Technical Report: EKF-based Observability Analysis for WINS

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We examine the observability properties of the linearized WiFi-inertial pose estimator by the EKF in the general case when the PAoA from a single AP is observed by the vehicle performing arbitrary motion. We analytically show that the linearized estimator has three unobservable directions, *i.e.*, the ones corresponding to the global translations.

In practice, this is the case when the vehicle moves along the direction of the line connecting itself to the AP since the AP is stationary. It is conceivable that the PAoAs do not change under such motions. Nevertheless, as long as the vehicle does not deliberately keep moving along the unobservable direction, we can update the vehicle's state as shown by the simulation in Section III and the experiments in Section VI.

When there is no PAoA estimation due to occlusion, there are two cases for WINS. The first is that the occlusion blocks the direct path PAoA but the WiFi card can still receive packets and estimate the AoA of a reflected path. This is still valid to correct the IMU drift to some extent but the system will be less accurate as shown in Section VI-D Obstacle Effects. The second case is that the occlusion is so strong that the WiFi connection is completely lost. There is no PAoA at all to the vehicle. In this case, the system cannot be updated so that it becomes completely unobservable.

We begin with an introduction of the propagation and measurement models of WINS. Then we elaborate on the observability analysis.

System state and propagation model. The EKF estimates the 3D pose, linear velocity, and the position of a WiFi AP. The state vector is the (12×1) vector.

$$\mathbf{x} = \begin{bmatrix} \mathbf{q}_w^b; & \mathbf{v}_b^w; & \mathbf{p}_b^w; & \mathbf{c}^w \end{bmatrix}, \tag{1}$$

where $\mathbf{q}_w^b$ is the unit quaternion that denotes the attitude of the world from $\{w\}$ in the body frame $\{b\}$. The body frame $\{b\}$ is attached to the IMU and $\{w\}$ is a reference frame whose origin coincides with the initial IMU position. $\mathbf{v}_b^w$ and $\mathbf{p}_b^w$ are the 3D position and velocity of $\{b\}$ in $\{w\}$. $\mathbf{c}^w$ denotes the map state, which is the APs' positions. We here focus on the case of a single AP for the purpose of simplifying the presentation. Extending this analysis to the case of multiple APs is straightforward.

The continuous-time system model can be expressed as (see **[R1]**)

$$\begin{aligned}
\dot{\mathbf{q}}_w^b(t) &= \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}(t))\mathbf{q}_w^b(t), \\
\dot{\mathbf{v}}_b^w(t) &= \mathbf{a}^w(t), \\
\dot{\mathbf{p}}_b^w(t) &= \mathbf{v}_b^w(t), \\
\dot{\mathbf{c}}^w(t) &= \mathbf{0}_{3\times 1},
\end{aligned} \tag{2}$$

where $\boldsymbol{\omega}(t)$ is the angular velocity of the IMU, expressed in the body frame, $\mathbf{a}^w(t)$ is the IMU acceleration expressed in the world frame, and

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor \boldsymbol{\omega}\times \rfloor & \boldsymbol{\omega} \\ -(\boldsymbol{\omega})^T & \mathbf{0} \end{bmatrix}, \tag{3}$$

where $\lfloor \boldsymbol{\omega}\times \rfloor$ is the skew-symmetric matrix from $\boldsymbol{\omega}$. The measurements of gyroscope and accelerometer, $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{a}}$, can be modelled as

$$\begin{aligned}
\hat{\boldsymbol{\omega}}(t) &= \boldsymbol{\omega}(t) + \mathbf{n}_\omega(t), \\
\hat{\boldsymbol{a}}(t) &= \mathbf{R}(\mathbf{q}_w^b(t))\left(\mathbf{a}^w(t) - \mathbf{g}^w\right) + \mathbf{n}_a(t),
\end{aligned} \tag{4}$$

where $\mathbf{n}_\omega(t)$ and $\mathbf{n}_a(t)$ follows zero-mean Gaussian distribution and $\mathbf{g}^w$ is the earth's gravity. The matrix $\mathbf{R}(\mathbf{q})$ denotes the rotation matrix corresponding to $\mathbf{q}$. The AP belongs to the static scene and thus its time derivative is zero.

