

Joint Distribution

- How can we model two rv's using probability models? For example, if we are interested in both weight and height.
- Is it enough if we just use a normal model for weight and another normal model for height?
- We need to introduce [joint probability distribution](#) in order to model multiple rv's.

Joint PMF

- Let X and Y be two discrete rv's defined on the sample space. The **joint probability mass function** $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = P(X=x, Y=y).$$

- As in the single rv case, we must have $p(x, y) \geq 0$ and $\sum_x \sum_y p(x, y) = 1$.

Example

Ex. We randomly put two different balls into 3 numbered (numbered as $\{1,2,3\}$) boxes. Let X be the number of empty boxes left; let Y be the minimum of the box number that has balls in it. What is the joint distribution of (X, Y) ?

X can take values from $\{1, 2\}$;

Y can take values from $\{1, 2, 3\}$;

It's not hard to see we have the following (why?):

$$p(2, j) = P(X=2, Y=j) = 1/9, \text{ for } j = 1, 2, 3.$$

$$p(1, 3) = P(X=1, Y=3) = 0.$$

$$p(1, 1) = P(X=1, Y=1) = 4/9.$$

$$p(1, 2) = P(X=1, Y=2) = 2/9.$$

p_{ij}	1	2	3
1	4/9	2/9	0
2	1/9	1/9	1/9

Marginal PMF

- The **marginal probability mass functions** of X and Y, denoted by $p_X(x)$ and $p_Y(y)$, respectively, are given by

$$p_X(x) = \sum_y p(x, y) \quad p_Y(y) = \sum_x p(x, y)$$

Ex.

p_{ij}	1	2	3		$p_X(x)$
1	4/9	2/9	0	→	2/3
2	1/9	1/9	1/9	→	1/3
	↓	↓	↓		
$p_Y(y)$	5/9	1/3	1/9		

- Notice that the marginal probability mass functions are automatically proper pmf's. (why?)

Two continuous rv's

- We would like to extend the same ideas to the continuous case. Let X and Y be continuous rv's. A **joint probability density function** $f(x, y)$ for these two variables is a function satisfying $f(x, y) \geq 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

- The **marginal probability density function** of X and Y , denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{for } -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad \text{for } -\infty < y < \infty$$

Remarks

- In the continuous case, roughly speaking, $f(x, y)dxdy$ can be treated as $P(X=x, Y=y)$.
- $P(a < X < b, c < Y < d) = \int_a^b \int_c^d f(x, y)dxdy$
- As in the discrete case, $f_X(x)$ and $f_Y(y)$ calculated from the joint distribution are automatically proper pdf's.
- Marginal distributions are, in fact, the distributions of the marginal random variables when they are treated as univariate random variables.

Example

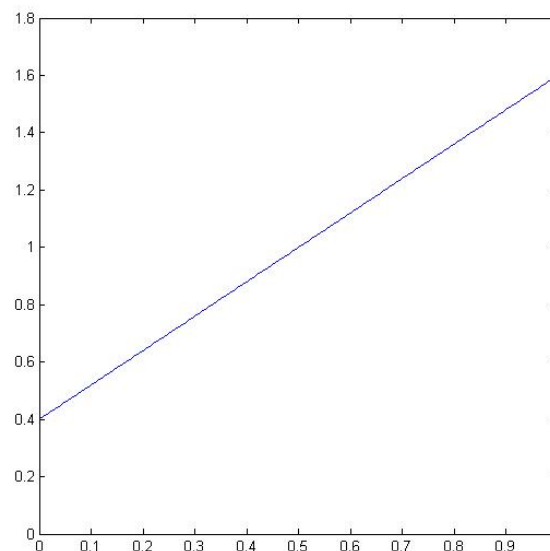
Ex. Suppose the joint pdf of the pair (X, Y) is given by

