Topic 8
Modular Networks

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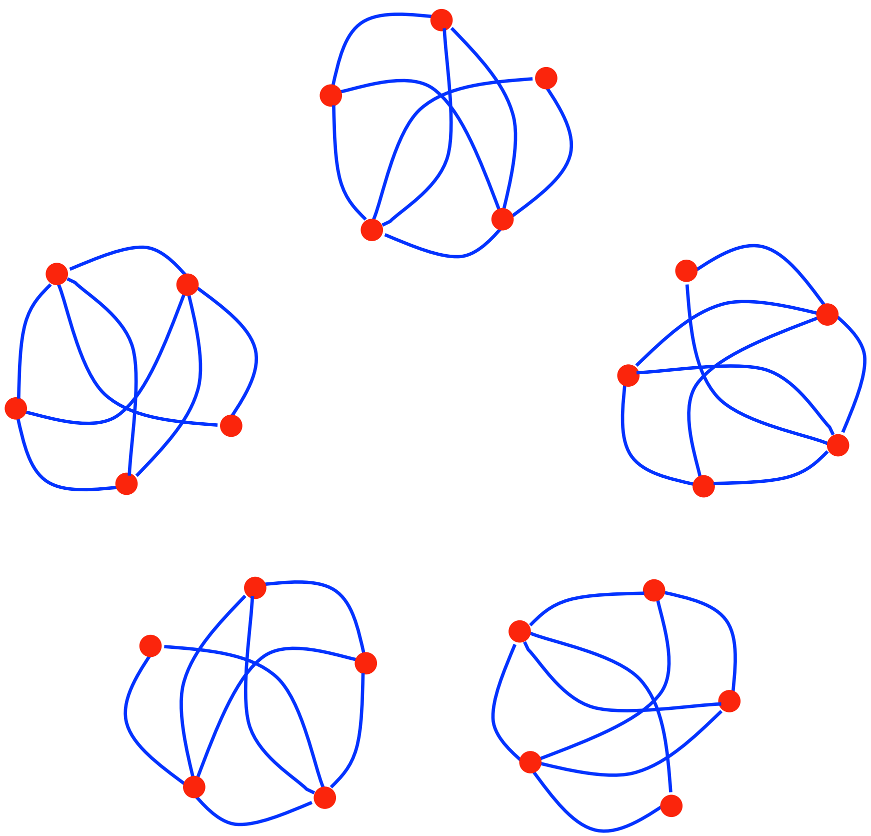
Overview

- Modularity
- Spatial embedding
- Hub nodes
- Connective cores

Modularity

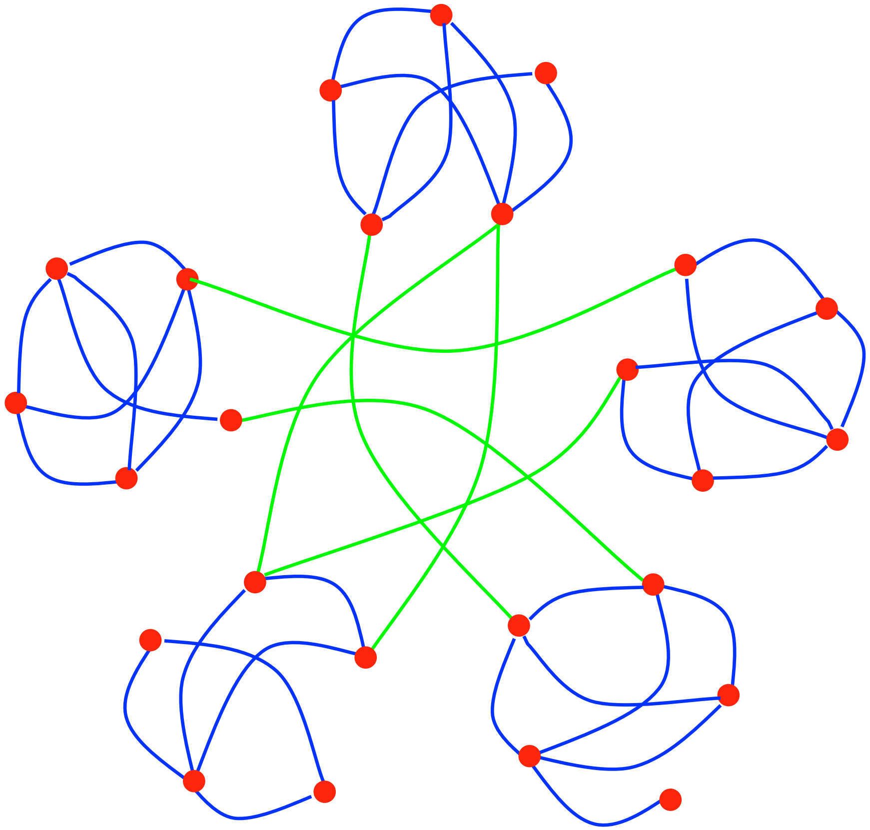
- The Watts-Strogatz procedure only produces one type of small-world network, and it isn't a very realistic one
- Moreover, there are other topological properties of a network that can be given precise definitions, and that are of interest to neuroscientists
- One example is modularity. A network is *modular* if its nodes can be partitioned into subsets (*modules*, or *communities*) that
  - are highly intra-connected (there are many internal connections within the module)
  - but sparsely inter-connected (there are relatively fewer connections between modules)
- Modular networks are typically (but not necessarily) also small-world networks

