ON THE CLASSIFICATION OF NONSEPARATING SUBSETS

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Abstract. In this paper, we investigate the existence, some general properties and the classification of nonseparating subsets of finite Abelian groups A generated by two elements with exponent at most 10 such that $A/2A \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

1. Introduction

A Thurston map $f : (S^2, P_f) \to (S^2, P_f)$ with postcritical set P_f is called NET map if each critical point is simple and $|P_f| = 4$. NET maps are simple generalizations of rational Lattès maps. From Lemma 1.3 of [2], it follows that every NET map f has the property that $f^{-1}(P_f)$ contains exactly four points which are not critical points. As a consequence, NET maps lift to maps of tori as follows.

Theorem 1.1. *Let f be a Thurston map. Then f is nearly Euclidean if and only if there exist branched covering maps $p_1 : T_1 \to S^2$ and $p_2 : T_2 \to S^2$ with degree 2 from the tori T_1 and T_2 to S^2 such that the set of branch of p_2 is P_f and there exists a continuous map $\tilde{f} : T_1 \to T_2$ such that $p_2 \circ \tilde{f} = f \circ p_1$.*

For $j \in \{1, 2\}$, let $P_j \subset S^2$ be the set of branched points of p_j and let $q_j : \mathbb{R}^2 \to T_j$ be a universal covering map. The map $p_j \circ q_j : \mathbb{R}^2 \to S^2$ is a regular branched covering map whose local degree at every ramification point is 2. We can choose q_j so that Γ_j (the set of deck transformations of $p_j \circ q_j$) is generated by the set of all Euclidean rotations of order 2 about the points of Λ_j (the set of ramification points of $p_j \circ q_j$). We may normalize so that $0 \in \Lambda_j$. It follows that Λ_j is a lattice in $\mathbb{R}^2$ and that $\Gamma_j = \{x \mapsto 2\lambda \pm x : \lambda \in \Lambda_j\}$. Furthermore, we can choose q_1 so that $\tilde{f}$ lifts to the identity map. Thus, $\Lambda_1 \subseteq \Lambda_2$ and $\Gamma_1 \subseteq \Gamma_2$. So we obtain the standard commutative diagram

$$\begin{array}{ccc}
\Lambda_1 & \xrightarrow{i_c} & \Lambda_2 \\
{\scriptstyle i_c}\downarrow & & \downarrow{\scriptstyle i_c} \\
\mathbb{R}^2 & \xrightarrow{id} & \mathbb{R}^2 \\
{\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\
T_1 & \xrightarrow{\tilde{f}} & T_2 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
S^2 & \xrightarrow{f} & S^2
\end{array}$$

where $id : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map and the maps from Λ_1 to Λ_2 are inclusion maps. The standard commutative diagram implies that $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ are both identified with S^2. It is furthermore true that p_2 branches over P_f; i.e., $P_2 = P_f$.

Topological Property. Let f be a NET map. A simple closed curve in $S^2 \setminus P_f$ is *peripheral* if it can be shrunk to a point in P_f. A simple closed curve in $S^2 \setminus P_f$ is *essential* if it is neither null homotopic nor peripheral in $S^2 \setminus P_f$. We would like to know when a NET map f verifies the following topological property.

Property(Δ): Every connected component of $f^{-1}(\delta)$ is inessential for all simple closed curve δ in $S^2 \setminus P_f$.

In terms of Teichmüller theory, this property is equivalent to say that the Teichmüller map induced by the map f is constant [1]. Roughly speaking, if a NET map f verifies the property (Δ), then we may regard f as a rational map on the Riemann sphere.

Basic Group Situation. Let A be a finite abelian group. Let H be a subset of A which is the disjoint union of four pairs $\{\pm h_1\}$, $\{\pm h_2\}$, $\{\pm h_3\}$, $\{\pm h_4\}$. Let B be a subgroup such that A/B is cyclic, and let a be an element of A so that $a + B$ generates A/B. Let n be the order of A/B. For each $k \in \{1, 2, 3, 4\}$ there exists a unique integer c with $0 \leq c \leq n/2$ such that $(ca + B) \cap \{\pm h_k\} \neq \emptyset$. Let c_1, c_2, c_3, c_4 be these four integers ordered so that $0 \leq c_1 \leq c_2 \leq c_3 \leq c_4$. These four numbers are called the *coset numbers* for H relative to B and the generator $a + B$ of A/B.

We are particularly interested in the coset numbers when $A = \Lambda_2/2\Lambda_1$, where Λ_1 and Λ_2 are the lattices given in the standard commutative diagram. If λ, μ form a basis of Λ_2, then $B = \langle \lambda + 2\Lambda_1 \rangle$ is a cyclic subgroup of A so that A/B is cyclic and $\mu + 2\lambda_1$ is an element of A whose image in A/B generates A/B. Since $q_1^{-1}(p_1^{-1}(P_2)) \subseteq \Lambda_2$, we may speak of coset numbers for $H = p_1^{-1}(P_2)$. The advantage of dealing with NET maps is that given δ, an essential simple closed curve in $S^2 \setminus P_2$, it is possible to count the number of essential components in $f^{-1}(\delta)$ in terms of the coset number for $H = p_1^{-1}(P_2)$. The following result can be found in [2].

Theorem 1.2. *Let δ be an essential simple closed curve in $S^2 \setminus P_2$ with slope p/q with respect to the basis (λ_2, μ_2) of Λ_2. Let $\lambda = q\lambda_2 + p\mu_2$. Choose $\mu \in \lambda_2$ so that (λ, μ) is a basis for Λ_2. Let c_1, c_2, c_3, c_4 be the coset numbers for the elements of $p_1^{-1}(P_2)$. Then, the number of essential components in $f^{-1}(\delta)$ is $c_3 - c_2$.*

The preceding theorem was the motivation of the following definition. In the basic group situation, H is a *nonseparating* subset of A if and only if $c_2 = c_3$ for every choice of B and a. As an immediate consequence we get an algebraic formulation of the property (Δ).

Theorem 1.3. *A NET map f verifies the property (Δ) if and only if $p_1^{-1}(P_2)$ is a nonseparating subset of the group $A = \Lambda_2/2\Lambda_1$.*

In the Buff-Epstein-Koch-Pilgrim paper [1], they characterize when the induced Teichmüller map of a Thurston map is constant. Unfortunately, checking whether or not the induced Teichmüller map is constant is very difficult primarily because there are infinitely many essential curves to consider. However, in the particular case of NET maps, Theorem 1.3 provides an algebraic characterization of those NET maps whose induced maps on Teichmüller space are constant. This characterization reduces to the existence of nonseparating subsets for finite Abelian groups generated by two elements. Our work is focused on the existence and classification of nonseparating subsets and the number of Hurwitz classes of nonseparating subsets for groups with exponents at most 10.

