S1211Q Introduction to Statistics
Lecture 11

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Joint Distribution

- How can we model two rv's using probability models? For example, if we are interested in both weight and height.
- Is it enough if we just use a normal model for weight and another normal model for height?
- We need to introduce joint probability distribution in order to model multiple rv's.

Joint PMF

- Let X and Y be two discrete rv's defined on the sample space. The joint probability mass function $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = P(X=x, Y=y).$$

- As in the single rv case, we must have $p(x, y) \geq 0$ and $\sum_x \sum_y p(x, y) = 1$.

Example

Ex. We randomly put two different balls into 3 numbered (numbered as {1,2,3}) boxes. Let X be the number of empty boxes left; let Y be the minimum of the box number that has balls in it. What is the joint distribution of (X, Y)?

X can take values from {1, 2};

Y can take values from {1, 2, 3};

It's not hard to see we have the following (why?):

$p(2, j) = P(X=2, Y=j) = 1/9$, for $j = 1, 2, 3$.

$p(1, 3) = P(X=1, Y=3) = 0$.

$p(1, 1) = P(X=1, Y=1) = 4/9$.

$p(1, 2) = P(X=1, Y=2) = 2/9$.

p_{ij}	1	2	3
1	$4/9$	$2/9$	0
2	$1/9$	$1/9$	$1/9$

Marginal PMF

- The marginal probability mass functions of X and Y, denoted by $p_X(x)$ and $p_Y(y)$, respectively, are given by

$$p_X(x) = \sum_y p(x,y) \qquad p_Y(y) = \sum_x p(x,y)$$

Ex.

p_{ij}	1	2	3	$p_X(x)$
1	$4/9$	$2/9$	0	→ 2/3
2	$1/9$	$1/9$	$1/9$	→ 1/3
$p_Y(y)$	↓ 5/9	↓ 1/3	↓ 1/9	

- Notice that the marginal probability mass functions are automatically proper pmf's. (why?)

Two continuous rv's

- We would like to extend the same ideas to the continuous case. Let X and Y be continuous rv's. A joint probability density function $f(x, y)$ for these two variables is a function satisfying $f(x, y) \geq 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

- The marginal probability density function of X and Y, denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{for} \quad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad \text{for} \quad -\infty < y < \infty$$

Remarks

- In the continuous case, roughly speaking, $f(x, y)\mathrm{d}x\mathrm{d}y$ can be treated as P(X=x,Y=y).
- $\mathrm{P}(a < \mathrm{X} < b, c < \mathrm{Y} < d) = \int_a^b \int_c^d f(x,y)dxdy$
- As in the discrete case, $f_X(x)$ and $f_Y(y)$ calculated from the joint distribution are automatically proper pdf's.
- Marginal distributions are, in fact, the distributions of the marginal random variables when they are treated as univariate random variables.

Example

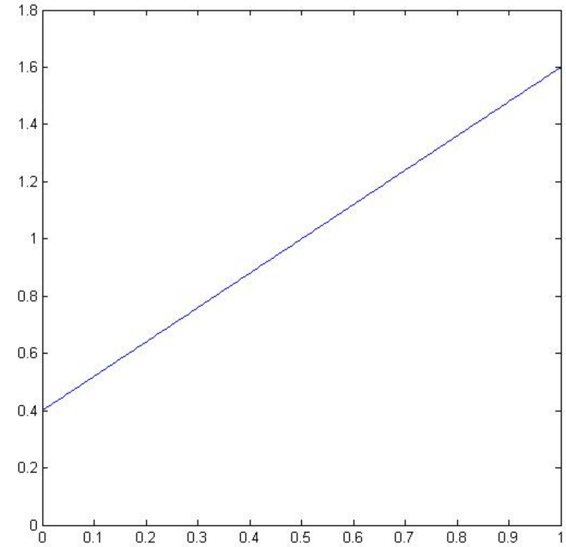
Ex. Suppose the joint pdf of the pair (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that this is a proper joint pdf.
2. What is $P(0 \leq X \leq 1/4, 0 \leq Y \leq 1/4)$?
3. What is $P(0 \leq Y \leq 1/4)$

Example cont.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{6}{5}(x+y^2)dy = \frac{6}{5}x + \frac{2}{5}$$



