



Kalman filtering for GNSS positioning and sensor integration: part l

AAE4203 – Guidance and Navigation

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Week 10, 7 Nov 2022





Outline

- >Overview About State Estimation
- >Kalman Filtering
- > GNSS Doppler based Velocity Estimation
- > Principle of Kalman Filtering
- >GNSS Positioning with Kalman Filtering
 - Loosely Coupled GNSS Positioning
 - Tightly Coupled GNSS Positioning





GNSS Benefits

> GNSS provides a high long-term position accuracy with errors limited to a few meters (stand-alone), while user equipment is available for less than \$100 (€80).

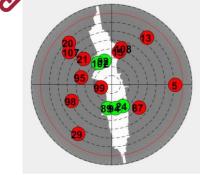




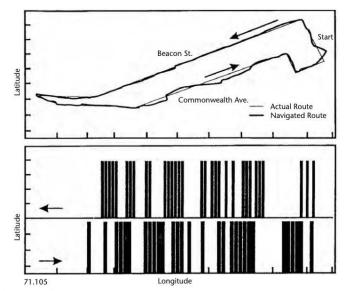


GPS Drawbacks

> One primary concern with using GPS as a standalone source for navigation is **signal interruption**.



If the number of usable satellites is <u>less</u> than three, some receivers have the option of not producing a solution or extrapolating the last position and velocity solution forward in what is called *dead-reckoning* (DR) navigation. Inertial navigation systems (INSs) can be used as a flywheel to provide navigation during shading outages.

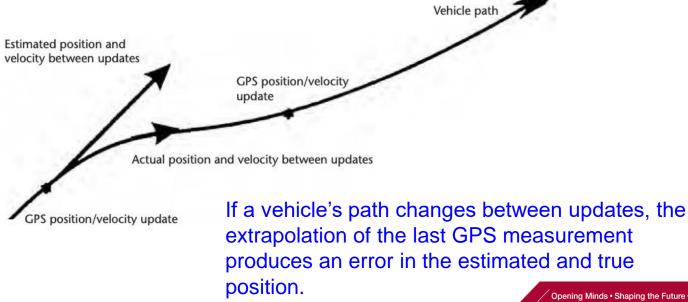






GPS Drawbacks

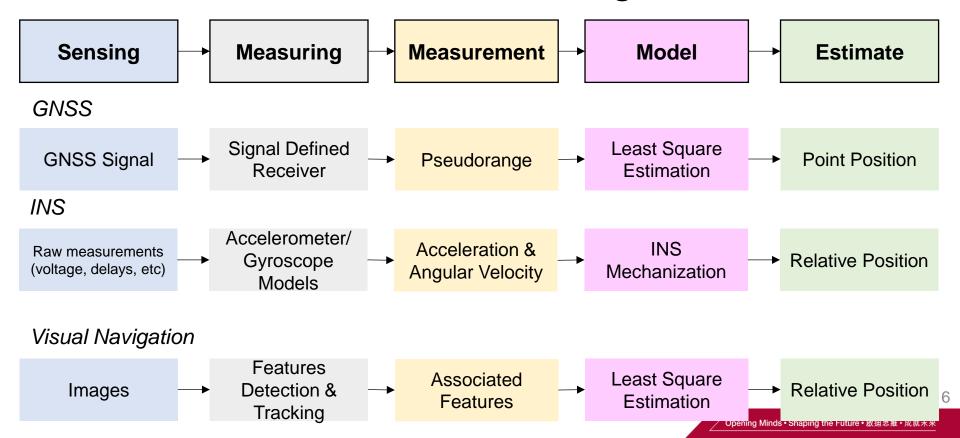
> The **low update rate** of the GPS observations in some equipment is also of concern in real-time applications, especially those related to vehicle control.







Framework of Sensors to Navigation





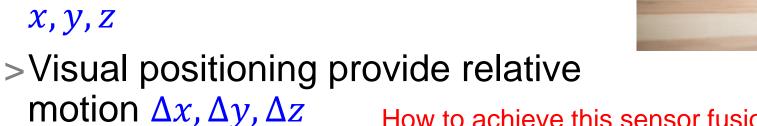
Sensor Integration Based on Kalman Filter





How the Sensor Fusion Problem Looks like...

- >GPS provide the position in x, y, z
- >IMU provide the linear angular velocity in x, y, z
- >GPS Doppler provide velocity in x, y, z

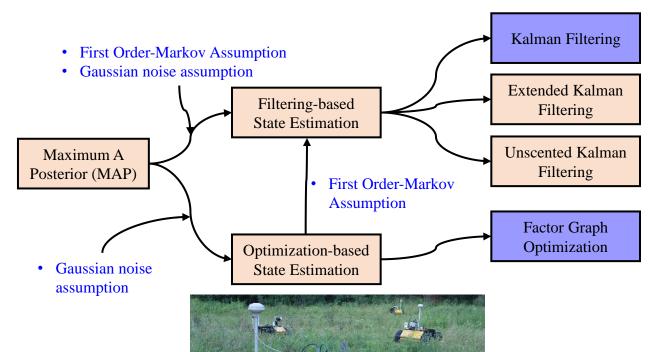


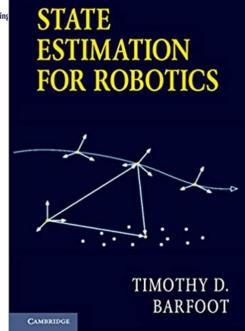
How to achieve this sensor fusion by combining the positioning from different sources?

GPS Antenna



State Estimation Methods









Basics for Probabilistic

Event A and B. P(A) denotes the probability that the event A happens.

$$P(B|A) = \frac{P(AB)}{P(A)} \qquad P(AB) = P(A|B)P(B) \qquad P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Event z denotes the measurements. P(z) denotes the probability that the event A happens.

$$P(\mathbf{x}_k|\mathbf{Z}_1, \dots, \mathbf{Z}_k) = \frac{P(\mathbf{Z}_0, \dots, \mathbf{Z}_k|\mathbf{x}_k)P(\mathbf{x}_k)}{P(\mathbf{Z}_0, \dots, \mathbf{Z}_k)}$$

The probabilistic view of the state estimation is: given a set of measurements $(\mathbf{z}_0, \dots, \mathbf{z}_k)$, can we find a best state \mathbf{x}_k to maximize the <u>conditional probabilistic</u> $P(\mathbf{x}_k | \mathbf{z}_1, \dots, \mathbf{z}_k)$?





How do we describe the uncertainty with **Gaussian Noise**?

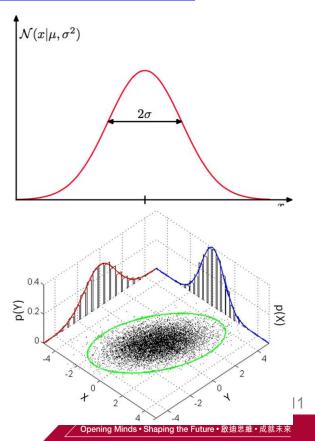
> Gaussian distribution

$$p(x) \sim N(\mu, \sigma^2)$$

1D (univariate)

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1(x-\mu)^2}{2\sigma^2}}$$

2D+ (multi variate)
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\mathbf{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x} - \mathbf{\mu})^t \mathbf{\Sigma}^{-1} (\mathbf{x} - \mathbf{\mu})}$$







How to understand the $P(\mathbf{z}_0, \dots, \mathbf{z}_n)$

 $\mathbf{Z}_{k}|\mathbf{X}_{k}$

Observation function for pseudorange (code) measurement

$$\underline{\rho_{r,t}^{S}} = \underline{r_{r,t}^{S}} + c(\delta_{r,t} - \delta_{r,t}^{S}) + \underline{I_{r,t}^{S}} + \underline{T_{r,t}^{S}} + \underline{\varepsilon_{r,t}^{S}}$$
Pseudorange Range Receiver clock Satellite clock ionospheric delay tropospheric delay

distance Bias (1~2m) bias

tropospheric delay Distance (1~2m)

$$||\mathbf{p}_{t}^{G,s} - \mathbf{p}_{r,t}^{G}||$$

$$\rho_{r,t}^s$$

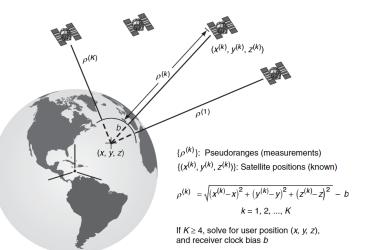
$$Z_{r,t}^{S}$$

$$\rho_{r,t}^s \longrightarrow z_{r,t}^s \qquad \mathbf{p}_{r,t}^G$$

$$P(\mathbf{x}_k|\mathbf{Z}_1, \dots, \mathbf{Z}_k) = \frac{P(\mathbf{Z}_0, \dots, \mathbf{Z}_k|\mathbf{x}_k)P(\mathbf{x}_k)}{P(\mathbf{Z}_0, \dots, \mathbf{Z}_k)}$$

Formulate the $P(\mathbf{x}_k|\mathbf{Z}_1, \dots, \mathbf{Z}_k)$ for the **GNSS** pseudorange measurements!

multipath effects, NLOS receptions, receiver noise, antenna phase-related noise $(0\sim100m)$

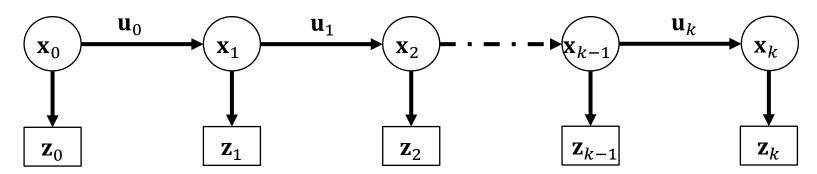






Estimation Formulation

u_i: IMU Measurementz_i: GNSS measurement



States set

$$\mathbf{\chi} = \{\mathbf{x}_o, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$$

The maximum a posteriori (MAP) estimate is given by

Optimal State Set

$$\hat{\mathbf{\chi}} = \arg \max_{\mathbf{\chi}} (P(\mathbf{\chi}|\mathbf{Z}, \mathbf{U})) \ P(\mathbf{\chi}|\mathbf{Z}, \mathbf{U}) = \prod_{k} P(\mathbf{z}_{k}|\mathbf{x}_{k}) P(\mathbf{x}_{0}) \prod_{k} P(\mathbf{x}_{k}|\mathbf{x}_{k-1}, \mathbf{u}_{k-1})$$

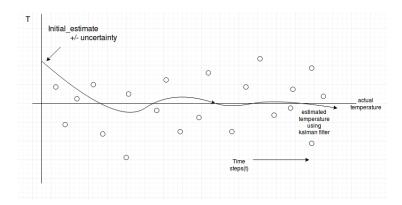
Bayesian theory

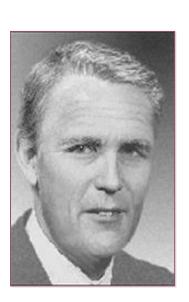




Kalman Filter

- In early 1960, American engineer R.E. Kalman discovered a linear minimum variance ESTIMATION method——Kalman Filter. Soon in space technology (such as flying Navigation system, missile guidance, and determination of satellite orbit and attitude) has been applied.
- > Optimal combination of MEASUREMENT and PROPAGATION





Rudolf Emil Kalman



http://www.cs.unc.edu/~welch/kalman/media/pdf/Kalman1960.pdf

R. E. KALMAN Research Institute for Advanced Study.2

A New Approach to Linear Filtering and Prediction Problems¹

The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the "state transition" method of analysis of dynamic systems. New results are:

(1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and infinite-

(2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the optimal estimation error. From the solution of this equation the coefficients of the difference (or differential) equation of the optimal linear filter are obtained without further calculations

(3) The filtering problem is shown to be the dual of the noise-free regulator problem The new method developed here is applied to two well-known problems, confirming and extending earlier results

The discussion is largely self-contained and proceeds from first principles; basic oncepts of the theory of random processes are reviewed in the Appendix.

