The Normal Distribution

- It's probably the most important distribution in the world!

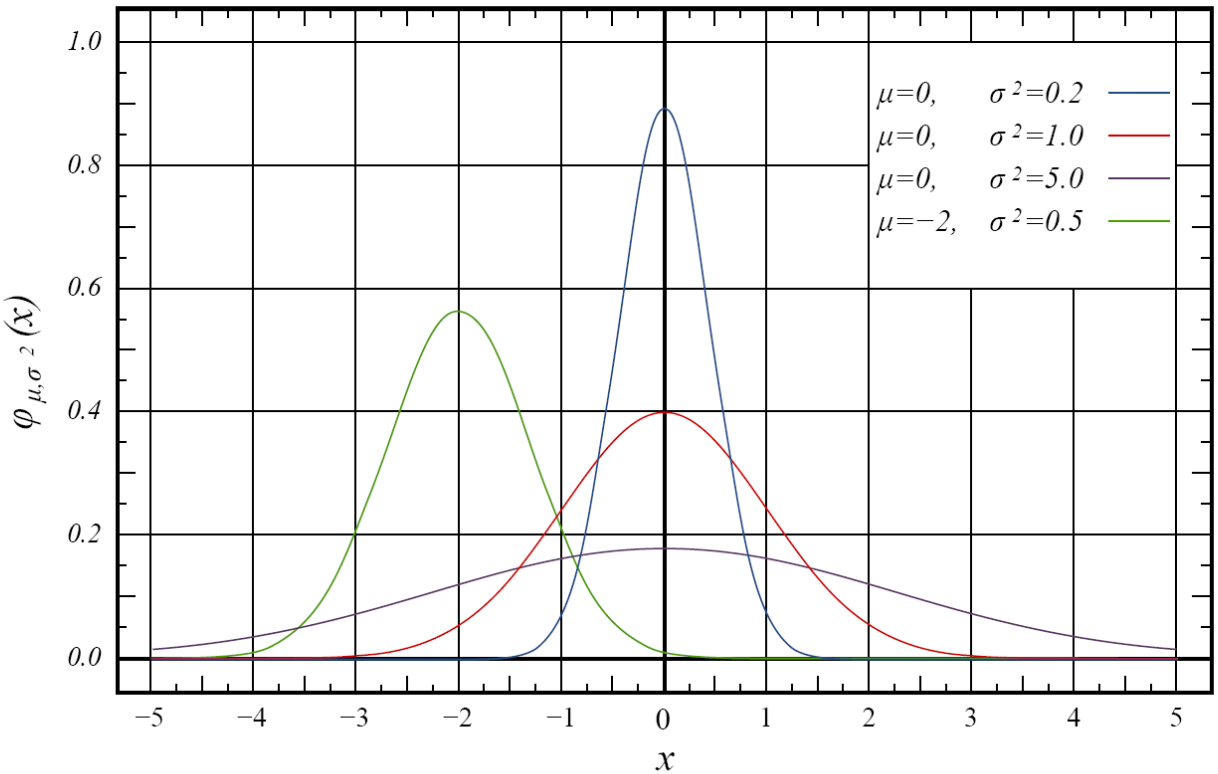
- Many numerical populations have distributions that can be fit very closely by an appropriate normal curve. (people's height/weight; testing scores; etc.) Even when the underlying distribution is discrete, (yearly number of customers to Wal-Mart; etc.) the normal curve often gives an excellent approximation.

- A continuous rv is said to have a normal (Gaussian) distribution with parameters μ and σ, where $-\infty<\mu<\infty$, and $0<\sigma$, if the pdf of X is

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \qquad -\infty < x < \infty$$

The Normal pdf

- Normal distribution is a bell-shaped, single peaked and symmetric distribution.



Parameters

- Clearly $f(x; \mu, \sigma) \geq 0$, but a somewhat complicated calculus argument must be used to verify that

$$\int_{-\infty}^{\infty} f(x; \mu, \sigma) dx = 1.$$

- Parameter μ, stands for the expected value of the normal distribution.
  Exercise: show that if $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$.

- Parameter σ, stands for the standard deviation of the normal distribution.
  Exercise: show that if $X \sim N(\mu, \sigma^2)$, then $Var(X) = \sigma^2$.

