Phase Transition in Ferromagnetic q-state Models: Contours, Long-Range Interactions and Decaying Fields

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Abstract

Using the group structure of the state space of q-state models and a new definition of contour for long-range spin-systems in \mathbb{Z}^d , with $d \geq 2$, a multidimensional version of Fröhlich-Spencer contours, we prove the phase transition for a class of ferromagnetic long-range systems which includes the Clock and Potts models. Our arguments work for the entire region of exponents of

includes the Clock and Potts models. Our arguments work for the entire region of exponents of regular power-law interactions, namely α > d, and for any q ≥ 2. As an application, we prove the phase transition for Potts models with decaying fields when the field decays fast enough.

1 Introduction

After the Ising model [31], one of the most studied models in statistical mechanics is its natural generalization when we have a q-state space (q ≥ 2), the Potts model [40] (for applications in several different areas of science see [42]). Since its appearance, a good amount of the literature was produced about the Potts model (we will mention a non-exhaustive list of papers), using several different tools like Reflection Positivity [10, 11, 12], mean-field theory [10, 11, 27, 36], random-cluster model [4, 7, 9, 17, 19, 20, 21, 22, 41], and Contours [35, 39, 43]. Many of the results have as their primary goal the description of the Gibbs measures at low temperatures, but, additionally, they have to put further restrictions, such as assuming that the dimension is d = 2, that the number of states q is big renough (with respect to the dimension d), or that the dimension d is sufficiently large. Most of the results consider nearest-neighbor interactions and, when long-range interactions are considered, they do not cover the entire region of the exponents of regular interactions, in the case of power-law decay [11, 34].

Since the emergence of Peierls' argument [37], contours have proven to be one of the most useful tools to get information about lattice systems at low-temperature, culminating in the celebrated Pirogov-Sinai theory [39, 43]. Nonetheless, the usual notion of connected contours has limited power to deal with long-range systems, due to the difficulty faced when trying to control the interaction between them [35].

In this paper, we will use the contours defined in [2] (see also [3]), which were inspired by the generalization of the one-dimension contours introduced by Fröhlich and Spencer [24] to dimension t

tween them is feasible (see section 4). With such control in hands, we can study the phase transition phenomenon for long-range lattice models over a finite state space with mild conditions. Our strategy combines our new definition of contour for long-range systems with an old approach which considers the group structure of the state space $\mathbb{Z}_q = \{1, \ldots, q\}$ as in Ginibre [26], and also in Gruber, Hintermann and Merlini [29]. The formalism allows us to explore the Fourier analysis on finite groups and then consider a class of interactions, including the Potts model and the Clock model.

In this paper we treat models whose the formal Hamiltonian can be written as

$$H(\sigma) = -\sum_{x,y} J_{xy} \varphi(\sigma_x - \sigma_y), \qquad (1.1)$$

where $\varphi: \mathbb{Z}_q \to \mathbb{R}$ is any function such that $\varphi(0) > \varphi(n), \forall n \neq 0$ (ferromagnetism) and J_{xy} obeys a polynomial decay with any exponent $\alpha > d$. We present below some well-known examples and their formal Hamiltonians that fit into these hypothesis:

<u>Clock model</u>: Also known as the vector Potts model, the clock model was introduced by Renfrey Potts in his PhD thesis [40], based on a suggestion by his advisor, Cyril Domb. The model generalizes the Ising model to describe situations where spins are not confined to a single direction but instead the q states are uniformly distributed over the circle S^1 . The formal Hamiltonian is given by

$$H = -\sum_{x,y} J_{xy} \cos\left(\frac{2\pi}{q}(\sigma_x - \sigma_y)\right). \tag{1.2}$$

<u>Potts model</u>: Appeared for the first time in [40]. The formal Hamiltonian is given by

$$H = -\sum_{x,y} J_{xy} \mathbb{1}_{\{\sigma_x = \sigma_y\}}.$$
(1.3)

After rescaling the energy, the Ising Hamiltonian corresponds to the case q=2 in the Potts model, but we have new phenomena. For example, when q is large enough, the magnetization becomes discontinuous at the critical temperature, and the q ordered phases coexist with the disordered one [11, 21, 22, 32, 41].

Most of the results concerning the Potts and Clock models are restricted to short-range interactions — that is, to cases where there exists R > 0 such that $J_{xy} = 0$ if |x - y| > R, some exceptions are [4, 30, 11, 34, 35].

Although it is possible to deduce the phase transition for the long-range case using information about the short-range one and correlation inequalities (like Griffiths' Inequalities presented in section 2), we will adopt a direct strategy and show the existence of the phase transition by means of contours and the Peierls' argument. It is undeniable that using contours brings many advantages, providing much information about the system and typical configurations. The phase transition results for the models without fields can be stated as follows.

Theorem 1.1. Let $q \geq 2$ be a natural number and consider the Hamiltonian (2.1) defined on the configuration space $\{1,...,q\}^{\mathbb{Z}^d}$, whose interactions decay according to a power-law with any exponent $\alpha > d$. Then, for every $C \in [0,1)$, there is $\beta_0 = \beta_0(C,\alpha,d,\varphi,q,J)$ such that the finite-volume Gibbs measure defined by Equation (2.2) satisfies

$$\mu_{\Lambda,\beta}^r(\sigma_0 = r) > C, \quad \forall \beta > \beta_0, \quad \forall r \in \mathbb{Z}_q.$$
 (1.4)

Corollary 1.2. Suppose that the Fourier transform $\widehat{\varphi}$ is non-negative. Then, for every $r, \ell \in \{1, ..., q\}$, $r \neq \ell$ implies that the thermodynamic limits μ_{β}^{r} and μ_{β}^{ℓ} obtained with monochromatic boundary conditions do exist and are different for every $\beta > \beta_{0}$.

A natural extension of this problem is the addition of a deterministic external field, a case where is not possible to be completely studied by correlation inequalities. For the Ising model, it is well-known by the Lee-Yang theory [33], that the phase transition is destroyed by any non-zero uniform field, no matter how small is its strength. The situation is much more complex when q is large enough [6, 8, 9, 28].

In the case of a non-translation invariant field, some results are known [5, 19, 38]. In the case of a decaying field, the modification in the hamiltonian does not change the free energy since the graph \mathbb{Z}^d is amenable. This class of fields was introduced for the Ising model (q=2) in [13], a collection of results for decaying fields in \mathbb{Z}^d is [1, 14, 13, 16]. There are some papers also on trees with fields as well, see [15, 18, 25].

To show the robustness of our methods, we prove the phase transition for the ferromagnetic Potts model with an external field with a sufficiently fast decay, both in the long-range (Theorem 1.3) and in the short-range case (Corollary 1.4). The proof produces the same region of exponents as in [1], but we use the contours defined in [2] to prove the following results.

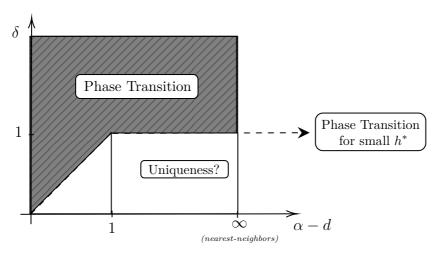


Figure 1: Phase Diagram.

Theorem 1.3. Suppose that there is $h^* \geq 0$ and $\delta > (\alpha - d) \wedge 1$ such that

$$h_{x,n} \le \frac{h^*}{|x|^{\delta}}, \forall x \in \mathbb{Z}^d, n \in \mathbb{Z}_q.$$
 (1.5)

Then, for any J > 0, $\alpha > d$ and J_{xy} defined as

$$J_{xy} := \begin{cases} \frac{J}{|x-y|^{\alpha}} & \text{if } x \neq y, \\ 0 & \text{otherwise,} \end{cases}$$
 (1.6)

there is phase transition for the Hamiltonian (1.7), when $\beta > 0$ is large enough.

$$H_{\Lambda,\mathbf{h}}^{q}(\sigma) = -\sum_{\{x,y\} \subset \Lambda} J_{xy} \mathbb{1}_{\{\sigma_x = \sigma_y\}} - \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} J_{xy} \mathbb{1}_{\{\sigma_x = q\}} - \sum_{x \in \Lambda} h_{x,\sigma_x}. \tag{1.7}$$

If $\delta = (\alpha - d) \wedge 1$, there is phase transition if h^* is small enough, and $\beta > 0$ is large enough.

Corollary 1.4. Consider the Hamiltonian (1.7) with short-range interactions given by

$$J_{xy} = \begin{cases} J & \text{if } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

As always, J > 0. Suppose, again, that there is $h^* \geq 0$ and $\delta > 1$ such that

$$h_{x,n} \le \frac{h^*}{|x|^{\delta}}, \forall x \in \mathbb{Z}^d, n \in \mathbb{Z}_q.$$
 (1.8)

Then, the model have phase transition when $\beta > 0$ is large enough. If $\delta = 1$, there is phase transition when h^* is small enough, and $\beta > 0$ is large enough.

This paper is divided as follows. In section 2, we present the relevant definitions. We also revisit correlation inequalities and the thermodynamic limit for q-state spin systems. The new contours are the protagonist of section 3, where the exponential growth in the number of possible contours is an important feature and can be found in [2]. The main computation is the energetic bound presented in section 4. These two ingredients are combined in section 5, which consists of the proof of Theorem 1.1. The application for models with decaying fields is proved in section 6. We finished the paper with section 7, where we mention possible consequences and problems for which this new notion of multi-scaled disconnected contours can be useful.

