S1211Q Introduction to Statistics Lecture 12

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Expectation of Functions

- Recall how we compute E[*h*(X)]. A similar result also holds for a function *h*(X, Y) of two jointly distributed rv's.

- Let X and Y be jointly distributed rv's with pmf *p*(*x*, *y*), if they are discrete; or pdf *f* (*x*, *y*), if they are continuous. The expected value of a function *h*(X, Y), denoted by E[*h*(X, Y)] is given by

$$\mathrm{E}[h(\mathrm{X}, \mathrm{Y})] = \begin{cases} \sum_x \sum_y h(x, y) \cdot p(x, y) & \text{if X and Y are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) \cdot f(x, y) dx dy & \text{if X and Y are continuous} \end{cases}$$

- This result can also be extended to multiple (>2) rv case.

Examples

Ex. (Important! Linearity of expectations) Show that for any two random variables X and Y, E(X+Y) = E(X) + E(Y).

Example

Ex. If two random variables X and Y are independent, what is E(XY)? What about E(g(X)h(Y))?

Expectation of Linear Function of Multiple RV's

- Linearity is well preserved in expectation.

$$E(a \cdot X + b \cdot Y + c) = a \cdot E(X) + b \cdot E(Y) + c$$

Expectation of Product of Multiple RV's

- Unlike the linear case, expectation of product in general doesn't equal to the product of expectations

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- And for independent RV's, in general

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Covariance

- When two random variables X and Y are not independent, it is often of interest to assess how strongly they are related to one another.

- A popular measurement to characterize the dependence of two rv's is called correlation. To calculate correlation of two rv's, we'll have calculate the covariance of the two rv's.

- The covariance between two rv's X and Y is

$$\begin{aligned} \mathrm{Cov}(\mathrm{X}, \mathrm{Y}) &= \mathrm{E}[(\mathrm{X} - \mu_X)(\mathrm{Y} - \mu_Y)] \\ &= \begin{cases} \sum_x \sum_y (x - \mu_X)(y - \mu_Y) \cdot p(x, y) & \mathrm{X, Y\ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) \cdot f(x, y) dx dy & \mathrm{X, Y\ continuous} \end{cases} \end{aligned}$$

Short cut

- Proposition:

$$\boxed{\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)}$$

- What happens if we set Y=X?

Example

Ex. Suppose the joint distribution of X and Y are

$$f(x,y) = \begin{cases} 24xy & 0 \le x \le 1, 0 \le y \le 1, x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the covariance of X and Y?

$$f_X(x) = \int_y f(x,y)dy = \int_0^{1-x} 24xydy = 12x(1-x)^2$$

$$f_Y(y) = 12y(1-y)^2$$

$$\mathrm{E(X)} = \int_0^1 x \cdot 12x(1-x)^2 dx = \frac{2}{5} = \mathrm{E(Y)}$$

$$\mathrm{E(XY)} = \int\int_{x,y} xyf(x,y)dxdy = \int_0^1 \int_0^{1-y} 24x^2y^2 dxdy = \frac{2}{15}$$

$$\mathrm{Cov(X,Y)} = \mathrm{E(XY)} - \mathrm{E(X)E(Y)} = \frac{2}{15} - \left(\frac{2}{5}\right)^2 = -\frac{2}{75}$$

Correlation

- The correlation coefficient of X and Y, denoted by Corr(X, Y) or $\rho_{X,Y}$ is defined by

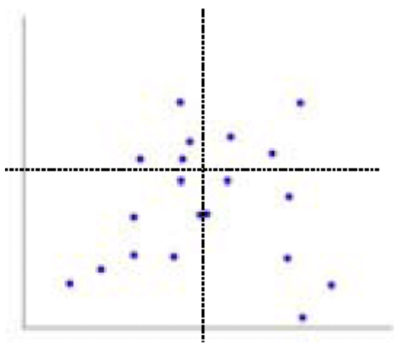
$$\rho_{X,Y} = \frac{\mathrm{Cov}(\mathrm{X}, \mathrm{Y})}{\sigma_X \cdot \sigma_Y}$$

