4.2

$f(x) = \frac{1}{10}$ for $-5 \le x \le 5$ and $= 0$ otherwise

a. $P(X < 0) = \int_{-5}^{0} \frac{1}{10}\,dx = .5$.

b. $P(-2.5 < X < 2.5) = \int_{-2.5}^{2.5} \frac{1}{10}\,dx = .5$.

c. $P(-2 \le X \le 3) = \int_{-2}^{3} \frac{1}{10}\,dx = .5$.

d. $P(k < X < k + 4) = \int_{k}^{k+4} \frac{1}{10}\,dx = \frac{1}{10}x\Big]_{k}^{k+4} = \frac{1}{10}[(k+4)-k] = .4$.

4.4

a. $\int_{-\infty}^{\infty} f(x;\theta)dx = \int_{0}^{\infty} \frac{x}{\theta^2} e^{-x^2/2\theta^2}\,dx = -e^{-x^2/2\theta^2}\Big]_{0}^{\infty} = 0-(-1) = 1$

b. $P(X \le 200) = \int_{-\infty}^{200} f(x;\theta)dx = \int_{0}^{200} \frac{x}{\theta^2} e^{-x^2/2\theta^2}\,dx = -e^{-x^2/2\theta^2}\Big]_{0}^{200} \approx -.1353+1 = .8647$.

$P(X < 200) = P(X \le 200) \approx .8647$, since X is continuous.

$P(X \ge 200) = 1 - P(X < 200) \approx .1353$.

c. $P(100 \le X \le 200) = \int_{100}^{200} f(x;\theta)dx = -e^{-x^2/20{,}000}\Big]_{100}^{200} \approx .4712$.

d. For $x > 0$, $P(X \le x) = \int_{-\infty}^{x} f(y;\theta)dy = \int_{0}^{x} \frac{y}{\theta^2} e^{-y^2/2\theta^2}\,dx = -e^{-y^2/2\theta^2}\Big]_{0}^{x} = 1-e^{-x^2/2\theta^2}$.

4.12

a. $P(X < 0) = F(0) = .5$.

b. $P(-1 \le X \le 1) = F(1) - F(-1) = .6875$.

c. $P(X > .5) = 1 - P(X \le .5) = 1 - F(.5) = 1 - .6836 = .3164$.

d. $f(x) = F'(x) = \frac{d}{dx}\left(\frac{1}{2}+\frac{3}{32}\left(4x-\frac{x^3}{3}\right)\right) = 0+\frac{3}{32}\left(4-\frac{3x^2}{3}\right) = .09375\left(4-x^2\right)$.

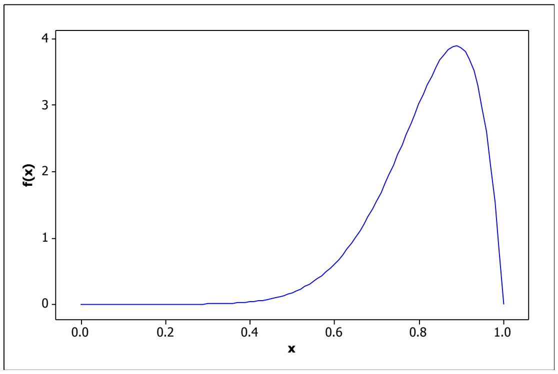
e. By definition, $F(\tilde{\mu}) = .5$. $F(0) = .5$ from **a** above, which is as desired.

