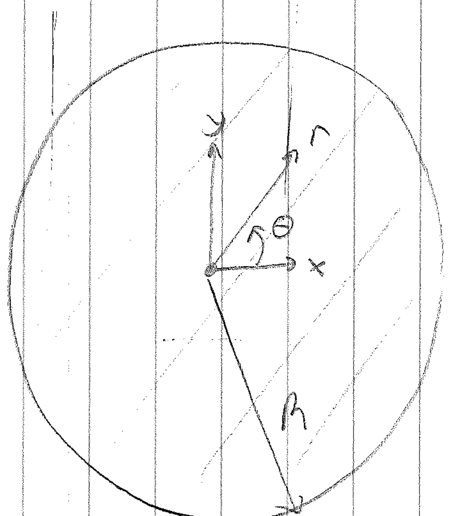
Example 2, Cylindrical Coordinates

Transient heat conduction in long cylinder w/ heat generation (constant)

T_∞

Cross-Section:



- Assume no θ dependence
- Surface of cylinder experiences same interaction w/surroundings for all θ.

Eqn: $$\rho \hat{C}_p \frac{\partial T}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + H_v$$

@ $t=0$, $T = T_0$

@ $r=0$, $\frac{\partial T}{\partial r} = 0$, or finite soln (r-θ coordinate system breaks down @ $r=0$!)

@ $r=R$: $T = T_\infty$, or $-k\frac{\partial T}{\partial r} = q$, or $-k\frac{\partial T}{\partial r} = h(T - T_\infty)$

System may be placed in dimensionless form, depending on problem.

$$\Theta = \frac{T - T_\infty}{T_0 - T_\infty}, \text{ for example}$$

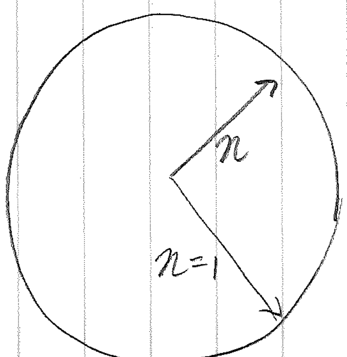
Eqn:
$$\frac{\partial \Theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Theta}{\partial r}\right) + 1$$

@ $t=0$, $\Theta = \Theta_0$

@ $r=0$, bounded soln, +/or $\frac{\partial \Theta}{\partial r} = 0$

@ $r=1$, $\Theta = \Theta_1$, or $\frac{\partial \Theta}{\partial r} = -\alpha$, or $\frac{\partial \Theta}{\partial r} = -\beta\Theta$

Dirichlet a) Neumann b) Robin c)



C2-2

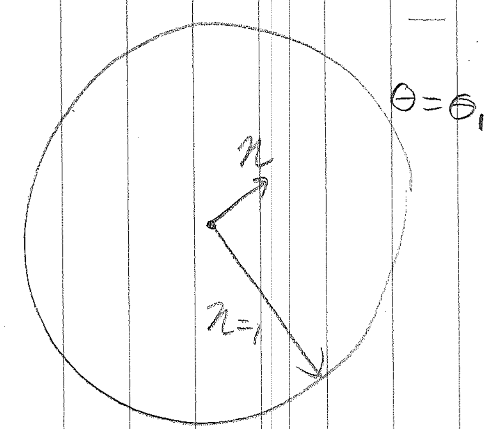
Case (a)

$$\frac{\partial \theta}{\partial \xi} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + 1$$

@ $\xi = 0$, $\theta = \theta_0$

@ $r = 0$, Bounded soln.

@ $r = 1$, $\theta = \theta_1$



① Choose direction for basis expansion. BCs must be homogeneous in that direction.

↳ If want *infinite* Fourier *sum* *solution*, must eigenfunction expand in *finite* *direction*.

↳ *Can* work w/ ξ direction, but as domain is ∞, need transforms, which are derived from Fourier series (as domain gets long, eigenvalues become continuous.

② Choose η direction:

· Need to make η direction homogeneous:

Find function $g(\eta) \ni$ $g(\eta) = \Theta_1$ @ $\eta = 1$

Bounded @ $\eta = 0$

Let $g(\eta) = \Theta_1$ ← we will see if good enough...

Write: $\boxed{\Theta = \Theta_1 + F(\xi, \eta)}$

$$\Rightarrow \quad \frac{\partial F}{\partial \xi} = \frac{1}{\eta} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial F}{\partial \eta}\right) + 1$$

@ $\xi = 0$, $\Theta = \Theta_0$ $\Rightarrow$ $\Theta_0 = \Theta_1 + F$ $\Rightarrow$ $F = \Theta_0 - \Theta_1$

@ $\eta = 1$, $\Theta = \Theta_1$ $\Rightarrow$ $\Theta_1 = \Theta_1 + F$ $\Rightarrow$ $F = 0$

@ $\eta = 0$, Bounded.

New System to solve:

$$\frac{\partial F}{\partial \xi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial F}{\partial \eta}\right) + 1$$

$\theta = \theta_1 + F(\xi, \eta)$

@ $\xi = 0$, $F = \theta_0 - \theta_1$

@ $\eta = 0$, F bounded

@ $\eta = 1$, $F = 0$

③ Now, solve for F:

$F = \sum_n A_n(\xi)\, B_n(\eta)$. Sub into PDE:

$$\sum_n \left[\frac{dA_n}{d\xi} B_n - \frac{1}{\eta}\frac{d}{d\eta}\left(\eta \frac{dB_n}{d\eta}\right) A_n(\xi)\right] = 1$$

Q3-5

$$\Rightarrow \sum_n \left[\frac{dA_n}{d\xi} - \left(\frac{1}{\eta} \frac{d}{d\eta}\left(\eta \frac{dB_n}{d\eta}\right) \frac{1}{B_n} \right) A_n(\xi) \right] B_n(\eta) = 1$$

For system to be separable, term in brackets must only be a function of ξ.

