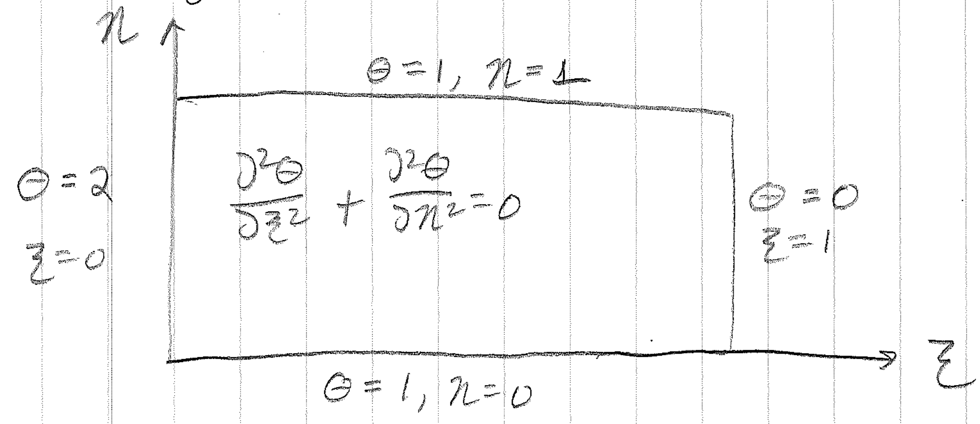
Ex 3: Steady Heat Conduction in a Square - Non Homogeneous BCs



① Choose direction for basis expansion. BCs <u>must</u> be homogeneous in that direction.

② Let's choose ξ direction for basis.

(2a) Before proceeding w/ assumed summation form, we need to make BCs in η direction homogeneous.

- We do this by superposition. (Linear system admits this)...

Superposition:

- Find function $g(\xi)$ that picks up boundary conditions at $\xi=0$, $\xi=1$.
- In general, choose least-order polynomial that does job – generally easiest

Any function will work!

- Here: $g(\xi)$ satisfies $g=2$ @ $\xi=0$, $g=0$ @ $\xi=1$
- Use simple polynomial: $g = a\xi + b \Rightarrow g = 2-2\xi$

Write: $\theta = 2-2\xi + F(\xi,\eta)$

Rewrite system: $\dfrac{\partial^2\theta}{\partial\xi^2} + \dfrac{\partial^2\theta}{\partial\eta^2} = 0 \Rightarrow \dfrac{\partial^2 F}{\partial\xi^2} + \dfrac{\partial^2 F}{\partial\eta^2} = 0$

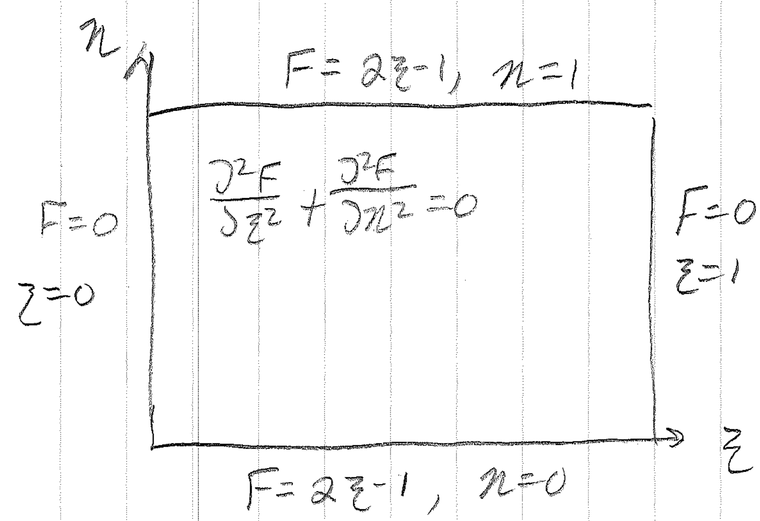
BCS: $\theta=2,\ \xi=0 \Rightarrow 2 = 2 + F(0,\eta) \Rightarrow F(0,\eta)=0$

$\theta=0,\ \xi=1 \Rightarrow 0 = F(1,\eta) \Rightarrow F(1,\eta)=0$

$\theta=1,\ \eta=1 \Rightarrow 1 = 2-2\xi + F(\xi,1) \Rightarrow F(\xi,1) = 2\xi-1$

$\theta=1,\ \eta=0 \Rightarrow 1 = 2-2\xi + F(\xi,0) \Rightarrow F(\xi,0) = 2\xi-1$

New System to Solve:



$$\boxed{\Theta = 2 - 2\xi + F(\xi, \eta)}$$

Now, solve for F...