Introduction

An important class of theoretical and practical problems in communication and control is of a statistical nature. Such problems are: (i) Prediction of random signals: (ii) separation of random signals from random noise; (iii) detection of signals of known form (pulses, sinusoids) in the presence of

In his pioneering work, Wiener [1]3 showed that problems (i) and (ii) lead to the so-called Wiener-Hopf integral equation; he also gave a method (spectral factorization) for the solution of this integral equation in the practically important special case of stationary statistics and rational spectra.

Many extensions and generalizations followed Wiener's basic work. Zadeh and Ragazzini solved the finite-memory case [2]. Concurrently and independently of Bode and Shannon [3], they also gave a simplified method [2] of solution. Booton discussed the nonstationary Wiener-Hopf equation [4]. These results are now in standard texts [5-6]. A somewhat different approach along these main lines has been given recently by Darlington [7]. For extensions to sampled signals, see, e.g., Franklin [8], Lees [9]. Another approach based on the eigenfunctions of the Wiener-Hopf equation (which applies also to nonstationary problems whereas the preceding methods in general don't), has been pioneered by Davis [10] and applied by many others, e.g., Shinbrot [11], Blum [12], Pugachev [13], Solodovnikov [14].

In all these works, the objective is to obtain the specification of a linear dynamic system (Wiener filter) which accomplishes the prediction, separation, or detection of a random signal.4

This research was supported in part by the U. S. Air Force Office of Scientific Research under Contract AF 49 (638)-382.

Numbers in brackets designate References at end of paper.

Of course, in general these tasks may be done better by nonlinear

filters. At present, however, little or nothing is known about how to obtain (both theoretically and practically) these nonlinear filters Contributed by the Instruments and Regulators Division and pre-

at the Instruments and Regulators Conference, March 29- April 2, 1959. of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Note: Statements and opinions advanced in papers are to be understood as individual expressions of their authors and not those of the Society. Manuscript received at ASME Headquarters, February 24, 1959. Paper

Present methods for solving the Wiener problem are subject to a number of limitations which seriously curtail their practical

(1) The optimal filter is specified by its impulse response. It is not a simple task to synthesize the filter from such data.

(2) Numerical determination of the ontimal impulse response is often quite involved and poorly suited to machine computation The situation gets rapidly worse with increasing complexity of (3) Important generalizations (e.g., growing-memory filters,

nonstationary prediction) require new derivations, frequently of considerable difficulty to the nonspecialist (4) The mathematics of the derivations are not transparent

Fundamental assumptions and their consequences tend to be This paper introduces a new look at this whole assemblage of

problems, sidestenning the difficulties just mentioned. The following are the highlights of the paper: (5) Optimal Estimates and Orthogonal Projections. The

Wiener problem is approached from the point of view of conditional distributions and expectations. In this way, basic facts of the Wiener theory are quickly obtained; the scope of the results and the fundamental assumptions appear clearly. It is seen that all statistical calculations and results are based on first and second order averages; no other statistical data are needed. Thus difficulty (4) is eliminated. This method is well known in probability theory (see pp. 75-78 and 148-155 of Doob [15] and pp. 455-464 of Loève [16]) but has not yet been used extensively in engineering.

(6) Models for Random Processes. Following in particular. Bode and Shannon [3], arbitrary random signals are represented (up to second order average statistical properties) as the output of a linear dynamic system excited by independent or uncorrelated random signals ("white noise"). This is a standard trick in the engineering applications of the Wiener theory [2-7]. The approach taken here differs from the conventional one only in the way in which linear dynamic systems are described. We shall emphasize the concepts of state and state transition; in other words, linear systems will be specified by systems of first-order difference (or differential) equations. This point of view is $= \Phi^*(t+1;t) \widetilde{\mathbf{x}} (t|t-1) + \mathbf{u}(t)$

hus Φ^* is also the transition matrix of the linear dynamic system From (23) we obtain at once a recursion relation for the coariance matrix $P^*(t)$ of the optimal error $\tilde{\mathbf{x}}(t|t-1)$. Noting that

(t) is independent of $\mathbf{x}(t)$ and therefore of $\widetilde{\mathbf{x}}(t|t-1)$ we get $f'''(t+1) = E \widetilde{\mathbf{x}} (t+1|t) \widetilde{\mathbf{x}} '(t+1|t)$

> $= \Phi^{*}(t+1; t)E\widetilde{\mathbf{x}}(t|t-1)\widetilde{\mathbf{x}}'(t|t-1)\Phi^{**}(t+1; t) + \mathbf{Q}(t)$ $=\Phi^*(t+1;t)E\widetilde{\mathbf{x}}(t|t-1)\widetilde{\mathbf{x}}'(t|t-1)\Phi'(t+1;t)+\mathbf{Q}(t)$

 $=\Phi^{*}(t+1;t)P^{*}(t)\Phi^{*}(t+1;t)+Q(t)$

here $\mathbf{Q}(t) = E\mathbf{u}(t)\mathbf{u}'(t)$. There remains the problem of obtaining an explicit formula for * (and thus also for **Φ***). Since,

 $\tilde{\mathbf{x}}(t+1)|Z(t)| = \mathbf{x}(t+1) - \hat{E}[\mathbf{x}(t+1)|Z(t)]$

orthogonal to $\tilde{\mathbf{v}}$ (t | t - 1), it follows that by (19) that $0 = E[\mathbf{x}(t+1) - \mathbf{\Delta}^{\bullet}(t)\widetilde{\mathbf{v}}(t|t-1)]\widetilde{\mathbf{v}}'(t|t-1)$

 $= E\mathbf{x}(t+1)\widetilde{\mathbf{y}}'(t|t-1) - \Delta^*(t)E\widetilde{\mathbf{y}}(t|t-1)\widetilde{\mathbf{y}}'(t|t-1).$

loting that $\overline{\mathbf{x}}(t+1|t-1)$ is orthogonal to Z(t), the definition of (t) given earlier, and (17), it follows further

 $= E\widetilde{\mathbf{x}}(t+1|t-1)\widetilde{\mathbf{v}}'(t|t-1) - \Delta^*(t)\mathbf{M}(t)\mathbf{P}^*(t)\mathbf{M}'(t)$

 $= E[\Phi(t+1;t)\widetilde{\mathbf{x}}(t|t-1) + \mathbf{u}(t|t-1)]\widetilde{\mathbf{x}}'(t|t-1)\mathbf{M}'(t)$ $-\Delta^*(t)\dot{\mathbf{M}}(t)\mathbf{P}^*(t)\mathbf{M}'(t).$

inally, since $\mathbf{u}(t)$ is independent of $\mathbf{x}(t)$, $0 = \Phi(t + 1; t)P^*(t)M'(t) - \Delta^*(t)M(t)P^*(t)M'(t).$

low the matrix $\mathbf{M}(t)\mathbf{P}^*(t)\mathbf{M}'(t)$ will be positive definite and hence vertible whenever $P^*(t)$ is positive definite, provided that none f the rows of M(t) are linearly dependent at any time, in other ords, that none of the observed scalar random variables $y_i(t)$, ..., "(t), is a linear combination of the others. Under these rcumstances we get finally:

 $\Delta^*(t) = \Phi(t+1; t)P^*(t)M'(t)[M(t)P^*(t)M'(t)]^{-1}$ (25) Since observations start at t_0 , $\widetilde{\mathbf{x}}(t_0|t_0-1)=\mathbf{x}(t_0)$; to begin the erative evaluation of $P^*(t)$ by means of equation (24), we must so that the estimate $\mathbf{x}^*(t+s|t)$ ($s \ge 0$) is also given by a linear dyby by by specify $\mathbf{P}^*(t_n) = E\mathbf{x}(t_n)\mathbf{x}^*(t_n)$. Assuming this matrix is ositive definite, equation (25) then yields $\Delta^*(t_0)$; equation (22) $P^*(t_0 + 1; t_0)$, and equation (24) $P^*(t_0 + 1)$, completing the cycle. now $\mathbf{Q}(t)$ is positive definite, then all the $\mathbf{P}^*(t)$ will be positive efinite and the requirements in deriving (25) will be satisfied at

rthogonal to V(t), we have

 $\mathbf{x}^*(t+1|t) = \hat{E} \left[\Phi(t+1;t)\mathbf{x}(t) + \mathbf{u}(t)|V(t) \right] = \Phi(t+1;t)\mathbf{x}^*(t|t)$ lence if $\Phi(t+1;t)$ has an inverse $\Phi(t;t+1)$ (which is always the ase when **Φ** is the transition matrix of a dynamic system escribable by a differential equation) we have

 $\mathbf{x}^*(t|t) = \mathbf{\Phi}(t; t+1)\mathbf{x}^*(t+1|t)$ $t_1 \ge t + 1$, we first observe by repeated application of (16) that

 $\Phi(t+s; t+r)\mathbf{u}(t+r)$

ince $\mathbf{u}(t+s-1)$, ..., $\mathbf{u}(t+1)$ are all orthogonal to $\mathbf{V}(t)$,

 $(t+s) = \Phi(t+s; t+1)\mathbf{x}(t+1)$

(23) $\mathbf{x}^*(t+s|t) = \hat{E} [\mathbf{x}(t+s)|\mathbf{y}(t)]$

 $=\hat{E} [\Phi(t+s;t+1)\mathbf{x}(t+1)]V(t)$ $= \Phi(t+s; t+1)x^*(t+1|t)$ $(s \ge 1)$

If s < 0, the results are similar, but $\mathbf{x}^*(t - s|t)$ will have (1 - s|t)s)(n-n) co-ordinates.

The results of this section may be summarized as follows: Theorem 3. (Solution of the Wiener Problem)

Consider Problem I. The optimal estimate $\mathbf{x}^*(t+1|t)$ of $\mathbf{x}(t+1|t)$ given \(\psi(t_0), ..., \(\psi(t) \) is generated by the linear dynamic system

 $\mathbf{x}^*(t+1|t) = \mathbf{\Phi}^*(t+1;t)\mathbf{x}^*(t|t-1) + \mathbf{\Delta}^*(t)\mathbf{v}(t)$ (21) The estimation error is given by

 $\widetilde{\mathbf{x}}(t+1|t) = \mathbf{\Phi}^*(t+1;t)\widetilde{\mathbf{x}}(t|t-1) + \mathbf{u}(t)$ The covariance matrix of the estimation error is

cov $\widetilde{\mathbf{x}}(t|t-1) = E\widetilde{\mathbf{x}}(t|t-1)\widetilde{\mathbf{x}}'(t|t-1) = \mathbf{P}^*(t)$

The expected quadratic loss is $\sum_{t} E \widetilde{x}_{t}^{2}(t|t-1) = \text{trace } \mathbf{P}^{*}(t)$

The matrices $\Delta^*(t)$, $\Phi^*(t + 1; t)$, $P^*(t)$ are generated by the recursion relations

 $\Delta^*(t) = \Phi(t+1; t)P^*(t)M'(t)[M(t)P^*(t)M'(t)]$ (28) $\Phi^*(t+1;t) = \Phi(t+1;t) - \Delta^*(t)M(t)$ $P^*(t+1) = \Phi^*(t+1;t)P^*(t)\Phi'(t+1;t)$ (30)

In order to carry out the iterations, one must specify the covariance $P^*(t_0)$ of $\mathbf{x}(t_0)$ and the covariance $\mathbf{Q}(t)$ of $\mathbf{u}(t)$. Finally, for any $s \ge 0$, if Φ is invertible

 $\mathbf{x}^*(t+s|t) = \mathbf{\Phi}(t+s; t+1)\mathbf{x}^*(t+1)|t\rangle$ $= \Phi(t+s; t+1)\Phi^*(t+1; t)\Phi(t; t+s-1)$

 $+\Phi(t+s;t+1)\Delta^*(t)\mathbf{y}(t)$

 $\times x^{*}(t+s-1|t-1)$

namic system of the type (21). Remarks. (h) Eliminating Δ* and Φ* from (28-30), a nonlinear difference equation is obtained for $P^*(t)$:

 $\Phi(t + 1; t) \{P^*(t) - P^*(t)M'(t)[M(t)P^*(t)M'(t)]^{-1}$ $\times \mathbf{P}^*(t)\mathbf{M}(t)\mathbf{\Phi}^*(t+1;t)+\mathbf{Q}(t)$ $t \ge t_0$ (32)

Now we remove the restriction that $t_1 = t + 1$. Since $\mathbf{u}(t)$ is This equation is linear only if $\mathbf{M}(t)$ is invertible but then the problem is trivial since all components of the random vector $\mathbf{x}(t)$ are observable $P^*(t+1) = Q(t)$. Observe that equation (32) plays a role in the present theory analogous to that of the Wiener-Hopf equation in the conventional theory.

