x_{i+h} - \bar{x})(x_i - \bar{x})$$

with $\widehat{\gamma}(-h) = \widehat{\gamma}(h)$ for $h = 0, 1, \ldots, n-1$

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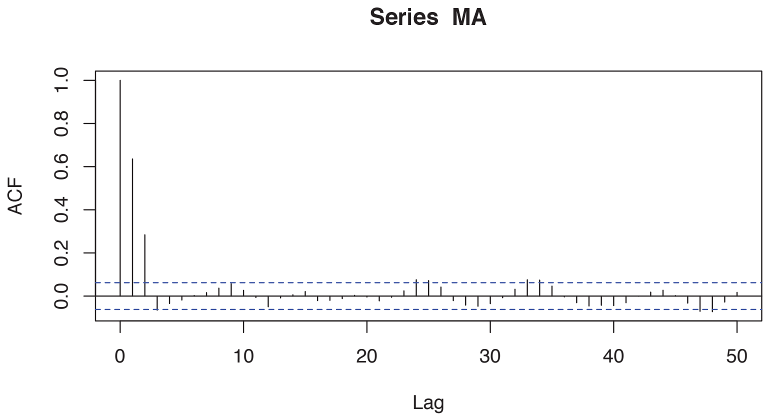
2. The *sample autocorrelation* is defined by

$$\widehat{\rho}(h) = \frac{\widehat{\gamma}(h)}{\widehat{\gamma}(0)}.$$

Repetition : Sample ACF of Simulated MA(5) Process



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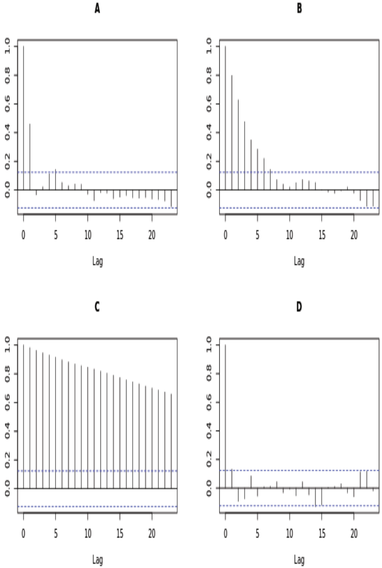
- These limits are automatically drawn by `R` or `Pyton` when calling `acf` (**blue lines**)

Clicker Question



A time series from each of the four models you have considered in this course was simulated and their sample autocorrelation functions (ACF) are shown in one of the four panels in the adjoining figure. They include the white noise (WN), random walk (RW), autoregressive (AR), and simple moving average (MA) models. Match each sample ACF plot with one of our models WN, RW, AR, MA.

- (A) MA, (B) RW, (C) AR, (D) WN
- (A) RW, (B) MA, (C) WN, (D) AR
- (A) WN, (B) RW, (C) AR, (D) MA
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- Goal : **predict** future values x_{n+k} with $k = 1, 2, \ldots$ given a time series up to the present time $\{x_1, \ldots, x_n\}$

Forecasting : Procedure

There are three steps that have to be carried out subsequently, in order to achieve this goal:

1. We need to be certain that the underlying process is **predictible**, i.e. in the future the process will not change dramatically but continues as it has up to the present (in a probabilistic sense)

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3. With the fitted model we **predict** future values of the process

Autoregressive Models

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The autoregressive model of order p is a discrete stochastic process that satisfies

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \cdots + a_p X_{n-p} + W_n$$

where $a_1, a_2, \ldots, a_n$ are the model parameters and $W_1, W_2, \ldots$ is a white noise process with variance σ^2.

