Imperial College London

Computational Neurodynamics

Topic 7
Small-world Networks

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Overview

- Brain networks
- The Watts-Strogatz procedure
- Small-world index
- Global and local efficiency
- Scale-free networks

Connectivity

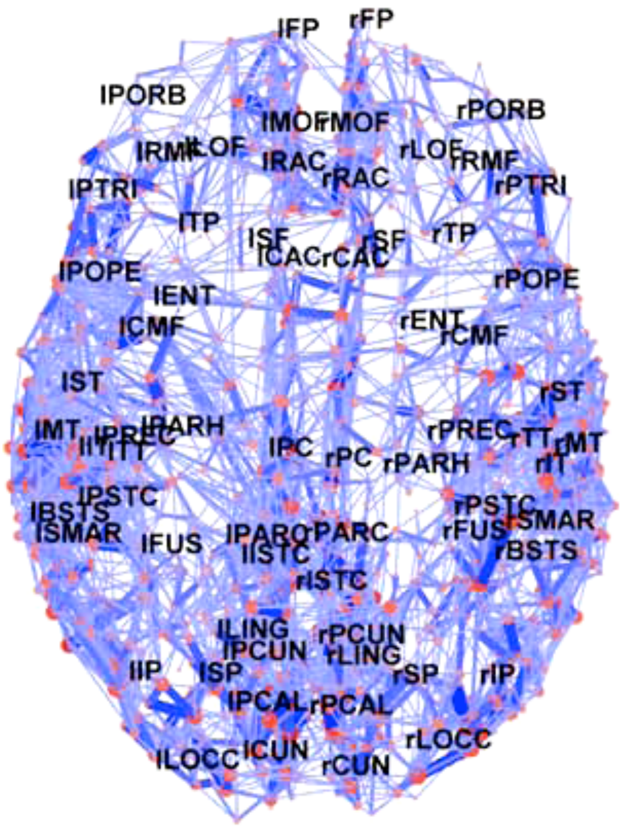
- All the networks of neurons we've looked at up to now have very simple patterns of connectivity
- They are partitioned into layers with no internal connections, and only feed-forward connections between the layers
- Connectivity patterns in real brains are much more complex, and the mathematical tools of network theory can be used to study them
  - Complex networks are a major area of study, because they arise in many contexts, such as social networks, the world wide web, and genetics, in addition to neuroscience
- Connectivity significantly affects neural dynamics, and one of the main issues we'll be concerned with is the relationship between the two

Structural Networks 1

Cortical white matter
From O'Donnell & Westin (2006)

- Using diffusion tensor imaging (DTI) (or diffusion spectrum imaging (DSI)), it's possible to build an atlas of the *white matter* tracts of the human brain
- These are the long-range myelinated fibres that form the brain's communications infrastructure
- This data can be turned into a network amenable to mathematical analysis

Structural Networks 2



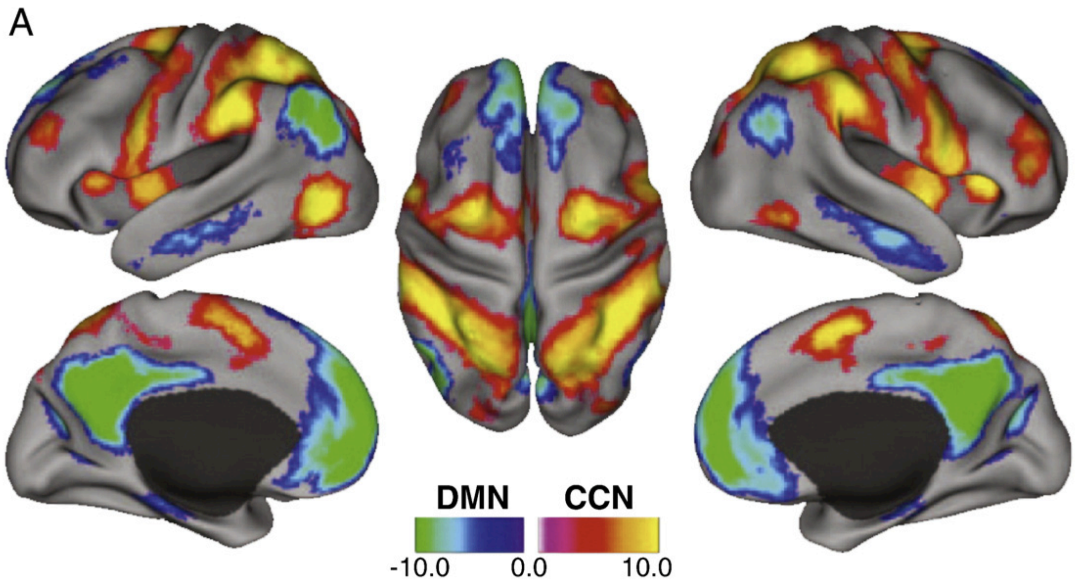
Structural connectivity
From Hagmann, *et al*. (2008)

