W1211 Introduction to Statistics
Lecture 3

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Measure of Variability

- Two samples that have same mean or median might have drastically different spreads.
- So to numerically summarize the sample, we need a measure of variability.

Standard deviation

- The variance and standard deviation are measures of spread that indicate how far values in the data set are from the mean, on average.

- Consider the observations $x_1, x_2, x_3, \ldots, x_n$.

- The deviations $(x_i - \bar{x})$ display the spread of x_i about their mean $\bar{x}$.

- The sum of the deviations is always 0, as some of the deviations are positive and others are negative.

- Squaring the deviations makes them all positive. Observations far from the mean will have large positive squared deviations.

- The variance is the 'average' squared deviation.

Standard deviation

- If we have *n* observations $x_1, x_2, x_3, \ldots, x_n$. The variance is defined as

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- The standard deviation, *s*, is the square root of the variance.
  1. *s* is a measure of spread about the mean and should be used when the mean is used as the measure of center.
  2. If *s*=0, then all the values in the data set are exactly the same (no spread). Why?
  3. The more spread out the data, the greater the standard deviation.
  4. *s* is always positive.
  5. *s* has the same unit of measurement as the original data

A short cut formula for s^2

Theorem

An alternative expression for variance s^2 is

$$s^2 = \frac{1}{n-1}\Big(\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2\Big)$$

Proof.

Do some algebra on the numerator.

$$\begin{aligned}
\sum_{i=1}^{n}(x_i - \bar{x})^2 &= \sum_{i=1}^{n}(x_i^2 - 2\bar{x}\cdot x_i + (\bar{x})^2) \\
&= \sum x_i^2 - 2\bar{x}\sum x_i + \sum(\bar{x})^2 \\
&= \sum x_i^2 - 2\bar{x}\cdot n\bar{x} + n(\bar{x})^2 \\
&= \sum x_i^2 - n(\bar{x})^2
\end{aligned}$$

$\square$

Fourth Spread

- Fourth Spread is a robust measure of variability based on quartiles.
- It is defined as
$$f_s = \text{upper fourth} - \text{lower fourth}$$

Boxplot

- A five number summary lists, in order, the minimum, Q1, the median, Q3, and the maximum.

- A boxplot is a graphical representation using a five number summary.
  1. Draw a vertical (horizontal) measurement scale.
  2. Place a rectangle to the right of (above) this axis; the lower (left) edge of the rectangle is at the lower fourth, and the upper (right) edge is at the upper fourth.
  3. Place a horizontal (vertical) line segment inside the rectangle at the location of the median.
  4. Draw "whiskers" out from either end of the rectangle to the smallest and largest observations that are NOT outliers.
  5. Using dots to represent outliers.

- R demo. `>boxplot(x)`

Probability

What is randomness?

- The world is full of random events that we seek to understand.
- An event is random if we know what outcomes could occur, but not the particular values that will happen.
- The outcome of these events is uncertain, but they follow a regular pattern.
- Deterministic models vs. Random models.
- Probability theory is the mathematical representation of random phenomena.

Notation

- An experiment is any action or process whose outcome is subject to uncertainty. e.g. tossing a coin once or several times; selecting a card or cards from a deck; weighing a loaf of bread; etc.

- The sample space of an experiment, denoted by S, is the set of all possible outcomes of that experiment.

Ex. Flip a coin. Two possible outcomes: Heads (H) or Tails (T). $S=\{H,T\}$.

Ex. Battery life. $S=\{x: 0\leq x <\infty\}$.

Notation

- An event is any collection of possible outcomes, that is, any subset of S (including S itself). An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcome.

- If the outcome of a random phenomenon is contained in an event A, then we say that A has occurred.

Ex. Flip a coin twice. Four possible outcomes, S={HH, HT, TH, TT}. Let A be the event that we obtain at least one H in the two flips. A={HH, HT, TH}. Let B be the event that we obtain two H's in the two flips. B={HH}.

Ex. Battery life example. The event that the battery lasts less than 3 hours is denoted as $A=\{x: 0 \leq x < 3\}$.