Linearizing at the current estimates and applying expectation operator on Eqn. (2), the state estimate propagation model can be written as

$$\begin{aligned}
\dot{\hat{\mathbf{q}}}_w^b(t) &= \frac{1}{2}\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}(t))\hat{\mathbf{q}}_w^b(t), \\
\dot{\hat{\mathbf{v}}}_b^w(t) &= \mathbf{R}^T(\hat{\mathbf{q}}_w^b(t))\left(\hat{\mathbf{a}}^w(t) + \mathbf{g}^w\right), \\
\dot{\hat{\mathbf{p}}}_b^w(t) &= \hat{\mathbf{v}}_b^w(t), \\
\dot{\hat{\mathbf{c}}}^w(t) &= \mathbf{0}_{3\times 1}.
\end{aligned} \tag{5}$$

The (12×1) error state vector can be defined as

$$\tilde{\mathbf{x}} = \left[\delta\boldsymbol{\theta}_w^b; \quad \tilde{\mathbf{v}}_b^w; \quad \tilde{\mathbf{p}}_b^w; \quad \tilde{\mathbf{c}}^w\right]. \tag{6}$$

For the vehicle position, velocity, and the AP's position, we use an additive error model, *i.e.*, $\tilde{\mathbf{v}}_b^w = \mathbf{v}_b^w - \hat{\mathbf{v}}_b^w$, $\tilde{\mathbf{p}}_b^w = \mathbf{p}_b^w - \hat{\mathbf{p}}_b^w$, and $\tilde{\mathbf{c}}^w = \mathbf{c}^w - \hat{\mathbf{c}}^w$. However, we use a multiplicative error model **[R1]**. The (3×1) angle-error vector $\delta\boldsymbol{\theta}_w^b$ is defined by the error quaternion

$$\delta\mathbf{q}_w^b = \mathbf{q}_w^b \otimes \left(\hat{\mathbf{q}}_w^b\right)^{-1} \simeq \left[\frac{1}{2}\delta\boldsymbol{\theta}_w^b; \quad 1\right], \tag{7}$$

where $\delta\mathbf{q}_w^b$ represents the small rotation that causes the true attitudes to coincide with the estimated attitudes. Thus, we can express the attitude uncertainty by a (3×3) covariance matrix $E\left[\delta\boldsymbol{\theta}_w^b\left(\delta\boldsymbol{\theta}_w^b\right)^T\right]$.

The linearized continuous-time error state equation can be written as

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} \mathbf{F}_p & \mathbf{0}_{9\times 3} \\ \mathbf{0}_{3\times 9} & \mathbf{0}_3 \end{bmatrix}\tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{G}_p \\ \mathbf{0}_{3\times 6} \end{bmatrix}\mathbf{n} = \mathbf{F}\tilde{\mathbf{x}} + \mathbf{G}\mathbf{n}, \tag{8}$$

where $\mathbf{0}_3$ denotes the 3×3 zero matrix, $\mathbf{n}$ is a vector comprising the IMU measurement noise terms,

$$\mathbf{n} = [\mathbf{n}_\omega; \quad \mathbf{n}_a] \,. \tag{9}$$

$\mathbf{F}_p$ denotes the error state transition matrix corresponding to the state of the vehicle platform, and $\mathbf{G}_p$ is the input noise matrix. They can be written as

$$\mathbf{F}_p = \begin{bmatrix} -\lfloor \hat{\boldsymbol{\omega}} \times \rfloor & \mathbf{0}_3 & \mathbf{0}_3 \\ -\mathbf{R}^T(\hat{\mathbf{q}}_w^b)\lfloor \hat{\mathbf{a}} \times \rfloor & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix}, \tag{10}$$

$$\mathbf{G}_p = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\mathbf{R}^T(\hat{\mathbf{q}}_w^b) \end{bmatrix}, \tag{11}$$

where $\mathbf{I}_3$ is the 3×3 identity matrix. We model the system noise as a zero-mean Gaussian distribution. The autocorrelation $E\left[\mathbf{n}(t)\mathbf{n}^T(\tau)\right] = \mathbf{Q}\delta(t-\tau)$, where $\mathbf{Q}$ represents the IMU noise statistical characteristics, which can be computed offline **[R1]**.