- Because of Cauchy-Schwarz inequality, we have

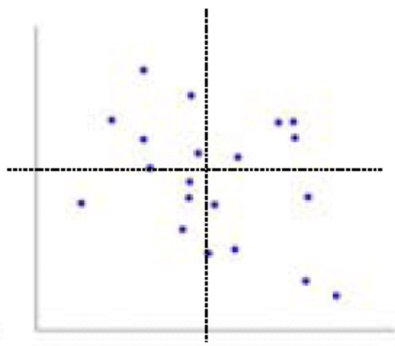
$$\mathrm{Cov}^2(\mathrm{X}, \mathrm{Y}) \leq \mathrm{Var}(\mathrm{X})\mathrm{Var}(\mathrm{Y}) \Longrightarrow |\rho_{X,Y}| \leq 1$$

- The correlation coefficient $\rho_{X,Y}$ is NOT a completely general measure of the strength of a relationship. $\rho_{X,Y}$ is actually a measure of the degree of *linear* relationship between X and Y.

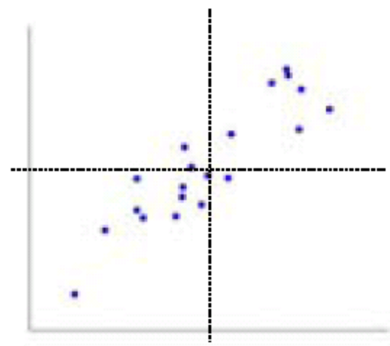
Remarks

- If X and Y are independent, then $\rho_{X,Y} = 0$ (why?). But $\rho_{X,Y} = 0$ does NOT imply independence.
- $\rho_{X,Y} = 1$ or -1 iff $Y = aX+b$ for some numbers a and b with $a \neq 0$.
- 

  $r \approx 0$

  $r \approx -0.3$

  $r \approx 0.9$

Relationship Between Correlation and Independence

- Independence leads to uncorrelatedness.

$$Cov(X, Y) = E(XY) - E(X)E(Y) = E(X)E(Y) - E(X)E(Y) = 0$$

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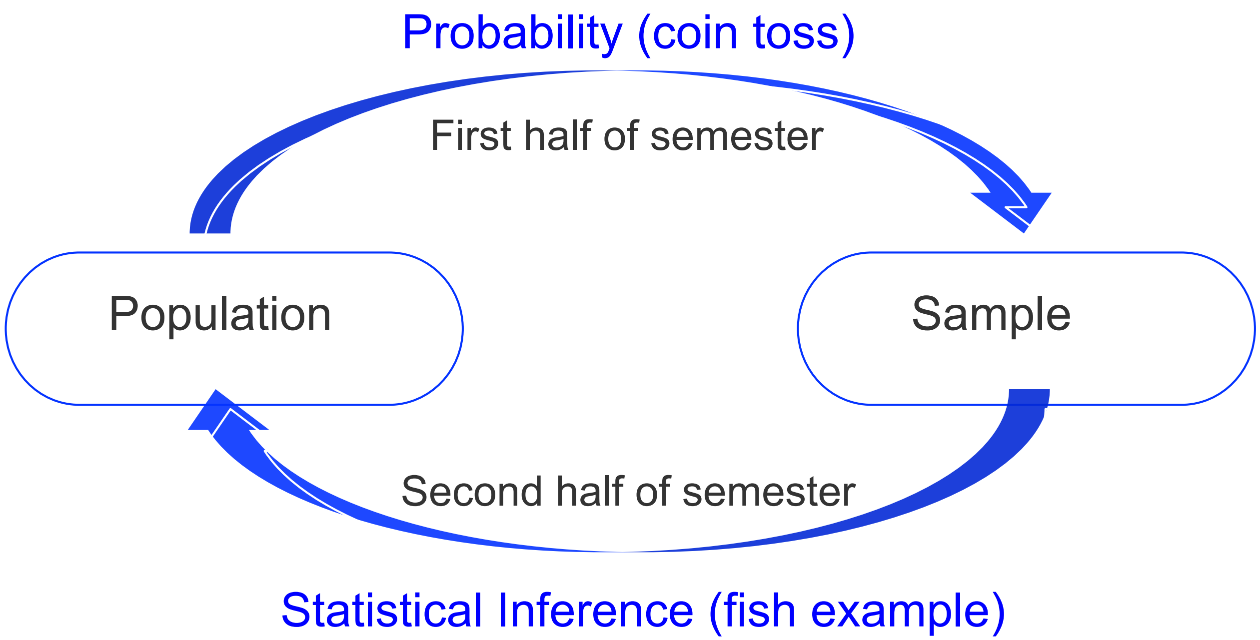
- But not vice versa!
- We will talk about this more in regression.

