S1211Q Introduction to Statistics
Lecture 14

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Estimator, Its Standard Error and Estimated Standard Error

- Now we are trying to estimate the probability of getting heads of a biased coin, so each flip X_i is a Bernoulli RV with parameter p, the estimator of parameter p is the sample mean/proportion
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- Also, we need to report how good our estimator is through its Standard Error. This is also related to the Interval Estimation.

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- The standard error is $Var(\hat{p}) = \frac{p(1-p)}{n}$, but we cannot report it since we don't know what p is.
- So we can only report the estimated standard error of the estimator $\hat{p}$

$$\widehat{Var(\hat{p})} = \frac{\hat{p}(1-\hat{p})}{n}$$

Another Example

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- $$\hat{\mu} = \bar{X}$$
- The standard error of $\hat{\mu}$ is $\sqrt{Var(\hat{\mu})} = \frac{\sigma}{\sqrt{n}}$. Can we report $\frac{\sigma}{\sqrt{n}}$?
- It really depends on whether or not we know σ. If we know it, then we can report $\frac{\sigma}{\sqrt{n}}$; otherwise, we can only report $\frac{\hat{\sigma}}{\sqrt{n}}$.

Methods of Point Estimation

- The definition of unbiasedness does not in general indicate how unbiased estimators can be derived.

- There are two commonly used “constructive” methods for obtaining point estimators: the method of moments and the method of maximum likelihood.

- Although maximum likelihood estimators are generally preferable to moment estimators because of certain efficiency properties, they often require significantly more computation than do moment estimators.

- It is NOT guaranteed that these two methods would yield unbiased estimators.

Population Moment and Sample Moment

- Let $X_1, \ldots, X_n$ be a random sample from a pmf or pdf $f(x)$. For $k = 1, 2, \ldots$, the kth population moment is $E(X^k)$. The kth sample moment is $(1/n)\sum_{i=1}^{n} X_i^k$.
- The essence of the Methods of Moment is to equate population moments with sample moments and solve the resulting equations.

Moment Estimators

- Definition:

Let X_1, X_2, …, X_n be an i.i.d. sample from a pmf or pdf $f(x)$. For k = 1, 2, 3, …, the moment estimator for the kth population moment, is the kth sample moment, i.e.,

$$\widehat{\mathrm{E}(\mathrm{X}^k)} = \frac{\sum_{i=1}^{n} \mathrm{X}_i^k}{n}$$

Remarks

- Notice that, the sample kth moment will converge to the population kth moment, as sample size $n \to \infty$, thanks to the LLN.

- For any finite sample size n, the sample kth moment in general will not be equal to the population kth moment. But it is a good candidate for estimation.

- The larger the n, the better the estimation is!

- In practice, if the underlying pmf or pdf has m parameters, namely, we have $f(x;\theta_1,\dots,\theta_m)$, where $\theta_1, \dots, \theta_m$ are parameters whose values are unknown. Then the moment estimators are obtained by equating the first m sample moments to the corresponding first m population moments and solving for $\theta_1,\dots,\theta_m$.

Example

Ex. Show that the sample proportion is the moment estimator of the population probability.

Example

Ex. Let X_1, X_2, …, X_n be an i.i.d. normal sample, and assume that the underlying normal distribution is $N(\mu,\sigma^2)$ where μ,σ^2 are unknown. How can we construct moment estimators to estimate the two unknown parameters?

As we already know if $X \sim N(\mu,\sigma^2)$, then $E(X) = \mu$, and $E(X^2) = \mu^2+\sigma^2$.

Therefore, we have two equations:

$$\begin{cases} \hat{\mu} = \sum_{i=1}^{n} X_i/n \\ \hat{\mu}^2 + \hat{\sigma}^2 = \sum_{i=1}^{n} X_i^2/n \end{cases} \longrightarrow \begin{cases} \hat{\mu} = \sum_{i=1}^{n} X_i/n \\ \hat{\sigma}^2 = \sum_{i=1}^{n} X_i^2/n - \bar{X}^2 \end{cases}$$

Is the variance estimator unbiased?