Then:

$$\boxed{\begin{gathered} \frac{1}{\eta} \frac{d}{d\eta}\left(\eta \frac{dB_n}{d\eta}\right) \frac{1}{B_n} = K_n \\ \text{BCs: } B_n \text{ bounded } @\, \eta = 0 \\ F = \sum_n A_n(\xi)\, B_n(\eta) = 0 \;@\; \eta = 1 \;\Rightarrow B_n(1) = 0 \end{gathered}}$$

Also:

$$\boxed{\sum_n \left[\frac{dA_n}{d\xi} - K_n A_n(\xi) \right] B_n(\eta) = 1}$$

$$\Rightarrow \boxed{\begin{aligned} &\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_n}{dr}\right) = K_n B_n \\ &B_n(0) \equiv \text{Bounded} \\ &B_n(1) = 0 \end{aligned}}$$

Sturm-Liouville Problem:

1. K_n real
2. $B_n \perp$ w.r.t. $\langle f, g\rangle = \int_0^1 r f g \, dr$
3. B_n forms basis
4. 1 B_n for each K_n

(2-7)

Perform Eigensearch:

$K_n < 0, \quad K_n = 0, \quad K_n > 0$ (Must be real).

$K_n = 0$:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_n}{dr}\right) = 0 \quad \Rightarrow \quad r\frac{dB_n}{dr} = C \quad \Rightarrow \frac{dB_n}{dr} = \frac{C}{r}.$$ For bounded soln, $C = 0$

$$\Rightarrow \frac{dB_n}{dr} = 0 \quad \Rightarrow B_n = D = \text{constant}.$$ But, $B_n = 0$ @ $r = 1$ $\Rightarrow D = 0$

So $B_n = 0 \ \forall r \Rightarrow$ No nontrivial soln.

$\Rightarrow$ No eigensolution

C2-8

$\underline{\underline{K_n < 0}}$

Let $K_n = -\lambda_n^2$, $\lambda_n > 0$ for definiteness

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dB_n}{dr}\right) = -\lambda_n^2 B_n$$

$$r\frac{d^2B_n}{dr^2} + \frac{dB_n}{dr} = -\lambda_n^2 B_n r$$

$$\Rightarrow \boxed{\begin{array}{l} r\frac{d^2B_n}{dr^2} + \frac{dB_n}{dr} + \lambda_n^2 B_n r = 0 \\ B_n(0) = \text{Bounded} \\ B_n(1) = 0 \end{array}}$$

Generally need to solve via power series...
↳ But: Most L-generating S.L. problems have been solved!

Go to handbook... & use
Abramowitz + Stegun (Knovel Data Base)

C2-9

Fr. Abramowitz + Stegun (Handbook of Mathematical Functions) (C2-9a)

9. Bessel Functions of Integer Order

Mathematical Properties

Notation

The tables in this chapter are for Bessel functions of integer order; the text treats general orders. The conventions used are:

$z=x+iy$; x, y real.

n is a positive integer or zero.

ν, μ are unrestricted except where otherwise indicated; ν is supposed real in the sections devoted to Kelvin functions **9.9, 9.10,** and **9.11.**

The notation used for the Bessel functions is that of Watson [9.15] and the British Association and Royal Society Mathematical Tables. The function $Y_\nu(z)$ is often denoted $N_\nu(z)$ by physicists and European workers.

Other notations are those of:

Aldis, Airey:

$$G_n(z) \text{ for } -\tfrac{1}{2}\pi Y_n(z), K_n(z) \text{ for } (-)^n K_n(z).$$

Clifford:

$$C_n(x) \text{ for } x^{-\frac{1}{2}n} J_n(2\sqrt{x}).$$

Gray, Mathews and MacRobert [9.9]:

$$Y_n(z) \text{ for } \tfrac{1}{2}\pi Y_n(z)+(\ln 2-\gamma)J_n(z),$$

$$\overline{Y}_\nu(z) \text{ for } \pi e^{\nu\pi i}\sec(\nu\pi)Y_\nu(z),$$

$$G_\nu(z) \text{ for } \tfrac{1}{2}\pi i H_\nu^{(1)}(z).$$

Jahnke, Emde and Lösch [9.32]:

$$\Lambda_\nu(z) \text{ for } \Gamma(\nu+1)(\tfrac{1}{2}z)^{-\nu}J_\nu(z).$$

Jeffreys:

$$Hs_\nu(z) \text{ for } H_\nu^{(1)}(z),\ Hi_\nu(z) \text{ for } H_\nu^{(2)}(z),$$

$$Kh_\nu(z) \text{ for } (2/\pi)K_\nu(z).$$

Heine:

$$K_n(z) \text{ for} -\tfrac{1}{2}\pi Y_n(z).$$

Neumann:

$$Y^n(z) \text{ for } \tfrac{1}{2}\pi Y_n(z)+(\ln 2-\gamma)J_n(z).$$

Whittaker and Watson [9.18]:

$$K_\nu(z) \text{ for } \cos(\nu\pi)K_\nu(z).$$

Bessel Functions J and Y

9.1. Definitions and Elementary Properties

Differential Equation

9.1.1
$$\boxed{z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=0}$$

Solutions are the Bessel functions of the first kind $J_{\pm\nu}(z)$, of the second kind $Y_\nu(z)$ (also called Weber's function) and of the third kind $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ (also called the Hankel functions). Each is a regular (holomorphic) function of z throughout the z-plane cut along the negative real axis, and for fixed $z(\neq 0)$ each is an entire (integral) function of ν. When $\nu=\pm n$, $J_\nu(z)$ has no branch point and is an entire (integral) function of z.

