S1211Q Introduction to Statistics
Lecture 4

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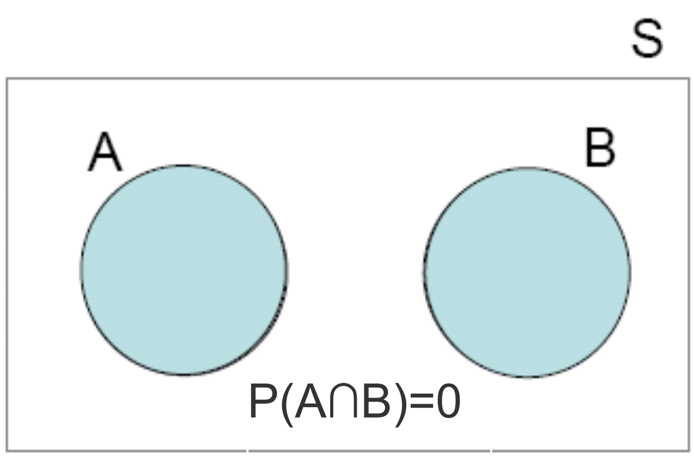
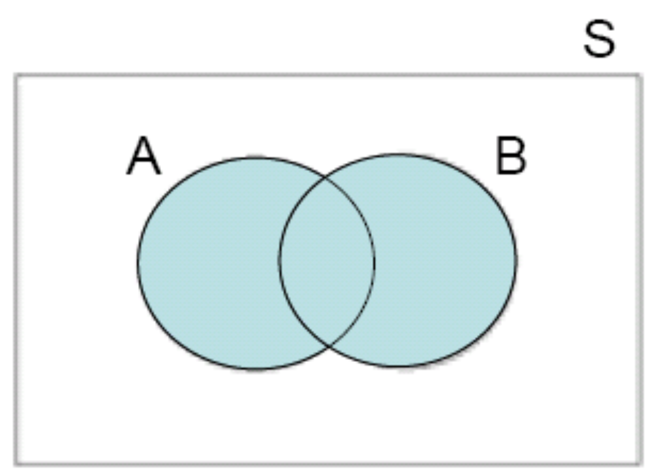
More Probability Properties

- Consider an experiment whose sample space is *S*. For each event A (B) in *S*, we assume that a number P(A) is defined and satisfies the following rules:
  1. $0 \le P(A) \le 1$.
  2. $P(S)=1$.
  3. $P(A^c)=1-P(A)$.
  4. If A and B are disjoint, then $P(A \cup B)=P(A)+P(B)$.
  5. For any two events A and B, $P(A \cup B)=P(A)+P(B)-P(A \cap B)$.





Example

Ex. A store accepts either VISA or Mastercard. 50% of the stores customers have VISA, 30% have Mastercard and 10% have both. What is the probability that a customer has a credit card the store accepts?

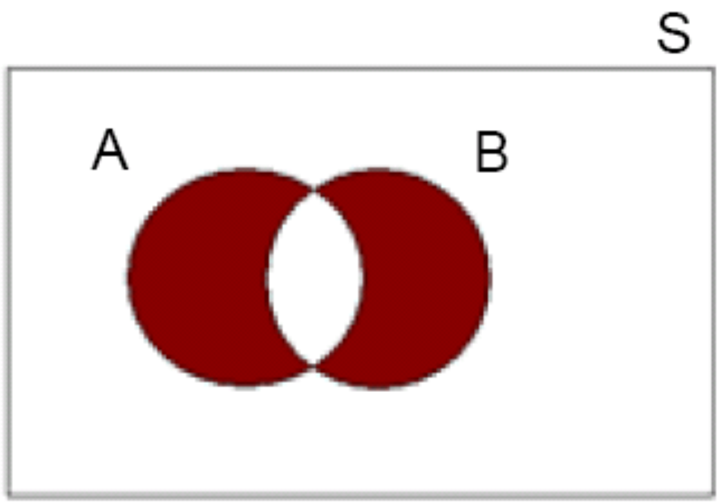
A = customers has VISA

B = customers has Mastercard

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.5 + 0.3 - 0.1 = 0.7$$

Example cont.

What is the probability that a customer has either a VISA or MC, but not both?



$$P(A \text{ or } B \text{ but not both}) = P(A) + P(B) - 2P(A \cap B)$$
$$= 0.5 + 0.3 - 0.2 = 0.6$$

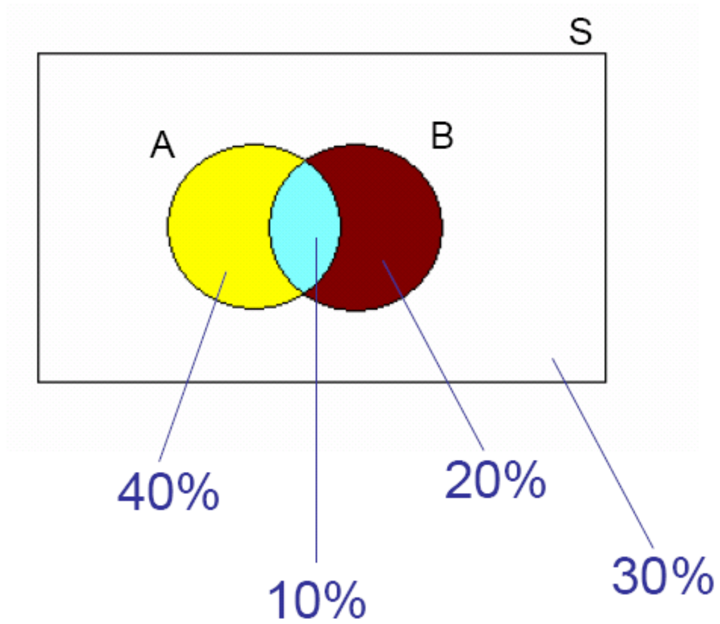
Example Cont.

