# Bayesian Hierarchical Models with Applications in Meta-analysis and Large-scale Survey

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# Bayesian Hierarchical Models

- The overarching theme of the following projects is the use of Bayesian hierarchical models (Multilevel models, or in Frequentist parlance, Empirical Bayes models, Random-effects/Mixed-effects models).
- In a nutshell, Bayesian hierarchical models regularize the fit of data with group structure through "partial pooling" across groups and provide more stable estimates.

# Overview

1. Hierarchical Non-parametric Models for Individual-level Meta-analysis
2. Accurate Prediction Using Highly Nonrepresentative Online Survey Data via MRP Correction
3. Use of Cross-validation in Comparing Hierarchical Models

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   - A Causal Framework for Meta-analysis
   - An Application to Vioxx Trials
   - A Bayesian Approach
   - Non-parametric Models for Causal Inference
2. Accurate Prediction Using Highly Nonrepresentative Online Survey Data via MRP Correction
   - Introduction
   - MRP
   - Xbox Data
   - Methods
3. Use of Cross-validation in Comparing Hierarchical Models
   - Model Comparison in A Decision Theoretic Framework
   - Data and Model Descriptions
   - Results

# Meta-analysis

- Meta-analyses synthesize evidence from multiple studies.
- Traditionally, meta-analysis researchers go through thorough literature reviews and data extractions.
- Increasingly, researchers begin to have access to original data from a suite of studies and individual-level meta-analysis becomes feasible.

# Potential Outcomes and Causal Inference

- Potential outcome framework is the standard tool for causal inference (Neyman, 1923; Rubin 1977).
- The key insight is to consider the individual's responses subject to all possible treatment assignments $\vec{Y} = (Y(1), Y(2), \ldots, Y(L))$.
- Consider the case of binary treatment, we define causal estimands such as unit causal effect $Y_i(1) - Y_i(0)$, population average causal effect $E(Y_i(1) - Y_i(0))$ and conditional average causal effect $E(Y_i(1) - Y_i(0)|X)$.

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- However, meta-analysis researchers rarely take on the potential outcome framework even though most of meta-analysis target at causal conclusions.

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- Heterogeneity of effects across studies attracts most methodological research.
- Random-effects model (DerSimonion and Laird 1986) is the most popular choice, which gives that effect sizes $\hat{\theta}_i$ from different studies a prior distribution.

$$\hat{\theta}_i = \theta_i + \varepsilon_i, \theta_i \sim \Psi(\cdot)$$

# An Extended Potential Outcome for Meta-analysis

- Sobel et al. (2013) lays out an extended potential outcome framework for meta-analysis.
- The extended potential outcome for meta-analysis is to consider all possible outcomes that individual $i$ can experience under every combination of study membership $s \in \{1, \ldots, G\}$ and treatment assignment $z \in \{1, \ldots, L\}$,

$$\vec{Y}_i = \begin{pmatrix} Y_i(1,1) & Y_i(1,2) & \cdots & Y_i(1,L) \\ Y_i(2,1) & Y_i(2,2) & \cdots & Y_i(2,L) \\ \vdots & \vdots & \ddots & \vdots \\ Y_i(G,1) & Y_i(G,2) & \cdots & Y_i(G,L) \end{pmatrix}.$$

# Consistency Assumptions

A1 Extended stable unit treatment value assumption:
$\vec{Y}_i(\vec{s}, \vec{z}) = Y_i(s_i, z_i)$ for all possible $s$ and $z$.

A2 Sampling assumption:

$$Y_i|s, z, x \overset{i.i.d.}{\sim} F(y|s, z, x).$$

A3 Response consistency:

$$p(y(s, z)|S = s'', Z = z, X = x) = p(y(s', z)|S = s'', Z = z, X = x), \forall s, s'$$

# Selection Assumptions

A6 Strongly ignorable treatment assignment:

$$\{Y(s,z) : s = 1, \ldots, G, z = 1, \ldots, L\} \perp\!\!\!\perp Z \mid S = s, X_1 = x_1$$

and

$0 < P(Z = z \mid S = s, X_1 = x_1) < 1$, for seen combinations of $s$ and $z$.

A7 Ignorable study selection:

$$\{Y(s,z) : s = 1, \ldots, G, z = 1, \ldots, L\} \perp\!\!\!\perp S \mid X_2$$

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- Extended potential outcome framework provides insights into the possible sources of heterogeneity that could be introduced into meta-analysis.
- First, the violation of the response consistency assumption (A3).
- Second, the violation of the no study selection assumption (A7).

# Two Sources of Heterogeneity Cont'd

- These two sources are inherently different and it might not be meaningful to use a model that blends these two.
- Traditional methods for meta-analysis make no distinction between these two sources.

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- Traditional methods for meta-analysis make no distinction between these two sources.
- As common in causal inference, assessing the validity of a particularly assumption is very important for practical modeling choice and analysis.

# Two Sources of Heterogeneity Cont'd

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- If (A3) is reasonable and (A7) is unlikely to hold, then the heterogeneity is the result of differential selection into studies. In this case, random-effects models are questionable.

# Vioxx Example

- In Sobel et al. (2013), an illustrative example of Vioxx is used to demonstrate the use of the potential outcome framework.
- Vioxx is a COX-2 selective, non-steroidal anti-inflammatory drug (NSAID) that was approved by the FDA for the relief of signs and symptoms of osteoarthritis, the management of acute pain in adults, and the treatment of menstrual symptoms.
- It was withdrawn from the market several years later as clinical trials began to reveal severe negative impact on cardiovascular system.

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- We conducted a meta-analysis of 29 Vioxx trials finished before the withdrawal of Vioxx. In particular, we separate different dosages in our analysis.
- In this context, we deem assumption (A3), the response consistency assumption, as reasonable; this means that any heterogeneity will be the results of differential selection into studies.
- This leads to the use of study indicators and fixed-effects models rather than random-effects models. The meta-analysis confirms the negative impact of Vioxx on cardiovascular system, and differentiation of dosage leads to the finding of dosage-response relation that was ignored in previous Vioxx studies.

# A Bayesian Approach

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- In this case, we would like to build a formal Hierarchical Bayesian framework.

# Hierarchical Bayesian Framework

- The group structure of the extended potential outcome naturally lends itself to Hierarchical Bayesian Framework. For one individual under one treatment assignment, a total number of $G$ outcomes could have been experienced. In machine learning literature, this is known as multi-task learning problem.

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- The key insight is that parietal pooling across studies should help us improve our estimates.
- A major problem is that the proportion of missingness is very high.

# Non-parametric Methods in Causal Inference

- Causal inference in observational studies often involves fitting two models: a model for treatment assignment given variables pertinent to the selection process and a model for the potential outcomes given treatment and confounding variables. Rubin (2005) refers to the former as the model for selection and the latter as the model for science.

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- Mostly, simple parametric models, e.g., linear least square regression, are used to fit the selection model and potential outcome model. Considerations for missepcification leads to the popular doubly robust method (Scharfstein, Rotnitzky and Robins 1999).

