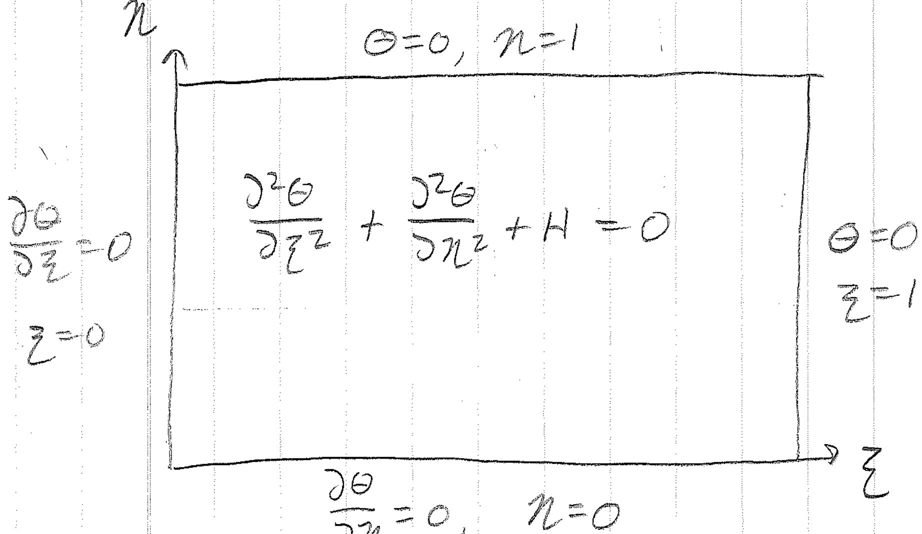
Ex 2: 2-D Steady Heat Conduction in a Square – Constant Heat Source



① Choose direction for Basis Expansion. BCs must be homogeneous in that direction. Either η, or ξ OK (we will deal w/ complications later).

② Let's choose η-direction....

③ Write: $\Theta = \sum_n A_n(\xi) B_n(\eta)$, n is index given definiteness later

Sub into PDE:

$$\sum_n \frac{d^2 A_n}{d\xi^2} B_n + \sum_n \frac{d^2 B_n}{d\eta^2} A_n = -H$$

$$\Rightarrow \sum_n \left[\frac{d^2 A_n}{d\xi^2} B_n + \frac{d^2 B_n}{d\eta^2} A_n \right] = -H$$

$$\sum_n \left[\frac{d^2 A_n}{d\xi^2} + \left(\frac{d^2 B_n}{d\eta^2} \frac{1}{B_n} \right) A_n \right] B_n(\eta) = -H$$

For system to be separable, term in Brackets must *only* be a function of ξ.

Thus $\Rightarrow$ $\boxed{\frac{d^2 B_n}{d\eta^2} \frac{1}{B_n} = K_n}$ & $\boxed{\sum_n \left[\frac{d^2 A_n}{d\xi^2} + K_n A_n \right] B_n(\eta) = -H}$

BCs on B_n? Since $\Theta = \sum_n A_n(\xi) B_n(\eta)$

$$@\ \eta=0, \quad \frac{\partial\theta}{\partial\eta}=0 \quad \Rightarrow \quad \sum_n A_n(\xi)\left.\frac{dB_n(\eta)}{d\eta}\right|_{\eta=0}=0.$$

. To assure this is the case $\forall\ \xi \Rightarrow \frac{dB_n}{d\eta}=0 \quad @\ \eta=0$

$$@\ \eta=1, \quad \theta=0 \quad \Rightarrow \quad \sum_n A_n(\xi)\,B_n(\eta)\Big|_{\eta=1}=0$$

$$\Rightarrow \quad B_n(\eta)=0 \quad @\ \eta=1$$

So... we have:

$$\boxed{\begin{aligned} &\frac{d^2B_n}{d\eta^2}=K_n B_n \\ &\frac{dB_n}{d\eta}=0 \quad @\ \eta=0 \\ &B_n=0 \quad @\ \eta=1 \end{aligned}}$$

Eigenvalue problem to solve for B_n.

