S1211Q Introduction to Statistics
Lecture 7

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Example

Ex. Flip three fair coins. (*Binomial*)

S = {HHH, HHT, HTH, HTT, THT, THH, TTH, TTT}. Let's define random variable X to be the number of heads in the experiment, i.e., X(HHH)=3, X(THT)=1, etc.

X	
0	TTT
1	TTH THT HTT
2	THH HTH HHT
3	HHH

Value of X	0	1	2	3
Probability	0.125	0.375	0.375	0.125

One can calculate the probability of an event by adding the probabilities p_i of the particular values of x_i that make up the event. For example, if we want to know the probability of getting less than 2 heads, we can use

$$P(X<2) = P(X=0) + P(X=1) = 0.125 + 0.375 = 0.5$$

Note: $P(X\leq2) = P(X=0) + P(X=1) + P(X=2) = 0.875$

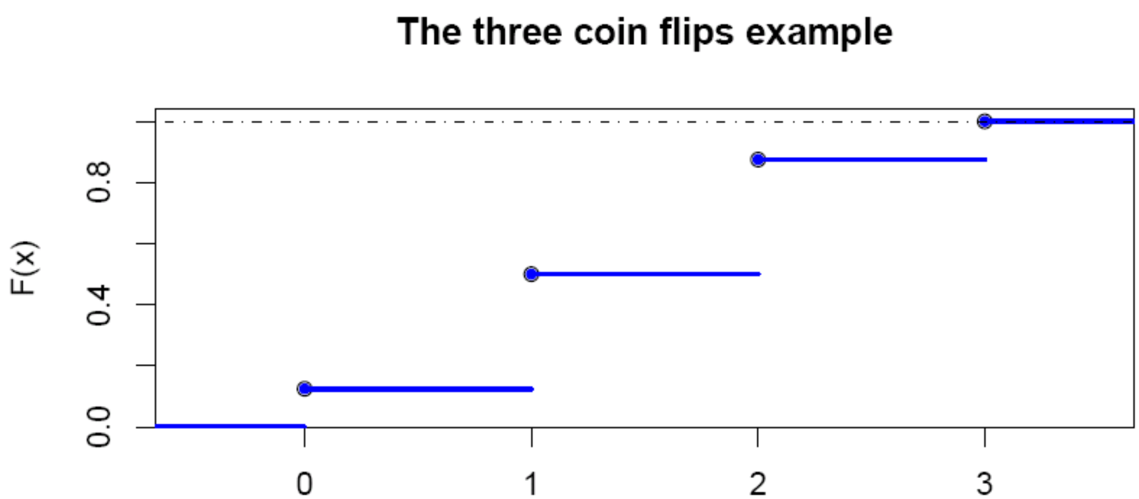
CDF

- The cumulative distribution function (cdf) F(x) of a discrete rv variable X with pmf $p(x)$ is defined for every number x by

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \leq x) = \sum_{y:y\leq x} p(y).$$

  For any number x, F(x) is the probability that the observed value of X will be at most x.

- For X a discrete rv, the graph of F(x) will have a jump at every possible value of X and will be flat between possible values. Such a graph is called a step function.



Parameter and Family

- Suppose $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a parameter of the distribution. The collection of all probability distributions for different values of the parameter is called a family of probability distributions.

Ex. For Bernoulli rv's, the parameter is the probability of being 1 (or 0), that is,

$$p = P(X=1)$$

Expectation and Variance

- Random variables have distributions, so they have centers and spreads.
- The expected value (mean value or expectation) of a random variable describes its theoretical long-run average value.
- We typically use μ or $E(X)$ to denote the mean, $Var(X)$ to denote the variance and σ or $SD(X)$ to denote the standard deviation of a rv X.

Motivating examples

Ex. How many heads would you expect if you flipped a fair coin twice?

S = {HH, HT, TH, TT}.

X = number of heads.