Important features of the various solutions are as follows: $J_\nu(z)(\mathscr{R}\nu\geq 0)$ is bounded as $z\to 0$ in any bounded range of arg z. $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent except when ν is an integer. $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all values of ν.

$H_\nu^{(1)}(z)$ tends to zero as $|z|\to\infty$ in the sector $0<\arg z<\pi$; $H_\nu^{(2)}(z)$ tends to zero as $|z|\to\infty$ in the sector $-\pi<\arg z<0$. For all values of ν, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linearly independent.

Relations Between Solutions

9.1.2
$$Y_\nu(z)=\frac{J_\nu(z)\cos(\nu\pi)-J_{-\nu}(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.1.3

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z)$$
$$=i\csc(\nu\pi)\{e^{-\nu\pi i}J_\nu(z)-J_{-\nu}(z)\}$$

9.1.4

$$H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z)$$
$$=i\csc(\nu\pi)\{J_{-\nu}(z)-e^{\nu\pi i}J_\nu(z)\}$$

9.1.5 $\quad J_{-n}(z)=(-)^nJ_n(z) \qquad Y_{-n}(z)=(-)^nY_n(z)$

9.1.6 $\quad H_{-\nu}^{(1)}(z)=e^{\nu\pi i}H_\nu^{(1)}(z) \qquad H_{-\nu}^{(2)}(z)=e^{-\nu\pi i}H_\nu^{(2)}(z)$

C2-9b

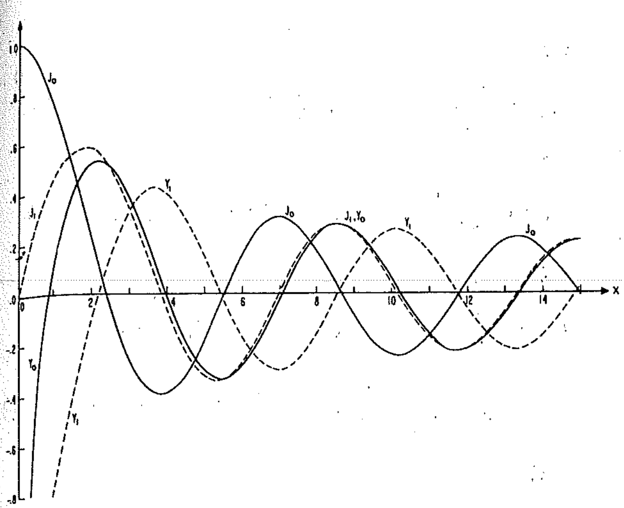


FIGURE 9.1. $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$.

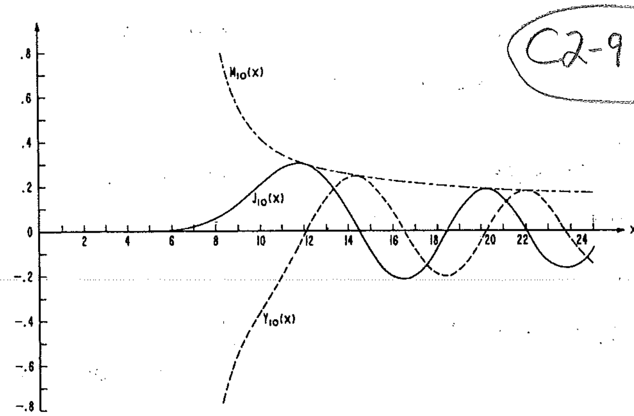


FIGURE 9.2. $J_{10}(x)$, $Y_{10}(x)$, *and* $M_{10}(x)=\sqrt{J_{10}^2(x)+Y_{10}^2(x)}$.

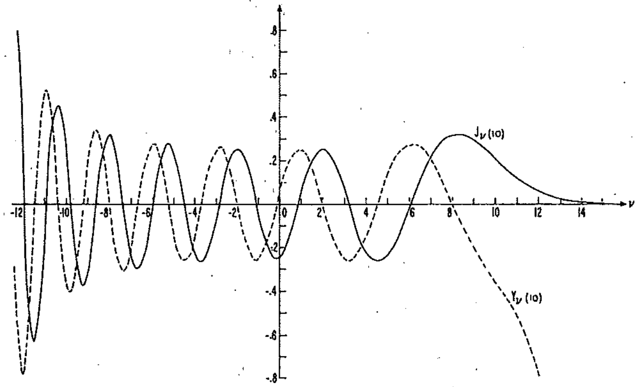


FIGURE 9.3. $J_\nu(10)$ *and* $Y_\nu(10)$.

