Hypothesis Testing for a Population Mean

- In this section, the null hypothesis is about a population mean $H_0 : \mu = \mu_0$ and there are three possible Alternative Hypotheses $H_a : \mu > \mu_0$ or $H_a : \mu < \mu_0$ or $H_a : \mu \neq \mu_0$.
- We will discuss three cases which parallel our discussion about Confidence Interval for a Population Mean.
- - Case I: Normal Distribution and Known σ (z Test)
  - Case II: General Distribution, Unknown σ but Large Sample (z Test)
  - Case III: Normal Distribution and Unknown σ (t Test)

Case I: Normal Distribution and Known σ (z Test)

- Under the null hypothesis, the test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

  follow a standard normal distribution.

Case I: Normal Distribution and Known σ (z Test)

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Case I: Normal Distribution and Known σ (z Test)

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$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$
follow a standard normal distribution.
- If the Alternative Hypothesis is $H_a : \mu > \mu_0$, then the Rejection Region is something like $\{z \geq c\}$, where c is a constant to be determined.
- c is determined by the level of the test α, if we set c as z critical value z_α then
$$\begin{aligned} P(\text{type I error}) &= P(H_0 \text{ is rejected when } H_0 \text{ is true}) \\ &= P(Z > c \text{ when } Z \sim N(0,1)) = \alpha \\ \Rightarrow c &= z_\alpha \end{aligned}$$

Case I: Normal Distribution and Known σ (z Test)

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic value: $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Alternative Hypothesis	Rejection Region for Level α Test
$H_a: \mu > \mu_0$	$z \geq z_\alpha$ (upper-tailed test)
$H_a: \mu < \mu_0$	$z \leq -z_\alpha$ (lower-tailed test)
$H_a: \mu \neq \mu_0$	either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

-

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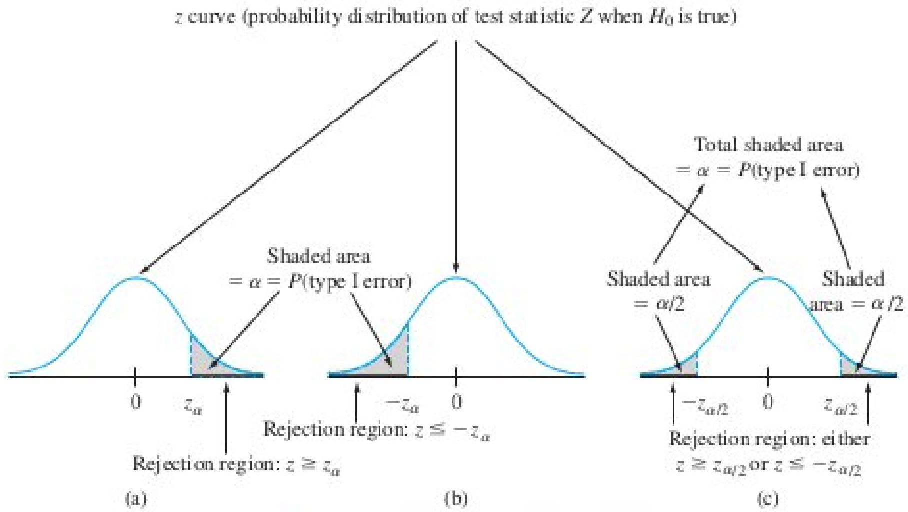
- 

  Figure 8.2 Rejection regions for z tests: (a) upper-tailed test; (b) lower-tailed test; (c) two-tailed test

Case I: Normal Distribution and Known σ (z Test)

- We can also compute Type II Error β and sample size n. Still we consider the upper-tailed test as a demonstration.
- Type II Error β will be a function of any particular number μ' that is larger than the null value μ_0.

$$\begin{aligned}
\beta(\mu') &= P(Z < z_\alpha \text{ when } \mu = \mu') \\
&= P(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < z_\alpha \text{ when } \mu = \mu') \\
&= P(\frac{\bar{X} - \mu'}{\sigma/\sqrt{n}} < z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \text{ when } \mu = \mu') \\
&= \Phi(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}) \leq 1 - \alpha
\end{aligned}$$

  $\Phi()$ is the CDF of standard normal.
- What is the power of the test?

