CLT

- Theorem:

The Central Limit Theorem (CLT)

Let X_1, X_2, …, X_n, be an i.i.d. sequence from a distribution with mean μ and variance σ^2. Then if n is sufficiently large, the sample mean $\bar{X}$ has approximately a normal distribution with $\mu_{\bar{X}} = \mu$ and $\sigma_{\bar{X}}^2 = \sigma^2/n$;
And the sample total has approximately a normal distribution with $\mu_T = n\mu$, $\sigma_T^2 = n\sigma^2$. The larger the value of n, the better the approximation.

- Rule of Thumb: if n>30, the CLT can be used.

Daily examples of statistics

Ex. This is an example of *statistical learning* as well as an example of *classification*. The following table is summarized from 4601 email messages, in a study to try to predict whether the email was junk email, or “spam.” For all 4601 email messages, the true outcome (email type) *email* or *spam* is available, along with the relative frequencies of 57 of the most commonly occurring words and punctuation marks in the email message.

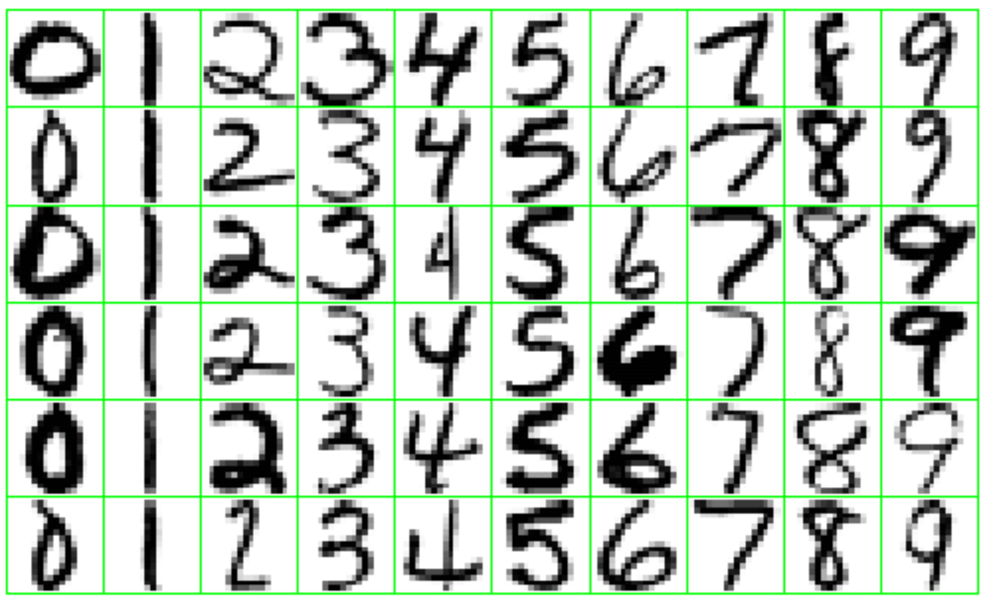
	george	you	your	hp	free	hpl	!	our	re	edu	remove
spam	0.00	2.26	1.38	0.02	0.52	0.01	0.51	0.51	0.13	0.01	0.28
email	1.27	1.27	0.44	0.90	0.07	0.43	0.11	0.18	0.42	0.29	0.01

Rule #1: if $(\%\texttt{george} < 0.6)$ & $(\%\texttt{you} > 1.5)$ then $\texttt{spam}$
else $\texttt{email}$.

Rule #2: if $(0.2 \cdot \%\texttt{you} - 0.3 \cdot \%\texttt{george}) > 0$ then $\texttt{spam}$
else $\texttt{email}$.

Daily examples of statistics

<u>Ex.</u> The data from this example come from the handwritten ZIP codes on envelopes from U.S. postal mail. The images are 16×16 eight-bit grayscale maps, with each pixel ranging in intensity from 0 to 255. Some sample images are shown in the following figure, The task is to predict, from the 16 × 16 matrix of pixel intensities, the identity of each image (0, 1, . . . , 9) quickly and accurately.



Statistical Inference

- From the previous two examples, we know that quite often, we need to infer the truth (population) from some partial information (sample).