2. Definitions and Known Results

In this section, we first review the definitions and facts on *coset numbers* and *nonseparating sets*. Then we will state several well-known facts about NET map and nonseparating set.

Let A be a finite abelian group. Let H be a subset of A which is the disjoint union of four pairs $\{\pm h_1\}$, $\{\pm h_2\}$, $\{\pm h_3\}$, $\{\pm h_4\}$. (It is possible that $h_i = -h_i$.) Let B be a subgroup such that A/B is cyclic, and let a be an element of A so that $a+B$ generates A/B. Let n be the order of A/B. For each $k \in \{1,2,3,4\}$ there exists a unique integer c with $0 \leq c \leq n/2$ such that $(ca+B) \cap \{\pm h_k\} \neq \emptyset$. Let c_1, c_2, c_3, c_4 be these four integers ordered so that $0 \leq c_1 \leq c_2 \leq c_3 \leq c_4$. These four numbers are called *coset numbers* for H relative to B and the generator $a+B$ of A/B.

Let A be a finite Abelian group. A subset H of A is called *nonseparating* if and only if it satisfies the following conditions:

- H is a disjoint union of the form $H = H_1 \coprod H_2 \coprod H_3 \coprod H_4$, where each H_i has the form $H_i = \{\pm h_i\}$. (It is possible that $h_i = -h_i$.)
- Let B be a cyclic subgroup of A such that A/B is cyclic. Let c_1, c_2, c_3, c_4 be the coset numbers for H relative to B and some generator of A/B. The main condition is that $c_2 = c_3$ for every such choice of B and generator of A/B.

The next two lemmas provide ways to produce nonseparating subsets from known ones from lower order. For details of the proofs, see Section 10 of [2].

Lemma 2.1. Let A be a finite Abelian group, and let H be a nonseparating subset of A. If $\varphi : A \to A$ is a group automorphism and h is an element of order 2 in A, then $\varphi(H) + h$ is a nonseparating subset of A.

Lemma 2.2. If A is a finite Abelian group and if A' is a subgroup of A, then every subset of A' which is nonseparating for A' is nonseparating for A.

The next lemma shows the converse of Lemma 2.2. For details of the proof, see Appendix A of [4].

Lemma 2.3. Let A be a finite Abelian group generated by two elements and let A' be a subgroup of A. If H is a subset of A' which is nonseparating for A, then H is nonseparating for A'.

Based on lemma 2.1 and lemma 2.2, given a finite Abelian group generated by two elements, we define an equivalence relation $\sim$ on the collection of nonseparating subsets of A as follows.

Definition 2.4. Let H_1, H_2 be two nonseparating subsets of A. We say that H_1 is related to H_2 and write $H_1 \sim H_2$ if and only if there exists φ an automorphism of A and an element $\tau \in A$ with $2\tau = 0$ such that $H_2 = \varphi(H_1) + \tau$. The equivalence classes under this equivalence relation will be called in this paper *Hurwitz* classes of nonseparating subsets.

Hurwitz classes of nonseparating subsets could be empty. Below is a result that supports this claim. For details of the proofs, see Section 10 of [2].

Theorem 2.5. *Let A be a finite Abelian group such that $A/2A \cong Z_2 \oplus \mathbb{Z}_2$ and $2A$ is a cyclic group of odd order. Then A does not contain a nonseparating subset.*

3. General Results

In this section we present some new results related to nonseparating subsets contained in particular classes of groups generated by two elements. We begin with a lemma. This lemma is due to professor Walter Parry.

Lemma 3.1. Let A be a finite Abelian group generated by two elements such that $A/2A \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, Let a, b, c and d be elements of A such that a and b have order 4, $2a = 2b = d - c$ and the four sets $\{\pm a\}, \{\pm b\}, \{\pm c\}, \{\pm d\}$ are distinct. Then $H = \{\pm a,\ \pm b,\ \pm c,\ \pm d\}$ is a nonseparating subset of A.

Proof. Let $e = 2a$. Since a has order 4, e has order 2. Note that $2(a - b) = 0$. Since $a \neq b$, $o(a - b) = 2$. Also, note that $e \neq a - b$, otherwise $a = -b$ which is a contradiction. So e and $a - b$ are distinct elements of A of order 2.

Let B be a cyclic subgroup of A such that A/B is cyclic. We first show that $|A/B|$ must be even. If $a + B$, $b + B$, $e + B$ were elements of order 1 in A/B, then $a - b$ and e would be two distinct elements of order 2 contained in the cyclic group B which is not possible. So the elements $a + B$, $b + B$, $e + B$ of the quotient group A/B cannot all have order 1. Since $o(a+B)|4$, $o(b+B)|4$, and $o(e+B)|2$, it follows that $|A/B| = 2n$ for some $n \in \mathbb{Z}^+$.

Now let C be the subgroup of order 2 in A/B. We analyze two cases.

Case 1: Suppose $e \in B$. Note that $2(a + B) = 2(b + B) = e + B = B$. Then $a + B$ and $b + B$ are elements of C. Since B is cyclic and e is the unique element of order 2 in B, $a - b \notin B$. Hence $C = \{a + B, b + B\}$. On the other hand, it is clear that $c + B = c + e + B = d + B$. So the coset numbers for H relative to B and any generator of A/B are $c_1 = 0$ and $c_4 = n$ for the elements of C and the value $c_2 = c_3$ corresponding to the coset of $c + B$. This completes the first case.

Case 2: Suppose $e \notin B$. In this case $e + B$ is the element of order 2 in A/B. So both $a + B$ and $b + B$ are elements of order 4 in the cyclic group A/B, hence $a + B = \pm(b + B)$. We have that 4 divides the order of A/B, and so $n = 2m$ for some integer m. It follows that both $a + B$ and $b + B$ have coset number m independent of the generator of A/B. Furthermore, if $c + B$ has coset number k, then $-d + B = -c - e + B = (e + B) - (c + B)$ has coset number $n - k$. Since k and $n - k$ are symmetric about m, we have $\{c_1, c_4\} = \{k, n - k\}$ and $c_2 = c_3 = m$. This completes the second case. $\square$

Based on the preceding lemma, Walter Parry found the following family of nonseparating subsets. Here we use his notation and terminology. Let $A = \mathbb{Z}_m \oplus \mathbb{Z}_n$, where m and n are even positive integers with $n|m$ and $m \geq 4$. Lemma 3.1 leads to 3 choices for a, b, c and d. Let m and n be even positive integers with m divisible by 4. Then we have the following 3 types of nonseparating sets:

Type 1: $m \neq 4, n = 2, a = (\frac{m}{4}, 0), b = (\frac{m}{4}, 1), c = (1, 0), d = (\frac{m}{2} + 1, 0) \in \mathbb{Z}_m \oplus \mathbb{Z}_2$.