Once $P^*(t)$ has been computed via (32) starting at $t = t_0$, the explicit specification of the optimal linear filter is immediately available from formulas (29-30). Of course, the solution of Equation (32), or of its differential-equation equivalent, is a much simpler task than solution of the Wiener-Hopf equation.

(i) The results stated in Theorem 3 do not resolve completely Problem I. Little has been said, for instance, about the physical significance of the assumptions needed to obtain equation (25), the convergence and stability of the nonlinear difference equation (32), the stability of the optimal filter (21), etc. This can actually be done in a completely satisfactory way, but must be left to a future paper. In this connection, the principal guide and

the matrices occurring in equation (31) and the covariance matrix of $\tilde{\mathbf{x}}(t|t)$ are found after simple calculations. We have, for all $t \ge$

 $tC_2(t)$ $\Phi(t; t+1)\Delta^*(t) = C_1(t)$ $C_1(t) - tC_2(t)$ $C_2(t)$

 $\Phi(t; t+1)\Phi^*(t+1; t)\Phi(t+1; t)$ $\begin{bmatrix} C_1(t) - tC_2(t) & C_1(t) - tC_2(t) & -\phi_{12}tC_2(t) \end{bmatrix}$ $-C_2(t)$ $C_1(t)-C_2(t)$ $-\phi_{11}C_2(t)$ $-C_1(t)+tC_2(t)$ $-C_1(t)+tC_2(t)$ $+\phi_{33}tC_2(t)$

$$\operatorname{cov} \widetilde{\mathbf{x}}(t|t) = E \widetilde{\mathbf{x}}(t|t) \widetilde{\mathbf{x}}'(t|t) = \frac{b^2}{C_1(t)} \begin{bmatrix} t^2 & t & -t^2 \\ t & 1 & -t \\ -t^2 & -t & t^2 \end{bmatrix}$$

To gain some insight into the behavior of this system, let us examine the limiting case $t \to \infty$ of a large number of observations. Then $C_1(t)$ obeys approximately the differential equation

$$dC_1(t)/dt \approx C_2^2(t)$$
 $(t >> 1)$

from which we find $C_1(t) \approx (1 - \phi_{11})^2 t^3 / 3 + \phi_{11} (1 - \phi_{11}) t^2 + \phi_{11}^2 t + b^2 / a^2$

Using (39), we get further,

$$\Phi^{-1}\Phi^{+}\Phi \approx \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$
 and $\Phi^{-1}\Delta^{+}\approx \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $(t >> 1)$

Thus as the number of observations becomes large, we depend almost exclusively on $x_1*(t|t)$ and $x_2*(t|t)$ to estimate $x_1*(t+1|t+1|t+1|t)$ 1) and $x_2^*(t + 1|t + 1)$. Current observations are used almost exclusively to estimate the noise

$$x_3*(t|t) \approx y_1*(t) - x_1*(t|t)$$
 (t>>1)

One would of course expect something like this since the problem is analogous to fitting a straight line to an increasing number

As a second check on the reasonableness of the results given, observe that the case t >> 1 is essentially the same as prediction based on continuous observations. Setting $\phi_{13} = 0$, we have

$$E \widetilde{x}_{1}^{2}(t|t) \approx \frac{a^{2}b^{2}t^{2}}{t^{2}-2\cdot3\cdot2}$$
 $(t>> 1; \phi_{33} = 0)$

which is identical with the result obtained by Shinbrot [11], Example 1, and Solodovnikov [14], Example 2, in their treatment of the Wiener problem in the finite-length, continuous-data case, using an approach entirely different from ours.

Conclusions

This paper formulates and solves the Wiener problem from the "state" point of view. On the one hand, this leads to a very general treatment including cases which cause difficulties when attacked by other methods. On the other hand, the Wiener problem is shown to be closely connected with other problems in the theory of control. Much remains to be done to exploit these

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Transactions of the ASME-Journal of Basic Engineering, 82 (Series D): 35-45, Copyright @ 1960 by ASME

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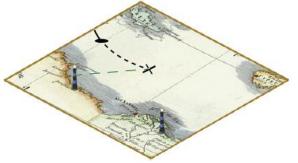


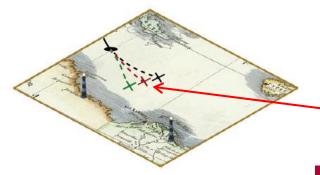


Kalman Filter——Navigation application using dead reckoning and visual measurement to landmark







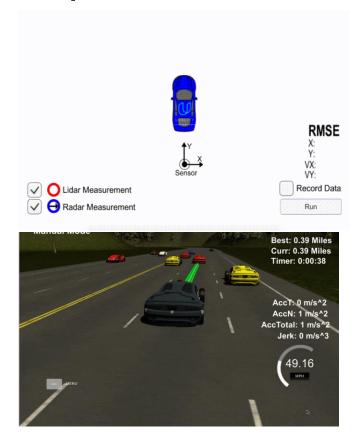


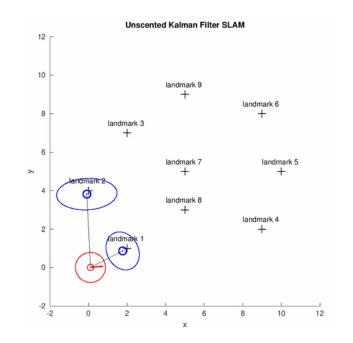
Predicted and corrected position of the ship





Examples of Kalman Filter in Navigation



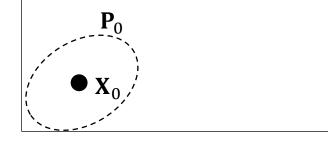


North

Definitions

X, The state vector is the set of parameters describing a system, known as states, which the Kalman filter estimates.

> P, Associated with the state vector is an error covariance matrix. This represents the uncertainties in the Kalman filter's state estimates and the degree of correlation between the errors in those estimates.

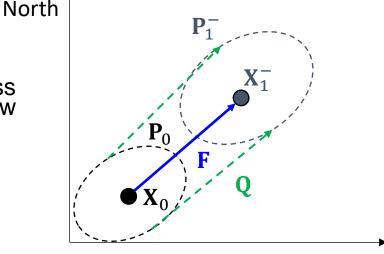


East



Definitions

- > F, The system model, also known as the process model or time-propagation model, describes how the Kalman filter states and error covariance matrix vary with time. (The system model is deterministic for the states as it is based on known properties of the system.)
- > Q, A state uncertainty should also be increased with time to account for unknown changes in the system that cause the state estimate to go out of date in the absence of new measurement information. This variation in the true values of the states is known as system noise or process noise.

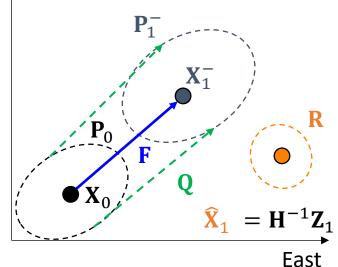


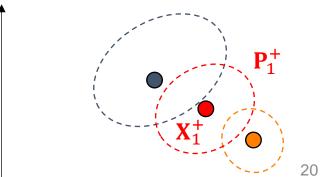




Definitions

- > Z, The *measurement vector* is a set of simultaneous measurements of properties of the system which are functions of the state vector.
- R, Associated with the measurement vector is a measurement noise covariance matrix which describes the statistics of the noise on the measurements.
 North
- > H, Z=H(X)=HX, The measurement model describes how the measurement vector varies as a function of the true state vector (as opposed to the state vector estimate) in the absence of measurement noise.





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Key Equations of Kalman Filter

$$>\mathbf{X}_{t}^{-} = \mathbf{F}\mathbf{X}_{t-1}^{+}$$

$$>\mathbf{P}_{t}^{-} = \mathbf{F}\mathbf{P}_{t-1}^{+}\mathbf{F}^{\mathrm{T}} + \mathbf{Q}$$

State Propagation



$$>\Delta \mathbf{Z}_{t} = \mathbf{Z}_{t} - \mathbf{H}\mathbf{X}_{t}^{-}$$

$$>\mathbf{S}_{t} = \mathbf{H}\mathbf{P}_{t}^{-}\mathbf{H}^{T} + \mathbf{R}$$

$$>\mathbf{K}_{t} = \mathbf{P}_{t}^{-}\mathbf{H}^{T}\mathbf{S}_{t}^{-}$$

$$>\mathbf{X}_{t}^{+} = \mathbf{X}_{t}^{-} + \mathbf{K}_{t}\Delta\mathbf{Z}_{t}$$

$$>\mathbf{P}_{t}^{+} = (\mathbf{I} - \mathbf{K}_{t}\mathbf{H})\mathbf{P}_{t}^{-}$$

Measurement Update





Measurement Innovation, $\Delta \mathbf{Z}_t$

$$> \Delta \mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H} \mathbf{X}_t^-$$

- >Meaning: The difference between propagation and measurement in the domain of **Z** (measurement)
- $> \Delta \mathbf{Z}_t = 0$, What does it mean?
- $> \Delta \mathbf{Z}_t \neq 0$, What shall we do?

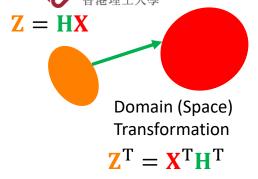




Kalman Filter Gain, K_t

$$>$$
 $\mathbf{K}_t = \mathbf{P}_{t-1}^{-} \mathbf{H}^{\mathrm{T}} \mathbf{S}_t^{-1}$

$$>$$
 $\mathbf{S}_t = \mathbf{H}\mathbf{P}_{t-1}^{-}\mathbf{H}^{\mathrm{T}} + \mathbf{R}$



If we forget there is no denominator in Matrix...

$$> K_{t} = \frac{P_{t-1}^{-}H^{T}}{HP_{t-1}^{-}H^{T}+R}$$

$$> K_{t} = \frac{(FP_{t-1}^{+}F^{T}+Q)H^{T}}{HP_{t-1}^{-}H^{T}+R}$$

The propagated state covariance (uncertainty) in the domain of the transpose of **Z** (measurement)

The propagated state covariance (uncertainty) in the domain of **Z** (measurement)

+ measurement covariance (uncertainty) R

The updated state covariance (uncertainty)





Kalman Filter Gain, K_t

$$>$$
 $\mathbf{K}_t = \frac{(\mathbf{FP}_{t-1}^+ \mathbf{F}^T + \mathbf{Q})\mathbf{H}^T}{\mathbf{HP}_{t-1}^- \mathbf{H}^T + \mathbf{R}}$

- >**R** \gg **Q**, what does it mean?
- >**R** \ll **Q**, what does it mean?
- >**R** = **Q**, what does it mean?