Making Modular Networks 1



- Here's an algorithm for generating a modular network with n nodes, m edges, and C communities. Like the Watts-Strogatz procedure, it has two steps
- First, a set of C disconnected communities is created with n/C nodes each
- Each community has m/C random edges

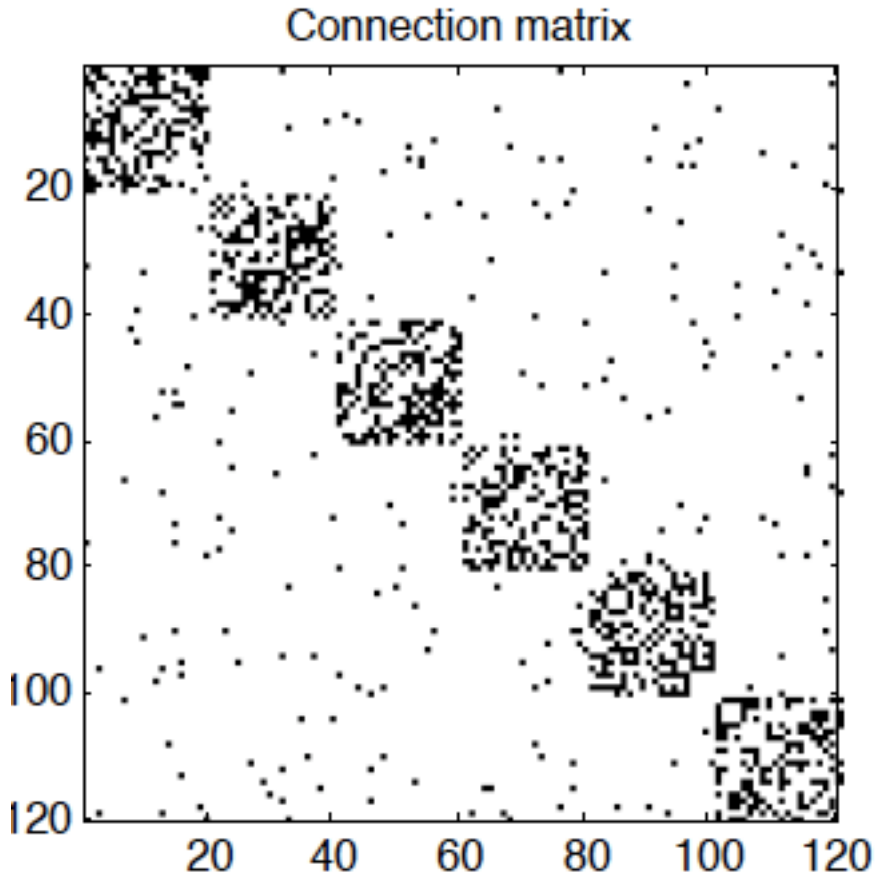
Making Modular Networks 2



- The second step is a rewiring process, like the Watts-Strogatz procedure again
- Each existing (intra-community) edge is considered, and with probability p is rewired as an edge between communities
- For rewired edges, the target community and target node within that community are randomly chosen

Visualising Modularity

- Here is a visualisation of the connectivity matrix of a network , comprising 6 modules of 20 nodes each, built using this procedure with rewiring probability p = 0.2
- The modular structure is clearly visible. But to bring it out visually the nodes have to be ordered so that members of the same module have similar numbers



Measuring Modularity 1

- Let $G = \langle V, E \rangle$ be a network with m edges
- A measure Q of how modular a given partitioning is for G is obtained by comparing the number of intra-community edges that actually occur in G with the number that would occur in a comparable network that was randomly connected

Q = (*actual fraction of edges that are within communities*) –
(*expected fraction of edges that are within communities*)

- If there are, so to speak, surprisingly many within-community edges then the network is modular (with respect to a given partitioning)

Measuring Modularity 2

- The actual fraction of intra-community edges is the sum of

$$\frac{A_{ij}}{2m}$$

In an undirected network, each edge counts twice

for all i and j in the same community, while the expected fraction of intra-community edges is the sum of

$$\frac{k_i}{2m} \times \frac{k_j}{2m}$$

The chances of an edge connecting with i *and* with j

for all i and j in the same community, where k_i is the degree of node i

Measuring Modularity 3

- In general, suppose the nodes in V are partitioned into communities, and let c_i denote the community to which node i belongs. Then we have

$$Q = \frac{1}{2m} \sum_{i,j} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta_{c_i c_j}$$

