# CLT

- Theorem:

> **The Central Limit Theorem (CLT)**
>
> Let $X_1$, $X_2$, …, $X_n$, be an i.i.d. sequence from a distribution with mean $\mu$ and variance $\sigma^2$. Then if n is sufficiently large, the sample mean $\bar{X}$ has approximately a normal distribution with $\mu_{\bar{X}} = \mu$ and $\sigma^2_{\bar{X}} = \sigma^2/n$;
> And the sample total has approximately a normal distribution with $\mu_T = n\mu$, $\sigma_T^2 = n\sigma^2$. The larger the value of n, the better the approximation.

- Rule of Thumb: if n>30, the CLT can be used.

# Distribution of a Linear Combination

- Sample mean is a particular case of linear combinations.
- The expectation and variance of a general linear combination

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

is given by the following result.

# A key result ***

Let $X_1, X_2, \ldots, X_n$, have mean values $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively.

- Whether or not the Xi's are independent,

$$\begin{aligned} \mathrm{E}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) &= a_1\mathrm{E}(\mathrm{X}_1) + a_2\mathrm{E}(\mathrm{X}_2) + \cdots + a_n\mathrm{E}(\mathrm{X}_n) \\ &= a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n \end{aligned}$$

- For any $X_1, X_2, \ldots, X_n$,

$$\mathrm{Var}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}(\mathrm{X}_i, \mathrm{X}_j)$$

  If they are independent, then

$$\begin{aligned} &\mathrm{Var}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) \\ =\ & a_1^2\mathrm{Var}(\mathrm{X}_1) + a_2^2\mathrm{Var}(\mathrm{X}_2) + \cdots + a_n^2\mathrm{Var}(\mathrm{X}_n) \\ =\ & a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2 \end{aligned}$$

# Special Cases

- $E(X+Y) = E(X) + E(Y)$;
- $E(X-Y) = E(X) - E(Y)$;
- $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- $Var(X-Y) = Var(X) + Var(Y) - 2Cov(X, Y)$
- If X and Y are independent, then $Cov(X, Y) = 0$, and
  $Var(X+Y) = Var(X) + Var(Y)$
  $Var(X - Y) = Var(X) + Var(Y)$

# Example

Ex. Show that if X ~ Bin($n$, $p$), then E(X) = $np$, and Var(X) = $np(1 - p)$.

Ex. Show that if X is a negative binomial rv with pmf $nb(x; r, p)$, then E(X) = $r(1-p)/p$, Var(X) = $r(1 - p)/p^2$.

# Statistical Inference

- From the previous two examples, we know that quite often, we need to infer the truth (population) from some partial information (sample).

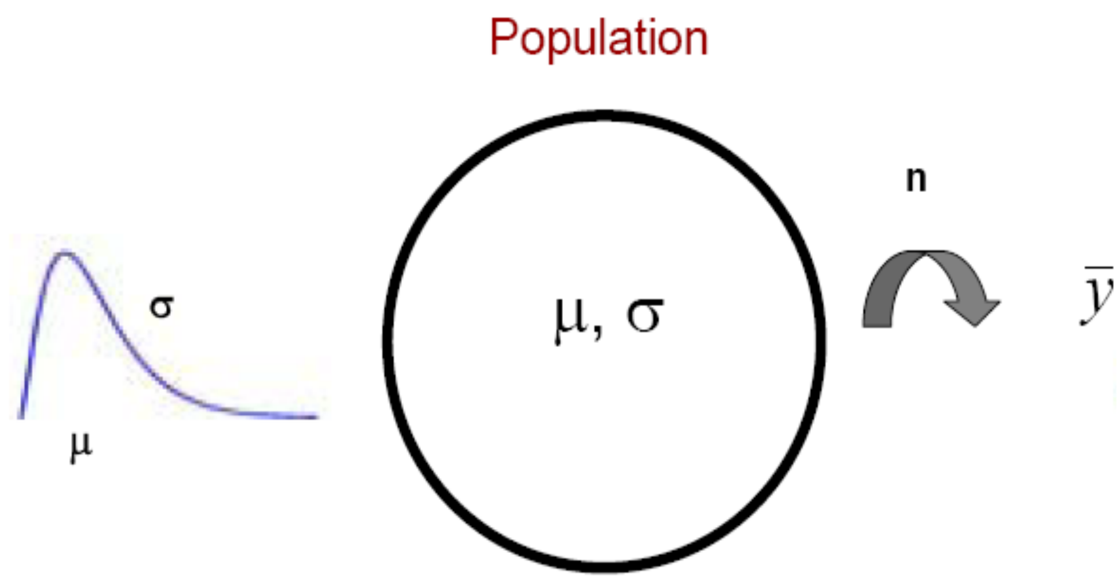
- Question: why do we need a model?

