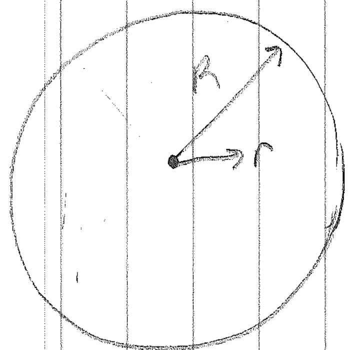
Transient Diffusion in a sphere



$C = C_\infty$ (maintained)

$$\frac{\partial c}{\partial t} = D \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right)$$

@ $t=0$, $C = 0$

@ $r=0$, C finite

@ $r=R$, $C = C_\infty$

Determine concentration profile in sphere.

Dimensionless form

$$\frac{\partial \tilde{c}}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial \tilde{c}}{\partial \eta}\right)$$

$\tilde{c} = \frac{c}{c_\infty}$, $\eta = \frac{r}{R}$, $\tau = \frac{Dt}{R^2}$

@ $\tau = 0$, $\tilde{c} = 0 \quad \forall \eta$

@ $\eta = 0$, Bounded soln

@ $\eta = 1$, $\tilde{c} = 1$

Soln:

① Choose direction for basis expansion... BCs must be homogeneous in that direction.

↳ If want infinite Fourier sum soln, must eigenfunction expand in finite direction.

↳ Can work w/ τ direction but since domain is ∞, need transform... transforms are derived from Fourier series (as domain gets long, eigenvalues/spectrum becomes continuous).

(2) Choose r direction

- Need to make r direction homogeneous...

Find function $g(r) \ni g(r)=1$ @ $r=1$

$g(r)$ Bounded @ $r=0$

Let $g(r)=1$

Write: $\hat{c} = 1 + F(\tau, r)$

$$\Rightarrow \frac{\partial F}{\partial \tau} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial F}{\partial r}\right)$$

@ $\tau=0$, $\hat{c}=0 \Rightarrow 0 = 1 + F(0,r) \Rightarrow F=-1$ @ $\tau=0$

@ $r=1$, $\hat{c}=1 \Rightarrow 1 = 1 + F(\tau,0) \Rightarrow F=0$ @ $r=1$

@ $r=0$, F bounded.

New System to solve:

$$\frac{\partial F}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial F}{\partial \eta}\right)$$

$\tilde{C} = 1 + F(\tau, \eta)$

@ $\tau = 0$, $F = -1$

@ $\eta = 1$, $F = 0$

@ $\eta = 0$, F bounded

(3) Now, solve for F

$F = \sum_n A_n(\tau) B_n(\eta)$. Sub into PDE:

$$\sum_n \left[\frac{dA_n}{d\tau} B_n - \frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2 \frac{dB_n}{d\eta}\right) A_n(\tau) \right] = 0$$

$$\Rightarrow \quad \sum_n \left[\frac{dA_n}{d\tau} - \left(\frac{1}{\eta^2} \frac{d}{d\eta}\left(\eta^2 \frac{dB_n}{d\eta}\right) \frac{1}{B_n} \right) A_n(\tau) \right] B_n(\eta) = 0$$

For system to be separable, term in brackets must only be a function of η.

Then:

$$\frac{1}{\eta^2} \frac{d}{d\eta}\left(\eta^2 \frac{dB_n}{d\eta}\right) \frac{1}{B_n} = K_n$$

BCs: B_n bounded @ $\eta = 0$

$F = \sum_n A_n(\tau) B_n(\eta) = 0$ @ $\eta = 1 \Rightarrow B_n(1) = 0$

Also:

$$\boxed{\sum_n \left[\frac{dA_n}{d\tau} - K_n A_n(\tau) \right] B_n(\eta) = 0}$$

$$\Rightarrow \boxed{\begin{array}{l} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dB_n}{dr}\right) = K_n B_n \\ \\ B_n(0) \text{ bounded} \\ \\ B_n(1) = 0 \end{array}}$$

<u>Sturm</u> - <u>Liouville</u> <u>Problem</u>

1. K_n real
2. $B_n \perp$ w.r.t. $\langle f, g\rangle = \int_0^1 r^2 f g \, dr$
3. B_n forms basis
4. 1 B_n for each K_n

Perform Eigensearch:

$K_n = 0, \quad K_n < 0, \quad K_n > 0$

$K_n = 0$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dB_n}{dr}\right) = 0 \quad \Rightarrow \quad r^2 \frac{dB_n}{dr} = C \quad \Rightarrow \quad \frac{dB_n}{dr} = \frac{C}{r^2}.$$

For bounded soln, $C = 0$

$\Rightarrow \frac{dB_n}{dr} = 0 \quad \Rightarrow B_n = D.$ But, $B_n = 0$ @ $r = 1 \Rightarrow D = 0.$

So, $B_n = 0 \ \forall r$

$\Rightarrow$ No nontrivial solutions (no eigensolution).

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$\underline{\underline{K_n < 0}}$

Let $K_n = -\lambda_n^2$, $\lambda_n > 0$ for definiteness

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dB_n}{dr}\right) = -\lambda_n^2 B_n$$

$$\boxed{\begin{aligned} & r^2 \frac{d^2 B_n}{dr^2} + 2r\frac{dB_n}{dr} + \lambda_n^2 r^2 B_n = 0 \\ & B_n(0) \equiv \text{bounded} \\ & B_n(1) = 0 \end{aligned}}$$

Generally need to solve via power series...

↳ But, most ⊥ generating Sturm-Liouville problems have been solved...

Go to handbook... I use Abramowitz + Stegun (Knovel Data Base).