Simplified the Models to Identity Matrix, I

$$>$$
 $\mathbf{X}_t^- = \mathbf{F}\mathbf{X}_{t-1}^+ + \mathbf{B}\mathbf{U}_t$

$$>\mathbf{P}_{t}^{-}=\mathbf{F}\mathbf{P}_{t-1}^{+}\mathbf{F}^{\mathrm{T}}+\mathbf{Q}$$

$$> \Delta \mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H} \mathbf{X}_t^-$$

$$>$$
 $\mathbf{S}_t = \mathbf{H} \mathbf{P}_{t-1}^{-} \mathbf{H}^{\mathrm{T}} + \mathbf{R}$

$$> \mathbf{K}_t = \mathbf{P}_{t-1}^{-} \mathbf{H}^{\mathrm{T}} \mathbf{S}_t^{-1}$$

$$> \mathbf{X}_t^+ = \mathbf{X}_t^- + \mathbf{K}_t \Delta \mathbf{Z}_t$$

$$> P_t^+ = (I - K_t H) P_{t-1}^-$$

$$>$$
 $\mathbf{X}_t^- = \mathbf{X}_{t-1}^+ + \mathbf{U}_t$

$$> P_t^- = P_{t-1}^+ + Q$$

$$> \Delta \mathbf{Z}_t = \mathbf{Z}_t - \mathbf{X}_t^-$$

$$>$$
 $\mathbf{S}_t = \mathbf{P}_{t-1}^- + \mathbf{R}$

$$> \mathbf{K}_t = \mathbf{P}_{t-1}^- \mathbf{S}_t^{-1}$$

$$>$$
 $\mathbf{X}_t^+ = \mathbf{X}_t^- + \mathbf{K}_t \Delta \mathbf{Z}_t$

$$> P_t^+ = (I - K_t)P_{t-1}^-$$

F = I H = I B = I



How do we describe the uncertainty?

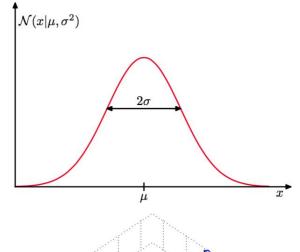
> Gaussian distribution

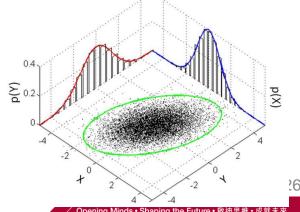
$$p(x) \sim N(\mu, \sigma^2)$$

1D (univariate)

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1(x-\mu)^2}{2\sigma^2}}$$

2D+ (multi variate)
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\mathbf{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x} - \mathbf{\mu})^t \mathbf{\Sigma}^{-1} (\mathbf{x} - \mathbf{\mu})}$$

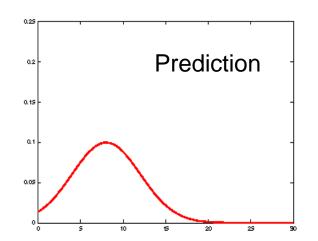


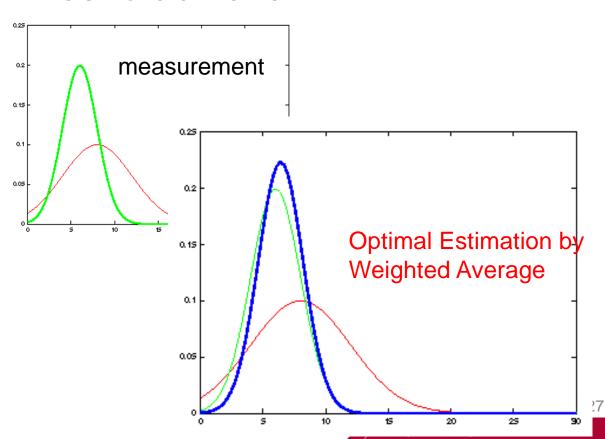






What Kalman Filter tries to achieve?









Fusion of Gaussian distribution in 1D

If $w_i = 1/\sigma_i^2$

Distribution of two measurements

$$\hat{q}_1 = q_1$$
 with variance σ_1^2
 $\hat{q}_2 = q_2$ with variance σ_2^2

> Weighted least-square

$$S = \sum_{i=1}^{n} w_i (\hat{q} - q_i)^2$$

> Finding minimum error

$$\frac{\partial S}{\partial \hat{q}} = \frac{\partial}{\partial \hat{q}} \sum_{i=1}^{n} w_i (\hat{q} - q_i)^2 = 2 \sum_{i=1}^{n} w_i (\hat{q} - q_i) = 0$$

After some calculation and rearrangements

Kalman Gain
$$\hat{q} = q_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} (q_2 - q_1)$$

$$f(q)$$

$$f(q) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(q - \mu)^2}{2\sigma^2}\right)$$

$$\hat{q} - q_i) = 0$$





Question: Why Kalman Filter is Optimal?

- >What does it mean to Optimal?
 - Maximum the Probability of the Estimation!

- >What assumptions are made in Kalman Filter so that it can achieve Optimal?
 - A. Gaussian Random Variable (Gaussian Noise)
 - B. 1st Order of Markov Chain





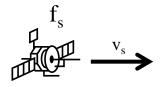
Kalman Filter in GNSS

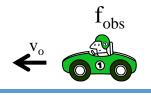
- > Example of the GNSS loosely-coupled pseudorange/Doppler integration using Kalman filter (this is done through the handwriting in visualizer)
- >Example of the GNSS tightly-coupled pseudorange/Doppler integration using Kalman filter





Doppler Effect





One dimension is assumed. Right direction is positive.

- > Receiver is set in the car.
- > Received frequency is
- > "cs" is speed of light.

$$f_{obs} = f_s \frac{cs - v_o}{cs - v_s}$$

- > Doppler frequency "f_D" is equal to "f_{obs} f_{source}"
- > FLL (frequency lock loop) tries to estimate "f_D".
- > Once we can estimate "f_D", "v_o" can be resolved.





Velocity Estimation

- > Velocity estimation in GPS is just same as shown in the previous slide.
- > The differences are as follows.
- * 3 dimension velocity $(v_{x_i} v_{y_i} v_z)$ have to be estimated.
- * Frequency in the receiver is based on on-board clock.
- * Measurement is pseudorange rate, which calculated from <u>Doppler</u> <u>frequency AND satellite velocity</u>.
- > 4 unknown variables (v_x, v_y, v_z, f_{clk}) have to be estimated using at least 4 visible satellites. DOP is also important.
- > Velocity estimation is same as position estimation.





Receiver Velocity Estimation from the Doppler Measurements

Measurements from Doppler

$$y = (-\lambda_i D_{r,i}^1, -\lambda_i D_{r,i}^2, -\lambda_i D_{r,i}^3, ..., -\lambda_i D_{r,i}^m)^T$$

Observation function

$$h(x) = \begin{pmatrix} r_r^1 + cd\dot{t}_r - cd\dot{T}^1 \\ r_r^2 + cd\dot{t}_r - cd\dot{T}^2 \\ r_r^3 + cd\dot{t}_r - cd\dot{T}^3 \\ \vdots \\ r_r^m + cd\dot{t}_r - cd\dot{T}^m \end{pmatrix} H = \begin{pmatrix} -e_r^{1T} & 1 \\ -e_r^{2T} & 1 \\ -e_r^{3T} & 1 \\ \vdots & \vdots \\ -e_r^{mT} & 1 \end{pmatrix}$$
(F.6.28)

Can you try to formulate the steps for GNSS velocity estimation similar to the position estimation? ©

The range-rage r_r^s between the receiver and the satellite in these equations is derived from:

$$r_r^s = e_r^{sT} \left(v^s(t^s) - v_r \right) + \frac{\omega_e}{c} \left(v_y^s x_r + y^s v_{x,r} - v_x^s v_r - x^s v_{y,r} \right)$$
 (F.6.29)







Thank you for your attention © Q&A

Dr. Weisong Wen

If you have any questions or inquiries, please feel free to contact me.

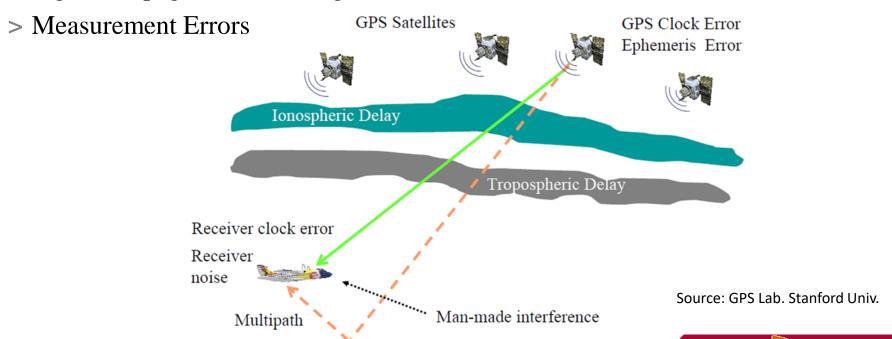
Email: welson.wen@polyu.edu.hk





Source of Measurements Errors

- > Control Segment Errors
- > Signal Propagation Modeling Errors







Why we discuss about measurement errors?

> Back to bias and noise errors discussion, noise errors of pseudorange can be mitigated to some degree using <u>carrier phase</u> <u>smoothing technique</u>.

> On the other hand, you have to estimate <u>bias errors</u> as accurate as possible <u>by yourself</u> to improve positioning performance.

> All kinds of improved techniques are essentially same in terms of estimating or eliminating bias or noise errors.



multipath effects, NLOS

receptions, receiver noise, antenna phase-related noise

 $(0\sim100m)$



Observation Model for Pseudorange/Carrier Measurements

Observation function for pseudorange (code) measurement

$$\underline{\rho_{r,t}^{S}} = \underline{r_{r,t}^{S}} + c\left(\delta_{r,t} - \delta_{r,t}^{S}\right) + I_{r,t}^{S} + T_{r,t}^{S} + \varepsilon_{r,t}^{S}$$
Pseudorange

Range
Receiver clock Satellite clock ionospheric delay bias

Distance (1~2m)

Distance (1~2m)

Pt - p_{r,t}^{G}

multipath effective properties and the properties of the properti

Observation function for carrier-phase measurement

$$\frac{\varphi_{r,t}^{S} = r_{r,t}^{S} + c(\delta_{r,t} - \delta_{r,t}^{S}) + I_{r,t}^{S} + T_{r,t}^{S} + \varepsilon_{r,t}^{S} + N_{r,t}^{S}}{\text{Carrier-phase}}$$
range
Range
Receiver clock
Bias (1~2m)
Bias (1~2m)
Bias Distance (1~2m)

Ambiguation

Contract Contract

tropospheric delay Distance (1~2m)

Distance (1~2m)

multipath effects, NLOS receptions, receiver noise, antenna phase-related noise $(0\sim100m)$

Ambiguity

To use the carrier-phase measurements, the ambiguity need to be resolved.

For each carrier-phase measurement, you got an unknown variable $N_{r,t}^{s}$!





RTK (Real Time Kinematic)

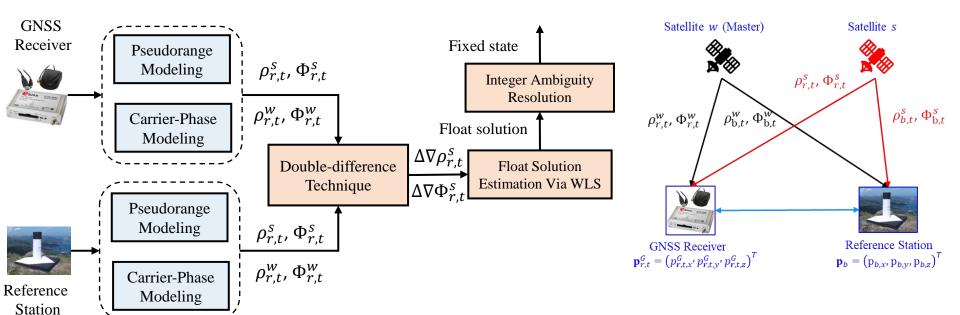
- The concept of RTK is same as DGPS.
- RTK uses carrier phase measurements. DGPS uses pseudo-range measurements.
- GPS receiver is able to measure 1/100 of wavelength of L1 frequency (19 cm).
- If you have high-end receiver, you know your position within 1-2cm accuracy as long as you have 5 or more LOS satellites.