2 Preliminaries

Given $\Lambda \subset \mathbb{Z}^d$, we define the local configuration space as $\Omega_{\Lambda} := (\mathbb{Z}_q)^{\Lambda}$. When $\Lambda = \mathbb{Z}^d$, we simply put $\Omega := (\mathbb{Z}_q)^{\mathbb{Z}^d}$. Fixed $\eta \in \Omega$, we also define Ω^{η}_{Λ} as the subset of Ω consisting of configurations such that $\sigma_x = \eta_x$ for each $x \notin \Lambda$. Finally, we write $\Lambda \in \mathbb{Z}^d$ to indicate that Λ is finite.

Given $\eta \in \Omega$ and $\Lambda \subseteq \mathbb{Z}^d$, we will be interested in models whose Hamiltonian can be written as follows:

$$H_{\Lambda,\mathbf{h}}^{\eta}(\sigma) = -\sum_{\{x,y\} \subset \Lambda} J_{xy} \varphi(\sigma_x - \sigma_y) - \sum_{\substack{x \in \Lambda \\ y \notin \Lambda}} J_{xy} \varphi(\sigma_x - \eta_y) - \sum_{x \in \Lambda} h_x \varphi(\sigma_x), \tag{2.1}$$

where $\mathbf{h} = (h_x)_{x \in \mathbb{Z}^d}$ is any family of real numbers and $\{J_{xy}\}_{x,y \in \mathbb{Z}^d}$ is defined by Equation (1.6) for some J > 0 and $\alpha > d$. Furthermore, we ask the function $\varphi : \mathbb{Z}_q \to \mathbb{R}$ to be such that $\varphi(0) > \varphi(n), \forall n \neq 0$ (ferromagnetism). We will restrict our attention to monochromatic boundary conditions, that is, when $\eta_x = r, \forall x \in \mathbb{Z}^d$, for some $r \in \mathbb{Z}_q$, in which case we will simply write $H^r_{\Lambda, \mathbf{h}}$.

Denote by \mathcal{F}_{Λ} the σ -algebra generated by the cylindrical sets supported on Λ and write $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^d}$.

Definition 2.1. For any $\Lambda \subseteq \mathbb{Z}^d$, $\eta \in \Omega$ and $\beta > 0$, we define the corresponding finite-volume Gibbs measure on (Ω, \mathcal{F}) by

$$\mu_{\Lambda,\beta,\mathbf{h}}^{\eta}(\sigma) := \mathbb{1}_{\Omega_{\Lambda}^{\eta}}(\sigma) \frac{e^{-\beta H_{\Lambda,\mathbf{h}}^{\eta}(\sigma)}}{Z_{\Lambda,\beta,\mathbf{h}}^{\eta}}, \tag{2.2}$$

where β has the physical meaning of the inverse temperature and the normalization factor $Z_{\Lambda,\beta,\mathbf{h}}^{\eta}$ is known as the partition function, defined by

$$Z_{\Lambda,\beta,\mathbf{h}}^{\eta} := \sum_{\sigma \in \Omega_{\Lambda}} e^{-\beta H_{\Lambda,\mathbf{h}}^{\eta}(\sigma)}.$$

Similarly to the Hamiltonian, we write $Z_{\Lambda,\beta,\mathbf{h}}^r$ and $\mu_{\Lambda,\beta,\mathbf{h}}^r$ for monochromatic boundary conditions. Moreover, when $\mathbf{h} \equiv 0$, we will omit the subscript \mathbf{h} . Notice that the collection of all finite subsets of \mathbb{Z}^d , $\mathcal{P}_f(\mathbb{Z}^d)$, has the structure of a directed set given by the inclusion.

Definition 2.2. Fixed $\beta > 0$ and $r \in \Omega_0$, the limit points of the net of the finite-volume Gibbs measures $(\mu_{\Lambda,\beta,\mathbf{h}}^{\eta})_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)}$, with respect to the weak-* topology, are called the *thermodynamic limits*.

Since the set of all probability measures in this case is compact, there exists some thermodynamic limit Gibbs measure for any $\beta > 0$ and $r \in \mathbb{Z}_q$. As a consequence of theorem 1.1, we know that limit points for different monochromatic boundary conditions must be different. In itself, this result already implies the existence of (at least) q different Gibbs measure. In some cases, however, it is possible to know uniqueness of the limit points for each boundary condition. For the Potts model, this statement was proven in [4] (see also [17]) using the representation in terms of the random-cluster model. A more general approach is to use the Griffiths inequalities, in the framework provided by Ginibre [26], this will be the subject until the end of this section. Before presenting the result, let's introduce some notation. Denote by $\mathcal{C}(\Omega_{\Lambda})$ the set of all complex continuous functions on Ω_{Λ} .

Definition 2.3 (Convex Cone). Let Q be a subset of a vector space V. The set Q is called a convex cone if, for every $v_1, v_2 \in V$ and every scalars $\lambda_1, \lambda_2 \geq 0$, we have $\lambda_1 v_1 + \lambda_2 v_2 \in Q$.

Remark 2.1. In what follows, we are going to use some basic facts about harmonic analysis on locally compact Abelian groups. We refer the reader to [23] for a good exposition on the subject.

Definition 2.4 (Positive Semi-Definite Function). Given a group G, a function $\varphi: G \to \mathbb{R}$ is said positive semi-definite if, for any finite family $g_1, ..., g_n \in G$, the matrix $(\varphi(g_i^{-1}g_j))_{ij}$ is positive semi-definite, that is, denoting by B the corresponding bilinear form, then $B(v, v) \geq 0$, for any $v \in \mathbb{R}^n$.

Given $S \subset \mathcal{C}(\Omega_{\Lambda})$, denote by Q(S) the closure of the intersection of all convex cones in $\mathcal{C}(\Omega_{\Lambda})$ containing S and closed under multiplication. Given $H \in \mathcal{C}(\Omega_{\Lambda})$ real, define

$$\langle f \rangle_H := \left[\sum_{\omega \in \Omega_{\Lambda}} e^{-\beta H(\omega)} \right]^{-1} \sum_{\omega \in \Omega_{\Lambda}} f(\omega) e^{-\beta H(\omega)}.$$

Theorem 2.5 (Ginibre, 1970 [26]). Let $S \subset \mathcal{C}(\Omega_{\Lambda})$ be a self-conjugate set and $-H \in Q(S)$. If, for any finite collection $f_1, ..., f_n \in S$ and any finite sequence $s_1, ..., s_n \in \{0, 1\}$,

$$\sum_{\sigma \in \Omega_{\Lambda}} \sum_{\omega \in \Omega_{\Lambda}} \prod_{i=1}^{n} \left(f_i(\sigma) + (-1)^{s_i} f_i(\omega) \right) \ge 0, \tag{2.3}$$

then the two Griffiths' inequalities hold. That is,

- 1. $\langle f \rangle_H \geq 0, \forall f \in Q(S),$
- 2. $\langle fg \rangle_H \langle f \rangle_H \langle g \rangle_H \ge 0, \ \forall f, g \in Q(S).$

The condition (2.3) is called (Q3) in [26]. By example 4 of [26], (2.3) holds if we take $S = S_{\Lambda}$ as the set of real positive semi-definite functions in Ω_{Λ} . Since $S_{\Lambda} = Q(S_{\Lambda})$, the Theorem above tells us that the Griffiths' Inequalities hold provided that -H is positive semi-definite. The following lemma gives us another characterization for positive semi-definite functions.

Lemma 2.6. Let $\varphi: G \to \mathbb{C}$ be a function in $L^1(G)$. If the Fourier Transform $\widehat{\varphi}$ is in $L^1(\widehat{G})$ and $\widehat{\varphi} \geq 0$, then φ is positive semi-definite.

Proof. Let $g_1, ..., g_n$ be a finite collection of elements in G. We want to show that the matrix $(\varphi(g_i^{-1}g_j))_{ij}$ is positive semi-definite. Since $\widehat{\varphi} \in L^1(\widehat{G})$, we can use the Fourier inversion formula, which tells us that

$$\varphi(g_i^{-1}g_j) = \int_{\widehat{G}} \widehat{\varphi}(\chi) \widehat{g_i^{-1}g_j}(\chi) d\widehat{\mu}(\chi) = \int_{\widehat{G}} \widehat{\varphi}(\chi) \widehat{g_i^{-1}}(\chi) \widehat{g_j}(\chi) d\widehat{\mu}(\chi),$$

where $\widehat{\mu}$ is the Pontryagin dual measure of some Haar measure μ in G, and $\widehat{g}(\chi) = \chi(g)$ is the evaluation map. Now, notice that the bilinear form $\langle \cdot, \cdot \rangle_h$ defined on span $\{\widehat{g_1}, ..., \widehat{g_n}\}$ by

$$\langle u, v \rangle_h = \int_{\widehat{C}} h(\chi) u(\chi) \overline{v(\chi)} d\widehat{\mu}(\chi)$$

is positive semi-definite provided that $h(\chi) \geq 0$, $\forall \chi \in \widehat{G}$. Recalling that a matrix given by $(\langle v_i, v_j \rangle)_{ij}$ is positive semi-definite if $\langle \cdot, \cdot \rangle$ is so, we have that the matrix $(\varphi(g_i^{-1}g_j))_{ij} = \langle \widehat{g}_i, \widehat{g}_j \rangle_{\widehat{\varphi}})_{ij}$ is positive semi-definite.

Remark 2.2. By the Bochner's Theorem (see Theorem 4.18 of [23]), a function is positive semi-definite if, and only if, its Fourier transform is non-negative.

Proposition 2.7. The Fourier transforms of the functions $\varphi_{cl}(n) = \cos(2\pi n/q)$ and $\varphi_p(n) = \mathbb{1}_{\{n=0\}}$ are non-negative.