- First, the cortical surface is parcellated into regions. These are the nodes of the network
- There is an edge between two nodes of the network if the white matter data shows a fibre tract
- The weight of the edge corresponds to the thickness of the tract
- The result turns out to be a *small-world modular network*

Functional Networks 1

- The material on the last two slides pertains to *structural networks* in the brain — that is to say to the physical connections
- Another important topic in contemporary neuroscience is *functional networks*
- In a functional network, the nodes are the same as in a structural network, ie: brain regions
- But there is an edge between two nodes if the regions in question are co-active (according to fMRI, for example) when the subject is engaging in a task, or is resting
- Combining functional and structural connectivity yields so-called *effective connectivity*

Functional Networks 2



From Cole & Schneider (2010)

- This figure shows two functional networks
- The default mode network (DMN) is active when the subject is resting
- The cognitive control network (CCN) is active when the subject is engaged in a task demanding attention and deliberate control of actions
- Functional networks are interesting, but in this course we'll concentrate on structural connectivity

Terminology 1

- A *network* (or graph) $G = \langle V, E \rangle$ comprises a set V of *nodes* (or vertices) and a set $E \subseteq V \times V$ of *edges* (or arcs or connections)
- The relation E can also be expressed as a two-dimensional *connectivity matrix* A, such that, for all $i, j \in V$

$$A(i,j) = \begin{cases} 1 \text{ if } (j,i) \in E \\ 0 \text{ otherwise} \end{cases}$$

- Note that $A(i,j)$ (sometimes written A_{ij}) is the connection from j to i
- The connectivity matrix is a good computer representation

Terminology 2

- We can generalise the connectivity matrix to account for different *connection strengths*, which is equivalent to adding numerical labels to the edges of the network
- For every edge $(i,j) \in E$, let $L(i,j)$ be the label of the edge from i to j. Then we have

$$A(i,j) = \begin{cases} L(j,i) \text{ if } (j,i) \in E \\ 0 \text{ otherwise} \end{cases}$$

- As before, $A(i,j)$ is the connection from j to i
- For now, we'll stick to *unlabelled* networks

Terminology 3

- Also, for now, we'll consider only *undirected* networks. These are represented as matrices with *symmetric* connections
  - So $A(i,j) = A(j,i)$
- And we'll forbid self-connections
  - So $A(j,j) = 0$
- In an undirected network, the *degree* k_i of a node i is the number of edges it participates in
- Often we're interested in a network's average degree: the average number of edges per node (usually denoted k)
- For an undirected network with n nodes and m edges $k = \dfrac{2m}{n}$
  (Note: each undirected edge has two matrix entries)

Confusing Conventions

- Note that we write $A(i,j)$ to denote the connection from j to i. This is the convention used by Newman (2010).
- But the Matlab Brain Connectivity toolbox uses the opposite convention. `CIJ(i,j)` represents the connection from `i` to `j`.
- This doesn't matter for undirected networks. But take care with Matlab code in the case of directed networks