Overview of GNSS Real-time Kinematic

WLS*: Weighted Least Squares DD*: Double-difference



 $\rho_{r,t}^s$: Pseudorange measurement $\Phi_{r,t}^s$: Carrier-phase measurement $\Delta \nabla \rho_{r,t}^s$: DD Pseudorange measurement $\Delta \nabla \Phi_{r,t}^s$: DD Carrier-phase measurement

Why GNSS Real-time Kinematic?

- Remove the error from receiver/satellite clock bias, atmosphere error using double-difference technique.
- Use the high-accuracy carrier-phase measurements.





Observation Model for Pseudorange/Carrier Measurements

Observation function for pseudorange (code) measurement

$$\underline{\rho_{r,t}^{S}} = \underline{r_{r,t}^{S}} + c\left(\delta_{r,t} - \delta_{r,t}^{S}\right) + I_{r,t}^{S} + T_{r,t}^{S} + \varepsilon_{r,t}^{S} + \varepsilon_{r,t$$

Observation function for carrier-phase measurement

$$\frac{\psi_{r,t}^{S} = r_{r,t}^{S} + c(\delta_{r,t} - \delta_{r,t}^{S}) + I_{r,t}^{S} + T_{r,t}^{S} + \varepsilon_{r,t}^{S} + N_{r,t}^{S}}{\text{Carrier-phase}}$$
range
Range
Range
Receiver clock
Bias (1~2m)
Bias Distance (1~2m)

Carrier-phase

Receiver clock
Bias (1~2m)
Bias (1~2m)
Bias (1~2m)
Bias (1~2m)

Receiver clock
Bias (1~2m)
Bias (1~2m)
Bias (1~2m)

Carrier-phase

Ambiguration

multipath effects, NLOS receptions, receiver noise, antenna phase-related noise $(0\sim100m)$

multipath effects, NLOS To use the carrier-phase receptions, receiver noise, antenna phase-related noise measurements, the $(0\sim100m)$ ambiguity need to be resolved.

Ambiguity





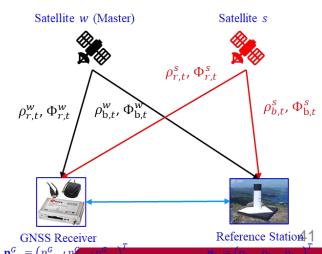
Single Difference Pseudorange Measurements

Single difference between the GNSS receiver and the reference station to remove the atmosphere errors:

$$\Delta \rho_{r,t}^s = \rho_{r,t}^s - \rho_{b,t}^s = r_{r,t}^s - r_{b,t}^s + c(\delta_{r,t} - \delta_{r,t}^s) - c(\delta_{b,t} - \delta_{b,t}^s)$$
 Satellite s

$$\begin{aligned}
\rho_{r,t}^{W} &= r_{r,t}^{W} + c(\delta_{r,t} - \delta_{r,t}^{W}) + I_{r,t}^{W} + T_{r,t}^{W} + \varepsilon_{r,t}^{W} \\
\rho_{b,t}^{W} &= r_{b,t}^{W} + c(\delta_{b,t} - \delta_{b,t}^{W}) + I_{b,t}^{W} + T_{b,t}^{W} + \varepsilon_{b,t}^{W} \\
\Delta \rho_{r,t}^{W} &= \rho_{r,t}^{W} - \rho_{b,t}^{W} = r_{r,t}^{W} - r_{b,t}^{W} + c(\delta_{r,t} - \delta_{r,t}^{W}) - c(\delta_{b,t} - \delta_{b,t}^{W}) & \text{Satellite} \\
\psi & \text{(Master)} \end{aligned}$$

Assumption: GNSS receiver and the reference station are close with the same atmosphere errors







Double Difference Pseudorange Measurements

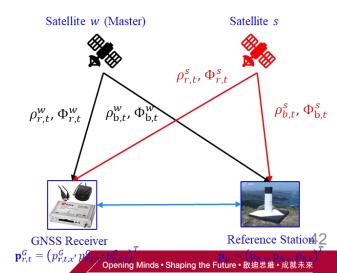
Second difference between the master satellite and the satellite *s* to remove the atmosphere errors:

$$\left\{ \Delta \rho_{r,t}^s = \rho_{r,t}^s - \rho_{b,t}^s = r_{r,t}^s - r_{b,t}^s + c(\delta_{r,t} - \delta_{r,t}^s) - c(\delta_{b,t} - \delta_{b,t}^s) \right\}$$
 Satellite s

Assumption: GNSS receiver and the reference station are close with the same atmosphere errors

$$\left[\Delta \rho_{r,t}^{w} = \rho_{r,t}^{w} - \rho_{b,t}^{w} = r_{r,t}^{w} - r_{b,t}^{w} + c \left(\delta_{r,t} - \delta_{r,t}^{w} \right) - c \left(\delta_{b,t} - \delta_{b,t}^{w} \right) \right] \qquad \text{Satellite}$$
w (Master)

$$\Delta \nabla \rho_{r,t}^s = \Delta \rho_{r,t}^s - \Delta \rho_{r,t}^w = \rho_{r,t}^s - \rho_{b,t}^s - \rho_{r,t}^w - \rho_{b,t}^w \qquad \text{DD}$$
measurements







Single Difference Carrier-phase Measurements

Single difference between the GNSS receiver and the reference station to remove the atmosphere errors:

$$\psi_{r,t}^{s} = r_{r,t}^{s} + c(\delta_{r,t} - \delta_{r,t}^{s}) + I_{r,t}^{s} + T_{r,t}^{s} + \varepsilon_{r,t}^{s} + N_{r,t}^{s}$$

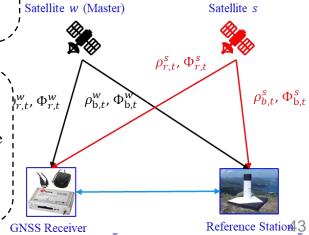
$$\psi_{b,t}^{s} = r_{b,t}^{s} + c(\delta_{b,t} - \delta_{b,t}^{s}) + I_{b,t}^{s} + T_{b,t}^{s} + \varepsilon_{b,t}^{s} + N_{b,t}^{s}$$
Satellite s

$$\Delta \psi_{r,t}^s = \psi_{r,t}^s - \psi_{b,t}^s = r_{r,t}^s - r_{b,t}^s + c(\delta_{r,t} - \delta_{r,t}^s) - c(\delta_{b,t} - \delta_{b,t}^s) + N_{r,t}^s - N_{b,t}^s$$

$$\psi_{r,t}^{w} = r_{r,t}^{w} + c(\delta_{r,t} - \delta_{r,t}^{w}) + I_{r,t}^{w} + T_{r,t}^{w} + \varepsilon_{r,t}^{w} + N_{r,t}^{w}$$

$$\psi_{b,t}^{w} = r_{b,t}^{w} + c(\delta_{b,t} - \delta_{b,t}^{w}) + I_{b,t}^{w} + T_{b,t}^{w} + \varepsilon_{b,t}^{w} + N_{b,t}^{w}$$
Satellite
$$\Delta \psi_{r,t}^{w} = \psi_{r,t}^{w} - \psi_{b,t}^{w} = r_{r,t}^{w} - r_{b,t}^{w} + c(\delta_{r,t} - \delta_{r,t}^{w}) - c(\delta_{b,t} - \delta_{b,t}^{w}) + N_{r,t}^{w} - N_{b,t}^{w}$$

Assumption: GNSS receiver and the reference station are close with the same atmosphere errors







Double Difference Pseudorange Measurements

Second difference between the master satellite and the satellite *s* to remove the atmosphere errors:

Satellite s

$$\left\{ \Delta \psi_{r,t}^{s} = \psi_{r,t}^{s} - \psi_{b,t}^{s} = r_{r,t}^{s} - r_{b,t}^{s} + c(\delta_{r,t} - \delta_{r,t}^{s}) - c(\delta_{b,t} - \delta_{b,t}^{s}) + N_{r,t}^{s} - N_{b,t}^{s} \right\}$$

Assumption: GNSS receiver and the reference station are close with the same atmosphere errors

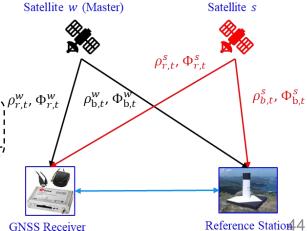
Satellite w (Master)

$$\left\{ \Delta \psi^{w}_{r,t} = \psi^{w}_{r,t} - \psi^{w}_{b,t} = r^{w}_{r,t} - r^{w}_{b,t} + c \left(\delta_{r,t} - \delta^{w}_{r,t} \right) - c \left(\delta_{b,t} - \delta^{w}_{b,t} \right) + N^{w}_{r,t} - N^{w}_{b,t} \right\}$$

DD measurements

$$\left| \Delta \nabla \psi^{s}_{r,t} = \Delta \psi^{s}_{r,t} - \Delta \psi^{w}_{r,t} = \rho^{s}_{r,t} - \rho^{s}_{b,t} - \rho^{w}_{r,t} - \rho^{w}_{b,t} + \left(N^{s}_{r,t} - N^{s}_{b,t}\right) - \left(N^{w}_{r,t} - N^{w}_{b,t}\right) \right|$$

DD Ambiguity: $\Delta \nabla N_{r,t}^s$







GNSS Real-time Kinematic Positioning: Float Solution Estimation

Estimate the float solution via weighted least

square positioning

$$\begin{bmatrix} \mathbf{p}_{r,t}^{G} \\ \Delta \nabla N_{r,t}^{1} \\ \Delta \nabla N_{r,t}^{2} \\ \dots \\ \Delta \nabla N_{r,t}^{m-1} \end{bmatrix} = \left(\mathbf{G}_{t}^{G^{T}} \mathbf{W}_{t} \mathbf{G}_{t}^{G} \right)^{-1} \mathbf{G}_{t}^{G^{T}} \mathbf{W}_{t} \begin{bmatrix} \Delta \nabla \rho_{r,t}^{2} \\ \vdots \\ \Delta \nabla \rho_{r,t}^{m-1} \\ \Delta \nabla \psi_{r,t}^{r} \\ \dots \\ \Delta \nabla \psi_{r,t}^{r} \end{bmatrix}$$

 $egin{array}{c} \Delta extstyle
ho_{r,t}^1 \ \Delta extstyle
ho_{r,t}^2 \ dots \ \Delta extstyle
ho_{r,t}^{m-1} \ \Delta extstyle \psi_{r,t}^1 \ \Delta extstyle \psi_{r,t}^{m-1} \ \end{bmatrix}$

 $\mathbf{p}_{r,t}^G$: Position of GNSS receiver

W_t: Weighting matrix

m: number of satellite

 \mathbf{G}_{t}^{G} : Observation matrix

 $\Delta \nabla \rho_{r,t}^{m-1}$: DD pseudorange measurements

 $\Delta \nabla \psi_{r,t}^{m-1}$: DD carrier-phase measurements

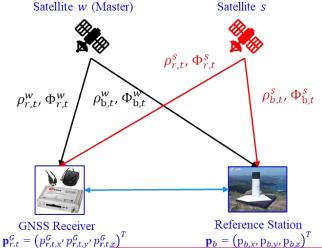
How to formulate the linear least square problem of the GNSS-RTK

Formulate it!