Case I: Normal Distribution and Known σ (z Test)

- For a given True Value μ', Type I Error level α and Type II Error β, we can determin the sample size n that we need with

$$\Phi(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}) = \beta$$

$$\Rightarrow -z_\beta = z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}$$

$$\Rightarrow n = \frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}$$

Case I: Normal Distribution and Known σ (z Test)

Alternative Hypothesis	Type II Error Probability $\beta(\mu')$ for a Level α Test
H_a: $\mu > \mu_0$	$\Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
H_a: $\mu < \mu_0$	$1 - \Phi\left(-z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
H_a: $\mu \neq \mu_0$	$\Phi\left(z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

where $\Phi(z) =$ the standard normal cdf.

The sample size n for which a level α test also has $\beta(\mu') = \beta$ at the alternative value μ' is

$$n = \begin{cases} \left[\dfrac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a one-tailed (upper or lower) test} \\ \left[\dfrac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a two-tailed test (an approximate solution)} \end{cases}$$

Case I: Normal Distribution and Known σ (z Test)

- Example

Let μ denote the true average tread life of a certain type of tire. Consider testing H_0: $\mu = 30{,}000$ versus H_a: $\mu > 30{,}000$ based on a sample of size $n = 16$ from a normal population distribution with $\sigma = 1500$. A test with $\alpha = .01$ requires $z_\alpha = z_{.01} = 2.33$. The probability of making a type II error when $\mu = 31{,}000$ is

$$\beta(31{,}000) = \Phi\left(2.33 + \frac{30{,}000 - 31{,}000}{1500/\sqrt{16}}\right) = \Phi(-.34) = .3669$$

Since $z_{.1} = 1.28$, the requirement that the level .01 test also have $\beta(31{,}000) = .1$ necessitates

$$n = \left[\frac{1500(2.33 + 1.28)}{30{,}000 - 31{,}000}\right]^2 = (-5.42)^2 = 29.32$$

The sample size must be an integer, so $n = 30$ tires should be used. ■

Case II: General Distribution, Unknown σ but Large Sample (z Test)

- As we discussed in Confidence Interval, under the null hypothesis, the test statistic
$$Z = \frac{\bar{X} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$
approximately follow a standard normal distribution.
- The rule of thumb is $n > 40$.
- All the procedure, e.g., Test Statistic, Rejection Region and formula for β and sample size, are the same except for substituting σ with its estimator $\hat{\sigma}$.

Case III: Normal Distribution and Unknown σ (t Test)

- Under the null hypothesis, the test statistic

$$T = \frac{\bar{X} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$

follows a t distribution with degrees of freedom $n - 1$

Case III: Normal Distribution and Unknown σ (t Test)

- Under the null hypothesis, the test statistic

$$T = \frac{\bar{X} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$

follows a t distribution with degrees of freedom $n - 1$

- Test Procedure

The One-Sample t Test

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic value: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$

Alternative Hypothesis	Rejection Region for a Level α Test
H_a: $\mu > \mu_0$	$t \geq t_{\alpha,n-1}$ (upper-tailed)
H_a: $\mu < \mu_0$	$t \leq -t_{\alpha,n-1}$ (lower-tailed)
H_a: $\mu \neq \mu_0$	either $t \geq t_{\alpha/2,n-1}$ or $t \leq -t_{\alpha/2,n-1}$ (two-tailed)

Case III: Normal Distribution and Unknown σ (t Test)

- The calculation of Type II Error β is much more difficult than z Test.

