S1211Q Introduction to Statistics
Lecture 3

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Standard deviation

- The variance and standard deviation are measures of spread that indicate how far values in the data set are from the mean, on average.

- Consider the observations $x_1, x_2, x_3, \ldots, x_n$.

- The deviations $(x_i - \bar{x})$ display the spread of x_i about their mean $\bar{x}$.

- The sum of the deviations is always 0, as some of the deviations are positive and others are negative.

- Squaring the deviations makes them all positive. Observations far from the mean will have large positive squared deviations.

- The variance is the 'average' squared deviation.

Standard deviation

- If we have *n* observations $x_1, x_2, x_3, \ldots, x_n$. The variance is defined as

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- The standard deviation, *s*, is the square root of the variance.
  1. *s* is a measure of spread about the mean and should be used when the mean is used as the measure of center.
  2. If *s*=0, then all the values in the data set are exactly the same (no spread). Why?
  3. The more spread out the data, the greater the standard deviation.
  4. *s* is always positive.
  5. *s* has the same unit of measurement as the original data

A short cut formula for s^2

Theorem

An alternative expression for variance s^2 is

$$s^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2\right)$$

Proof.

Do some algebra on the numerator.

$$\begin{aligned}
\sum_{i=1}^{n}(x_i - \bar{x})^2 &= \sum_{i=1}^{n}(x_i^2 - 2\bar{x}\cdot x_i + (\bar{x})^2) \\
&= \sum x_i^2 - 2\bar{x}\sum x_i + \sum(\bar{x})^2 \\
&= \sum x_i^2 - 2\bar{x}\cdot n\bar{x} + n(\bar{x})^2 \\
&= \sum x_i^2 - n(\bar{x})^2
\end{aligned}$$

$\square$

Probability

What is randomness?

- The world is full of random events that we seek to understand.
- An event is random if we know what outcomes could occur, but not the particular values that will happen.
- The outcome of these events is uncertain, but they follow a regular pattern.
- Deterministic models vs. Random models.
- Probability theory is the mathematical representation of random phenomena.

Notation

- An experiment is any action or process whose outcome is subject to uncertainty. e.g. tossing a coin once or several times; selecting a card or cards from a deck; weighing a loaf of bread; etc.

- The sample space of an experiment, denoted by S, is the set of all possible outcomes of that experiment.

Ex. Flip a coin. Two possible outcomes: Heads (H) or Tails (T). $S=\{H,T\}$.

Ex. Battery life. $S=\{x: 0\leq x <\infty\}$.

Notation

- An event is any collection of possible outcomes, that is, any subset of *S* (including *S* itself). An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcome.

- If the outcome of a random phenomenon is contained in an event A, then we say that A has occurred.

Ex. Flip a coin twice. Four possible outcomes, S={HH, HT, TH, TT}. Let A be the event that we obtain at least one H in the two flips. A={HH, HT, TH}. Let B be the event that we obtain two H's in the two flips. B={HH}.

Ex. Battery life example. The event that the battery lasts less than 3 hours is denoted as $A=\{x: 0 \leq x < 3\}$.

Set Operations

- Given any two events (or sets) A and B, we have the following elementary set operations:

  The union

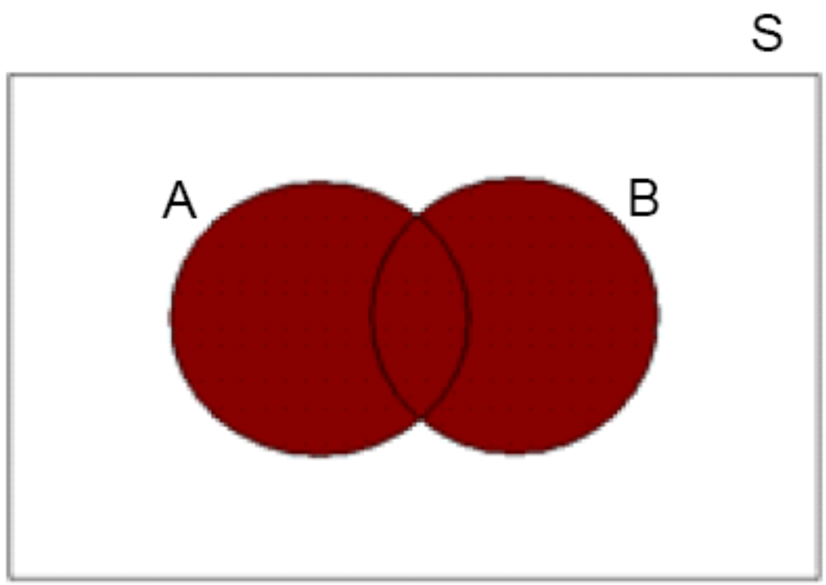
  The intersection

  The complement

- Venn diagrams are often used to illustrate relationships between sets.

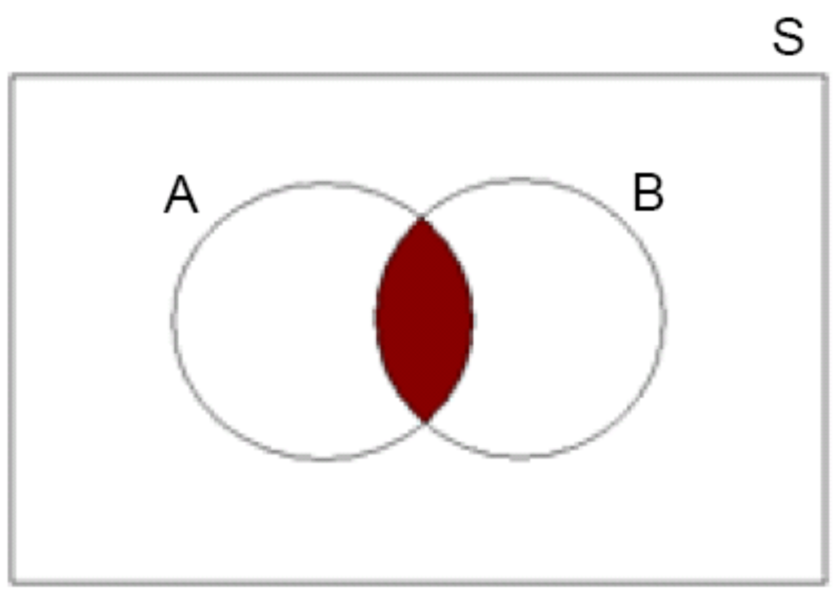
Union

- The union of A and B, written as AUB and read “A or B”, is the set of outcomes that belong to either A or B or both.



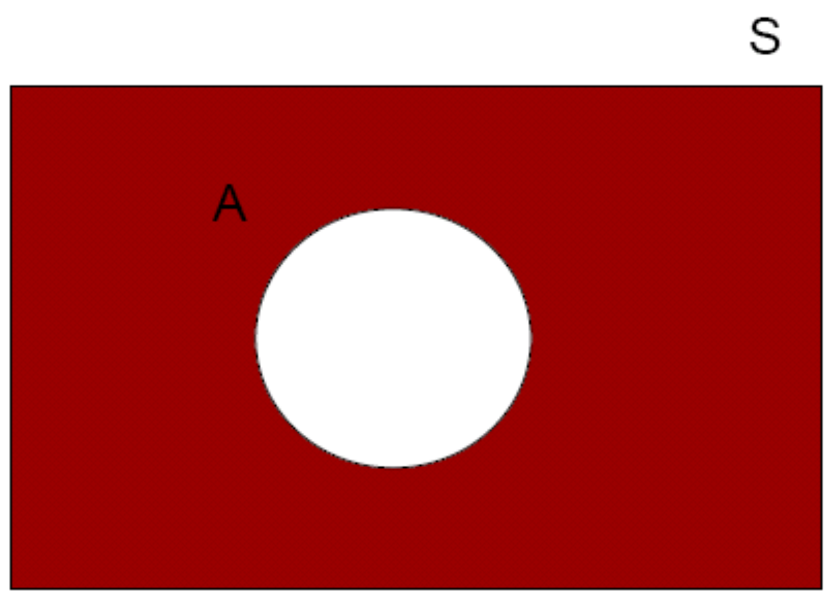
Intersection

- The intersection of A and B, written as $A \cap B$, read “A and B”, is the set of outcomes that belong to both A and B.