(3) Write: $F(\xi, \eta) = \sum_n B_n(\eta) A_n(\xi)$; ξ direction chosen for basis!

Suppose we wanted to choose η direction as basis?

- Find function that satisfies $\theta = 1$ @ $\eta = 0$ + $\eta = 1$. Let $g(\eta)$ be function
  - In general, choose least-order polynomial that does job... [Easiest, but any function will work]

Here $g(\eta) = 1$

- Write: $\theta = 1 + F(\xi, \eta)$

- Rewrite: System in terms of F:

$$\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2} = 0 \Rightarrow \frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} = 0$$

- BCs:

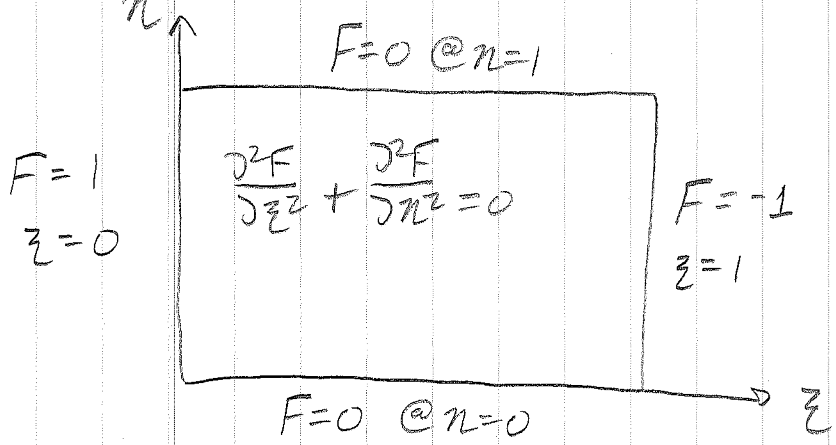
$\theta = 1, \eta = 0 \Rightarrow 1 = 1 + F(\xi, 0) \Rightarrow F(\xi, 0) = 0$

$\theta = 1, \eta = 1 \Rightarrow 1 = 1 + F(\xi, 1) \Rightarrow F(\xi, 1) = 0$

$\theta = 2, \xi = 0 \Rightarrow 2 = 1 + F(0, \eta) \Rightarrow F(0, \eta) = 1$

$\theta = 0, \xi = 1 \Rightarrow 0 = 1 + F(1, \eta) \Rightarrow F(1, \eta) = -1$

New System to solve:



$$\boxed{\Theta = 1 + F(\xi,\eta)}$$ ← Complete Soln.

Now, solve for F....

③ Write $F(\xi,\eta) = \sum_n A_n(\xi)\, B_n(\eta)$, + proceed....

- Note: • Choose direction whose BCs yield easiest eigenvalue problem based on form. (λs are analytic + not numerical!)
  • All directions are OK... just simpler result

Sub into PDE, page 3-3:

$$\sum_n \left[\frac{d^2 A_n}{d\xi^2} B_n + \frac{d^2 B_n}{d\eta^2} A_n \right] = 0$$

$$\sum_n \left[\left(\frac{d^2 A_n}{d\xi^2} \frac{1}{A_n} \right) B_n + \frac{d^2 B_n}{d\eta^2} \right] A_n(\xi) = 0$$

For system to be separable, term in Brackets must only be a function of η.

Then $\Rightarrow \quad \frac{d^2 A_n}{d\xi^2} \frac{1}{A_n} = K_n$, & $\boxed{\sum_n \left[\frac{d^2 B_n}{d\eta^2} + K_n B_n \right] A_n(\xi) = 0}$

BCS on B_n: $\quad F = 0$ @ $\xi = 0 \Rightarrow \sum_n B_n(\eta) A_n(0) = 0 \Rightarrow A_n(0) = 0 \quad \forall \eta.$

$F = 0$ @ $\xi = 1 \Rightarrow \sum_n B_n(\eta) A_n(1) = 0 \Rightarrow A_n(1) = 0 \quad \forall \eta$

So, Eigenvalue problem:

$$\boxed{\begin{aligned} &\frac{d^2 A_n}{d\xi^2} = K_n A_n \\ &A_n = 0 \ @ \ \xi = 0 \\ &A_n = 0 \ @ \ \xi = 1 \end{aligned}}$$