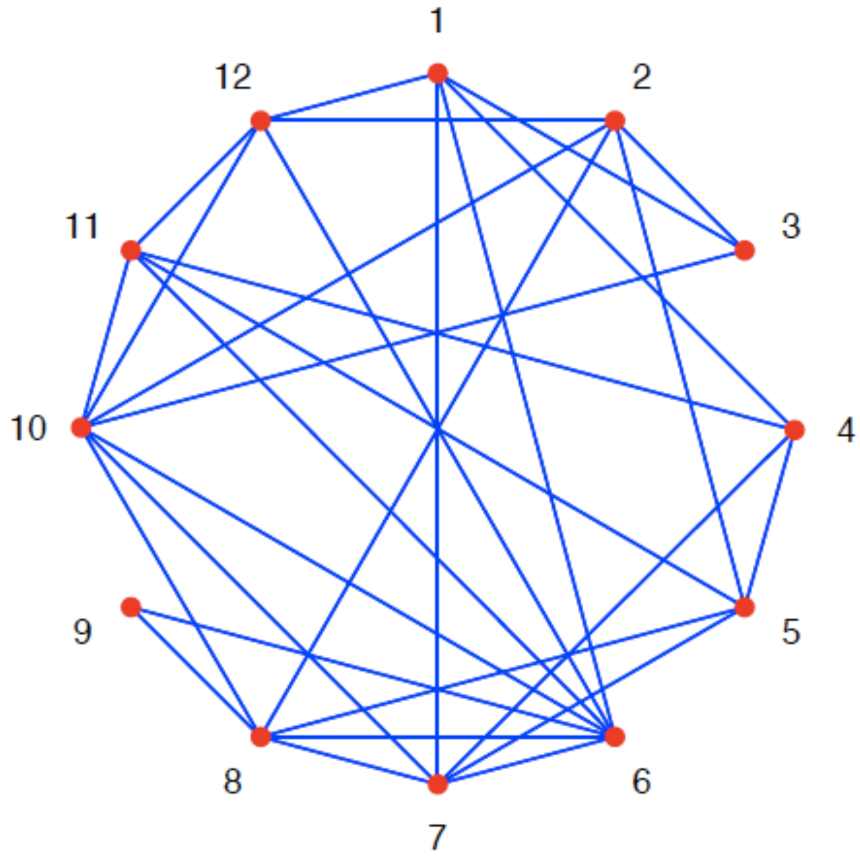
Random Networks

- A *random network* is one in which, for every pair of nodes i and j,

$$P(A(i,j) = x) = \begin{cases} p \text{ if } x = 1 \\ 1 - p \text{ if } x = 0 \end{cases}$$

where p is the connection probability

- In other words, a random network has a uniform distribution of randomly assigned edges



A random network for p = 0.36

Small-World Networks

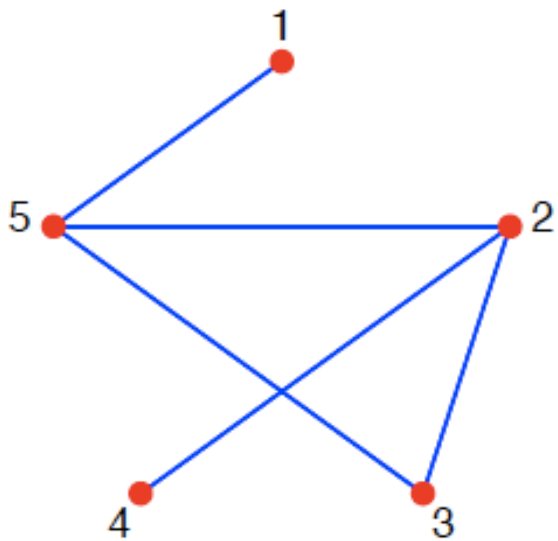
- Most networks found in nature or in the human environment are not random
  - Note: some authors use the term “random network” for any network whose construction is probabilistic. We’ll reserve it for networks defined as on the previous slide
- One common feature is high clustering. If a node x is connected to two nodes y and z then there is a better than chance probability that y and z are connected to each other
  - In a social network, for example, it’s quite likely that your friends know each other
- Another common feature is that it typically doesn’t take many hops to get from any node to any other node
  - In a social network, when you meet a complete stranger, you often find that you have mutual friends. Hence the exclamation, “What a small world!”

Path Length and Clustering 1

- These properties can be precisely quantified. Consider a network $G = \langle V, E \rangle$
- The *path length* between any pair of nodes in V is the number of edges in the shortest path between those nodes
- G' s *mean path length* λ_G is the path length averaged over every (distinct) pair of nodes in V
- The *clustering coefficient of a node* j in V is the fraction of the set of all possible edges between immediate neighbours of j that are actual edges
- The *clustering coefficient* γ_G *of the graph* G is the clustering coefficient averaged over all nodes in V

Path Length and Clustering 2

- Example: Let $G = \langle V, E \rangle$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\langle 1{,}5 \rangle, \langle 2{,}3 \rangle, \langle 2{,}4 \rangle, \langle 2{,}5 \rangle, \langle 3{,}5 \rangle\}$
- Then we have:



Pair	Path length
$\langle 1{,}2 \rangle$	2
$\langle 1{,}3 \rangle$	2
$\langle 1{,}4 \rangle$	3
$\langle 1{,}5 \rangle$	1
$\langle 2{,}3 \rangle$	1
$\langle 2{,}4 \rangle$	1
$\langle 2{,}5 \rangle$	1
$\langle 3{,}4 \rangle$	2
$\langle 3{,}5 \rangle$	1
$\langle 4{,}5 \rangle$	2

Node	Clustering coefficient
1	1
2	1/3=0.33
3	1/1=1
4	1
5	1/3=0.33

$$\lambda_G = 16/10 = 1.6$$

$$\gamma_G = 3.66/5 = 0.73$$

The Case of Leaf Nodes

- In the case of a leaf node, such as nodes 1 and 4 in the preceding example, what should the clustering coefficient be?
- The definition states that: “the fraction of the set of all possible edges between immediate neighbours of j that are actual edges”
- But a leaf node has just one neighbour, so there are zero possible edges between neighbours. This gives a denominator of zero, so the measure is not defined
- Some adopt the convention that the clustering coefficient for leaf nodes is zero. This is what the Matlab Brain Connectivity Toolbox does
- Others adopt the convention that the clustering coefficient for leaf nodes is one. *This is what we will do in this course*
- Kaiser has written a whole paper about this (*New Journal of Physics* 10 083042 (2008))

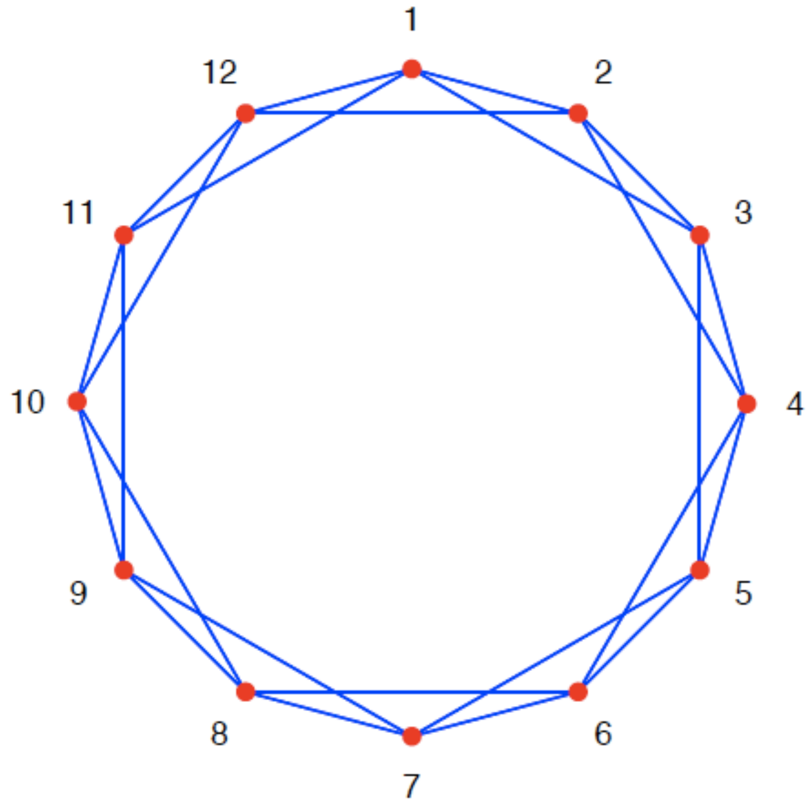
Small-World Index

- It can be shown that the mean path length λ_{rand} of a random network with n nodes and average degree k is (on average) $\ln(n)/\ln(k)$ and its clustering coefficient γ_{rand} is (on average) k/n
- A network G with n nodes and average degree k is a small-world network if
  - it is sparse ($k << n$)
  - its mean path length is comparable to that of a random network, and
  - its clustering coefficient is higher than that of a random network
- We can quantify this in terms of its *small-world index* σ_G