Fr. Abramowitz + Stegun, Handbook of Mathematical functions (S3-8a)

10. Bessel Functions of Fractional Order

Mathematical Properties

10.1. Spherical Bessel Functions

Definitions

Differential Equation

10.1.1

$$\boxed{z^2w''+2zw'+[z^2-n(n+1)]w=0 \qquad (n=0,\pm1,\pm2,\ldots)}$$

Particular solutions are the *Spherical Bessel functions of the first kind*

$$\boxed{j_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,J_{n+\frac{1}{2}}(z),}$$

the *Spherical Bessel functions of the second kind*

$$\boxed{y_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,Y_{n+\frac{1}{2}}(z),}$$

and the *Spherical Bessel functions of the third kind*

$$h_n^{(1)}(z)=j_n(z)+iy_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,H_{n+\frac{1}{2}}^{(1)}(z),$$

$$h_n^{(2)}(z)=j_n(z)-iy_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,H_{n+\frac{1}{2}}^{(2)}(z).$$

The pairs $j_n(z)$, $y_n(z)$ and $h_n^{(1)}(z)$, $h_n^{(2)}(z)$ are linearly independent solutions for every n. For general properties see the remarks after **9.1.1.**

Ascending Series (See 9.1.2, 9.1.10)

10.1.2

$$j_n(z)=\frac{z^n}{1\cdot3\cdot5\ldots(2n+1)}\left\{1-\frac{\frac{1}{2}z^2}{1!(2n+3)}+\frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)}-\ldots\right\}$$

10.1.3

$$y_n(z)=-\frac{1\cdot3\cdot5\ldots(2n-1)}{z^{n+1}}\left\{1-\frac{\frac{1}{2}z^2}{1!(1-2n)}+\frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)}-\ldots\right\}$$

$$(n=0,1,2,\ldots)$$

Limiting Values as $z\to0$

10.1.4 $$z^{-n}j_n(z)\to\frac{1}{1\cdot3\cdot5\ldots(2n+1)}$$

10.1.5

$$z^{n+1}y_n(z)\to-1\cdot3\cdot5\ldots(2n-1) \qquad (n=0,1,2,\ldots)$$

Wronskians

10.1.6 $$W\{j_n(z),y_n(z)\}=z^{-2}$$

10.1.7

$$W\{h_n^{(1)}(z),h_n^{(2)}(z)\}=-2iz^{-2} \qquad (n=0,1,2,\ldots)$$

Representations by Elementary Functions

10.1.8

$$j_n(z)=z^{-1}[P(n+\tfrac{1}{2},z)\sin(z-\tfrac{1}{2}n\pi)+Q(n+\tfrac{1}{2},z)\cos(z-\tfrac{1}{2}n\pi)]$$

10.1.9

$$y_n(z)=(-1)^{n+1}z^{-1}[P(n+\tfrac{1}{2},z)\cos(z+\tfrac{1}{2}n\pi)-Q(n+\tfrac{1}{2},z)\sin(z+\tfrac{1}{2}n\pi)]$$

$$P(n+\tfrac{1}{2},z)=1-\frac{(n+2)!}{2!\Gamma(n-1)}(2z)^{-2}+\frac{(n+4)!}{4!\Gamma(n-3)}(2z)^{-4}-\ldots$$

$$=\sum_0^{[\frac{1}{2}n]}(-1)^k(n+\tfrac{1}{2},2k)(2z)^{-2k}$$

$$Q(n+\tfrac{1}{2},z)=\frac{(n+1)!}{1!\Gamma(n)}(2z)^{-1}-\frac{(n+3)!}{3!\Gamma(n-2)}(2z)^{-3}+\frac{(n+5)!}{5!\Gamma(n-4)}(2z)^{-5}-\ldots$$

$$=\sum_0^{[\frac{1}{2}(n-1)]}(-1)^k(n+\tfrac{1}{2},2k+1)(2z)^{-2k-1}$$

$$(n=0,1,2,\ldots)$$

$$(n+\tfrac{1}{2},k)=\frac{(n+k)!}{k!\Gamma(n-k+1)}$$

n \ k	1	2	3	4	5
1	2				
2	6	12			
3	12	60	120		
4	20	180	840	1680	
5	30	420	3360	15120	30240

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10.1.10

$$j_n(z)=f_n(z)\sin z+(-1)^{n+1}f_{-n-1}(z)\cos z$$

$$f_0(z)=z^{-1}, \qquad f_1(z)=z^{-2}$$

$$f_{n-1}(z)+f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

$$(n=0, \pm1, \pm2, \ldots)$$

The Functions $j_n(z)$, $y_n(z)$ for $n=0, 1, 2$

10.1.11 $$\boxed{j_0(z)=\frac{\sin z}{z}}$$

$$j_1(z)=\frac{\sin z}{z^2}-\frac{\cos z}{z}$$

$$j_2(z)=\left(\frac{3}{z^3}-\frac{1}{z}\right)\sin z-\frac{3}{z^2}\cos z \qquad *$$

10.1.12

$$\boxed{y_0(z)=-j_{-1}(z)=-\frac{\cos z}{z}}$$

$$y_1(z)=j_{-2}(z)=-\frac{\cos z}{z^2}-\frac{\sin z}{z}$$

$$y_2(z)=-j_{-3}(z)=\left(-\frac{3}{z^3}+\frac{1}{z}\right)\cos z-\frac{3}{z^2}\sin z \qquad *$$

