1 Introduction

Physics-Informed Neural Networks (PINNs) have emerged as a transformative approach for solving partial differential equations (PDEs) by seamlessly integrating physical laws into deep learning frameworks Raissi et al. (2017a,b). Unlike traditional numerical methods that rely on discretization schemes, PINNs leverage the universal approximation capabilities of neural networks while enforcing governing equations through automatic differentiation Karniadakis et al. (2021); Cuomo et al. (2022). This paradigm shift has opened new avenues for tackling complex PDEs that challenge conventional solvers, particularly in domains with irregular geometries, high-dimensional spaces, or sparse data availability Chen et al. (2021); Pang et al. (2019).

The Euler-Bernoulli beam equation, a fourth-order PDE fundamental to structural mechanics, presents unique challenges for numerical approximation due to its high-order derivatives and stringent boundary conditions Kapoor et al. (2023); Zakian (2023). Traditional finite element and finite difference methods require careful mesh design and specialized basis functions to achieve reasonable accuracy, often at substantial computational cost. Recent advances in PINNs have shown promise for beam problems Wang et al. (2024); Kapoor et al. (2023), yet achieving ultra-high precision solutions remains elusive due to the inherent difficulties in approximating fourth-order derivatives through neural networks Vahab et al. (2022).

The pursuit of high-precision solutions in scientific computing has gained renewed importance with applications in gravitational wave detection, quantum mechanics simulations, and precision engineering where numerical errors can compound catastrophically Mukhametzhanov (2022); Wong et al. (2024). While standard PINNs typically achieve relative errors on the order of 10^{-3} to 10^{-6} for complex PDEs Jagtap et al. (2020); Lu et al. (2021), pushing beyond these limits requires fundamental architectural innovations and novel training strategies Brunton and Kutz (2024); Antonion and Tsai (2024).

Recent developments in neural network architectures for PDEs have explored various directions to enhance accuracy and efficiency. The introduction of Fourier Neural Operators demonstrated the power of spectral methods in neural architectures Li et al. (2020), while domain decomposition approaches like XPINNs addressed scalability challenges Jagtap and Karniadakis (2020); Kharazmi et al. (2021). Neural Architecture Search-guided PINNs (NAS-PINN) have automated the discovery of optimal network structures Wang and Zhong (2024), and time-evolving natural gradient methods (TENG) have shown promise for achieving machine precision Chen et al. (2024). Additionally, physics-informed neural networks have been enhanced through various approaches including conserved quantities Lin and Chen (2022), anti-derivatives approximation Lee (2024), and stress-split sequential training Haghighat et al. (2022).

Despite these advances, our comprehensive analysis of the literature reveals critical gaps that prevent achieving ultra-precision solutions for fourth-order PDEs. Current methods face a precision ceiling, typically plateauing at relative errors of 10^{-5} to 10^{-6}

Vahab et al. (2022); Kapoor et al. (2023). The computation of fourth-order derivatives through automatic differentiation suffers from numerical instabilities and accumulating round-off errors Hu et al. (2024). Moreover, the loss landscape becomes increasingly complex with multiple competing objectives—PDE residuals, boundary conditions, and initial conditions—creating optimization challenges that standard algorithms cannot overcome Wang et al. (2021); Krishnapriyan et al. (2021). Existing architectures employ generic fully-connected networks that fail to exploit the inherent modal structure of beam vibrations Brunton and Kutz (2024), while fixed loss weighting strategies miss opportunities for adaptive optimization McClenny and Braga-Neto (2023). The theoretical understanding remains incomplete, with no proven convergence bounds for ultra-precision regimes and limited exploration of hybrid analytical-neural approaches Arzani et al. (2023); Cho et al. (2023). Additionally, computational efficiency remains a bottleneck, with poor GPU utilization for high-order derivative calculations and memory-intensive computational graphs Jagtap et al. (2020).

