S1211Q Introduction to Statistics
Lecture 18

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Hypothesis Testing for a Population Mean

- In this section, the null hypothesis is about a population mean $H_0 : \mu = \mu_0$ and there are there possible Alternative Hypothesis $H_a : \mu > \mu_0$ or $H_a : \mu < \mu_0$ or $H_a : \mu \neq \mu_0$.
- We will discuss three cases which parallel our discussion about Confidence Interval for a Population Mean.
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    - Case I: Normal Distribution and Known σ (z Test)
    - Case II: General Distribution, Unknown σ but Large Sample (z Test)
    - Case III: Normal Distribution and Unknown σ (t Test)

Case I: Normal Distribution and Known σ (z Test)

- Under the null hypothesis, the test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma\sqrt{n}}$$

follow a standard normal distribution.

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- If the Alternative Hypo is $H_a : \mu > \mu_0$, then the Rejection Region is something like $\{z \geq z_0\}$.

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$$Z = \frac{\bar{X} - \mu_0}{\sigma\sqrt{n}}$$
follow a standard normal distribution.
- If the Alternative Hypo is $H_a : \mu > \mu_0$, then the Rejection Region is something like $\{z \geq z_0\}$.
- z_0 is determined by the level of the test α, if we set z_0 as z critical value z_α then
$$\begin{aligned} P(\text{type I error}) &= P(H_0 \text{ is rejected when } H_0 \text{ is true}) \\ &= P(Z > z_\alpha \text{ when } Z \sim N(0,1)) = \alpha \end{aligned}$$

Case I: Normal Distribution and Known σ (z Test)

- We can also compute Type II Error β and sample size n. Still we consider the upper-tailed test as a demonstration.
- Type II Error β will be a function of any particular number μ' that is larger than the null value μ_0.

$$
\begin{aligned}
\beta(\mu') &= P(Z < z_\alpha \text{ when } \mu = \mu') \\
&= P(\frac{\bar{X} - \mu_0}{\sigma\sqrt{n}} < z_\alpha \text{ when } \mu = \mu') \\
&= P(\frac{\bar{X} - \mu'}{\sigma\sqrt{n}} < z_\alpha + \frac{\mu_0 - \mu'}{\sigma\sqrt{n}} \text{ when } \mu = \mu') \\
&= \Phi(z_\alpha + \frac{\mu_0 - \mu'}{\sigma\sqrt{n}}) \leq 1 - \alpha
\end{aligned}
$$

$\Phi()$ is the CDF of standard normal.

- What is the power of the test?
- To add the table on p311 and figure on 312

Case I: Normal Distribution and Known σ (z Test)

- For a given True Value μ', Type I Error level α and Type II Error β, we can determined the sample size n that we need with

$$\Phi(z_\alpha + \frac{\mu_0 - \mu'}{\sigma\sqrt{n}}) = \beta$$

  Thus

$$-z_\beta = z_\alpha + \frac{\mu_0 - \mu'}{\sigma\sqrt{n}}$$

- To add the table on p314

Case II: General Distribution, Unknown σ but Large Sample (z Test)

- As we discussed in Confidence Interval, under the null hypothesis, the test statistic
$$Z = \frac{\bar{X} - \mu_0}{\hat{\sigma}\sqrt{n}}$$
approximately follow a standard normal distribution.
- The rule of thumb is $n > 40$.
- All the procedure, e.g., Test Statistic, Rejection Region and formula for β and sample size, are the same except for substituting σ with its estimator $\hat{\sigma}$.

Case III: Normal Distribution and Unknown σ (t Test)

- Under the null hypothesis, the test statistic
$$T = \frac{\bar{X} - \mu_0}{\hat{\sigma}\sqrt{n}}$$
follows a t distribution with degrees of freedom $n - 1$
- To add table on p317.