where

$$\delta_{xy} = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$$

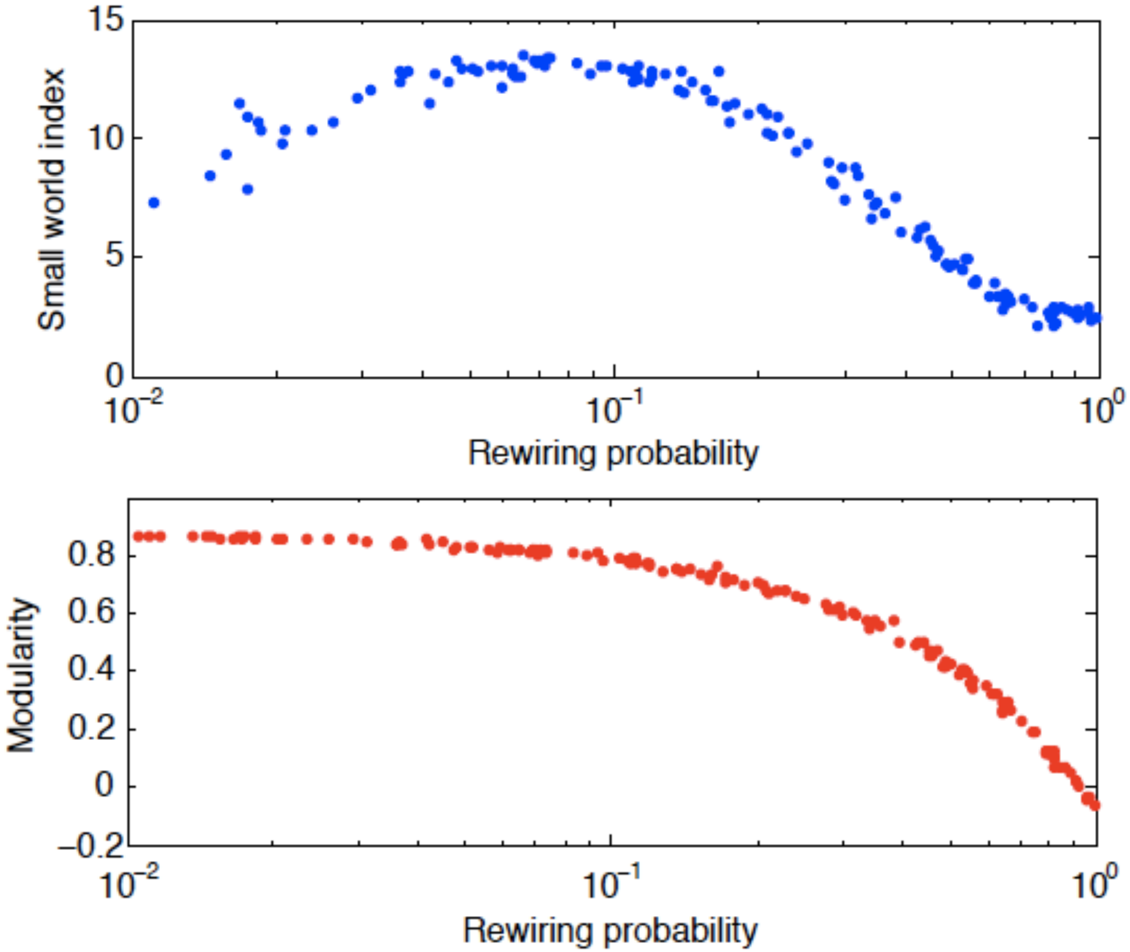
This function is called the Kronecker delta

- Q has a maximum value of 1
- Note that Q can be negative, if there are fewer connections within a "community" than there are between "communities"

Measuring Modularity 4

- The value Q is defined with respect to a given partitioning into communities
- Often the community structure is known in advance, so Q is easy to calculate
- The modularity of a network with unknown community structure is equal to the maximum obtainable value of Q
- To find it, we need to find search all possible partitionings, which is an NP-complete problem
- Various algorithms exist for finding community structure. But this topic is beyond the scope of these lectures

Measuring Modularity 5



- This is how small-world index and modularity vary with rewiring probability p using the algorithm described earlier, with 8 communities of 20 nodes and 100 edges each
- Note that modularity is negative for high values of p, because there are more connections between communities than within them
- With low values of p the network can become disconnected. These networks are ignored

Spatial Embedding

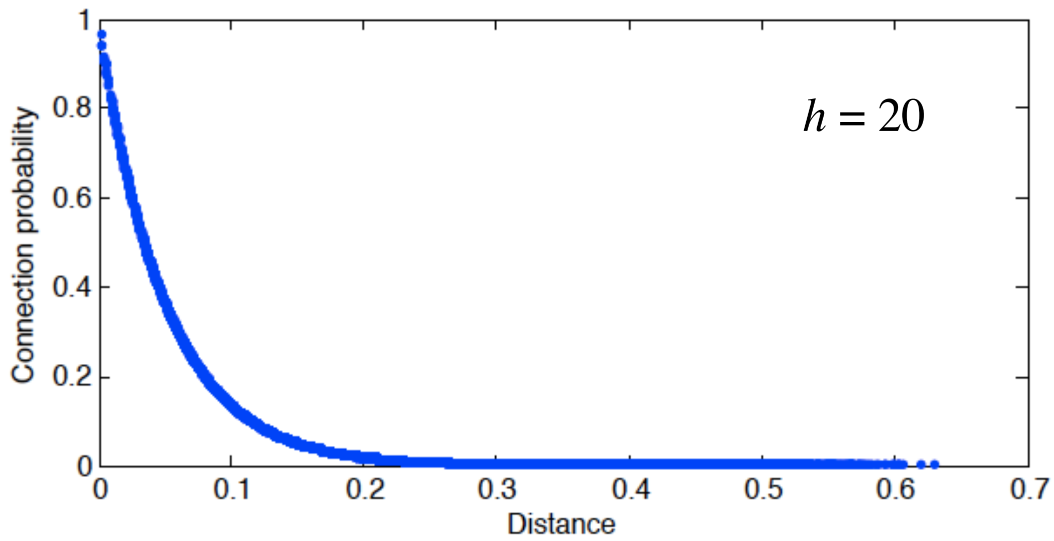
- Real networks, including brain networks, are often *spatially embedded*. This means the nodes have some spatial location, and the spatial locations of nodes influences their connectivity
- The ring lattice in the Watts-Strogatz procedure is one sort of spatial embedding. But it's only one-dimensional, and the distances are discrete rather than continuous
- A more realistic embedding results from assigning each node a location on a 2D plane
- By making the probability of connection depend on distance, we can construct another type of realistic network with a high small-world index, and medium to high modularity
- This is not such a good model of whole-brain connectivity, but it is a good model of a small patch of cortex