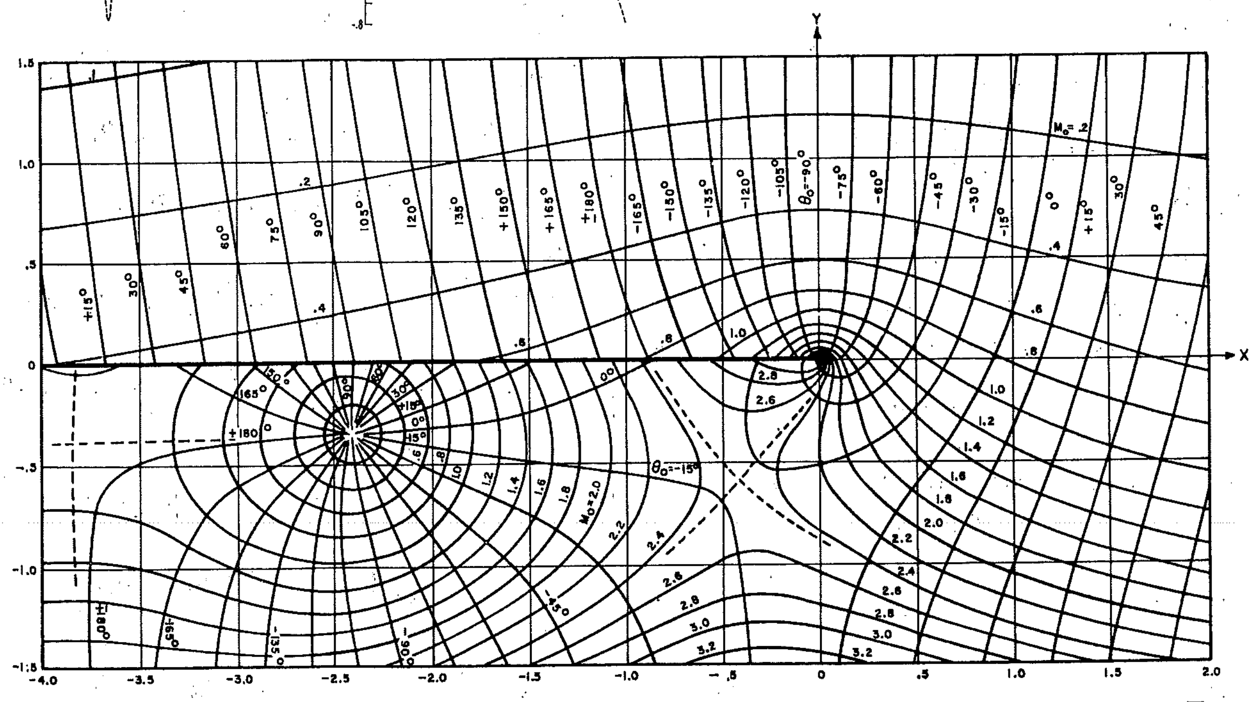


FIGURE 9.4. *Contour lines of the modulus and phase of the Hankel Function* $H_0^{(1)}(x+iy)=M_0e^{i\theta_0}$. From E. Jahnke, F. Emde, and F. Lösch, Tables of higher functions, McGraw-Hill Book Co., Inc., New York, N.Y., 1960 (with permission).

C2-9c

Limiting Forms for Small Arguments

When ν is fixed and $z\to 0$

9.1.7
$$J_\nu(z)\sim(\tfrac{1}{2}z)^\nu/\Gamma(\nu+1)\qquad(\nu\neq -1,-2,-3,\ldots)$$

9.1.8 $$Y_0(z)\sim -iH_0^{(1)}(z)\sim iH_0^{(2)}(z)\sim(2/\pi)\ln z$$

9.1.9
$$Y_\nu(z)\sim -iH_\nu^{(1)}(z)\sim iH_\nu^{(2)}(z)\sim -(1/\pi)\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}\qquad(\mathscr{R}\nu>0)$$

Ascending Series

9.1.10 $$J_\nu(z)=(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}\frac{(-\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}$$

9.1.11
$$Y_n(z)=-\frac{(\frac{1}{2}z)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(\tfrac{1}{4}z^2)^k+\frac{2}{\pi}\ln(\tfrac{1}{2}z)J_n(z)-\frac{(\frac{1}{2}z)^n}{\pi}\sum_{k=0}^{\infty}\{\psi(k+1)+\psi(n+k+1)\}\frac{(-\frac{1}{4}z^2)^k}{k!\,(n+k)!}$$

where $\psi(n)$ is given by **6.3.2.**

9.1.12 $$J_0(z)=1-\frac{\frac{1}{4}z^2}{(1!)^2}+\frac{(\frac{1}{4}z^2)^2}{(2!)^2}-\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

9.1.13
$$Y_0(z)=\frac{2}{\pi}\{\ln(\tfrac{1}{2}z)+\gamma\}J_0(z)+\frac{2}{\pi}\Big\{\frac{\frac{1}{4}z^2}{(1!)^2}-(1+\tfrac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{4}z^2)^3}{(3!)^2}-\ldots\Big\}$$

9.1.14
$$J_\nu(z)J_\mu(z)=(\tfrac{1}{2}z)^{\nu+\mu}\sum_{k=0}^{\infty}\frac{(-)^k\Gamma(\nu+\mu+2k+1)(\frac{1}{4}z^2)^k}{\Gamma(\nu+k+1)\Gamma(\mu+k+1)\Gamma(\nu+\mu+k+1)\,k!}$$

Wronskians

9.1.15
$$W\{J_\nu(z),J_{-\nu}(z)\}=J_{\nu+1}(z)J_{-\nu}(z)+J_\nu(z)J_{-(\nu+1)}(z)=-2\sin(\nu\pi)/(\pi z)$$

9.1.16
$$W\{J_\nu(z),Y_\nu(z)\}=J_{\nu+1}(z)Y_\nu(z)-J_\nu(z)Y_{\nu+1}(z)=2/(\pi z)$$

9.1.17
$$W\{H_\nu^{(1)}(z),H_\nu^{(2)}(z)\}=H_{\nu+1}^{(1)}(z)H_\nu^{(2)}(z)-H_\nu^{(1)}(z)H_{\nu+1}^{(2)}(z)=-4i/(\pi z)$$

Integral Representations

9.1.18
$$J_0(z)=\frac{1}{\pi}\int_0^\pi\cos(z\sin\theta)d\theta=\frac{1}{\pi}\int_0^\pi\cos(z\cos\theta)d\theta$$

9.1.19
$$Y_0(z)=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\cos(z\cos\theta)\{\gamma+\ln(2z\sin^2\theta)\}d\theta$$

9.1.20
$$J_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_0^\pi\cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta=\frac{2(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\cos(zt)dt\quad(\mathscr{R}\nu>-\tfrac{1}{2})$$