- Statistical inference comprises the use of statistics and random sampling to make inferences concerning some unknown aspect of a population.

- A point estimate of a parameter $\theta$ is a single number that can be regarded as a sensible value for $\theta$. A point estimate is obtained by selecting a suitable statistic and computing its value from the given sample data. The selected statistic is called the point estimator of $\theta$.

# Sampling scheme for a Mean

- Usually our problem set up will be as illustrated in the graph.




- The actual sample observations $y_1, y_2, \ldots, y_n$ (realizations) are assumed to be the result of a random sample $Y_1, Y_2, \ldots, Y_n$ (random variables) from a certain distribution.

# Estimating probability

Ex. A biased coin has probability $p$ of having heads and $p$ is unknown. Suppose we flipped the coin for 100 times and had 73 heads. What is your best guess for $p$?

Naturally, people would use $\text{estimator } \hat{p} = \frac{\text{number of heads}}{\text{number of flips}} = \frac{73}{100} = 0.73$

In other words, we are using the sample proportion to estimate the population probability.

Is this a good estimator? Are there any other estimators?

# Measure of a good Estimator

- Our estimator $\hat{\theta}$ is in fact a function of the sample $x_i$'s, therefore, it is also a random variable. For some samples, $\hat{\theta}$ may yield a value larger than $\theta$, whereas for other samples $\hat{\theta}$ may underestimate $\theta$.

- The quantity $\hat{\theta}$ - $\theta$ characterize the error of estimation. A good estimator should result in small estimation errors.

- A commonly used measure of accuracy is the mean square error.

$$\mathrm{MSE} = \mathrm{E}(\hat{\theta} - \theta)^2$$

- However, since MSE will generally depend on the value of $\theta$, finding an estimator with smallest MSE is typically NOT possible.

# Unbiased Estimators

- One way to find good estimators, is to restrict our attention just to estimators that have some specified desirable properties and then find the best in this restricted group.

- One popular property is *unbiasedness*.

- A point estimator $\hat{\theta}$ is said to be an unbiased estimator of $\theta$ if E( $\hat{\theta}$ ) = $\theta$ for every possible value of $\theta$. If $\hat{\theta}$ is not unbiased, the difference E( $\hat{\theta}$ ) – $\theta$ is called the bias of $\hat{\theta}$ .

# Example

Ex. Recall the unbiased coin example. Is the sample proportion an unbiased estimator of the population probability?

$$\text{estimator } \hat{p} = \frac{\text{number of heads}}{\text{number of flips}} = \frac{73}{100} = 0.73$$

What distribution does “number of heads” follow? What is its expectation?

# General Result

- Proposition:

  When X is a binomial rv with parameters $n$ and $p$, the sample proportion $\hat{p}$ = X/$n$ is an unbiased estimator of $p$.

# General Result

- Proposition:

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sequence of random samples from a distribution with mean $\mu$ and variance $\sigma^2$. Then the estimator

$$\hat{\sigma}^2 = S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$$

is an unbiased estimator of $\sigma^2$.

# General Result

- Proposition:

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sequence of random samples from a distribution with mean $\mu$. Then the sample mean $\bar{X}$ is an unbiased estimator of $\mu$. If in addition the distribution is continuous and symmetric, then the sample median M and any trimmed mean are also unbiased estimators of $\mu$.

# Estimator, Its Standard Error and Estimated Standard Error

- Now we are trying to estimate the probability of getting heads of a biased coin, so each flip $X_i$ is a Bernoulli RV with parameter $p$, the estimator of parameter $p$ is the sample mean/proportion
$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n}$$

# Estimator, Its Standard Error and Estimated Standard Error

- Now we are trying to estimate the probability of getting heads of a biased coin, so each flip $X_i$ is a Bernoulli RV with parameter $p$, the estimator of parameter $p$ is the sample mean/proportion
$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n}$$

- If we flipped 100 times and observed 75 heads, then our estimate of $p$ is
$$\hat{p} = \frac{75}{100} = 0.75$$

# Estimator, Its Standard Error and Estimated Standard Error

- Now we are trying to estimate the probability of getting heads of a biased coin, so each flip $X_i$ is a Bernoulli RV with parameter $p$, the estimator of parameter $p$ is the sample mean/proportion

$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n}$$

- If we flipped 100 times and observed 75 heads, then our estimate of $p$ is

$$\hat{p} = \frac{75}{100} = 0.75$$

- Also, we need to report how good our estimator is through its Standard Error. This is also related to the Interval Estimation.