Proof. Recall that every character of \mathbb{Z}_q can be written in the form $\chi_k(n) = \exp\left(i\frac{2\pi}{q}kn\right)$, for some $k \in \mathbb{Z}_q$. For φ_{cl} , we can write

$$\varphi_{\rm cl}(n) = \cos\left(\frac{2\pi}{q}n\right) = \frac{1}{2}\exp\left(i\frac{2\pi}{q}n\right) + \frac{1}{2}\exp\left(-i\frac{2\pi}{q}n\right) = \frac{1}{2}\chi_1(n) + \frac{1}{2}\chi_{q-1}(n).$$

Since the Fourier Transform must be proportional to these coefficients, we conclude that it must be non-negative.

For the Potts one, we have $\widehat{\varphi_p}(k) = \sum_n \varphi_p(n) \overline{\chi_k(n)} = \overline{\chi_k(0)} = 1$, so $\widehat{\varphi_p}$ is also non-negative. \square

Corollary 2.8. Let $\Lambda \subseteq \mathbb{Z}^d$ and S_{Λ} be the set of all real positive semi-definite functions on Ω_{Λ} . Then, for any Hamiltonian of the form (2.1), if φ is positive semi-definite, we have

1. $\langle f \rangle_{\Lambda,\beta,\mathbf{h}}^q \geq 0$;

2.
$$\langle fg \rangle_{\Lambda,\beta,\mathbf{h}}^q - \langle f \rangle_{\Lambda,\beta,\mathbf{h}}^q \langle g \rangle_{\Lambda,\beta,\mathbf{h}}^q \geq 0$$
,

for any $f, g \in S_{\Lambda}$.

Proof. In first place notice that, although the results in Theorem 2.5 (according to [26]) are restricted to finite volumes only, for any Λ -local function $f: \Omega \to \mathbb{C}$, we have

$$\langle f \rangle_{\Lambda,\beta,\mathbf{h}}^q = \langle f \rangle_{H_{\Lambda,\mathbf{h}}^q} := \left[\sum_{\omega \in \Omega_{\Lambda}} e^{-\beta H_{\Lambda,\mathbf{h}}^q(\omega)} \right]^{-1} \sum_{\omega \in \Omega_{\Lambda}} f(\omega) e^{-\beta H_{\Lambda,\mathbf{h}}^q(\omega)},$$

where on the right-hand side, both f and $H_{\Lambda,\mathbf{h}}^q$ are being regarded as functions on Ω_{Λ} , since they are Λ -local.

Due to the previous lemma, we only need to show that $-H^q_{\Lambda,\mathbf{h}}$ is non-negative. Given an Abelian and finite group G, define $\delta: G \times G \to G$ by $\delta(g,h) = g-h$. If $\varphi: G \to \mathbb{C}$ has a non-negative Fourier transform, then $\varphi \circ \delta: G \times G \to \mathbb{C}$ has a non-negative Fourier transform as well. Indeed, recall that the dual of the product of two groups is the product of the respective dual groups (see Proposition 4.6 of [23]). Thus,

$$\widehat{\varphi \circ \delta}(\chi_1, \chi_2) = \sum_{n_1, n_2} \varphi(n_1 - n_2) \overline{\chi_1(n_1)} \ \overline{\chi_2(n_2)}.$$

Using the Inversion Formula,

$$\varphi(n) = \sum_{\xi \in \widehat{G}} \alpha_{\xi} \xi(n),$$

where $\alpha_{\xi} \geq 0$, by the hypothesis that φ has a non-negative Fourier transform. Substituting,

$$\widehat{\varphi \circ \delta}(\chi_1, \chi_2) = \sum_{n_1, n_2} \sum_{\xi \in \widehat{G}} \alpha_{\xi} \xi(n_1 - n_2) \overline{\chi_1(n_1)} \ \overline{\chi_2(n_2)}$$

$$= \sum_{\xi \in \widehat{G}} \alpha_{\xi} \left[\sum_{n_1 \in G} \xi(n_1) \overline{\chi_1(n_1)} \right] \left[\sum_{n_2 \in G} \xi^{-1}(n_2) \overline{\chi_2(n_2)} \right].$$

Recall that the characters of an Abelian finite group satisfy the following orthogonality relation:

$$\sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} = \begin{cases} |G| & \text{if } \chi_1 = \chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

This means that the only term of the summation over ξ that will be non-zero is the term $\xi = \chi_1$, provided that $\chi_1 = \chi_2^{-1}$. In summary,

$$\widehat{\varphi \circ \delta}(\chi_1, \chi_2) = \begin{cases} \alpha_{\chi_1} |G|^2 & \text{if } \chi_1 = \chi_2^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $\widehat{\varphi \circ \delta}(\chi_1, \chi_2) \geq 0$, as desired.

Finally, we need to show that, if $\psi: G_1 \to \mathbb{C}$ has a non-negative Fourier transform, and $\pi_1(g_1, g_2) = g_1$, then $\psi \circ \pi_1: G_1 \times G_2 \to \mathbb{C}$ has a non-negative Fourier transform as well. In fact,

$$\widehat{\psi \circ \pi_1}(\chi_1, \chi_2) = \sum_{n_1 \in G_1} \sum_{n_2 \in G_2} (\psi \circ \pi_1)(n_1, n_2) \overline{\chi_1(n_1)} \ \overline{\chi_2(n_2)}$$

$$= \sum_{n_2 \in G_2} \left(\sum_{n_1 \in G_1} \psi(n_1) \overline{\chi(n_1)} \right) \overline{\chi_2(n_2)}$$

$$= \widehat{\psi}(\chi_1) \sum_{n_2 \in G_2} \overline{\chi_2(n_2)}.$$

Then,

$$\widehat{\psi \circ \pi_1}(\chi_1, \chi_2) = \begin{cases} |G_2|\widehat{\psi}(\chi_1) & \text{if } \chi_2 \equiv 1, \\ 0 & \text{otherwise.} \end{cases}$$

With the previous facts and using that the set of real positive semi-definite functions is a convex cone, we have that $J_{xy}\varphi(\sigma_x-\sigma_y)$ are positive semi-definite on Ω_{Λ} , and hence the whole Hamiltonian.

By standard methods we can prove the following proposition.

Proposition 2.9. For any Λ -local function $f: \Omega \to \mathbb{R}$ with non-negative Fourier transform and $\beta > 0$,

- 1. the mapping $h_z \mapsto \langle f \rangle_{\Lambda \beta \mathbf{h}}^q$ is non-decreasing for any $z \in \Lambda$.
- 2. $\langle f \rangle_{\Delta,\beta,\mathbf{h}}^q \leq \langle f \rangle_{\Lambda,\beta,\mathbf{h}}^q$, for any $\Lambda \subset \Delta \in \mathbb{Z}^d$.

Corollary 2.10. For any local function $f: \Omega \to \mathbb{R}$ and $r \in \mathbb{Z}_q$, $\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle f \rangle_{\Lambda,\beta}^r$ exists.

Proof. Let's start supposing that r is the identity q. The Proposition above, together with the first Griffiths' inequality shows us that $\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle f\rangle_{\Lambda,\beta}^q$ must exist whenever f is a local function with nonnegative Fourier transform. Now, let f be any real local function. If f is an odd function, the fact that H_{Λ}^q is even implies that $\langle f\rangle_{\Lambda,\beta}^q=0$. Thus, we may suppose without loss of generality that f is even. By the inversion formula, we can write $f=\sum_k a_k\chi_k$, where $(\chi_k)_k$ are the characters of $(\mathbb{Z}_q)^{\Lambda}$, for some Λ where f is local. Since f is even, the coefficients are real and we can split the previous sum in its positive and negative parts. Explicitly, we define $f_+:=\sum_k b_k\chi_k$ and $f_-:=\sum_k c_k\chi_k$, where $b_k:=\max(a_k,0)$ and $c_k:=\max(-a_k,0)$, so we can write $f=f_+-f_-$ such that both f_+ and f_- have a non-negative Fourier transform. Since f is real, we know that $k\mapsto a_k$ needs to be even. By construction, it is obvious that both $k\mapsto b_k$ and $k\mapsto c_k$ are even, so f_+ and f_- are real. By the last proposition,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle f\rangle_{\Lambda,\beta,\mathbf{h}}^q=\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle f_+\rangle_{\Lambda,\beta,\mathbf{h}}^q-\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle f_-\rangle_{\Lambda,\beta,\mathbf{h}}^q$$

so the conclusion follows. Now, take any $r \in \mathbb{Z}_q$ and define $\tau_r : \Omega_{\Lambda} \to \Omega_{\Lambda}$ by $(\tau_r(\sigma))_x = \sigma_x - r$. Notice that $H^r_{\Lambda}(\sigma) = H^q_{\Lambda}(\tau_r(\sigma))$. Hence,

$$\langle f \rangle_{\Lambda,\beta}^r = \sum_{\sigma \in \Omega_{\Lambda}} f(\sigma) e^{-\beta H_{\Lambda}^r(\sigma)} = \sum_{\sigma \in \Omega_{\Lambda}} f(\tau_r^{-1}(\tau_r(\sigma))) e^{-\beta H_{\Lambda}^q(\tau_r(\sigma))}.$$

Since τ_r is a bijection, we have $\langle f \rangle_{\Lambda,\beta}^r = \langle f \circ \tau_r^{-1} \rangle_{\Lambda,\beta}^q$. By what was already proven, the limit exist for $f \circ \tau_r^{-1}$ in the q-boundary condition, so the limit of f exists in the r-boundary condition. \square

Remark 2.3. (Proof of Phase Transition via Griffiths' Inequalities) If we highlight the dependence with respect to the coupling J_{xy} , writing $\langle f \rangle_{\Lambda,\beta,\mathbf{J}}^q$, we can prove by standard methods that $J_{xy} \mapsto \langle f \rangle_{\Lambda,\beta,\mathbf{J}}^q$ is non-decreasing for any $x,y \in \mathbb{Z}^d$ and any real, local function f that is positive semi-definite. We know that for $d \geq 2$, the nearest-neighbors Potts model presents phase transition at low temperatures. The monotonicity with respect to J_{xy} implies the phase transition for the long-range Potts model. However, our goal is to present the new contours and a direct proof of the phase transition; the approach with contours can be used for further applications as for dealing with models with decaying fields, and many other problems.