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where $a_1, a_2, \ldots, a_n$ are the model parameters and $W_1, W_2, \ldots$ is a white noise process with variance σ^2. For a given autoregressive process, the **characteristic polynomial** is defined as

$$\Phi(x) = 1 - a_1 x - a_2 x^2 - \cdots - a_p x^p.$$

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- AR(1) process is defined by

$$X_n = a_1 X_{n-1} + W_n$$

- Remark : random walk a special case of an AR(1) with $a_1 = 1$, but non-stationary

Example : AR(1)

- Expected value of the process by taking expectations on both sides of the equation: $X_n = a_1 X_{n-1} + W_n$

$$\mu = \mathrm{E}(X_n) = a_1 \mathrm{E}(X_{n-1}) + \mathrm{E}(W_n) = a_1 \mu + 0$$

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- Conclusion :
  - I.e. if the process is **stationary**, then the mean function is $\mu(i) = 0$
  - If the process is a **random walk**, i.e. non-stationary, with $a_1 = 1$, then μ may take on any value

Example : AR(1)

- Next we compute the **variance** of the process:

$$\sigma_X^2 = \mathrm{Var}(X_n) = a_1^2 \mathrm{Var}(X_{n-1}) + \mathrm{Var}(W_n) = a_1^2 \sigma_X^2 + \sigma^2.$$

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- Interpretation : in order to be **stationary**, the dependence of the process on past values should **not** be too strong.

General Condition for an AR(p) to be Weakly Stationary

Stationarity of AR(p)

An AR(p) stochastic process is weakly stationary, if all (complex) roots of the characteristic polynomial

$$\Phi(x) = 1 - a_1 x - a_2 x^2 - \cdots - a_p x^p.$$

exceed 1 in absolute value.

- Example AR(1) : characteristic polynomial is given by

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- Single root of the polynomial Φ is

$$x = 1/a_1$$

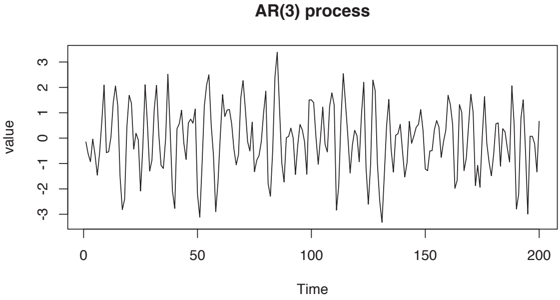
which exceeds 1 in absolute value if and only if $|a_1| < 1$

Example : AR(3) Process

- AR(3) process

$$X_n = 0.5X_{n-1} - 0.5X_{n-2} - 0.1X_{n-3} + W_n$$

- Roots of characteristic polynomial : $1.28, 1.28, 6.09 \rightarrow$ **stationary**
- Simulation of AR(3) process:



- See example 1.3 of Time Series Forecasting chapter

Autocorrelation of AR(p) processes

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- **Partial autocorrelation** is introduced as a further measure well suited for determining the **order** of an autoregressive model

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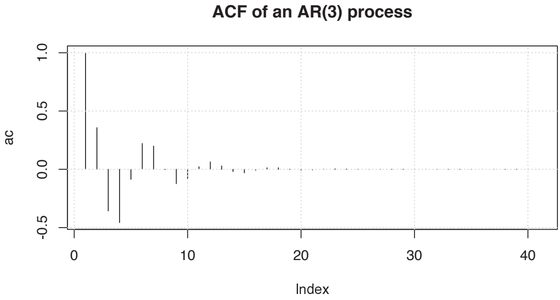
- Example : AR(3) process defined by coefficients $a_1 = 0.5$, $a_2 = -0.5$ and $a_3 = -0.1$
- Please check example `1.5` of `Forecasting` chapter

Example : Theoretical ACF of AR(3) Process



- As it can be seen in above figure, the theoretical autocorrelation of the given AR(3) is **oscillating** and **decreasing** - essentially following an exponential function
- This is the typical autocorrelation behaviour of an autoregressive process

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- This is due to propagation of correlation through the model: If X_k is strongly correlated with X_{k+1} and in turn X_{k+1} is strongly correlated with X_{k+2}, then X_k will likely be correlated with X_{k+2} as well
- If we want to study the **direct** correlation between X_k and X_{k+2}, i.e. the proportion of correlation that is **not** due to X_{k+1}, we have to compute the **partial autocorrelation**

Partial Autocorrelation

Partial autocorrelation

For a weakly stationary stochastic process $\{X_1, X_2, \dots\}$ the partial autocorrelation is defined as

$$\pi(h) = \mathrm{Cor}(X_k, X_{k+h} | X_{k+1}, \dots, X_{k+h-1})^a.$$

[a] The quantity $\mathrm{Cor}(X, Y|Z)$ denotes the conditional correlation of X and Y given the value of Z

Two important properties:

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Two important properties:

1. The p-th coefficient α_p of an AR(p) process equals $\pi(p)$, i.e. the partial autocorrelation value at lag p of the process
2. For an autoregressive process AR(p) the partial autocorrelation at lags greater than p is zero, i.e. $\pi(k) = 0$ for $k > p$.