2D Small World Networks 1

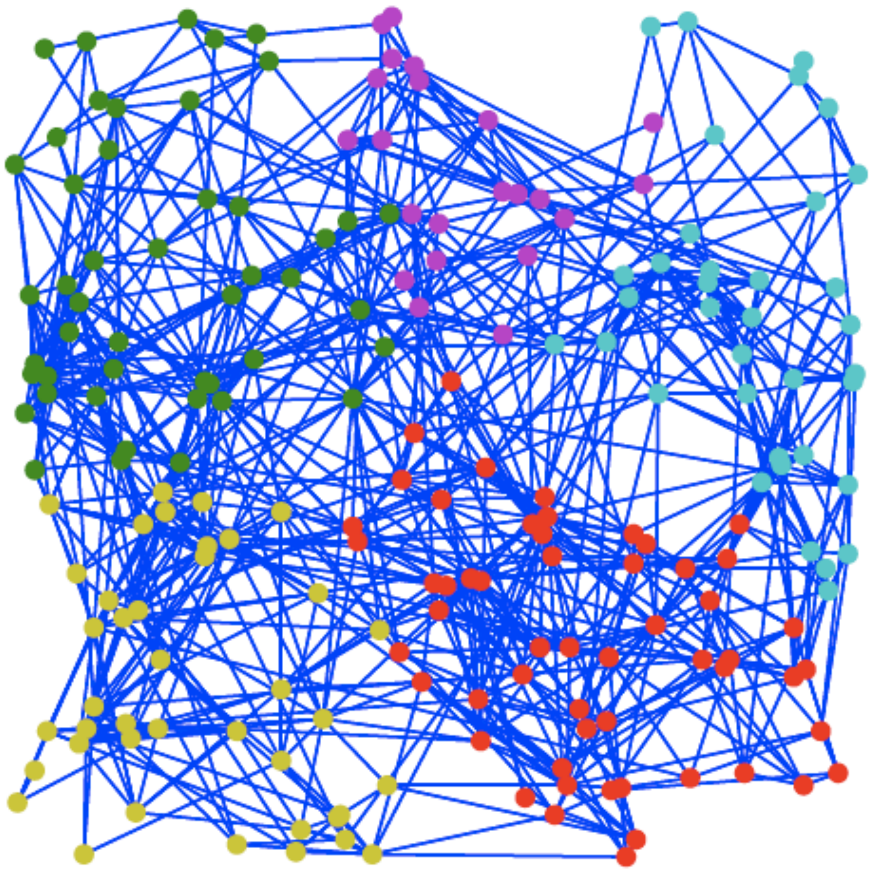
- Suppose the nodes are arranged on a 2D square, and each is assigned random co-ordinates in the range 0 to 0.5. This ensures that the distance between any two nodes is in the range 0 to 1 (actually less than $\sqrt{0.5}$)
- Then let the probability p of a connection between any two nodes be