# Estimator, Its Standard Error and Estimated Standard Error

- The standard error is $\sqrt{Var(\hat{p})} = \sqrt{\frac{p(1-p)}{n}}$, but we cannot report it since we don't know what $p$ is.
- So we can only report the estimated standard error of the estimator $\hat{p}$

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

# Another Example

- Now we have $X_1, X_2, \ldots, X_N$ IID with mean $\mu$ and variance $\sigma^2$, what's the estimator of $\mu$?

# Another Example

- Now we have $X_1, X_2, \ldots, X_N$ IID with mean $\mu$ and variance $\sigma^2$, what's the estimator of $\mu$?
- $$\hat{\mu} = \bar{X}$$

# Another Example

- Now we have $X_1, X_2, \ldots, X_N$ IID with mean $\mu$ and variance $\sigma^2$, what's the estimator of $\mu$?
- $$\hat{\mu} = \bar{X}$$
- The standard error of $\hat{\mu}$ is $\sqrt{Var(\hat{\mu})} = \frac{\sigma}{\sqrt{n}}$. Can we report $\frac{\sigma}{\sqrt{n}}$?

# Another Example

- Now we have $X_1, X_2, \ldots, X_N$ IID with mean $\mu$ and variance $\sigma^2$, what's the estimator of $\mu$?
- $$\hat{\mu} = \bar{X}$$
- The standard error of $\hat{\mu}$ is $\sqrt{Var(\hat{\mu})} = \frac{\sigma}{\sqrt{n}}$. Can we report $\frac{\sigma}{\sqrt{n}}$?
- It really depends on whether or not we know $\sigma$. If we know it, then we can report $\frac{\sigma}{\sqrt{n}}$; otherwise, we can only report $\frac{\hat{\sigma}}{\sqrt{n}}$.

# Methods of Point Estimation

- The definition of unbiasedness does not in general indicate how unbiased estimators can be derived.

- There are two commonly used “constructive” methods for obtaining point estimators: the method of moments and the method of maximum likelihood.

- Although maximum likelihood estimators are generally preferable to moment estimators because of certain efficiency properties, they often require significantly more computation than do moment estimators.

- It is NOT guaranteed that these two methods would yield unbiased estimators.

# Population Moment and Sample Moment

- Let $X_1, \ldots, X_n$ be a random sample from a pmf or pdf $f(x)$. For $k = 1, 2, \ldots$, the $k$th population moment is $E(X^k)$. The $k$th sample moment is $(1/n)\sum_{i=1}^{n} X_i^k$ .
- The essence of the Methods of Moment is to equate population moments with sample moments and solve the resulting equations.

# Moment Estimators

- Definition:

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a pmf or pdf $f(x)$. For $k = 1, 2, 3, \ldots$, the moment estimator for the $k$th population moment, is the $k$th sample moment, i.e.,

$$\widehat{\mathrm{E}(\mathrm{X}^k)} = \frac{\sum_{i=1}^{n} \mathrm{X}_i^k}{n}$$

# Example

<u>Ex.</u> Show that the sample proportion is the moment estimator of the population probability.

# Example

<u>Ex.</u> Let $X_1$, $X_2$, …, $X_n$ be an i.i.d. normal sample, and assume that the underlying normal distribution is $N(\mu,\sigma^2)$ where $\mu,\sigma^2$ are unknown. How can we construct moment estimators to estimate the two unknown parameters?

As we already know if $X \sim N(\mu,\sigma^2)$, then $E(X) = \mu$, and $E(X^2) = \mu^2+\sigma^2$.

Therefore, we have two equations:

$$\begin{cases} \hat{\mu} = \sum_{i=1}^{n} \mathrm{X}_i/n \\ \hat{\mu}^2 + \hat{\sigma}^2 = \sum_{i=1}^{n} \mathrm{X}_i^2/n \end{cases} \longrightarrow \begin{cases} \hat{\mu} = \sum_{i=1}^{n} \mathrm{X}_i/n \\ \hat{\sigma}^2 = \sum_{i=1}^{n} \mathrm{X}_i^2/n - \bar{\mathrm{X}}^2 \end{cases}$$

Is the variance estimator unbiased?