Population and Sample

- We will start changing our discussion from probability to statistics, which means we need to think about samples and how they relate to the underlying population.
- Recall the relationship between population and sample (probability and inference) that we visualized in the first lecture.

Probability and Inference



RV or a Particular Number

- In the first chapter, we use lowercase letters to represent the sample, $x_1, x_2, x_3, \ldots$. That means we have already observed the data and each of the letters can be replaced by a particular number.
- Before the data becoming available, there is uncertainty as to what value we will observe, so we view each observation as a RV, thus denoted by uppercase letter $X_1, X_2, X_3, \ldots$

Sample and Statistics

- A statistic is any quantity whose value can be calculated from sample data, such as Sample Mean and Sample Variance.
- Before obtaining data, a statistic is also a RV. The bulk of statistical inference is to find the distribution of the statistics, or the so-called *Sampling Distributions*.
- To make things easier, we often need to assume the observed data are *Simple Random Samples*, which means they are IID (Independently Identically Distributed).

Introduction to IID

- A sequence of random variables, X_1, X_2, …, X_n, is independent and identically distributed (i.i.d.) if each random variable has the same probability distribution as the others and all are mutually independent.

- In statistical analysis, we often assume the sampled data X_1, X_2, …, X_n, are i.i.d. from a common distribution f(x). And usually, we end up analyzing a linear combination of the X_i's, that is

$$\mathrm{Y} = a_1 \mathrm{X}_1 + \cdots + a_n \mathrm{X}_n = \sum_{i=1}^{n} a_i \mathrm{X}_i$$

Sample Mean***

- Let $X_1, X_2, \ldots, X_n$, be an i.i.d. sequence of rv's from a distribution with mean value μ and standard deviation σ.

- Notice that the sample mean or the sample total ($T = X_1+X_2+\ldots+X_n$) can also be viewed as a special case of linear combination of $X_1, X_2, \ldots, X_n$. In the i.i.d. case,

$$E(T) = E(X_1)+E(X_2)+\ldots+E(X_n) = n\mu$$

$$Var(T) = Var(X_1)+Var(X_2)+\ldots+Var(X_n) = n\sigma^2$$

- It is also easy to verify that for sample mean,

$$\mathrm{E}(\bar{\mathrm{X}}) = \mu_{\bar{X}} = \mu$$

$$\mathrm{Var}(\bar{\mathrm{X}}) = \sigma_{\bar{X}}^2 = \sigma^2/n \Longrightarrow \sigma_{\bar{X}} = \sigma/\sqrt{n}$$

Invariance under Summation

- When $X_1, X_2, \ldots, X_n$ are normally distributed, a linear combination of these random variables

$$Y = a_1 X_1 + \cdots + a_n X_n = \sum_{i=1}^{n} a_i X_i$$

  will still be normally distributed.

- Note that $X_1, X_2, \ldots, X_n$ do not have to be i.i.d.

- What are the parameters of Y?

- This phenomenon does NOT happen to every distribution, for example, sum of uniform random variables.

CLT

- Theorem:

The Central Limit Theorem (CLT)

Let X_1, X_2, …, X_n, be an i.i.d. sequence from a distribution with mean μ and variance σ^2. Then if n is sufficiently large, the sample mean $\bar{X}$ has approximately a normal distribution with $\mu_{\bar{X}} = \mu$ and $\sigma^2_{\bar{X}} = \sigma^2/n$;
And the sample total has approximately a normal distribution with $\mu_T = n\mu$, $\sigma_T^2 = n\sigma^2$. The larger the value of n, the better the approximation.

- Rule of Thumb: if n>30, the CLT can be used.

A key result ***

Let $X_1, X_2, \ldots, X_n$, have mean values $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively.