3 Contours

In this section we define the notion of (M, a)-partition, which allow us to define the analogous to the Fröhlich-Spencer contours in the multidimensional setting.

Definition 3.1. Given a configuration σ , a point $x \in \mathbb{Z}^d$ is r-correct for σ if $\sigma_y = r$ for every $y \in B_1(x)$, where $B_1(x)$ is the unit ball in the ℓ_1 -norm centered at $x \in \mathbb{Z}^d$. A point is called *incorrect* for σ if it's not r-correct for any $r \in \mathbb{Z}_q$. The *boundary* of a configuration σ is the set $\partial \sigma$ of all incorrect points for σ .

For systems with finite-range interactions, we can define the contours of a configuration as the connected components of its boundary. In our case, the contours will also be defined by a partition of the boundary, but taking connected components is no longer suitable. We need to introduce the following notion.

Definition 3.2. Let M>0 and a>d. For each $A \in \mathbb{Z}^d$, a set $\Gamma(A):=\{\overline{\gamma}:\overline{\gamma}\subset A\}$ is called an (M,a)-partition of A when the following two conditions are satisfied.

- (A) They form a partition of A, i.e., $\bigcup_{\overline{\gamma} \in \Gamma(A)} \overline{\gamma} = A$.
- (B) For all $\overline{\gamma}, \overline{\gamma}' \in \Gamma(A)$, $\operatorname{dist}(\overline{\gamma}, \overline{\gamma}') > M \min\{|V(\overline{\gamma})|, |V(\overline{\gamma}')|\}^{\frac{a}{d+1}}, \tag{3.1}$

where $V(\Lambda)$ denotes the *volume* of $\Lambda \subseteq \mathbb{Z}^d$, and is given by $V(\Lambda) := \mathbb{Z}^d \setminus \Lambda^{(0)}$ with $\Lambda^{(0)}$ being the unique unbounded connected component of Λ^c . For any $A \subseteq \mathbb{Z}^d$, we denote by |A| its cardinality.

Even after fixing the parameters M and a, there can be multiple partitions of a set that are (M, a)-partitions. However, there is always a finest (M, a)-partition and we pick this one (see [2] for details). The finest (M, a)-partition of $A \in \mathbb{Z}^d$ satisfies the following property (see [3]):

(A1) For any $\overline{\gamma}, \overline{\gamma}' \in \Gamma(A), \overline{\gamma}'$ is contained in only one connected component of $(\overline{\gamma})^c$.

In this paper we will use $a := a(\alpha, d) = \frac{3(d+1)}{(\alpha-d) \wedge 1}$. The constant M will be appropriately chosen later.

Definition 3.3 (Contours). Given a configuration σ with finite boundary, its *contours* γ are pairs $(\overline{\gamma}, \sigma_{\overline{\gamma}})$, where $\overline{\gamma} \in \Gamma(\partial \sigma)$. The *support of the contour* γ is defined as $\operatorname{sp}(\gamma) := \overline{\gamma}$, and its *size* is given by $|\gamma| := |\operatorname{sp}(\gamma)|$.

With this definition, every configuration $\sigma \in \Omega_{\Lambda}^q$ is naturally associated to the family of contours $\Gamma(\sigma) := \{\gamma_1, ..., \gamma_n\}$, where the respective supports are the (M, a)-partition of $\Gamma(\partial \sigma)$.

Given a subset $\Lambda \in \mathbb{Z}^d$ we define its interior as $I(\Lambda) := V(\Lambda) \setminus \Lambda$. For the special case of a contour γ , we write $I(\gamma)$ and $V(\gamma)$ instead of $I(\operatorname{sp}(\gamma))$ and $V(\operatorname{sp}(\gamma))$. Moreover, we define $V(\Gamma) := \bigcup_{\gamma \in \Gamma} V(\gamma)$. We also define the $edge\ boundary$ of Λ as $\partial \Lambda := \{\{x,y\} \subset \mathbb{Z}^d; |x-y| = 1, x \in \Lambda, y \in \Lambda^c\}$, the inner boundary as $\partial_{in}\Lambda := \{x \in \Lambda; |x-y| = 1 \text{ for some } y \in \Lambda^c\}$ and the exterior boundary as $\partial_{ex}\Lambda := \{x \in \Lambda^c; |x-y| = 1 \text{ for some } y \in \Lambda\}$.

Also, denoting by $I(\gamma)^{(k)}$, k=1,...,n, the connected components of $I(\gamma)$, we can define the label map $lab_{\overline{\gamma}}: \{sp(\gamma)^{(0)}, I(\gamma)^{(1)}, \ldots, I(\gamma)^{(n)}\} \to \mathbb{Z}_q$ by taking the label of $sp(\gamma)^{(0)}$ as the spin of σ in $\partial_{in}V(\gamma)$ and the label of $I(\gamma)^{(k)}$ as the spin of σ in $\partial_{ex}V(I(\gamma)^{(k)})$. Notice that there can be connected components of a contour sitting inside its own interior. However, the labels are well-defined, since the spin of σ is constant in the boundaries of $sp(\gamma)$. The following sets will be useful

$$I_n(\gamma) := \bigcup_{\substack{k \ge 1, \\ \operatorname{lab}_{\operatorname{SD}(\gamma)}(\operatorname{I}(\gamma)^{(k)}) = n}} \operatorname{I}(\operatorname{sp}(\gamma))^{(k)}, \quad \operatorname{I}(\gamma) = \bigcup_{\substack{n \in \mathbb{Z}_q \\ n \ne q}} \operatorname{I}_n(\gamma), \quad \operatorname{I}'(\gamma) = \bigcup_{\substack{n \in \mathbb{Z}_q \\ n \ne q}} \operatorname{I}_n(\gamma). \tag{3.2}$$

Definition 3.4 (External Contours). A contour γ is external with respect to a family Γ if $\operatorname{sp}(\gamma) \cap V(\gamma') = \emptyset$ for every $\gamma' \in \Gamma \setminus \{\gamma\}$. We will denote Γ^e the family of all external contours from a given family of contours Γ .

In the usual Peierls' argument, the spin-flip symmetry is exploited in order to extract the contribution of a contour to the energy of a configuration. We will do the same here, but the spin-flip will be replaced by a transformation in the configuration space. Given some $\sigma \in \Omega_{\Lambda}^q$ and a contour $\gamma \in \Gamma^e(\sigma)$, we define

$$\tau_{\gamma}(\sigma)_{x} := \begin{cases} q & \text{if } x \in \operatorname{sp}(\gamma), \\ \sigma_{x} - n & \text{if } x \in \operatorname{I}_{n}(\gamma), \\ \sigma_{x} & \text{if } x \in V(\gamma)^{c}. \end{cases}$$

Notice that the effect of τ_{γ} in σ is to erase the contour γ . The last feature of the contours we will need (and a very crucial one) is the exponential growth of the numbers of contours with a given size. Define

$$C_y(n) := \{ \gamma : y \in V(\gamma), |\gamma| = n \}.$$

Proposition 3.5. Let $d \geq 2$, $y \in \mathbb{Z}^d$ and $\Lambda \in \mathbb{Z}^d$. There exists $c_1 := c_1(d, M, \alpha) > 0$ such that

$$|\mathcal{C}_y(n)| \le e^{(\log q + c_1)n}, \quad \forall n \ge 1.$$
 (3.3)

Proof. Consider the projection sp : $\mathcal{C}_0(n) \to \mathcal{P}_f(\mathbb{Z}^d)$ given by $(\overline{\gamma}, \sigma_{\overline{\gamma}}) \mapsto \overline{\gamma}$. Then,

$$C_0(n) = \bigcup_{A \in \operatorname{sp}(C_0(n))} \operatorname{sp}^{-1}(A).$$

Therefore,

$$|\mathcal{C}_0(n)| = \sum_{A \in \operatorname{sp}(\mathcal{C}_0(n))} |\operatorname{sp}^{-1}(A)|.$$

Now, note that $|\operatorname{sp}^{-1}(A)| \leq |\Omega_A| = q^{|A|} = e^{|A| \log q}$. Hence,

$$|\mathcal{C}_0(n)| \le \sum_{A \in \operatorname{sp}(\mathcal{C}_0(n))} e^{|A| \log q} = e^{n \log q} |\operatorname{sp}(\mathcal{C}_0(n))|.$$

Using the Corollary (3.28) from [2], we know that $|\operatorname{sp}(\mathcal{C}_0(n))| \leq e^{c_1 n}$. Therefore,

$$|\mathcal{C}_0(n)| = |\mathcal{C}_y(n)| \le e^{n \log q} \cdot e^{c_1 n} = e^{(c_1 + \log q)n}$$

4 Energy Bounds

In this section we are going to prove the main bounds of this work. Before that, we present two useful lemmas. Without loss of generality, we may suppose, by the addition of a constant and a suitable redefinition of J, that we can rewrite the Hamiltonian as

$$H_{\Lambda,\mathbf{h}}^{\eta}(\sigma) = \sum_{\{x,y\} \subset \Lambda} J_{xy} \psi(\sigma_x - \sigma_y) + \sum_{\substack{x \in \Lambda \\ y \notin \Lambda}} J_{xy} \psi(\sigma_x - \eta_y) + \sum_{x \in \Lambda} h_x \psi(\sigma_x),$$

with ψ such that $0 \leq \psi(n) \leq 1$, for any $n \in \mathbb{Z}_q$ and $\psi(0) = 0$. Explicitly, we can take

$$\psi(n) = \frac{\varphi(0) - \varphi(n)}{\varphi(0) - \min_{n \neq 0} \varphi(n)}.$$

After this redefinition, we denote by $m := \min\{\psi(n); n \neq 0\}$ the minimum excitation. Observe that m > 0.