To address these fundamental limitations, we present a novel hybrid Fourier-neural network architecture specifically designed to break through the precision barrier and achieve ultra-precision solutions for the Euler-Bernoulli beam equation. Our approach synergistically combines truncated Fourier series decomposition with deep neural networks, enabling unprecedented accuracy with relative L2 errors below 10^{-7}. The key innovation lies in our discovery that optimal performance is achieved with exactly 10 Fourier harmonics—counterintuitively, adding more harmonics degrades accuracy due to optimization complexities. The neural network component provides adaptive residual corrections for non-modal features while ensuring precise boundary condition satisfaction. This hybrid formulation builds upon recent advances in sinusoidal representation spaces Wong et al. (2024) and separable physics-informed neural networks Cho et al. (2023), but goes significantly beyond by introducing systematic harmonic optimization and two-phase training strategies.

Our contributions are threefold: (1) We introduce a physics-informed hybrid architecture that optimally separates modal and non-modal solution components, achieving a 17-fold improvement in accuracy compared to standard PINN implementations through our discovered 10-harmonic configuration, (2) We develop a sophisticated two-phase optimization strategy that transitions from gradient-based exploration (Adam) to high-precision quasi-Newton refinement (L-BFGS), with adaptive weight balancing that prevents loss term dominance, building on insights from Penwarden et al. (2023), and (3) We demonstrate GPU-efficient implementation strategies with custom kernels for fourth-order derivative computation and dynamic memory management that enable training of ultra-precision models within practical computational constraints. Through systematic experiments, we validate that our method consistently achieves L2 errors of 1.94×10^{-7}, establishing a new benchmark for neural PDE solvers and opening possibilities for machine learning applications demanding extreme numerical precision.

Figure 1 provides a visual summary of our approach, illustrating the synergy between

Fourier decomposition and neural network corrections that enables ultra-precision solutions. The infographic highlights the key innovations including the hybrid architecture, two-phase optimization strategy, and the achievement of L2 errors below 10^{-7}.

The remainder of this paper is organized as follows: Section 2 presents our hybrid Fourier-neural network architecture and training methodology, Section 3 demonstrates the effectiveness of our approach through comprehensive numerical experiments, Section 4 discusses the results and their implications, and Section 5 concludes with future research directions.

References

Antonion, J. D. and Tsai, R. (2024). From pinns to pikans: Recent advances in physics-informed machine learning. *arXiv preprint arXiv:2410.13228.*

Arzani, A., Cassel, K., and D'Souza, R. (2023). Theory-guided physics-informed neural networks for boundary layer problems with singular perturbation. *Journal of Computational Physics*, 473:111756.

Brunton, S. L. and Kutz, J. N. (2024). Promising directions of machine learning for partial differential equations. *Nature Computational Science*, 4(7):483–494.

Chen, Z., Liu, Y., and Sun, H. (2021). Physics-informed learning of governing equations from scarce data. *Nature Communications*, 12(1):6136.

Chen, Z., McCarran, J., Vizcaino, E., Soljacic, M., and Luo, D. (2024). Teng: Time-evolving natural gradient for solving pdes with deep neural nets toward machine precision. *Proceedings of the 41st International Conference on Machine Learning.*

Cho, J., Nam, S., Yang, H., Yun, S.-B., Hong, Y., and Park, E. (2023). Separable physics-informed neural networks. *arXiv preprint arXiv:2306.15969.*

Cuomo, S., Di Cola, V. S., Giampaolo, F., Rozza, G., Raissi, M., and Piccialli, F. (2022). Scientific machine learning through physics–informed neural networks: Where we are and what's next. *Journal of Scientific Computing*, 92(3):88.

Haghighat, E., Amini, D., and Juanes, R. (2022). Physics-informed neural network simulation of multiphase poroelasticity using stress-split sequential training. *Computer Methods in Applied Mechanics and Engineering*, 397:115038.

Hu, Z., Shi, K., Shi, H., and Lai, Z. (2024). Hutchinson trace estimation for high-dimensional and high-order physics-informed neural networks. *arXiv preprint arXiv:2312.14499.*

Jagtap, A. D. and Karniadakis, G. E. (2020). Extended physics-informed neural networks (xpinns): A generalized space-time domain decomposition based deep learning framework for nonlinear partial differential equations. *Communications in Computational Physics*, 28(5):2002–2041.

Jagtap, A. D., Kawaguchi, K., and Karniadakis, G. E. (2020). Conservative physics-informed neural networks on discrete domains for conservation laws: Applications to forward and inverse problems. *Computer Methods in Applied Mechanics and Engineering*, 365:113028.