Set Operations

- Given any two events (or sets) A and B, we have the following elementary set operations:

  The union

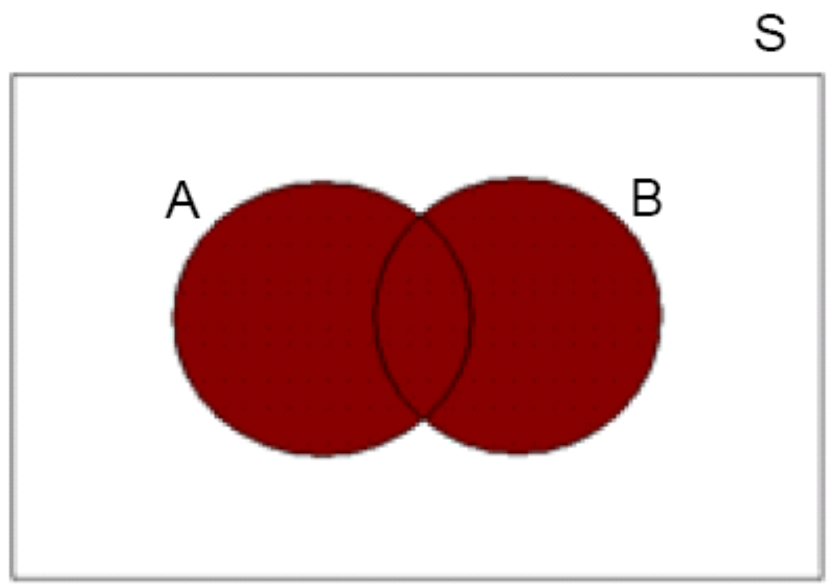
  The intersection

  The complement

- Venn diagrams are often used to illustrate relationships between sets.

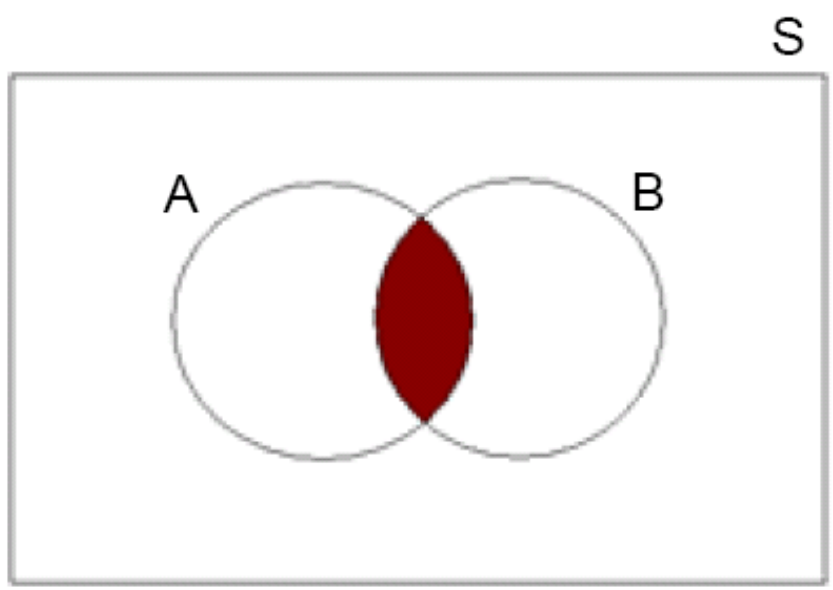
Union

- The union of A and B, written as AUB and read “A or B”, is the set of outcomes that belong to either A or B or both.