<u>Recognize</u> as <u>Sturm</u>-<u>Liouville</u> <u>problem</u> (weighting $=1$)

↳ <u>Guarantees</u>:

1. K_n real
2. $B_n \perp$ w.r.t. $\langle f,g\rangle=\int_0^1 fg\,dx$
3. B_n form Basis
4. 1 B_n for each K_n.

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Eigensearch ($K_n > 0, = 0, < 0$)

$K_n = \lambda^2, \quad \lambda > 0, \quad K_n > 0$

$\frac{d^2 B_n}{d x^2} = \lambda^2 B_n \Rightarrow \frac{d^2 B_n}{d x^2} - \lambda^2 B_n = 0$

$B_n = C \sinh \lambda x + D \cosh \lambda x$

$\frac{d B_n}{d x} = 0 \ @ \ x = 0 \qquad \frac{d B_n}{d x} = C \lambda \cosh \lambda x + D \lambda \sinh \lambda x$

$0 = C\lambda \Rightarrow C = 0$

$\Rightarrow B_n = D \cosh \lambda x$

$B_n = 0 \ @ \ x = 1 \Rightarrow D \cosh \lambda = 0 \Rightarrow D = 0$

So, $K_n > 0 \Rightarrow$ no positive eigenvalues/eigenvectors (No nontrivial solns.)

$\underline{K_n = 0}$

$$\frac{d^2 B_n}{dx^2} = 0$$

$$\frac{dB_n}{dx} = 0 \quad @ x = 0$$

$$B_n = 0 \quad @ x = 1$$

$$B_n = Cx + D \qquad \frac{dB_n}{dx} = 0 \ @ \ x = 0 \Rightarrow C = 0$$

$$B_n = 0 \ @ \ x = 1 \Rightarrow D = 0$$

$\Rightarrow B_n = 0 \Rightarrow$ <u>no zero eigenvalues/eigenvectors</u>

Eigensearch, $K_n < 0$

$$K_n = -\lambda^2$$

$$\frac{d^2 B_n}{d\eta^2} = -\lambda^2 B_n$$

$$\frac{dB_n}{d\eta} = 0 \quad @ \eta = 0$$

$$B_n = 0 \quad @ \eta = 1$$

$$B_n = C \sin \lambda \eta + D \cos \lambda \eta$$

$$\frac{dB_n}{d\eta} = 0 \quad @ \eta = 0 \Rightarrow C = 0$$

$$\Rightarrow B_n = D \cos \lambda \eta$$

$$B_n = 0 \quad @ \eta = 1 \Rightarrow D \cos \lambda = 0 \Rightarrow \lambda = (2n+1)\frac{\pi}{2}, \qquad n = 0, 1, 2, 3 \ldots$$

D — $0 \Rightarrow$ trivial soln - no good

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So: $\boxed{B_n(x) = \cos \lambda_n x, \qquad \lambda_n = (2n+1)\frac{\pi}{2},\ n = 0, 1, 2, 3 \qquad K_n = -\lambda_n^2}$

Eigenvalues + eigenfunctions

So, now we can formalize summation:

$$\Theta = \sum_n A_n(z)\, B_n(x)$$

$$\boxed{\Theta = \sum_{n=0}^{\infty} A_n(z) \cos \lambda_n x, \quad \lambda_n = (2n+1)\frac{\pi}{2}, \quad K_n = -\lambda_n^2}$$

From separated PDE, we had (page ②):

$$\sum_n \left[\frac{d^2A_n}{d\xi^2} + K_n A_n\right] B_n(x) = -H$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[\frac{d^2A_n}{d\xi^2} - \lambda_n^2 A_n\right] \cos \lambda_n x = -H$$