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- Mostly, simple parametric models, e.g., linear least square regression, are used to fit the selection model and potential outcome model. Considerations for missepcification leads to the popular doubly robust method (Scharfstein, Rotnitzky and Robins 1999).
- However, there have been a growing interest in developing flexible non-parametric models for the potential outcomes. Hill (2011) uses Bayesian Additive Regression Tree (BART) to model the potential outcomes without fitting the selection models.

# Gaussian Processes

- Gaussian Process is another highly-flexible and also computationally attractive non-parametric model.
- A random function $f$ follows a Gaussian Process if its arbitrary finite distribution follows a multivariate Gaussian distribution, i.e., $f(x_1, x_2, \ldots, x_n) \sim N(\mu(\vec{x}), K(\vec{x}, \vec{x})), \forall n$, where $\kappa$ is the kernel that describes the underlying Gaussian process and $K(\vec{x}, \vec{x}) = (\kappa(x_i, x_j)))_{n \times n}$ is the Gram matrix.
- Gaussian Process is also used to model the mean function of a regression $y = f(x) + \varepsilon$, while the observational noise $\varepsilon$ is often given a normal distribution $N(0, \sigma^2)$.

# Incorporating Group Structure into Gaussian Processes

- In the context of meta-analysis, each study has its own mean function $f_s$, which is a function of the treatment $z$ and confounders $x$, that follows a Gaussian process with kernel $\kappa_s$.

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- In the context of meta-analysis, each study has its own mean function $f_s$, which is a function of the treatment $z$ and confounders $x$, that follows a Gaussian process with kernel $\kappa_s$.
- Bonilla (2008) proposes to decompose $\kappa_s = \tilde{\kappa} \cdot \check{\kappa}$, in which $\tilde{\kappa}$ describes "inter-task similarities" based on task-level characteristics, and $\check{\kappa}$ describes the similarity resulting from individual characteristics.

# Hierarchical Gaussian Processes

- Another approach is to give the $\kappa_s$'s a prior distribution. This is discussed in Yu et al. (2004) and Schwaighofter et al. (2005).
- Restricting the infinite dimensional kernel function $\kappa_s$ on the $N$ individual observed in the data as the Gram matrix $K^s$, we can assign a standard Wishart prior with a base kernel $\kappa_0$, whose Gram matrix is $K_o$

$$Q(K_s) \sim \text{InvW}(\tau, K_0), s = 1, \ldots, G$$

# Future work

- Develop Hierarchical Gaussian Process models, specifically with the high "missingness" of the potential outcome framework in mind.

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- Develop Hierarchical Gaussian Process models, specifically with the high "missingness" of the potential outcome framework in mind.
- Gaussian process is natural for continuous responses, while there are other rapid development in Bayesian nonparametric models for data types such as count data and time-to-event data.

# References


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# Introduction

- National Survey data is another example that Bayesian Hierarchical Models could be a vital tool.
- Typically, national survey data is cross-tabulated by categorical geographical-demographic variables. (It is a common practice to discretize continuous variables such as age and income.) This creates a complex group structure.

# Introduction

- National Survey data is another example that Bayesian Hierarchical Models could be a vital tool.
- Typically, national survey data is cross-tabulated by categorical geographical-demographic variables. (It is a common practice to discretize continuous variables such as age and income.) This creates a complex group structure.
- Multilevel Regression and Poststratification (MRP) takes advantage of this structure and is popular in small-area estimation at state level and sub-state level (Park et al., 2004; Lax and Phillips, 2009; Ghitza and Gelman, 2013 ).
- We apply MRP technique to a highly-biased online survey data set to construct accurate state-level prediction for 2012 Presidential Election.

# Multilevel Regression and Poststratification

- The core idea of MRP is to partition the data set into small cells, use the sample to estimate the proportion in each cells via multilevel regression, and then aggregate the cell-level proportion to the population-level proportion by reweighting each cell according to the population information.

$$\hat{y}^{\text{PS}} = \frac{\sum_{j=1}^{J} N_j \hat{y}_j}{\sum_{j=1}^{J} N_j}$$

- However, there is a trade-off between how fine we want to partition the data and the sample sizes within the small cells. The multilevel regression facilitates "borrowing strength" across demographically/geographically similar cells, and thus producing more stable cell-level estimates.

# Description of the Xbox Data Set

- An opt-in poll was placed on Xbox gaming platform daily in the period leading up to the 2012 US Presidential Election.
- Users' daily voting intent as well as their demographic information was collected.
- In total, 750,148 interviews were conducted with 345,858 unique respondents—over 30,000 of whom completed five or more polls—making this one of the largest ever election panel studies.

# But Highly Nonrepresentative...

- However, the survey is highly biased in gender and age. 93% respondents are male and 65% are 18-to-29-year-olds. The raw result is drawn below

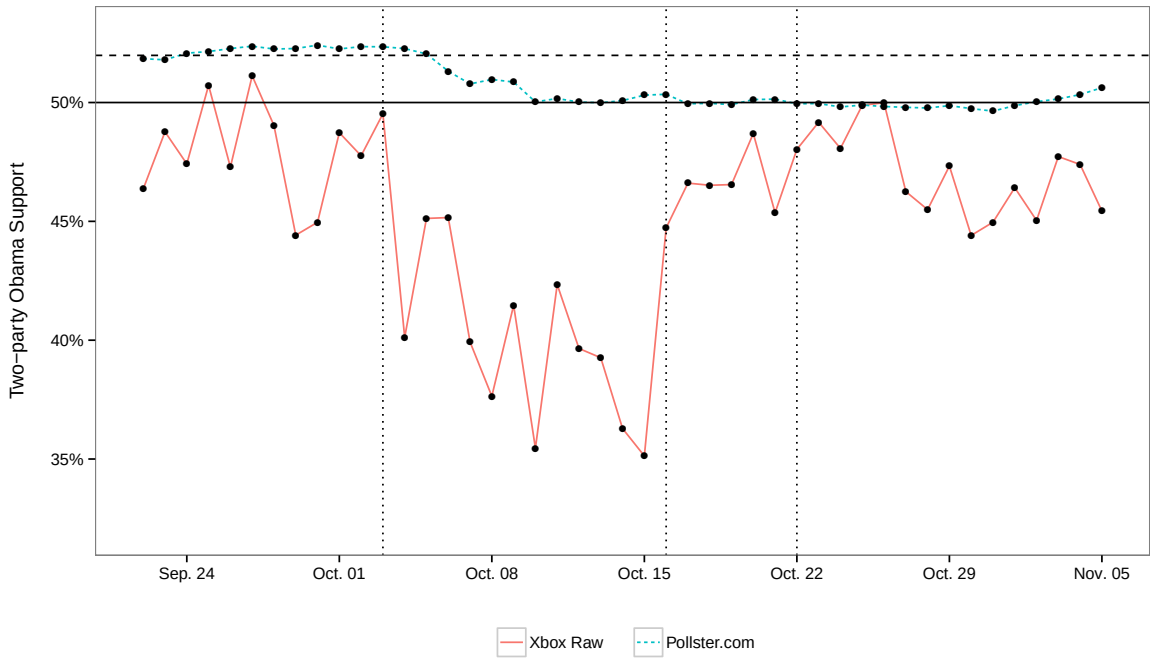


Figure: Raw results of the Xbox data.