The IMU measurements $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{a}}$ are sampled at a constant rate $1/\delta t$, where $\delta t = t_{k+1} - t_k$. Every time a new IMU measurement is received, the system propagates the state using Runge-Kutta numerical integration of Eqn. (5). In order to derive the covariance propagation matrix, we compute the discrete-time state transition matrix, $\boldsymbol{\Phi}_{k+1,k}$ from time t_k to t_{k+1}, as the solution to the following differential equation:

$$\begin{aligned} \dot{\boldsymbol{\Phi}}_{k+1,k} &= \mathbf{F}\boldsymbol{\Phi}_{k+1,k} \\ \text{initial condition: } \boldsymbol{\Phi}_{k,k} &= \mathbf{I}_{12}. \end{aligned} \tag{12}$$

Thus,

$$\boldsymbol{\Phi}_{k+1,k} = \exp\left(\int_{t_k}^{t_{k+1}} \mathbf{F}(\tau)\,\mathrm{d}\tau\right). \tag{13}$$

Then, the discrete-time noise covariance matrix $\mathbf{Q}_k$ can be computed as

$$\mathbf{Q}_k = \int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}_{k+1,\tau}\mathbf{G}\mathbf{Q}\mathbf{G}^T\boldsymbol{\Phi}_{k+1,\tau}^T\,\mathrm{d}\tau. \tag{14}$$

Finally, the covariance is propagated as

$$\mathbf{P}_{k+1|k} = \boldsymbol{\Phi}_{k+1,k}\mathbf{P}_{k|k}\boldsymbol{\Phi}_{k+1,k}^T + \mathbf{Q}_k. \tag{15}$$

Note that $\mathbf{P}_{i|j}$ denotes the estimates of the error-state covariance at time i computed using measurements up to time j.

Measurement update model. As the WiFi-inertial platform moves, the WiFi card observed PAoAs with respect to the AP. The PAoAs are used to simultaneously estimate the vehicle's motion and the AP's position.

To simplify the discussion, we consider the frame of the antenna array attached with a WiFi card is coincided with the IMU frame. The WiFi card measures $\mathbf{z}$, which is the unit vector $\mathbf{d}$ computed from the PAoAs that represents the direction of the vehicle to the AP, expressed in the body frame $\{b\}$,

$$\hat{\mathbf{z}} = \lambda\mathbf{d} + \boldsymbol{\eta}, \tag{16}$$

where

$$\lambda \mathbf{d} = \mathbf{c}^b = \mathbf{R}(\mathbf{q}_w^b)\left(\mathbf{c}^w - \mathbf{p}_b^w\right), \tag{17}$$

where λ is the distance of the AP to the vehicle, which is initially unknown. We can initialized it using multiple spatially different PAoAs. The measurement noise, $\boldsymbol{\eta}$, is modelled as zero-mean Gaussian distribution with covariance $\boldsymbol{\Gamma}$. The linearized error model is

$$\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}} \simeq \mathbf{H}\tilde{\mathbf{x}} + \boldsymbol{\eta}, \tag{18}$$

where the measurement Jacobian $\mathbf{H}$ can be trivially evaluated at the current state estimate, *i.e.*, $\mathbf{c}^w$, $\mathbf{q}_w^b$, and $\mathbf{p}_b^w$.

Then we apply the measurement model to update the filter. In particular, we compute the measurement residual $\mathbf{r}$ along with its covariance matrix $\mathbf{S}$ and the Kalman gain $\mathbf{K}$,

$$\begin{aligned}
\mathbf{r} &= \mathbf{z} - \hat{\mathbf{z}}, \\
\mathbf{S} &= \mathbf{H}\mathbf{P}_{k+1|k}\mathbf{H}^T + \boldsymbol{\Gamma}, \\
\mathbf{K} &= \mathbf{P}_{k+1|k}\mathbf{H}^T\mathbf{S}^{-1},
\end{aligned} \tag{19}$$

and the EKF state and covariance can be updated as

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1|k+1} &= \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}\mathbf{r}, \\
\mathbf{P}_{k+1|k+1} &= \mathbf{P}_{k+1|k} - \mathbf{K}\mathbf{S}\mathbf{K}^T.
\end{aligned} \tag{20}$$

Evert time a new PAoA is measured, we initialize it into the state vector. However, a single PAoA does not provide enough information to resolve the distance λ (see Eqn. (17)). We can utilize multiple PAoAs by solving a bundle-adjustment problem over a short time window.

At this stage, we have defined the system model. Next, we examine the observability properties of this model.