Example

Ex. Let X_1, X_2, …, X_n be an i.i.d. sample from exponential distribution with parameter λ which is unknown. How do we estimate λ using moment estimator?

As we already know if X ~ Exp(λ), then $E(X) = 1/\lambda$.

Thus, we have equation $1/\hat{\lambda} = \bar{X} \rightarrow \hat{\lambda} = 1/\bar{X}$.

Is this estimator unbiased?

Maximum Likelihood Est.

- The method of maximum likelihood was first introduced by R.A. Fisher, a geneticist and statistician, in the 1920s. It is by far the most commonly used method to obtain estimators.

- Likelihood function is just another way of looking at the *joint pmf or the pdf*. In particular, let $X_1, X_2, \ldots, X_n$ (not necessarily i.i.d.) have joint pmf or pdf

$$f(x_1, x_2, \ldots, x_n; \theta_1, \ldots, \theta_m)$$

  where $\theta_1, \ldots, \theta_m$ are parameters whose values are unknown. When $x_1, x_2, \ldots, x_n$ are the observed sample values and $f(.)$ is then regarded as a function of $\theta_1, \ldots, \theta_m$, it is called the likelihood function.

Example

Ex. A biased coin has been flipped for 10 times. Let $X_1, X_2, \ldots, X_{10}$ denote the outcomes of the coin flips. Assume the probability of having a head is *p* (parameter of interest), and the sample we observed is {0,1,1,0,0,0,1,0,0,0}. Write down the likelihood function for *p*.

$$f(x_1, x_2, \ldots, x_n; p) = f(x_1; p)\, f(x_2; p) \ldots f(x_n; p) = (1-p)\, p\, p\, (1-p) \ldots (1-p) = p^3(1-p)^7$$

Idea of Maximum Likelihood: can we find a *p* that can maximize the above function?

MLE

- The maximum likelihood estimates (mle's) $\hat{\theta}_1, \ldots, \hat{\theta}_m$ are those values of θ_i's that maximize the likelihood function, so that

  $$f(x_1, \ldots, x_n; \hat{\theta}_1, \ldots, \hat{\theta}_m) \geq f(x_1, \ldots, x_n; \theta_1, \ldots, \theta_m) \quad \text{for all } \theta_1, \ldots, \theta_m$$

  when the X_i's are substituted in place of the x_i's.

- Remark: the likelihood function tells us how likely the observed sample is as a function of the possible parameter values. Maximizing the likelihood gives the parameter values for which the observed sample is most likely to have been generated – that is, the parameter values that "agree most closely" with the observed data.

- In practice, in stead of maximizing the likelihood itself, people usually choose to maximize the log-likelihood function.

Example

Ex. Let X_1, X_2, …, X_n be an i.i.d. sample from exponential distribution with parameter λ which is unknown. Write down the likelihood function for λ. What is the MLE of λ? Is the MLE unbiased?

Since we have an i.i.d. sample, it is easy to see that the likelihood function is a product of the individual pdf's:

$$f(x_1, \ldots, x_n; \lambda) = (\lambda e^{-\lambda x_1}) \cdot \cdots \cdot (\lambda e^{-\lambda x_n}) = \lambda^n e^{-\lambda \sum x_i}$$

$$\log[f(x_1, \ldots, x_n; \lambda)] = n \log(\lambda) - \lambda \sum x_i$$

$$\hat{\lambda} = n / \sum \mathrm{X}_i$$

Example with Normal

- Let $X_1, X_2, \ldots, X_n$ be an IID sample from normal distribution with mean μ and variance σ^2, what is the likelihood function?