Lemma 4.1. For any $x, y \in \mathbb{Z}^d$ such that $x \neq y$, it holds

$$J_{xy} \ge \frac{1}{(2d+1)2^{\alpha}} \sum_{x' \in B_1(x)} J_{x'y}.$$

Proof. Firstly, notice that we have

$$\sum_{x' \in B_1(x)} J_{x'y} = J_{xy} \sum_{x' \in B_1(x)} \frac{J_{x'y}}{J_{xy}} = J_{xy} \sum_{x' \in B_1(x) \setminus \{y\}} \left(\frac{|x-y|}{|x'-y|} \right)^{\alpha}.$$

Using the triangle inequality, it follows that:

$$\sum_{x' \in B_1(x)} J_{x'y} \le J_{xy} \sum_{x' \in B_1(x) \setminus \{y\}} \left(\frac{|x - x'|}{|x' - y|} + \frac{|x' - y|}{|x' - y|} \right)^{\alpha} = J_{xy} \sum_{x' \in B_1(x) \setminus \{y\}} \left(\frac{1}{|x' - y|} + 1 \right)^{\alpha}.$$

Since $1/|x'-y| \le 1$,

$$\sum_{x' \in B_1(x)} J_{x'y} \le J_{xy} \sum_{x' \in B_1(x) \setminus \{y\}} 2^{\alpha} \le J_{xy} (2d+1) 2^{\alpha},$$

and the inequality is proven.

Lemma 4.2. For any contour γ , and $y \in \mathbb{Z}^d$, it holds that

$$\sum_{\substack{x \in \operatorname{sp}(\gamma) \\ x' \in B_1(x) \\ x \neq y}} J_{x'y} \psi(\sigma_x - \sigma_y) \ge m \sum_{z \in \operatorname{sp}(\gamma)} J_{zy}.$$

Proof. Let z be an element of $\operatorname{sp}(\gamma)$. There exists x=x(z) in $\operatorname{sp}(\gamma)\cap B_1(z)$ such that $\sigma_x\neq\sigma_y$. In fact, if $\sigma_z\neq\sigma_y$, we can simply take x(z)=z. If $\sigma_z=\sigma_y$, since z is an incorrect point, there exists $x\in B_1(z)$ such that $\sigma_x\neq\sigma_z$, so $\sigma_x\neq\sigma_y$. But x will also be an incorrect point, so x must be in $\operatorname{sp}(\gamma)$. Remembering that $m=\min\{\psi(z);z\neq0\}$, we conclude that, for each $z\in\operatorname{sp}(\gamma)$, there exists x(z) such that $J_{zy}\psi(\sigma_{x(z)}-\sigma_y)\geq mJ_{zy}$. Summing over z,

$$m \sum_{z \in \operatorname{sp}(\gamma)} J_{zy} \le \sum_{z \in \operatorname{sp}(\gamma)} J_{zy} \psi(\sigma_{x(z)} - \sigma_y).$$

Now, since every term is non-negative, we can get an upper bound by summing also over x, then

$$m \sum_{\substack{z \in \operatorname{sp}(\gamma) \\ x \in \operatorname{sp}(\gamma) \\ |x-z|=1}} J_{zy} \psi(\sigma_x - \sigma_y).$$

Notice that the sum in the right-hand side is a sum over all ordered pairs (x, z) such that $x, z \in \operatorname{sp}(\gamma)$ and |x-z|=1, but this is the same as summing over $x \in \operatorname{sp}(\gamma)$ and then over $z \in B_1(x) \cap \operatorname{sp}(\gamma)$. Again using that each term is non-negative, we can drop the last conditions and we have

$$m \sum_{z \in \operatorname{sp}(\gamma)} J_{zy} \le \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ z \in B_1(x)}} J_{zy} \psi(\sigma_x - \sigma_y).$$

The following proposition, which gives us the energy of erasing a contour, will be the core of the Peierls' argument in the next section.

Proposition 4.3. There is a constant $c_2 = c_2(J, m, \alpha, d)$ such that, for any configuration $\sigma \in \Omega^q_{\Lambda}$ and $\gamma \in \Gamma^e(\sigma)$,

$$H_{\Lambda}^{q}(\sigma) - H_{\Lambda}^{q}(\tau) \ge c_2 \left(|\gamma| + F_{\operatorname{sp}(\gamma)} + \sum_{n=1}^{q-1} F_{\operatorname{I}_n(\gamma)} + F_{\operatorname{I}'(\gamma)} \right),$$

where $\tau := \tau_{\gamma}(\sigma)$ and $I'(\gamma)$ is given by Equation (3.2).

Proof. In first place, let's investigate how to write the Hamiltonian in terms of the contours. To do this, we will write the Hamiltonian in terms of the function ψ . Given subsets $A, B \in \mathbb{Z}^d$ and some configuration $\sigma \in \Omega_{\Lambda}^q$, we define

$$\psi(A, B)[\sigma] = \sum_{\substack{x \in A \\ y \in B}} J_{xy} \psi(\sigma_x - \sigma_y)$$

and

$$\psi(A)[\sigma] = \frac{1}{2}\psi(A, A)[\sigma] = \sum_{\{x,y\} \subset A} J_{xy}\psi(\sigma_x - \sigma_y).$$

Then, for any partition $\Lambda = \bigcup_{k=1}^n \Lambda_k$ of Λ , the Hamiltonian decomposes as

$$H_{\Lambda}^{q}(\sigma) = \sum_{k=1}^{n} \psi(\Lambda_{k})[\sigma] + \sum_{\{i,j\}} \psi(\Lambda_{i}, \Lambda_{j})[\sigma] + \sum_{k=1}^{n} \psi(\Lambda_{k}, \Lambda^{c})[\sigma].$$

We are interested in finding a lower bound for $H_{\Lambda}^q(\sigma) - H_{\Lambda}^q(\tau)$ depending only on γ . In order to do so, we are going to start by partitioning Λ into $\operatorname{sp}(\gamma) \cup \bigcup_{n=1}^q \operatorname{I}_n(\gamma) \cup \Lambda \setminus V(\gamma)$.

The previous remark gives us

$$H_{\Lambda}^{q}(\sigma) = \psi(\operatorname{sp}(\gamma))[\sigma] + \sum_{n=1}^{q} \psi(\operatorname{I}_{n}(\gamma))[\sigma] + \psi(\Lambda \setminus V(\gamma))[\sigma] + \sum_{n=1}^{q} \psi(\operatorname{sp}(\gamma), \operatorname{I}_{n}(\gamma))[\sigma]$$

$$+ \psi(\operatorname{sp}(\gamma), \Lambda \setminus V(\gamma))[\sigma] + \sum_{n=1}^{q} \psi(\operatorname{I}_{n}(\gamma), \Lambda \setminus V(\gamma))[\sigma] + \sum_{n \neq n'} \psi(\operatorname{I}_{n}(\gamma), \operatorname{I}_{n'}(\gamma))[\sigma]$$

$$+ \psi(\operatorname{sp}(\gamma), \Lambda^{c})[\sigma] + \sum_{n=1}^{q} \psi(\operatorname{I}_{n}(\gamma), \Lambda^{c})[\sigma] + \psi(\Lambda \setminus V(\gamma), \Lambda^{c})[\sigma],$$

where $n \neq n'$ indicates a summation over unordered pairs $\{n, n'\}$ of distinct elements of $\{1, ..., q\}$. Now, since we are interested in the difference of the Hamiltonians, let's define $\Delta(A, B)$ as $\psi(A, B)[\sigma] - \psi(A, B)[\tau]$ and $\Delta(A) = \Delta(A, A)/2$. Since the τ map leaves $\Lambda^c, \Lambda \setminus V(\gamma)$ and $I_q(\gamma)$ invariant we know that any term which only depends on these regions will be cancelled out. In a less obvious fashion, notice that the τ map acts on each I_n as a translation and, since ψ only depends on the difference between spins, $\psi(\sigma_x - \sigma_y) = \psi(\tau_x - \tau_y)$ whenever $x, y \in I_n(\gamma)$. Thus, $\Delta(I_n(\gamma)) = 0$. We are then left with

$$H_{\Lambda}^{q}(\sigma) - H_{\Lambda}^{q}(\tau) = \psi(\operatorname{sp}(\gamma))[\sigma] + \sum_{n=1}^{q} \psi(\operatorname{sp}(\gamma), \operatorname{I}_{n}(\gamma))[\sigma] + \psi(\operatorname{sp}(\gamma), V(\gamma)^{c})[\sigma]$$

$$- \sum_{n=1}^{q} \psi(\operatorname{sp}(\gamma), \operatorname{I}_{n}(\gamma))[\tau] - \psi(\operatorname{sp}(\gamma), V(\gamma)^{c})[\tau]$$

$$+ \sum_{n=1}^{q-1} \Delta(\operatorname{I}_{n}(\gamma), \Lambda \backslash V(\gamma)) + \sum_{n=1}^{q-1} \Delta(\operatorname{I}_{n}(\gamma), \Lambda^{c}) + \sum_{n=1}^{q-1} \Delta(\operatorname{I}_{n}(\gamma), \operatorname{I}_{q}(\gamma))$$

$$+ \sum_{\{n,n'\} \subset \{1,\dots,q-1\}} \Delta(\operatorname{I}_{n}(\gamma), \operatorname{I}_{n'}(\gamma)).$$