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

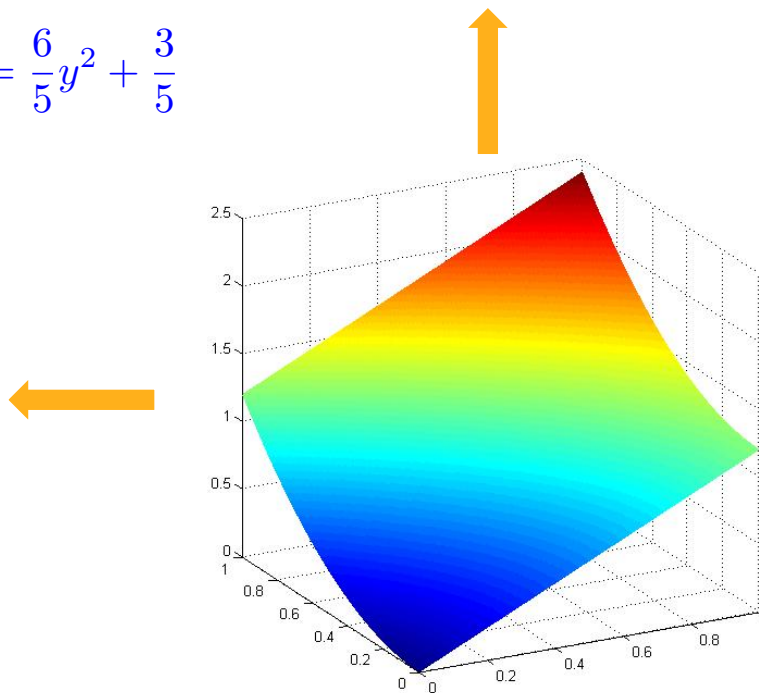
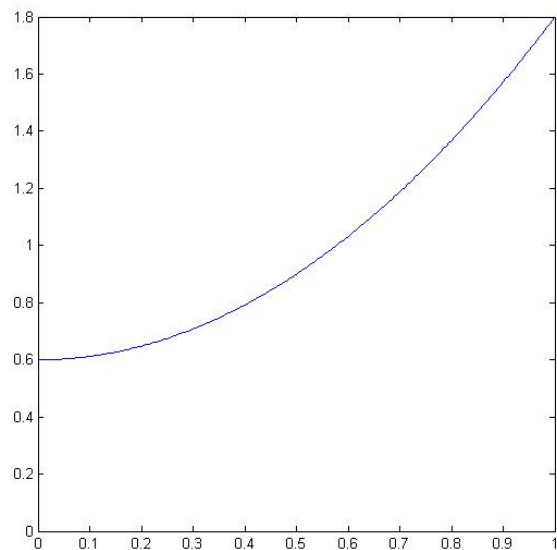
1. Show that this is a proper joint pdf.
2. What is $P(0 \leq X \leq 1/4, 0 \leq Y \leq 1/4)$?
3. What is $P(0 \leq Y \leq 1/4)$

Example cont.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 \frac{6}{5}(x + y^2) dy = \frac{6}{5}x + \frac{2}{5}$$

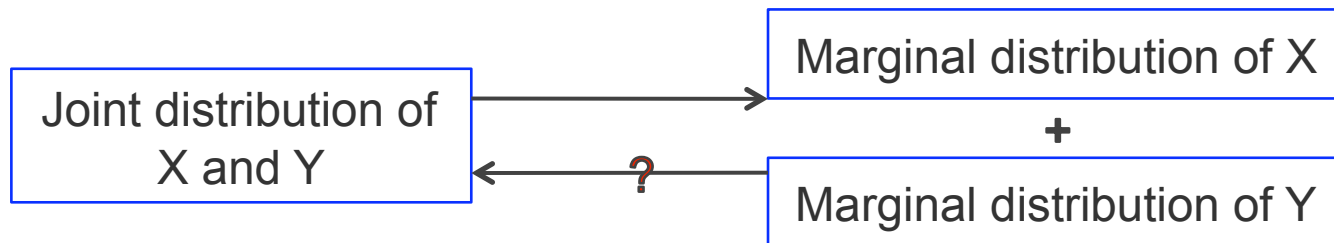


$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 \frac{6}{5}(x + y^2) dx = \frac{6}{5}y^2 + \frac{3}{5}$$



Joint and Marginal

- Now we have



- In general, we **CANNOT** go the other way around. Further information about the dependence structure of X and Y is needed to determine the joint distribution.

Example

Ex. Consider the following two joint distributions of X and Y.

p_{ij}	0	1
0	$3/10$	$3/10$
1	$3/10$	$1/10$

p_{ij}	0	1
0	$9/25$	$6/25$
1	$6/25$	$4/25$

It is easy to see that the marginal distributions of X and Y are the same in both cases. $P(X=0) = P(Y=0) = 3/5$; $P(X=1) = P(Y=1) = 2/5$.