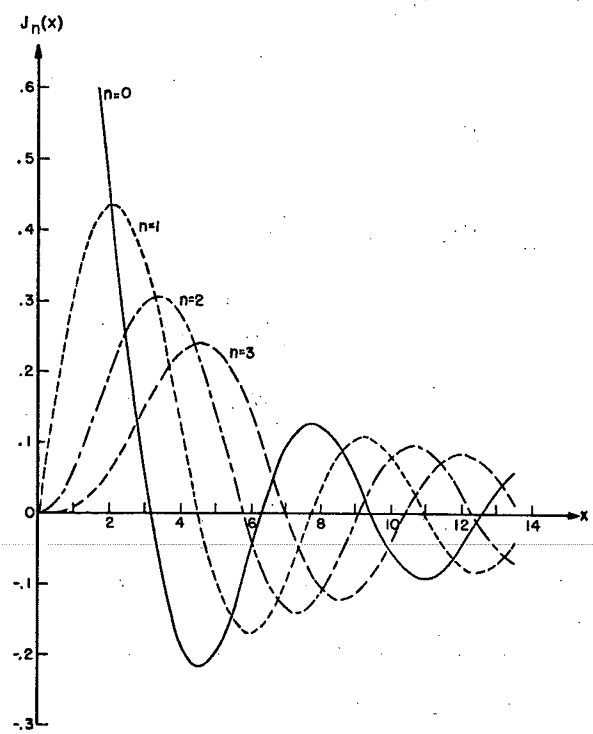

FIGURE 10.1. $j_n(x)$. $n=0(1)3$.

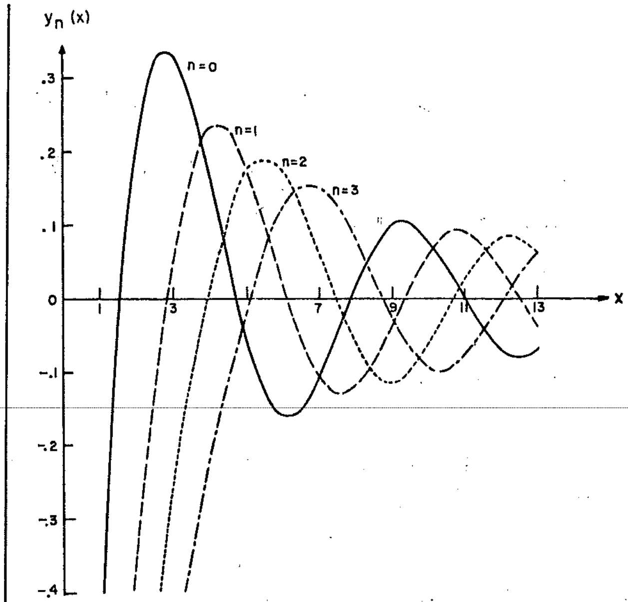

FIGURE 10.2. $y_n(x)$. $n=0(1)3$.

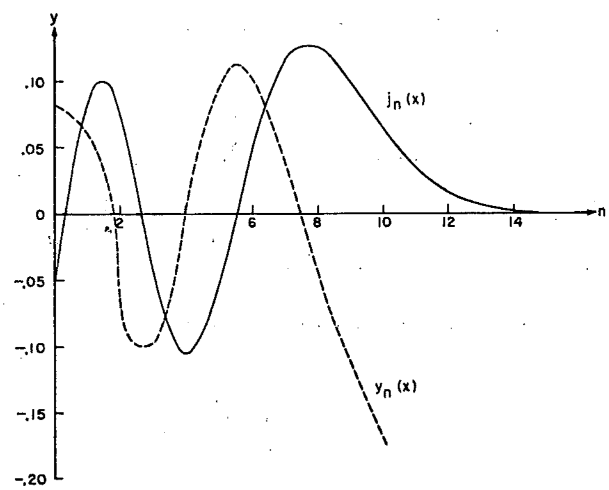

FIGURE 10.3. $j_n(x)$, $y_n(x)$. $x=10$.

Poisson's Integral and Gegenbauer's Generalization

10.1.13 $$j_n(z)=\frac{z^n}{2^{n+1}n!}\int_0^\pi \cos(z\cos\theta)\sin^{2n+1}\theta\,d\theta$$

(See **9.1.20.**)

10.1.14

$$=\frac{1}{2}(-i)^n\int_0^\pi e^{iz\cos\theta}P_n(\cos\theta)\sin\theta\,d\theta$$

$$(n=0, 1, 2, \ldots)$$

*See page II.

S3-8c

Spherical Bessel Functions of the Second and Third Kind

10.1.15

$$y_n(z)=(-1)^{n+1}j_{-n-1}(z) \qquad (n=0, \pm 1, \pm 2, \ldots)$$

10.1.16

$$h_n^{(1)}(z)=i^{-n-1}z^{-1}e^{iz}\sum_0^n (n+\tfrac{1}{2}, k)(-2iz)^{-k}$$

10.1.17

$$h_n^{(2)}(z)=i^{n+1}z^{-1}e^{-iz}\sum_0^n (n+\tfrac{1}{2}, k)\,(2iz)^{-k} \qquad *$$

10.1.18

$$h_{-n-1}^{(1)}(z)=i(-1)^n h_n^{(1)}(z)$$

$$h_{-n-1}^{(2)}(z)=-i(-1)^n h_n^{(2)}(z) \qquad (n=0, 1, 2, \ldots)$$

Elementary Properties

Recurrence Relations

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.19 $$f_{n-1}(z)+f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

10.1.20 $$nf_{n-1}(z)-(n+1)f_{n+1}(z)=(2n+1)\frac{d}{dz}f_n(z)$$

10.1.21 $$\frac{n+1}{z}f_n(z)+\frac{d}{dz}f_n(z)=f_{n-1}(z)$$

(See **10.1.23.**)

10.1.22 $$\frac{n}{z}f_n(z)-\frac{d}{dz}f_n(z)=f_{n+1}(z)$$

(See **10.1.24.**)

Differentiation Formulas

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.23 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{n+1}f_n(z)]=z^{n-m+1}f_{n-m}(z)$$

10.1.24 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{-n}f_n(z)]=(-1)^m z^{-n-m}f_{n+m}(z)$$

$(m=1, 2, 3, \ldots)$

Rayleigh's Formulas

10.1.25

$$j_n(z)=z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z}$$

10.1.26

$$y_n(z)=-z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cos z}{z} \qquad (n=0, 1, 2, \ldots)$$

*See page II.