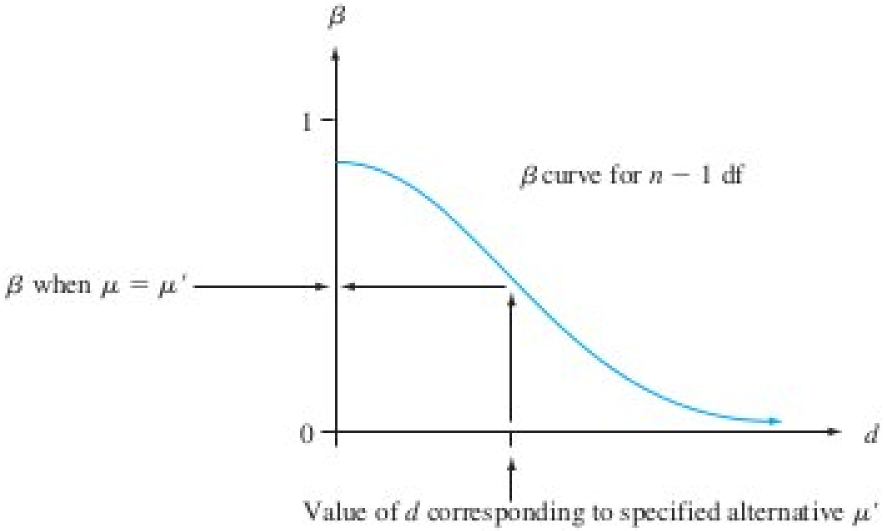
$$\beta(\mu') = P(T < t_{\alpha,n-1} \text{ when } \mu = \mu' \text{ rather than } \mu_0)$$

Case III: Normal Distribution and Unknown σ (t Test)

- The calculation of Type II Error β is much more difficult than z Test.

$$\beta(\mu') = P(T < t_{\alpha,n-1} \text{ when } \mu = \mu' \text{ rather than } \mu_0)$$

- A typical β curve



Hypothesis Testing for a Population Proportion

- Let p denote the proportion of individuals or objects in a population who possess a specified property (probability of success). In order to make inference about p, naturally we would look at the sample proportion, which is X/n. X is the number of Successes in the sample. In practice, X should follow a binomial distribution, and when n is large, it can further be approximated by a normal distribution.
- We first consider large sample tests.

Large-sample tests

- Thanks to the Central Limit Theorem, we have

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} \sim N(0, 1)$$

under the null hypothesis.

- Thus the rejection region is determined by

1. H_a: $p > p_0$: $Z > z_\alpha$
2. H_a: $p < p_0$: $Z < -z_\alpha$
3. H_a: $p \neq p_0$: $Z > z_{\alpha/2}$ or $Z_0 < -z_{\alpha/2}$

- The test procedures are valid provided that $np_0 \geq 10$ and $n(1-p_0) \geq 10$.

Example

<u>Ex.</u> (Defective rate cont.) A factory claims that less than 10% of the components they produce are defective. A consumer group is skeptical of the claim and checks a random sample of 300 components and finds that 39 are defective. Is there evidence that 10% of all components made at the factory are defective?

$H_0 : p = 0.10$ $H_a : p > 0.10$

$$\hat{p} = \frac{39}{300} = 0.13 \quad Z = \frac{0.13 - 0.1}{\sqrt{0.1(1-0.1)/300}} = 1.72$$

$z_{0.05} = 1.645$. $Z > z_{0.05}$, thus we would reject H_0 at level $\alpha = 0.05$.

Type II Error

- We can calculate Type II Error based on the large sample normal approximation

$$
\begin{aligned}
\beta(p') &= \mathrm{P}(H_0 \text{ is not rejected when } p = p') \\
&= \mathrm{P}\left(\frac{\hat{p}-p_0}{\sqrt{p_0(1-p_0)/n}} \leq z_\alpha | p = p'\right) \\
&= \mathrm{P}\left(\frac{\hat{p}-p'}{\sqrt{p_0(1-p_0)/n}} \leq z_\alpha + \frac{p_0-p'}{\sqrt{p_0(1-p_0)/n}} | p = p'\right) \\
&= \mathrm{P}\left(\frac{\hat{p}-p'}{\sqrt{p'(1-p')/n}} \leq \frac{z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}} + \frac{(p_0-p')}{\sqrt{p'(1-p')/n}} | p = p'\right) \\
&= \Phi\left(\frac{p_0-p'+z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)
\end{aligned}
$$

Determining sample size

- If we specify a particular alternative *p*' and specify a *β* value that can be tolerated (e.g. 0.1). Then from

$$\beta = \Phi\left(\frac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right) \Longrightarrow -z_\beta = \frac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}$$

- Therefore, in order to achieve the specified type I and type II error, one has to have a sample size of at least

$$n = \left(\frac{z_\alpha\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right)^2$$

- For two sided test, we have to change z_α to $z_{\alpha/2}$ in the above formula.