$$p = e^{-hd}$$

where d is the distance between the nodes and h is a constant



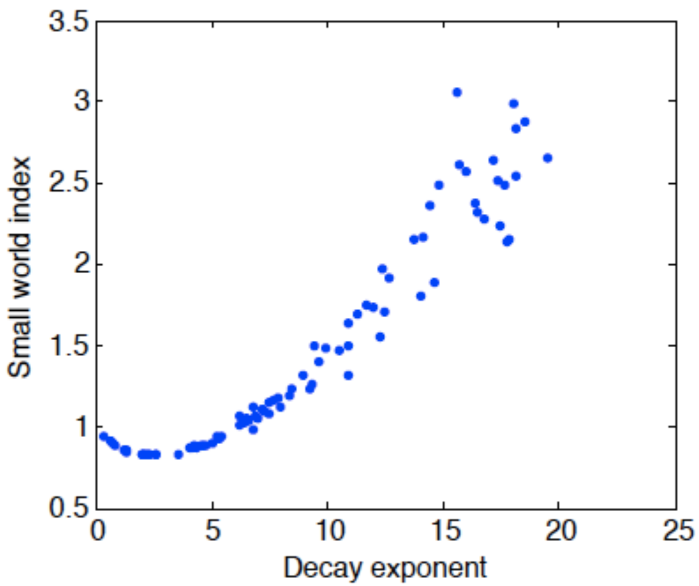
2D Small World Networks 2

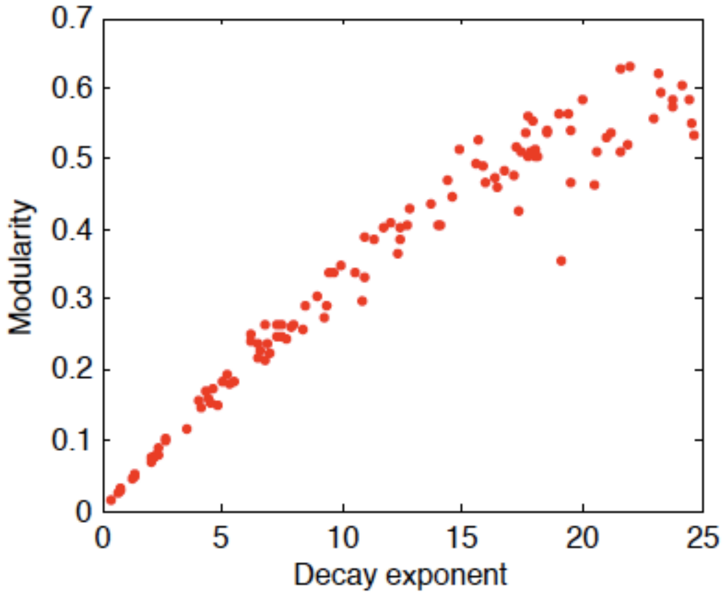


- Here's an example of a spatially embedded network generated in this way, with 200 nodes and h=20
- The small world index is 2.8585, which is fairly high
- The network is also quite modular. The colours are an assignment of nodes to modules, giving Q=0.5195

2D Small-World Networks 3

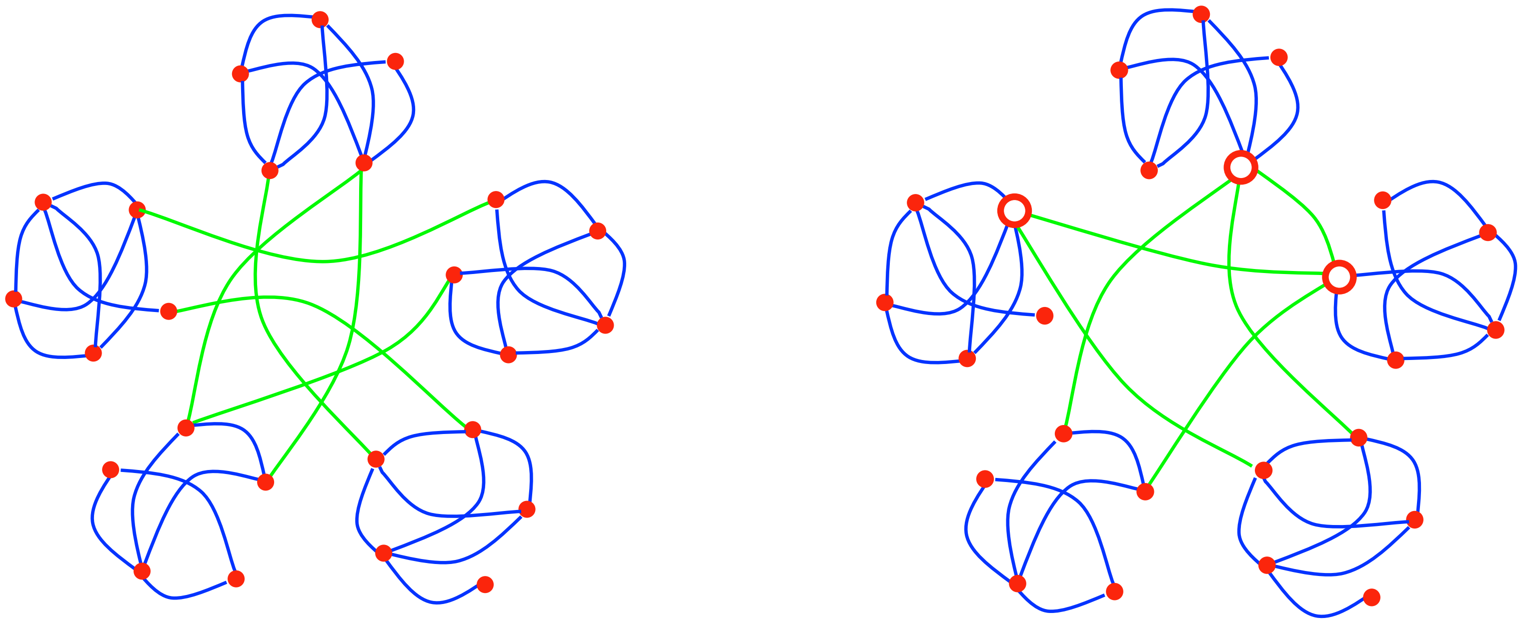
- If we randomly generate spatially embedded networks (with 100 nodes) we find that both small-world index and modularity increase with the exponent of decay h
- But with very high h, the network fragments (becomes disconnected), and the small-world index is undefined





Hub Nodes

- Compare the two networks below. Both are modular. But in the one on the right, the inter-modular connections involve only a small subset of the nodes
- These are called *connector hubs*