Modulus and Phase

$$j_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\cos\theta_{n+\frac{1}{2}}(z),$$

$$y_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\sin\theta_{n+\frac{1}{2}}(z)$$

(See **9.2.17.**)

10.1.27

$$(\tfrac{1}{2}\pi/z)\,M^2_{n+\frac{1}{2}}(z)=\frac{1}{z^2}\sum_0^n \frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2z)^{2k-2n}$$

(See **9.2.28.**)

10.1.28 $$(\tfrac{1}{2}\pi/z)M^2_{1/2}(z)=j_0^2(z)+y_0^2(z)=z^{-2}$$

10.1.29

$$(\tfrac{1}{2}\pi/z)M^2_{3/2}(z)=j_1^2(z)+y_1^2(z)=z^{-2}+z^{-4}$$

10.1.30

$$(\tfrac{1}{2}\pi/z)M^2_{5/2}(z)=j_2^2(z)+y_2^2(z)=z^{-2}+3z^{-4}+9z^{-6}$$

Cross Products

10.1.31 $$j_n(z)y_{n-1}(z)-j_{n-1}(z)y_n(z)=z^{-2}$$

10.1.32

$$j_{n+1}(z)y_{n-1}(z)-j_{n-1}(z)y_{n+1}(z)=(2n+1)z^{-3}$$

10.1.33

$$j_0(z)j_n(z)+y_0(z)y_n(z)$$

$$=z^{-2}\sum_0^{[\frac{1}{2}n]}(-1)^k 2^{n-2k}\left(k+\tfrac{1}{2}\right)_{n-2k}\binom{n-k}{k}z^{2k-n}$$

$(n=0, 1, 2, \ldots)$

Analytic Continuation

10.1.34 $$j_n(ze^{m\pi i})=e^{mn\pi i}j_n(z)$$

10.1.35 $$y_n(ze^{m\pi i})=(-1)^m e^{mn\pi i}y_n(z)$$

10.1.36 $$h_n^{(1)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(2)}(z)$$

10.1.37 $$h_n^{(2)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(1)}(z)$$

10.1.38 $$h_n^{(l)}(ze^{2m\pi i})=h_n^{(l)}(z)$$

$(l=1, 2;\ m, n=0, 1, 2, \ldots)$

Generating Functions

10.1.39

$$\frac{1}{z}\sin\sqrt{z^2+2zt}=\sum_0^\infty \frac{(-t)^n}{n!}y_{n-1}(z) \qquad (2|t|<|z|)$$

10.1.40 $$\frac{1}{z}\cos\sqrt{z^2-2zt}=\sum_0^\infty \frac{t^n}{n!}j_{n-1}(z)$$

Eqn. page (53-8)

$$\boxed{r^2 \frac{d^2 B_n}{dr^2} + 2r \frac{dB_n}{dr} + \lambda_n^2 r^2 B_n = 0}$$

$B_n(0) \equiv$ bounded

$B_n(1) = 0$

In form of eqn 10.1.1 of A+S w/ $n=0$

$$z^2 \frac{d^2 w}{dz^2} + 2z \frac{dw}{dz} + z^2 w = 0$$

Let $z = r\lambda_n$

$$\Rightarrow \boxed{z^2 \frac{d^2 B_n}{dz^2} + 2z \frac{dB_n}{dz} + z^2 B_n = 0}$$

Spherical Bessel function of 2nd Kind.

Soln: $B_n = C_n j_0(z) + D_n y_0(z)$

Spherical Bessel function of 1st Kind

$\Rightarrow B_n = C_n\, j_0(\lambda_n r) + D_n\, y_0(\lambda_n r)$

From A+S, page S3-8b of notes

$$j_0(z) = \frac{\sin z}{z}, \quad y_0(z) = \frac{-\cos z}{z}$$

So:

$$B_n = C_n \frac{\sin(\lambda_n r)}{\lambda_n r} - D_n \frac{\cos(\lambda_n r)}{\lambda_n r}.$$

Lump λ_n into C_n, D_n.

$$\boxed{B_n = E_n \frac{\sin(\lambda_n r)}{r} + G_n \frac{\cos(\lambda_n r)}{r}}$$

Apply BCs: @ $r=0$, soln bounded $\Rightarrow G_n = 0$

$$\Rightarrow B_n = E_n \frac{\sin(\lambda_n r)}{r}$$

@ $r=1$, $B_n = 0 \Rightarrow \sin(\lambda_n) = 0$

$\Rightarrow \lambda_n = n\pi, \quad n = 1, 2, 3, \ldots$

So, we have $K_n < 0$:

$$\boxed{B_n = \frac{\sin(\lambda_n x)}{x}, \quad \lambda_n = n\pi, \quad K_n = -\lambda_n^2, \quad n = 1, 2, 3, \ldots}$$

Negative Eigenvalues

$K_n > 0$:

Let $K_n = \lambda_n^2$ on page (S3-6)

=> Follow same procedure w/ $K_n < 0$. Sign flips on λ_n term, page (S3-8)

=> $$\boxed{\begin{aligned} & r^2 \frac{d^2 B_n}{dr^2} + 2r \frac{dB_n}{dr} - \lambda_n^2 r^2 B_n = 0 \\ & B_n(0) \text{ bounded} \\ & B_n(1) = 0 \end{aligned}}$$

Is in form of eqn 10.2.1 of A+S w/ $n=0$ (next page)

$$z^2 \frac{d^2 w}{dz^2} + 2z \frac{dw}{dz} - z^2 w = 0 \quad \text{for } z = \lambda_n r$$

Modified spherical Bessel function of 2nd kind.