Observability Analysis. The observability matrix $\mathbf{O}$ is defined as **[R2]**

$$\mathbf{O}(\mathbf{x}^*) = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2\boldsymbol{\Phi}_{2,1} \\ \vdots \\ \mathbf{H}_k\boldsymbol{\Phi}_{k,1} \end{bmatrix}, \tag{21}$$

where $\boldsymbol{\Phi}_{k,1} = \boldsymbol{\Phi}_{k,k-1}\cdots\boldsymbol{\Phi}_{2,1}$ is the state transition matrix from time 1 to k, and $\mathbf{H}_k$ is the measurement Jacobian matrix (refer to Eqn. (18)) for the PAoA at time k. Since all the Jacobians are evaluated at a particular state $\mathbf{x}^* = [\mathbf{x}_1^*; \cdots; \mathbf{x}_k^*]$, the observability matrix is a function of $\mathbf{x}^*$. If $\mathbf{O}(\mathbf{x}^*)$ is full column rank, then the model will be fully observable. However, as we will show below, $\mathbf{O}(\mathbf{x}^*)$ is rank deficient. The system in unobservable in general in three directions corresponding to 3D global translations. We first show that there are four unobservable directions, which corresponds to 3D global translations and global rotations of the vehicle and the AP about the gravity, when linearizing at the true state. Based on that, we then show the observability matrix $\mathbf{O}(\mathbf{x}^*)$ gains rank when linearizing at the estimated state, which has errors across time, reducing the number of unobservable directions to three.

When evaluating the system and measurement model at the true state, *i.e.*, $\mathbf{x}^* = \mathbf{x}$, the Jacobian $\mathbf{H}_k$ in $\mathbf{O}(\mathbf{x})$ can be written as

$$\mathbf{H}_k = \mathbf{H}_{\text{ap},k}\mathbf{R}(\mathbf{q}_w^b) \cdot \left[\lfloor \mathbf{c}^w - \mathbf{p}_{b_k}^w \times \rfloor \mathbf{R}(\mathbf{q}_w^b)^T \quad \mathbf{0}_3 \quad -\mathbf{I}_3 \quad \mathbf{I}_3 \right], \tag{22}$$

where $\mathbf{H}_{\text{ap},k}$ denotes the Jacobian of the PAoA with respect to the AP, which can be evaluated from Eqn. (18).

To compute the remaining blocks of $\mathbf{O}(\mathbf{x})$, *i.e.*, $\mathbf{\Phi}_{k,1}$, we need to solve the following differential equation:

$$\begin{gathered}\dot{\mathbf{\Phi}}_{k,1} = \mathbf{F}\mathbf{\Phi}_{k,1} \\ \text{initial condition: } \mathbf{\Phi}_{1,1} = \mathbf{I}_{12},\end{gathered} \tag{23}$$

where $\mathbf{F}$ is defined in Eqn. (8). By solving this, $\mathbf{\Phi}_{k,1}$ has the following structure:

$$\mathbf{\Phi}_{k,1} = \begin{bmatrix} \mathbf{\Phi}_{k,1}^{(1,1)} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{\Phi}_{k,1}^{(2,1)} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{\Phi}_{k,1}^{(3,1)} & \delta t_k \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, \tag{24}$$

where $\delta t_k = \delta t(k-1)$ is the time from time 1 to k. For the observability analysis, $\mathbf{\Phi}_{k,1}^{(1,1)}$ and $\mathbf{\Phi}_{k,1}^{(3,1)}$ are required. We start with $\mathbf{\Phi}_{k,1}^{(1,1)}$ from Eqn. (23).

$$\dot{\mathbf{\Phi}}_{k,1}^{(1,1)} = \mathbf{F}^{(1,:)}\mathbf{\Phi}_{k,1}^{(:,1)} = \begin{bmatrix} -\lfloor \boldsymbol{\omega} \times \rfloor & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{k,1}^{(1,1)} \\ \mathbf{\Phi}_{k,1}^{(2,1)} \\ \mathbf{\Phi}_{k,1}^{(3,1)} \\ \mathbf{0}_3 \end{bmatrix} = -\lfloor \boldsymbol{\omega} \times \rfloor \mathbf{\Phi}_{k,1}^{(1,1)}. \tag{25}$$

With the initial condition $\mathbf{\Phi}_{1,1}^{(1,1)} = \mathbf{I}_3$, we solve $\mathbf{\Phi}_{k,1}^{(1,1)}$ as

$$\mathbf{\Phi}_{k,1}^{(1,1)} = \mathbf{\Phi}_{1,1}^{(1,1)} \exp\left(\int_{t_1}^{t_k} -\lfloor \boldsymbol{\omega} \times \rfloor \mathrm{d}\tau \right) = \mathbf{R}(\mathbf{q}_{b_1}^{b_k}). \tag{26}$$