# Example

Ex. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from exponential distribution with parameter $\lambda$ which is unknown. How do we estimate $\lambda$ using moment estimator?

As we already know if $X \sim \text{Exp}(\lambda)$, then $E(X) = 1/\lambda$.

Thus, we have equation $1/\hat{\lambda} = \bar{X} \rightarrow \hat{\lambda} = 1/\bar{X}$.

Is this estimator unbiased?

# Maximum Likelihood Est.

- The method of maximum likelihood was first introduced by R.A. Fisher, a geneticist and statistician, in the 1920s. It is by far the most commonly used method to obtain estimators.

- Likelihood function is just another way of looking at the *joint pmf or the pdf*. In particular, let $X_1, X_2, \ldots, X_n$ (not necessarily i.i.d.) have joint pmf or pdf

$$f(x_1, x_2, \ldots, x_n; \theta_1, \ldots, \theta_m)$$

  where $\theta_1, \ldots, \theta_m$ are parameters whose values are unknown. When $x_1, x_2, \ldots, x_n$ are the observed sample values and $f(.)$ is then regarded as a function of $\theta_1, \ldots, \theta_m$, it is called the likelihood function.

# Example

Ex. A biased coin has been flipped for 10 times. Let $X_1, X_2, \ldots, X_{10}$ denote the outcomes of the coin flips. Assume the probability of having a head is $p$ (parameter of interest), and the sample we observed is {0,1,1,0,0,0,1,0,0,0}. Write down the likelihood function for $p$.

$$f(x_1, x_2, \ldots, x_n; p) = f(x_1; p)\, f(x_2; p) \ldots f(x_n; p) = (1-p)\, p\, p\, (1-p) \ldots (1-p) = p^3(1-p)^7$$

Idea of Maximum Likelihood: can we find a $p$ that can maximize the above function?

# MLE

- The maximum likelihood estimates (mle's) $\hat{\theta}_1, \ldots, \hat{\theta}_m$ are those values of $\theta_i$'s that maximize the likelihood function, so that

$$f(x_1, \ldots, x_n; \hat{\theta}_1, \ldots, \hat{\theta}_m) \geq f(x_1, \ldots, x_n; \theta_1, \ldots, \theta_m) \quad \text{for all } \theta_1, \ldots, \theta_m$$

  when the $X_i$'s are substituted in place of the $x_i$'s.

- Remark: the likelihood function tells us how likely the observed sample is as a function of the possible parameter values. Maximizing the likelihood gives the parameter values for which the observed sample is most likely to have been generated – that is, the parameter values that "agree most closely" with the observed data.

- In practice, in stead of maximizing the likelihood itself, people usually choose to maximize the log-likelihood function.

# Example

<u>Ex.</u> Let $X_1$, $X_2$, …, $X_n$ be an i.i.d. sample from exponential distribution with parameter $\lambda$ which is unknown. Write down the likelihood function for $\lambda$. What is the MLE of $\lambda$? Is the MLE unbiased?

Since we have an i.i.d. sample, it is easy to see that the likelihood function is a product of the individual pdf's:

$$f(x_1, \ldots, x_n; \lambda) = (\lambda e^{-\lambda x_1}) \cdot \cdots \cdot (\lambda e^{-\lambda x_n}) = \lambda^n e^{-\lambda \sum x_i}$$

$$\Downarrow$$

$$\log[f(x_1, \ldots, x_n; \lambda)] = n \log(\lambda) - \lambda \sum x_i$$

$$\Downarrow$$

$$\hat{\lambda} = n / \sum \mathrm{X}_i$$

# Example with Normal

- Let $X_1, X_2, \ldots, X_n$ be an IID sample from normal distribution with mean $\mu$ and variance $\sigma^2$, what is the likelihood function?

# Example with Normal

- Let $X_1, X_2, \ldots, X_n$ be an IID sample from normal distribution with mean $\mu$ and variance $\sigma^2$, what is the likelihood function?
- $$f(x_1, x_2, \ldots, x_n; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

  or in logarithm

  $$-\frac{n}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{n}[-(x_i - \mu)^2/\sigma^2]$$

# Example with Normal

- Let $X_1, X_2, \ldots, X_n$ be an IID sample from normal distribution with mean $\mu$ and variance $\sigma^2$, what is the likelihood function?
- $$f(x_1, x_2, \ldots, x_n; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

  or in logarithm

  $$-\frac{n}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{n}[-(x_i - \mu)^2/\sigma^2]$$
- Take derivative with respect to $\mu$ and $\sigma^2$ and solve the resulting equations

  $$\hat{\mu} = \bar{X}, \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$$

# Some complications

- The following is an example that MLE's can't be calculated analytically.

