W1211 Introduction to Statistics
Lecture 23

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What we talked about last lecture

- Large Sample Confidence Interval for a population mean μ.
- Large Sample Confidence Interval for a population proportion p.

CIs Based on the t Distribution

- The above discussions are based on the large-sample assumptions. But what can we do if we don't have a large sample?
- When the distribution under discussion is normal, we do have a solution, that is based on the so-called t distribution.
- Our assumption right now is $X_1, X_2, \ldots, X_n$ IID from *normal* distribution with unknown mean μ and unknown σ.

The t Distribution

- When $\bar{X}$ is the sample mean of a simple random sample from normal under the previous assumptions, then RV
$$T = \frac{\bar{X} - \mu}{\hat{\sigma}/\sqrt{n}}$$
has a probability distribution called a t distribution with $n - 1$ degrees of freedom (df). We write
$$\frac{\bar{X} - \mu}{\hat{\sigma}/\sqrt{n}} \sim t_{n-1}$$

The t Distribution Cont'd

- The property of the *t* distribution
  - Bell-shaped curve centered at 0.
  - More spread-out than standard normal curve (heavy-tail).
  - When the degrees of freedom approach infinity, *t* distribution converges to standard normal.

The t Distribution Cont'd

- The property of the *t* distribution
  - Bell-shaped curve centered at 0.
  - More spread-out than standard normal curve (heavy-tail).
  - When the degrees of freedom approach infinity, *t* distribution converges to standard normal.
- The shape of t density curves

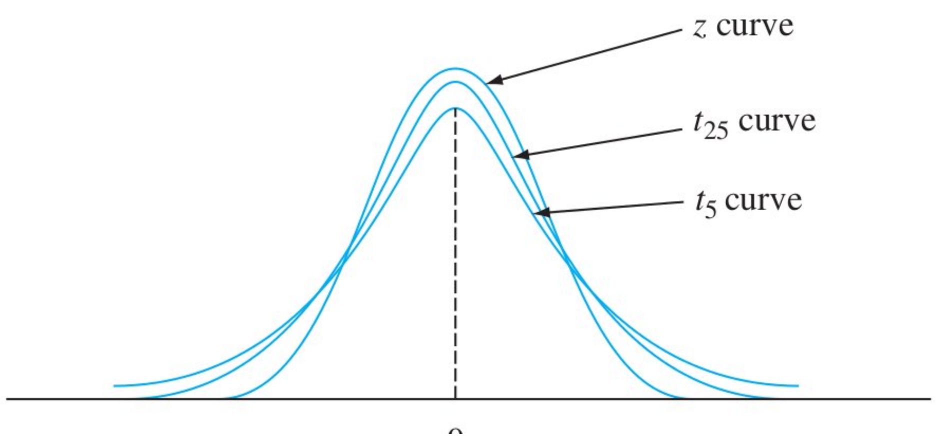

Figure: Comparison between normal density curve (z curve) and t density curves.

t distribution table

Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

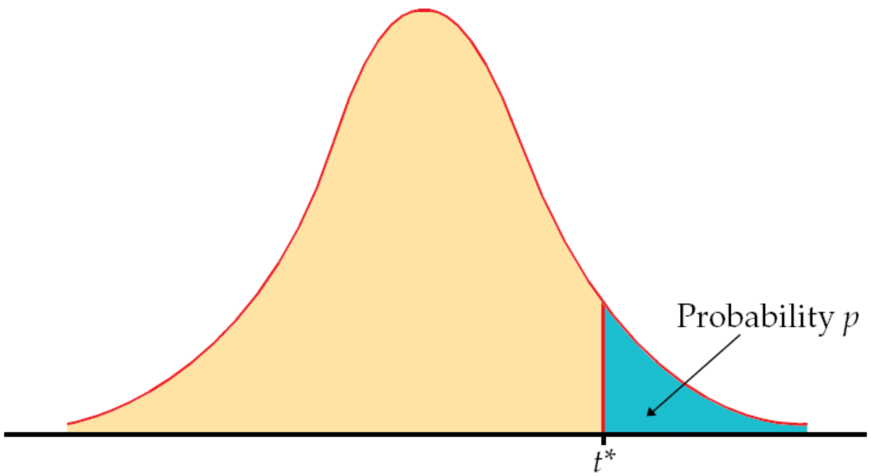

TABLE D

t distribution critical values

	Upper-tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408

Confidence Interval for μ

- Let $\bar{x}$ and s be the sample mean and sample standard deviation computed from a simple random sample from a normal population with mean μ, then a $100(1-\alpha)\%$ confidence interval for μ is

$$\left(\bar{x} - t_{\alpha/2,n-1}\frac{\hat{\sigma}}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1}\frac{\hat{\sigma}}{\sqrt{n}}\right)$$

- An upper confidence interval is

$$\bar{x} + t_{\alpha,n-1}\frac{\hat{\sigma}}{\sqrt{n}}$$

Confidence Interval Summary

- Confidence Interval is random, the parameter is fixed!
- The general strategy is to find a pivotal quantity and derive the CI from there.
- We have worked out the formula of CI's for population mean μ in the following 3 scenarios.