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- However, we will show that with sophisticated statistical adjustments, nonrepresentative survey can still yield accurate information about the population in question, and thus a solid alternative to traditional probabilistic sampling.
- It should be noted that representative sampling suffers heavily from nonresponse and collects much smaller sample size due to high expense.

# Multilevel Model for Voters Intent

- For each day in the 45 days period leading up to the presidential election, we fit two multilevel models, one with the support for a major party candidate as the outcome, and another with the support for Democratic candidate Barrack Obama given the voter supporting a major party candidate as the outcome,

$$\begin{aligned}\Pr(Y_i \in \{\text{Obama, Romney}\}) = \\ \text{logit}^{-1}\big(&\alpha_0 + \alpha_1(\text{state last vote share}) + a^{\text{state}}_{j[i]} + a^{\text{edu}}_{j[i]} \\ &+ a^{\text{sex}}_{j[i]} + a^{\text{age}}_{j[i]} + a^{\text{race}}_{j[i]} + a^{\text{party ID}}_{j[i]} + b^{\text{ideology}}_{j[i]} + b^{\text{last vote}}_{j[i]}\big)\end{aligned}$$

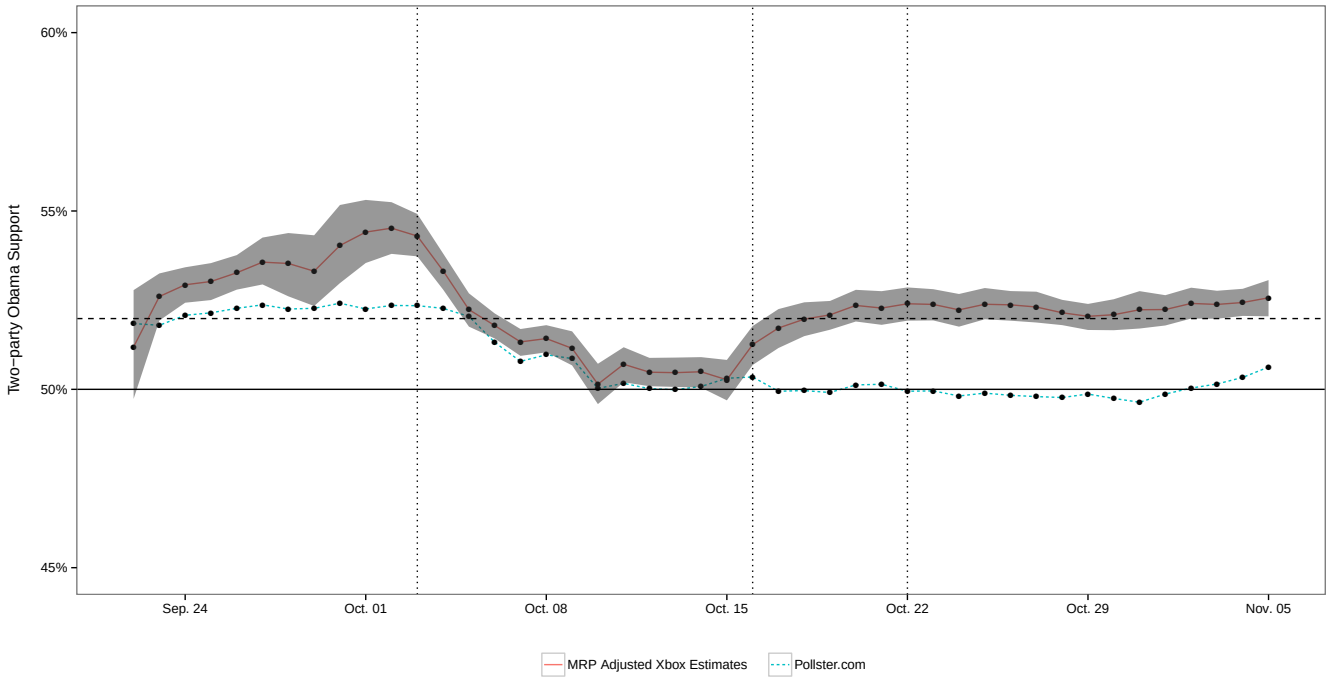
$$\begin{aligned}\Pr(Y_i = \text{Obama} \mid Y_i \in \{\text{Obama, Romney}\}) = \\ \text{logit}^{-1}\big(&\beta_0 + \beta_1(\text{state last vote share}) + b^{\text{state}}_{j[i]} + b^{\text{edu}}_{j[i]} + b^{\text{sex}}_{j[i]} \\ &+ b^{\text{age}}_{j[i]} + b^{\text{race}}_{j[i]} + b^{\text{party ID}}_{j[i]} + b^{\text{ideology}}_{j[i]} + b^{\text{last vote}}_{j[i]}\big)\end{aligned}$$

# Poststratification Population

- Another key ingredient in MRP is an appropriate poststratification population with cross-tabulation information.
- Typical choice in public opinion analysis is the Current Population Survey (CPS). However, since we are concerned about the electorate rather than the general population, we use the exit poll from the 2008 presidential election as our poststratification population.
- Admittedly, this choice ignores the demographic shift in the intervening years. A more principled choice is to combine 2008 exit poll with CPS.