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- Let $X_1, X_2, \ldots, X_n$ be an IID sample from normal distribution with mean μ and variance σ^2, what is the likelihood function?
- $$f(x_1, x_2, \ldots, x_n; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

  or in logarithm

  $$-\frac{n}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{n}[-(x_i - \mu)^2/\sigma^2]$$

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- Take derivative with respect to μ and σ^2 and solve the resulting equations

  $$\hat{\mu} = \bar{X}, \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$$

Some complications

- The following is an example that MLE's can't be calculated analytically.

<u>Ex.</u> Let X_1, X_2, …, X_n be an i.i.d. sample from Weibull distribution with parameters α and β and pdf

$$f(x;\alpha,\beta) = \begin{cases} \frac{\alpha}{\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-(x/\beta)^\alpha} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

by solving equations $\dfrac{\partial \log(f)}{\partial \alpha} = 0 \quad \dfrac{\partial \log(f)}{\partial \beta} = 0$

$$\hat{\alpha} = \left[\frac{\sum x_i^{\hat{\alpha}} \cdot \log(x_i)}{\sum x_i^{\hat{\alpha}}} - \frac{\sum \log(x_i)}{n} \right]^{-1} \qquad \hat{\beta} = \left(\frac{\sum x_i^{\hat{\alpha}}}{n} \right)^{1/\hat{\alpha}}$$

Some Complications

- Also, sometimes we cannot use calculus to get the MLE, such as when the density is not differentiable.
- Read Example 6.22 on textbook P.262.

The Invariance Principle

- One of the nice features of MLE's is that, the MLE of a function of parameters, is the function of the MLE's of the parameters.

- More specifically, we have

> Let $\hat{\theta}_1, \ldots, \hat{\theta}_m$ be the MLE's of the parameters $\theta_1, \ldots, \theta_m$. Then the MLE of any function $h(\theta_1, \ldots, \theta_m)$ of these parameters is $h(\hat{\theta}_1, \ldots, \hat{\theta}_m)$.

<u>Ex.</u> In the normal example, what is the MLE of σ?

Large Sample Behavior

- The following proposition says, for large samples, it is “optimal” to use MLE’s, because it is asymptotically unbiased and has the minimal variance among all unbiased estimators.

- Proposition:

  Under very general conditions on the joint distribution of the sample, When the sample size n is large, the maximum likelihood estimator is Approximately the MVUE of the parameter.

Confidence Intervals

- A point estimate, because it is a single number, by itself provides no information about the precision and reliability of estimation (the reason why we need standard error).

- An alternative to reporting a single sensible value for the parameter being estimated is to calculate and report an entire interval of plausible values – an *interval estimate* or *confidence interval* (*CI*).

- A confidence interval is always calculated by first selecting a *confidence level*, which is a measure of the degree of reliability of the interval.

- Construct a confidence interval for a standard normal random variable.

Illustration

- Let’s first consider a simple, somewhat unrealistic problem situation.
  1. We are interested in the population mean parameter μ.
  2. The population distribution is normal.
  3. The value of the population standard deviation σ is known. (unlikely!)

- Suppose we have a random sample $X_1, X_2, \ldots, X_n$ from a normal distribution with mean value μ and standard deviation σ. As we know, $\bar{X}$ also follows a normal distribution with mean value μ and standard deviation $\sigma/\sqrt{n}$. Thus, we could get a standard normal distribution by normalizing $\bar{X}$.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Construction

- The smallest interval that contains 95% of the possible outcomes of Z is (-1.96, 1.96).

$$-1.96 < \frac{\bar{\mathrm{X}} - \mu}{\sigma/\sqrt{n}} < 1.96$$

$$\Downarrow$$

$$-1.96 \cdot \frac{\sigma}{\sqrt{n}} < \bar{\mathrm{X}} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

$$\Downarrow$$

$$\bar{\mathrm{X}} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{\mathrm{X}} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