Type 2: $n = 4$, $a = (0, 1)$, $b = (\frac{m}{2}, 1)$, $c = (1, 0)$, $d = (1, 2) \in \mathbb{Z}_m \oplus \mathbb{Z}_4$.

Type 3: $m \neq 8$ $n = 2$, $a = (\frac{m}{4}, 0)$, $b = (\frac{m}{4}, 1)$, $c = (2, 0)$, $d = (\frac{m}{2} + 2, 0) \in \mathbb{Z}_m \oplus \mathbb{Z}_2$.

In the next section we will describe our classification in terms of these types and some exceptional cases that do not arise from these types.

Another family of nonseparating subsets can be found in Section 2 of [3]. The authors point out the next result. Since no proof has been published yet, we provide a proof following the spirit of the proof of Lemma 3.1.

Lemma 3.2. Let A be a finite Abelian group generated by two elements such that $A/2A \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, Let a, b, c and d be elements of A such that a and b have order 4, $2a \neq 2b$, $c = a + b$, $d = a - b$ and the four sets $\{\pm a\}, \{\pm b\}, \{\pm c\}, \{\pm d\}$ are distinct. Then $H = \{\pm a,\ \pm b,\ \pm c,\ \pm d\}$ is a nonseparating subset of A.

Proof. Let B be a cyclic subgroup of A such that A/B is cyclic. We first show that $|A/B|$ must be even. Since a and b both have order of 4, $o(2a + B)$ and $o(2b + B)$ are divisors 2. If both $2a + B$ and $2b + B$ have order 1, then $2a \in B$ and $2b \in B$, but that is impossible because B is cyclic, $o(2a) = o(2b) = 2$ and $2a \neq 2b$. So at least one of $2a + B$, $2b + B$ has order of 2. Thus, $|A/B| = 2n$ for some $n \in \mathbb{Z}^+$.

Now let C be the subgroup of order 2 of A/B. Then $2a+B \in C$ and $2b+B \in C$ and $C = \{B, nz + B\}$ for any generator $z + B$ of A/B. Since we have shown that $2a + B$ and $2b + B$ cannot both be equal to B, that leads us to 3 cases:

Case 1: Suppose $2a + B = B$ and $2b + B = nz + B$. Since $o(2b + B) = 2$ and $o(b) = 4$, it follows that $o(b + B) = 4$. So $|A/B|$ is divisible by 4. Then $n = 2m$ for some $m \in \mathbb{Z}^+$, and we without loss of generality we may assume that the coset number for $b + B$ is m. Since A/B is cyclic and $2a + B = B$, either $a + B = B$ or $a + B = nz + B$. We analyze two subcases:

- If $a + B = B$, then the coset number for $a + B$ is 0. Also, note that $c+B = a+b+B = b+B = b-a+B = -d+B$. Hence, regardless the choice of the generator of A/B, the coset numbers are $c_1 = 0$, $c_2 = c_3 = c_4 = m$.
- If $a + B = nz + B$, then the coset number for $a + B$ is n. Also, note that $-c + B = -a + B - b + B = nz + B - mz + B = mz + B$ and that $d+B = a+B-b+B = nz+B-mz+B = mz+B$. Hence, regardless the choice of the generator of A/B, the coset numbers are $c_1 = c_2 = c_3 = m$ and $c_4 = n$.

Case 2: Suppose $2a + B = nz + B$ and $2b + B = B$. The proof of this case is symmetric to the proof of the first case.

Case 3: Suppose $2a \notin B$ and $2b \notin B$. In this case, $2a + B = nz + B = 2b + B$. Since $o(2a + B) = 2$ and $o(a) = 4$, it follows that $o(a + B) = 4$. So $|A/B| = 2m$ for some $m \in \mathbb{Z}^+$ and $a + B = \pm b + B$. Without loss of generality we may assume that the coset numbers for $a + B$ and $b + B$ are both equal to m. This yields $c + B = a + b + B = 2mz + B = nz + B$ and $d + B = a - b + B = B$. So, regardless the choice of the generator of A/B, the coset numbers are $c_1 = 0$, $c_2 = c_3 = m$ and $c_4 = n$. □

We now provide new properties of nonseparating subsets contained in a particular class of finite Abelian groups. Let $A = \mathbb{Z}_{2n} \oplus \mathbb{Z}_{2n}$, where $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, p_i prime, $p_i \geq 3$, Assume that $H = \{\pm h_1, \pm h_2, \pm h_3, \pm h_4\}$ is a nonseparating subset of A. Here $h_i = (x_i, y_i)$ where $x_i, y_i \in \mathbb{Z}_{2n}$.

Lemma 3.3. Under the above settings $h_i \neq 0$ for all i.

Proof. Draw a square grid with vertices at $(0, 0)$, $(2n, 0)$, $(0, 2n)$ and $(2n, 2n)$. We now proceed by contradiction. Suppose H contains the point $(0, 0)$. Without loss of generality we may assume that $h_1 = (0, 0)$. Choose B, a cyclic subgroup of A, so that it contains h_2 and A/B is cyclic. By a group automorphism we may assume that $B = \langle(1,0)\rangle$. Then $c_1 = c_2 = 0$ and so $c_3 = 0$. We may and do assume that $h_3 \in B$ and $0 < x_2 < x_3 \leq n$. Let $h_4 = (x_4, y_4)$. Geometrically it follows that either $x_4 = x_2$ or $x_4 = 2n - x_2$. Assume that $x_4 = x_2$. Let $B' = \langle(1,1)\rangle$. Note that $h_4 \notin B'$; otherwise, h_2 or h_3 would be an element of B', which is impossible. Now, let $a = (1,0)$ and consider the generator $a + B'$ of A/B'. The coset $0a + B'$ contains h_1 while the coset $x_2a + B'$ contains h_2, since $h_3 \notin x_2a + B'$, $-h_4$ must be in the coset $x_2a + B'$. This forces $h_4 = (x_2, 2x_2) = x_2(1,2)$. So the cyclic subgroup $B'' = \langle(1,2)\rangle$ contains h_1 and h_4. Then h_2 or h_3 must be in B'' which is impossible. Similarly, the case $x_4 = 2n - x_2$ lead us to a contradiction. $\square$

Corollary 3.4. Under the above settings, H does not contain elements of order 2.

Proof. Argue by contradiction. Without loss of generality suppose $y_1 = 0$. Translating H by an element of order 2, we may assume that $h_1 = (0,0)$. This contradicts the above lemma. $\square$

Lemma 3.5. Let be A and n be as above where n is not a multiple of 3. If B be a cyclic subgroup of A such that A/B is cyclic, then B contains at most one of the elements h_1, h_2, h_3, h_4.

