W1211 Introduction to Statistics
Lecture 21

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The Invariance Principle

- One of the nice features of MLE's is that, the MLE of a function of parameters, is the function of the MLE's of the parameters.

- More specifically, we have

> Let $\hat{\theta}_1, \ldots, \hat{\theta}_m$ be the MLE's of the parameters $\theta_1, \ldots, \theta_m$. Then the MLE of any function $h(\theta_1, \ldots, \theta_m)$ of these parameters is $h(\hat{\theta}_1, \ldots, \hat{\theta}_m)$.

<u>Ex.</u> In the normal example, what is the MLE of σ?

Large Sample Behavior

- The following proposition says, for large samples, it is “optimal” to use MLE’s, because it is asymptotically unbiased and has the minimal variance among all unbiased estimators.

- Proposition:

  > Under very general conditions on the joint distribution of the sample,
  > When the sample size n is large, the maximum likelihood estimator is
  > Approximately the MVUE of the parameter.

Confidence Intervals

- A point estimate, because it is a single number, by itself provides no information about the precision and reliability of estimation (the reason why we need standard error).
- An alternative to reporting a single sensible value for the parameter being estimated is to calculate and report an entire interval of plausible values – an *interval estimate* or *confidence interval* (*CI*).
- A confidence interval is always calculated by first selecting a *confidence level*, which is a measure of the degree of reliability of the interval.
- Construct a confidence interval for a standard normal random variable.

Illustration

- Let's first consider a simple, somewhat unrealistic problem situation.
    1. We are interested in the population mean parameter μ.
    2. The population distribution is normal.
    3. The value of the population standard deviation σ is known. (unlikely!)

- Suppose we have a random sample $X_1, X_2, \ldots, X_n$ from a normal distribution with mean value μ and standard deviation σ. As we know, $\bar{X}$ also follows a normal distribution with mean value μ and standard deviation $\sigma/\sqrt{n}$. Thus, we could get a standard normal distribution by normalizing $\bar{X}$.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Construction

- The smallest interval that contains 95% of the possible outcomes of Z is (-1.96, 1.96).

$$-1.96 < \frac{\bar{\mathrm{X}} - \mu}{\sigma/\sqrt{n}} < 1.96$$

$$\Downarrow$$

$$-1.96 \cdot \frac{\sigma}{\sqrt{n}} < \bar{\mathrm{X}} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

$$\Downarrow$$

$$\bar{\mathrm{X}} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{\mathrm{X}} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

Interpretation

- Thus we have $\mathrm{P}\left(\bar{\mathrm{X}} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{\mathrm{X}} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) = 0.95$.

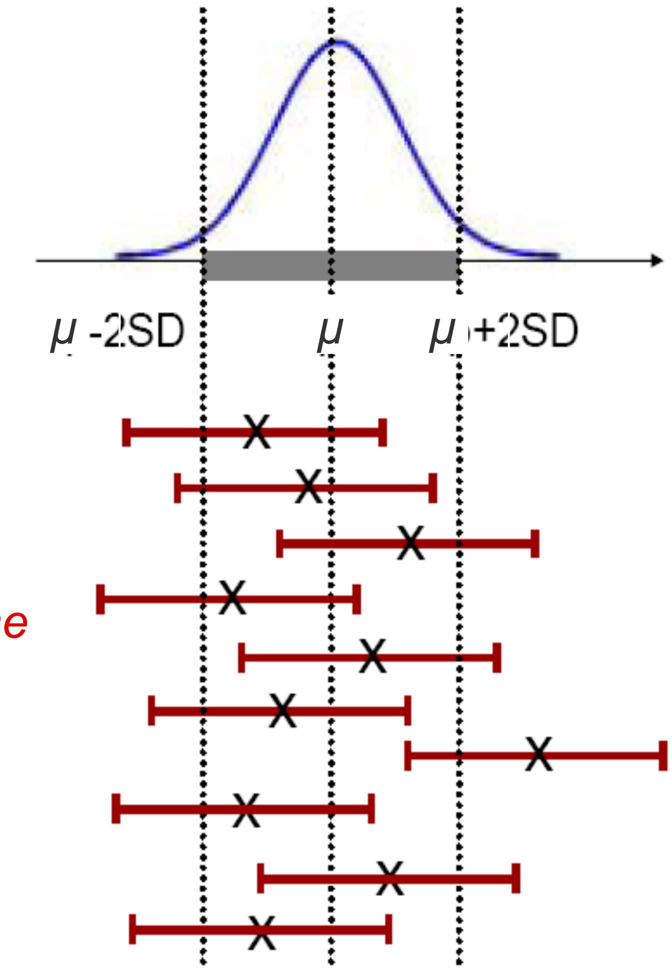
- Some people interpreted this as: the true parameter *μ* has 95% chance of falling in the interval of $(\bar{\mathrm{X}} - 1.96 \cdot \sigma/\sqrt{n}, \bar{\mathrm{X}} + 1.96 \cdot \sigma/\sqrt{n})$. Is it right?

- In fact, the two boundaries of the interval given above are random! Thus every time we sample n observations from the same population, we will get a different confidence interval!

Random Interval

- By constructing a confidence interval like this, we never be sure whether μ actually lies in our confidence interval. However, we know that about 95 out of 100 times intervals constructed using this method will capture the true parameter.

- Interpreted as: "*the probability is .95 that the random interval includes or covers the true value of μ.*"



Confidence Interval for the Mean of a Normal Population when Variance is assumed known

- A $100(1-\alpha)\%$ confidence interval for the mean μ of a normal population when the value of σ is known is given by
$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right)$$
- $z_{\alpha/2}$ is the upper $(100 \cdot \alpha/2)\%$ percentile of a standard normal distribution, i.e., $P(Z > z_{\alpha/2}) = \alpha/2$.
- z_α's are usually refered to as z critical values.

Connection between Interval Width, Confidence Level and Sample Size

- When constructing a confidence interval, confidence level, interval width, and sample size are closely related.
- Mathematically,
$$w = 2 \cdot z_{\alpha/2} \cdot \sigma/\sqrt{n}$$
.
- So lower confidence level and larger sample size result in narrower interval width.
- Sometimes, we might want to know how many observations we need to collect to achieve a certain precision (width) under a fixed confidence level. This is called Sample Size Calculation.

Sample Size Calculation

- The general formula for the sample size n necessary to ensure an interval width w is obtained from $w = 2 \cdot z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

$$n = \left(2 \cdot z_{\alpha/2} \cdot \frac{\sigma}{w}\right)^2$$

- Ex. A new operating system has been installed, and we wish to estimate the true average response time μ to a particular editing command. Assuming that response times are normally distributed with $\sigma = 25$ millisec. How many tests should we do to ensure that the resulting 95% CI has a width of at most 10?

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- Plug in into the formula

$$n = (2 \cdot 1.96 \cdot \frac{25}{10})^2 = 96.04$$

So we need at least 97 tests.