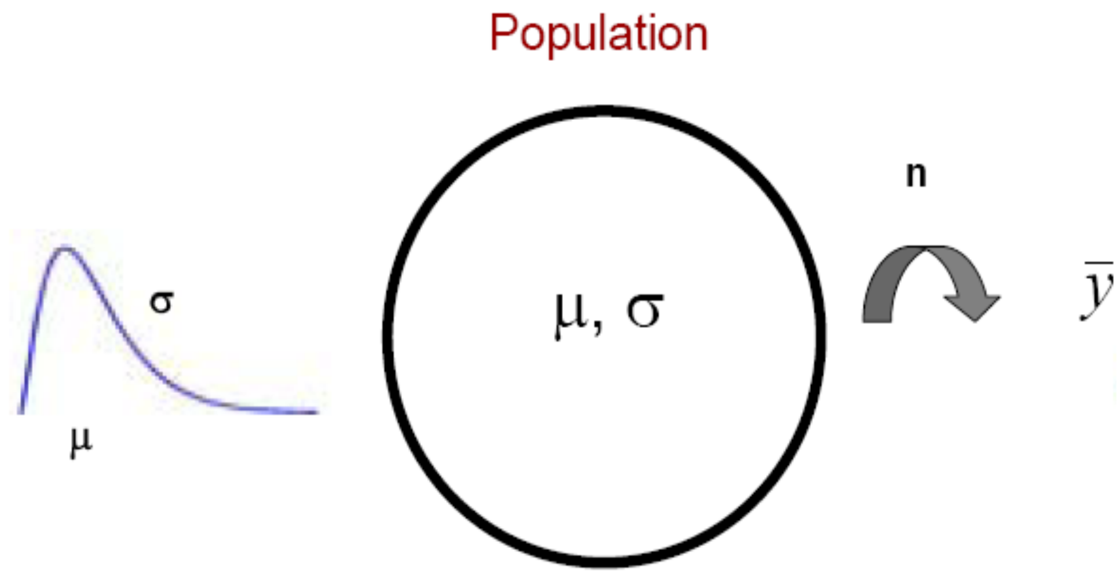
- Question: why do we need a model?

- Statistical inference comprises the use of statistics and random sampling to make inferences concerning some unknown aspect of a population.

- A point estimate of a parameter θ is a single number that can be regarded as a sensible value for θ. A point estimate is obtained by selecting a suitable statistic and computing its value from the given sample data. The selected statistic is called the point estimator of θ.

Sampling scheme for a Mean

- Usually our problem set up will be as illustrated in the graph.



- The actual sample observations $y_1, y_2, \ldots, y_n$ (realizations) are assumed to be the result of a random sample $Y_1, Y_2, \ldots, Y_n$ (random variables) from a certain distribution.

Estimating probability

<u>Ex.</u> A biased coin has probability *p* of having heads and *p* is unknown. Suppose we flipped the coin for 100 times and had 73 heads. What is your best guess for *p*?

Naturally, people would use $\text{estimator } \hat{p} = \dfrac{\text{number of heads}}{\text{number of flips}} = \dfrac{73}{100} = 0.73$

In other words, we are using the sample proportion to estimate the population probability.

Is this a good estimator? Are there any other estimators?

Guidelines on Estimation

1. Decide what (population mean/variance, proportion, etc.) to estimate and how (sample mean/variance, proportion, etc.) to estimate.

2. Obtain the sample, and calculate the estimator based on “how” from 1.

3. Report the estimator as well as its associated error.

More estimators

- In most problems, there will be more than one reasonable estimators.

Ex. Use `rnorm` to generate 20 normal random variables with mean 3 and standard deviation 1. Let's denote them as x_1, x_2, …, x_{20}. Now we pretend we do NOT know anything about the true underlying distribution. Come up with some estimators for the mean parameter.

(mean, trimmed-mean, median, partial mean)

Measure of a good Estimator

- Our estimator $\hat{\theta}$ is in fact a function of the sample x_i's, therefore, it is also a random variable. For some samples, $\hat{\theta}$ may yield a value larger than θ, whereas for other samples $\hat{\theta}$ may underestimate θ.

- The quantity $\hat{\theta}$ - θ characterize the error of estimation. A good estimator should result in small estimation errors.