$$\Rightarrow B_n = \underbrace{C_n \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} I_{\frac{1}{2}}(z)}_{\text{Modified spherical Bessel function of 1st kind}} + D_n \left(\frac{\pi}{2z}\right) I_{-\frac{1}{2}}(z)$$

(53-12a)

10.2. Modified Spherical Bessel Functions

Definitions

Differential Equation

10.2.1

$$\boxed{z^2w''+2zw'-[z^2+n(n+1)]w=0 \qquad (n=0, \pm 1, \pm 2, \ldots)}$$

Particular solutions are the *Modified Spherical Bessel functions of the first kind,*

10.2.2

$$\boxed{\begin{aligned}\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)&=e^{-n\pi i/2}j_n(ze^{\pi i/2}) &&(-\pi<\arg z\leq\tfrac{1}{2}\pi)\\ &=e^{3n\pi i/2}j_n(ze^{-3\pi i/2}) &&(\tfrac{1}{2}\pi<\arg z\leq\pi)\end{aligned}}$$

of the second kind,

10.2.3

$$\boxed{\begin{aligned}\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)&=e^{3(n+1)\pi i/2}y_n(ze^{\pi i/2}) &&(-\pi<\arg z\leq\tfrac{1}{2}\pi)\\ &=e^{-(n+1)\pi i/2}y_n(ze^{-3\pi i/2}) &&(\tfrac{1}{2}\pi<\arg z\leq\pi)\end{aligned}}$$

of the third kind,

10.2.4

$$\sqrt{\tfrac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)=\tfrac{1}{2}\pi(-1)^{n+1}\sqrt{\tfrac{1}{2}\pi/z}\,[I_{n+\frac{1}{2}}(z)-I_{-n-\frac{1}{2}}(z)]$$

The pairs

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)$$

and

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)$$

are linearly independent solutions for every n.

Most properties of the Modified Spherical Bessel functions can be derived from those of the Spherical Bessel functions by use of the above relations.

Ascending Series

10.2.5

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=\frac{z^n}{1\cdot 3\cdot 5\ldots(2n+1)}\left\{1+\frac{\frac{1}{2}z^2}{1!(2n+3)}+\frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)}+\ldots\right\}$$

10.2.6

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{(-1)^nz^{n+1}}\left\{1+\frac{\frac{1}{2}z^2}{1!(1-2n)}+\frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)}+\ldots\right\} \qquad (n=0, 1, 2, \ldots)$$

Wronskians

10.2.7

$$W\{\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)\}=(-1)^{n+1}z^{-2}$$

10.2.8

$$W\{\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)\}=-\tfrac{1}{2}\pi z^{-2}$$

Representations by Elementary Functions

10.2.9

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=(2z)^{-1}[R(n+\tfrac{1}{2},-z)e^z-(-1)^nR(n+\tfrac{1}{2},z)e^{-z}]$$

10.2.10

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)=(2z)^{-1}[R(n+\tfrac{1}{2},-z)e^z+(-1)^nR(n+\tfrac{1}{2},z)e^{-z}]$$

10.2.11

$$\begin{aligned}R(n+\tfrac{1}{2},z)&=1+\frac{(n+1)!}{1!\Gamma(n)}(2z)^{-1}+\frac{(n+2)!}{2!\Gamma(n-1)}(2z)^{-2}+\ldots\\ &=\sum_0^n(n+\tfrac{1}{2},k)(2z)^{-k} \qquad (n=0, 1, 2, \ldots)\end{aligned}$$

(See **10.1.9.**)

10.2.12

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=g_n(z)\sinh z+g_{-n-1}(z)\cosh z$$

$$g_0(z)=z^{-1},\quad g_1(z)=-z^{-2}$$

$$g_{n-1}(z)-g_{n+1}(z)=(2n+1)z^{-1}g_n(z) \qquad (n=0, \pm 1, \pm 2, \ldots)$$

The Functions $\sqrt{\tfrac{1}{2}\pi/z}\,I_{\pm(n+\frac{1}{2})}(z)$, $n=0, 1, 2$

10.2.13

$$\boxed{\sqrt{\tfrac{1}{2}\pi/z}\,I_{1/2}(z)=\frac{\sinh z}{z}}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{3/2}(z)=-\frac{\sinh z}{z^2}+\frac{\cosh z}{z}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{5/2}(z)=\left(\frac{3}{z^3}+\frac{1}{z}\right)\sinh z-\frac{3}{z^2}\cosh z \quad *$$

10.2.14

$$\boxed{\sqrt{\tfrac{1}{2}\pi/z}\,I_{-1/2}(z)=\frac{\cosh z}{z}}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-3/2}(z)=\frac{\sinh z}{z}-\frac{\cosh z}{z^2}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-5/2}(z)=-\frac{3}{z^2}\sinh z+\left(\frac{3}{z^3}+\frac{1}{z}\right)\cosh z \quad *$$

*See page II.

S3-12h

Modified Spherical Bessel Functions of the Third Kind

10.2.15

$$\sqrt{\tfrac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)=\tfrac{1}{2}\pi i e^{(n+1)\pi i/2}h_n^{(1)}(ze^{\frac{1}{2}\pi i})\qquad(-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

$$=-\tfrac{1}{2}\pi i e^{-(n+1)\pi i/2}h_n^{(2)}(ze^{-\frac{1}{2}\pi i})\qquad(\tfrac{1}{2}\pi<\arg z\leq\pi)$$

$$=(\tfrac{1}{2}\pi/z)e^{-z}\sum_{0}^{n}(n+\tfrac{1}{2},k)(2z)^{-k}$$

10.2.16

$$K_{n+\frac{1}{2}}(z)=K_{-n-\frac{1}{2}}(z)\qquad(n=0,1,2,\ldots)$$

The Functions $\sqrt{\frac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z), n=0, 1, 2$