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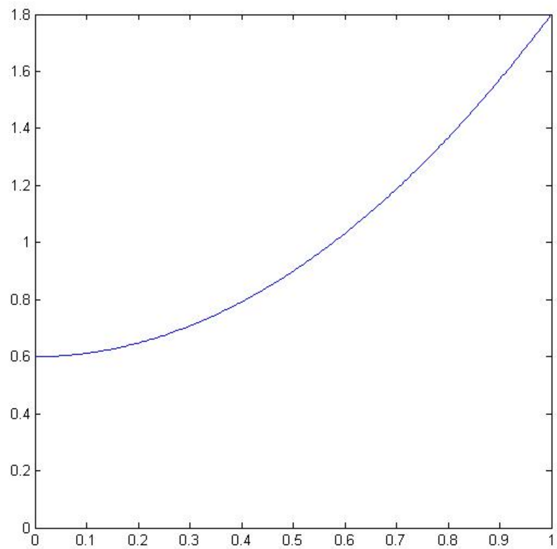
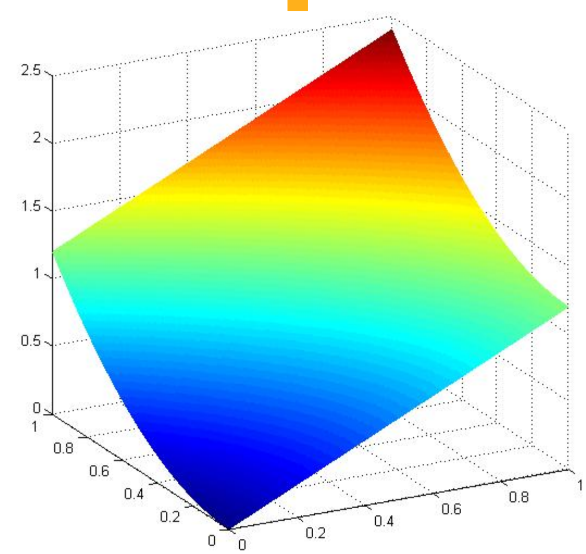
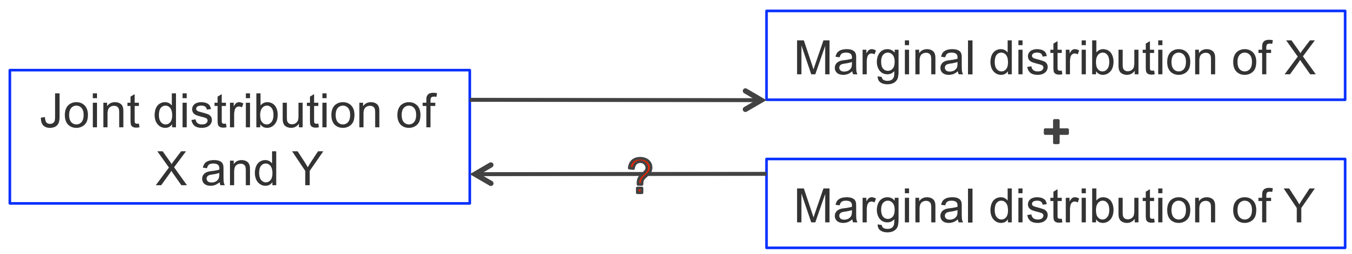




Joint and Marginal

- Now we have



- In general, we CANNOT go the other way around. Further information about the dependence structure of X and Y is needed to determine the joint distribution.

Example

Ex. Consider the following two joint distributions of X and Y.

p_{ij}	0	1
0	$3/10$	$3/10$
1	$3/10$	$1/10$

p_{ij}	0	1
0	$9/25$	$6/25$
1	$6/25$	$4/25$

It is easy to see that the marginal distributions of X and Y are the same in both cases. $P(X=0) = P(Y=0) = 3/5$; $P(X=1) = P(Y=1) = 2/5$.

This is the example that *different* joint distributions may have the *same* marginal distributions.

Independent rv's

- Recall the definition of independence of two random events A and B.

$$P(A \cap B) = P(A)\,P(B)$$

- We say two random variables X and Y are independent if and only if

$$P(X=x, Y=y) = P(X=x)\,P(Y=y), \text{ for any } x \text{ and } y.$$

- More specifically, two random variables X and Y are said to be independent if for every pair x and y values,

$$p(x, y) = p_X(x)\,p_Y(y), \text{ when X and Y are discrete;}$$

  or

$$f(x, y) = f_X(x)\,f_Y(y), \text{ when X and Y are continuous.}$$

Ex. The second case of the previous example.