Proof. Draw a square grid with vertices at $(0,0)$, $(2n,0)$, $(0,2n)$ and $(2n,2n)$ and proceed by contradiction. Suppose there is a cyclic subgroup B such that A/B is cyclic and $h_1, h_2 \in B$. Since H is nonseparating, $h_3 \in B$. By a group automorphism we may assume that $B = \langle(1,0)\rangle$. Since H has no elements of order 2, we may assume that $h_1 = (x_1, 0)$, $h_2 = (x_2, 0)$, $h_3 = (x_3, 0)$ with $0 < x_1 < x_2 < x_3 < n$. Geometrically, either $x_4 = x_2$ or $x_4 = 2n - x_2$. Assume $x_4 = x_2$. We know that cyclic groups do not contain nonseparating subsets, so $0 < y_4 < 2n$. Let $B' = \langle(1,1)\rangle$ and consider the generator $a + B'$ with $a = (1,0)$. Because H is nonseparating, $h_4 \notin B'$; otherwise, $c'_1 = 0$, $c'_2 = x_1$ and $c'_3 = x_2$. Since x_1, x_2 and x_3 are distinct coset numbers, $-h_4$ must be in the coset $x_2a + B'$. This forces $-h_4 = (2n - x_2, 2n - 2x_2)$ and so $h_4 = (x_2, 2x_2)$. Now let $B'' = \langle(-1,1)\rangle$ and consider the generator $a + B''$ with $a = (-1,0)$. Because x_1, x_2 and x_3 are distinct coset numbers we must have $c''_2 = c''_3 = x_2$. It follows that $h_4 \in x_2a + B$. This forces $h_4 = (x_2, 2n - x_2)$. Then $2n = 3x_2$ and so $3|n$. This contradicts our main assumption on n. Similarly, the case $x_4 = 2n - x_2$ lead us to a contradiction. $\square$

Corollary 3.6. Let A' be a proper subgroup of A generated by two elements. Suppose that $H = \{\pm h_1, \pm h_2, \pm h_3, \pm h_4\}$ is a nonseparating subset of A contained in A'. If B' is a cyclic subgroup of A' such that A'/B' is cyclic, then B' contains at most one of the elements h_1, h_2, h_3, h_4.

Proof. There exists a cyclic subgroup B of A such that A/B is cyclic and $A' \cap B = B'$. By Lemma 3.5, B contains at most one of the elements h_1, h_2, h_3, h_4. Since $A' \cap B = B'$, B' contains at most one of the elements h_1, h_2, h_3, h_4. $\square$

4. Classifying Nonseparating Subsets

The first author wrote a computer program to find all the nonseparating subsets of finite Abelian group A generated by 2 elements of the form $A = \mathbb{Z}_m \oplus \mathbb{Z}_n$ where n, m are even positive integers with $n|m$ and $m \geq 4$. The program outputs every nonseparating subset for $m \leq 100$. Using the data obtained by the program, we classify all nonseparating subsets contained in subgroups of exponent at most 10.

Classifying nonseparating subsets in $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Lemma 4.1. The group $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not contain any nonseparating subset.

Proof. For any cyclic subgroup B of A for which A/B is cyclic it follows that $c_1 = c_2 = 0$ and $c_3 = c_4 = 1$. □

Classifying nonseparating subsets in $A = \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Lemma 4.2. There does exist only one Hurwitz class of nonseparating subsets for $A = \mathbb{Z}_4 \oplus \mathbb{Z}_2$. A representative for this class is

$$H = \{(0,0), \pm(1,0), \pm(2,0), \pm(1,1)\}.$$

Proof. To show that $\{(0,0), \pm(1,0), \pm(2,0), \pm(1,1)\}$ is a nonseparating subset of A it suffices to apply Lemma 3.1 to $a = (1,0)$, $b = (2,0)$, $d = (2,0)$, $c = (0,0)$. We now show that there is only one Hurwitz class. Group inversion generates an equivalence relation on A whose equivalence classes are

$$\{(0,0)\} \quad \{(2,0)\} \quad \{(0,1)\} \quad \{(2,1)\} \quad \{\pm(1,0)\} \quad \{\pm(1,1)\}$$

Let H be a nonseparating subset of A. From Lemma 4.1 and Lemma 2.3 it follows that H must contain at least one element of order 4. If H contains no elements of order 2, then $H \subseteq \{(0,0), \pm(1,0), \pm(1,1)\}$ which lead us to a contradiction. So H must contain at least one element of order 2, say τ. By Lemma 2.1, $H' = H + \tau$ is a nonseparating subset of A. It follows that $(0,0) \in H'$. Also, note that H' must contain at least one element of order 4; otherwise H' would be the subgroup $\langle 2 \rangle \oplus \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ which contradicts Lemma 4.1. By a group automorphism of A we may assume that $(1,0) \in H'$. So without loss of generality we may assume that $(0,0) \in H$ and that $(1,0) \in H$. Let $B = \langle (1,0) \rangle$, it is clear that A/B is cyclic. Since $c_2 = c_3$, it follows that $(2,0) \in H'$. Now let $B' = \langle (0,1) \rangle$ and consider the generator of A/B' as $(1,0) + B'$. Since $c_2' = c_3'$, it follows that $(1,1) \in H$. This completes the proof. □

Classifying nonseparating subsets in $A = \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Lemma 4.3. There are exactly three Hurwitz classes of nonseparating subsets for $A = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. The following subsets are representatives for each Hurwitz class:

$$H_1 = \{(0,0); \pm(1,0); \pm(1,2); (2,0)\}, \; H_2 = \{\pm(1,0), \pm(0,1), \pm(2,1), \pm(1,2)\},$$

$$H_3 = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(3,1)\}.$$

Proof. Note that $H_1 \subseteq \mathbb{Z}_4 \oplus \langle 2 \rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and that H_1 is a nonseparating subset of $\mathbb{Z}_4 \oplus \langle 2 \rangle$. By Lemma 2.2 it follows that H_1 is a nonseparating subset of A. To show that H_2 is a nonseparating subset of A it suffices to apply Lemma 3.1 to $a = (1,2)$, $b = (1,0)$, $d = (2,1)$, $c = (0,1)$. To show that H_3 is a nonseparating subset of A it suffices to apply Lemma 3.2 to $a = (1,0)$, $b = (0,1)$, $d = (3,1)$, $c = (1,1)$. Moreover, H_1 is a nonseparating subset of type 3 while H_2 is a nonseparating subset of type 2.