- Difference between the sample size calculation formula in chapter 7 and the one above.

Type II Error and Sample Size calculation

- In general Type II Error and Sample Size formulas are give below

Alternative Hypothesis	$\beta(p')$
$H_a:\ p > p_0$	$\Phi\left[\dfrac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$
$H_a:\ p < p_0$	$1 - \Phi\left[\dfrac{p_0 - p' - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$
$H_a:\ p \neq p_0$	$\Phi\left[\dfrac{p_0 - p' + z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$ $-\Phi\left[\dfrac{p_0 - p' - z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$

The sample size n for which the level α test also satisfies $\beta(p') = \beta$ is

$$n = \begin{cases} \left[\dfrac{z_\alpha\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right]^2 & \text{one-tailed test} \\[2ex] \left[\dfrac{z_{\alpha/2}\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right]^2 & \text{two-tailed test (an approximate solution)} \end{cases}$$

Example

<u>Ex.</u> A package-delivery service advertises that at least 90% of all packages brought to its office by 9 a.m. for delivery in the same city are delivered by noon that day. Let p denote the true proportion of such packages that are delivered as advertised and consider the hypothesis $H_0: p = 0.9$ versus $H_a: p < 0.9$. If only 80% of the packages are delivered, how likely is it that a level .01 test based on n=225 packages will detect such departure from H_0? What should the sample size be to ensure that $\beta(0.8) = 0.01$? With $\alpha = .01$, $p_0 = .9$, $p' = .8$, and $n = 225$.

Type II error:
$$\begin{aligned}\beta(p') &= 1 - \Phi\left(\frac{p_0 - p' - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)\\ &= 1 - \Phi\left(\frac{.9 - .8 - 2.33\sqrt{(.9)(.1)/225}}{\sqrt{(.8)(.2)/225}}\right)\\ &= 1 - \Phi(2.00) = .0228\end{aligned}$$

Example cont.

- Using $z_{.01}$=2.33, the sample size can then be calculated from

$$n = \left(\frac{z_\alpha\sqrt{p_0(1-p_0)/n} + z_\beta\sqrt{p'(1-p')/n}}{p'-p_0}\right)^2$$
$$= \left(\frac{2.33\sqrt{(.9)(.1)} + 2.33\sqrt{(.8)(.2)}}{.8-.9}\right)^2 \approx 266$$

- 1-β is often referred to as the power of a test. It is the probability that the test can actually detect the alternative given the alternative is true! For α-level tests, the bigger the power the better!

Small sample tests

- For testing population proportions, when the sample size is small, the normal approximation is no longer appropriate. Thus a more accurate test should be used.

- As mentioned before, the sample proportion is X/n. X is the number of *S*'s in the sample and can be treated as a binomial random variable. Thus a rejection region can be constructed using binomial cdf/pmf.

- Can we get an exact α-level test using binomial?

Notes on Normal Probability Plot

- Because of the important role that Normal Distribution plays in statistical inference, we often want to assess whether a given sample is roughly normal distributed. Normal Probability Plot is used for this purpose.

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- The basic strategy is to compare sample features with population features. In probability plot, we are using sample percentile(quantile) and population percentile(quantile), so it is also known as Q-Q plot.