We now obtain ODE for $A_n(\xi)$: Take inner product of both sides of eqn w/ $\cos \lambda_m x$

$$\langle f, g \rangle = \int_0^1 f g \, dx$$

$$\sum_{n=0}^{\infty} \left\langle \left[\frac{d^2A_n}{d\xi^2} - \lambda_n^2 A_n\right] \cos \lambda_n x, \cos \lambda_m x \right\rangle = \langle -H, \cos \lambda_m x \rangle$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[\frac{d^2 A_n}{d\xi^2} - \lambda_n^2 A_n \right] \langle \cos\lambda_n x, \cos\lambda_m x \rangle = -H \langle 1, \cos\lambda_m x \rangle$$

For $n \neq m$, $\langle \cos\lambda_n x, \cos\lambda_m x \rangle = 0$ (orthogonality)

$$\Rightarrow \left[\frac{d^2 A_m}{d\xi^2} - \lambda_m^2 A_m \right] \langle \cos\lambda_m x, \cos\lambda_m x \rangle = -H \langle 1, \cos\lambda_m x \rangle$$

$$\Rightarrow \frac{d^2 A_m}{d\xi^2} - \lambda_m^2 A_m = -H \frac{\langle 1, \cos\lambda_m x \rangle}{\langle \cos\lambda_m x, \cos\lambda_m x \rangle}$$

m is dummy $\Rightarrow$

$$\boxed{\frac{d^2 A_n}{d\xi^2} - \lambda_n^2 A_n = -H \frac{\langle 1, \cos\lambda_n x \rangle}{\langle \cos\lambda_n x, \cos\lambda_n x \rangle}, \quad \langle 1, \cos\lambda_n x \rangle = \int_0^1 \cos\lambda_n x \, dx, \quad \langle \cos\lambda_n x, \cos\lambda_n x \rangle = \int_0^1 \cos^2\lambda_n x \, dx}$$

Note: Right-hand side of Boxed eqn, page ⑨, is a constant.

Let: $\gamma_n = H \dfrac{\langle 1, \cos \lambda_n \eta \rangle}{\langle \cos \lambda_n \eta, \cos \lambda_n \eta \rangle}$

$$\Rightarrow \frac{d^2 A_n}{d \xi^2} - \lambda_n^2 A_n = -\gamma_n$$

$A_n = A_{n_h} + A_{n_p}$. $\quad \dfrac{d^2 A_{n_h}}{d \xi^2} - \lambda_n^2 A_{n_h} = 0 \quad A_{n_h} = C_n \sinh(\lambda_n \xi) + D_n \cosh(\lambda_n \xi)$

A_{n_p}: Method of undetermined coefficients: $A_{n_p} = E_n$

$\Rightarrow$ Sub into ODE: $\; -\lambda_n^2 E_n = -\gamma_n \quad \Rightarrow E_n = \dfrac{\gamma_n}{\lambda_n^2}$

So, $\boxed{A_n = C_n \sinh(\lambda_n \xi) + D_n \cosh(\lambda_n \xi) + \dfrac{\gamma_n}{\lambda_n^2}}$

BCs: on $A_n(\xi)$: $\Theta = \sum_{n=0}^{\infty} A_n(\xi) \cos \lambda_n \eta$

@ $\xi = 0$, $\frac{\partial \Theta}{\partial \xi} = 0$

$$\frac{\partial \Theta}{\partial \xi} = \sum_{n=0}^{\infty} \frac{dA_n}{d\xi} \cos \lambda_n \eta \quad \Rightarrow \quad 0 = \sum_{n=0}^{\infty} \left.\frac{dA_n}{d\xi}\right|_{\xi=0} \cos \lambda_n \eta$$

- Since $\cos \lambda_n \eta$ is basis, coefficients must each be zero as $\cos \lambda_n \eta$ are linearly independent.
- Or: Take inner product of both sides w/ $\cos \lambda_m \eta$. Only Term in sum that remains is when $n = m$ (previous manipulation)