We can consider the union $Q(\gamma) = I_q(\gamma) \cup V(\gamma)^c$ and we rewrite the difference as

$$H^q_{\Lambda}(\sigma) - H^q_{\Lambda}(\tau) = (I) + (II) + (III),$$

where

$$(I) = \psi(\operatorname{sp}(\gamma))[\sigma] + \sum_{n=1}^{q} \psi(\operatorname{sp}(\gamma), \operatorname{I}_{n}(\gamma))[\sigma] + \psi(\operatorname{sp}(\gamma), V(\gamma)^{c})[\sigma]$$

$$(II) = -\sum_{n=1}^{q} \psi(\operatorname{sp}(\gamma), \operatorname{I}_{n}(\gamma))[\tau] - \psi(\operatorname{sp}(\gamma), V(\gamma)^{c})[\tau]$$

$$(III) = \sum_{n=1}^{q-1} \Delta(\operatorname{I}_{n}(\gamma), Q(\gamma)) + \sum_{\{n,n'\} \subset \{1,\dots,q-1\}} \Delta(\operatorname{I}_{n}(\gamma), \operatorname{I}_{n'}(\gamma)).$$

Now, we will bound (I), (II) e (III). The first line is equal to

$$\frac{1}{2} \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in \mathbb{Z}^d}} J_{xy} \psi(\sigma_x - \sigma_y) + \frac{1}{2} \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in \operatorname{sp}(\gamma)^c}} J_{xy} \psi(\sigma_x - \sigma_y), \tag{4.1}$$

so we face the task to provide a lower bound for the expression above. Clearly, many terms above will be zero — always that we have a pair of equal spins. However, we can use the fact that the contour is composed of incorrect points to see that, given a pair $\{x,y\}$ of sites with the same spin and $x \in \operatorname{sp}(\gamma)$, there exists a $x' \in B_1(x)$ such that $\{x',y\}$ is a pair of sites with different spins. Hence, it will be useful to consider averages of interactions across balls.

Now, using the previous inequality and the Lemma 4.2, we have

$$(I) = \frac{1}{2} \sum_{y \in \mathbb{Z}^d} \sum_{x \in \operatorname{sp}(\gamma)} J_{xy} \psi(\sigma_x - \sigma_y) + \frac{1}{2} \sum_{y \in \operatorname{sp}(\gamma)^c} \sum_{x \in \operatorname{sp}(\gamma)} J_{xy} \psi(\sigma_x - \sigma_y)$$

$$\geq \frac{1}{2} \sum_{y \in \mathbb{Z}^d} \frac{1}{(2d+1)2^{\alpha}} \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ x' \in B_1(x) \\ x \neq y}} J_{x'y} \psi(\sigma_x - \sigma_y) + \frac{1}{2} \sum_{y \in \operatorname{sp}(\gamma)^c} \frac{1}{(2d+1)2^{\alpha}} \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ x' \in B_1(x) \\ x \neq y}} J_{x'y} \psi(\sigma_x - \sigma_y)$$

$$\geq \frac{1}{2} \sum_{y \in \mathbb{Z}^d} \frac{m}{(2d+1)2^{\alpha}} \sum_{z \in \operatorname{sp}(\gamma)} J_{zy} + \frac{1}{2} \sum_{y \in \operatorname{sp}(\gamma)^c} \frac{m}{(2d+1)2^{\alpha}} \sum_{z \in \operatorname{sp}(\gamma)} J_{zy}.$$

Thus

$$(\mathrm{I}) \ge \frac{m}{(2d+1)2^{\alpha+1}} \left(\sum_{z \in \mathrm{sp}(\gamma)} \sum_{y \in \mathbb{Z}^d} J_{zy} + \sum_{\substack{z \in \mathrm{sp}(\gamma) \\ y \in \mathrm{sp}(\gamma)^c}} J_{zy} \right) \ge \frac{m}{(2d+1)2^{\alpha+1}} \left(Jc_{\alpha} |\gamma| + F_{\mathrm{sp}(\gamma)} \right),$$

where $c_{\alpha} := \sum_{y \neq 0} |y|^{-\alpha}$.

For the second term, we have

$$(II) = -\sum_{n=1}^{q} \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in I_n(\gamma)}} J_{xy} \psi(\tau_x - \tau_y) - \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in V(\gamma)^c}} J_{xy} \psi(\tau_x - \tau_y)$$

$$= -\sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in I(\gamma)}} J_{xy} \psi(\tau_x - \tau_y) - \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in V(\gamma)^c}} J_{xy} \psi(\tau_x - \tau_y)$$

$$= -\sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in \operatorname{sp}(\gamma)^c}} J_{xy} \psi(\tau_x - \tau_y) = -\sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in \operatorname{sp}(\gamma)^c}} J_{xy} \psi(q - \tau_y),$$

where we used the definition of the τ map.

Now, putting $\Gamma = \Gamma(\sigma)$, it's not difficult to see that $\tau_y \neq q$ implies that $y \in V(\Gamma \setminus \gamma)$. Thus, the summation is zero for any $y \notin V(\Gamma \setminus \gamma)$. This observation, together with Corollary 2.9 of [2], gives us

$$\sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in \operatorname{sp}(\gamma)^c}} J_{xy} \psi(\tau_x - \tau_y) \le \sum_{\substack{x \in \operatorname{sp}(\gamma) \\ y \in V(\Gamma \setminus \gamma)}} J_{xy} \le \kappa_\alpha^{(2)} \frac{F_{\operatorname{sp}(\gamma)}}{M^{(\alpha - d) \wedge 1}}, \tag{4.2}$$

where

$$\kappa_{\alpha}^{(2)} := (1 + J^{-1}) \left[\frac{J 2^{d-1+\alpha} e^{d-1}}{(\alpha - d)} + 3\zeta \left(\frac{a}{d+1} - 1 \right) \right].$$

Hence,

$$(II) = -\sum_{\substack{x \in sp(\gamma) \\ y \in sp(\gamma)^c}} J_{xy} \psi(\tau_x - \tau_y) \ge -\sum_{\substack{x \in sp(\gamma) \\ y \in V(\Gamma \setminus \gamma)}} J_{xy} \ge -\kappa_\alpha^{(2)} \frac{F_{\operatorname{sp}(\gamma)}}{M^{(\alpha - d) \wedge 1}}$$

As for the third term,

$$(III) = \frac{1}{2} \sum_{n=1}^{q-1} \sum_{\substack{n'=1\\n' \neq n}}^{q-1} \Delta(I_n(\gamma), I_{n'}(\gamma)) + \sum_{n=1}^{q-1} \Delta(I_n(\gamma), Q(\gamma))$$

$$= \frac{1}{2} \sum_{n=1}^{q-1} \Delta(I_n(\gamma), (I_n(\gamma) \cup \operatorname{sp}(\gamma))^c) + \frac{1}{2} \sum_{n=1}^{q-1} \Delta(I_n(\gamma), Q(\gamma))$$

$$= \frac{1}{2} \sum_{n=1}^{q-1} A_n(\gamma) + \frac{1}{2} B(\gamma),$$

where

$$A_n(\gamma) = \sum_{\substack{x \in I_n(\gamma) \\ y \notin I_n(\gamma) \cup \text{sp}(\gamma)}} J_{xy} \left(\psi(\sigma_x - \sigma_y) - \psi(\tau_x - \tau_y) \right),$$

and

$$B(\gamma) = \sum_{\substack{x \in I'(\gamma) \\ y \in Q(\gamma)}} J_{xy} \left(\psi(\sigma_x - \sigma_y) - \psi(\tau_x - \tau_y) \right).$$

Fixed some n, in order to bound $A_n(\gamma)$ we use Γ' to denote the set of contours inside $I_n(\gamma)$ and Γ'' to denote the set of contours outside $I_n(\gamma)$ (except for γ). Outside of the volumes of Γ' and Γ'' ,



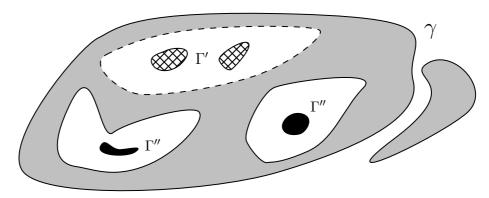


Figure 2: The grouping of contours performed to bound $A_n(\gamma)$. The contour γ is painted gray. The interior $I_n(\gamma)$ is highlighted with a dashed line, and the family of contours inside it, denoted by Γ' , is filled with a checkered background. In contrast, the family of contours Γ'' , outside $I_n(\gamma)$ is filled with a solid black color.

the spins are controllable, that is,

$$\sigma_y = \begin{cases} n & \text{if } y \in I_n(\gamma) \backslash V(\Gamma'), \\ n' & \text{if } y \in I_{n'}(\gamma) \backslash V(\Gamma''), \\ q & \text{if } y \in Q(\gamma) \backslash V(\Gamma''). \end{cases}$$

This motivates us to split $A_n(\gamma)$ in terms of this sets. Explicitly,

$$\begin{split} A_{n}(\gamma) &= \sum_{\substack{x \in \mathcal{I}_{n}(\gamma) \\ y \notin \mathcal{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma)}} J_{xy} \left(\psi(\sigma_{x} - \sigma_{y}) - \psi(\tau_{x} - \tau_{y}) \right) \\ &= \sum_{\substack{x \in \mathcal{I}_{n}(\gamma) \\ y \in V(\Gamma'')}} J_{xy} \left(\psi(\sigma_{x} - \sigma_{y}) - \psi(\sigma_{x} - \sigma_{y}) \right) + \sum_{\substack{x \in \mathcal{I}_{n}(\gamma) \\ y \notin \mathcal{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} \left(\psi(\sigma_{x} - \sigma_{y}) - \psi(\sigma_{x} - \sigma_{y}) \right) \\ &= \sum_{\substack{x \in \mathcal{I}_{n}(\gamma) \\ y \in V(\Gamma'')}} J_{xy} \left(\psi(\sigma_{x} - \sigma_{y}) - \psi(\sigma_{x} - \sigma_{y}) \right) + \sum_{\substack{x \in V(\Gamma') \\ y \notin \mathcal{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} \left(\psi(\sigma_{x} - \sigma_{y}) - \psi(\sigma_{x} - \sigma_{y}) \right) \\ &+ \sum_{\substack{x \in \mathcal{I}_{n}(\gamma) \setminus V(\Gamma') \\ y \notin \mathcal{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} \left(\psi(\sigma_{x} - \sigma_{y}) - \psi(\sigma_{x} - \sigma_{y}) \right) . \end{split}$$