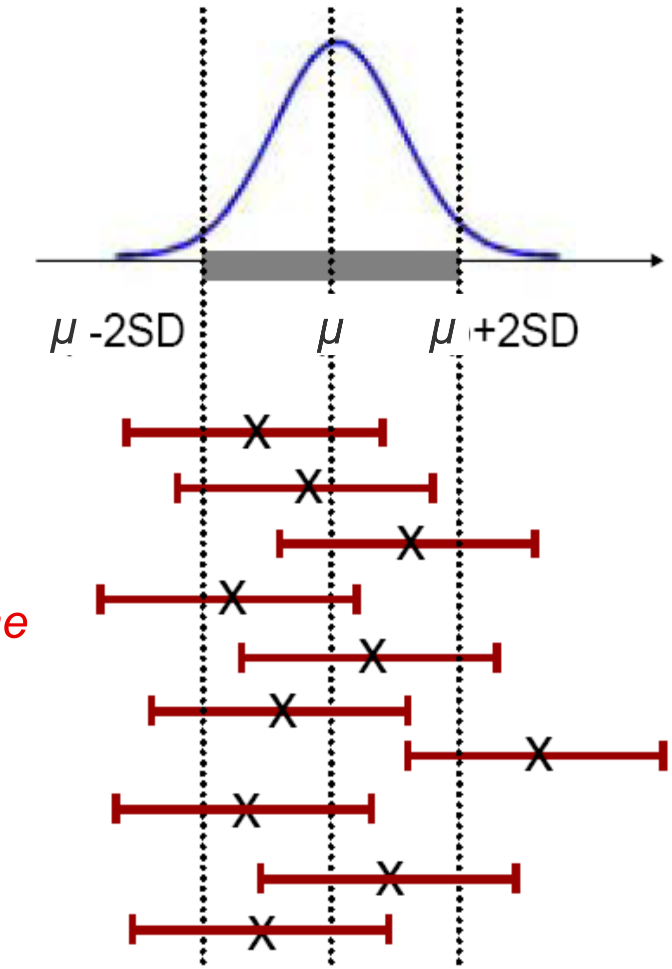
Interpretation

- Thus we have $\mathrm{P}\left(\bar{\mathrm{X}} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{\mathrm{X}} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) = 0.95$.
- Some people interpreted this as: the true parameter *μ* has 95% chance of falling in the interval of $(\bar{\mathrm{X}} - 1.96 \cdot \sigma/\sqrt{n}, \bar{\mathrm{X}} + 1.96 \cdot \sigma/\sqrt{n})$. Is it right?
- In fact, the two boundaries of the interval given above are random! Thus every time we sample n observations from the same population, we will get a different confidence interval!

Random Interval

- By constructing a confidence interval like this, we never be sure whether *μ* actually lies in our confidence interval. However, we know that about 95 out of 100 times intervals constructed using this method will capture the true parameter.

- Interpreted as: "*the probability is .95 that the random interval includes or covers the true value of μ*."



Confidence Interval

- Definition:

A $100(1-\alpha)\%$ confidence interval for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

- $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of a standard normal distribution, i.e., $P(Z>z_{\alpha/2})=\alpha/2$.

Remarks

- When constructing a confidence interval, *confidence level*, *precision*, and *sample size* are closely related. Is there a finite 100% confidence interval?

- The precision, or the width of the confidence interval when σ is known is, $2z_{\alpha/2}\sigma/\sqrt{n}$. Thus we can see, the confidence level of the interval is inversely related to its precision.

- The precision is also inversely related to the sample size.

- An appealing strategy is to specify both the desired confidence level and interval width and then determine the necessary sample size.

Sample Size Calculation

- The general formula for the sample size *n* necessary to ensure an interval width *w* is obtained from $w = 2 \cdot z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

$$n = \left(2 \cdot z_{\alpha/2} \cdot \frac{\sigma}{w}\right)^2$$

<u>Ex.</u> A new operating system has been installed, and we wish to estimate the true average response time μ to a particular editing command. Assuming that response times are normally distributed with σ=25 millisec. How many tests should we do to ensure that the resulting 95% CI has a width of at most 10?

Non-normal and Unknown Variance

- Previously we constructed a confidence interval for normal population mean with known variance. The next question would then be, what if we don't have normality and what if we don't know the underlying variance?