Mean and Variance of Normal Distribution

- $$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} (x-\mu+\mu) \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) d(x-\mu) \\
&= \mu + \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt \text{ (Density integrates to 1)} \\
&= \mu \text{ (Symmetry)}
\end{aligned}
$$

Mean and Variance of Normal Distribution

- $$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\, dx \\
&= \int_{-\infty}^{\infty} [(x-\mu)^2 + 2x\mu - \mu^2] \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\, d(x-\mu) \\
&= \int_{-\infty}^{\infty} t^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{t^2}{2\sigma^2}\Big)\, dt + 2\mu \cdot E(X) - \mu^2 \\
&= \int_{-\infty}^{\infty} -\frac{t\sigma}{\sqrt{2\pi}}\, d\exp\Big(-\frac{t^2}{2\sigma^2}\Big) + \mu^2 \\
&= 0 + \int_{-\infty}^{\infty} \frac{\sigma}{\sqrt{2\pi}} \exp\Big(-\frac{t^2}{2\sigma^2}\Big)\, dt + \mu^2 \\
&= \sigma^2 + \mu^2
\end{aligned}$$

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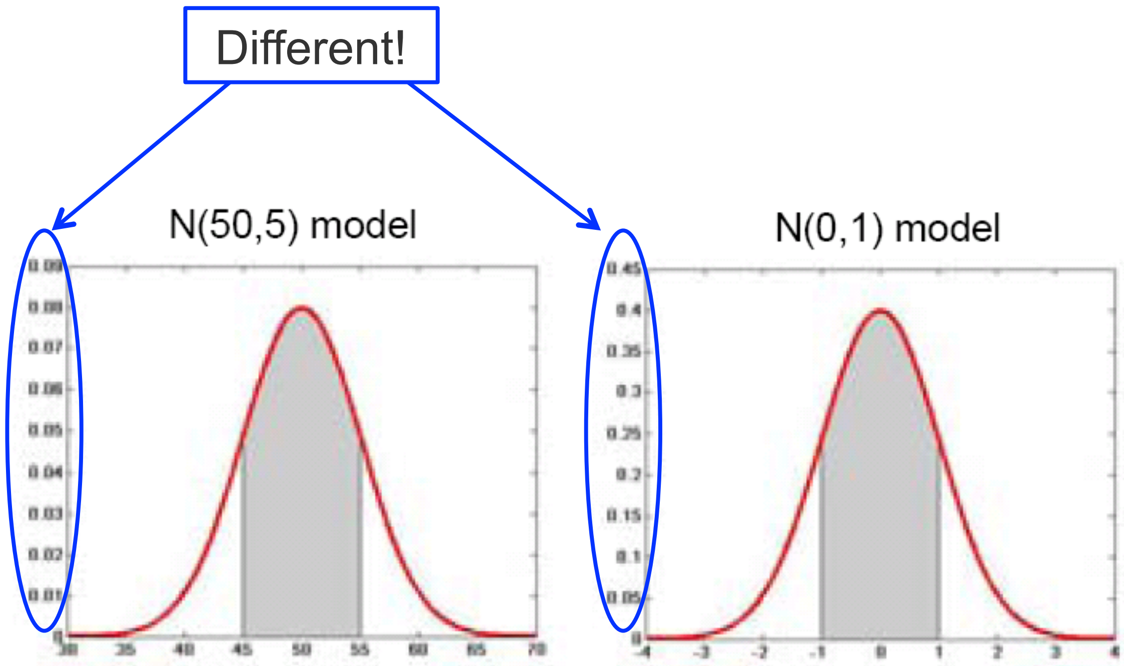
&= \sigma^2 + \mu^2
\end{aligned}$$