$\Rightarrow$ Sturm-Liouville Problem

1. K_n real
2. $B_n \perp$ w.r.t $\langle f, g \rangle = \int_0^1 f g \, d\xi$
3. B_n forms basis
4. 1 B_n for each K_n

Eigensearch for $K_n > 0$, $K_n = 0$, $K_n < 0$. Real K_n only! – * Follow procedure as in Ex 2!

$$\boxed{A_n = \sin \lambda_n \xi \ , \quad \lambda_n = n\pi, \ K_n = -\lambda_n^2, \ n = 1, 2, 3, \ldots}$$

Formalize summation

$$F = \sum_n B_n(\eta)\, A_n(\xi)$$

$$\boxed{F = \sum_{n=1}^{\infty} B_n(\eta) \sin \lambda_n \xi \,, \quad \lambda_n = n\pi, \quad K_n = -\lambda_n^2 .}$$

Separated PDE (looped, page 3-4)

$$\sum_{n=1}^{\infty} \left[\frac{d^2 B_n}{d\eta^2} - \lambda_n^2 B_n \right] \sin(\lambda_n \xi) = 0$$

- Use orthogonality of sin by taking inner product w/ $\sin \lambda_m \xi$. Or use fact that $\sin(\lambda_n \xi)$ form basis so only way sum can be zero is if all coefficients are zero (def of linearly independent vectors).

Then:

$$\frac{d^2B_n}{d\eta^2} - \lambda_n^2 B_n = 0$$

Soln: $\boxed{B_n = C_n \sinh \lambda_n \eta + D_n \cosh \lambda_n \eta}$

Find constants... Use BCS.

@ $\eta = 0$, $F = 2\xi - 1$, $F = \sum_{n=1}^{\infty} B_n(\eta) \sin \lambda_n \xi$

$$\Rightarrow \quad 2\xi - 1 = \sum_{n=1}^{\infty} B_n(0) \sin \lambda_n \xi$$

Take inner product of both sides w/ $\sin \lambda_m \xi$, $\langle f, g \rangle = \int_0^1 fg\,dx$

$\langle 2\xi - 1, \sin \lambda_m \xi \rangle = B_m(0) \langle \sin \lambda_m \xi, \sin \lambda_m \xi \rangle$. m is dummy.

$$\Rightarrow \quad \boxed{B_n(0) = \frac{\langle 2\xi - 1, \sin \lambda_n \xi \rangle}{\langle \sin \lambda_n \xi, \sin \lambda_n \xi \rangle}}$$

@ $x=1$, $F = 2\xi - 1$, $F = \sum_{n=1}^{\infty} B_n(x) \sin \lambda_n \xi$

$$\Rightarrow 2\xi - 1 = \sum_{n=1}^{\infty} B_n(1) \sin \lambda_n \xi$$

Inner product of both sides w/ $\sin \lambda_m \xi$, $\langle f, g \rangle = \int_0^1 fg\,dx$

$$\langle 2\xi - 1, \sin \lambda_m \xi \rangle = B_m(1) \langle \sin \lambda_m \xi, \sin \lambda_m \xi \rangle$$

$\underline{m \text{ is dummy}}$

$$\Rightarrow \boxed{B_n(1) = \frac{\langle 2\xi - 1, \sin \lambda_n \xi \rangle}{\langle \sin \lambda_n \xi, \sin \lambda_n \xi \rangle}}$$

Let $\boxed{\gamma_n = \frac{\langle 2\xi - 1, \sin \lambda_n \xi \rangle}{\langle \sin \lambda_n \xi, \sin \lambda_n \xi \rangle}}$

$\Rightarrow$ $B_n(0) = B_n(1) = \gamma_n$, $B_n = C_n \sinh \lambda_n x + D_n \cosh \lambda_n x$

@ $x=0$, $B_n(0) = \gamma_n = C_n \sinh \lambda_n(0) + D_n \cosh \lambda_n(0)$

$$\Rightarrow \boxed{D_n = \gamma_n}$$

$$\Rightarrow \quad B_n = C_n \sinh \lambda_n x + \gamma_n \cosh \lambda_n x$$

@ $x=1$, $B_n(1) = \gamma_n$

$$\gamma_n = C_n \sinh \lambda_n + \gamma_n \cosh \lambda_n$$

$$C_n = \frac{\gamma_n (1-\cosh \lambda_n)}{\sinh \lambda_n}$$

So: $$B_n = \gamma_n \left\{ \frac{(1-\cosh \lambda_n)}{\sinh \lambda_n} \sinh \lambda_n x + \cosh(\lambda_n x) \right\}$$