$$\sigma_G = \frac{\gamma_G / \gamma_{rand}}{\lambda_G / \lambda_{rand}}$$

The Watts-Strogatz Method 1

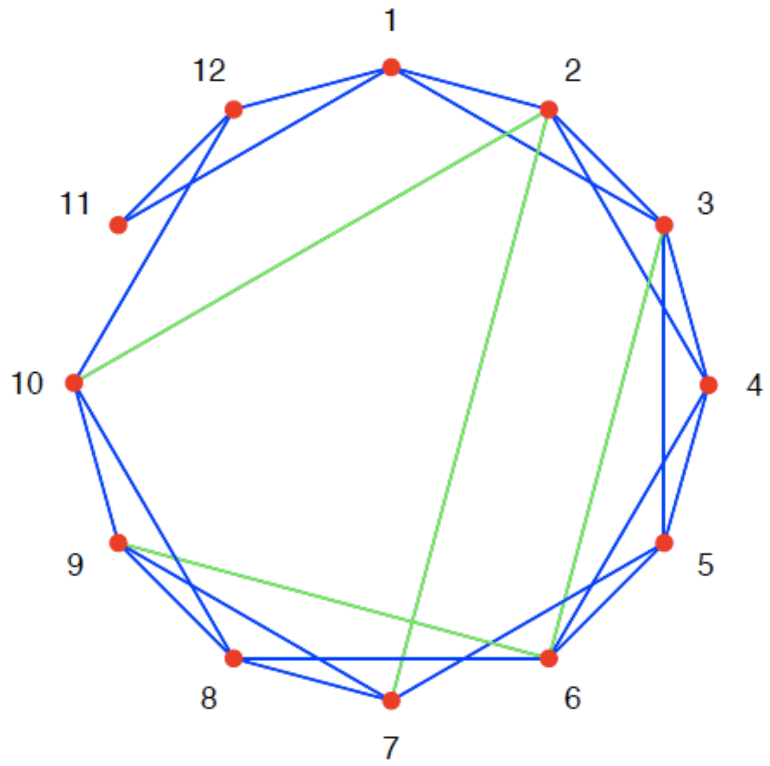
- Watts and Strogatz proposed a simple algorithm for constructing small-world networks
- There are two steps. First, a ring lattice is constructed
- A *ring lattice* with degree k is a set of nodes notionally arranged in a circle, where each node is connected to all its (spatial) neighbours that are less than or equal to $k/2$ nodes away



A ring lattice with degree k = 4

The Watts-Strogatz Method 2

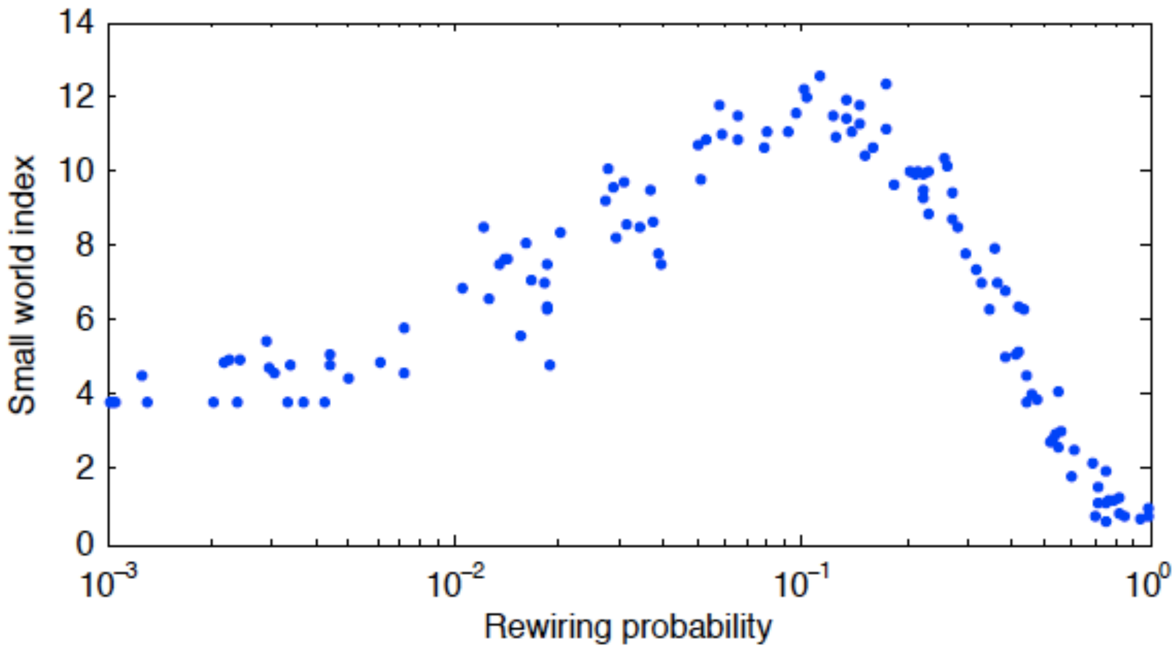
- The second step of the algorithm involves rewiring some of the connections of the ring lattice
- The algorithm is parameterised with a probability p
- Each edge is considered in turn, and with probability p it is rewired
- Rewiring an edge (j,i) means deleting (j,i) from E and adding (h,i) for some randomly chosen h