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  1. Compute the partial autocorrelation
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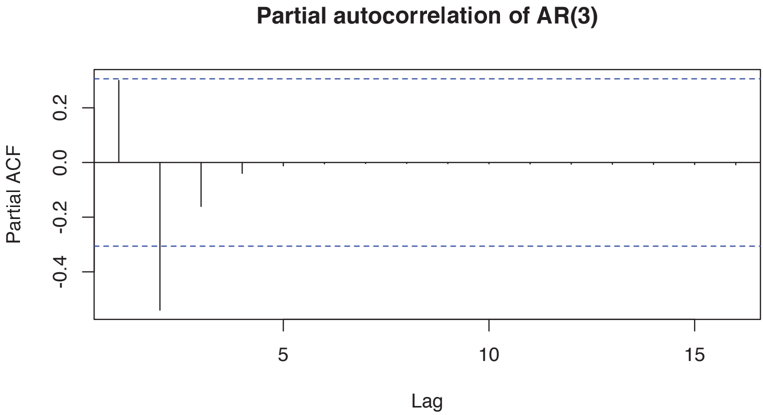
- Please check example `1.6` of the **`Forecasting`** chapter

Example : AR(3) Process



Partial autocorrelation coefficients **larger** than 3 are almost zero :

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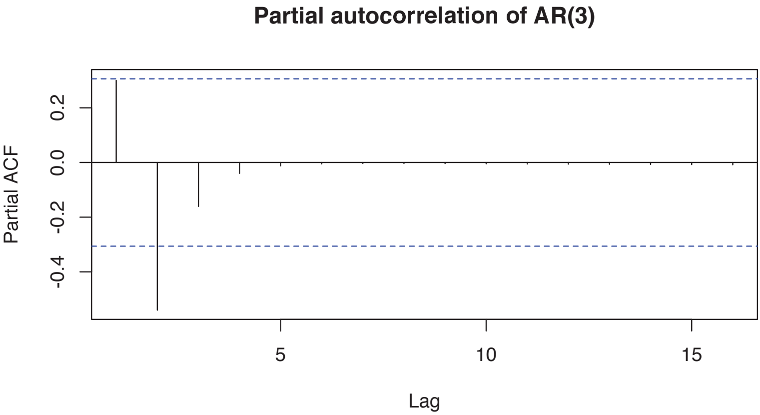
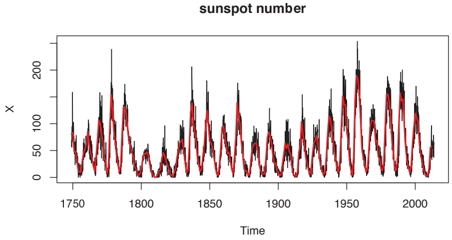


Partial autocorrelation coefficients **larger** than 3 are almost zero : choose an autoregressive model of **order** 3 for the present sequence

Example : Sunspot number



- Forcasting solar activity is important for satellite drag, telecommunication outages and solar winds in connection with blackout of powerplants

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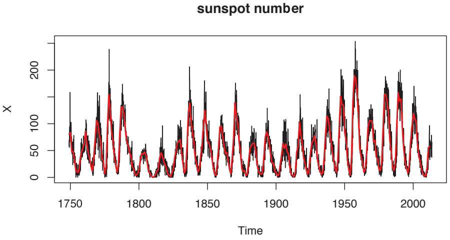

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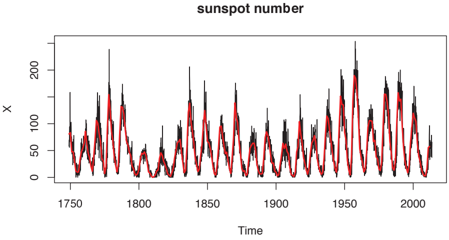