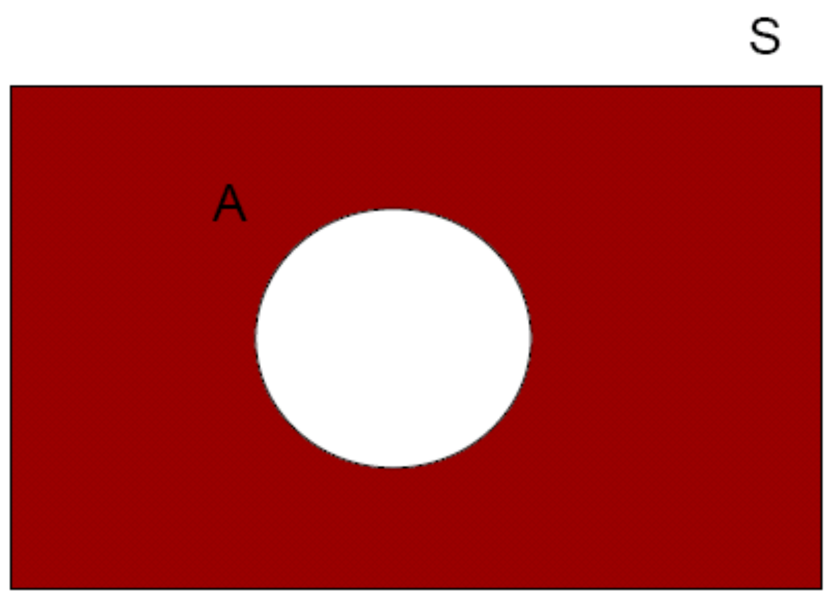
Intersection

- The intersection of A and B, written as A∩B, read "A and B", is the set of outcomes that belong to both A and B.



Complement

- The complement of A, written as A’ or A^c, is the set of all outcomes in *S* that are not in A.



Example

Ex. Select a card at random from a standard deck of cards, and note its suit: clubs (Cl), diamonds (D), hearts (H) or spades (Sp).

The sample space is S={Cl, D, H, Sp}.

Let: A={Cl, D}, B={D, H, Sp} and C={H}.

AUB={Cl, D, H, Sp}=S

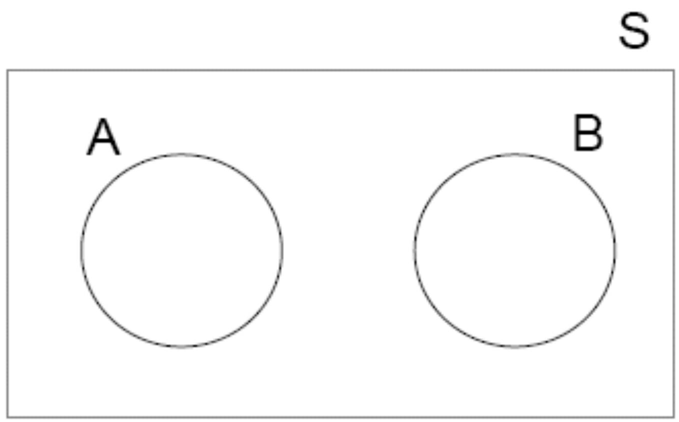
$A \cap B$={D}

A^c={H, Sp}

$A \cap C = \emptyset$ (null event – event consisting of no outcomes)

Disjoint events

- If $A \cap B = \emptyset$ then A and B are said to be mutually exclusive or disjoint events.



- Any event and its complement are disjoint!

Probability models

- A probability model consists of a sample space and the assignment of probabilities to each possible outcome.

- Probability that event A occurs is written as P(A), which will give a precise measure of the chance that A will occur.

- To ensure the probability assignments will be consistent with our intuitive notions of probability, all assignments should satisfy the following axioms (basic properties) of probability.
    1. For any event A, $P(A) \geq 0$.
    2. $P(S)=1$.
    3. If $A_1, A_2, A_3, \ldots$ is an infinite (finite) collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum P(A_i)$$

Propositions

- For any event A, $0 \leq P(A) \leq 1$.
- $P(A) + P(A^c) = 1$.
- If event A is contained in event B, in the sense that every outcome in A is also in B, then
$$P(A) \leq P(B)$$
- $P(\emptyset) = 0$.