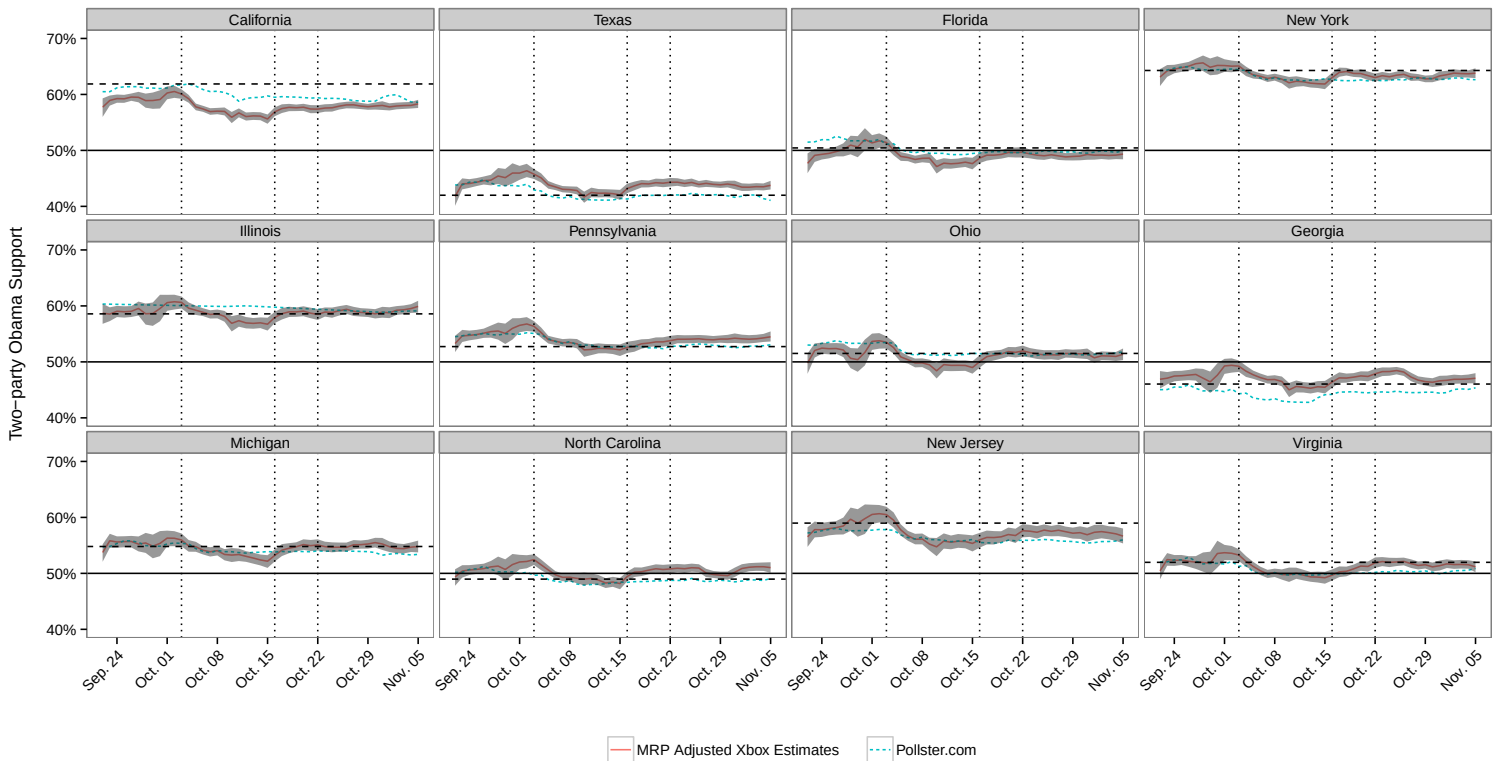
# Results of MRP adjustments

- MRP adjustment produces a daily snapshot of the voters' intent before the 2012 presidential election. The national voters' intent of Obama's two-party support is plotted below



# State-level Results of MRP adjustments

- Furthermore, due to the flexibility of MRP, we can break down the daily snapshots by states. The results from 12 largest states are plotted below



# Demographic Subgroups Results of MRP adjustments

- Comparison between the 2012 exit poll estimates and MRP last day snapshots for demographic subgroups and two-way interacted demographic subgroups are also shown

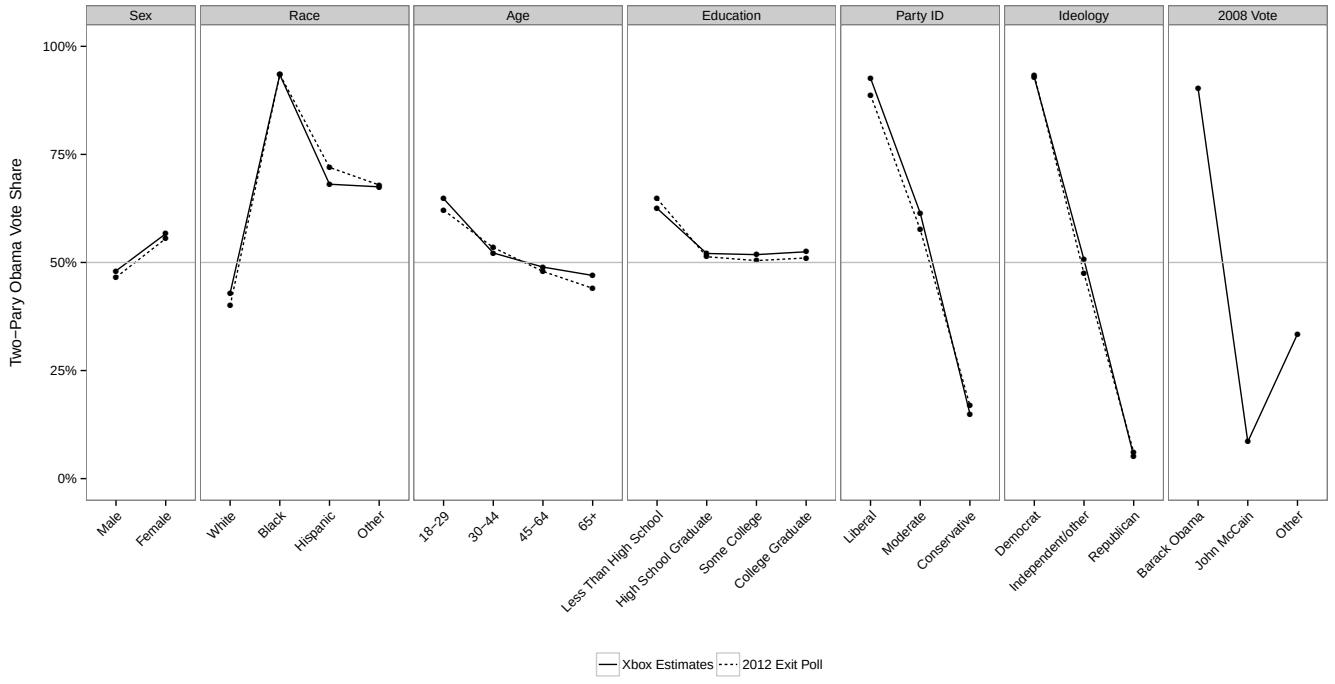



Figure: Demographic subgroups two-party Obama support.

