Three Distributions

- In statistical inference, there are three distributions playing very important roles. They are: *Chi-square distribution*, *t – distribution* and *F – distribution*.

- They arise naturally, when one deals with normal population. More specifically, each of the three distributions can be constructed by using i.i.d. normal random variables.

Chi-square

- Chi-square distribution was first brought up by K. Pearson. We often denote a Chi-square random variable with *n* degrees of freedom χ_n^2 .

- Theorem: if we have $\xi_1, \xi_2, \ldots, \xi_n$ i.i.d. ~ N(0,1), let

$$\eta = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$$

  then η has a χ_n^2 distribution.

- From the above theorem, if we have $\eta_1 \sim \chi_n^2$, $\eta_2 \sim \chi_m^2$, and they are independent, what is the distribution of $\eta_1+\eta_2$?

Examples

- The following are two very important examples of Chi-square distributed rv's.

Ex. If we have $X_1, X_2, \ldots, X_n$ i.i.d. from a normal distribution $N(\mu,\sigma^2)$, then we have

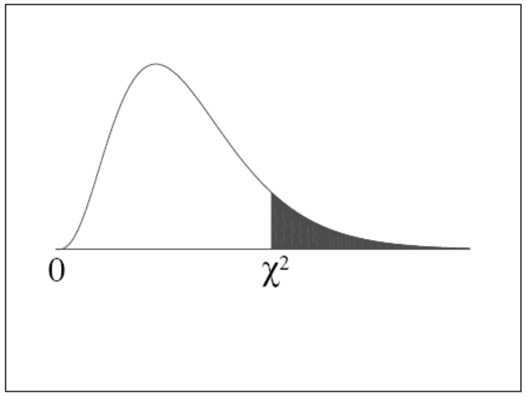
$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \chi^2_{n-1}$$

Ex. If we have $Y_1, Y_2, \ldots, Y_n$ i.i.d. from an exponential distribution with parameter $\lambda>0$. then

$$2\lambda\sum_{i=1}^{n} Y_i \sim \chi^2_{2n}$$

- Construct CI for σ^2 and λ using the chi-square table in the appendix.

Chi-square distribution table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278

t-distribution

- t-distribution was first proposed by W. Gosset and published under the pseudonym “Student”. Thus, the t-distribution is also sometimes referred to as the “Student’s t-distribution”.

- The t-distribution density is symmetric around the origin and looks quite similar to the standard normal density. In fact, when the degree of freedom of t-distribution is large, it is indeed close to the standard norm density.

- Theorem: if $\eta_1 \sim \chi_n^2$, $\eta_2 \sim$ N(0,1), and η_1 and η_2 are independent, let

  $$\zeta = \frac{\eta_2}{\sqrt{\eta_1/n}}$$

  then $\zeta \sim t_n$.

Example

- The following result is one of the most important results in statistical inference.

Ex. If we have X_1, X_2, …, X_n i.i.d. from a normal distribution $N(\mu,\sigma^2)$, then we have

$$\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t_{n-1}$$

- The above example also points out a very important fact that the sample mean from a normal population is independent with the sample variance (sd)!

- Construct CI for μ using the t-table in the appendix.