Complement

- The complement of A, written as A’ or A^c, is the set of all outcomes in *S* that are not in A.



Example

Ex. Select a card at random from a standard deck of cards, and note its suit: clubs (Cl), diamonds (D), hearts (H) or spades (Sp).

The sample space is S={Cl, D, H, Sp}.

Let: A={Cl, D}, B={D, H, Sp} and C={H}.

A∪B={Cl, D, H, Sp}=S

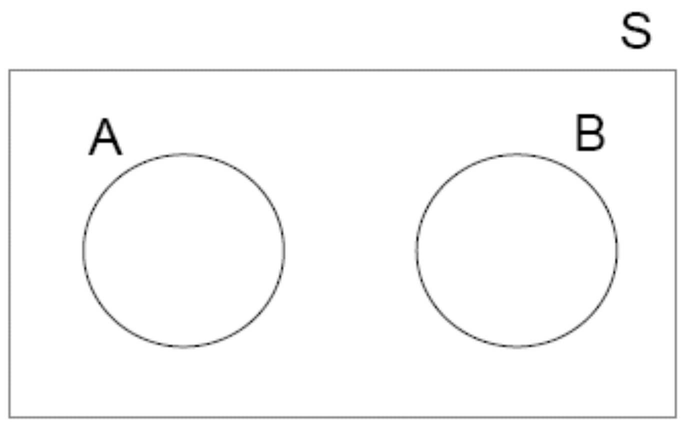
A∩B={D}

A^c={H, Sp}

A∩C= ∅ (null event – event consisting of no outcomes)

Disjoint events

- If $A \cap B = \emptyset$ then A and B are said to be mutually exclusive or disjoint events.



- Any event and its complement are disjoint!

Probability models

- A probability model consists of a sample space and the assignment of probabilities to each possible outcome.

- Probability that event A occurs is written as P(A), which will give a precise measure of the chance that A will occur.

- To ensure the probability assignments will be consistent with our intuitive notions of probability, all assignments should satisfy the following axioms (basic properties) of probability.
    1. For any event A, $P(A) \geq 0$.
    2. $P(S)=1$.
    3. If $A_1, A_2, A_3, \ldots$ is an infinite (finite) collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum P(A_i)$$

Propositions

- For any event A, $0 \leq P(A) \leq 1$.
- $P(A) + P(A^c) = 1$.
- If event A is contained in event B, in the sense that every outcome in A is also in B, then
$$P(A) \leq P(B)$$
- $P(\emptyset) = 0$.