# Demographic Subgroups Results Cont'd

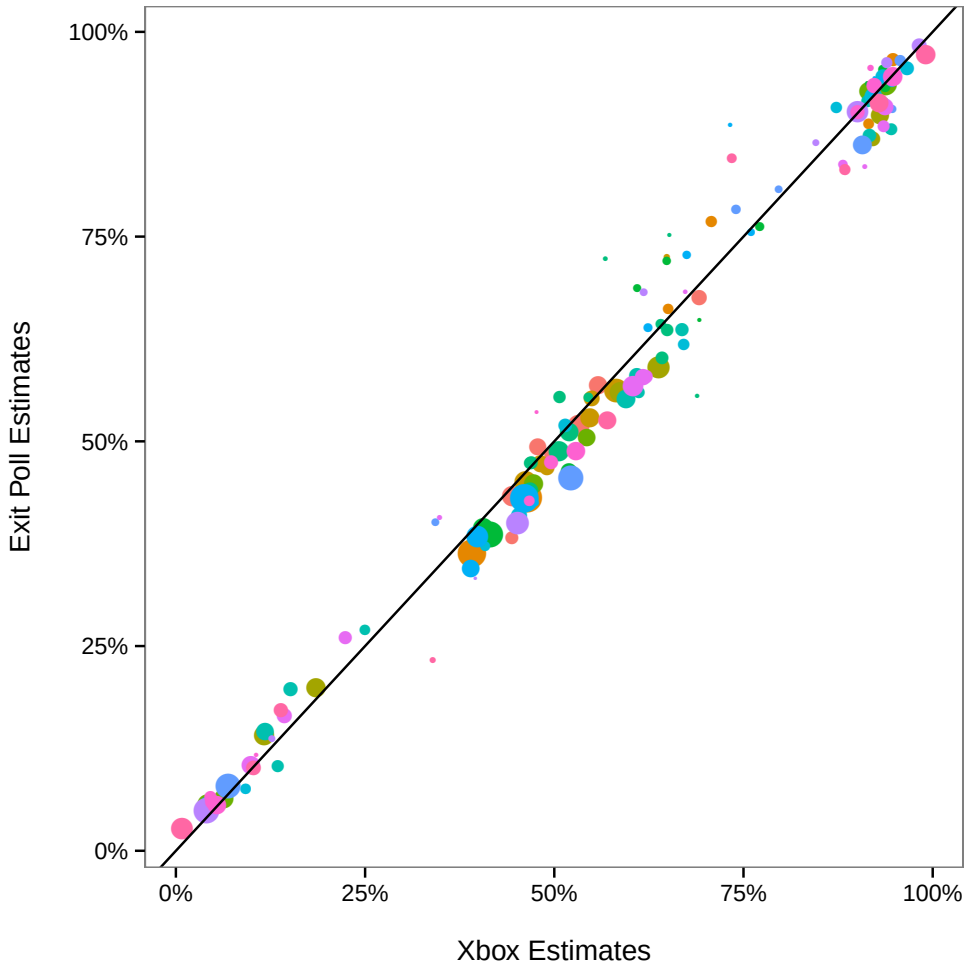


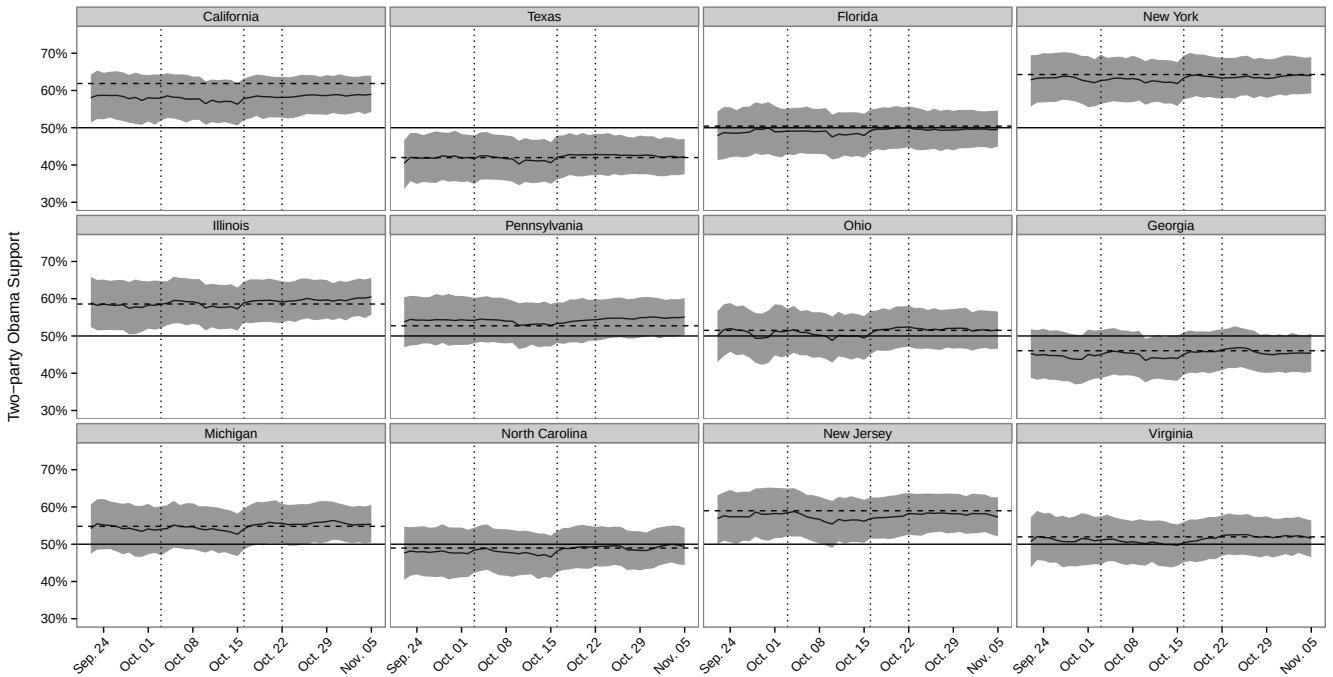

Figure: Two-way interaction demographic subgroups two-party Obama support.

# From Daily Snapshots to Election Day Prediction

- One final step is required to convert the daily snapshots to election day prediction.
- We follow the approach of Erikson and Wlezien (2008). A regression is fit on historical top-line data and actual election outcomes (from 2000, 2004 and 2008 elections) and projection is made with the MRP adjusted snapshots for 2012.

