S1211Q Introduction to Statistics
Lecture 6

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Midterm Practice Question 2.101

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- $P(B|A) = \frac{P(A \cap B)}{P(A)}$

Bayes Theorem and Tree Diagram

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- We can also find this probability on a Tree Diagram.

Independence

- Two events A and B are independent, *if and only if*

$$P(A \cap B) = P(A) \bullet P(B).$$

- Recall that independence implies:

  $P(A|B) = P(A)$

  $P(B|A) = P(B)$

  Therefore, by the multiplication rule for $P(A \cap B)$, we have

$$\begin{aligned} P(A \cap B) &= P(A|B) \bullet P(B) \\ &= P(B|A) \bullet P(A) \\ &= P(A) \bullet P(B) \end{aligned}$$

Example

Ex. A box contains 8 blue balls and 4 red balls. We draw two balls from the box without replacement. What is the probability that both are red?

A = first ball is red.

B = second ball is red.

$$
\begin{aligned}
P(\text{both balls are red}) = P(A \cap B) &= P(A)\, P(B|A) \\
&= 4/12 * 3/11 \\
&= 1/11
\end{aligned}
$$

More general "multiplication rule": $P(A \cap B \cap C) = P(C|A \cap B)\, P(B|A)\, P(A)$

Question: Draw three balls without replacement, what is the probability that all are red?

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- Finite Population v.s. Infinite Population

Multiple Events

- Events $A_1, \ldots, A_n$ are mutually independent if for every k ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, i_2, \ldots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})\, P(A_{i_2}) \ldots P(A_{i_k}).$$

- Independence is very very important!

Example

Ex. You recently bought a new set of tires from a manufacturer who just announced a recall because 2% of that particular brand were defective. What is the probability that at least one of your tires is defective? You may assume that the tires are defective independently of one another.

P(at least one defective tire) = 1 – P(no defective tire)

Let A_i = tire *i* is not defective

$P(A_i) = 1-0.02 = 0.98$

$$P(\text{no defective tire}) = P(A_1 \cap A_2 \cap A_3 \cap A_4)$$
$$= P(A_1)\,P(A_2)\,P(A_3)\,P(A_4) = (0.98)^4$$

$P(\text{at least one defective tire}) = 1-(0.98)^4 = 0.0776$

Random variables

- A random variable is a variable whose value is a numerical outcome of a random phenomenon.

- For a given sample space *S* of some experiment, a random variable (rv) is any rule that associates a number with each outcome in *S*.

- To put it more mathematically, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

- Remark: a random variable is NOT a sample space.

Discrete vs. Continuous

- X is a discrete random variable if its possible values either constitute a finite set or else can be listed in an infinite sequence in which there is a first element, a second element, and so on ("countably" infinite).

- X is a continuous random variable if it takes all possible values in an interval of numbers or all numbers in a disjoint union of such intervals. No possible value of the variable has positive probability, that is, $P(X=c) = 0$ for any possible value c.

- X can also be a random variable with a mixture distribution of both discrete and continuous components.

PMF

- The probability model for a discrete random variable X, lists its possible values and their probabilities.

Value of X	x_1	x_2	……..	x_k
Probability	p_1	p_2	……..	p_k

- Every probability, p_i, is a number between 0 and 1.
- $p_1+p_2+\ldots+p_k=1$

- The probability distribution or probability mass function (pmf) of a discrete rv is defined for every number x by $p(x) = P(X=x) = P(\text{all } s \in S\text{: } X(s)=x)$.

- How to check if some function $p(x)$ is a proper PMF?

Bernoulli RV

- The arguably simplest probability model is Bernoulli. Any random variable whose possible values are only 0 and 1 is called a Bernoulli random variable.

Ex. Flip a coin. S={H, T}. X is a Bernoulli random variable. X(H)=1, X(T)=0.

$$P(X=1) = 0.5,\ P(X=0) = 0.5.$$

Ex. Roll a die. S={1, 2, 3, 4, 5, 6}. X is a Bernoulli random variable. X(1)=1, X(2)=1, X(3)=0, X(4)=0, X(5)=0, X(6)=0.

$$P(X=1) = 1/3,\ P(X=0) = 2/3.$$

Example

<u>Ex.</u> Flip three fair coins. (*Binomial*)

S = {HHH, HHT, HTH, HTT, THT, THH, TTH, TTT}. Let's define random variable X to be the number of heads in the experiment, i.e., X(HHH)=3, X(THT)=1, etc.

X
0 TTT
1 TTH THT HTT
2 THH HTH HHT
3 HHH

Value of X	0	1	2	3
Probability	0.125	0.375	0.375	0.125

One can calculate the probability of an event by adding the probabilities p_i of the particular values of x_i that make up the event. For example, if we want to know the probability of getting less than 2 heads, we can use

