

S1211Q Introduction to Statistics

Lecture 5

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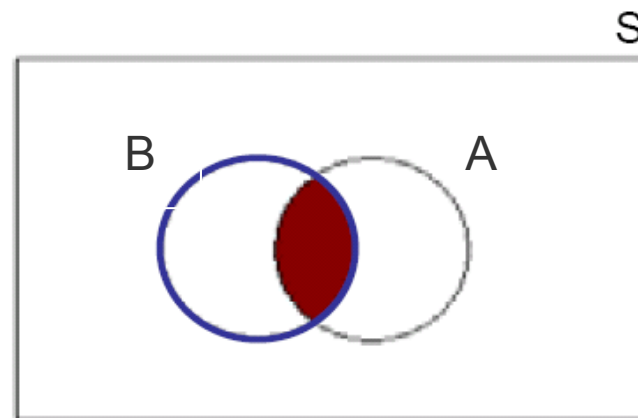
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Conditional Probabilities

- If we are given information that an event A has occurred, then the probability of another event B occurring may change.
- We use $P(A|B)$ to represent the conditional probability of A given that the event B has occurred. B is the “conditioning event”.
- Definition: for any two events A and B with $P(B) > 0$, the conditional probability of A given B has occurred is defined by

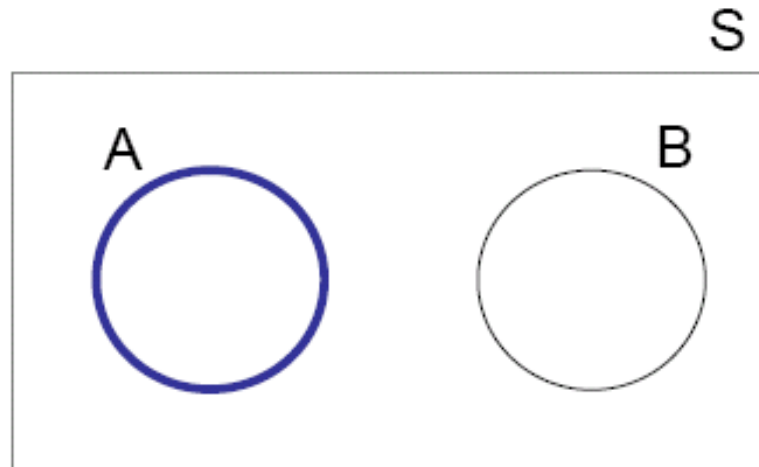
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Proportion of outcomes in B that are also in A.



Revisit disjoint events

- If A and B are disjoint then $P(A \cap B) = 0$.
- Hence, $P(A|B) = 0$. If B occurs, then A cannot occur.



Example

Ex. 1,000 voters were asked about their party affiliation and position on death penalty.

	Favor	Oppose	Total
Republican	260	40	300
Democrat	120	240	360
Other	240	100	340
Total	620	380	1000

R = A randomly chosen voter is a Republican.

D = A randomly chosen voter is a Democrat.

DP = A randomly chosen voter favors the death penalty.

What is the probability that a randomly chosen voter favors the death penalty given he is a Republican? ($26/30 = 0.867$)

What is the probability that a voter is a Democrat given he favors the death penalty? ($12/62 = 0.194$)

The Multiplication Rule

- ▶ The definition of conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$ points us another way to calculate probability of intersection of events.

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- ▶ It can be generalized to multiple events.

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$$

Example

Ex. A box contains 8 blue balls and 4 red balls. We draw two balls from the box **without replacement**. What is the probability that both are red?

A = first ball is red.

B = second ball is red.

$$\begin{aligned} P(\text{both balls are red}) &= P(A \cap B) = P(A) P(B|A) \\ &= 4/12 * 3/11 \\ &= 1/11 \end{aligned}$$

More general “**multiplication rule**”: $P(A \cap B \cap C) = P(C|A \cap B) P(B|A) P(A)$

Question: Draw three balls without replacement, what is the probability that all are red?

Tree diagram

- Often a problem involves several stages of using the **multiplication and addition rules**.
- A **tree diagram** is a useful way of keeping track of the multiplication and addition rules.
- It helps us think about conditional probabilities by showing the sequences of events as paths that resemble the branches of a tree.

Tree example

Ex. A company purchases 60% of its parts from vendor A and the rest from vendor B. Past experience indicates that 10% of vendor A's parts are defective, while 20% of vendor B's parts are defective.