Rewiring the ring lattice with p = 0.2
Rewired connections are shown in green

The Watts-Strogatz Method 3

- If we randomly generate networks using the Watts-Strogatz procedure, we see that just a few rewirings are sufficient to confer a high small-world index



- This plot was produced for n = 200 and k = 4
- There is a clear peak in small-world index at approximately p = 0.1
- Note the log scale on the x-axis

Global and Local Efficiency 1

- Latora & Marchiori proposed a different statistic for complex networks. Intuitively, it characterises the *efficiency* with which information can be propagated through the network
- Consider a network $G = \langle V, E \rangle$ with n nodes
- Let $Eff(i,j)$ be $1/\lambda$, where λ is the path length in G from node i to node j. This captures the efficiency with which information can be propagated from i to j, the maximum being 1 if i and j are neighbours
- The *global efficiency* of G is then the efficiency averaged over the whole network, defined as

$$Eff_{\mathrm{glob}}(G) = \frac{1}{n(n-1)} \sum_{i \neq j} Eff(i,j)$$

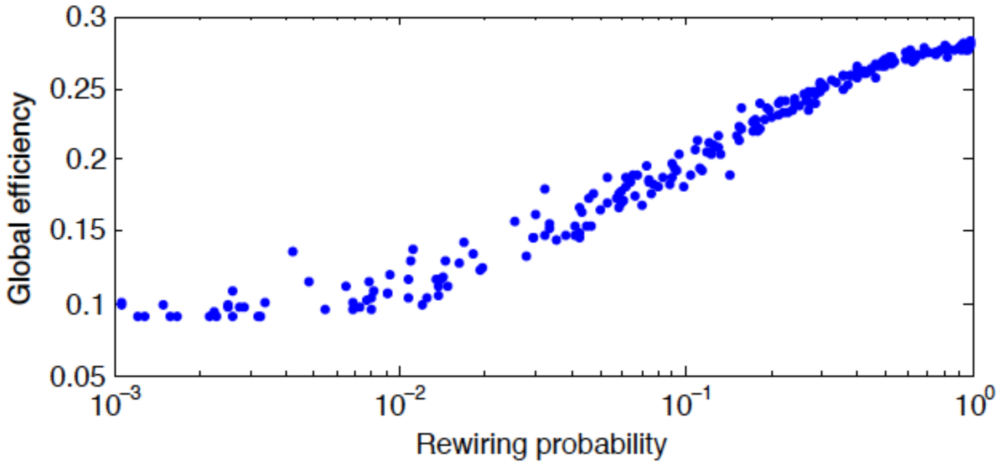
Global and Local Efficiency 2

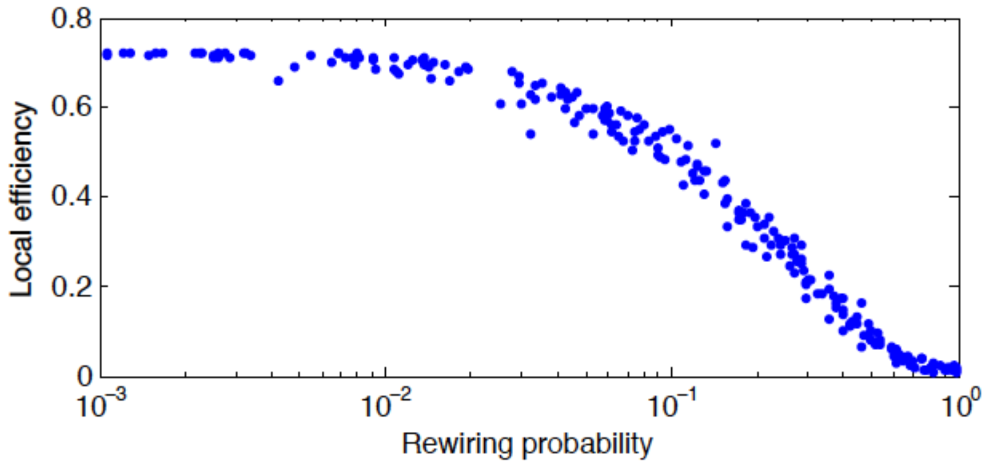
- We can also define the efficiency of the neighbourhood of a given node
- Let $G_i = \langle V', E' \rangle$ be a sub-network of G such that $V' \subseteq V$ is the set of neighbours of i, and $E' \subseteq E$ is the edges that join nodes in V'
- The efficiency of the neighbourhood of node i is then $Eff_{\text{glob}}(G_i)$