In the first two summations we will use $\psi(\sigma_x - \sigma_y) - \psi(\tau_x - \tau_y) \ge -1$. In the last one, we know that $\psi(\sigma_x - \sigma_y) - \psi(\tau_x - \tau_y) = \psi(n - n') - 0 \ge m$. Thus,

$$A_{n}(\gamma) \geq \sum_{\substack{x \in \mathcal{I}_{n}(\gamma) \setminus V(\Gamma') \\ y \notin I_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} mJ_{xy} - \sum_{\substack{x \in \mathcal{I}_{n}(\gamma) \\ y \in V(\Gamma'')}} J_{xy} - \sum_{\substack{x \in V(\Gamma') \\ y \notin I_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy}. \tag{4.3}$$

Now, notice that

$$F_{\mathbf{I}_{n}(\gamma)} = \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \in \mathbf{I}_{n}(\gamma)^{c}}} J_{xy} = \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \in V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy}.$$

$$= \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \setminus V(\Gamma') \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma)}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \notin \mathbf{I}_{n}(\gamma)}} J_{xy} + \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \cup V(\Gamma'')}} J_{xy} + \sum_{$$

Rearranging, we are left with

$$\sum_{\substack{x \in \mathcal{I}_n(\gamma) \backslash V(\Gamma') \\ y \notin \mathcal{I}_n(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} = F_{\mathcal{I}_n(\gamma)} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in V(\Gamma'')}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \notin \mathcal{I}_n(\gamma) \cup \operatorname{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in \operatorname{sp}(\gamma)}}} J_{xy} - \sum_{\substack{x \in \mathcal{I}_n(\gamma) \\ y \in$$

Now, substituting the last expression in Equation (4.3),

$$A_{n}(\gamma) \geq m \left(F_{\mathbf{I}_{n}(\gamma)} - \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \in \mathrm{sp}(\gamma)}} J_{xy} \right) - (1+m) \left(\sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \in V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in V(\Gamma') \\ y \notin \mathbf{I}_{n}(\gamma) \cup \mathrm{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} \right)$$

$$\geq \frac{m}{(2d+1)2^{\alpha+2}} \left(F_{\mathbf{I}_{n}(\gamma)} - \sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \in \mathrm{sp}(\gamma)}} J_{xy} \right) - (1+m) \left(\sum_{\substack{x \in \mathbf{I}_{n}(\gamma) \\ y \in V(\Gamma'')}} J_{xy} + \sum_{\substack{x \in V(\Gamma') \\ y \notin \mathbf{I}_{n}(\gamma) \cup \mathrm{sp}(\gamma) \cup V(\Gamma'')}} J_{xy} \right)$$

Again using Corollary 2.9 from [2],

$$A_{n}(\gamma) \geq \frac{m}{(2d+1)2^{\alpha+2}} \left(F_{I_{n}(\gamma)} - \sum_{\substack{x \in I_{n}(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} \right) - (1+m) \left(\kappa_{\alpha}^{(2)} \frac{F_{I_{n}(\gamma)}}{M^{(\alpha-d)\wedge 1}} + \kappa_{\alpha}^{(2)} \frac{F_{I_{n}(\gamma)}}{M} \right)$$

$$\geq \frac{m}{(2d+1)2^{\alpha+2}} \left(F_{I_{n}(\gamma)} - \sum_{\substack{x \in I_{n}(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy} \right) - 2(1+m) \kappa_{\alpha}^{(2)} \frac{F_{I_{n}(\gamma)}}{M^{(\alpha-d)\wedge 1}}$$

$$\geq \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{4\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}} \right) F_{I_{n}(\gamma)} - \frac{m}{(2d+1)2^{\alpha+2}} \sum_{\substack{x \in I_{n}(\gamma) \\ y \in \operatorname{sp}(\gamma)}} J_{xy}.$$

Then,

$$\begin{split} \sum_{n=1}^{q-1} A_n(\gamma) & \geq \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{4\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}}\right) \sum_{n=1}^{q-1} F_{\mathbf{I}_n(\gamma)} - \frac{m}{(2d+1)2^{\alpha+2}} \sum_{n=1}^{q-1} \sum_{\substack{x \in \mathbf{I}_n(\gamma) \\ y \in \mathrm{sp}(\gamma)}} J_{xy} \\ & = \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{4\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}}\right) \sum_{m=1}^{q-1} F_{\mathbf{I}_n(\gamma)} - \frac{m}{(2d+1)2^{\alpha+2}} \sum_{\substack{x \in \bigcup_{n=1}^{q-1} \mathbf{I}_n(\gamma) \\ y \in \mathrm{sp}(\gamma)}} J_{xy} \\ & \geq \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{4\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}}\right) \sum_{m=1}^{q-1} F_{\mathbf{I}_n(\gamma)} - \frac{m}{(2d+1)2^{\alpha+2}} F_{\mathrm{sp}(\gamma)}. \end{split}$$

The bound for $B(\gamma)$ is completely analogous, yielding

$$B(\gamma) \ge \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{4\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}}\right) F_{\mathrm{I}'(\gamma)} - \frac{m}{(2d+1)2^{\alpha+2}} F_{\mathrm{sp}(\gamma)}.$$

Finally, we are left with

(III) =
$$\frac{1}{2} \sum_{m=1}^{q-1} A_n(\gamma) + \frac{1}{2} B(\gamma)$$

$$\geq \frac{1}{2} \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{4\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}} \right) \sum_{n=1}^{q-1} F_{\mathbf{I}_{n}(\gamma)} - \frac{m}{(2d+1)2^{\alpha+3}} F_{\mathrm{sp}(\gamma)}$$

$$+ \frac{1}{2} \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{4\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}} \right) F_{\mathbf{I}'(\gamma)} - \frac{m}{(2d+1)2^{\alpha+3}} F_{\mathrm{sp}(\gamma)}$$

$$\geq \left(\frac{m}{(2d+1)2^{\alpha+3}} - \frac{2\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}} \right) \left(\sum_{n=1}^{q-1} F_{\mathbf{I}_{n}(\gamma)} + F_{\mathbf{I}'(\gamma)} \right) - \frac{m}{(2d+1)2^{\alpha+2}} F_{\mathrm{sp}(\gamma)}.$$

Since

$$H^q_{\Lambda}(\sigma) - H^q_{\Lambda}(\tau) = (I) + (II) + (III),$$

we obtain that

$$\begin{split} H_{\Lambda}^{q}(\sigma) - H_{\Lambda}^{q}(\tau) &\geq \frac{m}{(2d+1)2^{\alpha+1}} \left(Jc_{\alpha} |\gamma| + F_{\mathrm{sp}(\gamma)} \right) - \kappa_{\alpha}^{(2)} \frac{F_{\mathrm{sp}(\gamma)}}{M^{(\alpha-d)\wedge 1}} \\ &+ \left(\frac{m}{(2d+1)2^{\alpha+3}} - \frac{2\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\Lambda 1}} \right) \left(\sum_{n=1}^{q-1} F_{\mathrm{I}_{n}(\gamma)} + F_{\mathrm{I}'(\gamma)} \right) - \frac{m}{(2d+1)2^{\alpha+2}} F_{\mathrm{sp}(\gamma)} \\ &\geq \frac{Jmc_{\alpha}}{(2d+1)2^{\alpha+1}} |\gamma| + \left(\frac{m}{(2d+1)2^{\alpha+1}} - \frac{m}{(2d+1)2^{\alpha+2}} - \frac{\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}} \right) F_{\mathrm{sp}(\gamma)} \\ &+ \left(\frac{m}{(2d+1)2^{\alpha+3}} - \frac{2\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}} \right) \left(\sum_{n=1}^{q-1} F_{\mathrm{I}_{n}(\gamma)} + F_{\mathrm{I}'(\gamma)} \right). \end{split}$$

Thus, we conclude that

$$H_{\Lambda}^{q}(\sigma) - H_{\Lambda}^{q}(\tau) \ge \frac{Jmc_{\alpha}}{(2d+1)2^{\alpha+1}} |\gamma| + \left(\frac{m}{(2d+1)2^{\alpha+2}} - \frac{\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}}\right) F_{\text{sp}(\gamma)} + \left(\frac{m}{(2d+1)2^{\alpha+3}} - \frac{2\kappa_{\alpha}^{(2)}}{M^{(\alpha-d)\wedge 1}}\right) \left(\sum_{n=1}^{q-1} F_{I_{n}(\gamma)} + F_{I'(\gamma)}\right).$$

Taking $M^{(\alpha-d)\wedge 1} > 2^{\alpha+5}(2d+1)\kappa_{\alpha}^{(2)}m^{-1}$ and $c_2 = \frac{m}{(2d+1)2^{\alpha+1}}\min\left\{Jc_{\alpha}, \frac{1}{8}\right\}$, the result of the demonstration follows.

5 Phase Transition

In this section we prove Theorem 1.1, that is, the long-range Potts model with zero field undergoes a phase transition at low temperature. More precisely, we are going to prove that, for any $r, \ell \in \{1, \ldots, q\}$, if $r \neq \ell$, then the thermodynamic limits, μ_{β}^r and μ_{β}^{ℓ} , are also different when β is large enough.