Ex. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from Weibull distribution with parameters $\alpha$ and $\beta$ and pdf

$$f(x;\alpha,\beta) = \begin{cases} \frac{\alpha}{\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-(x/\beta)^\alpha} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

by solving equations $\dfrac{\partial \log(f)}{\partial \alpha} = 0 \quad \dfrac{\partial \log(f)}{\partial \beta} = 0$

$$\hat{\alpha} = \left[ \frac{\sum x_i^{\hat{\alpha}} \cdot \log(x_i)}{\sum x_i^{\hat{\alpha}}} - \frac{\sum \log(x_i)}{n} \right]^{-1} \qquad \hat{\beta} = \left( \frac{\sum x_i^{\hat{\alpha}}}{n} \right)^{1/\hat{\alpha}}$$

# Some Complications

- Also, sometimes we cannot use calculus to get the MLE, such as when the density is not differentiable.
- Read Example 6.22 on textbook P.262.

# The Invariance Principle

- One of the nice features of MLE's is that, the MLE of a function of parameters, is the function of the MLE's of the parameters.

- More specifically, we have

> Let $\hat{\theta}_1, \ldots, \hat{\theta}_m$ be the MLE's of the parameters $\theta_1$, …, $\theta_m$. Then the MLE of any function $h(\theta_1, \ldots, \theta_m)$ of these parameters is $h(\hat{\theta}_1, \ldots, \hat{\theta}_m)$.

<u>Ex.</u> In the normal example, what is the MLE of $\sigma$?

# Large Sample Behavior

- The following proposition says, for large samples, it is “optimal” to use MLE’s, because it is asymptotically unbiased and has the minimal variance among all unbiased estimators.

- Proposition:

> Under very general conditions on the joint distribution of the sample,
> When the sample size n is large, the maximum likelihood estimator is
> Approximately the MVUE of the parameter.

# Confidence Intervals

- A point estimate, because it is a single number, by itself provides no information about the precision and reliability of estimation (the reason why we need standard error).

- An alternative to reporting a single sensible value for the parameter being estimated is to calculate and report an entire interval of plausible values – an *interval estimate* or *confidence interval* (*CI*).

- A confidence interval is always calculated by first selecting a *confidence level*, which is a measure of the degree of reliability of the interval.

- Construct a confidence interval for a standard normal random variable.

# Illustration

- Let's first consider a simple, somewhat unrealistic problem situation.
    1. We are interested in the population mean parameter $\mu$.
    2. The population distribution is normal.
    3. The value of the population standard deviation $\sigma$ is known. (unlikely!)

- Suppose we have a random sample $X_1, X_2, \ldots, X_n$ from a normal distribution with mean value $\mu$ and standard deviation $\sigma$. As we know, $\bar{X}$ also follows a normal distribution with mean value $\mu$ and standard deviation $\sigma/\sqrt{n}$. Thus, we could get a standard normal distribution by normalizing $\bar{X}$.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

# Construction

- The smallest interval that contains 95% of the possible outcomes of Z is (-1.96, 1.96).

$$-1.96 < \frac{\bar{\mathrm{X}} - \mu}{\sigma / \sqrt{n}} < 1.96$$

$$\Downarrow$$

$$-1.96 \cdot \frac{\sigma}{\sqrt{n}} < \bar{\mathrm{X}} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

$$\Downarrow$$

$$\bar{\mathrm{X}} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{\mathrm{X}} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

# Interpretation

- Thus we have $\mathrm{P}\left(\bar{\mathrm{X}} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{\mathrm{X}} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) = 0.95$.

- Some people interpreted this as: the true parameter $\mu$ has 95% chance of falling in the interval of $(\bar{\mathrm{X}} - 1.96 \cdot \sigma/\sqrt{n}, \bar{\mathrm{X}} + 1.96 \cdot \sigma/\sqrt{n})$. Is it right?

- In fact, the two boundaries of the interval given above are random! Thus every time we sample n observations from the same population, we will get a different confidence interval!