Multiple Random Variables

- If $X_1, X_2, \ldots, X_n$ are all discrete random variables, the joint pmf of the variables is the function

  $$p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

  If the variables are continuous, the joint pdf of $X_1, X_2, \ldots, X_n$ is the function $f(x_1, x_2, \ldots, x_n)$ such that for any n intervals $[a_1, b_1], \ldots, [a_n, b_n]$,

  $$P(a_1 \leq X_1 \leq b_1, \ldots, a_n \leq X_n \leq b_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

- What should be the regularity conditions for $p(x_1, x_2, \ldots, x_n)$ and $f(x_1, x_2, \ldots, x_n)$?

- How do get the marginal distributions of $X_1, X_2, \ldots$ by using $p(x_1, x_2, \ldots, x_n)$ and $f(x_1, x_2, \ldots, x_n)$?

Independence

- Proposition:

The random variables $X_1, X_2, \ldots, X_n$, are said to be independent if for every subset $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$, of the variables (each pair, each triple, and so on), the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

- $$p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$

- $$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

Example

Ex. Two people each arrive independently at the station at some random time between 5:00 am and 6:00 am (arrival time for either person is uniformly distributed). They stay exactly five minutes and then leave. What is the probability they will meet on a given day.

Conditional dist.

- Using the marginal distributions, one can calculate the conditional distribution of one rv given the other.

- Let X and Y be two conditional rv's with joint pdf $f(x, y)$ and marginal X pdf $f_X(x)$. Then for any X value x for which $f_X(x)>0$, the conditional probability density function of Y given that X=x is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad -\infty < y < \infty.$$

- If X and Y are discrete, replace pdf's by pmf's in this definition gives the conditional probability mass function of Y when X=x.

Example

Ex. Let (X, Y) have the joint density

$$f(x, y) = 24y(1 - x - y),\ x, y \geq 0,\ x+y < 1.$$

1. What is the conditional density of X given Y=1/2?
2. What is the conditional density of Y given X=1/2?

Example

Ex. For some $\lambda>0$, random variable X has the density function

$$f(x) = \lambda^2 x e^{-\lambda x}, \ x > 0,$$

and given X, Y is a uniform random variable on the interval [0, X].

1. What is the joint distribution of X and Y?
2. What is the distribution of Y?

Expectation of Functions

- Recall how we compute E[*h*(X)]. A similar result also holds for a function *h*(X, Y) of two jointly distributed rv's.

- Let X and Y be jointly distributed rv's with pmf *p*(*x*, *y*), if they are discrete; or pdf *f* (*x*, *y*), if they are continuous. The expected value of a function *h*(X, Y), denoted by E[*h*(X, Y)] is given by

$$\mathrm{E}[h(\mathrm{X},\mathrm{Y})] = \begin{cases} \sum_x \sum_y h(x,y) \cdot p(x,y) & \text{if X and Y are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y) dx dy & \text{if X and Y are continuous} \end{cases}$$

- This result can also be extended to multiple (>2) rv case.

Examples

Ex. (Important! Linearity of expectations) Show that for any two random variables X and Y, E(X+Y) = E(X) + E(Y).

Example

Ex. If two random variables X and Y are independent, what is E(XY)? What about E (g(X)h(Y))?

Covariance

- When two random variables X and Y are not independent, it is often of interest to assess how strongly they are related to one another.

- A popular measurement to characterize the dependence of two rv's is called correlation. To calculate correlation of two rv's, we'll have calculate the covariance of the two rv's.

- The covariance between two rv's X and Y is

$$
\begin{aligned}
\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) &= \mathrm{E}[(\mathrm{X} - \mu_X)(\mathrm{Y} - \mu_Y)] \\
&= \begin{cases} \sum_x \sum_y (x - \mu_X)(y - \mu_Y) \cdot p(x, y) & \mathrm{X, Y\ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) \cdot f(x, y) dx dy & \mathrm{X, Y\ continuous} \end{cases}
\end{aligned}
$$

Short cut

- Proposition:

$$\boxed{\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)}$$

- What happens if we set Y=X?