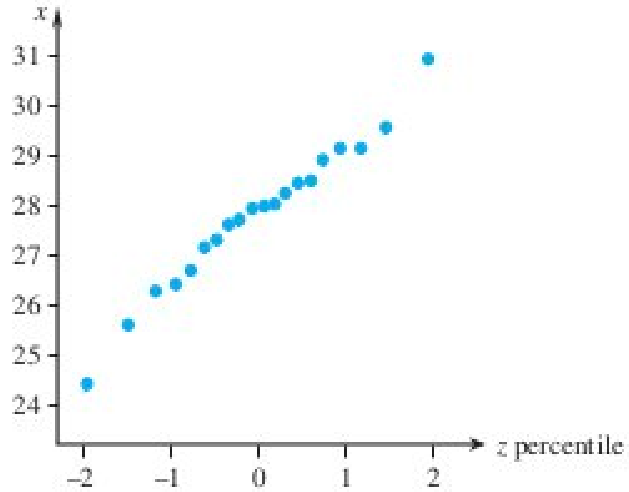
Notes on Normal Probability Plot

- Because of the important role that Normal Distribution plays in statistical inference, we often want to assess whether a given sample is roughly normal distributed. Normal Probability Plot is used for this purpose.
- The basic strategy is to compare sample features with population features. In probability plot, we are using sample percentile(quantile) and population percentile(quantile), so it is also known as Q-Q plot.
- The definition of a normal probability plot

> A plot of the n pairs
>
> ([100(i − .5)/n]th z percentile, ith smallest observation)
>
> on a two-dimensional coordinate system is called a **normal probability plot.** If the sample observations are in fact drawn from a normal distribution with mean value μ and standard deviation σ, the points should fall close to a straight line with slope σ and intercept μ. Thus a plot for which the points fall close to some straight line suggests that the assumption of a normal population distribution is plausible.

Examples of Normal Probability Plot

- A Normal Sample



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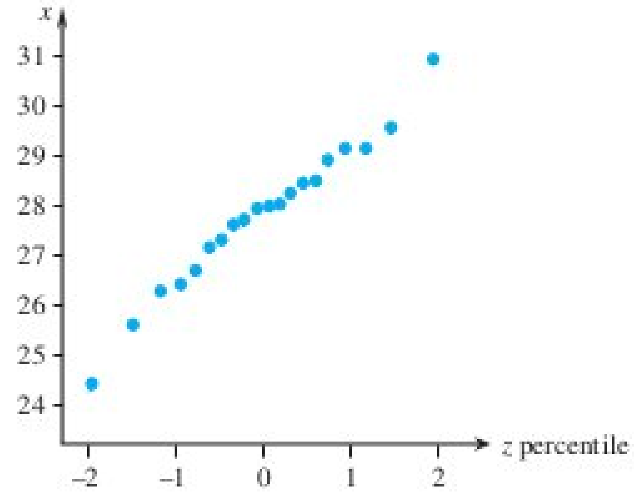