Dummy Index

$$\Rightarrow \langle 0, \cos \lambda_m \eta \rangle = \sum_{n=0}^{\infty} \left.\frac{dA_n}{d\xi}\right|_{\xi=0} \langle \cos \lambda_n \eta, \cos \lambda_m \eta \rangle \quad \Rightarrow \quad \left.\frac{dA_m}{d\xi}\right|_{\xi=0} = 0$$

$$\Rightarrow \boxed{\left.\frac{dA_n}{d\xi}\right|_{\xi=0} = 0}$$

@ $\xi = 1, \ \theta = 0$

$$\Rightarrow \ 0 = \sum_{n=0}^{\infty} A_n(1) \cos \lambda_n \eta$$

Inner product or basis argument:

$$\Rightarrow \boxed{A_n(1) = 0}$$

So: Now can solve for A_n

@ $\xi = 0 \ \ \frac{dA_n}{d\xi} = 0$

$$\frac{dA_n}{d\xi} = C_n \lambda_n \cosh(\lambda_n \xi) + D_n \lambda_n \sinh(\lambda_n \xi)$$

$$0 = C_n \lambda_n \ \Rightarrow \boxed{C_n = 0}$$

$$\Rightarrow A_n = D_n \cosh(\lambda_n \xi) + \frac{\gamma_n}{\lambda_n^2}$$

$$A_n(1) = 0 \ \Rightarrow \ 0 = D_n \cosh(\lambda_n) + \frac{\gamma_n}{\lambda_n^2}$$

$$\boxed{D_n = \frac{-\gamma_n}{\lambda_n^2 \cosh(\lambda_n)}}$$

So:

$$A_n = \frac{\gamma_n}{\lambda_n^2}\left[1 - \frac{\cosh(\lambda_n \xi)}{\cosh(\lambda_n)}\right]$$

So, $\theta = \sum_{n=0}^{\infty} A_n(\xi) \cos \lambda_n \eta$, $\xi, \eta \in [0,1]$

$$\lambda_n = (2n+1)\frac{\pi}{2}, \quad \gamma_n = H \frac{\langle 1, \cos \lambda_n \eta \rangle}{\langle \cos \lambda_n \eta, \cos \lambda_n \eta \rangle}$$

$$\langle 1, \cos \lambda_n \eta \rangle = \int_0^1 \cos \lambda_n \eta \, d\eta = \frac{1}{\lambda_n} \sin \lambda_n \eta \Big|_0^1 = \frac{1}{\lambda_n} \sin \lambda_n = \frac{1}{\lambda_n} \sin\left[(2n+1)\frac{\pi}{2}\right]$$

$$= \frac{1}{\lambda_n}(-1)^n$$

$$\langle \cos \lambda_n \eta, \cos \lambda_n \eta \rangle = \int_0^1 \cos^2 \lambda_n \eta \, d\eta = \int_0^1 \left(\frac{1}{2} + \frac{1}{2}\cos 2\lambda_n \eta\right) d\eta$$

$$= \frac{1}{2}\eta\Big|_0^1 + \frac{1}{4\lambda_n}\sin 2\lambda_n \eta\Big|_0^1 = \frac{1}{2}$$

$$\Rightarrow \boxed{\gamma_n = H \frac{\frac{1}{\lambda_n}(-1)^n}{\frac{1}{2}} = \frac{2H(-1)^n}{\lambda_n}}$$

Final Soln:

$$\boxed{\begin{aligned} \Theta &= \sum_{n=0}^{\infty} A_n(\xi) \cos \lambda_n \eta , \qquad \xi, \eta \in [0,1] \\ A_n(\xi) &= \frac{2H(-1)^n}{\lambda_n^3}\left[1 - \frac{\cosh(\lambda_n \xi)}{\cosh(\lambda_n)}\right] \end{aligned}}$$