- And thus $Var(X) = E(X^2) - [E(x)]^2 = \sigma^2$

Basic Properties

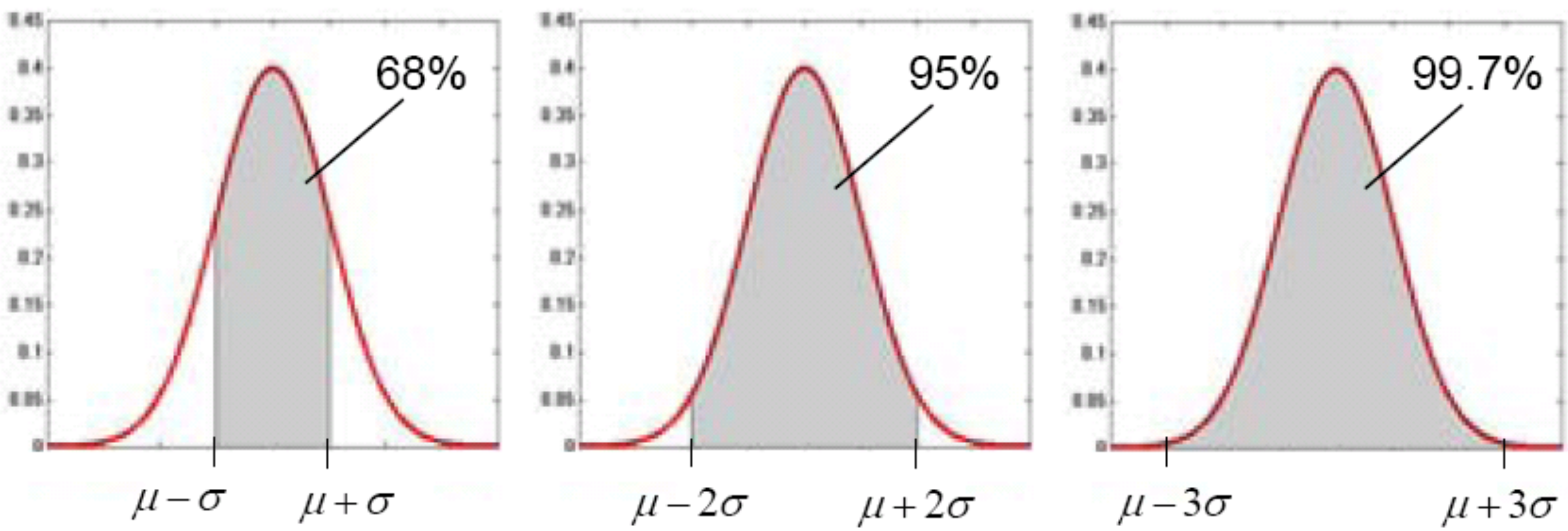
- All normal models have the same shape and the same area within *x* standard deviations of its mean.



Area between (45,55) = Area between (-1,1)

The 68-95-99.7 Rule

- For any normal distribution, we have the following result:



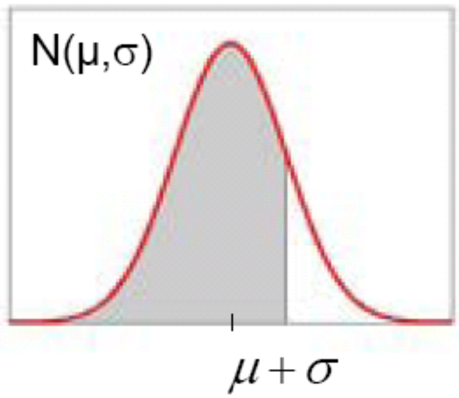
Example

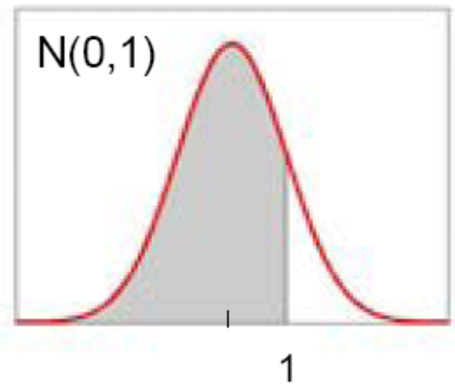
Ex. On an exam the scores followed an approximate normal model with μ = 72 and σ = 8.

- 68% of the students scored between 72±8 or (64, 80).
- 95% of the scores were between 72±2*8 or (56, 88).
- 99.7% of the scores were between 72±3*8 or (48, 96).

- What proportion scored below 84?

Key Result





$$area\{y < \mu + \sigma\} \quad = \quad area\{z < 1\}$$

Standard Normal

- If $Z \sim N(0, 1)$, i.e., if Z is a normal random variable with $\mu=0$, $\sigma=1$. Then Z is said to have a standard normal distribution.

- Any normally distributed rv's could be obtained by using standard normal rv's. To put it more mathematically, if $X \sim N(\mu, \sigma^2)$, then X could be written as

$$X = \mu + \sigma \cdot Z$$

  where Z is a standard normal rv.

- Conversely, if $X \sim N(\mu, \sigma^2)$, then

$$Z = (X - \mu) / \sigma$$

  has a standard normal distribution. And Z is often called the "*z-score*" of X.

Example cont.

<u>Ex.</u> The exam scores followed a N(72,8) model.