- Two Non-normal Samples

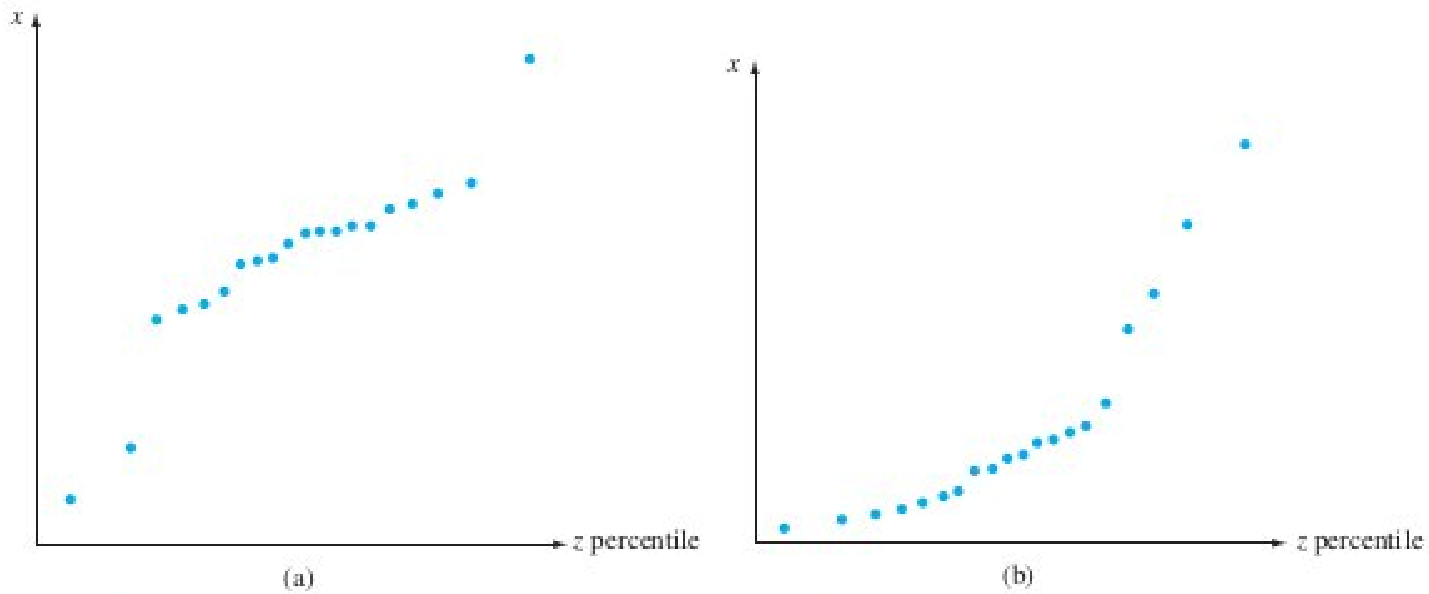


Figure 4.37 Probability plots that suggest a nonnormal distribution: (a) a plot consistent with a heavy-tailed distribution; (b) a plot consistent with a positively skewed distribution

P-Value

- To report the result of a hypothesis-testing analysis is to simply say whether the null hypothesis was rejected at a specified level of significance. This type of statement is somewhat inadequate because it says nothing about whether the conclusion was a very close call or quite clear cut.

- P-value is a quantity that conveys much information about the strength of evidence against H_0 and allows an individual decision maker to draw a conclusion at any specified level α.

- The P-value (*observed significance level*) is the probability, under the null hypothesis, that the test statistic is more ***extreme*** than the observed statistic.

What P-Values are not

- The P-value is not the probability that H_0 is true.
- The P-value is not Type I Error α.
- The P-value is not the significance level.
- The P-value is not Type II Error β

Comparison Between P-value and Type I Error α

- P-value=P(Test Statistic is more extreme than observed Test Statistic Value under Null Hypothesis)
- Type I Error=P(Test Statistic falls into Rejection Region under Null Hypothesis)