We now show that there are exactly three Hurwitz classes of nonseparating subsets. Group inversion generates an equivalence relation on A whose equivalence classes are

$$\{(0,0)\} \quad \{(2,0)\} \quad \{\pm(1,0)\} \quad \{\pm(1,1)\} \quad \{\pm(1,2)\}$$

$$\{(0,2)\} \quad \{(2,2)\} \quad \{\pm(0,1)\} \quad \{\pm(2,1)\} \quad \{\pm(1,3)\}$$

Let H be a nonseparating subset of A. The subset H must contain at least one element of order 4; otherwise H would be the subset $\langle 2\rangle \oplus \langle 2\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We first show that if H contains at least one element of order 2, then H is in the equivalence class of $H_1 = \{(0,0); \pm(1,0); \pm(2,0); (1,2)\}$. Let b an element of order 2 in H, then $H' = H + b$ is a nonseparating set containing $(0,0)$ and an element of order 4. By a group automorphism on A, we may assume that H' contains $\pm(1,0)$. So without loss of generality we may assume that H contains $(0,0)$ and $\pm(1,0)$. Let $B = \langle(1,0)\rangle$, one verifies that $(2,0) \in H$. Now take $B' = \langle(0,1)\rangle$ and consider the generator $(1,0) + B'$ of A/B'. Then exactly one of the following must be contained in H: $\{\pm(1,1)\}$, $\{\pm(1,2)\}$, $\{\pm(1,3)\}$. Now let $B'' = \langle(1,2)\rangle$. Since H is a nonseparating subset, any generator for A/B'' lead us to $c_1 = c_2 = c_3 = 0$. Thus, $\pm(1,2) \in H$.

We show that if H contains no elements of order 2, then H is either in the Hurwitz class of H_2 or in the Hurwitz class of H_3. First of all, notice that H does not contain $(0,0)$; otherwise, it would be in the equivalence class of H_1. So H contains exactly four of the following sets

$$\{\pm(1,0)\} \quad \{\pm(0,1)\} \quad \{\pm(1,1)\} \quad \{\pm(1,2)\} \quad \{\pm(2,1)\} \quad \{\pm(1,3)\}$$

Then H contains a basis of A. By a group automorphism of A, we may assume that H contains $\pm(1,0)$ and $\pm(0,1)$, so H is in the Hurwitz class of at least one of the following subsets

$$J_1 = \{\pm(1,0), \pm(0,1), \pm(1,2), \pm(2,1)\} \qquad J_2 = \{\pm(1,0), \pm(0,1), \pm(1,2), \pm(1,1)\}$$

$$J_3 = \{\pm(1,0), \pm(0,1), \pm(1,2), \pm(3,1)\} \qquad J_4 = \{\pm(1,0), \pm(0,1), \pm(2,1), \pm(1,1)\}$$

$$J_5 = \{\pm(1,0), \pm(0,1), \pm(2,1), \pm(3,1)\} \qquad J_6 = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(3,1)\}.$$

In this paragraph we reduce the number of representatives. The automorphism $\varphi(x,y) = (y,x)$ takes J_2 into J_4 and the affine mapping $\phi(x,y) = (x+y,y) + (2,0)$ takes J_4 into J_6. The automorphism $f(x,y) = (y,x)$ takes J_3 into J_5 and $J_5 + (2,0) = J_4$. This reduces to two representatives: J_1 and J_6. It remains to show that J_1 and J_6 are not in the same Hurwitz class. To do so, let $L = (a_{ij})$ be a 2×2 matrix with integer coefficients that induces an automorphism on A and let $b = (b_1, b_2) \in A$ such that $2b = 0$. Then,

$$L\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \equiv \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} \pmod 2 \qquad L\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \equiv \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} \pmod 2$$

$$L\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \equiv \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} \pmod 2 \qquad L\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \equiv \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} \pmod 2$$

On the other hand, the elements of J_6 verifies the following

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pmod 2 \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pmod 2$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} \pmod 2 \qquad \begin{bmatrix} 3 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} \pmod 2$$

Since $\{(1,0);(0,1);(1,1)\}$ and $\{(m_{11},m_{21});(m_{21},m_{22})\}$ have different cardinalities, we conclude that J_6 cannot be an element of the Hurwitz class of J_1. This completes the proof. The computer program also verifies the result. □

Classifying nonseparating Sets in $A = \mathbb{Z}_6 \oplus \mathbb{Z}_6$.

We use the isomorphism $T : \mathbb{Z}_6 \oplus \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ defined by $T(x,y) = (x,y,x,y)$ to identify $A = \mathbb{Z}_6 \oplus \mathbb{Z}_6$ with the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. By abuse of notation we denote $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ by A and assume that $H = \{\pm h_1, \pm h_2, \pm h_3, \pm h_4\}$ is a nonseparating subset of A. Here $h_i = (a_i, b_i)$ where $a_i \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $b_i \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Corollary 3.4 tells us that any nonseparating subset of A contains no elements of order 2. So $\{\pm b_1, \pm b_2, \pm b_3, \pm b_4\}$ is a subset of $(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \setminus \{(0,0)\}$.

Theorem 4.4. $\{\pm b_1, \pm b_2, \pm b_3, \pm b_4\} = (\mathbb{Z}_3 \oplus \mathbb{Z}_3) \setminus \{(0,0)\}$.

Proof. Argue by contradiction. Suppose $b_1 = b_2$. Translating H by an element of order 2, we may assume that $h_1 = (0, b_1)$. Since $b_1 = b_2$ and $a_1 = 0$, there exists a cyclic subgroup $B = \langle \alpha \rangle \oplus \langle b_1 \rangle$ of order 6 containing h_1 and h_2. Then $c_1 = c_2 = c_3 = 0$ and so $h_3 \in B$ or $h_4 \in B$. Without loss of generality $h_3 \in B$, whence $h_3 = (a_3, b_1)$. Since $h_1 \neq \pm h_3$ it follows that $a_3 \neq 0$. So the subgroup B is generated by $h_3 = (a_3, b_1)$. This forces $h_2 = (0, -b_1)$ or $h_2 = (a_3, 0)$. Since $h_2 = (a_2, b_1)$ and $o(b_1) = 3$ we finally reach a contradiction. □

As an immediate consequence any nonseparating set of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ has the form $H = \{\pm(a_1,(1,0)), \pm(a_2,(0,1)), \pm(a_3,(2,1)), \pm(a_4,(2,2))\}$. Since the subset $(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \setminus \{(0,0)\}$ is invariant under automorphisms of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, the Hurwitz class of a nonseparating subset H of A depends only on the choice of the elements a_1, a_2, a_3, a_4. We now classify the nonseparating subsets of $A = \mathbb{Z}_6 \oplus \mathbb{Z}_6$.

Case 1 All the elements of the set $\{a_1, a_2, a_3, a_4\}$ are the same. Translating by an element of order two we obtain the set

$$H_1 = \{\pm(0,0,1,0), \pm(0,0,0,1), \pm(0,0,2,1), \pm(0,0,2,2)\}.$$

Note that $T^{-1}(H_1) = \{\pm(0,2), \pm(2,0), \pm(2,4), \pm(2,2)\}$, which is actually a nonseparating subset of $A = \mathbb{Z}_6 \oplus \mathbb{Z}_6$ (see Example 10.4 of [2]).