9.1.21
$$J_n(z)=\frac{1}{\pi}\int_0^\pi\cos(z\sin\theta-n\theta)d\theta=\frac{i^{-n}}{\pi}\int_0^\pi e^{iz\cos\theta}\cos(n\theta)d\theta$$

9.1.22
$$J_\nu(z)=\frac{1}{\pi}\int_0^\pi\cos(z\sin\theta-\nu\theta)d\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\sinh t-\nu t}dt\quad(|\arg z|<\tfrac{1}{2}\pi)$$

$$Y_\nu(z)=\frac{1}{\pi}\int_0^\pi\sin(z\sin\theta-\nu\theta)d\theta-\frac{1}{\pi}\int_0^\infty\{e^{\nu t}+e^{-\nu t}\cos(\nu\pi)\}e^{-z\sinh t}dt\quad(|\arg z|<\tfrac{1}{2}\pi)$$

9.1.23
$$J_0(x)=\frac{2}{\pi}\int_0^\infty\sin(x\cosh t)dt\quad(x>0)$$

$$Y_0(x)=-\frac{2}{\pi}\int_0^\infty\cos(x\cosh t)dt\quad(x>0)$$

9.1.24
$$J_\nu(x)=\frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty\frac{\sin(xt)dt}{(t^2-1)^{\nu+\frac{1}{2}}}\quad(|\mathscr{R}\nu|<\tfrac{1}{2},x>0)$$

$$Y_\nu(x)=-\frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty\frac{\cos(xt)dt}{(t^2-1)^{\nu+\frac{1}{2}}}\quad(|\mathscr{R}\nu|<\tfrac{1}{2},x>0)$$

9.1.25
$$H_\nu^{(1)}(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i}e^{z\sinh t-\nu t}dt\quad(|\arg z|<\tfrac{1}{2}\pi)$$

$$H_\nu^{(2)}(z)=-\frac{1}{\pi i}\int_{-\infty}^{\infty-\pi i}e^{z\sinh t-\nu t}dt\quad(|\arg z|<\tfrac{1}{2}\pi)$$

9.1.26
$$J_\nu(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-t)(\frac{1}{2}x)^{\nu+2t}}{\Gamma(\nu+t+1)}dt\quad(\mathscr{R}\nu>0,x>0)$$

In the last integral the path of integration must lie to the left of the points $t=0,1,2,\ldots$

Eqn page C2-9

$$\boxed{\begin{array}{l} r^2 \dfrac{d^2 B_n}{dr^2} + r \dfrac{dB_n}{dr} + \lambda_n^2 r^2 B_n = 0 \\ \\ B_n(0) \equiv \text{bounded} \\ B_n(1) = 0 \end{array}}$$

In form of eqn 9.1.1 of A+S w/ $\nu = 0$

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + z^2 w = 0$$

Let $z = \lambda_n r$

$$\Rightarrow \boxed{z^2 \frac{d^2 B_n}{dz^2} + z \frac{dB_n}{dz} + z^2 B_n = 0}$$

Soln:

$$B_n = C_n J_0(z) + D_n Y_0(z)$$

$$\Rightarrow \boxed{B_n = C_n J_0(\lambda_n r) + D_n Y_0(\lambda_n r)}$$

Bessel Function of 2nd Kind (order 0)

Bessel function of 1st Kind (order 0)

BCs: $B_n(0) =$ Bounded

$B_n(r=1) = 0$

Look at properties of J_0 + Y_0 on page C2-9b

Y_0 unbounded @ $r=0$, $J_0 = 1$ @ $r=0$

$\Rightarrow D_n = 0$

$\Rightarrow \boxed{B_n = C_n J_0(\lambda_n r)}$ @ $r=1$, $B_n = 0$

$\Rightarrow \boxed{J_0(\lambda_n) = 0}$ ⇐ Determines λ_n... these are tabulated

$\lambda_1 = 2.405$, $\lambda_2 = 5.520$, $\lambda_3 = 8.654$ ...

So, $\boxed{\begin{array}{l} B_n = J_0(\lambda_n x) \\ J_0(\lambda_n) = 0 \\ B_n \perp \text{ w.r.t. } \langle f, g \rangle = \int_0^1 x f g \, dx \\ K_n = -\lambda_n^2, \quad n = 1, 2, 3, \ldots \end{array}}$

Negative eigenvalues

$\lambda_1 = 2.405,$

$\lambda_2 = 5.520$

$\lambda_3 = 8.654\ldots$

$\vdots$

C2-12

$\underline{K_n > 0}$:

Let $K_n = \lambda_n^2$ on page C2-7.

$\Rightarrow$ Follow same procedure as w/ $K_n < 0$. Sign flips on λ_n term, page C2-10

$$\boxed{\begin{gathered} \eta^2 \frac{d^2 B_n}{d\eta^2} + \eta \frac{dB_n}{d\eta} - \lambda_n^2 \eta^2 B_n = 0 \\ B_n(0) \text{ bounded} \\ B_n(1) = 0 \end{gathered}}$$

Is form of eqn 9.6.1 of A+S w/ $\nu = 0$ (Next Page)

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - z^2 w = 0 \quad \text{w/} \quad z = \lambda_n \eta$$

$$\Rightarrow B_n = C_n I_0(z) + D_n K_0(z).$$

C-13a

There are n zeros of each function near the finite curve extending from $z=-n$ to $z=n$; the asymptotic expansions of these zeros for large n are given by the right side of **9.5.22** or **9.5.24** with $\nu=n$ and $\zeta=e^{-2\pi i/3}n^{-2/3}a_s$ or $\zeta=e^{-2\pi i/3}n^{-2/3}a_s'$.