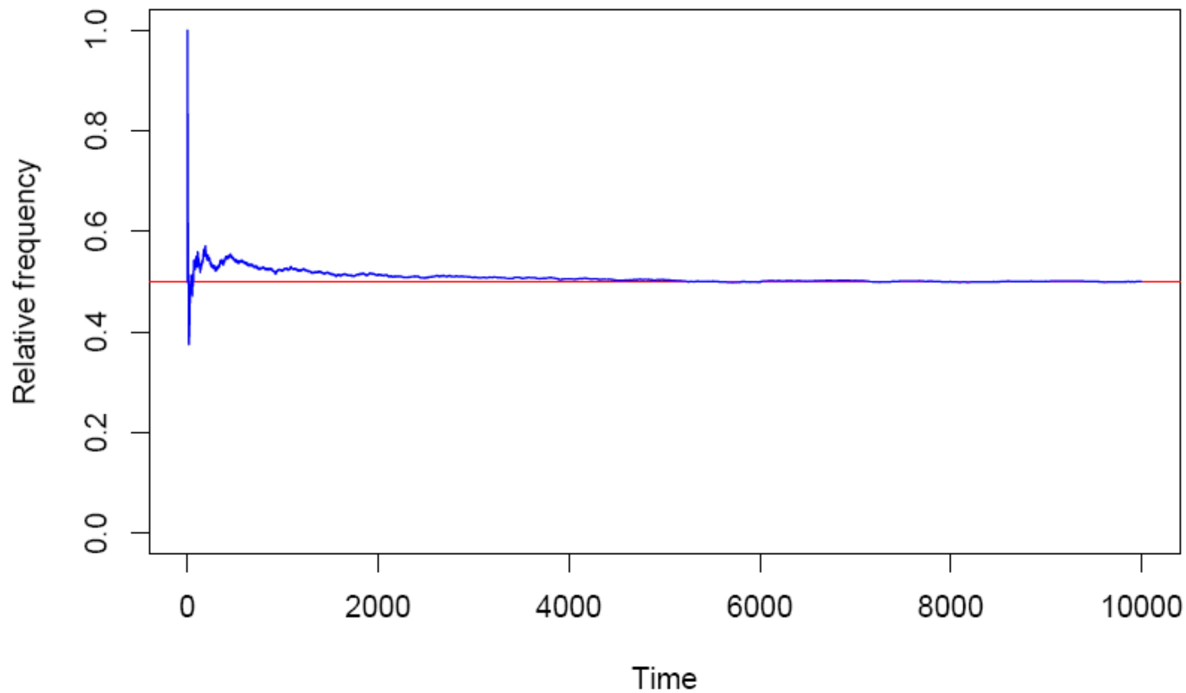
Interpreting Probability

- What does it mean when we say we have 50% chance of having a head when flipping a coin? Or what does it mean when we put $P(H)=0.5$?

- Probability is often treated as the *long-term relative frequency* or the *limiting relative frequency*.

Interpreting Probability

<u>Ex.</u> Flip a fair coin *n* times and calculate the proportion of heads.



- R demo. (Function: `sample(x, size)`; `rbinom(x, size, prob)`)

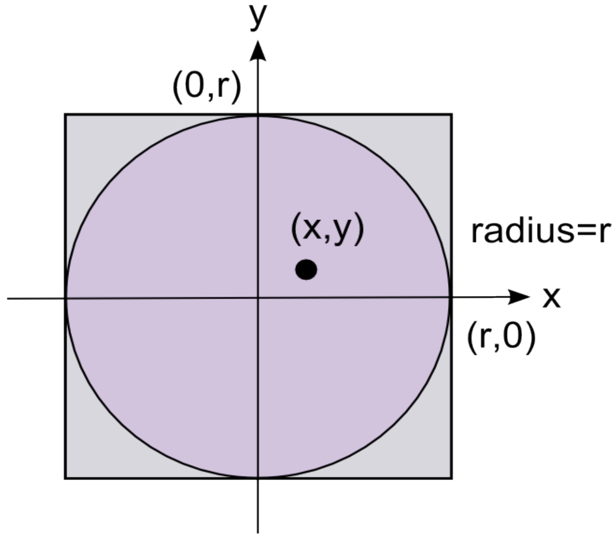
Law of Large Numbers

- The law of large numbers says that the long-run relative frequency of repeated independent events gets closer and closer to the true relative frequency as the number of trials increases.

$$\frac{\text{\# of occurrence of event A}}{\text{\# of trials } (n)} \xrightarrow{n \to \infty} P(A)$$

How to calculate Pi

- An interesting application of Law of Large Numbers is to calculate Pi through simulations.
- If we spread a large quantity of seeds randomly but evenly on this square, what percentage of the seeds will lie inside the circle?



- R Demo.
- This type of simulation-based methods has a fancy name: Monte Carlo methods.

Assigning Probabilities

- The assignment of probabilities can often be derived from the physical set-up of an experiment.

- Suppose we have N outcomes in our sample space, each equally likely to occur. The each has a probability of $1/N$, and the probability of any event A is,

$$P(A) = \frac{\text{number of outcomes in A}}{N}$$

Ex. Roll a fair die. $S=\{1,2,3,4,5,6\}$. Our sample space consists of 6 points, each of which is equally likely to occur.

P(roll a 1) = 1/6.

Let A = roll a 4 or less = {1,2,3,4}. P(A) = 4/6.