Case 2 All the elements of the set $\{a_1, a_2, a_3, a_4\}$ are distinct. Without loss of generality, we may assume that $a_1 = (1,0)$, $a_2 = (0,1)$, $a_3 = (1,1)$ and $a_4 = (0,0)$. So we obtain

$$H_2 = \{\pm(1,0,1,0), \pm(0,1,0,1), \pm(1,1,2,1), \pm(0,0,2,2)\}.$$

Note that $T^{-1}(H_2) = \{\pm(1,0), \pm(0,1), \pm(5,1), \pm(2,2)\}$, which is actually a nonseparating subset of A. This was verified with the assistance of the computer program (see Appendix A).

Case 3 Two pair of elements of $\{a_1, a_2, a_3, a_4\}$ are the same but not all the elements are the same. Without loss of generality, we may assume that $a_1 = a_3 = (1,0)$ and $a_2 = a_4 = (0,1)$. So we obtain

$$H_3 = \{\pm(1,0,1,0), \pm(0,1,0,1), \pm(1,0,2,1), \pm(0,1,2,2)\}.$$

Note that $T^{-1}(H_3) = \{\pm(1,0), \pm(0,1), \pm(5,4), \pm(2,5)\}$, which is actually a nonseparating subset of A. This was verified with the assistance of the computer program (see Appendix A).

Case 4 Exactly two elements of $\{a_1, a_2, a_3, a_4\}$ are the same and the other two are distinct. Without loss of generality, we may assume that $a_1 = a_2 = 0$, $a_3 = (1,0)$ and $a_4 = (0,1)$. So we obtain

$$H_4 = \{\pm(0,0,1,0), \pm(0,0,0,1), \pm(1,0,2,1), \pm(0,1,2,2)\}.$$

Note that $T^{-1}(H_4) = \{\pm(2,0); \pm(0,2); \pm(5,4); \pm(2,5)\}$. It is not difficult to show that $T^{-1}(H_4)$ is not a nonseparating subset. To do so, consider the cyclic subgroup $B = \langle(0,1)\rangle$ of $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ and the generator $(1,0) + B$ of A/B. It follows that $c_1 = 0$, $c_2 = 1$, $c_3 = c_4 = 2$. So H_4 is not a nonseparating subset. Therefore, there does not exist a nonseparating subset in this case.

Case 5 Exactly three elements of $\{a_1, a_2, a_3, a_4\}$ are the same. Without loss of generality, we may assume that $a_1 = a_2 = a_3 = 0$ and $a_4 = (1,0)$. So we obtain

$$H_5 = \{\pm(0,0,1,0), \pm(0,0,0,1), \pm(0,0,2,1), \pm(1,0,2,2)\}.$$

Note that $T^{-1}(H_5) = \{\pm(2,0); \pm(0,2); \pm(4,2); \pm(1,4)\}$. It is not difficult to show that $T^{-1}(H_5)$ is not a nonseparating subset. To do so, consider the cyclic subgroup $B = \langle(0,1)\rangle$ of $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ and the generator $(1,0) + B$ of A/B. It follows that $c_1 = 0$, $c_2 = 1$, $c_3 = c_4 = 2$. So H_5 is not a nonseparating subset. Therefore, there does not exist a nonseparating subset in this case.

Classifying nonseparating subsets in $A = \mathbb{Z}_8 \oplus \mathbb{Z}_2$.

With the assistance of the computer program we obtained the following information. Plus/minus are omitted in the table below.

Hurwitz class representative	number of elements in the Hurwitz class
$\{(0,0), (2,0), (4,0), (2,1)\}$	2
$\{(1,0), (2,0), (3,0), (2,1)\}$	2

Note that the subset $H_1 = \{(0,0), \pm(2,0), (4,0), \pm(2,1)\}$ is a nonseparating subset of the subgroup $A' = \langle 2 \rangle \oplus \mathbb{Z}_2$ which is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. On the other hand, $H_2 = \{\pm(0,0), \pm(2,0), \pm(2,1), \pm(4,0)\}$ is a nonseparating subset of Type 1.

Classifying nonseparating subsets in $A = \mathbb{Z}_8 \oplus \mathbb{Z}_4$.

In this case, the order of the group A is 32 and there are several subgroups B for which A/B is cyclic, so it is convenient to use the computer program designed by the first author. The program outputs one representative of for each Hurtwiz class.

Almost all of these representatives correspond to either type 1, type 2 or type 3; however, there is one special subset that does not correspond to any of these types, we call this nonseparating subset exceptional. We provide an alternative proof for this exceptional case. There are exactly 41 nonseparating subsets in $\mathbb{Z}_8 \oplus \mathbb{Z}_4$ and 9 distinct Hurwitz classes. Plus/minus are omitted in the table below.

Hurwitz class representative	cardinality of Hurwitz class	Type
$\{(0,0),(2,0),(4,0),(2,2)\}$	2	3
$\{(0,0),(0,1),(4,1),(0,2)\}$	4	3
$\{(1,0),(2,0),(3,0),(2,2)\}$	4	1
$\{(1,1),(2,1),(3,1),(6,1)\}$	8	2
$\{(1,0),(1,1),(1,2),(3,1)\}$	8	Exceptional
$\{(2,0),(0,1),(2,1),(4,1)\}$	8	Lemma 3.2
$\{(2,0),(0,1),(6,1),(2,2)\}$	4	Lemma 3.2
$\{(2,0),(0,1),(4,1),(2,2)\}$	2	2
$\{(0,1),(2,1),(4,1),(6,1)\}$	1	2

We now provide a topological-algebraic proof of the nonseparability the exceptional subset $\{\pm(1,0);\pm(1,1);\pm(1,2);\pm(3,1)\}$. We begin with a basic claim.

Claim 4.5. If B is a cyclic subgroup of $A = \mathbb{Z}_8 \oplus \mathbb{Z}_4$ such that $|B| = 4$ and A/B is cyclic, then $(4,0) \notin B$.