A = part from vendor A

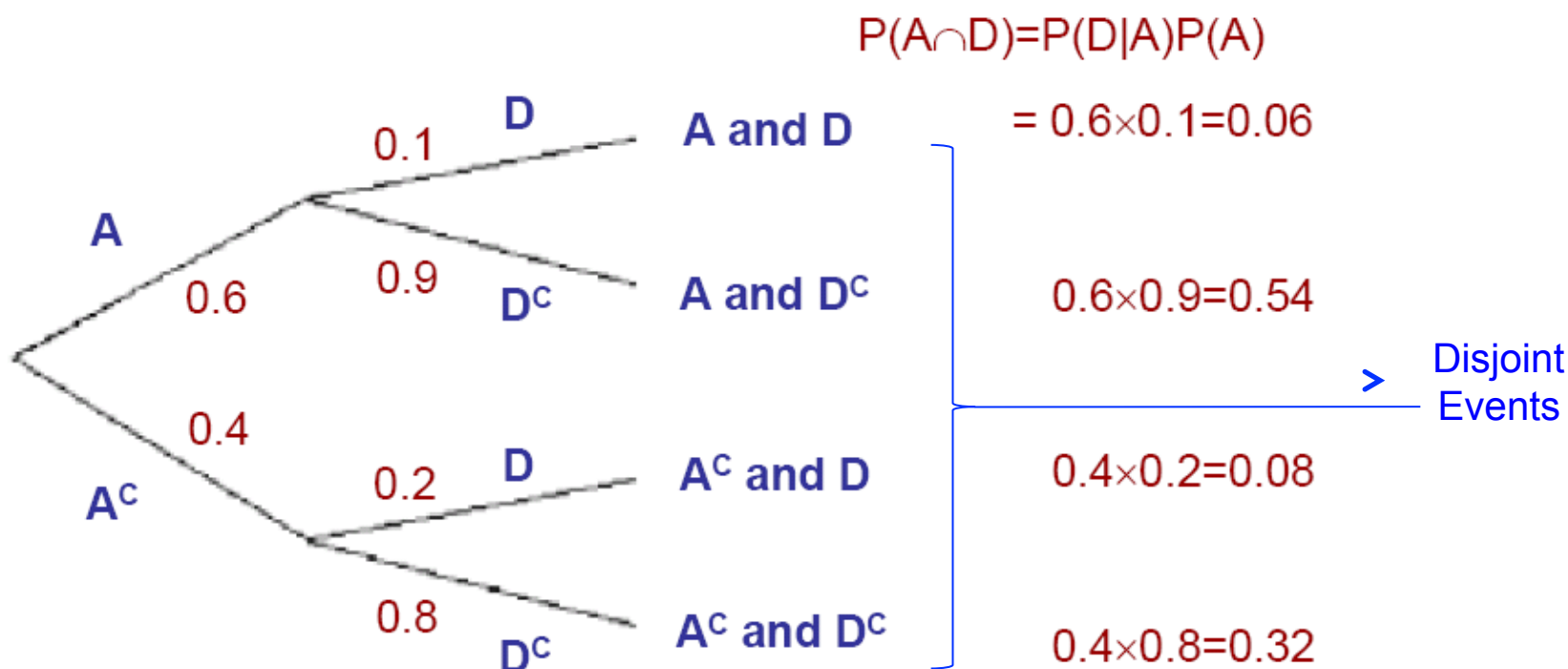
$$P(A) = 0.6$$

$$P(A^c) = 0.4$$

D = part defective

$$P(D|A) = 0.1$$

$$P(D|A^c) = 0.2$$



Example cont.

- A part is selected at random, what is the probability that it's defective?

$$\begin{aligned}P(D) &= P(D|A) P(A) + P(D|A^c) P(A^c) \\ &= 0.06 + 0.08 = 0.14\end{aligned}$$

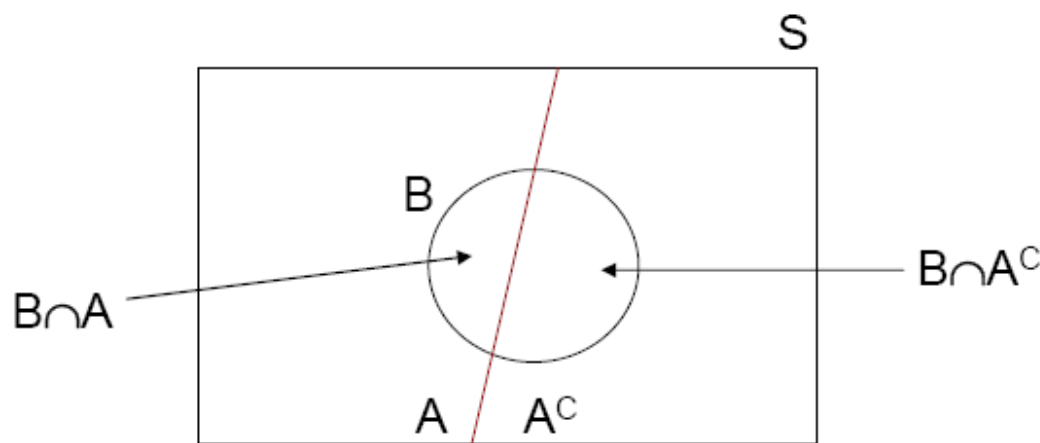
- Suppose a part is defective, what is the probability that it's from vendor A?

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.06}{0.14} = \frac{3}{7}$$

- In these two examples, we used **rule of total probability** and **Bayes rule**.

Motivations

- We can write any event B as the union of the portion contained in another event A and the portion contained in A^c , i.e., $B = (B \cap A) \cup (B \cap A^c)$.



- Because A and A^c are disjoint, so are $B \cap A$ and $B \cap A^c$.

The Law of Total Probability

- The **Law of Total Probability** states, Let A_1, \dots, A_k be mutually exclusive and *exhaustive* events. Then for any other event B .

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + \dots + P(B|A_k)P(A_k) \\ &= \sum P(B|A_i)P(A_i) \end{aligned}$$

- A_1, \dots, A_k are *exhaustive*, if one A_i must occur, so that $A_1 \cup \dots \cup A_k = S$.
- Proof: when $k=2$,

$$\begin{aligned} P(B) &= P((B \cap A) \cup (B \cap A^c)) \\ &= P(B \cap A) + P(B \cap A^c) \\ &= P(B|A)P(A) + P(B|A^c)P(A^c) \end{aligned}$$

Bayes Theorem

- With the help of the Law of Total Probability, we can state the Bayes Rule, which says, let A_1, \dots, A_k be a collection of k mutually exclusive and exhaustive events with *prior* probabilities $P(A_i)$ ($i=1, \dots, k$). Then for any other event B for which $P(B) > 0$, the *posterior* probability of A_j given that B has occurred is,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^k P(B|A_i)P(A_i)}$$

- When $k=2$, we have,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

- Bayes Rule can be used to “*reverse*” the probability from the conditional probability that was originally given, or *to find the cause given the result*.