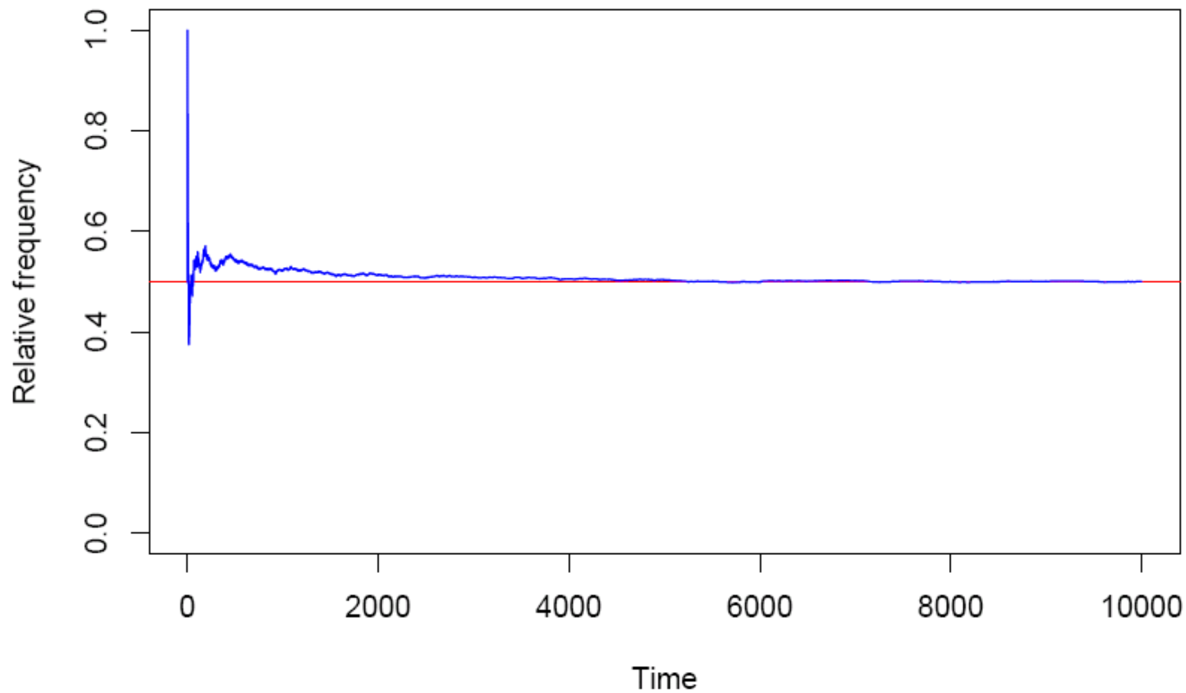
Interpreting Probability

- What does it mean when we say we have 50% chance of having a head when flipping a coin? Or what does it mean when we put $P(H)=0.5$?

- Probability is often treated as the *long-term relative frequency* or the *limiting relative frequency*.

Interpreting Probability

<u>Ex.</u> Flip a fair coin *n* times and calculate the proportion of heads.



- R demo. (Function: `sample(x, size)`; `rbinom(x, size, prob)`)

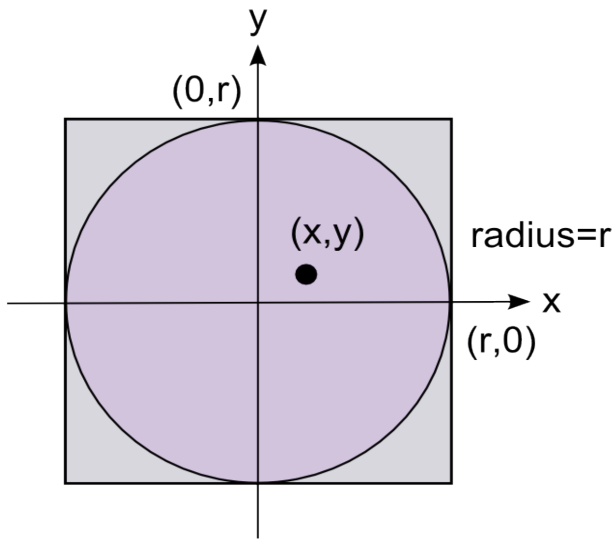
Law of Large Numbers

- The law of large numbers says that the long-run relative frequency of repeated independent events gets closer and closer to the true relative frequency as the number of trials increases.

$$\frac{\text{\# of occurrence of event A}}{\text{\# of trials } (n)} \xrightarrow{n \to \infty} P(A)$$

How to calculate Pi

- An interesting application of Law of Large Numbers is to calculate Pi through simulations.
- If we spread a large quantity of seeds randomly but evenly on this square, what percentage of the seeds will lie inside the circle?



- R Demo.
- This type of simulation-based methods has a fancy name: Monte Carlo methods.

Assigning Probabilities

- The assignment of probabilities can often be derived from the physical set-up of an experiment.

- Suppose we have N outcomes in our sample space, each equally likely to occur. The each has a probability of $1/N$, and the probability of any event A is,

$$P(A) = \frac{\text{number of outcomes in A}}{N}$$

Ex. Roll a fair die. $S=\{1,2,3,4,5,6\}$. Our sample space consists of 6 points, each of which is equally likely to occur.

P(roll a 1) = 1/6.