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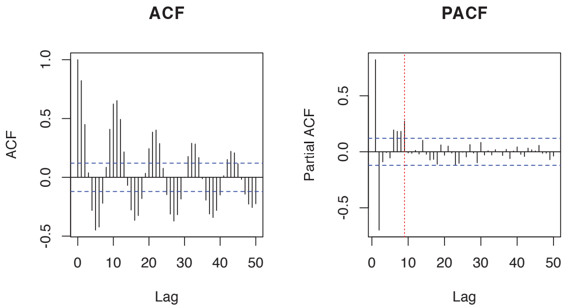
- Swiss astronomer Johann Rudolph Wolf introduced the sunspot number in 1848 and the number of sunspots has been recorded on a monthly basis back to the year 1749

Example : Sunspot Number



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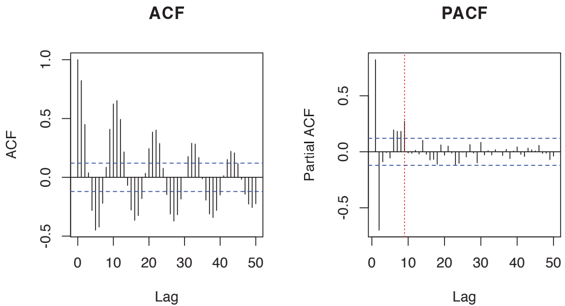
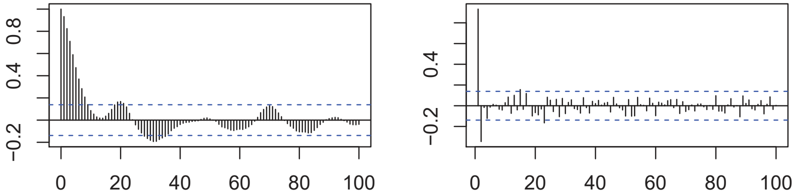


- Autocorrelation shows typical behaviour of an autoregressive process: an **oscillating pattern with an exponential decay**
- Partial autocorrelation : **maximum lag of** 9 which we will use as our model parameter p
- Please check example 1.7 of the Forecasting chapter and solve exercise 1

Clicker Question 1

In the following there is a pair of sample autocorrelation/partial autocorrelation plots belonging to a time series that is not shown. Decide whether the process is autoregressive or not. If yes, estimate the model order p.

a) 

Clicker Question 2

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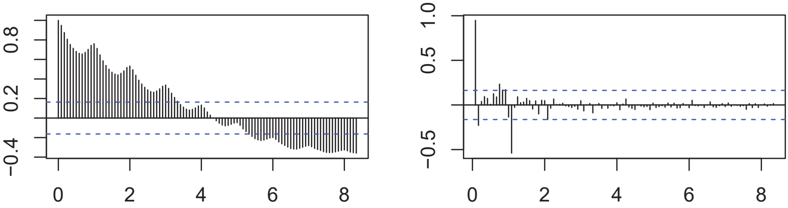

b)

Clicker Question 3

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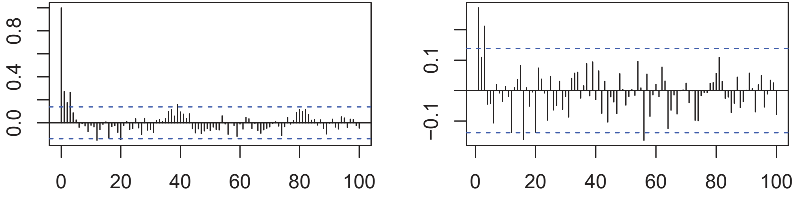

c)

Model Fitting Procedure

- *Task*: estimation of an AR(p) model from given time series data

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  3. Choose **parameters** $a_1, \ldots, a_p$ such that given data are likely to be realizations of corresponding autoregressive process