I. Normal Distribution, Known σ, Any Sample Size

- Under these assumptions, a $100(1-\alpha)\%$ CI of population mean μ is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

II. General Distribution, Unknown σ, Large Sample

- Under these assumptions, an approximate $100(1-\alpha)\%$ CI of population mean μ is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}\right)$$

- Notice a special case if the distribution is Bernoulli, we have a more accurate but very complicated formula.

III. Normal Distribution, Unknown σ, Any Sample Size

- Under these assumptions, a $100(1-\alpha)\%$ CI of population mean μ is given by

$$\left(\bar{x} - t_{\alpha/2,n-1} \cdot \frac{\hat{\sigma}}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1} \cdot \frac{\hat{\sigma}}{\sqrt{n}}\right)$$

- Here we utilize the *t* distribution.

Confirmatory v.s. Exploratory Data Analysis

- There are two traditions in statistics: Exploratory Data Analysis and Confirmatory Data Analysis.
- In Confirmatory Data Analysis (Hypothesis Testing), we have a null-hypothesis that we are testing against, which represents some form of our prior belief about the world. Example: Popularity of violent games and movies has no effect on crime rate.
- In Exploratory Data Analysis, there is no null-hypothesis. In some sense, our job is to discover new null-hypothesis that we can test against. Example: Collecting various variables from different countries and investigate which variables are most closely associated with crime rate.

Hypothesis Testing

- A *statistical hypothesis*, or just *hypothesis*, is a claim or assertion either about the value of a single parameter (population characteristic or characteristic of a probability distribution), about the values of several parameters, or about the form of an entire probability distribution.

- A testing problem usually contains two hypotheses: the null hypothesis, denoted by H_0, is the claim that is initially assumed to be true (the “prior belief” claim). The alternative hypothesis, denoted by H_a, is the assertion that is contradictory to H_0.

- The null hypothesis will be rejected in favor of the alternative only if sample evidence suggests that H_0 is false. If the sample does not strongly contradict H_0, we will continue to believe in the truth of the null hypothesis. The two possible conclusions from a testing analysis are then *reject* H_0 or *fail to reject* H_0.

Examples

Ex. A factory claims that less than 10% of the components they produce are defective. A consumer group is skeptical of the claim and checks a random sample of 300 components and finds that 39 are defective. Is there evidence that more than 10% of all components made at the factory are defective?

$H_0 : p \leq 0.10 \qquad H_a : p > 0.10$

Ex. We are interested in height of all Columbia students. In a sample of 12 students, the sample mean is 66.30 inches, and the sample s.d. is 4.35 inches. Should we reject the null hypothesis $H_0 : \mu = 68$ vs $H_a : \mu \neq 68$?

Remarks

- In our treatment of hypothesis testing, H_0 will always be stated as an equality claim. If θ denotes the parameter of interest, the null hypothesis will have the form H_0: $\theta = \theta_0$.

- The alternative to the null hypothesis H_0: $\theta = \theta_0$ will usually look like one of the following three forms:

1. H_a: $\theta > \theta_0$ (in which case the implicit null hypothesis is $\theta \leq \theta_0$).
2. H_a: $\theta < \theta_0$ (in which case the implicit null hypothesis is $\theta \geq \theta_0$).
3. H_a: $\theta \neq \theta_0$.
4. H_a: $\theta = \theta_1 \neq \theta_0$ (simple alternative).

- The value θ_0 separates the alternative from the null and is called the null value. The null and alternative are not treated equivalently, once a statement is in the null hypothesis, we will not easily reject it unless we have enough evidence.

Motivating example

Ex. Suppose we have a biased coin, we believe that it has probability 95% of having a head in a flip. Alternatively, it could also have probability 5% of having a head. Can you design a simple test to see if the coin has probability 95% of having heads?

Simple alternative: $H_0 : p = 0.95$ $H_a : p = 0.05$

Test Procedures

A test procedure is specified by the following:

1. Find a test statistic, a function of the sample data on which the decision (reject H_0 or do not reject H_0).
2. Construct a rejection region, the set of all test statistic values for which H_0 will be rejected.

The null hypothesis will then be rejected if and only if the observed or computed test Statistic value falls in the rejection region.

Can you construct a test procedure for the previous example?

Example cont.

Ex. (Biased coin cont.) In order to test if $p = 0.95$ we decide to conduct one experiment. We are going to flip this biased coin once, if it comes out a head, we will accept the null hypothesis, if it comes out a tail, we will reject the null hypothesis.

Test statistic: X = outcome of the first flip (Bernoulli rv.)

Rejection region: $\{X: X = 0\}$

Any other test statistics?

What are the odds that we'll make a mistake in our decision?