Zeros of Cross-Products

If ν is real and λ is positive, the zeros of the function

9.5.27 $$J_\nu(z)Y_\nu(\lambda z)-J_\nu(\lambda z)Y_\nu(z)$$

are real and simple. If $\lambda>1$, the asymptotic expansion of the sth zero is

9.5.28 $$\beta+\frac{p}{\beta}+\frac{q-p^2}{\beta^3}+\frac{r-4pq+2p^3}{\beta^5}+\ldots$$

where with $4\nu^2$ denoted by μ,

9.5.29
$$\beta=s\pi/(\lambda-1)$$
$$p=\frac{\mu-1}{8\lambda},\qquad q=\frac{(\mu-1)(\mu-25)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)}$$
$$r=\frac{(\mu-1)(\mu^2-114\mu+1073)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)}$$

The asymptotic expansion of the large positive zeros (not necessarily the sth) of the function

9.5.30 $$J_\nu'(z)Y_\nu'(\lambda z)-J_\nu'(\lambda z)Y_\nu'(z)\qquad(\lambda>1)$$

is given by **9.5.28** with the same value of β, but instead of **9.5.29** we have

9.5.31
$$p=\frac{\mu+3}{8\lambda},\qquad q=\frac{(\mu^2+46\mu-63)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)}$$
$$r=\frac{(\mu^3+185\mu^2-2053\mu+1899)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)}$$

The asymptotic expansion of the large positive zeros of the function

9.5.32 $$J_\nu'(z)Y_\nu(\lambda z)-Y_\nu'(z)J_\nu(\lambda z)$$

is given by **9.5.28** with

9.5.33
$$\beta=(s-\tfrac{1}{2})\pi/(\lambda-1)$$
$$p=\frac{(\mu+3)\lambda-(\mu-1)}{8\lambda(\lambda-1)}$$
$$q=\frac{(\mu^2+46\mu-63)\lambda^3-(\mu-1)(\mu-25)}{6(4\lambda)^3(\lambda-1)}$$
$$5(4\lambda)^5(\lambda-1)r=(\mu^3+185\mu^2-2053\mu+1899)\lambda^5-(\mu-1)(\mu^2-114\mu+1073)$$

Modified Bessel Functions *I* and *K*

9.6. Definitions and Properties

Differential Equation

9.6.1 $$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-(z^2+\nu^2)w=0$$

Solutions are $I_{\pm\nu}(z)$ and $K_\nu(z)$. Each is a regular function of z throughout the z-plane cut along the negative real axis, and for fixed $z(\neq0)$ each is an entire function of ν. When $\nu=\pm n$, $I_\nu(z)$ is an entire function of z.

$I_\nu(z)$ $(\mathscr{R}\nu\geq0)$ is bounded as $z\to0$ in any bounded range of arg z. $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent except when ν is an integer. $K_\nu(z)$ tends to zero as $|z|\to\infty$ in the sector $|\arg z|<\frac{1}{2}\pi$, and for all values of ν, $I_\nu(z)$ and $K_\nu(z)$ are linearly independent. $I_\nu(z)$, $K_\nu(z)$ are real and positive when $\nu>-1$ and $z>0$.

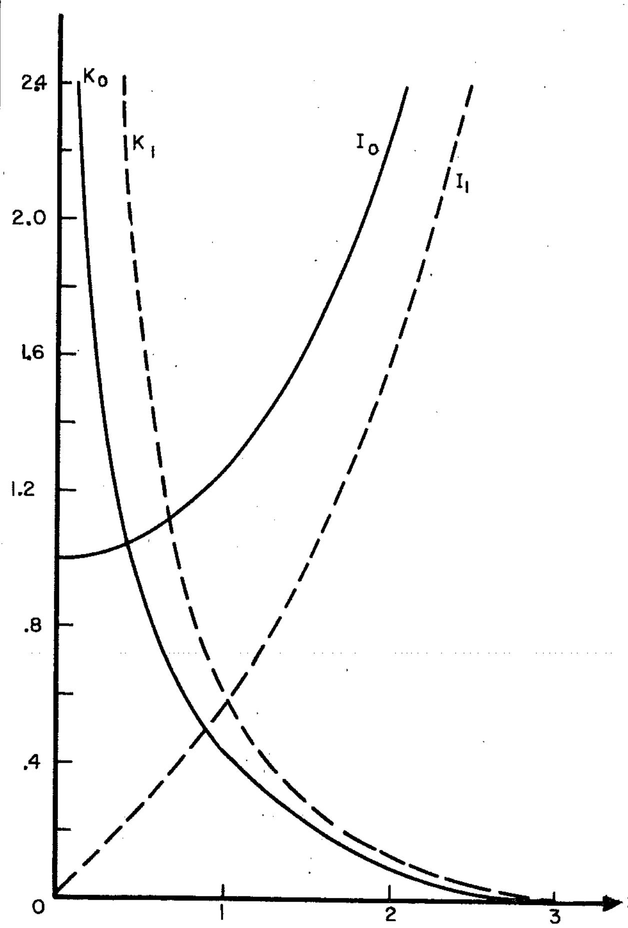


FIGURE 9.7. $I_0(x)$, $K_0(x)$, $I_1(x)$ *and* $K_1(x)$.