Identifying Connector Hubs

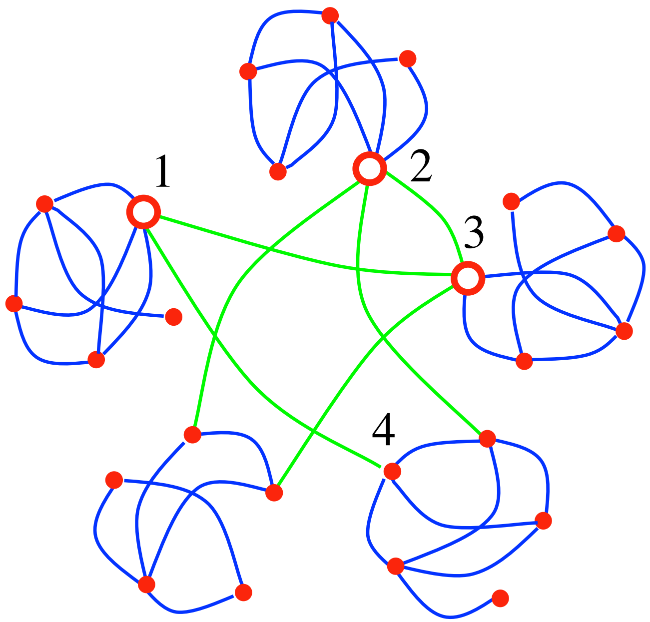
- The *participation index* P_i of node i is defined as follows

$$P_i = 1 - \sum_c \left(\frac{k_i^c}{k_i} \right)^2$$

where k_i^c is the number of edges from node i to nodes in community c, and k_i is the overall degree of node i

- A node i is a *connector hub* if
  - k_i is greater than the mean node degree k in the network plus the standard deviation sd of node degree in the network, and
  - P_i is greater than a threshold (usually 0.3)

Hub Nodes Example



- For the overall network here, we have

  k = 3.2
  sd = 1.19
  k + sd = 4.39

- For the indicated nodes, we have

  $k_1 = 5$, $k_2 = 6$, $k_3 = 5$, $k_4 = 4$

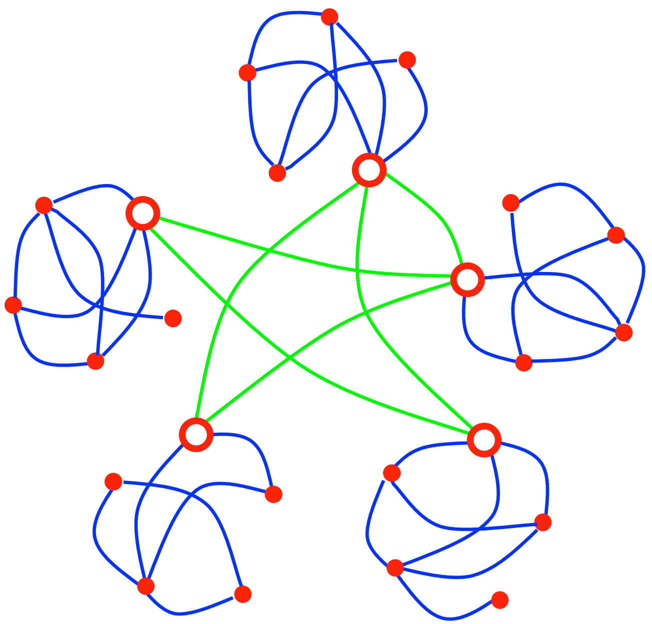
- So nodes 1, 2, and 3 are candidate connector hubs. Node 4 is not

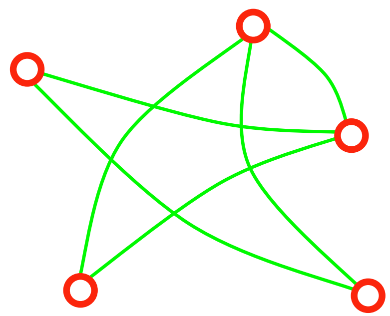
- We have $$P_1 = 1 - \left(\frac{1}{25} + \frac{1}{25} + \frac{9}{25}\right) = 0.56 > 0.3$$

- So node 1 is a connector hub. Similarly for nodes 2 and 3

Connective Cores

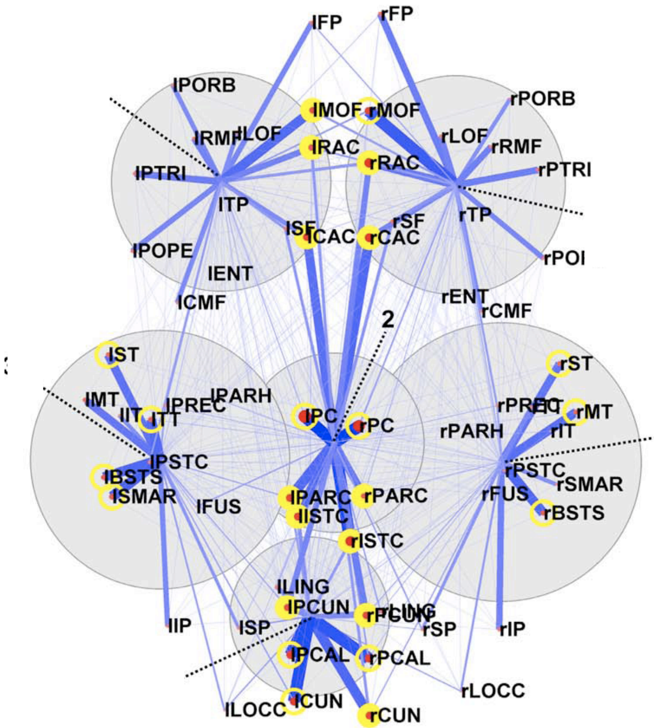
- If we have a modular network with connector hubs, then we can extract its *connective core*
- This is the set of connector hubs plus the connections between them



