S1211Q Introduction to Statistics Lecture 5

Wei Wang

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Conditional Probabilities

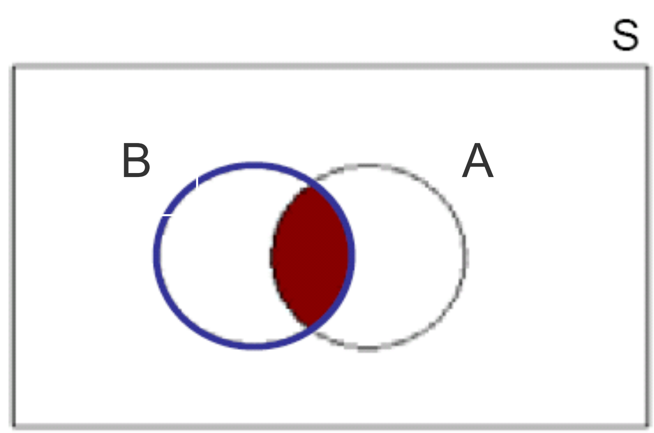
- If we are given information that an event A has occurred, then the probability of another event B occurring may change.

- We use P(A|B) to represent the conditional probability of A given that the event B has occurred. B is the “conditioning event”.

- Definition: for any two events A and B with P(B)>0, the conditional probability of A given B has occurred is defined by

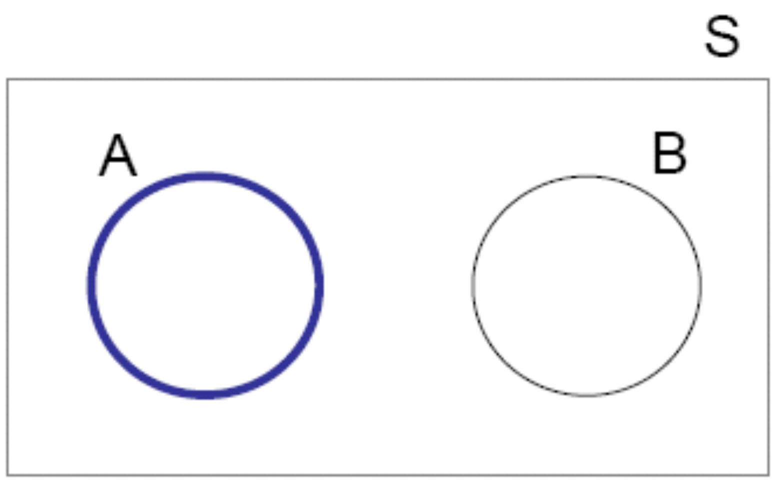
$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \bigcap B)}{\mathrm{P}(B)}$$

Proportion of outcomes in B that are also in A.



Revisit disjoint events

- If A and B are disjoint then $P(A \cap B) = 0$.
- Hence, $P(A|B) = 0$. If B occurs, then A cannot occur.



Example

Ex. 1,000 voters were asked about their party affiliation and position on death penalty.

	Favor	Oppose	*Total*
Republican	260	40	300
Democrat	120	240	360
Other	240	100	340
Total	620	380	1000

R = A randomly chosen voter is a Republican.

D = A randomly chosen voter is a Democrat.

DP = A randomly chosen voter favors the death penalty.

What is the probability that a randomly chosen voter favors the death penalty given he is a Republican? (26/30 = 0.867)

What is the probability that a voter is a Democrat given he favors the death penalty? (12/62 = 0.194)

The Multiplication Rule

- The definition of conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$ points us another way to calculate probability of intersection of events.

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$$P(A \cap B) = P(A|B) \cdot P(B)$$

  or alternatively

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- It can be generalized to multiple events.

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$$

Example

Ex. A box contains 8 blue balls and 4 red balls. We draw two balls from the box without replacement. What is the probability that both are red?

A = first ball is red.

B = second ball is red.

$$
\begin{aligned}
P(\text{both balls are red}) = P(A \cap B) &= P(A)\, P(B|A) \\
&= 4/12 * 3/11 \\
&= 1/11
\end{aligned}
$$

More general "multiplication rule": $P(A \cap B \cap C) = P(C|A \cap B)\, P(B|A)\, P(A)$

Question: Draw three balls without replacement, what is the probability that all are red?

Tree diagram

- Often a problem involves several stages of using the multiplication and addition rules.