10.2.17
$$\sqrt{\tfrac{1}{2}\pi/z}\,K_{1/2}(z)=(\tfrac{1}{2}\pi/z)e^{-z}$$
$$\sqrt{\tfrac{1}{2}\pi/z}\,K_{3/2}(z)=(\tfrac{1}{2}\pi/z)e^{-z}(1+z^{-1})$$
$$\sqrt{\tfrac{1}{2}\pi/z}\,K_{5/2}(z)=(\tfrac{1}{2}\pi/z)e^{-z}(1+3z^{-1}+3z^{-2})$$

Elementary Properties

Recurrence Relations

$f_n(z)$: $\sqrt{\frac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)$, $(-1)^{n+1}\sqrt{\frac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)$ $\qquad(n=0,\pm1,\pm2,\ldots)$

10.2.18 $$f_{n-1}(z)-f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

10.2.19 $$nf_{n-1}(z)+(n+1)f_{n+1}(z)=(2n+1)\frac{d}{dz}f_n(z)$$

10.2.20 $$\frac{n+1}{z}f_n(z)+\frac{d}{dz}f_n(z)=f_{n-1}(z)$$
(See **10.2.22.**)

10.2.21 $$-\frac{n}{z}f_n(z)+\frac{d}{dz}f_n(z)=f_{n+1}(z)$$
(See **10.2.23.**)

Differentiation Formulas

$f_n(z)$: $\sqrt{\frac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)$, $(-1)^{n+1}\sqrt{\frac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)$ $\qquad(n=0,\pm1,\pm2,\ldots)$

10.2.22 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m[z^{n+1}f_n(z)]=z^{n-m+1}f_{n-m}(z)$$

10.2.23 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m[z^{-n}f_n(z)]=z^{-n-m}f_{n+m}(z)\qquad(m=1,2,3,\ldots)$$

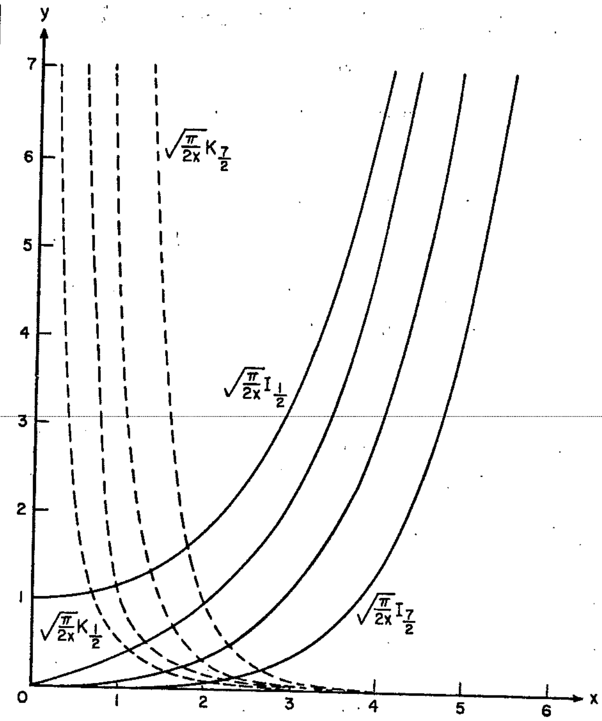

FIGURE 10.4. $\sqrt{\frac{\pi}{2x}}\,I_{n+\frac{1}{2}}(x)$, $\sqrt{\frac{\pi}{2x}}\,K_{n+\frac{1}{2}}(x)$. $n=0(1)3$.

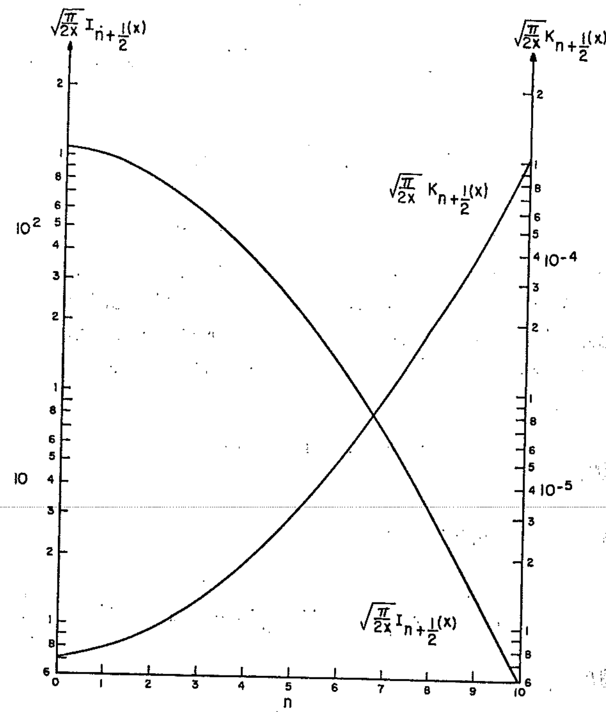

FIGURE 10.5. $\sqrt{\frac{\pi}{2x}}\,I_{n+\frac{1}{2}}(x)$, $\sqrt{\frac{\pi}{2x}}\,K_{n+\frac{1}{2}}(x)$. $x=10$.