- If we have large enough sample size, the celebrated CLT can help us construct a confidence interval for the mean parameter of a population with unknown distribution and unknown variance. Consider the following quantity

$$\frac{\bar{\mathrm{X}}-\mu}{\hat{\sigma}/\sqrt{n}} = \underset{\boxed{N(0,1)}}{\underset{\text{CLT}\,\Big\downarrow}{\boxed{\frac{\bar{\mathrm{X}}-\mu}{\sigma/\sqrt{n}}}}} \cdot \underset{\boxed{1}}{\underset{\Big\downarrow\,\text{LLN}}{\boxed{\frac{\sigma}{\hat{\sigma}}}}}$$

General Results

- Proposition:

  A 100(1-α)% confidence interval for the mean μ of any population when the value of σ is unknown and sample size n is sufficiently large is given by

  $$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}\right)$$

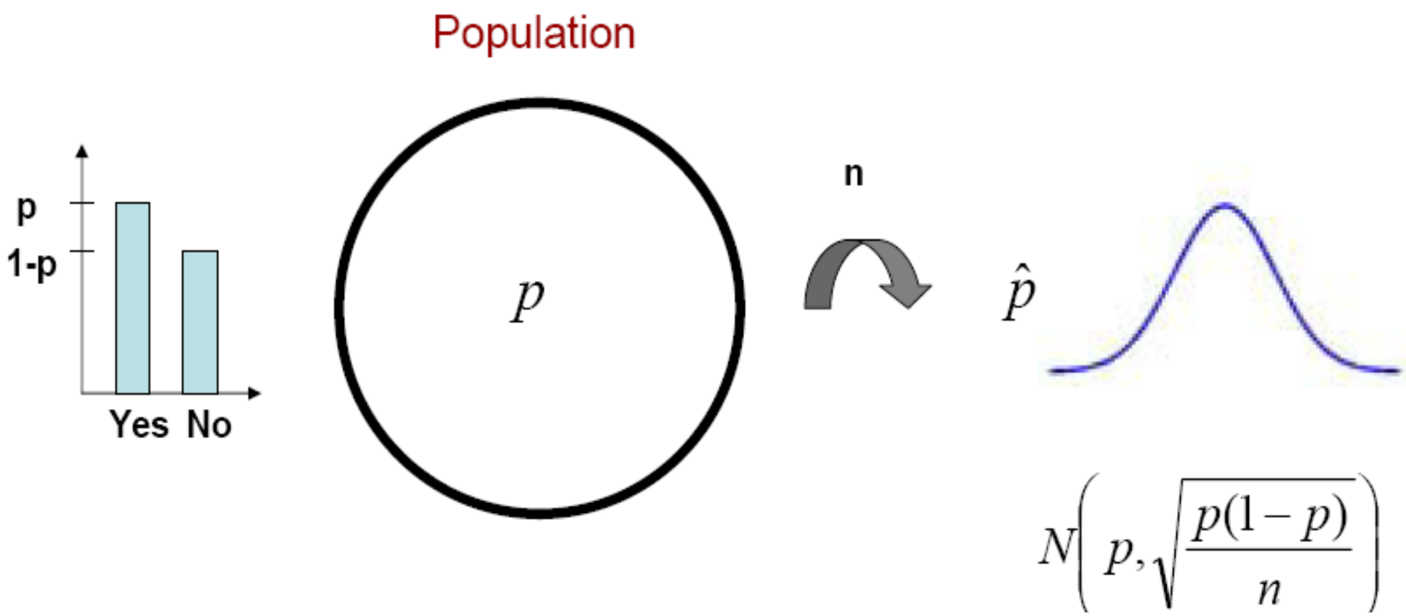
- Rule of Thumb: generally speaking, n>40 will be sufficient to justify the use of this interval. This is somewhat more conservative than the rule of thumb for the CLT, because of the additional randomness coming from $\hat{\sigma}$.