- Whether or not the Xi's are independent,

$$\begin{aligned} \mathrm{E}(a_1 \mathrm{X}_1 + a_2 \mathrm{X}_2 + \cdots + a_n \mathrm{X}_n) &= a_1 \mathrm{E}(\mathrm{X}_1) + a_2 \mathrm{E}(\mathrm{X}_2) + \cdots + a_n \mathrm{E}(\mathrm{X}_n) \\ &= a_1 \mu_1 + a_2 \mu_2 + \cdots + a_n \mu_n \end{aligned}$$

- For any $X_1, X_2, \ldots, X_n$,

$$\mathrm{Var}(a_1 \mathrm{X}_1 + a_2 \mathrm{X}_2 + \cdots + a_n \mathrm{X}_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \mathrm{Cov}(\mathrm{X}_i, \mathrm{X}_j)$$

  If they are independent, then

$$\begin{aligned} & \mathrm{Var}(a_1 \mathrm{X}_1 + a_2 \mathrm{X}_2 + \cdots + a_n \mathrm{X}_n) \\ = & \; a_1^2 \mathrm{Var}(\mathrm{X}_1) + a_2^2 \mathrm{Var}(\mathrm{X}_2) + \cdots + a_n^2 \mathrm{Var}(\mathrm{X}_n) \\ = & \; a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \cdots + a_n^2 \sigma_n^2 \end{aligned}$$

Special Cases

- $E(X+Y) = E(X) + E(Y)$;
- $E(X-Y) = E(X) - E(Y)$;
- $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- $Var(X-Y) = Var(X) + Var(Y) -2Cov(X, Y)$
- If X and Y are independent, then $Cov(X, Y) = 0$, and
  $Var(X+Y) = Var(X) + Var(Y)$
  $Var(X - Y) = Var(X) + Var(Y)$

Example

Ex. Show that if X ~ Bin(n, p), then E(X) = np, and Var(X) = $np(1 - p)$.

Ex. Show that if X is a negative binomial rv with pmf $nb(x; r, p)$, then E(X) = $r(1-p)/p$, Var(X) = $r(1 - p)/p^2$.

Statistical Inference

- From the previous two examples, we know that quite often, we need to infer the truth (population) from some partial information (sample).

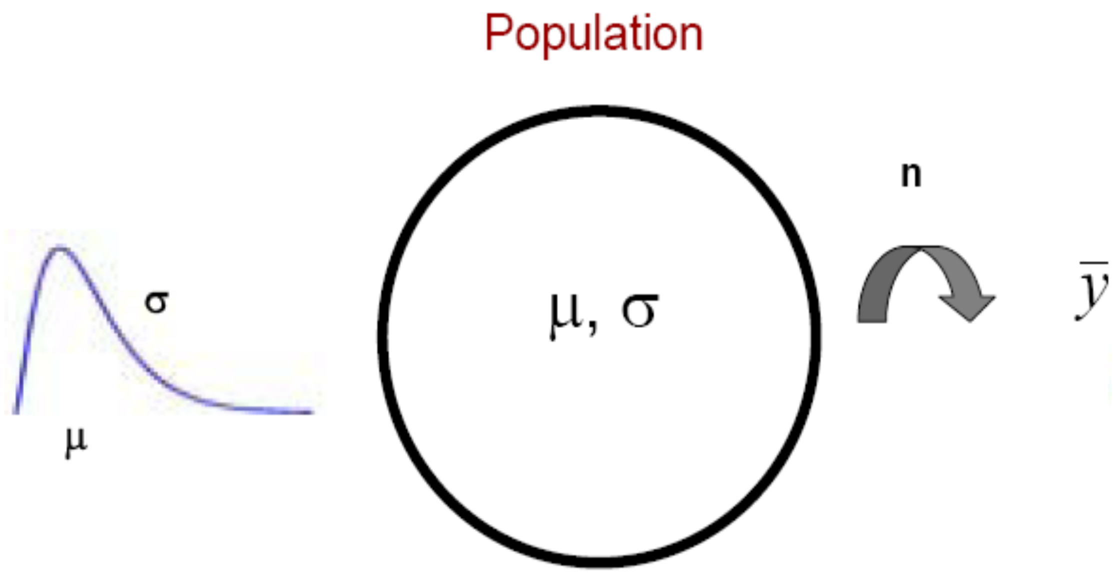
- Question: why do we need a model?

- Statistical inference comprises the use of statistics and random sampling to make inferences concerning some unknown aspect of a population.

- A point estimate of a parameter θ is a single number that can be regarded as a sensible value for θ. A point estimate is obtained by selecting a suitable statistic and computing its value from the given sample data. The selected statistic is called the point estimator of θ.

Sampling scheme for a Mean

- Usually our problem set up will be as illustrated in the graph.



- The actual sample observations $y_1, y_2, \ldots, y_n$ (realizations) are assumed to be the result of a random sample $Y_1, Y_2, \ldots, Y_n$ (random variables) from a certain distribution.