- The *local efficiency* of G is then the neighbourhood efficiency averaged over the whole network, defined as

$$Eff_{\text{loc}}(G) = \frac{1}{n} \sum_{i \in G} Eff_{\text{glob}}(G_i)$$

- Note that if there is no path between two nodes i and j then $Eff(i,j) = 0$

Global and Local Efficiency 3

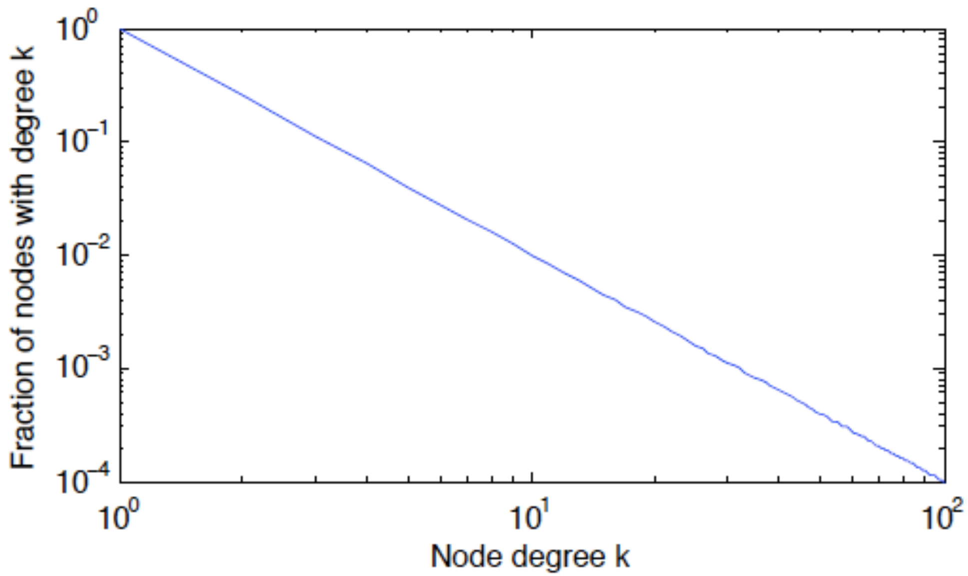




- As with the small-world index, we can use efficiency to assess networks generated with the Watts-Strogatz procedure for different values of p
- These plots were produced for n = 200 and k = 4 as before
- Where the small-world index peaks (at around p = 0.1), we have a good balance of local and global efficiency

Scale-Free Networks

- A *scale-free* network is one in which the distribution of node degrees follows a “power law”
- That is to say, if node degree is plotted against the number of nodes having that degree, you get a curve that decays slowly (or a straight line on a log-log plot)
- There are lots of nodes with low degree, and some, but very few, with very high degree
- A well-known procedure for generating such networks is the “preferential attachment” method of Barbási & Albert
- We won’t look into scale-free networks further



Related Reading

Latora, V. & Marchiori, M. (2001). Efficient Behavior of Small-World Networks. *Physical Review Letters* 87 (19).

Newman, M.E.J. (2010) *Networks: An Introduction*. Oxford University Press

Sporns, O. (2010). *Networks of the Brain*. MIT Press.

Watts, D.J. & Strogatz, S.H. (1998). Collective Dynamics of 'Small-world' Networks. *Nature* 393, 440–442.