- One can also derive a similar sample size calculation formula in this case

  $$n = \left(2 \cdot z_{\alpha/2} \cdot \frac{\hat{\sigma}}{w}\right)^2$$

Proportions

- A special case of non-normal population is Bernoulli population. And the parameter of interest is the population proportion *p*.



Large Sample CI

- One can directly apply the proposition from the large sample case to construct the CI for the population proportion *p*.

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}\right)$$

- In this case $\bar{x} = \hat{p}$, $\hat{\sigma}^2 = \hat{p}(1-\hat{p})$.

- If we set *q*=1-*p*, then the large sample confidence interval for *p* should be

$$\left(\hat{p} - z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}, \hat{p} + z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}\right)$$

- To calculate sample size: $n = \left(2 \cdot z_{\alpha/2} \cdot \frac{\sqrt{\hat{p}\hat{q}}}{w}\right)^2$

Another way

- The large sample confidence interval works fine if we have enough data. But for finite samples we can construct a better CI.

- Since in this case, we only have 1 parameter *p*, by CLT, we have

$$\mathrm{P}\left(-z_{\alpha/2} < \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} < z_{\alpha/2}\right) \approx 1 - \alpha$$

- If we solve the resulting quadratic function, we'll have a new confidence interval for *p*.

$$\left(\frac{\hat{p} + \frac{z_{\alpha/2}^2}{2n} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + z_{\alpha/2}^2/n}, \frac{\hat{p} + \frac{z_{\alpha/2}^2}{2n} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + z_{\alpha/2}^2/n}\right)$$

Remarks

- The latter confidence interval looks complicated, but it "can be recommended for use with nearly all sample sizes and parameter values". Therefore we don't have to check for large sample conditions.

- In the latter case, we can also derive a new sample size calculation formula

$$n = \frac{2z_{\alpha/2}^2\hat{p}\hat{q} - z_{\alpha/2}^2 w^2 \pm \sqrt{4z_{\alpha/2}^4\hat{p}\hat{q}(\hat{p}\hat{q} - w^2) + w^2 z_{\alpha/2}^4}}{w^2}$$

  "+" sign is used!

- When sample size is large, the confidence interval we just constructed and the sample size calculation formula will be equivalent to

$$\left(\hat{p} - z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}, \hat{p} + z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}\right) \quad \text{and} \quad n = \left(2 \cdot z_{\alpha/2} \cdot \frac{\sqrt{\hat{p}\hat{q}}}{w}\right)^2$$

One-sided CI

- In some situations, an investigator will want only one upper bound or one lower bound for the parameter.

- Follow a similar argument as in the two-sided case, we have the following result

A large sample 100(1-α)% confidence upper bound for the mean μ is

$$\mu < \bar{x} + z_\alpha \cdot \frac{\hat{\sigma}}{\sqrt{n}}$$

and a lower bound is

$$\mu > \bar{x} - z_\alpha \cdot \frac{\hat{\sigma}}{\sqrt{n}}$$

A one-sided confidence bound for p results from replacing $z_{\alpha/2}$ by z_α.

Constructing a CI

- The previous examples show the general procedure of constructing confidence intervals. Suppose $X_1, X_2, \ldots, X_n$ are the sample on which the CI for a parameter θ is to be based. Then we construct a so-called "pivotal" quantity whose distribution does not depend on parameters.

- In other words, the pivotal quantity is a function of both samples and parameters, i.e., $h(X_1, X_2, \ldots, X_n, \theta)$, and the distribution of $h(\cdot)$ does not depend on θ or any other unknowns.

- Then one can find *a* and *b* to satisfy $\mathrm{P}(a < h(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n; \theta) < b) = 1 - \alpha$, by the pivotal property, *a* and *b* do not depend on θ. Then the inequality can be manipulated to isolate θ, giving the equivalent probability statement

$$\mathrm{P}(l(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n) < \theta < u(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n)) = 1 - \alpha$$