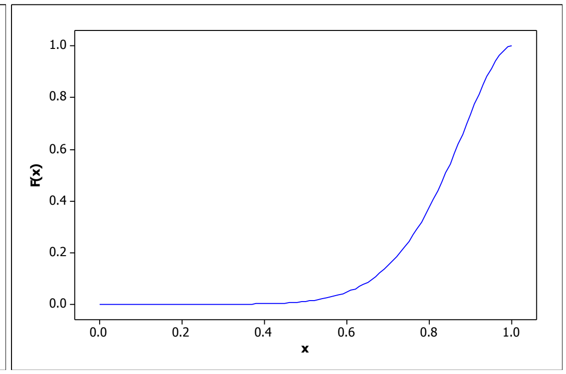
4.15

a. Since X is limited to the interval (0, 1), $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$.
For $0 < x < 1$,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_0^x 90y^8(1-y)dy = \int_0^x (90y^8 - 90y^9)dy = 10y^9 - 9y^{10}\Big]_0^x = 10x^9 - 9x^{10} \ .$$

The graphs of the pdf and cdf of X appear below.





b. $F(.5) = 10(.5)^9 - 9(.5)^{10} = .0107$.

c. $P(.25 < X \le .5) = F(.5) - F(.25) = .0107 - [10(.25)^9 - 9(.25)^{10}] = .0107 - .0000 = .0107$.
Since X is continuous, $P(.25 \le X \le .5) = P(.25 < X \le .5) = .0107$.

d. The 75th percentile is the value of x for which $F(x) = .75$: $10x^9 - 9x^{10} = .75 \Rightarrow x = .9036$ using software.

e. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x \cdot 90x^8(1-x)dx = \int_0^1 (90x^9 - 90x^{10})dx = 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = 9 - \frac{90}{11} = \frac{9}{11} = .8182$.

Similarly, $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_0^1 x^2 \cdot 90x^8(1-x)dx = \ldots = .6818$, from which $V(X) = .6818 - (.8182)^2 = .0124$ and $\sigma_X = .11134$.

f. $\mu \pm \sigma = (.7068, .9295)$. Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068) = .8465 - .1602 = .6863$, and the probability X is <u>more</u> than 1 standard deviation from its mean value equals $1 - .6863 = 3137$.

4.19

a. $P(X \le 1) = F(1) = .25[1 + \ln(4)] = .597$.

b. $P(1 \le X \le 3) = F(3) - F(1) = .966 - .597 = .369$.

c. For $x < 0$ or $x > 4$, the pdf is $f(x) = 0$ since X is restricted to (0, 4). For $0 < x < 4$, take the first derivative of the cdf:

$$F(x) = \frac{x}{4}\left[1 + \ln\left(\frac{4}{x}\right)\right] = \frac{1}{4}x + \frac{\ln(4)}{4}x - \frac{1}{4}x\ln(x) \Rightarrow$$

$$f(x) = F'(x) = \frac{1}{4} + \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) - \frac{1}{4}x\frac{1}{x} = \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) = .3466 - .25\ln(x)$$

4.28

e. $\Phi(1.37) = .9147$.

f. $P(-1.75 < Z) + [1 - P(Z < -1.75)] = 1 - \Phi(-1.75) = .9599$.

g. $\Phi(2) - \Phi(-1.50) = .9104$.

4.29

a. .9838 is found in the 2.1 row and the .04 column of the standard normal table so $c = 2.14$.

c. $P(c \le Z) = .121 \Rightarrow 1 - P(Z < c) = .121 \Rightarrow 1 - \Phi(c) = .121 \Rightarrow \Phi(c) = .879 \Rightarrow c = 1.17$.

d. $P(-c \le Z \le c) = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1 = .668 \Rightarrow \Phi(c) = .834 \Rightarrow$ $c = 0.97$.

4.36

a. $P(X < 1500) = P(Z < 3) = \Phi(3) = .9987$; $P(X \ge 1000) = P(Z \ge -.33) = 1 - \Phi(-.33) = 1 - .3707 = .6293$.

b. $P(1000 < X < 1500) = P(-.33 < Z < 3) = \Phi(3) - \Phi(-.33) = .9987 - .3707 = .6280$

c. From the table, $\Phi(z) = .02 \Rightarrow z = -2.05 \Rightarrow x = 1050 - 2.05(150) = 742.5$ μm. The smallest 2% of droplets are those smaller than 742.5 μm in size.

d. P(at least one droplet in 5 that exceeds 1500 μm) = 1 – P(all 5 are less than 1500 μm) = $1 - (.9987)^5 = 1 - .9935 = .0065$.