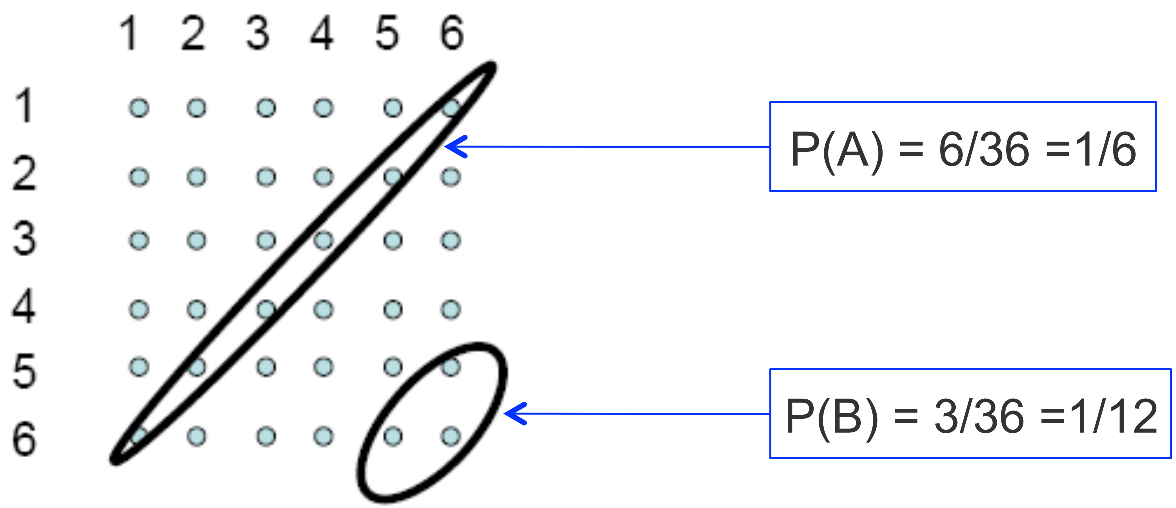
Let B = roll an even number = {2,4,6}. P(B) = 3/6.

Example

<u>Ex.</u> Roll two fair dice.

There are 36 possible outcomes: {(1,1),(1,2),(1,3),…,(6,5),(6,6)}.

Let A = sum of two rolls is 7; B = sum of two rolls is 11 or more. What are P(A) and P(B)?



Counting Techniques

- In the previous example, we used brute force to calculate the probability for event A and event B.

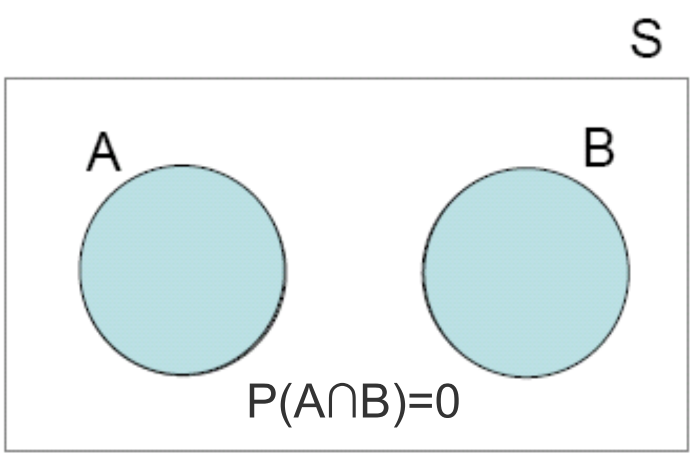
- In this class, quite often, we need better ways to count how many outcomes there are in a particular event.

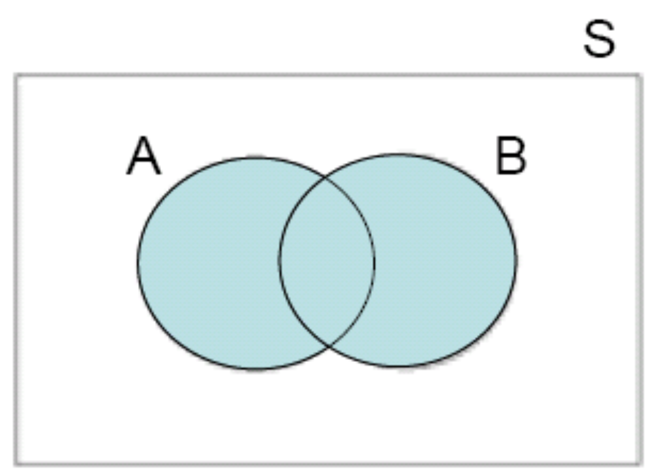
- Permutation: $P_{k,n} = n!/(n-k)!$

- Combination: $C_{k,n} = P_{k,n}/k!$ Also denoted as $\binom{n}{k}$.