Can rewrite as:

$$B_n = \gamma_n \left\{ \frac{\sinh \lambda_n x}{\sinh \lambda_n} - \frac{\sinh \lambda_n x \cosh \lambda_n}{\sinh \lambda_n} + \frac{\cosh(\lambda_n x) \sinh \lambda_n}{\sinh \lambda_n} \right\}$$

$$B_n = \frac{\gamma_n}{\sinh \lambda_n} \left\{ \sinh \lambda_n x + \sinh(\lambda_n - \lambda_n x) \right\}$$

$$\boxed{B_n = \frac{\gamma_n}{\sinh \lambda_n} \left\{ \sinh \lambda_n x + \sinh[\lambda_n (1-x)] \right\}}$$

$$\gamma_n = \frac{\langle 2\xi - 1, \sin \lambda_n \xi \rangle}{\langle \sin \lambda_n \xi, \sin \lambda_n \xi \rangle} \qquad \langle \sin \lambda_n \xi, \sin \lambda_n \rangle = \int_0^1 \sin^2 \lambda_n \xi = \frac{1}{2}$$

$$\langle 2\xi-1, \sin\lambda_n\xi\rangle = \int_0^1 2\xi\sin\lambda_n\xi\,d\xi - \int_0^1 \sin\lambda_n\xi\,d\xi$$

$$\int_0^1 2\xi\sin\lambda_n\xi\,d\xi = 2\xi\cdot\frac{-1}{\lambda_n}\cos\lambda_n\xi\Big|_0^1 + \int_0^1 \frac{2}{\lambda_n}\cos\lambda_n\xi\,d\xi$$

$u = 2\xi \quad dv = \sin\lambda_n\xi\,d\xi$

$du = 2\,d\xi \quad v = \frac{-1}{\lambda_n}\cos\lambda_n\xi$

$$\int_0^1 2\xi\sin\lambda_n\xi\,d\xi = \frac{-2}{\lambda_n}\cos\lambda_n + \frac{2}{\lambda_n^2}\sin\lambda_n\xi\Big|_0^1$$

$$= \frac{-2}{\lambda_n}\cos\lambda_n + \cancel{\frac{2}{\lambda_n^2}\sin\lambda_n} \; 0 \qquad \lambda_n = n\pi$$

$$\boxed{\int_0^1 2\xi\sin\lambda_n\xi\,d\xi = \frac{-2}{\lambda_n}(-1)^n}$$

$$\int_0^1 \sin\lambda_n\xi\,d\xi = \frac{-1}{\lambda_n}\cos\lambda_n\xi\Big|_0^1 = \frac{-1}{\lambda_n}\left[\cos\lambda_n - 1\right]$$

$$\int_0^1 \sin\lambda_n\xi\,d\xi = \frac{-1}{\lambda_n}\left[(-1)^n - 1\right]$$

So:

$$\boxed{\langle 2\xi-1, \sin\lambda_n\xi\rangle = \frac{-2}{\lambda_n}(-1)^n - \left(\frac{-1}{\lambda_n}\left[(-1)^n-1\right]\right) = \frac{1}{\lambda_n}\left[-2(-1)^n + \left[(-1)^n-1\right]\right]}$$

$$= \frac{1}{\lambda_n}\left[-(-1)^n-1\right] = \frac{-1}{\lambda_n}\left[1+(-1)^n\right]$$

So, $$\boxed{\gamma_n = \frac{\langle 2\xi-1, \sin\lambda_n\xi\rangle}{\langle \sin\lambda_n\xi, \sin\lambda_n\xi\rangle} = \frac{-\frac{1}{\lambda_n}\left[1+(-1)^n\right]}{\frac{1}{2}} = \frac{-2}{\lambda_n}\left[1+(-1)^n\right]}$$

So:

$$\boxed{B_n = \frac{-2\left[1+(-1)^n\right]}{\lambda_n \sinh\lambda_n}\left\{\sinh(\lambda_n x) + \sinh\left[\lambda_n(1-x)\right]\right\}}$$

Soln for $F(\xi, \eta)$:

$$\boxed{\begin{aligned} F &= \sum_{n=1}^{\infty} B_n(\eta) \sin \lambda_n \xi, \qquad \lambda_n = n\pi \\ B_n &= \frac{-2\left[1+(-1)^n\right]}{\lambda_n \sinh \lambda_n} \left\{ \sinh(\lambda_n \eta) + \sinh\left[\lambda_n(1-\eta)\right] \right\} \end{aligned}}$$

Don't forget superposition to find Θ!

$$\boxed{\Theta = 2 - 2\xi + F(\xi, \eta)}$$

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