 $\mathbf{p}_{r,t}^G$: Float solution of position of GNSS receiver

 $\Delta \nabla N_{r,t}^1, \Delta \nabla N_{r,t}^2, \dots$: Float ambiguity

 $\left(\mathbf{G}_{t}^{G^{T}}\mathbf{W}_{t}\mathbf{G}_{t}^{G}\right)^{-1}$: Covariance matrix







GNSS Real-time Kinematic Positioning: Ambiguity Resolution

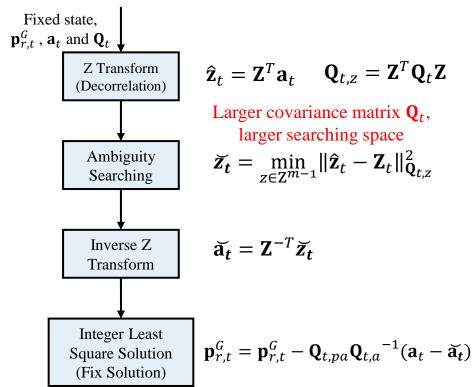
 $\mathbf{p}_{r,t}^G$: Float solution of position of GNSS receiver

 $\mathbf{a}_t = \Delta \nabla N_{r,t}^1, \Delta \nabla N_{r,t}^2, \dots$: Float ambiguity

$$\mathbf{Q}_t = \left(\mathbf{G}_t^G^T \mathbf{W}_t \mathbf{G}_t^G\right)^{-1}$$
: Covariance matrix

$$\mathbf{G}_{t}^{G} = \begin{bmatrix} \frac{p_{t,x}^{G,1} - p_{r,t,x}^{G}}{\|\mathbf{p}_{t}^{G,1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,y}^{G,1} - p_{r,t,y}^{G}}{\|\mathbf{p}_{t}^{G,1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,z}^{G,1} - p_{r,t,z}^{G}}{\|\mathbf{p}_{t}^{G,1} - \mathbf{p}_{r,t}^{G}\|} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \frac{p_{t,x}^{G,m-1} - p_{r,t,x}^{G}}{\|\mathbf{p}_{t}^{G,m-1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,y}^{G,m-1} - p_{r,t,y}^{G}}{\|\mathbf{p}_{t}^{G,m-1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,z}^{G,m-1} - p_{r,t,z}^{G}}{\|\mathbf{p}_{t}^{G,1} - \mathbf{p}_{r,t,z}^{G}\|} & 0 & \dots & 0 \\ \frac{p_{t,x}^{G,1} - p_{r,t,x}^{G}}{\|\mathbf{p}_{t}^{G,1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,y}^{G,1} - p_{r,t,y}^{G}}{\|\mathbf{p}_{t}^{G,1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,z}^{G,1} - p_{r,t,z}^{G}}{\|\mathbf{p}_{t}^{G,1} - \mathbf{p}_{r,t}^{G}\|} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \frac{p_{t,x}^{G,m-1} - p_{r,t,x}^{G}}{\|\mathbf{p}_{t,x}^{G,m-1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,y}^{G,m-1} - p_{r,t,y}^{G}}{\|\mathbf{p}_{t}^{G,m-1} - \mathbf{p}_{r,t}^{G}\|} & \frac{p_{t,z}^{G,m-1} - p_{r,t,z}^{G}}{\|\mathbf{p}_{t}^{G,m-1} - \mathbf{p}_{r,t}^{G}\|} & 0 & \dots & 1 \end{bmatrix}$$

The correct fixed solution relies on the accuracy of float solution \mathbf{p}_{rt}^{G} and the covariance matrix \mathbf{Q}_{t} .







Statistical Methods for Estimation and Optimization

Statistical method

States set

$$\mathbf{\chi} = \{\mathbf{x}_o, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$$

$$\mathbf{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_m \end{bmatrix}$$

Measurement set

$$\mathbf{Z} = \{\mathbf{z}_o, \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_k\}$$

$$\mathbf{z} = \begin{bmatrix} z_0 \\ \vdots \\ z_n \end{bmatrix}$$

Frequentist

 $P(\mathbf{z}|\mathbf{x})$

Maximum Likelihood Estimation (MLE)

$$\hat{\mathbf{x}} = \arg \max_{\mathbf{x} \in \mathbf{X}} P_{(\mathbf{Z}|\mathbf{\chi})}(\mathbf{z}|\mathbf{x})$$

event-centric estimation, fully based on data. Should be used in Big Data application.

Bayesians

Maximum a Posterior Estimation (MAP)

$$P(\mathbf{x}|\mathbf{z}) = \frac{P(\mathbf{z}|\mathbf{x})P(\mathbf{x})}{P(\mathbf{z})} \qquad \hat{\mathbf{x}} = \arg\max_{\mathbf{x} \in \mathbf{X}} P(\mathbf{z}|\mathbf{x})^{(\mathbf{x}|\mathbf{z})}$$

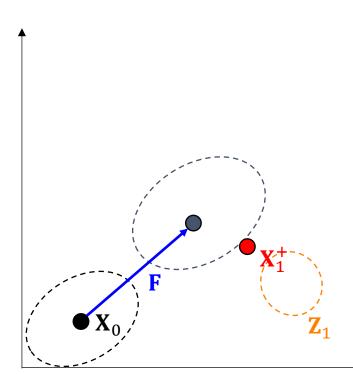
observer-centric estimation, required a knowledgeful observer (good prior-information)

Optimization, i.e., Kalman filter, factor graph optimization, etc.





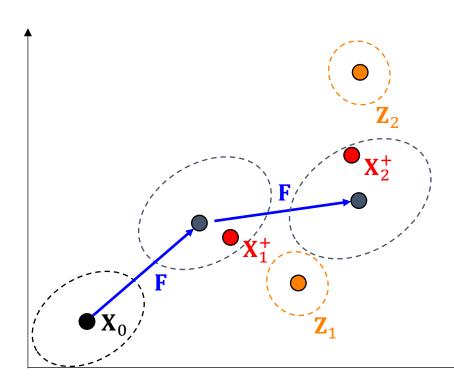
Batch (including all the data in the past) optimization







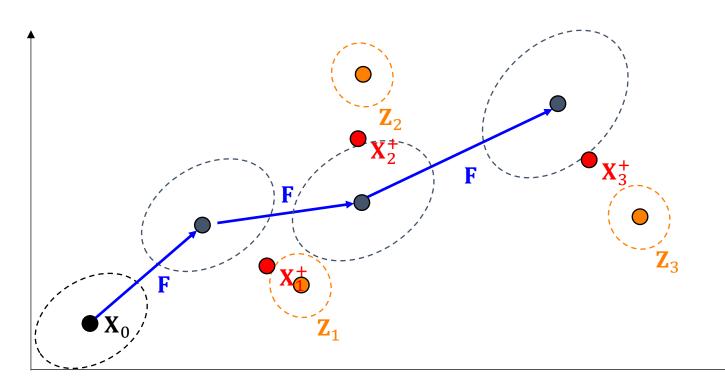
Batch (including all the data in the past) optimization





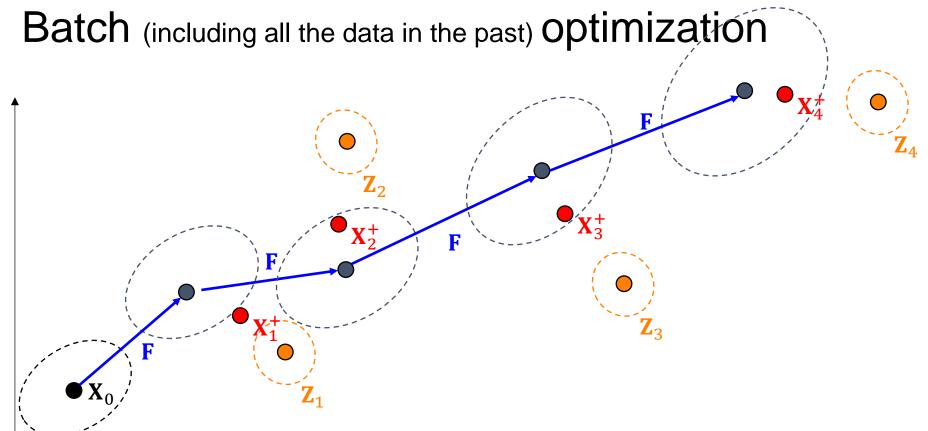


Batch (including all the data in the past) Optimization



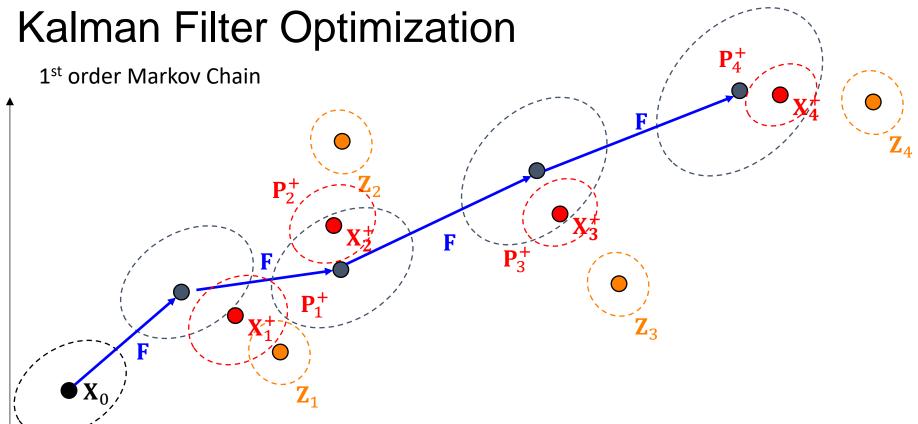








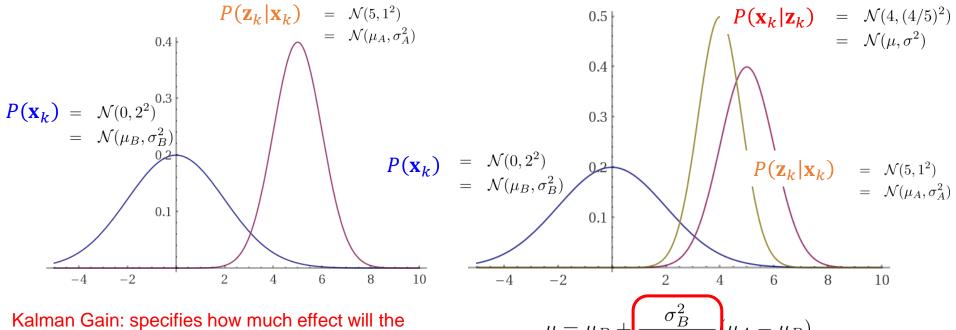








Kalman Filter with 1D state——Update step



measurement have in the posterior, compared to the prediction prior.

Which one do you trust more, your prior or your measurement?