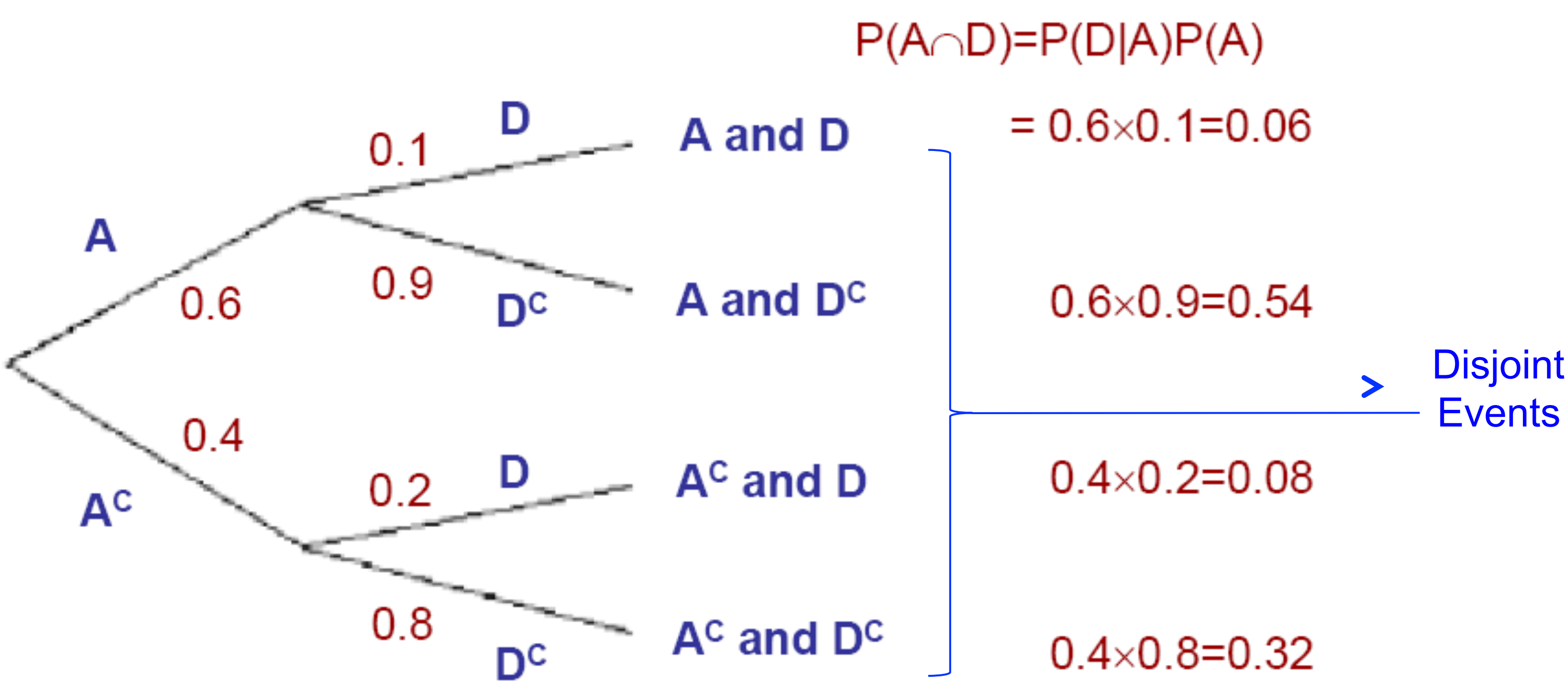
- A tree diagram is a useful way of keeping track of the multiplication and addition rules.

- It helps us think about conditional probabilities by showing the sequences of events as paths that resemble the branches of a tree.

Tree example

<u>Ex.</u> A company purchases 60% of its parts from vendor A and the rest from vendor B. Past experience indicates that 10% of vendor A's parts are defective, while 20% of vendor B's parts are defective.

A = part from vendor A	$P(A) = 0.6$	$P(A^c) = 0.4$
D = part defective	$P(D\|A) = 0.1$	$P(D\|A^c) = 0.2$



Example cont.

- A part is selected at random, what is the probability that it's defective?

$$P(D) = P(D|A)\,P(A) + P(D|A^c)\,P(A^c)$$
$$= 0.06 + 0.08 = 0.14$$

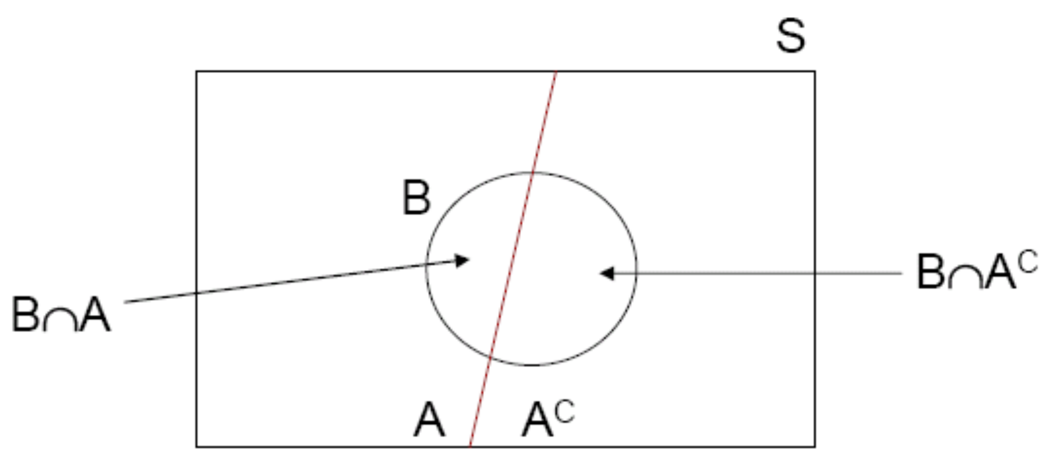
- Suppose a part is defective, what is the probability that it's from vendor A?

