HOMEWORK 5

4.

a. $\int_{-\infty}^{\infty} f(x;\theta)dx = \int_0^{\infty} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx = -e^{-x^2/2\theta^2}\Big]_0^{\infty} = 0-(-1) = 1$

b. $P(X \le 200) = \int_{-\infty}^{200} f(x;\theta)dx = \int_0^{200} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx = -e^{-x^2/2\theta^2}\Big]_0^{200} \approx -.1353+1 = .8647$.

$P(X < 200) = P(X \le 200) \approx .8647$, since X is continuous.

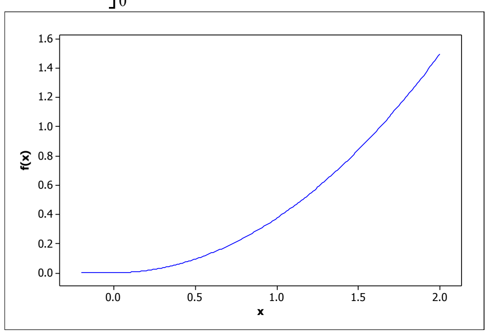
$P(X \ge 200) = 1 - P(X < 200) \approx .1353$.

c. $P(100 \le X \le 200) = \int_{100}^{200} f(x;\theta)dx = -e^{-x^2/20{,}000}\Big]_{100}^{200} \approx .4712$.

d. For $x > 0$, $P(X \le x) = \int_{-\infty}^{x} f(y;\theta)dy = \int_0^x \frac{y}{\theta^2} e^{-y^2/2\theta^2} dx = -e^{-y^2/2\theta^2}\Big]_0^x = 1-e^{-x^2/2\theta^2}$.

5.

a. $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 kx^2 dx = \frac{kx^3}{3}\Big]_0^2 = \frac{8k}{3} \Rightarrow k = \frac{3}{8}$.



b. $P(0 \le X \le 1) = \int_0^1 \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_0^1 = \frac{1}{8} = .125$.

c. $P(1 \le X \le 1.5) = \int_1^{1.5} \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_1^{1.5} = \frac{1}{8}\left(\frac{3}{2}\right)^3 - \frac{1}{8}(1)^3 = \frac{19}{64} = .296875$.

d. $P(X \ge 1.5) = 1 - \int_{1.5}^{2} \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_{1.5}^{2} = \frac{1}{8}(2)^3 - \frac{1}{8}(1.5)^3 = .578125$.

12.

a. $P(X < 0) = F(0) = .5$.

b. $P(-1 \le X \le 1) = F(1) - F(-1) = .6875$.

c. $P(X > .5) = 1 - P(X \le .5) = 1 - F(.5) = 1 - .6836 = .3164$.

d. $f(x) = F'(x) = \frac{d}{dx}\left(\frac{1}{2} + \frac{3}{32}\left(4x - \frac{x^3}{3}\right)\right) = 0 + \frac{3}{32}\left(4 - \frac{3x^2}{3}\right) = .09375\left(4-x^2\right)$.

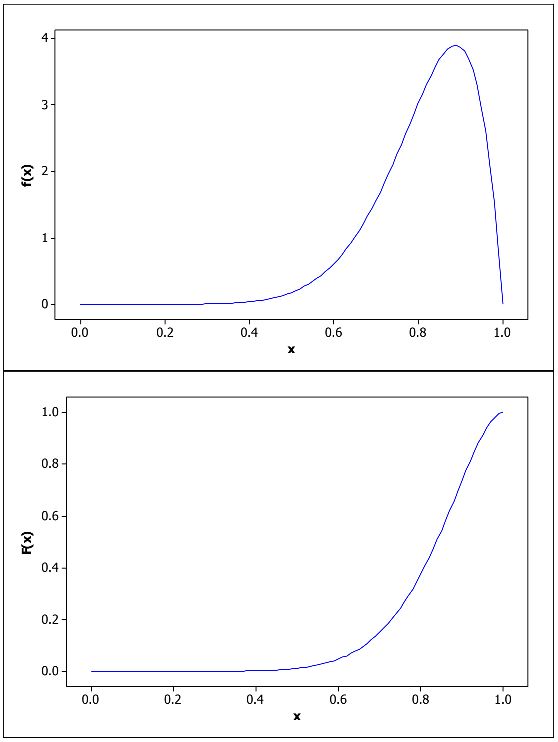
e. By definition, $F(\tilde{\mu}) = .5$. $F(0) = .5$ from **a** above, which is as desired.