- A commonly used measure of accuracy is the mean square error.

$$\mathrm{MSE} = \mathrm{E}(\hat{\theta} - \theta)^2$$

- However, since MSE will generally depend on the value of θ, finding an estimator with smallest MSE is typically NOT possible.

Unbiased Estimators

- One way to find good estimators, is to restrict our attention just to estimators that have some specified desirable properties and then find the best in this restricted group.

- One popular property is *unbiasedness*.

- A point estimator $\hat{\theta}$ is said to be an unbiased estimator of θ if $E(\hat{\theta}) = \theta$ for every possible value of θ. If $\hat{\theta}$ is not unbiased, the difference $E(\hat{\theta}) - \theta$ is called the bias of $\hat{\theta}$.

Example

Ex. Recall the unbiased coin example. Is the sample proportion an unbiased estimator of the population probability?

$$\text{estimator } \hat{p} = \frac{\text{number of heads}}{\text{number of flips}} = \frac{73}{100} = 0.73$$

What distribution does “number of heads” follow? What is its expectation?

General Result

- Proposition:

  When X is a binomial rv with parameters n and p, the sample proportion $\hat{p}$ = X/n is an unbiased estimator of p.

Example

Ex. Suppose that X, the reaction time to a certain stimulus, has a uniform distribution on the interval from 0 to an unknown upper limit θ. It is desired to estimate θ based on random sample $x_1, x_2, \ldots, x_n$. What is the best guess of θ? Is your estimator unbiased?

General Result

- Proposition:

Let X_1, X_2, …, X_n be an i.i.d. sequence of random samples from a distribution with mean μ and variance σ^2. Then the estimator

$$\hat{\sigma}^2 = S^2 = \frac{\sum_{i=1}^{n}(\mathrm{X}_i - \bar{\mathrm{X}})^2}{n-1}$$

is an unbiased estimator of σ^2.

General Result

- Proposition:

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sequence of random samples from a distribution with mean μ. Then the sample mean $\bar{X}$ is an unbiased estimator of μ. If in addition the distribution is continuous and symmetric, then the sample median M and any trimmed mean are also unbiased estimators of μ.

MVUE

- For unbiased estimators, what are their MSE's?

$$\mathrm{E}(\hat{\theta}-\theta)^2 = \mathrm{E}(\hat{\theta}-\mathrm{E}(\hat{\theta}))^2 = \mathrm{Var}(\hat{\theta})$$

- Among all estimators of θ that are unbiased, choose the one that has minimum variance. The resulting $\hat{\theta}$ is called the minimum variance unbiased estimator (MVUE) of θ.

- One needs more knowledge to actually identify if some estimator is really MVUE. But in a special case, we have the following theorem.

> Let X_1, X_2, …, X_n be an i.i.d. sequence of random samples from a normal distribution with mean μ and σ. Then the estimator $\hat{\mu} = \bar{\mathrm{X}}$ is the MVUE for μ.

The Standard Error

- When reporting a point estimator, one also reports the standard error associated with it.

- The standard error of an estimator $\hat{\theta}$ is its standard deviation $\sigma_{\hat{\theta}} = \sqrt{\mathrm{Var}(\hat{\theta})}$. If the standard error itself involves unknown parameters whose values can be estimated, substitution of these estimates into $\sigma_{\hat{\theta}}$ yields the estimated standard error of the estimator, which we denote as $\hat{\sigma}_{\hat{\theta}}$.

- The associated standard error gives us an idea of how good/accurate the estimators are.

Examples

<u>Ex.</u> What is the standard error of the sample proportion?

<u>Ex.</u> What is the standard error of the sample mean? Given that we know the variance.

<u>Ex.</u> What is the standard error of the sample mean? Given that we don’t know the variance.

Methods of Point Estimation

- The definition of unbiasedness does not in general indicate how unbiased estimators can be derived.

- There are two commonly used “constructive” methods for obtaining point estimators: the method of moments and the method of maximum likelihood.

- Although maximum likelihood estimators are generally preferable to moment estimators because of certain efficiency properties, they often require significantly more computation than do moment estimators.

- It is NOT guaranteed that these two methods would yield unbiased estimators.