S3-12c

Formulas of Rayleigh's Type

10.2.24 $$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=z^n\left(\frac{1}{z}\frac{d}{dz}\right)^n\frac{\sinh z}{z}$$

10.2.25

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)=z^n\left(\frac{1}{z}\frac{d}{dz}\right)^n\frac{\cosh z}{z}\qquad(n=0,1,2,\ldots)$$

Formulas for $I^2_{n+\frac{1}{2}}(z)-I^2_{-n-\frac{1}{2}}(z)$

10.2.26

$$(\tfrac{1}{2}\pi/z)[I^2_{n+\frac{1}{2}}(z)-I^2_{-n-\frac{1}{2}}(z)]=\frac{1}{z^2}\sum_0^n(-1)^{k+1}\frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2z)^{2k-2n}\qquad(n=0,1,2,\ldots)$$

10.2.27 $$(\tfrac{1}{2}\pi/z)[I^2_{1/2}(z)-I^2_{-1/2}(z)]=-z^{-2}$$

10.2.28 $$(\tfrac{1}{2}\pi/z)[I^2_{3/2}(z)-I^2_{-3/2}(z)]=z^{-2}-z^{-4}$$

10.2.29

$$(\tfrac{1}{2}\pi/z)[I^2_{5/2}(z)-I^2_{-5/2}(z)]=-z^{-2}+3z^{-4}-9z^{-6}$$

Generating Functions

10.2.30

$$\frac{1}{z}\sinh\sqrt{z^2-2izt}=\sum_0^\infty\frac{(-it)^n}{n!}[\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n+\frac{1}{2}}(z)]\qquad(2|t|<|z|)$$

10.2.31

$$\frac{1}{z}\cosh\sqrt{z^2+2izt}=\sum_0^\infty\frac{(it)^n}{n!}[\sqrt{\tfrac{1}{2}\pi/z}\,I_{n-\frac{1}{2}}(z)]$$

Derivatives With Respect to Order

10.2.32

$$\left[\frac{\partial}{\partial\nu}I_\nu(x)\right]_{\nu=\frac{1}{2}}=-\frac{1}{2\pi x}[\mathrm{Ei}(2x)e^{-x}-E_1(-2x)e^x]$$

10.2.33

$$\left[\frac{\partial}{\partial\nu}I_\nu(x)\right]_{\nu=-\frac{1}{2}}=\frac{1}{2\pi x}[\mathrm{Ei}(2x)e^{-x}+E_1(-2x)e^x]$$

10.2.34 $$\left[\frac{\partial}{\partial\nu}K_\nu(x)\right]_{\nu=\pm\frac{1}{2}}=\mp\sqrt{\pi/2x}\,\mathrm{Ei}(-2x)e^x$$

For $E_1(x)$ and $\mathrm{Ei}(x)$, see **5.1.1, 5.1.2.**

*See page II.

Addition Theorems and Degenerate Forms

r,ρ,θ,λ arbitrary complex; $R=\sqrt{r^2+\rho^2-2r\rho\cos\theta}$

10.2.35

$$\frac{e^{-\lambda R}}{\lambda R}=\frac{2}{\pi}\sum_0^\infty(2n+1)[\sqrt{\tfrac{1}{2}\pi/\lambda r}\,I_{n+\frac{1}{2}}(\lambda r)][\sqrt{\tfrac{1}{2}\pi/\lambda\rho}\,K_{n+\frac{1}{2}}(\lambda\rho)]P_n(\cos\theta)$$

10.2.36

$$e^{z\cos\theta}=\sum_0^\infty(2n+1)[\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)]P_n(\cos\theta)$$

10.2.37

$$e^{-z\cos\theta}=\sum_0^\infty(-1)^n(2n+1)[\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)]P_n(\cos\theta)$$

Duplication Formula

10.2.38

$$K_{n+\frac{1}{2}}(2z)=n!\pi^{-\frac{1}{2}}z^{n+\frac{1}{2}}\sum_0^n\frac{(-1)^k(2n-2k+1)}{k!(2n-k+1)!}K^2_{n-k+\frac{1}{2}}(z)$$

10.3. Riccati-Bessel Functions

Differential Equation

10.3.1

$$z^2w''+[z^2-n(n+1)]w=0$$

$$(n=0,\pm1,\pm2,\ldots)$$

Pairs of linearly independent solutions are

$$zj_n(z),\ zy_n(z)$$

$$zh_n^{(1)}(z),\ zh_n^{(2)}(z)$$

All properties of these functions follow directly from those of the Spherical Bessel functions.

The Functions $zj_n(z)$, $zy_n(z)$, $n=0,1,2$

10.3.2

$$zj_0(z)=\sin z,\qquad zj_1(z)=z^{-1}\sin z-\cos z$$

$$zj_2(z)=(3z^{-2}-1)\sin z-3z^{-1}\cos z\quad *$$

10.3.3

$$zy_0(z)=-\cos z,\qquad zy_1(z)=-\sin z-z^{-1}\cos z$$

$$zy_2(z)=-3z^{-1}\sin z-(3z^{-2}-1)\cos z\quad *$$

Wronskians

10.3.4 $$W\{zj_n(z),zy_n(z)\}=1$$

10.3.5 $$W\{zh_n^{(1)}(z),zh_n^{(2)}(z)\}=-2i$$

$$(n=0,1,2,\ldots)$$

From A+S, page 53-12a of notes:

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} I_{\frac{1}{2}}(z) = \frac{\sinh z}{z}, \quad \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} I_{-\frac{1}{2}}(z) = \frac{\cosh z}{z}$$

So: $B_n = C_n \frac{\sinh z}{z} + D_n \frac{\cosh z}{z}$. $z = \lambda_n r$

$$\Rightarrow B_n = C_n \frac{\sinh(\lambda_n r)}{\lambda_n r} + D_n \frac{\cosh(\lambda_n r)}{\lambda_n r}$$ Lump λ_n into C_n, D_n..

$$B_n = E_n \frac{\sinh(\lambda_n r)}{r} + G_n \frac{\cosh(\lambda_n r)}{r}$$

Apply BCs. @ $r=0$, Soln bounded $\Rightarrow G_n = 0$

$$\Rightarrow B_n = E_n \frac{\sinh(\lambda_n r)}{r}$$

@ $r=1$, $B_n = 0 \Rightarrow \sinh(\lambda_n) = 0$.

$\sinh(\lambda_n) = 0$ if $\lambda_n = 0$. (only!)