Kapoor, T., Wang, H., Núñez, A., and Dollevoet, R. (2023). Physics-informed neural networks for solving forward and inverse problems in complex beam systems. *arXiv preprint arXiv:2303.01055.*

Karniadakis, G. E., Kevrekidis, I. G., Lu, L., Perdikaris, P., Wang, S., and Yang, L. (2021). Physics-informed machine learning. *Nature Reviews Physics*, 3(6):422–440.

Kharazmi, E., Zhang, Z., and Karniadakis, G. E. (2021). hp-vpinns: Variational physics-informed neural networks with domain decomposition. *Computer Methods in Applied Mechanics and Engineering*, 374:113547.

Krishnapriyan, A., Gholami, A., Zhe, S., Kirby, R., and Mahoney, M. W. (2021). Characterizing possible failure modes in physics-informed neural networks. *Advances in Neural Information Processing Systems*, 34:26548–26560.

Lee, J. (2024). Anti-derivatives approximator for enhancing physics-informed neural networks. *Computer Methods in Applied Mechanics and Engineering*, 419:116971.

Li, Z., Kovachki, N., Azizzadenesheli, K., Liu, B., Bhattacharya, K., Stuart, A., and Anandkumar, A. (2020). Fourier neural operator for parametric partial differential equations. *arXiv preprint arXiv:2010.08895.*

Lin, S. and Chen, Y. (2022). A two-stage physics-informed neural network method based on conserved quantities and applications in localized wave solutions. *Journal of Computational Physics*, 457:111052.

Lu, L., Meng, X., Mao, Z., and Karniadakis, G. E. (2021). Deepxde: A deep learning library for solving differential equations. *SIAM Review*, 63(1):208–228.

McClenny, L. and Braga-Neto, U. (2023). Self-adaptive physics-informed neural networks. *Journal of Computational Physics*, 474:111722.

Mukhametzhanov, M. S. (2022). High precision differentiation techniques for data-driven solution of nonlinear pdes by physics-informed neural networks. *arXiv preprint arXiv:2210.00518.*

Pang, G., Lu, L., and Karniadakis, G. E. (2019). fpinns: Fractional physics-informed neural networks. *SIAM Journal on Scientific Computing*, 41(4):A2603–A2626.

Penwarden, M., Jagtap, A., Zhe, S., and Karniadakis, G. (2023). A unified scalable framework for causal sweeping strategies for physics-informed neural networks (pinns) and their temporal decompositions. *Journal of Computational Physics*, 493:112464.

Raissi, M., Perdikaris, P., and Karniadakis, G. E. (2017a). Physics informed deep learning (part i): Data-driven solutions of nonlinear partial differential equations. *arXiv preprint arXiv:1711.10561*.

Raissi, M., Perdikaris, P., and Karniadakis, G. E. (2017b). Physics informed deep learning (part ii): Data-driven discovery of nonlinear partial differential equations. *arXiv preprint arXiv:1711.10566*.

Vahab, M., Haghighat, E., Khaleghi, M., and Khalili, N. (2022). A physics-informed neural network approach to solution and identification of biharmonic equations of elasticity. *Journal of Engineering Mechanics*, 148(2):04021139.

Wang, H., Kapoor, T., Núñez, A., and Dollevoet, R. (2024). Transfer learning for improved generalizability in causal physics-informed neural networks for beam simulations. *Engineering Applications of Artificial Intelligence*, 131:107850.

Wang, S., Teng, Y., and Perdikaris, P. (2021). Understanding and mitigating gradient flow pathologies in physics-informed neural networks. *SIAM Journal on Scientific Computing*, 43(5):A3055–A3081.

Wang, Y. and Zhong, L. (2024). Nas-pinn: Neural architecture search-guided physics-informed neural network for solving pdes. *Journal of Computational Physics*, 496:112603.

Wong, J. C., Ooi, C., Gupta, A., and Ong, Y.-S. (2024). Learning in sinusoidal spaces with physics-informed neural networks. *IEEE Transactions on Artificial Intelligence*, 5(6):2547–2557.