Model Fitting

- Given the data $\{x_1, x_2, \ldots, x_n\}$ and a model order p, we fit the AR(p) process to the data by solving the following linear equation system

$$
\begin{aligned}
x_{p+1} &= a_1 x_p + a_2 x_{p-1} + \cdots + a_p x_1 + W_{p+1} \\
x_{p+2} &= a_1 x_{p+1} + a_2 x_p + \cdots + a_p x_2 + W_{p+2} \\
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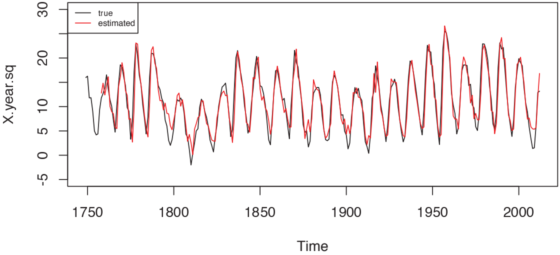
  3. Maximum Likelihood Method

Example : Sunspot Number



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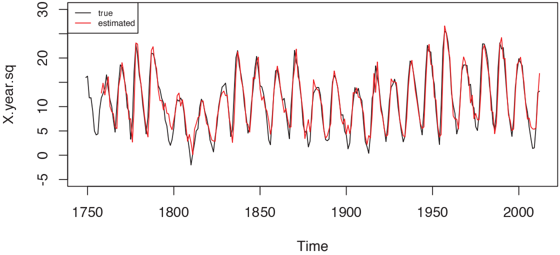

- Plot shows the annually averaged time series in black and the output of the model in red
- Please check example 1.8 of `Forecasting` chapter

Forecasting AR(p) processes

- The general methodology of forecasting stationary time series can be summarized as follows:

k-step ahead forecast

Assume that $\{X_1, X_2, \ldots,\}$ is a stationary process and that we have observed a times series $\{x_1, x_2, \ldots, x_n\}$. The k-step ahead forecast is an estimate of the random variable X_{n+k} given by

$$\widehat{X}_{n+k} = \mathrm{E}[X_{n+k} | X_1 = x_1, \ldots, X_n = x_n].$$

Here $\mathrm{E}[X|Y = y]$ denotes the conditional expectation of X given that $Y = y$

Example : AR(1)

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- For $k > 1$ we plug-in the model equation several times and obtain

$$\begin{aligned}
\widehat{X}_{n+k} &= \mathrm{E}[X_{n+k} | X_1 = x_1, \ldots, X_n = x_n] \\
&= \mathrm{E}[a_1 X_{n+k-1} + W_{n+k} | X_1 = x_1, \ldots, X_n = x_n] \\
&= a_1 \mathrm{E}[X_{n+k-1} | X_1 = x_1, \ldots, X_n = x_n] \\
&= \cdots = a_1^k x_n.
\end{aligned}$$

- k-step ahead forecast of an AR(1) process is **exponentially decaying to zero** from last observation x_n and not dependent on earlier observations

Confidence Intervals

- The standard error σ_k is the square root of the conditional variance

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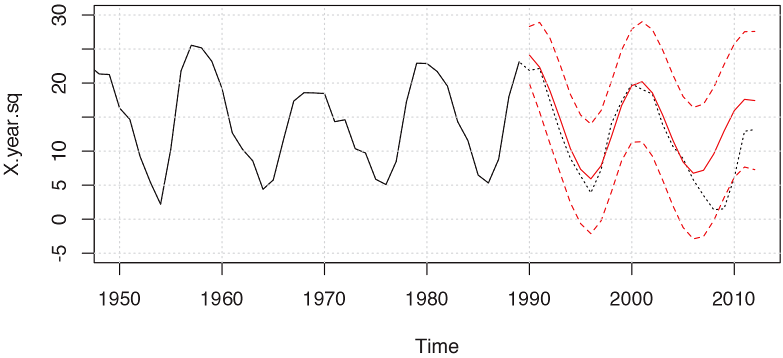
- With this standard error a 95% **confidence interval** for the conditional expectation $\mathrm{E}[X_{n+k}|X_1 = x_1, \ldots, X_n = x_n]$ can be computed

$$\widehat{X}_{n+k} \pm 1.96\sigma_k.$$

Example : Sunspot Number Prediction with Confidence Intervals



Please check example 1.11 of the Forecasting chapter and solve exercises 3