What proportion of the students scored below 84?

$$z = \frac{y-\mu}{\sigma} = \frac{84-72}{8} = 1.5$$

Answer: 93.32%

TABLE A Standard normal probabilities (*continued*)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	$\boxed{.9332}$	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Simplification

- Thus, any problem about any normal rv $X \sim N(\mu, \sigma^2)$, can be translated to a problem about a standard normal rv Z.

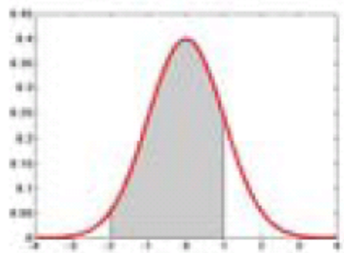
Ex. $P(a \le X \le b) = P[(a-\mu)/\sigma \le (X-\mu)/\sigma \le (b-\mu)/\sigma] = P[(a-\mu)/\sigma \le Z \le (b-\mu)/\sigma]$.

- The cumulative distribution function of standard normal distribution, that is $\Phi(z) = P(Z \le z)$, is already known! (Appendix Table.)

- Check Table A.3 to determine $P(Z \le 0.76)$; $P(Z > 0.76)$; $P(-1.32 \le Z \le 0.76)$.

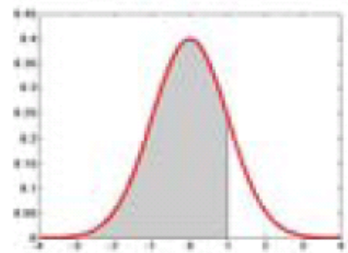
- Question: How to get the *p*-th percentile of the standard normal from A.3?

Using the Normal Table



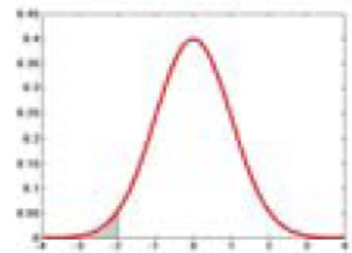
0.8185

=



0.8413

-



0.0228

0.0228

=

1.00

-

0.9772

R instead of tables

- R command: `dnorm(x, mean = 0, sd = 1)`,
  `pnorm(q, mean = 0, sd = 1)`,
  `qnorm(p, mean = 0, sd = 1)`.

Example

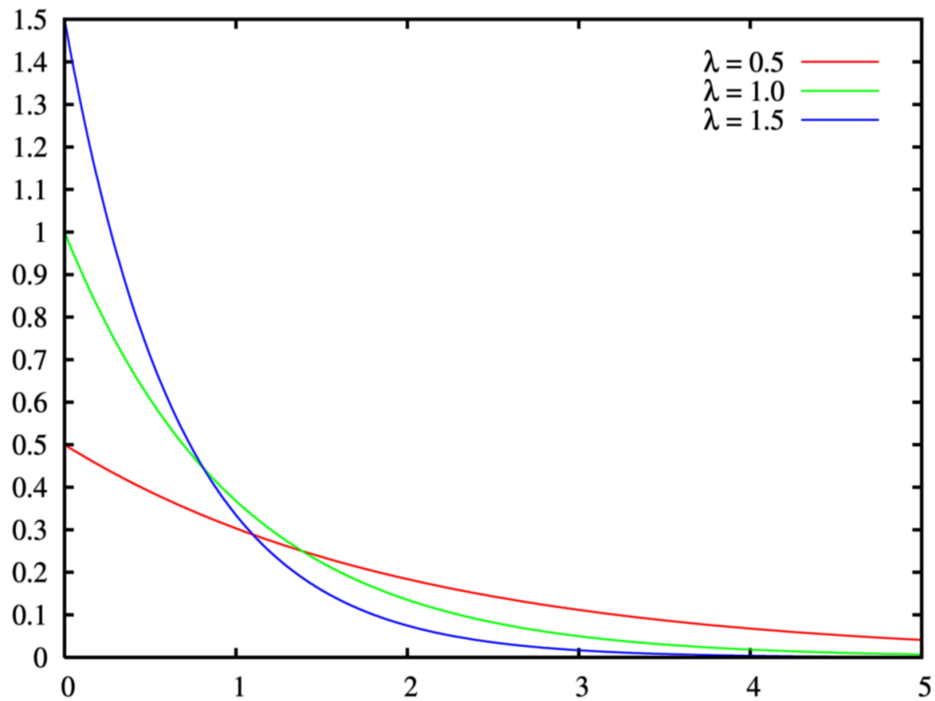
Ex. Suppose the height of all Columbia students can be described by a N(68, 4) model.