C-13b

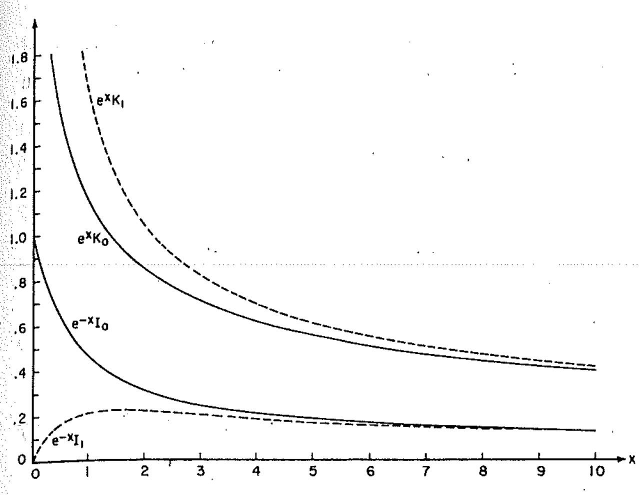


FIGURE 9.8. $e^{-x}I_0(x), e^{-x}I_1(x), e^xK_0(x)$ *and* $e^xK_1(x)$.

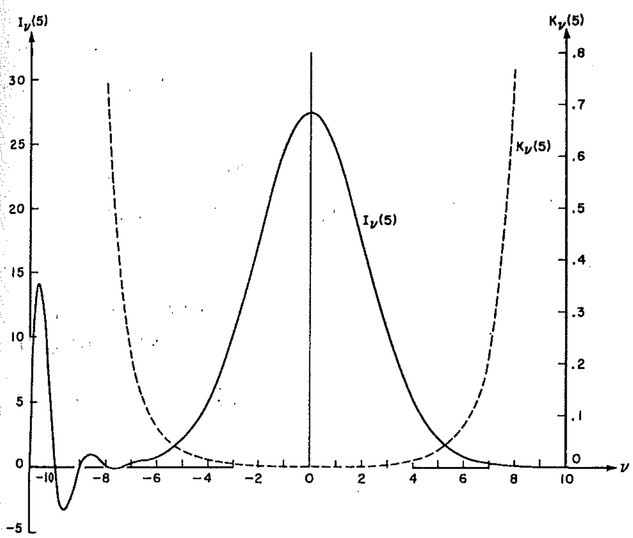


FIGURE 9.9. $I_\nu(5)$ *and* $K_\nu(5)$.

Relations Between Solutions

9.6.2 $$K_\nu(z)=\tfrac{1}{2}\pi\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.6.3

$$I_\nu(z)=e^{-\frac{1}{2}\nu\pi i}J_\nu(ze^{\frac{1}{2}\pi i})\qquad(-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

$$I_\nu(z)=e^{3\nu\pi i/2}J_\nu(ze^{-3\pi i/2})\qquad(\tfrac{1}{2}\pi<\arg z\leq\pi)$$

9.6.4

$$K_\nu(z)=\tfrac{1}{2}\pi ie^{\frac{1}{2}\nu\pi i}H_\nu^{(1)}(ze^{\frac{1}{2}\pi i})\qquad(-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

$$K_\nu(z)=-\tfrac{1}{2}\pi ie^{-\frac{1}{2}\nu\pi i}H_\nu^{(2)}(ze^{-\frac{1}{2}\pi i})\qquad(-\tfrac{1}{2}\pi<\arg z\leq\pi)$$

9.6.5

$$Y_\nu(ze^{\frac{1}{2}\pi i})=e^{\frac{1}{2}(\nu+1)\pi i}I_\nu(z)-(2/\pi)e^{-\frac{1}{2}\nu\pi i}K_\nu(z)\qquad(-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

9.6.6 $$I_{-n}(z)=I_n(z),\; K_{-\nu}(z)=K_\nu(z)$$

Most of the properties of modified Bessel functions can be deduced immediately from those of ordinary Bessel functions by application of these relations.

Limiting Forms for Small Arguments

When ν is fixed and $z\rightarrow 0$

9.6.7

$$I_\nu(z)\sim(\tfrac{1}{2}z)^\nu/\Gamma(\nu+1)\qquad(\nu\neq-1,-2,\ldots)$$

9.6.8 $$K_0(z)\sim-\ln z$$

9.6.9 $$K_\nu(z)\sim\tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}\qquad(\mathscr{R}\nu>0)$$

Ascending Series

9.6.10 $$I_\nu(z)=(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}$$

9.6.11

$$K_n(z)=\tfrac{1}{2}(\tfrac{1}{2}z)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-\tfrac{1}{4}z^2)^k+(-)^{n+1}\ln(\tfrac{1}{2}z)I_n(z)$$
$$+(-)^n\tfrac{1}{2}(\tfrac{1}{2}z)^n\sum_{k=0}^{\infty}\{\psi(k+1)+\psi(n+k+1)\}\frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by **6.3.2.**

9.6.12 $$I_0(z)=1+\frac{\frac{1}{4}z^2}{(1!)^2}+\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

9.6.13

$$K_0(z)=-\{\ln(\tfrac{1}{2}z)+\gamma\}I_0(z)+\frac{\frac{1}{4}z^2}{(1!)^2}$$
$$+(1+\tfrac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

Wronskians

9.6.14

$$W\{I_\nu(z),I_{-\nu}(z)\}=I_\nu(z)I_{-(\nu+1)}(z)-I_{\nu+1}(z)I_{-\nu}(z)$$
$$=-2\sin(\nu\pi)/(\pi z)$$

9.6.15

$$W\{K_\nu(z),I_\nu(z)\}=I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z)=1/z$$

$$\Rightarrow \quad B_n = C_n \underbrace{I_0(\lambda_n r)}_{1^{st}\text{ kind}} + D_n \underbrace{K_0(\lambda_n r)}_{2^{nd}\text{ kind}}$$

Modified Bessel functions of 1^{st} + 2^{nd} kind

BCS: $B_n(0)$ bounded, $B_n(r=1) = 0$ } Look at properties of I_0 + K_0, page C-13a, figure 9.7