15.

a. Since X is limited to the interval (0, 1), $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$.
For $0 < x < 1$,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_{0}^{x} 90y^8(1-y)dy = \int_{0}^{x}(90y^8 - 90y^9)dy = 10y^9 - 9y^{10}\Big]_0^x = 10x^9 - 9x^{10}.$$

The graphs of the pdf and cdf of X appear below.



b. $F(.5) = 10(.5)^9 - 9(.5)^{10} = .0107$.

c. $P(.25 < X \le .5) = F(.5) - F(.25) = .0107 - [10(.25)^9 - 9(.25)^{10}] = .0107 - .0000 = .0107$.
Since X is continuous, $P(.25 \le X \le .5) = P(.25 < X \le .5) = .0107$.

d. The 75th percentile is the value of x for which $F(x) = .75$: $10x^9 - 9x^{10} = .75$ $\Rightarrow x = .9036$ using software.

e. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_{0}^{1} x \cdot 90x^8(1-x)dx = \int_{0}^{1}(90x^9 - 90x^{10})dx = 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = 9 - \frac{90}{11}$ $= \frac{9}{11} = .8182$. Similarly, $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_{0}^{1} x^2 \cdot 90x^8(1-x)dx = \ldots = .6818$, from which $V(X) = .6818 - (.8182)^2 = .0124$ and $\sigma_X = .11134$.

f. $\mu \pm \sigma = (.7068, .9295)$. Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068) = .8465 - .1602 = .6863$, and the probability X is <u>more</u> than 1 standard deviation from its mean value equals $1 - .6863 = 3137$.

28.

e. $\Phi(1.37) = .9147$.

f. $P(-1.75 < Z) + [1 - P(Z < -1.75)] = 1 - \Phi(-1.75) = .9599$.

g. $\Phi(2) - \Phi(-1.50) = .9104$.

29.

c. $P(c \le Z) = .121 \Rightarrow 1 - P(Z < c) = .121 \Rightarrow 1 - \Phi(c) = .121 \Rightarrow \Phi(c) = .879 \Rightarrow c = 1.17$.

d. $P(-c \le Z \le c) = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1 = .668 \Rightarrow \Phi(c) = .834 \Rightarrow$
$c = 0.97$.

e. $P(c \le |Z|) = 1 - P(|Z| < c) = 1 - [\Phi(c) - \Phi(-c)] = 1 - [2\Phi(c) - 1] = 2 - 2\Phi(c) = .016 \Rightarrow \Phi(c) = .992 \Rightarrow c = 2.41$.

42. The probability X is within .1 of its mean is given by $P(\mu - .1 \le X \le \mu + .1) = P\left(\frac{(\mu-.1)-\mu}{\sigma} < Z < \frac{(\mu+.1)-\mu}{\sigma}\right) = \Phi\left(\frac{.1}{\sigma}\right) - \Phi\left(-\frac{.1}{\sigma}\right) = 2\Phi\left(\frac{.1}{\sigma}\right) - 1$. If we require this to equal 95%, we find $2\Phi\left(\frac{.1}{\sigma}\right) - 1 = .95 \Rightarrow \Phi\left(\frac{.1}{\sigma}\right) = .975 \Rightarrow \frac{.1}{\sigma} = 1.96$ from the standard normal table. Thus, $\sigma = \frac{.1}{1.96} = .0510$.
Alternatively, use the empirical rule: 95% of all values lie within 2 standard deviations of the mean, so we want $2\sigma = .1$, or $\sigma = .05$. (This is not quite as precise as the first answer.)

45. With $\mu = .500$ inches, the acceptable range for the diameter is between .496 and .504 inches, so unacceptable bearings will have diameters smaller than .496 or larger than .504.
The new distribution has $\mu = .499$ and $\sigma = .002$.
$P(X < .496 \text{ or } X > .504) = P\left(Z < \frac{.496 - .499}{.002}\right) + P\left(Z > \frac{.504 - .499}{.002}\right) = P(Z < -1.5) + P(Z > 2.5) =$
$\Phi(-1.5) + [1 - \Phi(2.5)] = .073$. 7.3% of the bearings will be unacceptable.

60.

a. $P(X \le 100) = 1 - e^{-(100)(.01386)} = 1 - e^{-1.386} = .7499$.
$P(X \le 200) = 1 - e^{-(200)(.01386)} = 1 - e^{-2.772} = .9375$.
$P(100 \le X \le 200) = P(X \le 200) - P(X \le 100) = .9375 - .7499 = .1876$.

b. First, since X is exponential, $\mu = \frac{1}{\lambda} = \frac{1}{.01386} = 72.15$, $\sigma = 72.15$. Then

$P(X > \mu + 2\sigma) = P(X > 72.15 + 2(72.15)) = P(X > 216.45) = 1 - (1 - e^{-.01386(216.45)}) = e^{-3} = .0498$.

c. Remember the median is the solution to $F(x) = .5$. Use the formula for the exponential cdf and solve for x: $F(x) = 1 - e^{-.01386x} = .5 \Rightarrow e^{-.01386x} = .5 \Rightarrow -.01386x = \ln(.5) \Rightarrow x = -\frac{\ln(.5)}{.01386} = 50.01$ m.

70. To find the $(100p)$th percentile, set $F(x) = p$ and solve for x: $p = F(x) = 1 - e^{-\lambda x} \Rightarrow e^{-\lambda x} = 1 - p \Rightarrow$

$-\lambda x = \ln(1 - p) \Rightarrow x = -\frac{\ln(1-p)}{\lambda}$.

To find the median, set $p = .5$ to get $\tilde{\mu} = -\frac{\ln(1-.5)}{\lambda} = \frac{.693}{\lambda}$.

88. The data values and z percentiles provided result in the probability plot below. The plot shows some non-trivial departures from linearity, especially in the lower tail of the distribution. This indicates a normal distribution might not be a good fit to the population distribution of clubhead velocities for female golfers.