But $\lambda_n > 0$ assumed $\Rightarrow E_n = 0$

Conclusion: No nontrivial solutions for $K_n > 0$.

Eigensearch results summary:

$$\boxed{\begin{array}{l} K_n < 0 \text{ Only} \\ B_n = \dfrac{\sin(\lambda_n x)}{x}, \quad \lambda_n = n\pi, \quad K_n = -\lambda_n^2; \quad n = 1, 2, 3, \ldots \\ B_n \perp \text{ w.r.t. } \langle f, g \rangle = \int_0^1 x^2 f g \, dx \end{array}}$$

Formalize Summation:

$$F = \sum_n A_n(\tau)\, B_n(x)$$

$$\Rightarrow \quad F = \sum_{n=1}^{\infty} A_n(\tau) \frac{\sin(\lambda_n x)}{x} \quad , \quad \lambda_n = n\pi$$

Separated PDE: Boxed Eqn, page (S3-5):

$$\sum_{n=1}^{\infty} \left[\frac{dA_n}{d\tau} + \lambda_n^2 A_n(\tau) \right] \frac{\sin(\lambda_n x)}{x} = 0$$

Use orthogonality of $\frac{\sin(\lambda_n x)}{x}$ by taking inner product w/ $\frac{\sin(\lambda_m x)}{x}$, or recognizing that since $\frac{\sin(\lambda_n x)}{x}$ $\forall n$ forms basis, the basis functions are linearly independent, + quantity in Brackets must be zero $\forall n$.

$$\Rightarrow \quad \boxed{\frac{dA_n}{d\tau} + \lambda_n^2 A_n = 0}$$

Soln: $A_n = C_n e^{-\lambda_n^2 \tau}$

Need BC @ $\tau = 0$

@ $\tau=0$, $F = -1$ (Page 53-4)

$$\Rightarrow F = \sum_{n=1}^{\infty} A_n(\tau) \frac{\sin(\lambda_n x)}{x}$$

$$-1 = \sum_{n=1}^{\infty} A_n(0) \frac{\sin(\lambda_n x)}{x}$$

Orthogonality: $\left\langle -1, \frac{\sin(\lambda_n x)}{x} \right\rangle = A_n(0) \left\langle \frac{\sin(\lambda_n x)}{x}, \frac{\sin(\lambda_n x)}{x} \right\rangle$

$$\Rightarrow A_n(0) = \frac{-\left\langle 1, \frac{\sin(\lambda_n x)}{x} \right\rangle}{\left\langle \frac{\sin(\lambda_n x)}{x}, \frac{\sin(\lambda_n x)}{x} \right\rangle} = -\gamma_n$$

Defined here.

Apply BC : $A_n(0) = -\gamma_n$

$\Rightarrow A_n = -\gamma_n e^{-\lambda_n^2 \tau}$

Soln for F :

$$F = \sum_{n=1}^{\infty} -\gamma_n e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x)}{x}$$

$$\gamma_n = \frac{\left\langle 1, \frac{\sin(\lambda_n x)}{x} \right\rangle}{\left\langle \frac{\sin(\lambda_n x)}{x}, \frac{\sin(\lambda_n x)}{x} \right\rangle} \qquad \langle f, g \rangle = \int_0^1 x^2 f g \, dx$$

Evaluate γ_n : $\left\langle 1, \frac{\sin(\lambda_n x)}{x} \right\rangle = \int_0^1 x^2 (1) \left(\frac{\sin \lambda_n x}{x} \right) dx = \int_0^1 x \sin(\lambda_n x) \, dx$

$$\left\langle \frac{\sin(\lambda_n x)}{x}, \frac{\sin(\lambda_n x)}{x} \right\rangle = \int_0^1 x^2 \frac{\sin^2(\lambda_n x)}{x^2} dx = \int_0^1 \sin^2(\lambda_n x) \, dx$$

$\lambda_n = n\pi$

$$\left\langle \frac{\sin(\lambda_n x)}{x}, \frac{\sin(\lambda_n x)}{x} \right\rangle = \int_0^1 \sin^2(\lambda_n x)\,dx = \frac{1}{2}$$

$$\left\langle 1, \frac{\sin(\lambda_n x)}{x} \right\rangle = \int_0^1 x \sin(\lambda_n x)\,dx = \frac{-x}{\lambda_n}\cos(\lambda_n x)\Big|_0^1 + \frac{1}{\lambda_n}\int_0^1 \cos(\lambda_n x)\,dx$$

$u = x \qquad dv = \sin(\lambda_n x)\,dx$

$du = dx \qquad v = \frac{-1}{\lambda_n}\cos(\lambda_n x)$

$$\left\langle 1, \frac{\sin(\lambda_n x)}{x} \right\rangle = -\frac{\cos(\lambda_n)}{\lambda_n} + \cancel{\frac{1}{\lambda_n^2}\sin(\lambda_n x)\Big|_0^1} \qquad \lambda_n = n\pi$$

$$= -\frac{\cos(n\pi)}{\lambda_n} = -\frac{(-1)^n}{\lambda_n}$$

$$\text{So, } \gamma_n = \frac{\left\langle 1, \frac{\sin(\lambda_n x)}{x} \right\rangle}{\left\langle \frac{\sin(\lambda_n x)}{x}, \frac{\sin(\lambda_n x)}{x} \right\rangle} = \frac{\frac{-(-1)^n}{\lambda_n}}{\frac{1}{2}} = \frac{-2(-1)^n}{\lambda_n}$$

S3-18

Our final soln for F is thus

$$F = \sum_{n=1}^{\infty} -\gamma_n \, e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x)}{x} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n} \, e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x)}{x}$$

Our desired solution for $\tilde{C}$ on page S3-2 is given as:

$$\tilde{C} \quad 1 + F$$

So, $$\boxed{\tilde{C} = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda^n} e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x)}{x} , \quad \lambda_n = n\pi}$$

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