Remarks

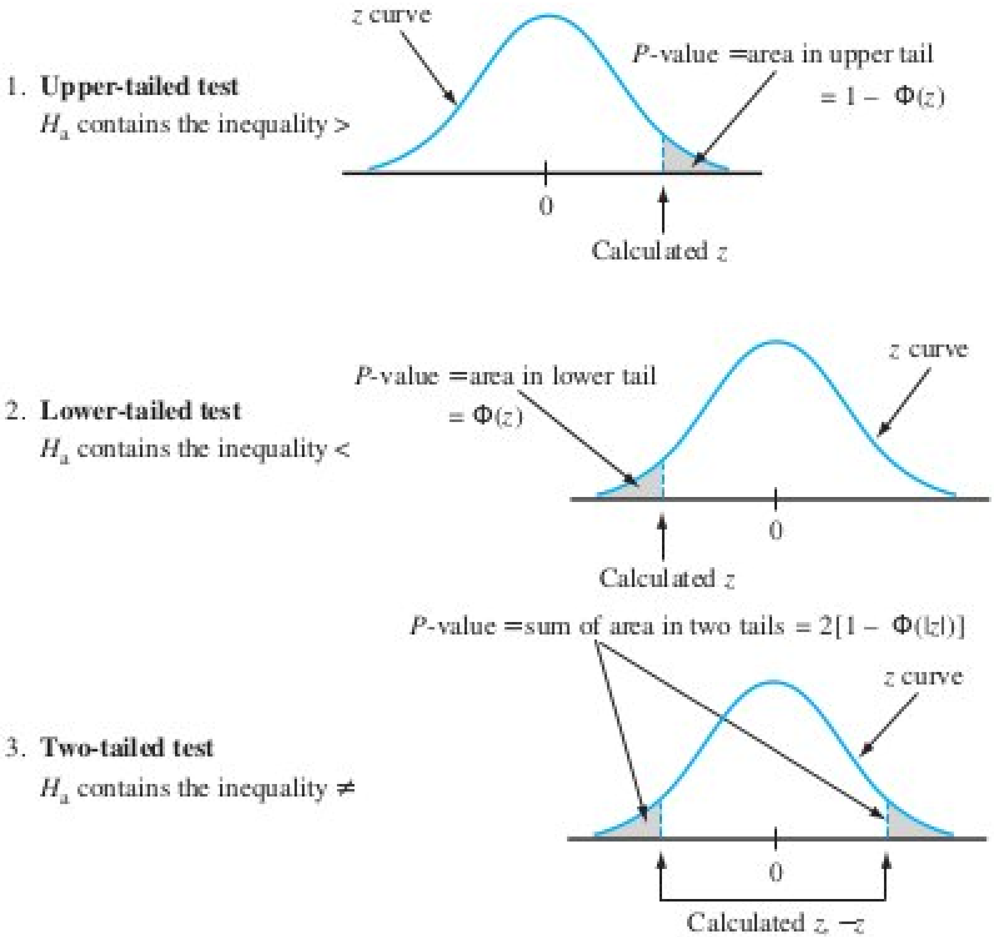
- **The smaller the P-value, the more evidence there is in the sample data against the null hypothesis and for the alternative hypothesis.**
- P-values can be seen as a more flexible procedure of Hypothesis Testing. The practical advantage is that it is easier to switch to a test of different significance level
- The decision rule based on P-values

> **Decision rule based on the *P*-value**
>
> Select a significance level α (as before, the desired type I error probability). Then
>
> $$\text{reject } H_0 \text{ if P-value} \leq \alpha$$
> $$\text{do not reject } H_0 \text{ if P-value} > \alpha$$

- The P-value is the smallest significance level α at which the null hypothesis can be rejected.

P-values and Tails

- Like Rejection Region, P-values are also related to the type of test we are concerning, uppper-tailed, lower-tailed or two-tailed.



Two sample tests

- A new drug is claimed to significantly reduce the blood pressure for high blood pressure patients. What kind of tests can we use to verify the claim?

- A new drug is claimed to perform much better in terms of reducing blood pressure than an old drug. What kind of tests can we use to verify the claim?

Things to cover

- As in the one sample testing problem, we will cover the following cases:
  1. Two normal populations with known variance.
  2. Two populations with unknown distribution and large sample size.
  3. Two normal populations with unknown variance.
  4. Two population proportions with large sample size.
  5. Tests about variances. (NOT required.)

- Basic assumptions for comparing population means:
  1. $X_1, X_2, \ldots, X_m$ is a random sample (i.i.d.) from a population with mean μ_1 and variance σ_1^2.
  2. $Y_1, Y_2, \ldots, Y_n$ is a random sample (i.i.d.) from a population with mean μ_2 and variance σ_2^2.
  3. The X and Y samples are independent of one another.