Similarly, $\mathbf{\Phi}_{k,1}^{(3,1)}$ can be computed as

$$\mathbf{\Phi}_{k,1}^{(3,1)} = \lfloor \mathbf{p}_{b_1}^w + \mathbf{v}_{b_1}^w \delta t_k - \frac{1}{2}\mathbf{g}\delta t_k^2 - \mathbf{p}_{b_k}^w \times \rfloor \mathbf{R}(\mathbf{q}_{b_1}^w). \tag{27}$$

Based on these expressions, we obtain k^{th}block row, $\mathbf{O}_k$, of $\mathbf{O}(\mathbf{x})$, $\forall k > 1$,

$$\mathbf{O}_k = \mathbf{H}_k\mathbf{\Phi}_{k,1} = \mathbf{\Lambda}_1 \left[\mathbf{\Lambda}_2 \quad -\delta t_k \mathbf{I}_3 \quad -\mathbf{I}_3 \quad \mathbf{I}_3 \right], \tag{28}$$

where

$$\begin{aligned} \mathbf{\Lambda}_1 &= \mathbf{H}_{\text{ap},k}\mathbf{R}(\mathbf{q}_w^{b_k}), \\ \mathbf{\Lambda}_2 &= \lfloor \mathbf{c}^w - \mathbf{p}_{b_1}^w - \mathbf{v}_{b_1}^w \delta t_k + \frac{1}{2}\mathbf{g}^w \delta t_k^2 \times \rfloor \mathbf{R}^T(\mathbf{q}_w^{b_1}). \end{aligned} \tag{29}$$

At this stage, we can find the right nullspace $\mathbf{N}$ of the linearized model $\mathbf{O}(\mathbf{x}^*)$ as

$$\mathbf{N} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{R}(\mathbf{q}_w^{b_1})\mathbf{g}^w \\ \mathbf{0}_3 & -\lfloor \mathbf{v}_{b_1}^w \times \rfloor \mathbf{g}^w \\ \mathbf{I}_3 & -\lfloor \mathbf{p}_{b_1}^w \times \rfloor \mathbf{g}^w \\ \mathbf{I}_3 & -\lfloor \mathbf{c}^w \times \rfloor \mathbf{g}^w \end{bmatrix} = \left[\mathbf{N}_t \quad | \quad \mathbf{N}_r \right]. \tag{30}$$

The nullspace $\mathbf{N}$ can be easily verified by multiplying each of its blocks with $\mathbf{O}(\mathbf{x}^*)$ as $\mathbf{O}(\mathbf{x}^*)\mathbf{N} = \mathbf{0}$.

From Eqn. (28) and Eqn. (30), we can see the following properties.

- The (12×3) block column $\mathbf{N}_t$ corresponds to 3D global translations, *i.e.*, translating both the vehicle and the AP by the same amount.
- The (12×1) column $\mathbf{N}_r$ corresponds to global rotations of the vehicle and the AP about the gravity.

However, in practice, we can only linearize at the estimated state, $\hat{\mathbf{x}}$. Due to errors in the estimates, the observability matrix, $\mathbf{O}(\hat{\mathbf{x}})$, gains rank. Specifically, the last two block elements of $\mathbf{O}_k$ are identity matrices, indicating that they are not a function of linearization point. Thus, the nullspace corresponding to translation are preserved, *i.e.*, $\mathbf{O}(\hat{\mathbf{x}})\mathbf{N}_t = \mathbf{0}$. On the contrary, $\mathbf{O}(\hat{\mathbf{x}})\hat{\mathbf{N}}_r \neq \mathbf{0}$ in general since $\hat{\mathbf{N}}_r$, $\mathbf{\Lambda}_2$, as well as $\mathbf{O}(\hat{\mathbf{x}})$ depend on the linearization point. In another words, the directions in which the estimator gains information are altered as we use different state estimates to evaluate the system and measurement Jacobians. This makes $\hat{\mathbf{N}}_r$, which corresponds to global rotations, not in the nullspace of $\mathbf{O}(\hat{\mathbf{x}})$. As a result, the rank of the observability matrix increases by one.