t distribution table

Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

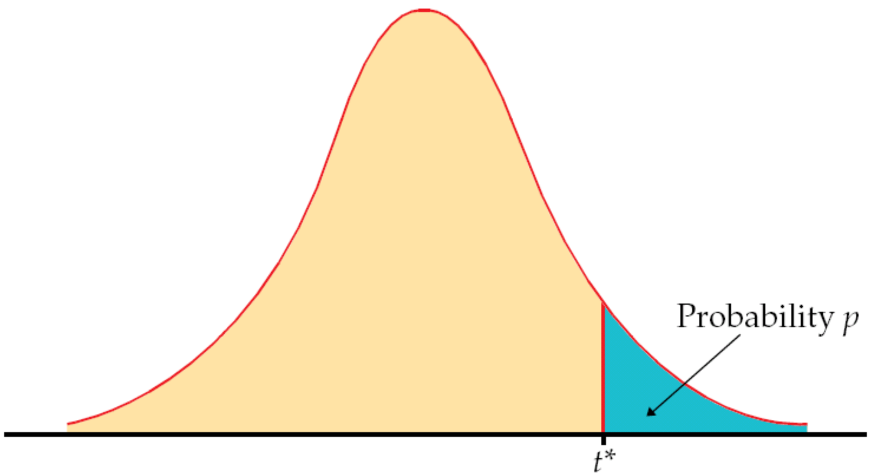


TABLE D

t distribution critical values

	Upper-tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408

Remarks about t

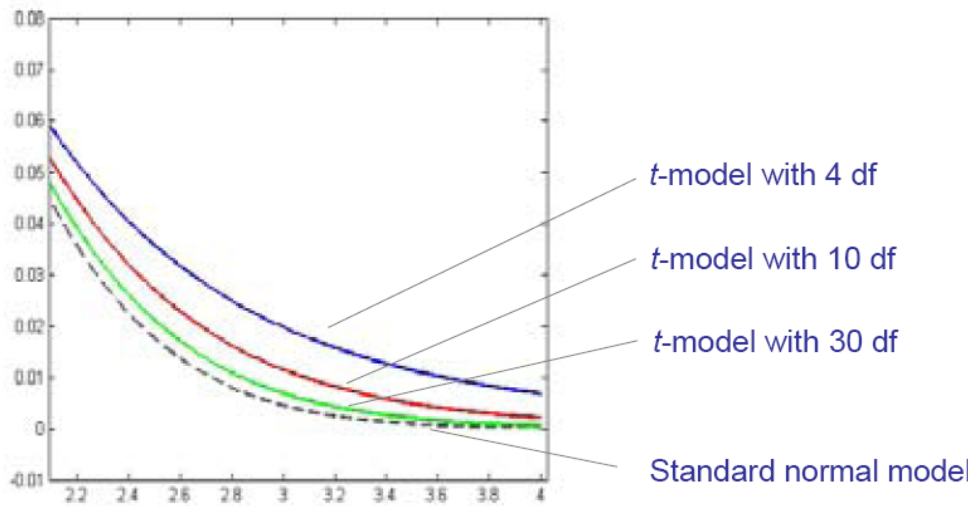
- There is a separate t-model corresponding to each degree of freedom.

- The spread of the t-model is slightly larger than that of the standard normal model. Consider the quantity

$$\frac{\sqrt{n}(\bar{\mathrm{X}} - \mu)}{\mathrm{S}}$$

  using S instead of σ introduces more variation into the statistic.

- Heavier tails:



Largest percentage changes of DJI

Largest daily percentage gains

Rank	Date	Close	Net Change	% Change
1	1933-03-15	62.10	+8.26	+15.34
2	1931-10-06	99.34	+12.86	+14.87
3	1929-10-30	258.47	+28.40	+12.34
4	1932-09-21	75.16	+7.67	+11.36
5	2008-10-13	9,387.61	+936.42	+11.08
6	2008-10-28	9,065.12	+889.35	+10.88
7	1987-10-21	2,027.85	+186.84	+10.15
8	1932-08-03	58.22	+5.06	+9.52
9	1932-02-11	78.60	+6.80	+9.47
10	1929-11-14	217.28	+18.59	+9.36
11	1931-12-18	80.69	+6.90	+9.35
12	1932-02-13	85.82	+7.22	+9.19
13	1932-05-06	59.01	+4.91	+9.08

Largest daily percentage losses

Rank	Date	Close	Net Change	% Change
1	1987-10-19	1,738.74	-508.00	-22.61
2	1929-10-28	260.64	-38.33	-12.82
3	1929-10-29	230.07	-30.57	-11.73
4	1929-11-06	232.13	-25.55	-9.92
5	1899-12-18	58.27	-5.57	-8.72
6	1932-08-12	63.11	-5.79	-8.40
7	1907-03-14	76.23	-6.89	-8.29
8	1987-10-26	1,793.93	-156.83	-8.04
9	2008-10-15	8,577.91	-733.08	-7.87
10	1933-07-21	88.71	-7.55	-7.84
11	1937-10-18	125.73	-10.57	-7.75
12	2008-12-01	8,149.09	-679.95	-7.70
13	2008-10-09	8,579.19	-678.91	-7.33

F-distribution

- The F-distribution is named after the famous statistician R.A. Fisher.

- Unlike Chi-square and t distribution, the F distribution has two degrees of freedom *n* and *m*. And these two parameters are NOT symmetric.

- Theorem: if we have $\eta_1 \sim \chi^2_n$, $\eta_2 \sim \chi^2_m$, and η_1 and η_2 are independent, let

$$\zeta = \frac{\eta_1/n}{\eta_2/m}$$

  then $\zeta \sim F_{n,m}$.

Example

Ex. Let $X_1, \ldots, X_m$ be an IID sample from a normal distribution with variance σ_1^2, let $Y_1, \ldots, Y_n$ be another IID sample from a normal distribution with variance σ_2^2. Let S_1^2 and S_2^2 denote the two sample variances, then the rv

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

has an F distribution with v_1=m-1 and v_2=n-1.

- How to construct a confidence interval for σ_1^2 / σ_2^2?

Hypothesis Testing

- A *statistical hypothesis*, or just *hypothesis*, is a claim or assertion either about the value of a single parameter (population characteristic or characteristic of a probability distribution), about the values of several parameters, or about the form of an entire probability distribution.

- A testing problem usually contains two hypotheses: the null hypothesis, denoted by H_0, is the claim that is initially assumed to be true (the “prior belief” claim). The alternative hypothesis, denoted by H_a, is the assertion that is contradictory to H_0.

- The null hypothesis will be rejected in favor of the alternative only if sample evidence suggests that H_0 is false. If the sample does not strongly contradict H_0, we will continue to believe in the truth of the null hypothesis. The two possible conclusions from a testing analysis are then *reject* H_0 or *fail to reject* H_0.