0 TT

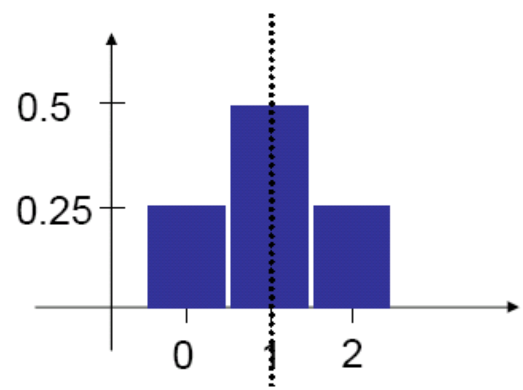
1 HT TH

2 HH

$p(X=0) = 0.25$; $p(X=1) = 0.5$; $p(X=2) = 0.25$.

Each outcome is weighted by its probability.

$\mu = 0 \times 0.25 + 1 \times 0.5 + 2 \times 0.25 = 1$



Example

Ex. How many heads would you expect if you flipped a coin three times?

$$\mu = 0 \times 0.125 + 1 \times 0.375 + 2 \times 0.375 + 3 \times 0.125 = 1.5$$

This can never occur in a single trial of 3 flips. However, on average we would expect to get 1.5 heads if we repeated the experiment many times.

Definition

- Suppose X is a discrete random variable whose probability model is given by

Value of X	x_1	x_2	……..	x_k
Probability	p_1	p_2	……..	p_k

The expected value of X is given by

$$\mathrm{E}(\mathrm{X}) = \mu_X = \sum_{x \in D} x \cdot p(x) = x_1 p_1 + x_2 p_2 + \cdots x_k p_k$$

Example

Ex. Expectation of a Bernoulli rv.

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & x \neq 0,1 \end{cases}$$

$\mu = 0 \times (1-p) + 1 \times p = p$.

Example

Ex. The general form for the pmf of X = number of children born up to and including the first boy is,

$$p(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

1. Verify that this is a proper pmf.
2. Calculate the expected value of X.

The Expected Value of a Function

- A bookstore purchases ten copies of a books at \$60 each to sell at \$120, and any unsold copies after three months can be redeemed for \$20. If the number of copies sold is X, what is the profit of the bookstore?

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- The profit is $h(X) = 100X - 400$. Is $h(X)$ a Random Variable? Then what is the expectation of $h(X)$?
- If Random Variable X has range D and pmf $p(x)$, then the expected value of function $h(X)$ is given by

$$E(h(X)) = \sum_{x \in D} \left(h(x) \cdot p(x)\right)$$

The Linear Function Case

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$$E(aX + b) = a \cdot E(X) + b$$
- To prove,
$$E(aX+b) = \sum_{x \in D} (ax+b) \cdot p(x) = a \sum_{x \in D} x \cdot p(x) + b \sum_{x \in D} p(x) = aE(X) + b$$

Difference Between Expectation and Measure of Center

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- In another word, expectation is a concept in probability, and sample mean is a concept in statistics. In inference, we use $\bar{x}$ to infer $E(X)$

Variance of Random Variables

- We defined the concept of sample variance in the first chapter. Similarly we can define the variance of random variables, which is still a measure of deviation from the center.
- X has pmf $p(x)$ and expected value μ, then the variance of X, denoted as $V(X)$ or σ^2 is

$$V(X) = E[(X-\mu)^2] = \sum_D (x-\mu)^2$$

The standard deviation of X is

$$\sigma = \sqrt{\sigma^2}$$

Example

- A reader can check out at most 6 videos from a library at one time. Consider only those who check out videos, let X denote the number of videos checked out to a randomly selected individual. The pmf is

x	1	2	3	4	5	6
p(x)	.3	.25	.15	.05	.1	.15

- How to calculate variance and standard deviation of X?

