W1211 Introduction to Statistics
Lecture 13

Wei Wang

Oct 17, 2012

Independent rv's

- Recall the definition of independence of two random events A and B.

$$P(A \cap B) = P(A)\, P(B)$$

- We say two random variables X and Y are independent if and only if

$$P(X=x, Y=y) = P(X=x)\, P(Y=y), \text{ for any } x \text{ and } y.$$

- More specifically, two random variables X and Y are said to be independent if for every pair x and y values,

$$p(x, y) = p_X(x)\, p_Y(y), \text{ when X and Y are discrete;}$$

or

$$f(x, y) = f_X(x)\, f_Y(y), \text{ when X and Y are continuous.}$$

<u>Ex.</u> The second case of the previous example.

Multiple Random Variables

- If $X_1, X_2, \ldots, X_n$ are all discrete random variables, the joint pmf of the variables is the function

  $$p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

  If the variables are continuous, the joint pdf of $X_1, X_2, \ldots, X_n$ is the function $f(x_1, x_2, \ldots, x_n)$ such that for any n intervals $[a_1, b_1], \ldots, [a_n, b_n]$,

  $$\mathrm{P}(a_1 \leq \mathrm{X}_1 \leq b_1, \ldots, a_n \leq \mathrm{X}_n \leq b_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

- What should be the regularity conditions for $p(x_1, x_2, \ldots, x_n)$ and $f(x_1, x_2, \ldots, x_n)$?

- How do get the marginal distributions of $X_1, X_2, \ldots$ by using $p(x_1, x_2, \ldots, x_n)$ and $f(x_1, x_2, \ldots, x_n)$?

Independence

- Proposition:

  The random variables X_1, X_2, …, X_n, are said to be independent if for every subset X_{i_1}, X_{i_2}, …, X_{i_k}, of the variables (each pair, each triple, and so on), the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

- $$p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$

- $$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

Conditional dist.

- Using the marginal distributions, one can calculate the conditional distribution of one rv given the other.

- Let X and Y be two conditional rv's with joint pdf $f(x, y)$ and marginal X pdf $f_X(x)$. Then for any X value x for which $f_X(x)>0$, the conditional probability density function of Y given that X=x is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \qquad -\infty < y < \infty.$$

- If X and Y are discrete, replace pdf's by pmf's in this definition gives the conditional probability mass function of Y when X=x.

Expectation of Functions

- Recall how we compute E[*h*(X)]. A similar result also holds for a function *h*(X, Y) of two jointly distributed rv's.

- Let X and Y be jointly distributed rv's with pmf *p*(*x*, *y*), if they are discrete; or pdf *f* (*x*, *y*), if they are continuous. The expected value of a function *h*(X, Y), denoted by E[*h*(X, Y)] is given by

$$\mathrm{E}[h(\mathrm{X},\mathrm{Y})] = \begin{cases} \sum_x \sum_y h(x,y)\cdot p(x,y) & \text{if X and Y are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\cdot f(x,y)dxdy & \text{if X and Y are continuous} \end{cases}$$

- This result can also be extended to multiple (>2) rv case.

Expectation of Linear Function of Multiple RV's

- Linearity is well preserved in expectation.

$$E(a \cdot X + b \cdot Y + c) = a \cdot E(X) + b \cdot E(Y) + c$$

Expectation of Product of Multiple RV's

- Unlike the linear case, expectation of product in general doesn't equal to the product of expectations

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- But if X and Y are independent, then

$$\begin{aligned} E(XY) &= \int\int xyf(x,y)dxdy = \int\int xyf_X(x)f_Y(y)dxdy \\ &= \int xf_X(x)dx \int yf_Y(y)dy = E(X)E(Y) \end{aligned}$$

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- And for independent RV's, in general

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Covariance

- When two random variables X and Y are not independent, it is often of interest to assess how strongly they are related to one another.

- A popular measurement to characterize the dependence of two rv's is called correlation. To calculate correlation of two rv's, we'll have calculate the covariance of the two rv's.

- The covariance between two rv's X and Y is

$$\begin{aligned}\mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= \mathrm{E}[(\mathrm{X}-\mu_X)(\mathrm{Y}-\mu_Y)] \\ &= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y)\cdot p(x,y) & \mathrm{X},\ \mathrm{Y}\ \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)\cdot f(x,y)dxdy & \mathrm{X},\ \mathrm{Y}\ \text{continuous}\end{cases}\end{aligned}$$

Short cut

- Proposition:

$$\boxed{\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)}$$

- What happens if we set Y=X?

Covariance and Variance

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- As we can see, variance is a special case of covariance, where $X = Y$.
- Variance of linear function of multiple RV's is given by

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab \cdot Cov(X, Y)$$

Example

Ex. Suppose the joint distribution of X and Y are

$$f(x,y) = \begin{cases} 24xy & 0 \le x \le 1, 0 \le y \le 1, x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the covariance of X and Y?

$$f_X(x) = \int_y f(x,y)dy = \int_0^{1-x} 24xydy = 12x(1-x)^2$$

$$f_Y(y) = 12y(1-y)^2$$

$$\mathrm{E(X)} = \int_0^1 x \cdot 12x(1-x)^2 dx = \frac{2}{5} = \mathrm{E(Y)}$$

$$\mathrm{E(XY)} = \int\int_{x,y} xyf(x,y)dxdy = \int_0^1 \int_0^{1-y} 24x^2y^2dxdy = \frac{2}{15}$$

$$\mathrm{Cov(X,Y)} = \mathrm{E(XY)} - \mathrm{E(X)E(Y)} = \frac{2}{15} - \left(\frac{2}{5}\right)^2 = -\frac{2}{75}$$

Correlation

- The correlation coefficient of X and Y, denoted by Corr(X, Y) or $\rho_{X,Y}$ is defined by

$$\rho_{X,Y} = \frac{\mathrm{Cov}(\mathrm{X},\mathrm{Y})}{\sigma_X \cdot \sigma_Y}$$

- Because of Cauchy-Schwarz inequality, we have

$$\mathrm{Cov}^2(\mathrm{X},\mathrm{Y}) \leq \mathrm{Var}(\mathrm{X})\mathrm{Var}(\mathrm{Y}) \Longrightarrow |\rho_{X,Y}| \leq 1$$

- The correlation coefficient $\rho_{X,Y}$ is NOT a completely general measure of the strength of a relationship. $\rho_{X,Y}$ is actually a measure of the degree of *linear* relationship between X and Y.