This is the example that *different* joint distributions may have the *same* marginal distributions.

Independent rv's

- Recall the definition of independence of two random events A and B.

$$P(A \cap B) = P(A) P(B)$$

- We say two random variables X and Y are **independent** if and only if

$$P(X=x, Y=y) = P(X=x) P(Y=y), \text{ for any } x \text{ and } y.$$

- More specifically, two random variables X and Y are said to be independent if for every pair x and y values,

$$p(x, y) = p_X(x) p_Y(y), \text{ when } X \text{ and } Y \text{ are discrete;}$$

or

$$f(x, y) = f_X(x) f_Y(y), \text{ when } X \text{ and } Y \text{ are continuous.}$$

Ex. The second case of the previous example.

Multiple Random Variables

- If X_1, X_2, \dots, X_n are all discrete random variables, the joint pmf of the variables is the function

$$p(x_1, x_2, \dots, x_n) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

If the variables are continuous, the joint pdf of X_1, X_2, \dots, X_n is the function

$f(x_1, x_2, \dots, x_n)$ such that for any n intervals $[a_1, b_1], \dots, [a_n, b_n]$,

$$P(a_1 \leq X_1 \leq b_1, \dots, a_n \leq X_n \leq b_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_1 \dots dx_n$$

- What should be the regularity conditions for $p(x_1, x_2, \dots, x_n)$ and $f(x_1, x_2, \dots, x_n)$?
- How do get the marginal distributions of X_1, X_2, \dots by using $p(x_1, x_2, \dots, x_n)$ and $f(x_1, x_2, \dots, x_n)$?

Independence

- Proposition:

The random variables X_1, X_2, \dots, X_n , are said to be independent if for every subset $X_{i_1}, X_{i_2}, \dots, X_{i_k}$, of the variables (each pair, each triple, and so on), the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

- $$p(x_1, x_2, \dots, x_n) = \prod_{i=1}^n p_{X_i}(x_i)$$

- $$f(x_1, x_2, \dots, x_n) = \prod_{i=1}^n f_{X_i}(x_i)$$

Example

Ex. Two people each arrive independently at the station at some random time between 5:00 am and 6:00 am (arrival time for either person is **uniformly distributed**). They stay exactly five minutes and then leave. What is the probability they will meet on a given day.

Conditional dist.

- Using the marginal distributions, one can calculate the conditional distribution of one rv given the other.
- Let X and Y be two conditional rv's with joint pdf $f(x, y)$ and marginal X pdf $f_X(x)$. Then for any X value x for which $f_X(x) > 0$, the **conditional probability density function** of Y given that $X=x$ is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} \quad -\infty < y < \infty.$$

- If X and Y are discrete, replace pdf's by pmf's in this definition gives the **conditional probability mass function** of Y when $X=x$.

Expectation of Functions

- Recall how we compute $E[h(X)]$. A similar result also holds for a function $h(X, Y)$ of two jointly distributed rv's.
- Let X and Y be jointly distributed rv's with pmf $p(x, y)$, if they are discrete; or pdf $f(x, y)$, if they are continuous. The expected value of a function $h(X, Y)$, denoted by $E[h(X, Y)]$ is given by

$$E[h(X, Y)] = \begin{cases} \sum_x \sum_y h(x, y) \cdot p(x, y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) \cdot f(x, y) dx dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

- This result can also be extended to multiple (>2) rv case.

Examples

Ex. (Important! **Linearity of expectations**) Show that for any two random variables X and Y , $E(X+Y) = E(X) + E(Y)$.

Example

Ex. If two random variables X and Y are independent, what is $E(XY)$? What about $E(g(X)h(Y))$?

Expectation of Linear Function of Multiple RV's

- ▶ Linearity is well preserved in expectation.

$$E(a \cdot X + b \cdot Y + c) = a \cdot E(X) + b \cdot E(Y) + c$$

Expectation of Product of Multiple RV's

- ▶ Unlike the linear case, expectation of product in general doesn't equal to the product of expectations

$$E(XY) \neq E(X)E(Y)$$

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- ▶ But if X and Y are independent, then

$$\begin{aligned} E(XY) &= \int \int xyf(x, y)dxdy = \int \int xyf_X(x)f_Y(y)dxdy \\ &= \int xf_X(x)dx \int yf_Y(y)dy = E(X)E(Y) \end{aligned}$$

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- ▶ And for independent RV's, in general

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$