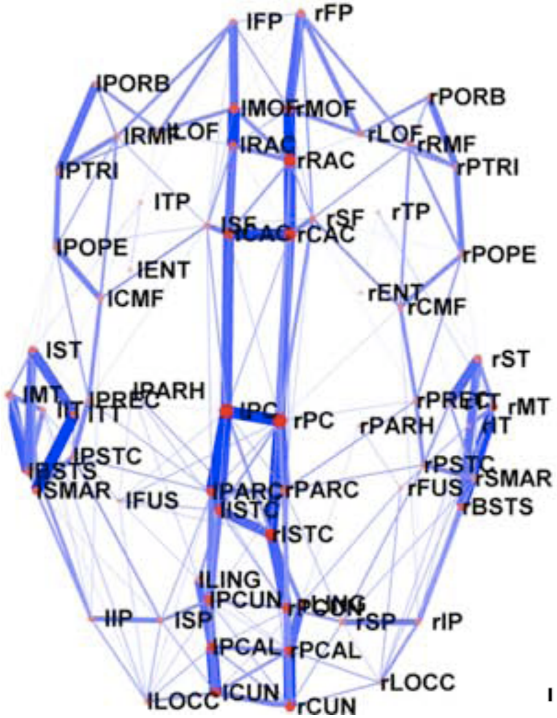


- The connective core is a set of densely interconnected, topologically central nodes
- Information funnels in to and fans out from the connective core

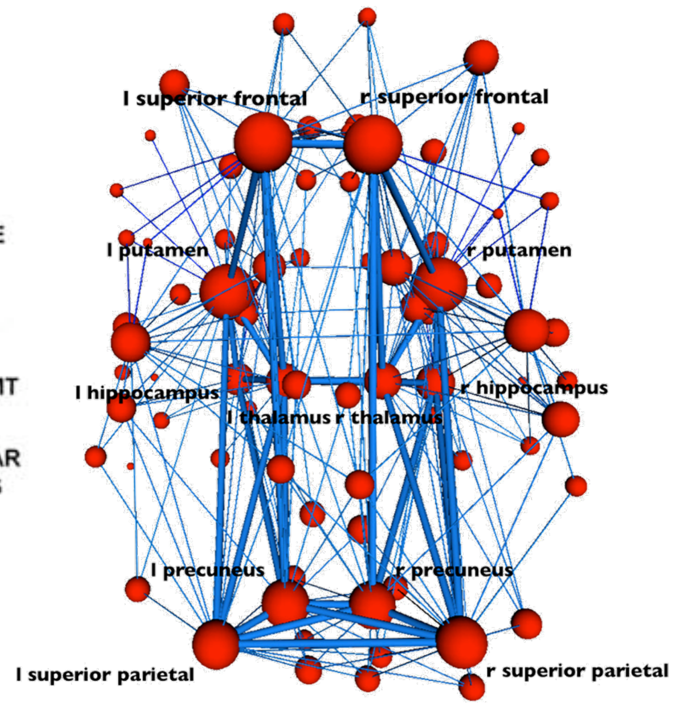
Cortical Networks



Modules
(Hagmann, *et al*., 2008)



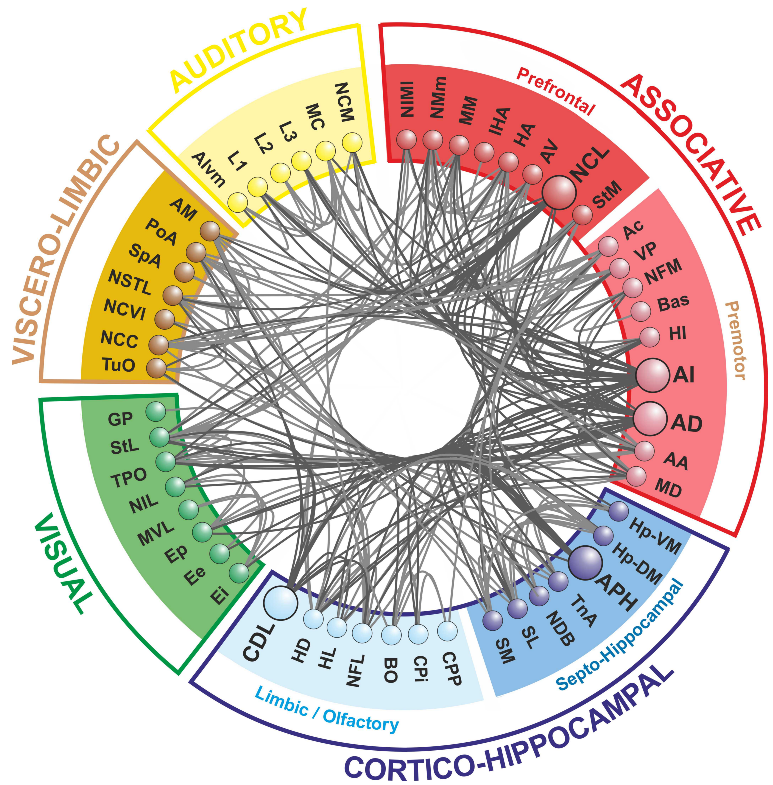
Hub nodes
(Hagmann, *et al*., 2008)



“Rich club” nodes
(van den Heuvel &
Sporns, 2011)