Let A = roll a 4 or less = {1,2,3,4}. P(A) = 4/6.

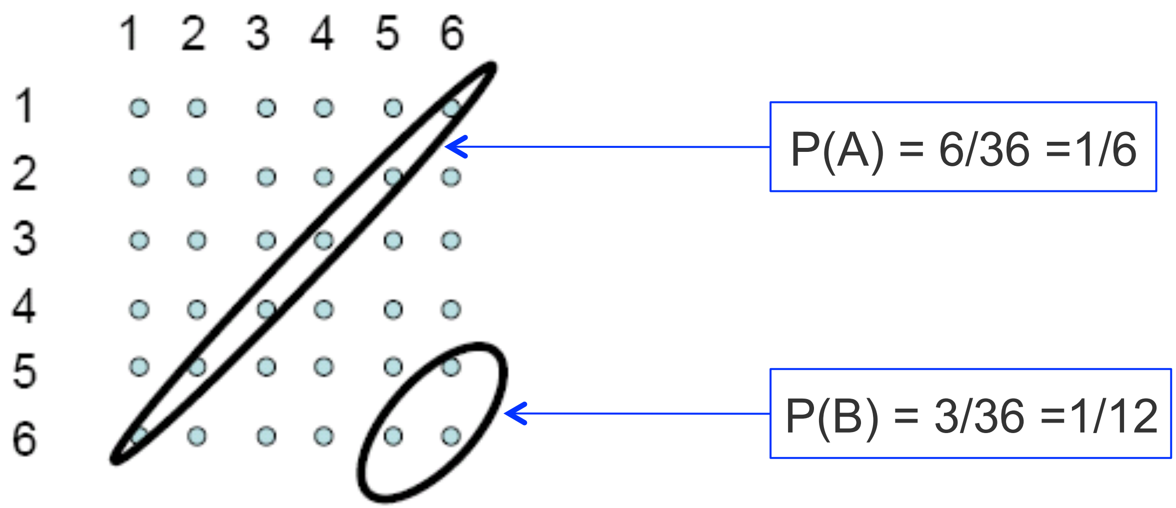
Let B = roll an even number = {2,4,6}. P(B) = 3/6.

Example

Ex. Roll two fair dice.

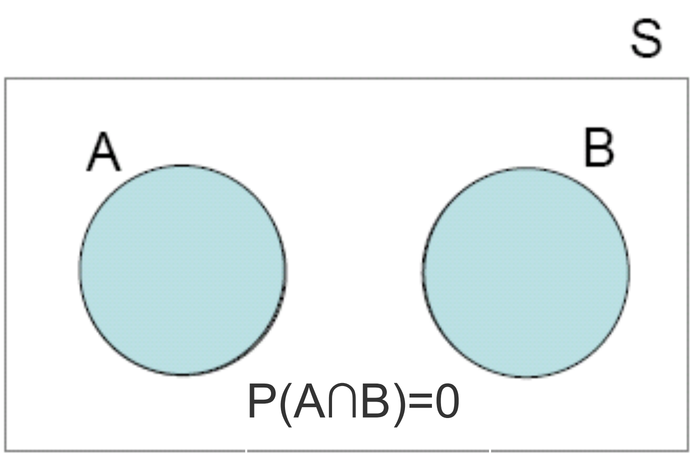
There are 36 possible outcomes: {(1,1),(1,2),(1,3),…,(6,5),(6,6)}.

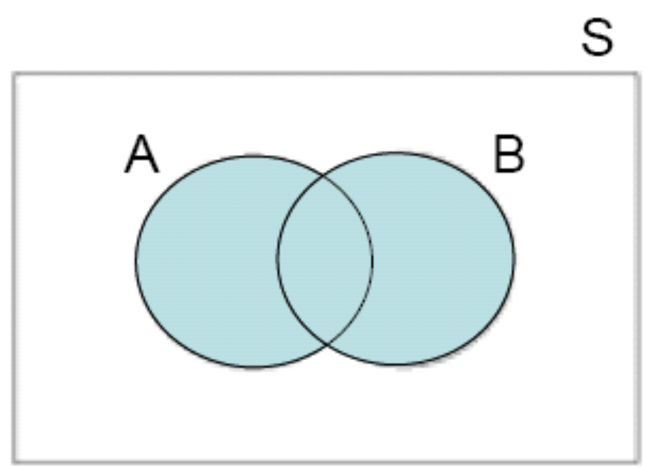
Let A = sum of two rolls is 7; B = sum of two rolls is 11 or more. What are P(A) and P(B)?