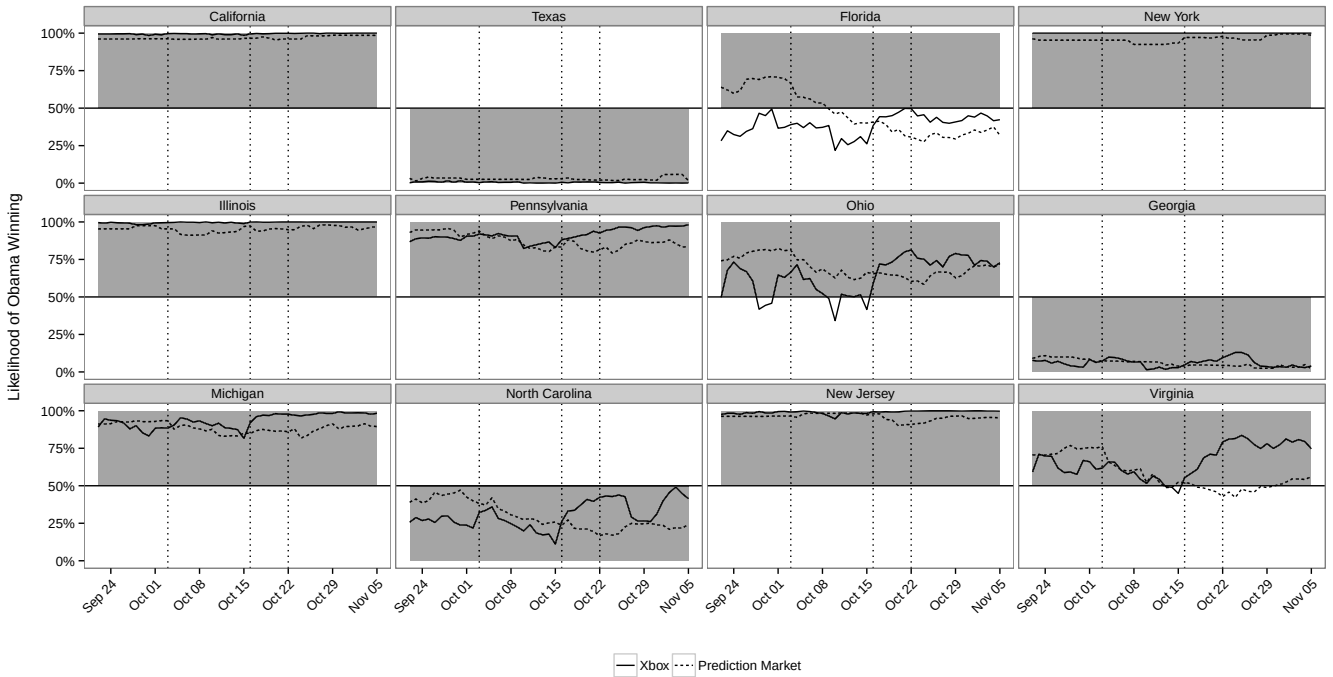
# Results of Prediction

- The daily predictions of the 12 largest states are plotted below



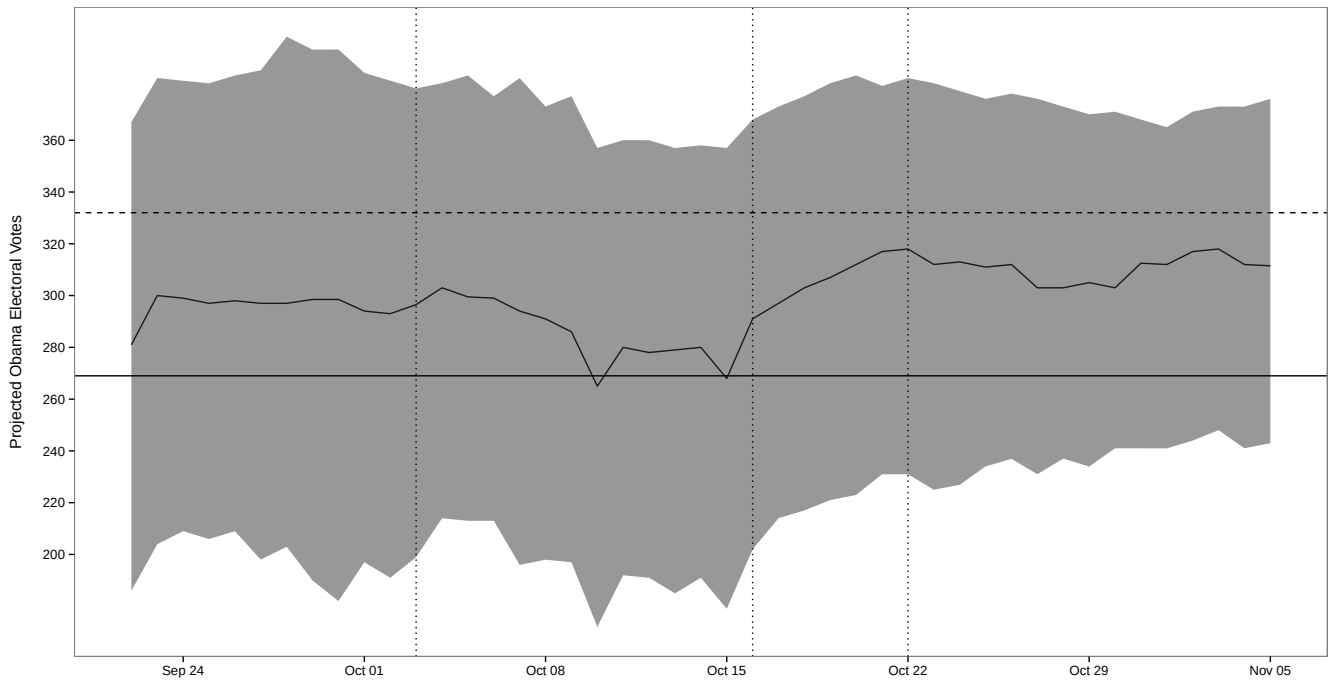
# Results of Prediction Cont'd

- Comparison with Prediction Market Data



# Results of Prediction Cont'd

- The daily predictions of Obama's electoral votes.



# Future Work

- Apart from the prediction of final election outcomes, there are other interesting questions concerning the presidential campaign, such as convention bounce.
- Based on MRP adjusted daily snapshots, we can quantify the proportion of convention bounce due to actual voting intent shift and the proportion due to partisan response rate shift resulting from voters' enthusiasm.
- Historical data could be traced in public opinion database to study the historical changes.

# References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# Introduction

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- We want to compare the performance of multilevel models and traditional models using cross-validatory metrics.

# A Formal Model Comparison Framework

- Our focus is on predictive power/generalization error.
- To take a decision theoretic view, the predictive loss incurred by a decision action $\alpha$ based on a model $M$ in face of future observation $\tilde{y}$ under loss function $l$ is given below

$$PL(p^t, M, D) = E_{p^t} l(\tilde{y}, a_M) = \int l(\tilde{y}, a_M) p^t(\tilde{y}) d\tilde{y},$$

- We use the whole predictive distribution $p(\tilde{y}|D, M)$ as the decision action $\alpha$ and log loss, the predictive loss materializes as

$$PL(p^t, M, D) = E_{p^t} \log p(\tilde{y}|D, M) = -\int p^t(\tilde{y}) \log p(\tilde{y}|D, M) d\tilde{y}$$

# Predictive Errors and Empirical Counterparts

- We can subtract the entropy of the true distribution $p^t(\tilde{y})$ from the predictive loss to obtain the predictive error

$$\begin{aligned} PE(p^t, M, D) &= PL(p^t, M, D) - LB(p^t) \\ &= -\int p^t(\tilde{y}) \log p(\tilde{y}|D, M) d\tilde{y} + \int p^t(\tilde{y}) \log p^t(\tilde{y}) d\tilde{y}. \end{aligned}$$

- The scale of the predictive error is more interpretable.
- We need to estimate the two parts in the predictive error.

# $k-$fold Cross-validation for Predictive Loss

- $k-$fold cross-validation estimate of the predictive loss is given by

$$\widehat{PL}^{\text{CV}}(M, D) = -\frac{1}{N}\sum_{k=1}^{K}\sum_{i\in \text{test}_k} \log p(y_i|D^k, M)$$
$$= -\frac{1}{N}\sum_{i=1}^{N} \log p(y_i|D^{(\setminus i)}, M),$$

where $D^k$ represents the $k^{\text{th}}$ training set and $D^{(\setminus i)}$ denotes the training set that excludes the $i^{\text{th}}$ observation.

# Lower Bound and Data Partition

- The lower bound of the predictive loss, i.e., the entropy of the true distribution, is approximated by the in-sample training loss the of saturated model $M_s$, which gives an estimates of the predictive error

$$\begin{aligned}\widehat{PE}(M,D) &= \widehat{PL}^{\text{CV}}(M,D) - TL(M_s,D) \\ &= -\frac{1}{N}\sum_{i=1}^{N}\log p(y_i|D^{(\setminus i)},M) + \frac{1}{N}\sum_{y\in D}\log p(y|D,M_s).\end{aligned}$$

- Partition of structured data in cross-validation could be tricky. One possibility is to do cluster sampling, in which we fit the model on training cells and test on hold-out cells. We adopt a stratified sampling approach and partition each cell into training and testing sets.