$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2} (\mu_A - \mu_B)$$

$$\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2} \sigma_B^2$$

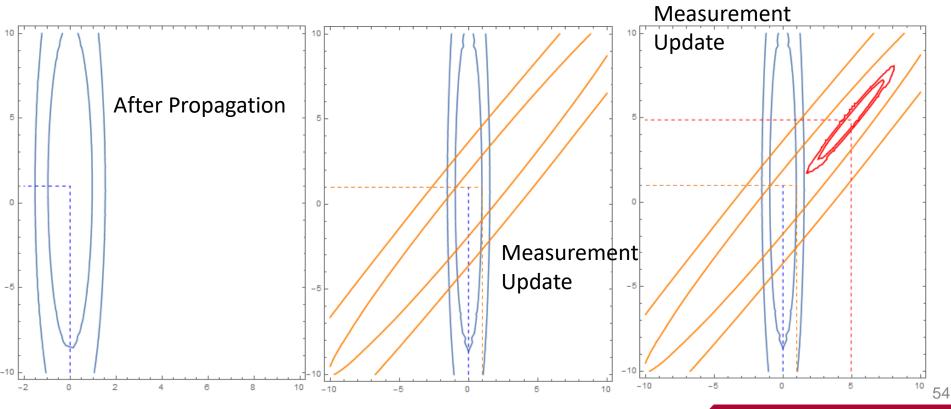
53



After











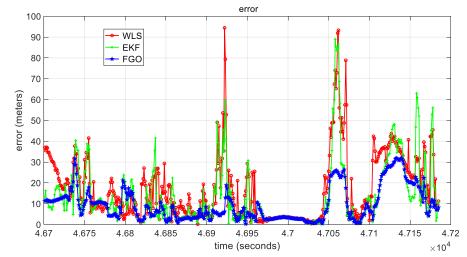
Kalman filter in GNSS

- >Example of the GNSS loosely-coupled pseudorange/Doppler integration using Kalman filter
- Example of the GNSS tightly-coupled pseudorange/Doppler integration using Kalman filter

Evaluation of GNSS Positioning

GNSS positioning performance using the three listed methods

All data	WLS	EKF	FGO
MEAN (m)	17.39	13.61	9.45
STD (m)	16.01	15.19	8.06
MAX (m)	94.43	88.97	31.94
Availability	100%	100%	100%





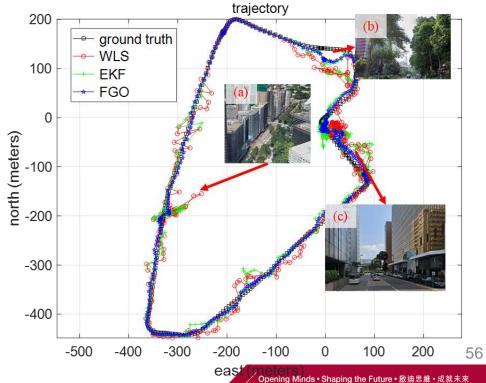
WLS*: weighted least square with pseudorange

EKF*: Pseudorange/Doppler fusion with extended

Kalman filter

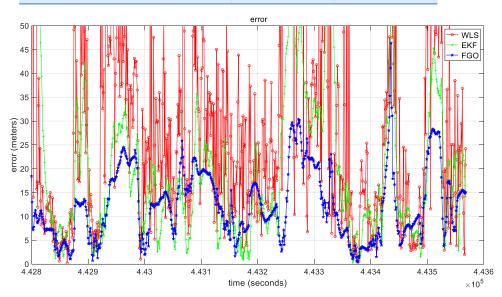
FGO*: Pseudorange/Doppler fusion with factor

graph optimization



Evaluation with Huawei P40 Pro

All data	WLS	EKF	FGO
MEAN (m)	31.98	19.84	12.541
STD (m)	38.22	15.78	7.48
MAX (m)	701.7	77.28	46.36



✓ Interdisciplinary Division of Aeronautical and Aviation Engineering 航空工程跨領域學部 THE HONG KONG
POLYTECHNIC UNIVERSITY
香港理工大學

WLS*: weighted least square with

pseudorange

Huawei P40 Pro

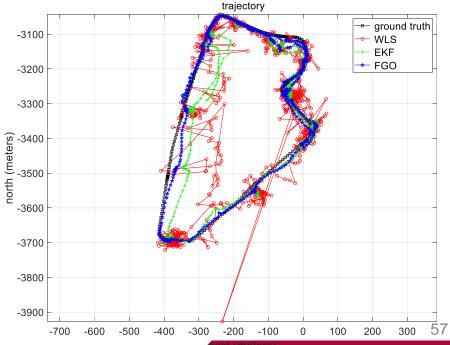
Phone

EKF*: Pseudorange/Doppler fusion

with extended Kalman filter

FGO*: Pseudorange/Doppler fusion

with factor graph optimization





Supplementary: GNSS/INS Integration Using Kalman Filtering





Inertial navigation system

 $\hat{\mathbf{a}}_t$, $\hat{\boldsymbol{\omega}}_t$ are the raw accelerometer and gyroscope measurements in the body frame \mathbf{a}_t , $\boldsymbol{\omega}_t$ are expected measurements

The cap ^ denotes the noisy measurement or estimation of a certain quantity

$$\hat{\mathbf{a}}_t = \mathbf{a}_t + \mathbf{R}_w^t \mathbf{g}^w + \mathbf{b}_{a_t} + \mathbf{n}_a \tag{1}$$

$$\widehat{\mathbf{\omega}}_t = \mathbf{\omega}_t + \mathbf{b}_{\omega_t} + \mathbf{n}_{\omega} (2)$$

$$\mathbf{n}_a \sim \mathcal{N}(0, \sigma_a^2), \mathbf{n}_\omega \sim \mathcal{N}(0, \sigma_\omega^2)$$



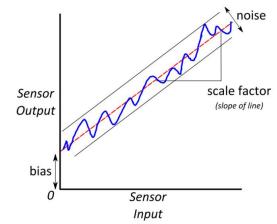






Error analysis of inertial navigation system

- > The errors of accelerometer and gyroscope can be divided into: deterministic error & random error.
- > Deterministic errors can be calibrated in advance including bias, scale...
- > Random error usually assumes that noise obeys Gaussian distribution, including Gaussian white noise, bias random walk...



Common systematic errors in IMU bias noise

scale factor





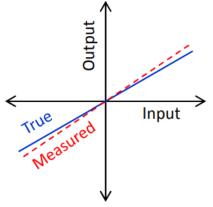
Deterministic error

(sourcing imperfectness of electrical/mechanical components)

> Bias: In theory, the output of the IMU sensor should be 0 when there is no external action. However, there is a bias b to the international data. Influence of accelerometer bias on orientation estimation:

$$\mathbf{V_{error}} = \mathbf{b_a}t, \mathbf{P_{error}} = \frac{1}{2}\mathbf{b_a}t^2$$

> Scale: The ratio between the actual value and the sensor output value.







Deterministic error (sourcing imperfectness of installation)

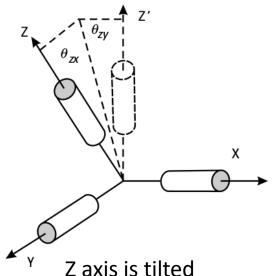
Nonorthogonality/Misalignment Errors: When manufacturing multi-axis IMU sensors, due to the manufacturing process, the xyz axis may not be vertical.

Scale + Misalignment

$$\begin{bmatrix} l_{ax} \\ l_{ay} \\ l_{az} \end{bmatrix} = \begin{bmatrix} s_{xx} & m_{xy} & m_{xz} \\ m_{yz} & s_{yy} & m_{yz} \\ m_{zx} & m_{zy} & s_{zz} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Measured Acc

True Acc







Deterministic error calibration method—Accelerometer

The six-sided method means that the three axes of the accelerometer are placed horizontally up or down for a period of time, and data on the six sides are collected to complete the calibration.

If the axes are orthogonal, it is easy to get bias and scale:

$$\mathbf{b_a} = \frac{\mathbf{l_f^{up} + l_f^{down}}}{2}$$

$$\mathbf{s_a} = \frac{\mathbf{l_f^{up} - l_f^{down}}}{2 \cdot \mathbf{g}} = \begin{bmatrix} S_{a,xx} \\ S_{a,yy} \\ S_{a,zz} \end{bmatrix}$$

$$\mathbf{l_f^{up}}$$

$$\mathbf{l_f^{dowb}}$$

 ${f l}$ is the measured value of a certain axis of the accelerometer, ${f g}$ is the local gravity acceleration





Deterministic error calibration method—Accelerometer

> When considering the inter-axis error, the relationship between the actual acceleration and the measured value is:

$$\begin{bmatrix} l_{ax} \\ l_{ay} \\ l_{az} \end{bmatrix} = \begin{bmatrix} s_{xx} & m_{xy} & m_{xz} \\ m_{yz} & s_{yy} & m_{yz} \\ m_{zx} & m_{zy} & s_{zz} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} b_{ax} \\ b_{ay} \\ b_{az} \end{bmatrix}$$

When placed horizontally and statically on 6 sides, the theoretical value of acceleration is

$$\mathbf{a_1} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}, \mathbf{a_4} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}, \mathbf{a_5} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}, \mathbf{a_6} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

> Corresponding measurement value matrix L

$$\mathbf{L} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{bmatrix}$$

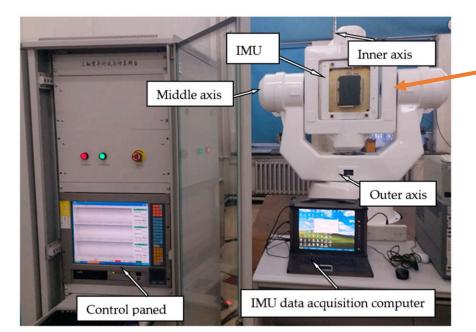
> 12 variables can be obtained by using least squares.





Deterministic error calibration method——Gyroscope

> Unlike the six-sided method of accelerometer, the true value of the gyroscope is provided by a high-precision turntable. The 6 faces in this refer to the clockwise and counterclockwise rotation of each axis



high-precision three-axis turntable

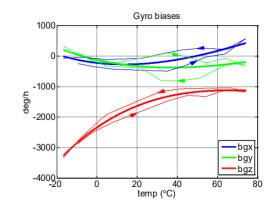


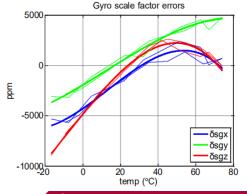


Random error – Unstableness of electrical and mechanical component due to temperature

- > We can calibrate to do temperature compensation on the bias and scale estimated by the sensor, and to obtain the values of bias and scale at different temperatures and draw them into a curve.
- Soak method: control the temperature value of the constant temperature room, and then read the sensor value for calibration.

The thin solid lines are the results under separated heating and cooling processes The thick lines are the final curve fitted result.









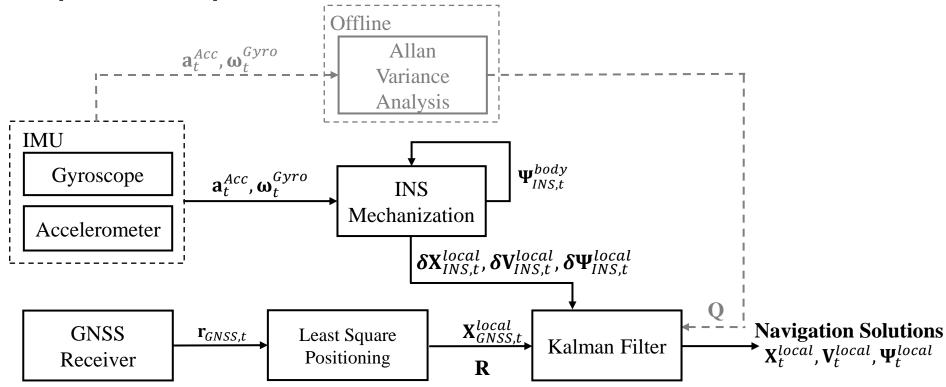
Nomenclature

- $> a_t^{Acc}$: measured 3-axis accelerations by the accelerometers at epoch t
- $> \omega_t^{Gyro}$: measured 3-axis rotation by the gyroscopes at epoch t
- $> \mathbf{X}_{INS,t}^{body}$: estimated 3-axis position in body frame by INS at epoch t
- $> V_{INS,t}^{body}$: estimated 3-axis velocity in body frame by INS at epoch t
- $> \Psi_{INS,t}^{body}$: estimated 3-axis orientation in body frame (Euler angles) by INS at epoch t
- $> \mathbf{B}_{a,t}^{body}$: estimated 3-axis biaes of accelerometers in body frame at epoch t
- $> \mathbf{B}_{\omega,t}^{body}$: estimated 3-axis biaes of gyroscopes in body frame at epoch t
- $> W_{b_a}$: estimated 3-axis random walk noise of accelerometers in body frame
- $> W_{b_a}$:estimated 3-axis random walk noise of gyroscopes in body frame





Open Loop





INS Mechanization

Sola, Joan. "Quaternion kinematics for the error-state Kalman filter." *arXiv preprint arXiv:1711.02508* (2017).