Example

Ex. Suppose the joint distribution of X and Y are

$$f(x,y) = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1, x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the covariance of X and Y?

$$f_X(x) = \int_y f(x,y)dy = \int_0^{1-x} 24xydy = 12x(1-x)^2$$

$$f_Y(y) = 12y(1-y)^2$$

$$\mathrm{E(X)} = \int_0^1 x \cdot 12x(1-x)^2 dx = \frac{2}{5} = \mathrm{E(Y)}$$

$$\mathrm{E(XY)} = \int\int_{x,y} xyf(x,y)dxdy = \int_0^1 \int_0^{1-y} 24x^2y^2dxdy = \frac{2}{15}$$

$$\mathrm{Cov(X,Y)} = \mathrm{E(XY)} - \mathrm{E(X)E(Y)} = \frac{2}{15} - \left(\frac{2}{5}\right)^2 = -\frac{2}{75}$$

Correlation

- The correlation coefficient of X and Y, denoted by Corr(X, Y) or $\rho_{X,Y}$ is defined by

$$\rho_{X,Y} = \frac{\mathrm{Cov}(\mathrm{X},\mathrm{Y})}{\sigma_X \cdot \sigma_Y}$$

- Because of Cauchy-Schwarz inequality, we have

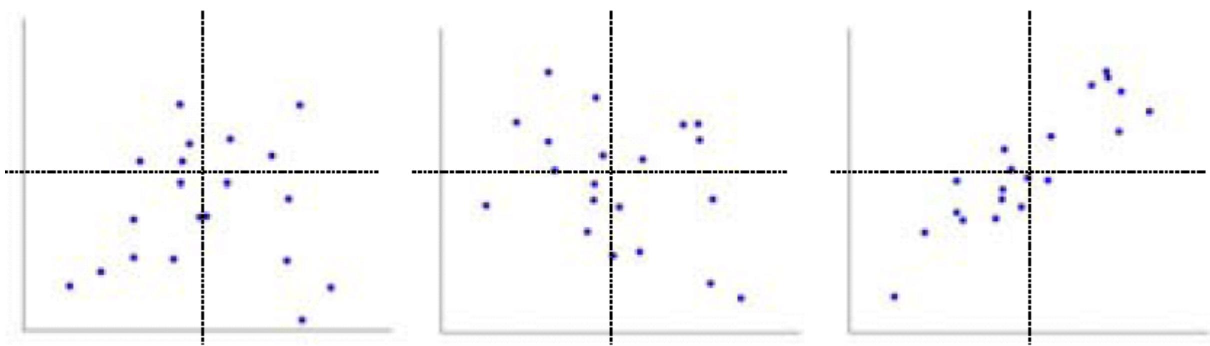
$$\mathrm{Cov}^2(\mathrm{X},\mathrm{Y}) \leq \mathrm{Var}(\mathrm{X})\mathrm{Var}(\mathrm{Y}) \Longrightarrow |\rho_{X,Y}| \leq 1$$

- The correlation coefficient $\rho_{X,Y}$ is NOT a completely general measure of the strength of a relationship. $\rho_{X,Y}$ is actually a measure of the degree of *linear* relationship between X and Y.

Remarks

- If X and Y are independent, then $\rho_{X,Y} = 0$ (why?). But $\rho_{X,Y} = 0$ does NOT imply independence.

- $\rho_{X,Y} = 1$ or -1 iff $Y = aX+b$ for some numbers a and b with $a \neq 0$.

- 

$r \approx 0$ $\quad$ $r \approx -0.3$ $\quad$ $r \approx 0.9$

Relationship Between Correlation and Independence

- Independence leads to uncorrelatedness.

$$Cov(X, Y) = E(XY) - E(X)E(Y) = E(X)E(Y) - E(X)E(Y) = 0$$

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- But not vice versa!

Relationship Between Correlation and Independence

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- But not vice versa!
- We will talk about this more in regression.

Statistics and Sampling Distributions

- We will start changing our discussion from probability to statistics, which means we need to think about data.
- We often need to assume the observed data are *simple random samples*, which means they are IID (Independently Identically Distributed).