More Probability Properties

- Consider an experiment whose sample space is *S*. For each event A (B) in *S*, we assume that a number P(A) is defined and satisfies the following rules:
  1. $0 \leq P(A) \leq 1$.
  2. $P(S)=1$.
  3. $P(A^c)=1-P(A)$.
  4. If A and B are disjoint, then $P(A \cup B)=P(A)+P(B)$.
  5. For any two events A and B, $P(A \cup B)=P(A)+P(B)-P(A \cap B)$.





Example

Ex. A store accepts either VISA or Mastercard. 50% of the stores customers have VISA, 30% have Mastercard and 10% have both. What is the probability that a customer has a credit card the store accepts?

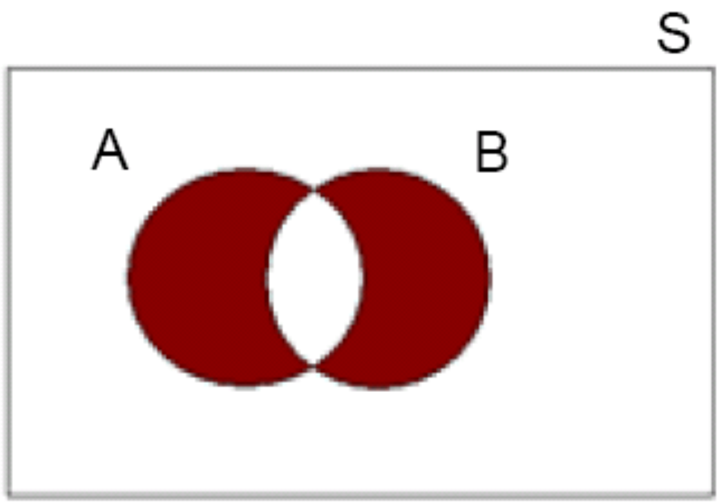
A = customers has VISA

B = customers has Mastercard

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.5 + 0.3 - 0.1 = 0.7$$

Example cont.

What is the probability that a customer has either a VISA or MC, but not both?



$$P(\text{A or B but not both}) = P(A) + P(B) - 2P(A \cap B)$$
$$= 0.5 + 0.3 - 0.2 = 0.6$$

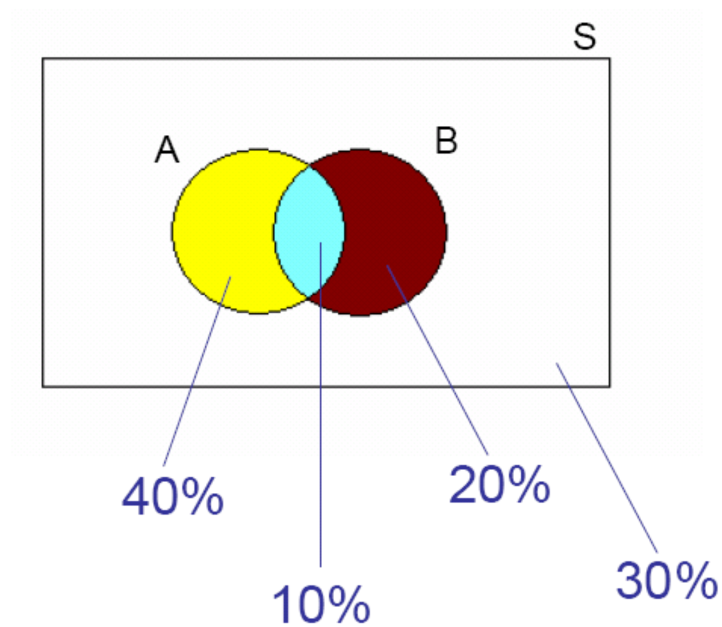
Example Cont.

What is the probability that a customer has a VISA but no MC?

$$P(\text{A but not both}) = P(A) - P(A \cap B)$$
$$= 0.5 - 0.1 = 0.4$$

What is the probability that a customer has a MC but no VISA?

$$P(\text{B but not both}) = P(B) - P(A \cap B)$$
$$= 0.3 - 0.1 = 0.2$$



Three Events

- For any three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$