$$\mathrm{P}(A|D) = \frac{\mathrm{P}(A \cap D)}{\mathrm{P}(D)} = \frac{0.06}{0.14} = \frac{3}{7}$$

- In these two examples, we used rule of total probability and Bayes rule.

Motivations

- We can write any event B as the union of the portion contained in another event A and the portion contained in A^c, i.e., $B = (B \cap A) \cup (B \cap A^c)$.



- Because A and A^c are disjoint, so are $B \cap A$ and $B \cap A^c$.

The Law of Total Probability

- The Law of Total Probability states, Let $A_1, \ldots, A_k$ be mutually exclusive and *exhaustive* events. Then for any other event B.

$$P(B) = P(B|A_1)P(A_1) + \ldots + P(B|A_k)P(A_k)$$
$$= \sum P(B|A_i)P(A_i)$$

- $A_1, \ldots, A_k$ are *exhaustive*, if one A_i must occur, so that $A_1 \cup \ldots \cup A_k = S$.

- Proof: when k=2,

$$P(B) = P((B \cap A) \cup (B \cap A^c))$$
$$= P(B \cap A) + P(B \cap A^c)$$
$$= P(B|A)P(A) + P(B|A^c)P(A^c)$$

Bayes Theorem

- With the help of the Law of Total Probability, we can state the Bayes Rule, which says, let A_1, …, A_k be a collection of *k* mutually exclusive and exhaustive events with *prior* probabilities $P(A_i)$ ($i=1,\ldots,k$). Then for any other event B for which P(B) >0, the *posterior* probability of A_j given that B has occurred is,

$$\mathrm{P}(A_j|B) = \frac{\mathrm{P}(A_j \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(B|A_j)\mathrm{P}(A_j)}{\sum_{i=1}^{k} \mathrm{P}(B|A_i)\mathrm{P}(A_i)}$$

- When *k*=2, we have,

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B|A)\mathrm{P}(A) + \mathrm{P}(B|A^c)\mathrm{P}(A^c)}$$

- Bayes Rule can be used to "*reverse*" the probability from the conditional probability that was originally given, or *to find the cause given the result*.

Bayes Theorem Example

- One percent of all individuals in a certain population are carriers of a particular disease. A diagnostic test for this disease has a 90% detection rate for carriers and a 5% detection rate for noncarriers. If a person is tested positive, what's the probability that this person is a carrier?

Bayes Theorem Example

- One percent of all individuals in a certain population are carriers of a particular disease. A diagnostic test for this disease has a 90% detection rate for carriers and a 5% detection rate for non-carriers. If a person is tested positive, what's the probability that this person is a carrier?
-

$$
\begin{aligned}
&P(\text{is a carrier}|\text{tested positive}) \\
&= \frac{P(\text{carrier} \cap \text{tested positive})}{P(\text{tested positive})} \\
&= \frac{P(\text{positive}|\text{carrier})P(\text{carrier})}{P(\text{positive}|\text{carrier})P(\text{carrier}) + P(\text{positive}|\text{non-carrier})P(\text{non-carrier})}
\end{aligned}
$$

Independence

- Definition: Two events A and B are independent if $P(A|B) = P(A)$ (or alternatively $P(B|A) = P(B)$).
- A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

- Independent Events $\neq$ Disjoint Events.

Multiple Events

- Events $A_1, \ldots, A_n$ are mutually independent if for every k ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, i_2, \ldots, i_k$,
$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})\, P(A_{i_2}) \ldots P(A_{i_k}).$$

- Independence is very very important!

Example

Ex. You recently bought a new set of tires from a manufacturer who just announced a recall because 2% of that particular brand were defective. What is the probability that at least one of your tires is defective? You may assume that the tires are defective independently of one another.

P(at least one defective tire) = 1 – P(no defective tire)

Let A_i = tire *i* is not defective

$P(A_i) = 1-0.02 = 0.98$

$$P(\text{no defective tire}) = P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

$$= P(A_1)\, P(A_2)\, P(A_3)\, P(A_4) = (0.98)^4$$

$$P(\text{at least one defective tire}) = 1-(0.98)^4 = 0.0776$$