The Avian Connective Core

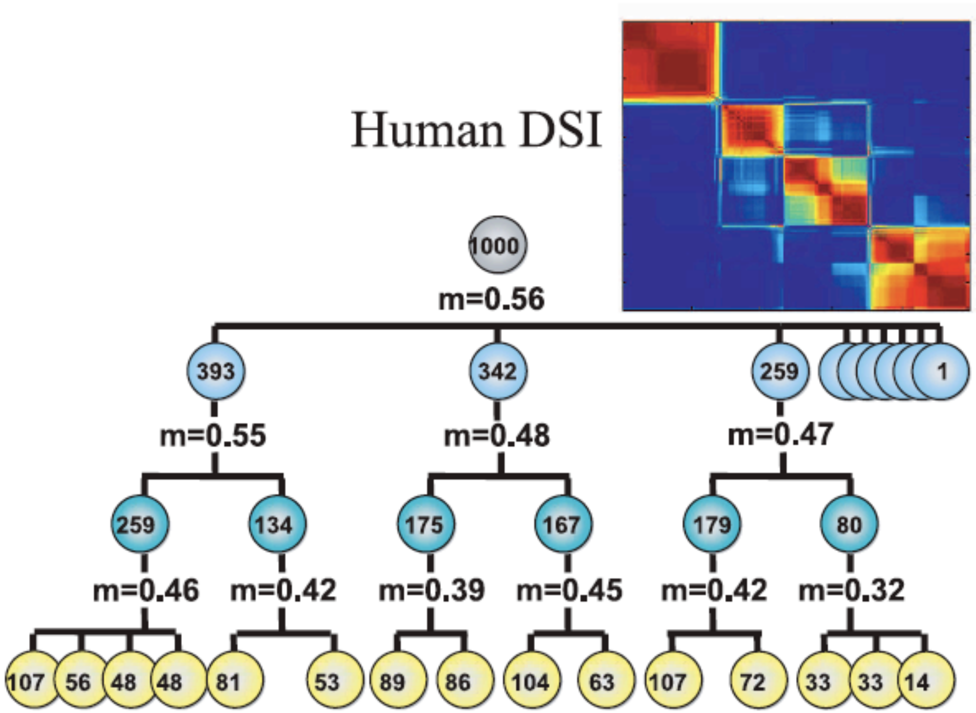
- We find a connective core in the brains of birds as well as mammals
- Here is the connectome of the pigeon forebrain
- Connections funnel in to and fan out from five hub nodes
- These include cognitively important areas (NCL and APH)



The pigeon connectome
(Shanahan, *et al*., 2013)

Hierarchical Modularity 1

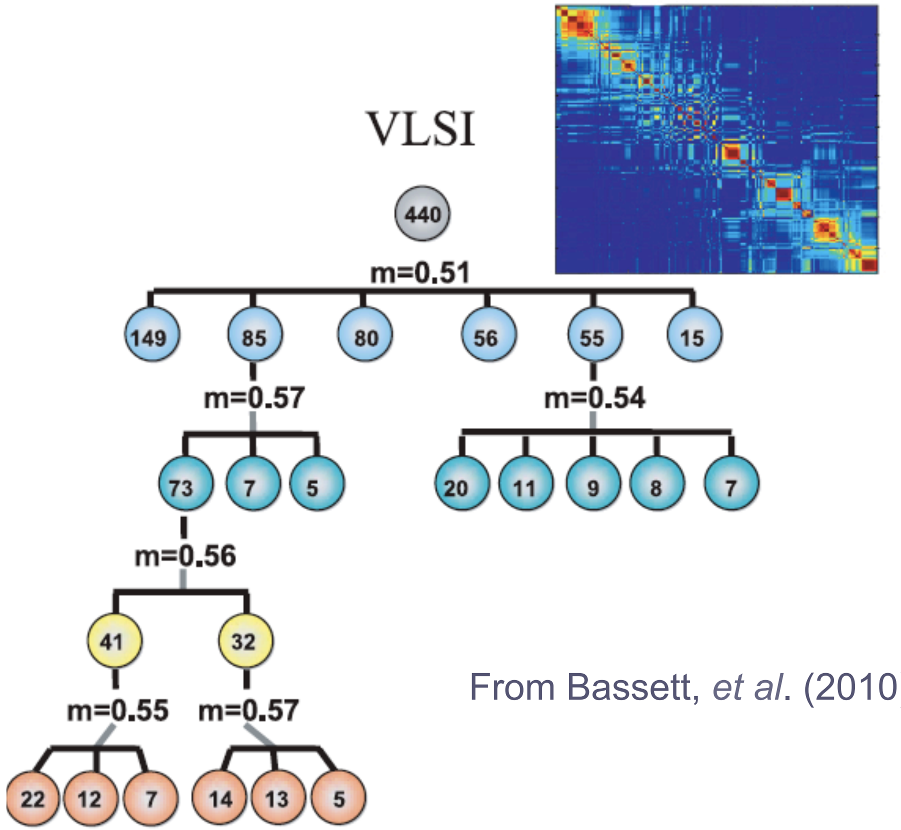
- A *hierarchically modular* network is a modular network in which the modules also have modular structure
- Whole-brain human structural networks are hierarchically modular



From Bassett, *et al*. (2010)

Hierarchical Modularity 2

- Hierarchical modularity is an effective way to organise wiring in general
- Large-scale integrated circuits are also hierarchically modular



From Bassett, *et al*. (2010)

Related Reading

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Sporns, O. (2010). *Networks of the Brain*. MIT Press.

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