1. What proportion of students is shorter than 74 inches?
2. What proportion of students is taller than 74 inches?
3. How tall does a student have to be to be among the 10% tallest students?

The Exponential Distribution

- X is said to have an exponential distribution with parameter λ (λ>0) if the pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



More on Exponential

- Note that an exponential rv X can only take positive values. And the cdf of X is

$$F(x;\lambda) = \begin{cases} \int_0^x \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- Thus $P(X>x) = 1 - F(x;\lambda) = e^{-\lambda x}$

- Proposition: (proof?)

If X is an exponential rv with parameter λ, then
$E(X) = 1/\lambda$, $Var(X) = 1/\lambda^2$

- R command:

```
dexp(x, lamda=1),
pexp(q, lamda=1),
qexp(p, lamda=1).
```

Exponential Distribution and Poisson Distribution

- Suppose the number of events occurring in a time interval of length t has Poisson Distribution with parameter αt, and the numbers of occurrences in non-overlapping intervals are independent of one another. Then the distribution of elapsed time between the occurrence of two successive events is exponential with parameter $\lambda = \alpha$.

Example

Ex. Suppose you are waiting for a bus at a bus station. And the distribution of the length of the time you have to wait to get on the bus after you arrive at the bus station is exponentially distributed with parameter λ. Assume you have already waited for s minutes, how much longer do you expect to wait?

First, we have to figure out the conditional probability distribution of the additional waiting time given we have waited for s minutes. For any $t > 0$

$$
\begin{aligned}
P(X \geq s + t \mid X \geq s) &= P[(X \geq s + t) \cap (X \geq s)] / P(X \geq s) \\
&= P(X \geq s + t) / P(X \geq s) \\
&= e^{-\lambda t}
\end{aligned}
$$

which is again an exponential distribution! Thus the expected additional waiting time is $1/\lambda$.

Memoryless Property

- From the previous example, we know that if a waiting time (or lifetime of something) follows an exponential distribution, *the distribution of additional waiting time (lifetime) is exactly the same as the original distribution of waiting time (lifetime)*. In other words, the exponential distributed waiting time does NOT remember how much time you have waited, it starts afresh at any time!

- It is popular to model the distribution of component lifetime using the exponential distribution. However, the memoryless property may not be realistic in may applied problems. More general lifetime models can be furnished by the gamma, Weibull, and lognormal distributions. (Book: p159 – p168).

Joint Distribution

- How can we model two rv's using probability models? For example, if we are interested in both weight and height.

- Is it enough if we just use a normal model for weight and another normal model for height?

- We need to introduce joint probability distribution in order to model multiple rv's.

Joint PMF

- Let X and Y be two discrete rv's defined on the sample space. The joint probability mass function $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = P(X=x, Y=y).$$

- As in the single rv case, we must have $p(x, y) \geq 0$ and $\sum_x \sum_y p(x, y) = 1$.

Example

<u>Ex.</u> We randomly put two different balls into 3 numbered (numbered as {1,2,3}) boxes. Let X be the number of empty boxes left; let Y be the minimum of the box number that has balls in it. What is the joint distribution of (X, Y)?

X can take values from {1, 2};

Y can take values from {1, 2, 3};

It's not hard to see we have the following (why?):

$p(2, j) = P(X=2, Y=j) = 1/9$, for $j = 1, 2, 3$.

$p(1, 3) = P(X=1, Y=3) = 0$.

$p(1, 1) = P(X=1, Y=1) = 4/9$.

$p(1, 2) = P(X=1, Y=2) = 2/9$.

p_{ij}	1	2	3
1	4/9	2/9	0
2	1/9	1/9	1/9

Marginal PMF

- The marginal probability mass functions of X and Y, denoted by $p_X(x)$ and $p_Y(y)$, respectively, are given by

$$p_X(x) = \sum_y p(x,y) \qquad p_Y(y) = \sum_x p(x,y)$$

Ex.

p_{ij}	1	2	3	$p_X(x)$
1	$4/9$	$2/9$	0	2/3
2	$1/9$	$1/9$	$1/9$	1/3
$p_Y(y)$	5/9	1/3	1/9	

- Notice that the marginal probability mass functions are automatically proper pmf's. (why?)