Shortcut Formula for σ^2

- Similar to sample variance, we have a shortcut formula for variance of random variables too
- $$V(X) = \sigma^2 = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$$
- Proof:
$$\sigma^2 = \sum_D x^2 \cdot p(x) - 2\mu \cdot \sum_D x \cdot p(x) + \mu^2 \sum_D p(x) = E(X^2) - \mu^2$$

Variance of Linear Function of Random Variables

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- $$V(aX + b) = a^2 \cdot V(X)$$

  and

  $$\sigma_{aX+b} = |a| \cdot \sigma_X$$

Discrete Probability Models

- We often conduct trials/experiments repeatedly. Today we will discuss probability models that allow us to answer questions regarding repeated trials. In particular we will be interested in series of n repeated trials of a random phenomena with two possible outcomes.

- Many popular discrete models are motivated by coin tosses, or more specifically a series of n Bernoulli trials. A series of n trials are Bernoulli trials if:
    1. The n trials are identical.
    2. The trials are independent (the outcome on any particular trial does not influence the outcome on any other trial).
    3. Each trial has two possible outcomes: success or failure.
    4. The probability of success, denoted by p, is the same for each trial (identical).

Binomial Experiment

- If in Bernoulli trials, the number of trials *n* is fixed in advance of the experiment. This experiment is called a binomial experiment.

Ex. The same coin is tossed successively and independently 10 times.

Ex. Suppose there are 50 colored socks in the drawer, of which 16 are red and the other 34 are blue. We are going to randomly draw 10 socks out of the drawer *without replacement*. We label the *i*th trial as a success if the *i*th sock is blue. (Is this a binomial experiment? What if it's *with replacement*?)

Ex. The previous example, what if we have 500,000 socks, of which 400,000 are blue. A sample of 10 socks are drawn without replacement.

$$P(\text{success on } 2 \mid \text{success on } 1) = 399{,}999/499{,}999 = .80000$$

$$P(\text{success on } 10 \mid \text{success on first } 9) = 399{,}991/499{,}991 = .799996 \approx .80000$$

A Rule of Thumb

- For drawing without replacement (*hypergeometric*), as the previous example suggests, although the trials are not exactly independent, the conditional probability differ so slightly from one another that for practical purposes the trials can be regarded as independent. Thus, to a very good approximation, the previous experiment is binomial with $n = 10$ and $p = .8$.

- As a rule of thumb: consider sampling without replacement from a dichotomous population of size N. If the sample size (number of trials) n is at most 5% of the population size, the experiment can be analyzed as though it were exactly a binomial experiment.

Binomial RV

- The binomial random variable X associated with a binomial experiment consisting of *n* trials is defined as

  X = the number of successes among the *n* trials.

- The pmf of a binomial rv X depends on the two parameters *n* and *p,* we denote the pmf by $b(x; n, p)$. The cdf will be denoted by

  $$P(X \leq x) = B(x; n, p) = \sum_{y=0}^{x} b(y; n, p).$$

  Note that *x* can only take values in {0,1,…, n}.

Example

Ex. Roll a ten-sided die four times. What is the probability of getting exactly one three?

S = rolling a three.

F = rolling something other than a three.

P(S) = p = 0.1 and P(F) = 1-p = 0.9

Let X = the number of threes, then X is Bin(4, 0.1) and we want to calculate P(X=1). There are four possible ways of rolling a three: SFFF, FSFF, FFSF, FFFS.

P(SFFF) = P(S)P(F)P(F)P(F) = $(1-p)^3 p = (.9)^3(.1) = 0.0729$

Similarly, P(FSFF) = P(FFSF) = P(FFFS) = 0.0729.

P(X=1) = P(SFFF) + P(FSFF) +P(FFSF) + P(FFFS)

= 4(0.0729) = 0.2916

Binomial pmf

- From the previous example, we see that
  P(X=1) = $b(1; 4, p) = 4(1-p)^3 p$
  = {# of outcomes with X=1} • {prob. of any particular outcome with X=1}

- Thus more generally, we have
  $b(x; n, p)$ = {# of outcomes with X=x} • {prob. of any particular outcome with X=x}

- The pmf of a binomial rv is

$$b(x;n,p) = \begin{cases} \boxed{\binom{n}{x}}\boxed{p^x(1-p)^{n-x}} & x = 0, 1, 2, 3, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Example