What is the probability that a customer has a VISA but no MC?

$$P(A \text{ but not both}) = P(A) - P(A \cap B)$$
$$= 0.5 - 0.1 = 0.4$$

What is the probability that a customer has a MC but no VISA?

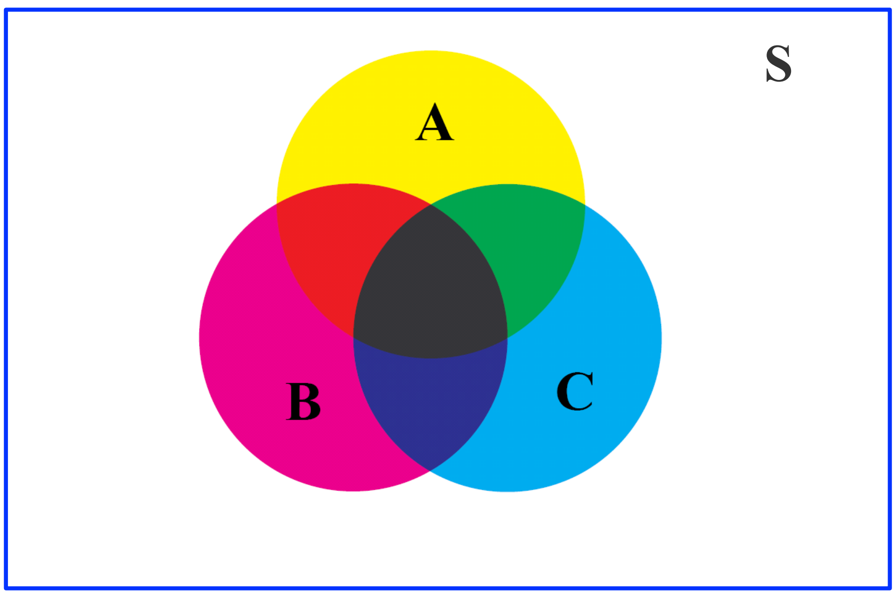
$$P(B \text{ but not both}) = P(B) - P(A \cap B)$$
$$= 0.3 - 0.1 = 0.2$$



Three Events

- For any three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



- Recall that for discrete variables and all outcomes are equally likely to occur, the probability of event A is given by

$$P(A) = \frac{\text{Number of outcomes in A}}{\text{Number of outcome in Sample Space S}}$$

- Counting Techniques are essential to efficiently calculate the numerator and denominator. Specifically, we will talk about Permutations and Combinations.
- Again, remember that in the language of Probability, set and event are interchangeable.

Product Rule

- A general situation is that a set consists of ordered pairs of objects and we wish to count the numbers of such pairs.
- If the first object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second objects can be selected in n_2 ways, then the number of pairs is $n_1 n_2$
- This rule applies when we have multiple stages.
- Here the key is that the stages are independent of each other.

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- There is a more general way to count. Think about positions of the ordered pair one at a time.
- Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters then shows that there are

$$n(n-1)(n-2)\ldots 3\cdot 2\cdot 1 = n!$$

  different permutations of the n objects.

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- **Solution:** There are $9! = 362880$ possible batting orders.

Example 2

- Ms. Davis has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are physics books, 2 are literature books and 1 is a language book. Ms. Mortimer wants to arrange her books so that all the books dealing with the same subject are together on her shelf. How many different arrangements are possible?

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- **Solution:** There are $4!3!2!1!$ arrangements such that the mathematics books are first in line, then the physics books, then the history books, and then the language books. Similarly, for each possible ordering of the subjects, there are $4!3!2!1!$ possible arrangements. Hence, as there are $4!$ possible ordering of the subjects, the desired answer is $4!4!3!2!1! = 6912$.

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- In particular, $P_{n,n} = n!$ and $P_{1,n} = n$

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- The denominator is $P_{9,20}$, what is the numerator?

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- There are thus $P_{3,5} = 5 \times 4 \times 3$ ways of selecting a group of 3 when the order in which the items are selected is relevant.
- But every group is counted $P_{3,3} = 3 \times 2 \times 1$ times. So there are
  $$\frac{P_{3,5}}{3!} = 10$$
  groups.

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- $\binom{n}{k} = P_{k,n}/k!$
- In particular, $\binom{n}{n} = 1$ and $\binom{n}{1} = n$.

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- **Solution:** There are $\binom{20}{3} = \frac{20\times 19\times 18}{3\times 2\times 1} = 1140$ possible committees.

Example 2

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- **Solution:** As there $\binom{5}{2}$ possible groups of 2 women and $\binom{7}{3}$ possible groups of 3 men, if follows that there are $\binom{5}{2}\binom{7}{3} = 350$ possible committees consisting of 2 women and 3 men.

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- In the last example, what if 2 of the men are feuding and refuse to serve on the committee together?
- **Solution:** Now suppose that 2 of them refuse to serve together, because a total of $\binom{2}{2}\binom{5}{1}$ out of the $\binom{7}{3} = 35$ possible groups of 3 men contain both of the feuding men, it follows that there are $35 - 5 = 30$ groups that do not contain both of the feuding men. Because there are still $\binom{5}{2}$ ways to choose 2 women, there are $30 \cdot 10 = 300$ possible committees in this case.

Example 3 Contd

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- **Solution:** $\frac{350-300}{350} = 1/7$