Proof. Let $z + B$ be a generator of A/B. It follows that the order of z is a multiple of 8. Since the maximum order of an element in A is 8, the order of $z = (a,b)$ must be 8. This forces a to be odd. Note that $4z = 4(a,b) = (4a,0) = (4,0)$. If $(4,0)$ were an element of B, we would have $4(z+B) = B$. So $z+B$ would not be a generator of A/B which lead us to a contradiction. Therefore $(4,0) \notin B$. $\square$

Theorem 4.6. *The set $H = \{\pm(1,0);\pm(1,1);\pm(1,2);\pm(3,1)\}$ is a nonseparating subset of the group $A = \mathbb{Z}_8 \oplus \mathbb{Z}_4$.*

Proof. Consider the lattices $\Lambda_2 = \mathbb{Z}^2$ and $\Lambda_1 = \langle (4,0),(0,2)\rangle$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined by $\Phi(x,y) = (4x, 2y)$. It is clear that $\Phi(\Lambda_2) = \Lambda_1$ so the mapping Φ induces a NET map $g : \mathbb{R}^2/\Gamma_1 \to \mathbb{R}^2/\Gamma_1$, where Γ_1 is the group of isometries on $\mathbb{R}^2$ generated by rotations 180° about elements of Λ_1. We identify the quotient space $\mathbb{R}^2/\Gamma_1$ with S^2. Bracket notation indicates a point in S^2; i.e., $[u,v]$ represents a point in S^2. It is not difficult to see the critical values of g are $\mathbf{a} = [0,0]$, $\mathbf{b} = [4,0]$, $\mathbf{c} = [4,2]$ and $\mathbf{d} = [0,2]$. It is furthermore true that each critical point of g is a simple. Below is part of the critical portrait of g.

$$\begin{tikzcd}
{[1,2],[1,0],[3,0],[3,2]} \arrow[r, "{2,2,2,2}"] & \mathbf{b} \arrow[dr] & \\
{[1,1],[3,1],[5,1],[7,1]} \arrow[r, "{2,2,2,2}"] & \mathbf{c} \arrow[r] & \mathbf{a} \arrow[loop right] \\
{[0,1],[2,1],[4,1],[6,1]} \arrow[r, "{2,2,2,2}"] & \mathbf{d} \arrow[ur] &
\end{tikzcd}$$

Note that $g([2,0]) = g([2,2]) = \mathbf{a}$ and that $[2,0],[2,2]$ are critical points of g. Also, note that $g = g_1 \circ g_2$ where $g_1 : S^2 \to S^2$ and $g_2 : S^2 \to S^2$ are the branched covering mappings induced by the linear mappings $L_1(x,y) = (2x, 2y)$ and $L_2(x,y) = (2x, y)$ respectively. Since $P_g = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and g_1 induce the identity mapping on Teichmüller space, we can easily use a core arc argument. Let γ be an arc in connecting the points $\mathbf{a}$ and $\mathbf{c}$ and suppose that $\gamma \setminus \{\mathbf{a}, \mathbf{c}\} \subset S^2 \setminus P_g$. The preimage of γ under g has two connected components say γ_1, γ_2. Exactly one of these components contains the point $[1,1]$, the other component contains the point $[3,1]$ while the points $[1,0]$ and $[1,2]$ are not contained in the disjoint union of γ_1, γ_2. Similarly, let δ be an arc connecting the points $\mathbf{a}$ and $\mathbf{b}$ and suppose that $\delta \setminus \{\mathbf{a}, \mathbf{b}\} \subset S^2 \setminus P_g$. The preimage of δ under g has two connected components say δ_1, δ_2. Exactly one of these components contains the point $[1,2]$, the other component contains the point $[1,0]$ while the points $[1,1]$ and $[1,3]$ are not contained in the disjoint union of δ_1, δ_2. Note that $\deg(g : \gamma_i \to \gamma) = \deg(g : \delta_i \to \delta) = 4$. By Theorem 4.1 of [2] it follows that $c_2 = c_3$ for any cyclic subgroup B of A with $|B| = 8$ such that A/B is cyclic.

Now let B be a cyclic subgroup of A with $|B| = 4$ such that A/B is cyclic. We show that $c_2 = c_3$ regardless the choice of the generator of A/B. Let $h_1 = (1,0)$, $h_2 = (1,1)$, $h_3 = (1,2)$, and $h_4 = (3,1)$. It is clear that $o(h_i + B) = 8$ for all i. So given any generator $z + B$ of A/B, we have $\pm h_i + B \in \{z + B, 3z + B\}$ for all i. We now go by cases.

Case 1: Suppose $(1,1) + B = (3,1) + B$. Then $(2,0) \in B$ and so $(4,0) \in B$. This contradicts claim 4.5.

Case 2: Suppose $(1,1) + B = -(3,1) + B$. Then $(4,2) \in B$. If $(1,2) + B = (1,0) + B$, then $(0,2) \in B$. Hence $(4,0) \in B$ which contradicts claim 4.5. If $(1,2) + B = -(1,0) + B$, then $(2,2) \in B$ and again $(4,0) \in B$ which contradicts claim 4.5. So in this case, $(1,2) + B \neq (1,0) + B$ and $(1,2) + B \neq -(1,0) + B$. Thus $c_2 = c_3$.

Case 3: Suppose $(1,1) + B = (1,0) + B$. Then $(0,1) \in B$, so $(1,2) + B = (1,1) + B$. This shows that $(1,1)$,$(1,0)$, and $(1,2)$ are in the same coset. Therefore $c_2 = c_3$.

Case 4: Suppose $(1,1) + B = -(1,0) + B$. Then $(2,1) \in B$, so $(5,3) + B = (1,1) + B$. This shows that $(1,1)$,$-(1,0)$, and $-(3,1)$ are in the same coset. Therefore $c_2 = c_3$.

Case 5: Suppose $(1,1) + B = (1,2) + B$. Then $(0,1) \in B$, so $(1,0) + B = (1,1) + B$. This shows $(1,1)$,$(1,0)$, and $(1,2)$ are in the same coset. Therefore $c_2 = c_3$.

Case 6: Suppose $(1,1) + B = -(1,2) + B$. Then $(2,3) \in B$, which implies that $(2,1) \in B$. So $(5,3) + B = (1,1) + B$. This shows that $(1,1)$,$-(1,2)$, and $-(3,1)$ are in the same coset. Therefore $c_2 = c_3$.

This completes the proof. □

5. Summary

The table below shows the findings of our work. The groups $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_{10} \oplus \mathbb{Z}_2$ are of the form $\mathbb{Z}_{2p} \oplus \mathbb{Z}_2$ where p is a prime number, so by Theorem 2.5 these groups do not contain nonseparating subsets. On the other hand, Dr. Edgar Saenz proved that groups of the form $\mathbb{Z}_{2p} \oplus \mathbb{Z}_{2p}$ -where p is a prime number- cannot contain nonseparating subsets, for further details see Chapter 3 of [4]. Hence, $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$ does not contain any nonseparating subset.