Introduction to IID

- A sequence of random variables, X_1, X_2, …, X_n, is independent and identically distributed (i.i.d.) if each random variable has the same probability distribution as the others and all are mutually independent.

- In statistical analysis, we often assume the sampled data X_1, X_2, …, X_n, are i.i.d. from a common distribution *f*(x). And usually, we end up analyzing a linear combination of the X_i's, that is

$$\mathrm{Y} = a_1 \mathrm{X}_1 + \cdots + a_n \mathrm{X}_n = \sum_{i=1}^{n} a_i \mathrm{X}_i$$

A key result ***

Let $X_1, X_2, \ldots, X_n$, have mean values $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively.

- Whether or not the Xi's are independent,

$$\begin{aligned} \mathrm{E}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) &= a_1\mathrm{E}(\mathrm{X}_1) + a_2\mathrm{E}(\mathrm{X}_2) + \cdots + a_n\mathrm{E}(\mathrm{X}_n) \\ &= a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n \end{aligned}$$

- For any $X_1, X_2, \ldots, X_n$,

$$\mathrm{Var}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}(\mathrm{X}_i, \mathrm{X}_j)$$

  If they are independent, then

$$\begin{aligned} & \mathrm{Var}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) \\ = \; & a_1^2\mathrm{Var}(\mathrm{X}_1) + a_2^2\mathrm{Var}(\mathrm{X}_2) + \cdots + a_n^2\mathrm{Var}(\mathrm{X}_n) \\ = \; & a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2 \end{aligned}$$

Special Cases

- $E(X+Y) = E(X) + E(Y)$;
- $E(X-Y) = E(X) – E(Y)$;
- $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- $Var(X-Y) = Var(X) + Var(Y) -2Cov(X, Y)$
- If X and Y are independent, then $Cov(X, Y) = 0$, and
  $Var(X+Y) = Var(X) + Var(Y)$
  $Var(X – Y) = Var(X) + Var(Y)$

Example

Ex. Show that if X ~ Bin(n, p), then E(X) = np, and Var(X) = $np(1 - p)$.

Ex. Show that if X is a negative binomial rv with pmf $nb(x; r, p)$, then E(X) = $r(1–p)/p$,
Var(X) = $r(1 - p)/p^2$.

Sample Mean***

- Let X_1, X_2, …, X_n, be an i.i.d. sequence of rv's from a distribution with mean value μ and standard deviation σ.

- Notice that the sample mean or the sample total ($T = X_1+X_2+\ldots+X_n$) can also be viewed as a special case of linear combination of X_1, X_2, …, X_n. In the i.i.d. case,

$$E(T) = E(X_1)+E(X_2)+\ldots+E(X_n) = n\mu$$

$$Var(T) = Var(X_1)+Var(X_2)+\ldots+Var(X_n) = n\sigma^2$$

- It is also easy to verify that for sample mean,

$$\mathrm{E}(\bar{\mathrm{X}}) = \mu_{\bar{X}} = \mu$$

$$\mathrm{Var}(\bar{\mathrm{X}}) = \sigma_{\bar{X}}^2 = \sigma^2/n \Longrightarrow \sigma_{\bar{X}} = \sigma/\sqrt{n}$$

Invariance under Summation

- When $X_1, X_2, \ldots, X_n$ are normally distributed, a linear combination of these random variables

$$Y = a_1 X_1 + \cdots + a_n X_n = \sum_{i=1}^{n} a_i X_i$$

  will still be normally distributed.

- Note that $X_1, X_2, \ldots, X_n$ do not have to be i.i.d.

- What are the parameters of Y?

- This phenomenon does NOT happen to every distribution, for example, sum of uniform random variables.

CLT

- Theorem:

The Central Limit Theorem (CLT)

Let X_1, X_2, …, X_n, be an i.i.d. sequence from a distribution with mean μ and variance σ^2. Then if n is sufficiently large, the sample mean $\bar{X}$ has approximately a normal distribution with $\mu_{\bar{X}} = \mu$ and $\sigma^2_{\bar{X}} = \sigma^2/n$;
And the sample total has approximately a normal distribution with $\mu_T = n\mu$, $\sigma_T^2 = n\sigma^2$. The larger the value of n, the better the approximation.

- Rule of Thumb: if n>30, the CLT can be used.

Example

<u>Ex.</u> Why is the normal approximation to Binomial distribution working?