Proof of Theorem 1.1. The proof of Equation (1.4) is the standard Peierls' argument. In fact, if $\sigma_{\Lambda^c} = r$ and $\sigma_0 \neq r$, then there must exist a contour γ with $0 \in V(\gamma)$. Then,

$$\mu_{\Lambda,\beta}^{r}(\sigma_{0} \neq r) \leq \mu_{\Lambda,\beta}\left(\left\{\sigma \in \Omega_{\Lambda}^{r}; \ \exists \ \gamma \in \Gamma^{e}(\sigma), \ 0 \in V(\gamma)\right\}\right) \leq \sum_{\gamma: \ 0 \in V(\gamma)} \mu_{\Lambda,\beta}^{r}\left(\left\{\sigma \in \Omega_{\Lambda}^{r}; \ \gamma \in \Gamma^{e}(\sigma)\right\}\right).$$

Let $\Omega(\gamma) = \{ \sigma \in \Omega^r_{\Lambda}; \ \gamma \in \Gamma^e(\sigma) \}$. Using Proposition 4.3, we have $\mu^r_{\Lambda,\beta}(\Omega(\gamma)) \leq e^{-\beta c_2|\gamma|}$. Then,

$$\mu_{\Lambda,\beta}^{r}(\sigma_{0} \neq r) \leq \sum_{\gamma: 0 \in V(\gamma)} e^{-\beta c_{2}|\gamma|} = \sum_{n \geq 1} e^{-\beta c_{2}n} |\{\gamma; 0 \in V(\gamma), |\gamma| = n\}|.$$

By Proposition 3.5,

$$\mu_{\Lambda,\beta}^r(\sigma_0 \neq r) \leq \sum_{n \geq 1} e^{-\beta c_2 n} \cdot e^{(c_1 + \log q)n} = \sum_{n \geq 1} e^{-(\beta c_2 - c_1 - \log q)n} = \frac{e^{-(\beta c_2 - c_1 - \log q)}}{1 - e^{-(\beta c_2 - c_1 - \log q)}}.$$

Then,

$$\mu_{\Lambda,\beta}^r(\sigma_0 = r) \le 1 - \frac{e^{-(\beta c_2 - c_1 - \log q)}}{1 - e^{-(\beta c_2 - c_1 - \log q)}},$$

which goes to 1 when $\beta \to \infty$.

Proof of Corollary 1.2. By the proposition above, there is β_0 such that

$$\mu_{\Lambda,\beta}^r(\sigma_0 \neq r) < \frac{1}{4}, \forall \beta > \beta_0, \forall r \in \mathbb{Z}_q.$$
 (5.1)

Since $\mu_{\Lambda,\beta}^r(\sigma_0=r) + \mu_{\Lambda,\beta}^r(\sigma_0\neq r) = 1$, we have that $\mu_{\Lambda,\beta}^r(\sigma_0=r) > \frac{3}{4}$. By Equation (5.1) and taking the thermodynamic limit (which exists by Corollary 2.10), $\mu_{\beta}^{\ell}(\sigma_0=r) \leq \mu_{\beta}^{\ell}(\sigma_0\neq \ell) \leq 1/4$, while $\mu_{\beta}^r(\sigma_0=r) \geq 3/4$.

Remark 5.1. We can take $\beta_0 = (c_1 + \ln 5 + \log q)/c_2$.

6 Application: Potts Model with Decaying Field

As an example of the robustness of our methods for proving phase transition, this section will present the occurrence of phase transition for the Potts model in the presence of a decaying field as an application.

The Hamiltonian of the Potts model with a general external field can be written as follows.

$$H_{\Lambda,\mathbf{h}}^{q}(\sigma) = -\sum_{\{x,y\} \subset \Lambda} J_{xy} \mathbb{1}_{\{\sigma_x = \sigma_y\}} - \sum_{\substack{x \in \Lambda \\ x \in \Lambda^c}} J_{xy} \mathbb{1}_{\{\sigma_x = q\}} - \sum_{x \in \Lambda} h_{x,\sigma_x}, \tag{6.1}$$

where $\mathbf{h} = (h_{x,n})_{\substack{x \in \mathbb{Z}^d \\ n \in \mathbb{Z}_a}}$ is a family of non-negative real numbers.

Proof of Theorem 1.3. Let $\sigma \in \Omega^q_{\Lambda}$ be any configuration and $\gamma \in \Gamma^e(\sigma)$. Define as before $\tau := \tau_{\gamma}(\sigma)$. Using Proposition 4.3, we have

$$H_{\Lambda,\mathbf{h}}^{q}(\sigma) - H_{\Lambda,\mathbf{h}}^{q}(\tau) = H_{\Lambda}^{q}(\sigma) - H_{\Lambda}^{q}(\tau) - \left(\sum_{x \in \operatorname{sp}(\gamma) \cup \operatorname{I}'(\gamma)} h_{x,\sigma_{x}} - h_{x,\tau_{x}}\right)$$

$$\geq c_{2} \left(|\gamma| + F_{\operatorname{sp}(\gamma)} + \sum_{n=1}^{q-1} F_{\operatorname{I}_{n}(\gamma)}\right) - \sum_{x \in \operatorname{sp}(\gamma) \cup \operatorname{I}'(\gamma)} h_{x,\sigma_{x}}$$

$$= \left(c_{2}|\gamma| - \sum_{x \in \operatorname{sp}(\gamma)} h_{x,\sigma_{x}}\right) + \sum_{n=1}^{q-1} \left(c_{2}F_{\operatorname{I}_{n}(\gamma)} - \sum_{x \in \operatorname{I}_{n}(\gamma)} h_{x,\sigma_{x}}\right).$$

Proceeding similarly to [1], we refer to the Theorem 7.33 of [25], which allows us to replace the original field by a truncated one given by

$$\widehat{h}_{x,n} = \begin{cases} h_{x,n} & \text{if } |x| \ge R, \\ 0 & \text{if } |x| < R, \end{cases}$$

where R will be chosen later, without compromising the existence (or not) of the phase transition. Notice that, by Equation (1.5),

$$\sum_{x \in \Lambda} \widehat{h}_{x,n} \le \frac{h^* |\Lambda|}{R^{\delta}},\tag{6.2}$$

so that $R^{\delta} > 2h^*/c_2$ gives us

$$|c_2|\gamma| - \sum_{x \in \operatorname{sp}(\gamma)} h_{x,\sigma_x} \ge \frac{c_2}{2} |\gamma|.$$

Now, using again Equation (1.5), the only remaining thing to be shown is that, for any finite subset $\Lambda \subseteq \mathbb{Z}^d$,

$$c_2 F_{\Lambda} - \sum_{x \in \Lambda} \widehat{h}_{x,n} \ge 0. \tag{6.3}$$

This analysis was already performed in Proposition 4.7 from [1], and is guaranteed for $\delta > (\alpha - d) \wedge 1$ or $\delta = (\alpha - d) \wedge 1$ if h^* is large enough.

Although in this paper we have been mainly concerned with the long-range case, the methods developed here are also useful for the short-range one. Notice that the nearest-neighbour Potts model consists of the interactions given by (1.6) when $\alpha \to +\infty$, so it is natural to expect that the Theorem above also holds in this case for $\delta > 1$. The proof is very similar to the long-range case and the sketch of the proof is presented below.

Proof of Corollary 1.4. The proof starts by following the same lines as the proof of Theorem 1.3. The difference is that, in the short-range case, we have $F_{\Lambda} = J|\partial\Lambda|$. A quick computation can show us that we still have

$$H_{\Lambda}^{q}(\sigma) - H_{\Lambda}^{q}(\tau) \ge c_2' \left(|\gamma| + \sum_{n=1}^{q-1} |\partial I_n(\gamma)| \right),$$

for some constant c'_2 . The unique inequality left to be proven is, thus,

$$c_2'|\partial I_n(\gamma)| - \sum_{x \in I_n(\gamma)} \widehat{h}_x \ge 0.$$

Now, notice that, in the case $\alpha > d+1$, the analysis done for (6.3) in [1] uses that $F_{\Lambda} \geq K|\partial\Lambda|$ for some constant K > 0, so the computation performed is exactly the same.

7 Concluding Remarks

In the proof of the phase transition, only a portion of Proposition 4.3 was used — specifically, the requirement that the difference between the Hamiltonian is bounded below by something proportional to $|\gamma|$. However, the full estimate is crucial for further applications, such as establishing the convergence of the cluster expansion at low temperatures. This is certainly achievable for the models addressed in this paper by a straightforward adaptation of the methods from [3].

Another natural direction for future works is the improvement of the results by Park [34, 35] concerning a Pirogov-Sinai theory for long-range interactions to every $\alpha > d$, which in his case is valid only for $\alpha > 3d + 1$. These results will be presented out in subsequent papers already in preparation.

The phase diagram for Ising and Potts models with decaying fields is not complete even in the short-range case, where it remains without an answer the question if for the critical exponent $\delta = 1$ we can have uniqueness when h^* is big enough, see [13]. The proof of the uniqueness when the field decays slowly is not standard, and the only known argument is a combination of results from [13] and

[19], only in the nearest-neighbor case. Since the uniqueness of the Gibbs state was obtained via a contour argument in the short-range case (see [13]), it is natural to investigate if the arguments work for disconnected contours in long-range models. The phase transition was proved for these models via the Peierls argument for the one-dimensional case in [16], and for what seems to be a sharp region for the exponents in the multidimensional case in [1] (the same region obtained in the present paper).

Many results for short-range q—state models depend on a notion of contour. For the moment, in addition to the present paper, we have in the multidimensional setting the proof of the phase transition for the long-range ferromagnetic Ising model with a random field Ising model [2] ($d \ge 3$), and with a decaying field [1]. Also, the proof of convergence of the cluster expansion at low temperatures for the long-range ferromagnetic Ising model, which implies the control of the decay of correlations, see [3].

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