$$P(X<2) = P(X=0) + P(X=1) = 0.125 + 0.375 = 0.5$$

Note: $P(X\leq 2) = P(X=0) + P(X=1) + P(X=2) = 0.875$

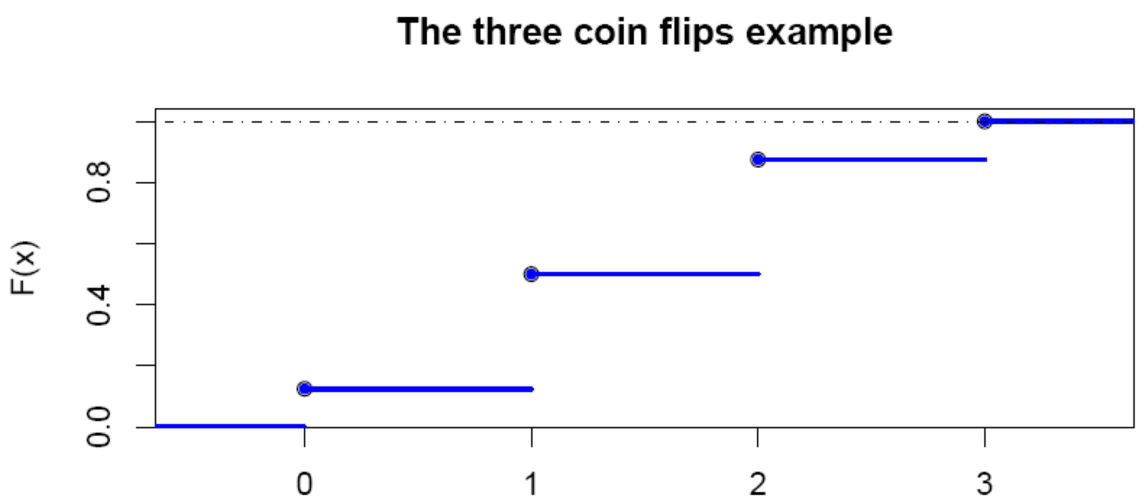
CDF

- The cumulative distribution function (cdf) F(*x*) of a discrete rv variable X with pmf *p*(*x*) is defined for every number *x* by

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \leq x) = \sum_{y: y \leq x} p(y).$$

  For any number *x*, F(*x*) is the probability that the observed value of X will be at most *x*.

- For X a discrete rv, the graph of F(*x*) will have a jump at every possible value of X and will be flat between possible values. Such a graph is called a step function.



Parameter and Family

- Suppose $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a parameter of the distribution. The collection of all probability distributions for different values of the parameter is called a family of probability distributions.

Ex. For Bernoulli rv's, the parameter is the probability of being 1 (or 0), that is,

$$p = P(X=1)$$

Expectation and Variance

- Random variables have distributions, so they have centers and spreads.
- The expected value (mean value or expectation) of a random variable describes its theoretical long-run average value.
- We typically use μ or $E(X)$ to denote the mean, $Var(X)$ to denote the variance and σ or $SD(X)$ to denote the standard deviation of a rv X.

Motivating examples

Ex. How many heads would you expect if you flipped a fair coin twice?

S = {HH, HT, TH, TT}.

X = number of heads.

0 TT

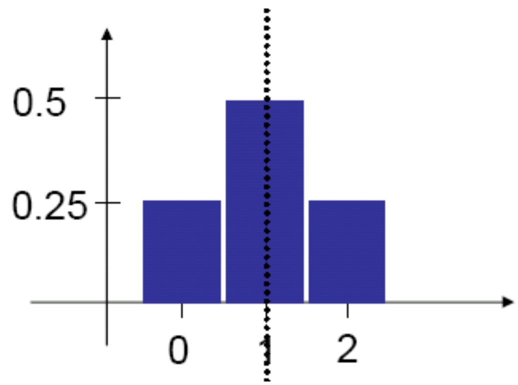
1 HT TH

2 HH

$p(X=0) = 0.25$; $p(X=1) = 0.5$; $p(X=2) = 0.25$.

Each outcome is weighted by its probability.

$\mu = 0 \times 0.25 + 1 \times 0.5 + 2 \times 0.25 = 1$



Example

Ex. How many heads would you expect if you flipped a coin three times?

$$\mu = 0 \times 0.125 + 1 \times 0.375 + 2 \times 0.375 + 3 \times 0.125 = 1.5$$

This can never occur in a single trial of 3 flips. However, on average we would expect to get 1.5 heads if we repeated the experiment many times.

Definition

- Suppose X is a discrete random variable whose probability model is given by

Value of X	x_1	x_2	……..	x_k
Probability	p_1	p_2	……..	p_k

The expected value of X is given by

$$\mathrm{E}(\mathrm{X}) = \mu_X = \sum_{x \in D} x \cdot p(x) = x_1 p_1 + x_2 p_2 + \cdots x_k p_k$$

Example

Ex. Expectation of a Bernoulli rv.

$$p(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & x \neq 0, 1 \end{cases}$$

$\mu = 0 \times (1-p) + 1 \times p = p$.

Example

Ex. The general form for the pmf of X = number of children born up to and including the first boy is,

$$p(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

1. Verify that this is a proper pmf.
2. Calculate the expected value of X.

The Expected Value of a Function

- A bookstore purchases ten copies of a books at \$60 each to sell at \$120, and any unsold copies after three months can be redeemed for \$20. If the number of copies sold is X, what is the profit of the bookstore?

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- The profit is $h(X) = 100X - 400$. Is $h(X)$ a Random Variable? Then what is the expectation of $h(X)$?
- If Random Variable X has range D and pmf $p(x)$, then the expected value of function $h(X)$ is given by

$$E(h(X)) = \sum_{x \in D} \big(h(x) \cdot p(x)\big)$$

The Linear Function Case

- In the case of linear function, we have a much more convenient formula

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- To prove,
$$E(aX+b) = \sum_{x \in D} (ax+b) \cdot p(x) = a \sum_{x \in D} x \cdot p(x) + b \sum_{x \in D} p(x) = aE(X) + b$$