Test statistics

- Since we are comparing the population means, a natural test statistic to use would be the difference of two sample means. Because of independence we have,

$$\begin{aligned} E(\bar{X} - \bar{Y}) &= \mu_1 - \mu_2 \\ Var(\bar{X} - \bar{Y}) &= \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n} \end{aligned}$$

Case I: normal, known variance

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

Test statistic: $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^{\,2}}{m} + \dfrac{\sigma_2^{\,2}}{n}}} \sim \mathrm{N}(0,1)$

vs Alternative Hypothesis:

$H_a : \mu_1 - \mu_2 > \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^{\,2}}{m} + \dfrac{\sigma_2^{\,2}}{n}}} > Z_\alpha$

$H_a : \mu_1 - \mu_2 < \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^{\,2}}{m} + \dfrac{\sigma_2^{\,2}}{n}}} < -Z_\alpha$

$H_a : \mu_1 - \mu_2 \neq \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^{\,2}}{m} + \dfrac{\sigma_2^{\,2}}{n}}} < -Z_{\alpha/2}$ or $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^{\,2}}{m} + \dfrac{\sigma_2^{\,2}}{n}}} > Z_{\alpha/2}$

Questions

- How to compute P-value for case I?
- How to compute type II errors for case I?
- In a balanced design, derive the sample size calculation formula (for alternative ">"):

$$m = n = \frac{(\sigma_1^2 + \sigma_2^2)(Z_\alpha + Z_\beta)^2}{(\Delta' - \Delta_0)^2}$$

Case II: large sample

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

Test statistic: $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} \sim A\mathrm{N}(0,1)$

vs Alternative Hypothesis:

$H_a : \mu_1 - \mu_2 > \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} > Z_\alpha$

$H_a : \mu_1 - \mu_2 < \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} < -Z_\alpha$

$H_a : \mu_1 - \mu_2 \neq \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} < -Z_{\alpha/2}$ or $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} > Z_{\alpha/2}$

Questions

- How to construct confidence interval for $\mu_1 - \mu_2$ in case II?

Case III: normal, unknown variance

$H_0 : \mu_1 - \mu_2 = \Delta_0$

Test statistic: $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} \sim t_\nu$, ν is the df of the t-distribution and it's approximately estimated

by the sampled data: $\nu = \dfrac{\left(\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}\right)^2}{\dfrac{(s_1^2/m)^2}{m-1} + \dfrac{(s_2^2/n)^2}{n-1}}$, and round ν town to the nearest integer.

Case III cont.

vs Alternative Hypothesis:

$$H_a: \mu_1 - \mu_2 > \Delta_0 \text{, reject if } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} > t_{\alpha,\nu}$$

$$H_a: \mu_1 - \mu_2 < \Delta_0 \text{, reject if } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} < -t_{\alpha,\nu}$$

$$H_a: \mu_1 - \mu_2 \neq \Delta_0 \text{, reject if } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} < -t_{\alpha/2,\nu} \text{ or } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} > t_{\alpha/2,\nu}$$

Questions

- How to compute P-values of the test?
- How to construct confidence interval for $\mu_1 - \mu_2$ in case III?
- What if we know that $\sigma_1^2 = \sigma_2^2$?

  The *pooled estimator* of $\sigma^2 = \sigma_1^2 = \sigma_2^2$ is given by

$$S_p^2 = \frac{m-1}{m+n-2} \cdot S_1^2 + \frac{n-1}{m+n-2} \cdot S_2^2$$

Case IV

$H_0 : p_1 - p_2 = 0$

Test statistic: $\dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{m}+\dfrac{1}{n}\right)}}$, $\hat{p} = \dfrac{m}{m+n}\hat{p}_1 + \dfrac{n}{m+n}\hat{p}_2$ (the *weighted* average of $\hat{p}_1$

and $\hat{p}_2$)

Case IV cont.

vs Alternative Hypothesis:

$$H_a: p_1 - p_2 > 0 \text{, reject if } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} > Z_\alpha$$

$$H_a: p_1 - p_2 < 0 \text{, reject if } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} < -Z_\alpha$$

$$H_a: p_1 - p_2 \neq 0 \text{, reject if } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} > Z_{\alpha/2} \text{ or } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} < -Z_{\alpha/2}$$

Paired t-test

- As in the previous example, the data is paired, the two scores (before and after) recorded for each individual are dependent, but the between individuals the pairs are independent.

- Thus in order to test H_0: $\mu_1 - \mu_2 = 0$, one has to look at the difference of each pair. The problem eventually becomes a one sample t-test problem.