Group	Hurwitz class representative	cardinality of the class	Type
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	NONE	0	-
$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\{(1,0),(0,1),(1,1),(2,1)\}$	2	3
$\mathbb{Z}_6 \oplus \mathbb{Z}_2$	NONE	0	-
$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	$\{(0,0),(1,0),(1,2),(2,0)\}$	6	3
$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	$\{(0,1),(1,0),(1,2),(2,1)\}$	3	2
$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	$\{(1,0),(0,1),(1,1),(3,1)\}$	12	Lemma 3.2
$\mathbb{Z}_6 \oplus \mathbb{Z}_6$	$\{(0,2),(1,0),(1,1),(2,1)\}$	24	Exceptional
$\mathbb{Z}_6 \oplus \mathbb{Z}_6$	$\{(0,1),(1,2),(1,4),(2,3)\}$	36	Exceptional
$\mathbb{Z}_6 \oplus \mathbb{Z}_6$	$\{(0,1),(2,1),(2,3),(2,5)\}$	4	Exceptional
$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\{(2,0),(0,1),(2,1),(4,1)\}$	2	3
$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\{(1,0),(2,0),(3,0),(2,1)\}$	2	1
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(0,0),(2,0),(4,0),(2,2)\}$	2	3
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(0,0),(0,1),(4,1),(0,2)\}$	4	3
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(1,0),(2,0),(3,0),(2,2)\}$	4	1
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(1,1),(2,1),(3,1),(6,1)\}$	8	2
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(1,0),(1,1),(1,2),(3,1)\}$	8	Exceptional
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(2,0),(0,1),(2,1),(4,1)\}$	8	Lemma 3.2
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(2,0),(0,1),(6,1),(2,2)\}$	4	Lemma 3.2
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(2,0),(0,1),(4,1),(2,2)\}$	2	2
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\{(0,1),(2,1),(4,1),(6,1)\}$	1	2
$\mathbb{Z}_{10} \oplus \mathbb{Z}_2$	NONE	0	-
$\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	NONE	0	-

The interested reader can verify the above information in [5].

Appendix A. Exceptional nonseparating subsets of $A = \mathbb{Z}_6 \oplus \mathbb{Z}_6$

We use the computer program to verify that the sets $H_2 = \{\pm(1,0), \pm(0,1), \pm(5,1), \pm(2,2)\}$ and $H_3 = \{\pm(1,0), \pm(0,1), \pm(5,4), \pm(2,5)\}$ are nonseparating subsets of $\mathbb{Z}_6 \oplus \mathbb{Z}_6$.

The following data corresponds to $H_2 = \{\pm(1,0), \pm(0,1), \pm(5,1), \pm(2,2)\}$:

B	generator of A/B	coset numbers
$\langle(0,1)\rangle$	$(1,4)+B$	$0,1,1,2$
$\langle(0,1)\rangle$	$(5,4)+B$	$0,1,1,2$
$\langle(1,0)\rangle$	$(4,1)+B$	$0,1,1,2$
$\langle(1,0)\rangle$	$(1,5)+B$	$0,1,1,2$
$\langle(1,1)\rangle$	$(0,1)+B$	$0,1,1,2$
$\langle(1,1)\rangle$	$(5,4)+B$	$0,1,1,2$
$\langle(1,2)\rangle$	$(5,5)+B$	$1,2,2,3$
$\langle(1,2)\rangle$	$(4,1)+B$	$1,2,2,3$
$\langle(1,3)\rangle$	$(1,4)+B$	$1,2,2,3$
$\langle(1,3)\rangle$	$(4,5)+B$	$1,2,2,3$
$\langle(1,4)\rangle$	$(1,5)+B$	$0,1,1,2$
$\langle(1,4)\rangle$	$(4,3)+B$	$0,1,1,2$

B	generator of A/B	coset numbers
$\langle(1,5)\rangle$	$(0,1)+B$	$0,1,1,2$
$\langle(1,5)\rangle$	$(1,4)+B$	$0,1,1,2$
$\langle(2,1)\rangle$	$(5,5)+B$	$1,2,2,3$
$\langle(2,1)\rangle$	$(1,4)+B$	$1,2,2,3$
$\langle(2,3)\rangle$	$(1,4)+B$	$1,2,2,3$
$\langle(2,3)\rangle$	$(1,5)+B$	$1,2,2,3$
$\langle(2,5)\rangle$	$(5,4)+B$	$0,1,1,2$
$\langle(2,5)\rangle$	$(1,5)+B$	$0,1,1,2$
$\langle(3,1)\rangle$	$(1,4)+B$	$1,2,2,3$
$\langle(3,1)\rangle$	$(5,4)+B$	$1,2,2,3$
$\langle(3,2)\rangle$	$(1,5)+B$	$1,2,2,3$
$\langle(3,2)\rangle$	$(5,5)+B$	$1,2,2,3$

The following data corresponds to $H_3 = \{\pm(1,0), \pm(0,1), \pm(5,4), \pm(2,5)\}$:

B	generator of A/B	coset numbers
$\langle(0,1)\rangle$	$(1,4)+B$	$0,1,1,2$
$\langle(0,1)\rangle$	$(5,4)+B$	$0,1,1,2$
$\langle(1,0)\rangle$	$(4,1)+B$	$0,1,1,2$
$\langle(1,0)\rangle$	$(1,5)+B$	$0,1,1,2$
$\langle(1,1)\rangle$	$(0,1)+B$	$1,1,1,3$
$\langle(1,1)\rangle$	$(5,4)+B$	$1,1,1,3$
$\langle(1,2)\rangle$	$(5,5)+B$	$0,1,1,2$
$\langle(1,2)\rangle$	$(4,1)+B$	$0,1,1,2$
$\langle(1,3)\rangle$	$(1,4)+B$	$1,1,1,3$
$\langle(1,3)\rangle$	$(4,5)+B$	$1,1,1,3$
$\langle(1,4)\rangle$	$(1,5)+B$	$1,2,2,3$
$\langle(1,4)\rangle$	$(4,3)+B$	$1,2,2,3$

B	generator of A/B	coset numbers
$\langle(1,5)\rangle$	$(0,1)+B$	$1,1,1,3$
$\langle(1,5)\rangle$	$(1,4)+B$	$1,1,1,3$
$\langle(2,1)\rangle$	$(5,5)+B$	$1,2,2,3$
$\langle(2,1)\rangle$	$(1,4)+B$	$1,2,2,3$
$\langle(2,3)\rangle$	$(1,4)+B$	$1,2,2,3$
$\langle(2,3)\rangle$	$(1,5)+B$	$1,2,2,3$
$\langle(2,5)\rangle$	$(5,4)+B$	$0,1,1,2$
$\langle(2,5)\rangle$	$(1,5)+B$	$0,1,1,2$
$\langle(3,1)\rangle$	$(1,4)+B$	$1,1,1,3$
$\langle(3,1)\rangle$	$(5,4)+B$	$1,1,1,3$
$\langle(3,2)\rangle$	$(1,5)+B$	$1,2,2,3$
$\langle(3,2)\rangle$	$(5,5)+B$	$1,2,2,3$

List of Symbols.

$\mathbb{Z}^+$ is the set of positive integers.
S^2 topological 2-sphere.
$\mathbb{Z}^2$ is the 2-dimensional integer lattice.
P_f is postcritical set of the Thurston map f.
$|A|$ is the order of the group A.
$o(g)$ is the order of the element g.

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