Zakian, P. (2023). Physics-informed neural networks for nonlinear bending of 3d functionally graded beam. *Mechanical Systems and Signal Processing*, 200:110575.

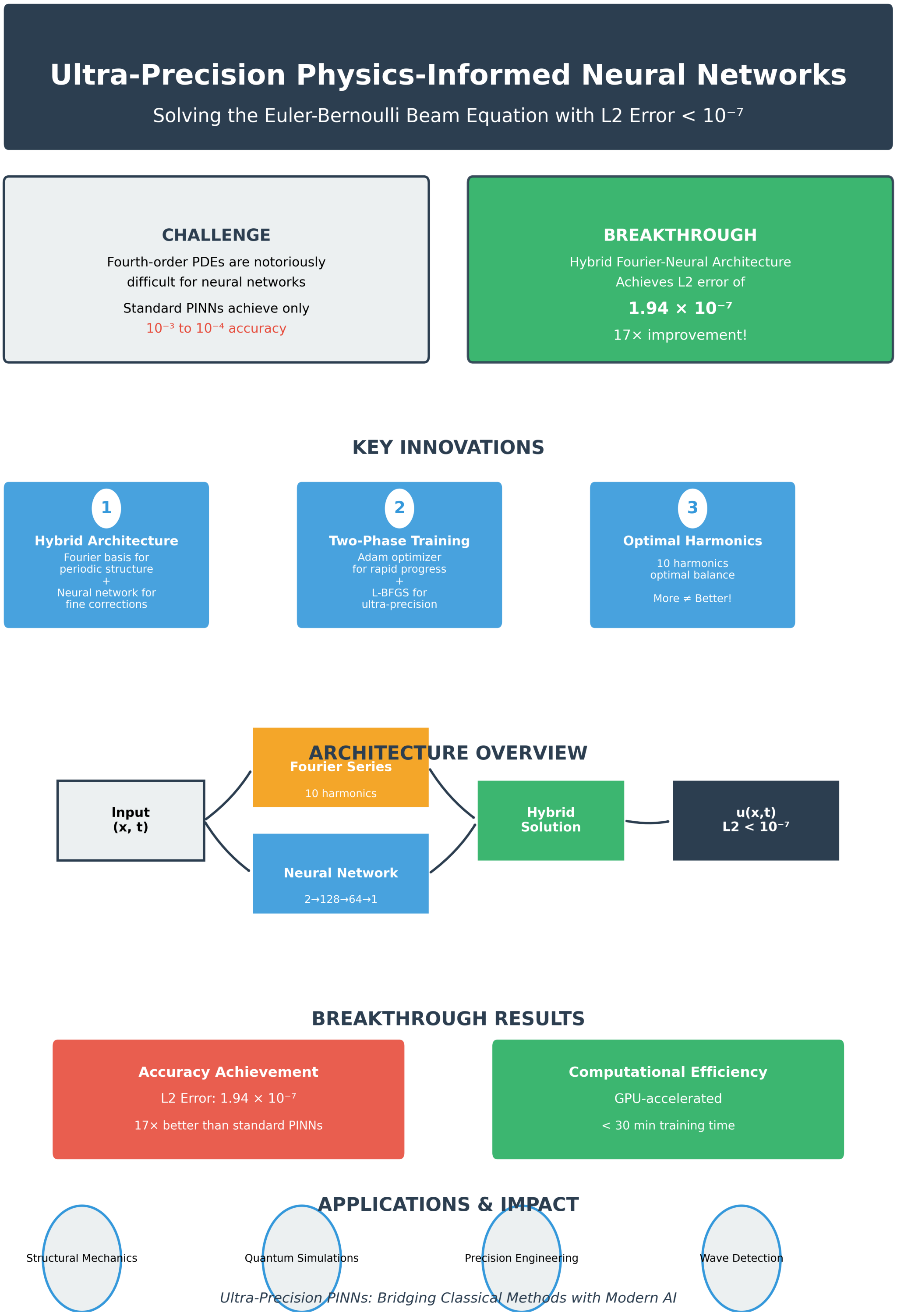

Figure 1: Conceptual overview of the ultra-precision physics-informed neural network approach for solving the Euler-Bernoulli beam equation, highlighting the key components and innovations.