4.45

With $\mu = .500$ inches, the acceptable range for the diameter is between .496 and .504 inches, so unacceptable bearings will have diameters smaller than .496 or larger than .504.
The new distribution has $\mu = .499$ and $\sigma = .002$.

$P(X < .496 \text{ or } X > .504) = P\left(Z < \frac{.496 - .499}{.002}\right) + P\left(Z > \frac{.504 - .499}{.002}\right) = P(Z < -1.5) + P(Z > 2.5) =$

$\Phi(-1.5) + [1 - \Phi(2.5)] = .073$. 7.3% of the bearings will be unacceptable.

4.60

a. $P(X \le 100) = 1 - e^{-(100)(.01386)} = 1 - e^{-1.386} = .7499$.
$P(X \le 200) = 1 - e^{-(200)(.01386)} = 1 - e^{-2.772} = .9375$.
$P(100 \le X \le 200) = P(X \le 200) - P(X \le 100) = .9375 - .7499 = .1876$.

b. First, since X is exponential, $\mu = \dfrac{1}{\lambda} = \dfrac{1}{.01386} = 72.15$, σ = 72.15. Then
$P(X > \mu + 2\sigma) = P(X > 72.15 + 2(72.15)) = P(X > 216.45) = 1 - (1 - e^{-.01386(216.45)}) = e^{-3} = .0498$.

c. Remember the median is the solution to $F(x) = .5$. Use the formula for the exponential cdf and solve for x: $F(x) = 1 - e^{-.01386x} = .5 \Rightarrow e^{-.01386x} = .5 \Rightarrow -.01386x = \ln(.5) \Rightarrow x = -\dfrac{\ln(.5)}{.01386} = 50.01$ m.

5.7

a. $p(1,1) = .030$.

b. $P(X \le 1 \text{ and } Y \le 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .120$.

c. $P(X = 1) = p(1,0) + p(1,1) + p(1,2) = .100$; $P(Y = 1) = p(0,1) + \ldots + p(5,1) = .300$.

d. $P(\text{overflow}) = P(X + 3Y > 5) = 1 - P(X + 3Y \le 5) = 1 - P((X,Y)=(0,0)$ or …or $(5,0)$ or $(0,1)$ or $(1,1)$ or $(2,1)) = 1 - .620 = .380$.

e. The marginal probabilities for X (row sums from the joint probability table) are $p_X(0) = .05$, $p_X(1) = .10$, $p_X(2) = .25$, $p_X(3) = .30$, $p_X(4) = .20$, $p_X(5) = .10$; those for Y (column sums) are $p_Y(0) = .5$, $p_Y(1) = .3$, $p_Y(2) = .2$. It is now easily verified that for every (x,y), $p(x,y) = p_X(x) \cdot p_Y(y)$, so X and Y are independent.

5.22

a. $E(X+Y) = \sum\sum (x+y)p(x,y) = (0 + 0)(.02) + (5 + 0)(.04) + \ldots + (10 + 15)(.01) = 14.10$.
Note: It can be shown that $E(X + Y)$ always equals $E(X) + E(Y)$, so in this case we could also work out the means of X and Y from their marginal distributions: $E(X) = 5.55$, $E(Y) = 8.55$, so $E(X + Y) = 5.55 + 8.55 = 14.10$.

b. For each coordinate, we need the maximum; e.g., max(0,0) = 0, while max(5,0) = 5 and max(5,10) = 10. Then calculate the sum: $E(\max(X,Y)) = \sum\sum \max(x,y)?p(x,y) =$
max(0,0)(.02) + max(5,0)(.04) + … + max(10,15)(.01) = 0(.02) + 5(.04) + … + 15(.01) = 9.60.