Ex. (Ten-sided die cont.) Use binomial pmf to verify P(X=1) we have calculated.

$$\mathrm{P(X=1)} = \binom{4}{1}(1/10)^1(9/10)^3 = \frac{4!}{1!3!}(1/10)^1(9/10)^3 = 0.2916$$

What is the probability of getting less than two 3's in four rolls?

$$\begin{aligned}\mathrm{P(X<2)} &= \mathrm{P(X=0)} + \mathrm{P(X=1)} \\ &= \binom{4}{0}(1/10)^0(9/10)^4 + \binom{4}{1}(1/10)^1(9/10)^3 \\ &= 0.6561 + 0.2916 = 0.9477\end{aligned}$$

- Try using `dbinom(); pbinom()` to calculate the things above.

Example

Ex. Suppose we are searching for new apartments in the city, and our goal is to find an apartment among the top 5% (based on some criteria). Our strategy is to randomly sample 20 apartments from the pool, and choose the best out of these 20. What is the probability that we will accomplish our goal?

Mean and Variance of Binomial

- Proposition:

If $X \sim \text{Bin}(n, p)$, then $E(X) = np$, $\text{Var}(X) = np(1 - p) = npq$, and $\sigma_X = \sqrt{npq}$ (where $q = 1 - p$).

We'll show an easy proof in chapter 5.

Hypergeometric Distribution

Ex. (Socks example cont.) Suppose there are 50 colored socks in the drawer, of which 16 are red and the other 34 are blue. We are going to randomly draw 10 sock out of the drawer without replacement. What is the probability that we will have exactly 2 blue socks?

- As pointed out in the socks example, when we have a *finite* or *small* population, and we sample without replacement, the binomial approximation will not be appropriate.

- Notice that any subset of 10 socks in this example is equally likely to be chosen.

- Again, we use X = the number of successes (blue socks) in the sample we draw, then X is said to have the hypergeometric distribution.

Parameters

- It is easy to see that the probability distribution of X depends on three parameters:

  n = sample size (10 in socks example).

  M = total number of successes in the population (34 in socks example).

  N = total number of individuals in the population (50 in socks example).

  We wish to obtain $P(X=x) = h(x; n, M, N)$.

- $P(X=2) = h(2; 10, 34, 50)$ = {# of outcomes with X=2} / {# of possible outcomes}.

- Thus we have

  # of ways of selecting 2 blue socks

  # of ways of selecting 8 red socks

$$h(2; 10, 34, 50) = \frac{\boxed{\binom{34}{2}}\boxed{\binom{16}{8}}}{\boxed{\binom{50}{10}}}$$

- To compute one can use R command: `choose(n,k)`,

  pmf: `dhyper(x,M,N-M,n)`, cdf: `phyper(x,M,N-M,n)`.

Hypergeometric pmf and statistics

- If X is the number of successes in a completely random sample of size n drawn from a population consisting of M successes and $(N - M)$ failures, then the distribution of X is given by

$$\mathrm{P}(\mathrm{X} = x) = h(x; n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

  for x an integer satisfying $\max(0, n - N + M) \leq x \leq \min(n, M)$.

- Proposition:

  If X ~ hypergeometric with pmf $h(x; n, M, N)$, then $E(X) = n(M/N)$,
  $\mathrm{Var}(X) = (N - n)/(N - 1)\, n\, (M/N)\, (1 - M/N)$.

Connection with Binomial

- From the proposition, notice that if we let *p*=*M/N*, we get

$$\begin{aligned} \mathrm{E(X)} &= np \\ \mathrm{Var(X)} &= \left(\frac{N-n}{N-1}\right) \cdot np(1-p) \end{aligned}$$

- Notice that if we fix n, and let N be sufficiently large, Var(X) → $np(1 - p)$ which is the variance of a binomial rv. This is the reason why we can use a binomial model to approximate hypergeometric when population is large.

- $\left(\frac{N-n}{N-1}\right)$ is often called finite population correction factor.