# CCES 2006 Survey Data

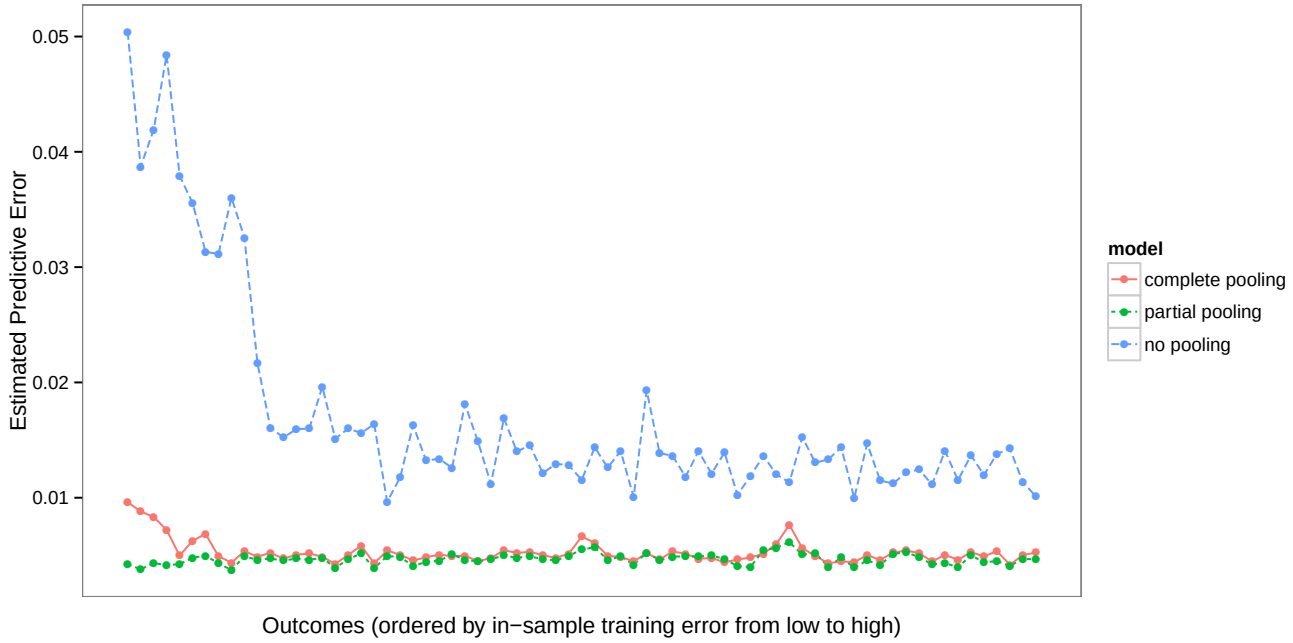
- Collaborative Congressional Election Survey 2006 is a national stratified sample of size 30,000, with a wide variety of response outcomes.
- We convert all outcomes (71 in total) into binary, and fit multilevel models for each of the outcomes.
- This provides an ideal setting to evaluate cross-validation.

# Model Descriptions

- For simplicity, we only consider two predictors, state and income. The data is cross-tabulated by these two variables, and for respondents in cell $(j_1, j_2)$, the probability that they give a positive response is $\pi_{j_1,j_2}$, which is modeled with a logistic regression $\text{logit}(\pi_{j_1 j_2}) = Z\beta$, in which $Z$ is the design matrix and $\beta$ includes the main and interaction effects.
- We compare three models
  - Complete pooling: $\pi_{j_1 j_2} = \text{logit}^{-1}\left(\beta_{j_1}^{\text{state}} + \beta_{j_2}^{\text{inc}}\right)$
  - No pooling (saturated model): $\pi_{j_1 j_2} = \text{logit}^{-1}\left(\beta_{j_1}^{\text{state}} + \beta_{j_2}^{\text{inc}} + \beta_{j_1 j_2}^{\text{state*inc}}\right)$
  - Partial pooling: $\pi_{j_1 j_2} = \text{logit}^{-1}\left(\beta_{j_1}^{\text{state}} + \beta_{j_2}^{\text{inc}} + \beta_{j_1 j_2}^{\text{state*inc}}\right)$, with $\beta_{j_1 j_2}^{\text{state*inc}} \overset{i.i.d.}{\sim} \Phi(\cdot)$.

# The Real Data

- The estimated predictive errors for all 71 outcomes are plotted below. The x-axis is ordered by the training loss of the saturated model. The no pooling model gives a bad fit. Partial pooling does best but in most cases is almost indistinguishable from complete pooling under the cross-validation criterion.

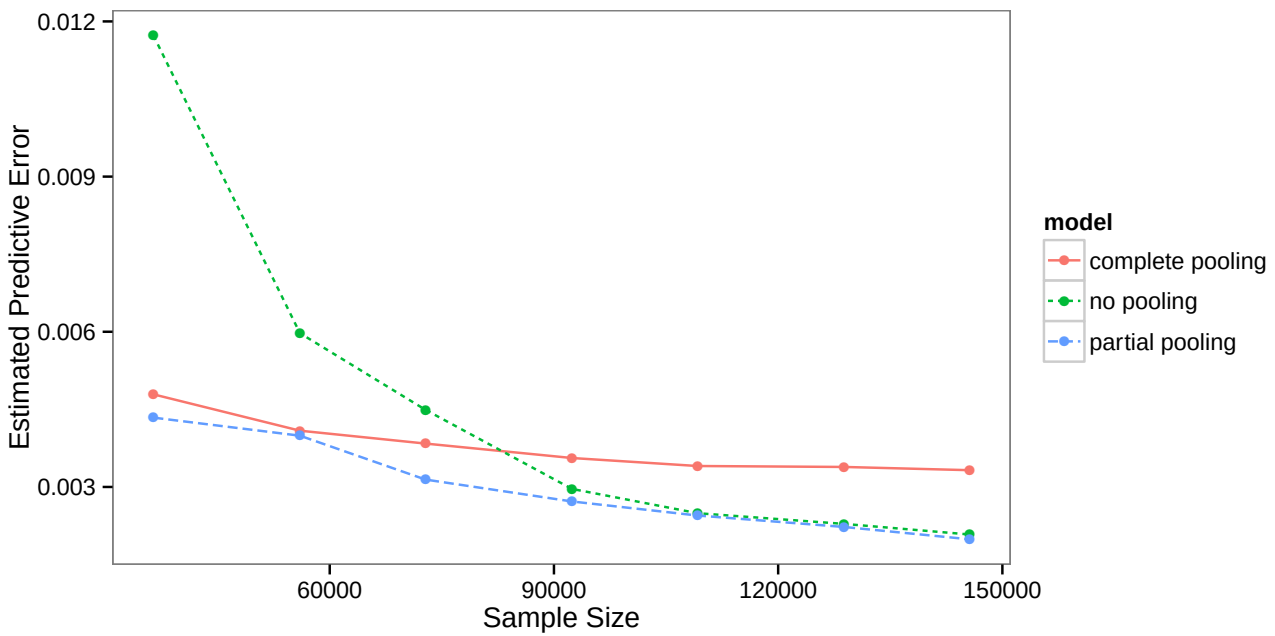


# Simulated Data: Changing Sample Size

- From the previous graph, it seems that the multilevel model fails to pick up the interaction between income and state. We augment the data set by replicating each observation multiple times.