$K_0(0)$ not bounded @ $r=0$

$$\Rightarrow B_n = C_n I_0(\lambda_n r)$$

$\Rightarrow I_0(\lambda_n) = 0$. From Fig 9.7, A+S: $I_0(\lambda_n) \neq 0 \quad \forall \lambda_n \Rightarrow$

$C_n = 0$

So, for $k_n > 0$, $B_n = 0$, + no eigenfunctions

Eigensearch Results:

So, only eigenfunctions are $K_n < 0$:

$B_n = J_0(\lambda_n r)$

$J_0(\lambda_n) = 0, \quad \lambda_1 = 2.405, \; \lambda_2 = 5.520, \; \lambda_3 = 8.654 \ldots; \quad K_n = -\lambda_n^2, \; n = 1, 2, 3, \ldots$

$B_n \perp$ w.r.t. $\langle f, g \rangle = \int_0^1 r f g \, dr$

Formalize summation:

$$F = \sum_{n} A_n(\xi)\, B_n(r)$$

$$\Rightarrow F = \sum_{n=1}^{\infty} A_n(\xi)\, J_0(\lambda_n r), \quad \lambda_n \in J_0(\lambda_n) = 0$$

Separated PDE: Boxed eqn, page C2-6

$$\sum_{n=1}^{\infty} \left[\frac{dA_n}{d\xi} + \lambda_n^2 A_n \right] J_0(\lambda_n r) = 1$$

Use orthogonality of J_0's: by taking inner product w/ $J_0(\lambda_m r)$

$$\Rightarrow \left(\frac{dA_m}{d\xi} + \lambda_m^2 A_m \right) \langle J_0(\lambda_m r), J_0(\lambda_m r) \rangle = \langle 1, J_0(\lambda_m r) \rangle$$

m is dummy $\Rightarrow$ $$\boxed{\frac{dA_n}{d\xi} + \lambda_n^2 A_n = \frac{\langle 1, J_0(\lambda_n r) \rangle}{\langle J_0(\lambda_n r), J_0(\lambda_n r) \rangle}}$$

Let $\boxed{\gamma_n = \dfrac{\langle 1, J_0(\lambda_n x)\rangle}{\langle J_0(\lambda_n x), J_0(\lambda_n x)\rangle} \equiv \underline{\text{Constant}}}$

$$\langle 1, J_0(\lambda_n x)\rangle = \int_0^1 x J_0(\lambda_n x)\, dx$$

$$\langle J_0(\lambda_n x), J_0(\lambda_n x)\rangle = \int_0^1 x J_0^2(\lambda_n x)\, dx$$

$$\Rightarrow \frac{dA_n}{d\xi} + \lambda_n^2 A_n = \gamma_n$$

Soln: $\boxed{A_n = \dfrac{\gamma_n}{\lambda_n^2} + C_n e^{-\lambda_n^2 \xi}}$

Need BC @ $\xi = 0$

@ $\xi = 0$, $F = \Theta_0 - \Theta_1$ $\qquad F = \sum_{n=1}^{\infty} A_n(\xi) J_0(\lambda_n x)$

$$\Rightarrow \Theta_0 - \Theta_1 = \sum_{n=1}^{\infty} A_n(0) J_0(\lambda_n x)$$

Orthogonality! $\langle \Theta_0 - \Theta_1, J_0(\lambda_n x)\rangle = A_n(0) \langle J_0(\lambda_n x), J_0(\lambda_n x)\rangle$

$$A_n(0) = (\Theta_0 - \Theta_1) \frac{\langle 1, J_0(\lambda_n x)\rangle}{\langle J_0(\lambda_n x), J_0(\lambda_n x)\rangle}$$

So, $A_n(0) = (\Theta_0 - \Theta_1)\gamma_n$ ← previous page

Apply BC:

$$A_n = \frac{\gamma_n}{\lambda_n^2} + C_n e^{-\lambda_n^2 \xi} \qquad @\ \xi = 0 \Rightarrow$$

$$(\Theta_0 - \Theta_1)\gamma_n = \frac{\gamma_n}{\lambda_n^2} + C_n$$

$$C_n = \gamma_n\left[(\Theta_0 - \Theta_1) - \frac{1}{\lambda_n^2}\right]$$

$$\Rightarrow \boxed{A_n = \frac{\gamma_n}{\lambda_n^2} + \gamma_n\left[(\Theta_0 - \Theta_1) - \frac{1}{\lambda_n^2}\right] e^{-\lambda_n^2 \xi}}$$

Soln for F:

$$F = \sum_{n=1}^{\infty} A_n(\xi)\, J_0(\lambda_n \eta)$$

$$A_n(\xi) = \gamma_n \left\{ \frac{1}{\lambda_n^2} + \left[(\Theta_0 - \Theta_1) - \frac{1}{\lambda_n^2} \right] e^{-\lambda_n^2 \xi} \right\}$$

$$\gamma_n = \frac{\langle 1, J_0(\lambda_n \eta) \rangle}{\langle J_0(\lambda_n \eta), J_0(\lambda_n \eta) \rangle}, \quad \langle 1, J_0(\lambda_n \eta) \rangle = \int_0^1 \eta\, J_0(\lambda_n \eta)\, d\eta$$

$$\langle J_0(\lambda_n \eta), J_0(\lambda_n \eta) \rangle = \int_0^1 \eta\, J_0^2(\lambda_n \eta)\, d\eta$$

$$J_0(\lambda_n) = 0, \quad \lambda_1 = 2.405, \quad \lambda_2 = 5.520, \quad \lambda_3 = 8.654 \ldots$$

Final soln!

$$\boxed{\Theta = \Theta_1 + F(\xi, \eta)}$$

//

Note: Other BCs associated w/ Neuman + Roben constraints on page C2-2 lead to eigenfunction expressions on page LN23-4 of Paul's handout.