Example

Ex. For some $\lambda>0$, random variable X has the density function

$$f(x) = \lambda^2 x e^{-\lambda x}, \; x > 0,$$

and given X, Y is a uniform random variable on the interval [0, X].

1. What is the joint distribution of X and Y?
2. What is the distribution of Y?

Expectation of Functions

- Recall how we compute E[*h*(X)]. A similar result also holds for a function *h*(X, Y) of two jointly distributed rv's.

- Let X and Y be jointly distributed rv's with pmf *p*(*x*, *y*), if they are discrete; or pdf *f* (*x*, *y*), if they are continuous. The expected value of a function *h*(X, Y), denoted by E[*h*(X, Y)] is given by

$$\mathrm{E}[h(\mathrm{X},\mathrm{Y})] = \begin{cases} \sum_x \sum_y h(x,y) \cdot p(x,y) & \text{if X and Y are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y) dx dy & \text{if X and Y are continuous} \end{cases}$$

- This result can also be extended to multiple (>2) rv case.

Examples

Ex. (Important! Linearity of expectations) Show that for any two random variables X and Y, $E(X+Y) = E(X) + E(Y)$.

Example

Ex. If two random variables X and Y are independent, what is E(XY)? What about E(g(X)h(Y))?

Covariance

- When two random variables X and Y are not independent, it is often of interest to assess how strongly they are related to one another.

- A popular measurement to characterize the dependence of two rv's is called correlation. To calculate correlation of two rv's, we'll have calculate the covariance of the two rv's.

- The covariance between two rv's X and Y is

$$\begin{aligned}\mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= \mathrm{E}[(\mathrm{X}-\mu_X)(\mathrm{Y}-\mu_Y)] \\ &= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y)\cdot p(x,y) & \mathrm{X},\ \mathrm{Y}\ \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)\cdot f(x,y)\,dx\,dy & \mathrm{X},\ \mathrm{Y}\ \text{continuous} \end{cases}\end{aligned}$$

Short cut

- Proposition:

$$\boxed{\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)}$$

- What happens if we set Y=X?

Example

<u>Ex.</u> Suppose the joint distribution of X and Y are

$$f(x,y) = \begin{cases} 24xy & 0 \le x \le 1, 0 \le y \le 1, x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the covariance of X and Y?

$$f_X(x) = \int_y f(x,y)dy = \int_0^{1-x} 24xydy = 12x(1-x)^2$$

$$f_Y(y) = 12y(1-y)^2$$

$$\mathrm{E(X)} = \int_0^1 x \cdot 12x(1-x)^2 dx = \frac{2}{5} = \mathrm{E(Y)}$$

$$\mathrm{E(XY)} = \int\int_{x,y} xyf(x,y)dxdy = \int_0^1 \int_0^{1-y} 24x^2y^2dxdy = \frac{2}{15}$$

$$\mathrm{Cov(X,Y)} = \mathrm{E(XY)} - \mathrm{E(X)E(Y)} = \frac{2}{15} - \left(\frac{2}{5}\right)^2 = -\frac{2}{75}$$

Correlation

- The correlation coefficient of X and Y, denoted by Corr(X, Y) or $\rho_{X,Y}$ is defined by

$$\rho_{X,Y} = \frac{\mathrm{Cov}(\mathrm{X}, \mathrm{Y})}{\sigma_X \cdot \sigma_Y}$$

- Because of Cauchy-Schwarz inequality, we have

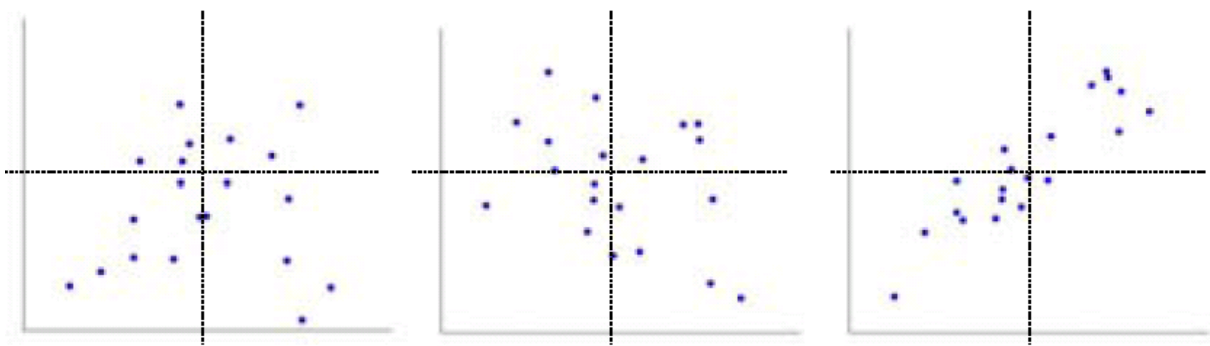
$$\mathrm{Cov}^2(\mathrm{X}, \mathrm{Y}) \leq \mathrm{Var}(\mathrm{X})\mathrm{Var}(\mathrm{Y}) \Longrightarrow |\rho_{X,Y}| \leq 1$$

- The correlation coefficient $\rho_{X,Y}$ is NOT a completely general measure of the strength of a relationship. $\rho_{X,Y}$ is actually a measure of the degree of *linear* relationship between X and Y.

Remarks

- If X and Y are independent, then $\rho_{X,Y} = 0$ (why?). But $\rho_{X,Y} = 0$ does NOT imply independence.

- $\rho_{X,Y} = 1$ or -1 iff $Y = aX+b$ for some numbers a and b with $a \neq 0$.

- 

$r \approx 0$ $r \approx -0.3$ $r \approx 0.9$

Example

Ex. If X has a symmetric distribution centered at 0 with finite 3rd moment, $E(|X|^3)<\infty$, show that X^2 and X are uncorrelated.