The Euler angle rates obtained by angular velocity:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \dot{\Psi}_{INS,t}^{body} = \begin{bmatrix} 1 & sin(\phi) tan(\theta) & cos(\phi) tan(\theta) \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) sec(\theta) & cos(\phi) sec(\theta) \end{bmatrix} \boldsymbol{\omega}_{t}^{Gyro} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$oldsymbol{\delta\Psi}_{INS,t}^{body} = \dot{\Psi}_{INS,t}^{body} \Delta t = egin{bmatrix} \delta\phi \ \delta\theta \ \delta\psi \end{bmatrix}$$

Rotate the Euler angles from Body to Local

$$\Psi_{INS,t}^{local} = \mathbf{R}_{body}^{local} \Psi_{INS,t}^{body}$$

Update the current Euler angles

$$\Psi_{INS,t}^{body} = \Psi_{INS,t-1}^{body} + \delta \Psi_{INS,t}^{body}$$

$$\mathbf{R}_{body}^{local} = \mathbf{R}(X, \phi) \mathbf{R}(Y, \theta) \mathbf{R}(Z, \psi)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} 9$$





INS Mechanization

To remove the gravity from the acceleration

$$\mathbf{a}_{INS,t}^{local} = \mathbf{R}_{body}^{local} \mathbf{a}_{INS,t}^{body} - \mathbf{g}$$

To obtain the change of aircraft in terms of position, velocity and orientation

$$\delta \mathbf{X}_{INS,t}^{local} = \mathbf{X}_{INS,t}^{local} - \mathbf{X}_{INS,t-1}^{local} = \mathbf{V}_{INS,t-1}^{local} \Delta t + \frac{\mathbf{a}_{INS,t}^{local} \Delta t^2}{2}$$

$$\boldsymbol{\delta V_{INS,t}^{local} = V_{INS,t}^{local} - V_{INS,t-1}^{local} = \boldsymbol{a}_{INS,t}^{local} \Delta t}$$

$$\boldsymbol{\delta\Psi}_{INS,t}^{local} = \boldsymbol{\Psi}_{INS,t}^{local} - \boldsymbol{\Psi}_{INS,t-1}^{local}$$





Kalman Filter——GNSS/INS(Open Loop)

System States:

$$\begin{aligned} \mathbf{X}_{t} &= \left(\mathbf{X}_{t}^{local}, \mathbf{V}_{t}^{local}, \mathbf{\Psi}_{t}^{local}\right) \\ \mathbf{X}_{t}^{local} &= \left(x_{t}^{local}, y_{t}^{local}, z_{t}^{local}\right) \\ \mathbf{V}_{t}^{local} &= \left(vx_{t}^{local}, vy_{t}^{local}, vz_{t}^{local}\right) \\ \mathbf{\Psi}_{t}^{local} &= \left(\phi_{roll}, \theta_{pitch}, \psi_{yaw}\right) \end{aligned}$$

Propagation model:

$$\mathbf{X}_t^- = \mathbf{F} \mathbf{X}_{t-1}^+ + \mathbf{B} \mathbf{U}_t$$

$$\mathbf{F} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{B}$$

$$\mathbf{B} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{U}_t = \begin{bmatrix} \boldsymbol{\delta} \mathbf{X}_{INS,t}^{local} \\ \boldsymbol{\delta} \mathbf{V}_{INS,t}^{local} \\ \boldsymbol{\delta} \mathbf{\Psi}_{INS,t}^{local} \end{bmatrix}$$

Measurement model:

$$\Delta \mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H} \mathbf{X}_t^-$$

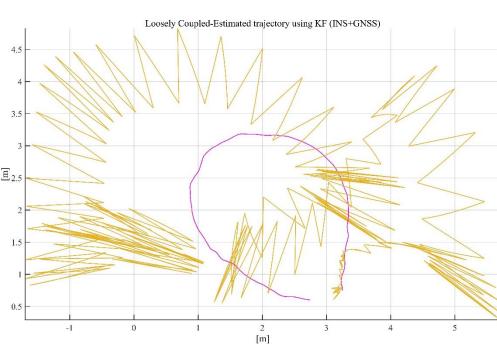
$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

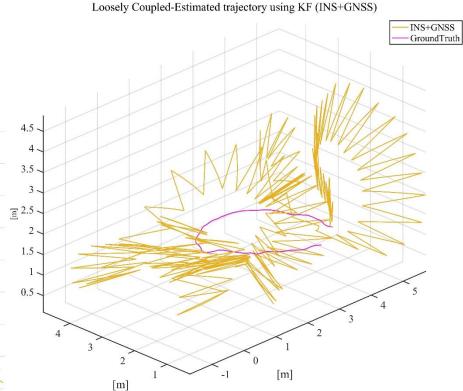
$$\mathbf{Z}_t = \mathbf{X}_{GNSS,t}^{local} = egin{bmatrix} x_{GNSS,t}^{local} \ y_{GNSS,t}^{local} \ z_{GNSS,t}^{local} \end{bmatrix}$$















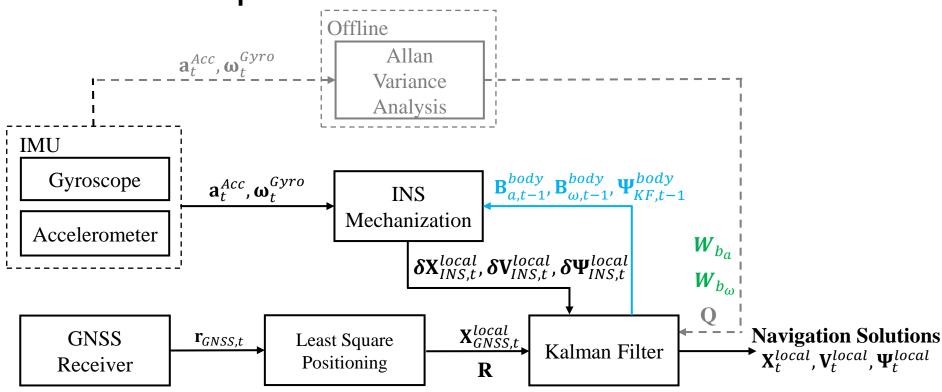
Closed-loop Correction

- > The <u>estimated position</u>, <u>velocity</u>, <u>and attitude errors</u> are fed back <u>to</u> <u>the inertial navigation processor</u>, where they are used to correct the inertial navigation solution itself.
- > any accelerometer and gyro errors estimated by the Kalman filter are fed back to correct the IMU measurements, as they are input to the inertial navigation equations.
- Unlike the position, velocity, and attitude corrections, the accelerometer and gyro corrections must be applied on <u>every</u> <u>iteration</u>





Closed Loop





INS Mechanization

The Euler angle rates obtained by angular velocity:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \dot{\Psi}_{INS,t}^{body} = \begin{bmatrix} 1 & sin(\phi) tan(\theta) & cos(\phi) tan(\theta) \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) sec(\theta) & cos(\phi) sec(\theta) \end{bmatrix} \begin{pmatrix} \mathbf{\omega}_t^{Gyro} - \mathbf{B}_{\omega,t-1}^{body} \end{pmatrix}$$

$$\boldsymbol{\delta\Psi}_{INS,t}^{body} = \dot{\boldsymbol{\Psi}}_{INS,t}^{body} \Delta t = \begin{vmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{vmatrix}$$

Rotate the Euler angles from Body to Local

$$\Psi_{INS,t}^{local} = \mathbf{R}_{body}^{local} \Psi_{INS,t}^{body}$$

Update the current Euler angles

$$\Psi_{INS,t}^{body} = \Psi_{KF,t-1}^{body} + \delta \Psi_{INS,t}^{body}$$

$$\Psi_{KF,t-1}^{local} = \mathbf{R}_{body}^{local} \Psi_{KF,t-1}^{body}$$

$$\begin{split} \mathbf{R}_{body}^{local} &= \mathbf{R}(X,\phi)\mathbf{R}(Y,\theta)\mathbf{R}(Z,\psi) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{split}$$





INS Mechanization

To remove the gravity from the acceleration

$$\mathbf{a}_{INS,t}^{local} = \mathbf{R}_{body}^{local} (\mathbf{a}_{INS,t}^{body} - \mathbf{B}_{a,t-1}^{body}) - \mathbf{g}$$

To obtain the change of aircraft in terms of position, velocity and orientation

$$\delta \mathbf{X}_{INS,t}^{body} = \mathbf{V}_{INS,t-1}^{local} \Delta t + \frac{\boldsymbol{a}_{INS,t}^{local} \Delta t^2}{2}$$

$$\delta \mathbf{V}_{INS,t}^{local} = \boldsymbol{a}_{INS,t}^{local} \Delta t$$

$$\delta \Psi_{INS,t}^{local} = \Psi_{KF,t-1}^{local} - \Psi_{INS,t-1}^{local}$$





Kalman Filter——GNSS/INS (Closed Loop)

System States:

$$\begin{split} \mathbf{X}_t &= \left(\mathbf{X}_t^{local}, \mathbf{V}_t^{local}, \mathbf{\Psi}_t^{local}, \mathbf{B}_{a,t}^{body}, \mathbf{B}_{\omega,t}^{body}\right) \\ \mathbf{X}_t^{local} &= \left(x_t^{local}, y_t^{local}, z_t^{local}\right) \\ \mathbf{V}_t^{local} &= \left(vx_t^{local}, vy_t^{local}, vz_t^{local}\right) \\ \mathbf{\Psi}_t^{local} &= \left(\phi_{roll}, \theta_{pitch}, \psi_{yaw}\right) \\ \mathbf{B}_{a,t}^{body} &= \left(b_{ax,t}^{body}, b_{ay,t}^{body}, b_{az,t}^{body}\right) \\ \mathbf{B}_{\omega,t}^{body} &= \left(b_{\omega x,t}^{body}, b_{\omega y,t}^{body}, b_{\omega z,t}^{body}\right) \end{split}$$

$$\mathbf{U}_t = egin{bmatrix} \boldsymbol{\delta \mathbf{X}_{INS,t}^{local}} \ \boldsymbol{\delta \mathbf{V}_{INS,t}^{local}} \ \boldsymbol{\delta \mathbf{Y}_{INS,t}^{local}} \ \boldsymbol{\delta \mathbf{W}_{bocal}} \ \boldsymbol{\delta \mathbf{W}_{b\omega}} \ \boldsymbol{W}_{b_{a}} \end{bmatrix}$$

Measurement model:

$$\Delta \mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H} \mathbf{X}_t^-$$

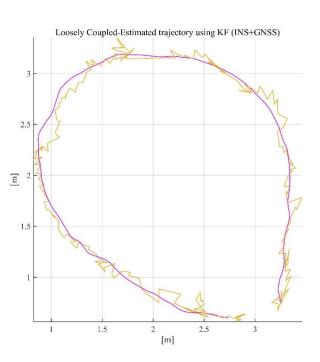
$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

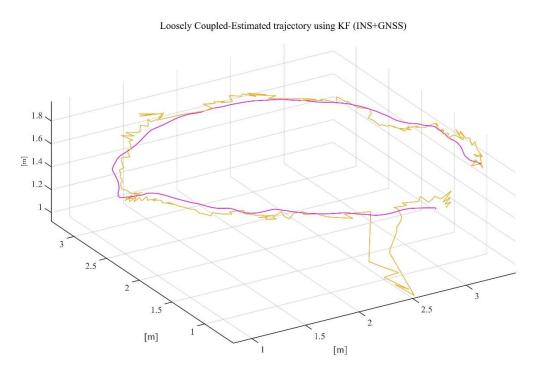
$$\mathbf{Z}_t = \mathbf{X}_{GNSS,t}^{local} = egin{bmatrix} x_{GNSS,t}^{local} \ y_{GNSS,t}^{local} \ z_{GNSS,t}^{local} \end{bmatrix}$$





Closed Loop





INS+GNSS
GroundTruth





References

>Chapters 3 and 5, Paul D. Groves, *Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems, 2nd Edition*, Artech House, 2013.