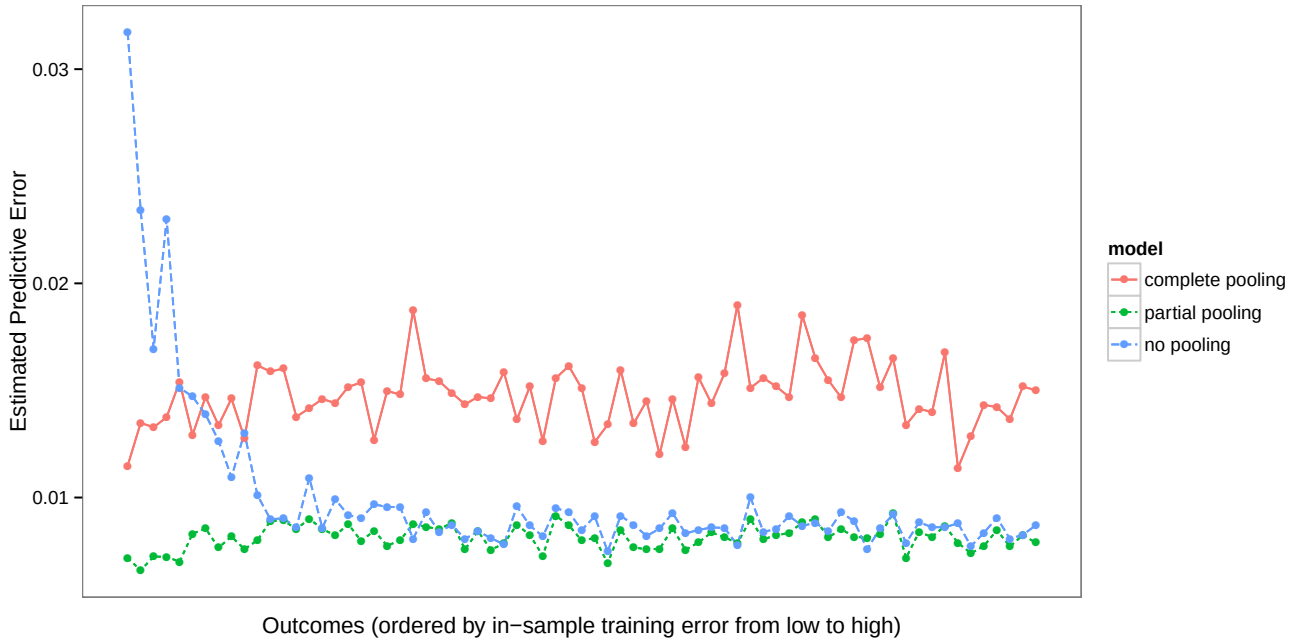
# Simulated Data: Changing Sample Size Cont'd

- We also look at one particular outcome, two-party republican support in the congressional election. Again, partial pooling performs the best, but its performance is roughly matched by no pooling in large sample sizes and by complete pooling in small sample sizes.



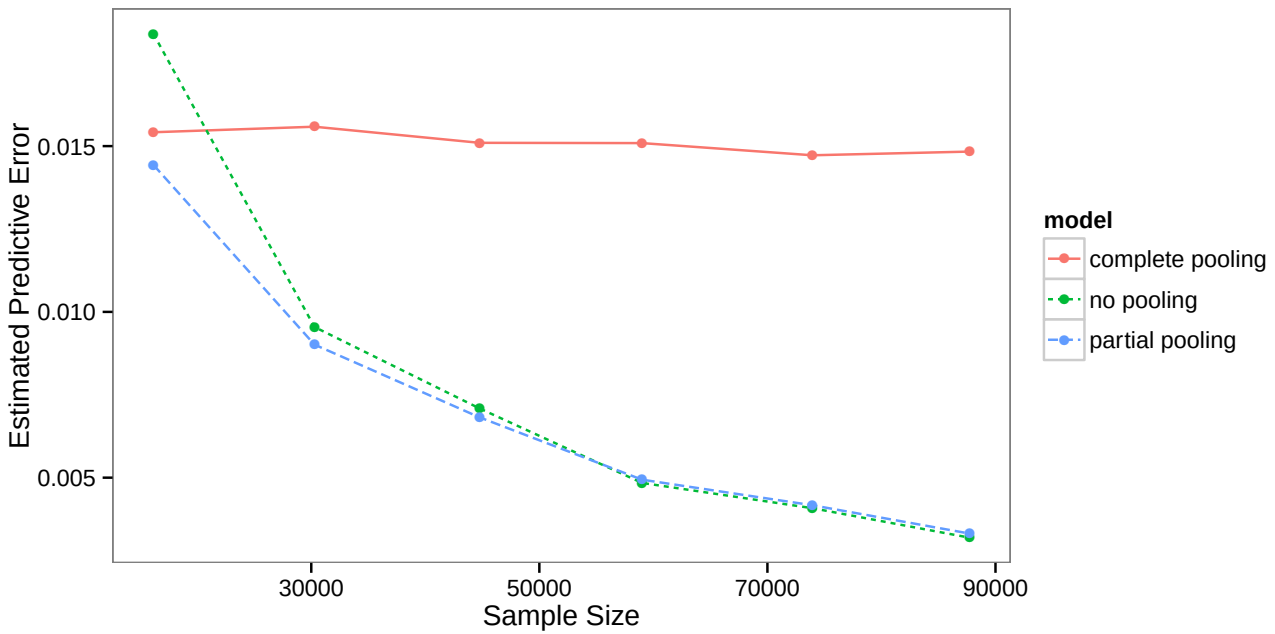
# Simulated Data: Balancedness of Cells

- Another factor underlying the relative performance of different models might be the highly-unbalanced structure of the survey data. We simulate data of the same sample size and cell proportion as the original data but with balanced cell sample sizes.



# Simulated Data: Balancedness of Cells Cont'd

- In this case, we also let sample size grow and see the relative performance of the three models for the congressional election republican vote.



# Discussion

- Multilevel models capture the important interactions that are not included in the complete pooling model, while at the same time avoiding the inevitable overfitting from the no pooling model.
- However, the improvement of the multilevel model as given by cross-validation is surprisingly tiny, almost negligible to unsuspecting eyes.
- Simulations based on real data show that sample size and structure of the cross-tabulated cells play important roles in the relative margins of different models in cross-validation based model selection.
- Caution should be exercised in applying cross-validation for model selection with structured data.

# References


- Wang, Wei and Gelman, Andrew, 2013. A problem with the use of cross-validation for selecting among multilevel models. Technical Report.
- Vehtari, Aki and Ojanen, Janne, 2013. A survey of Bayesian predictive methods for model assessment, selection and comparison. Statistics Survey.