More Probability Properties

- Consider an experiment whose sample space is *S*. For each event A (B) in *S*, we assume that a number P(A) is defined and satisfies the following rules:
  1. $0 \leq P(A) \leq 1$.
  2. $P(S)=1$.
  3. $P(A^c)=1-P(A)$.
  4. If A and B are disjoint, then $P(A \cup B)=P(A)+P(B)$.
  5. For any two events A and B, $P(A \cup B)=P(A)+P(B)-P(A \cap B)$.





Example

Ex. A store accepts either VISA or Mastercard. 50% of the stores customers have VISA, 30% have Mastercard and 10% have both. What is the probability that a customer has a credit card the store accepts?

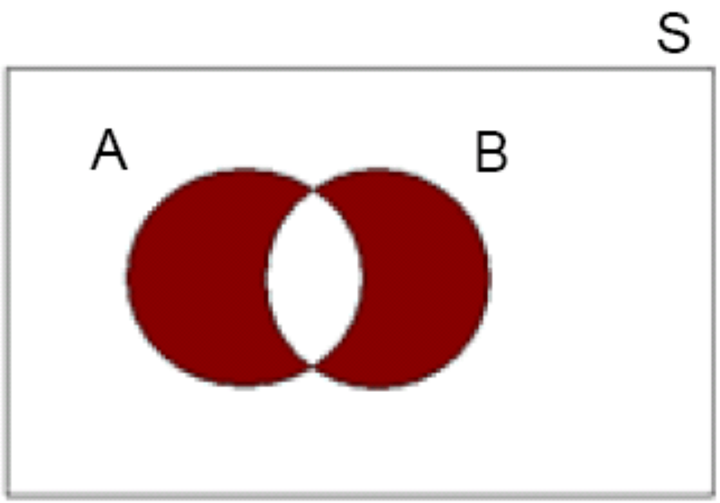
A = customers has VISA

B = customers has Mastercard

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.5 + 0.3 - 0.1 = 0.7$$

Example cont.

What is the probability that a customer has either a VISA or MC, but not both?



$$P(\text{A or B but not both}) = P(A) + P(B) - 2P(A \cap B)$$
$$= 0.5 + 0.3 - 0.2 = 0.6$$

Example Cont.

What is the probability that a customer has a VISA but no MC?

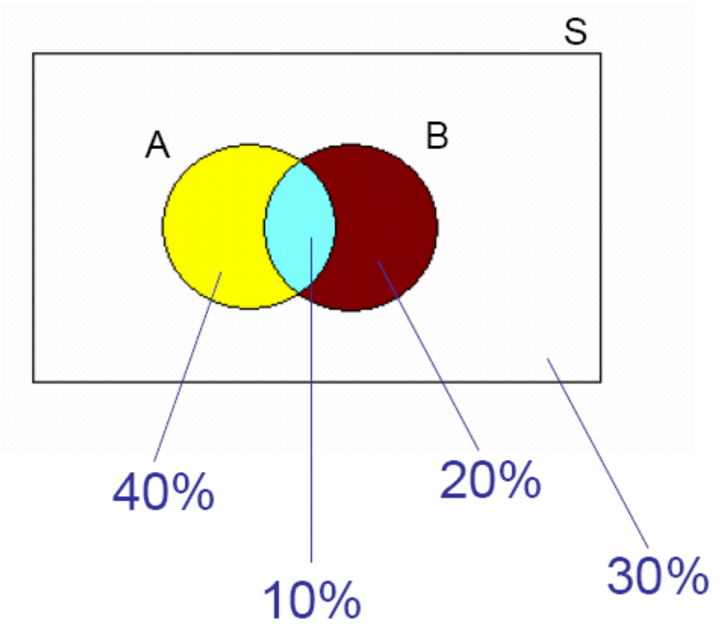
$P(\text{A but not both}) = P(A) - P(A \cap B)$

$= 0.5 - 0.1 = 0.4$

What is the probability that a customer has a MC but no VISA?

$P(\text{B but not both}) = P(B) - P(A \cap B)$

$= 0.3 - 0.1 = 0.2$



Three Events

- For any three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$