W1211 Introduction to Statistics
Lecture 12

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Normal Probability Plot

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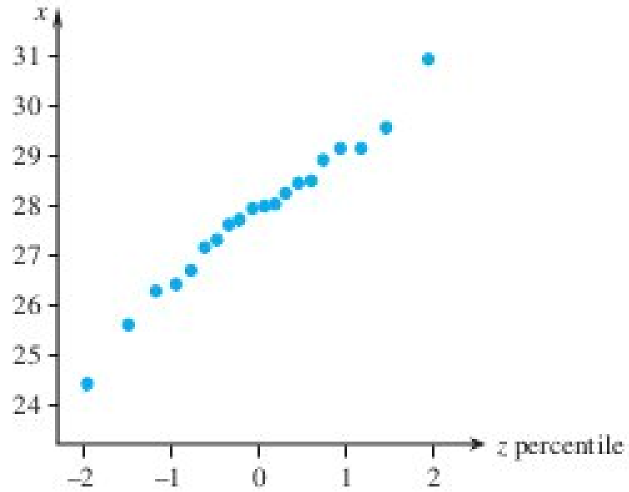
- The definition of a normal probability plot

> A plot of the n pairs
>
> $$([100(i - .5)/n]\text{th } z \text{ percentile, } i\text{th smallest observation})$$
>
> on a two-dimensional coordinate system is called a **normal probability plot.** If the sample observations are in fact drawn from a normal distribution with mean value μ and standard deviation σ, the points should fall close to a straight line with slope σ and intercept μ. Thus a plot for which the points fall close to some straight line suggests that the assumption of a normal population distribution is plausible.

Examples of Normal Probability Plot

- A Normal Sample



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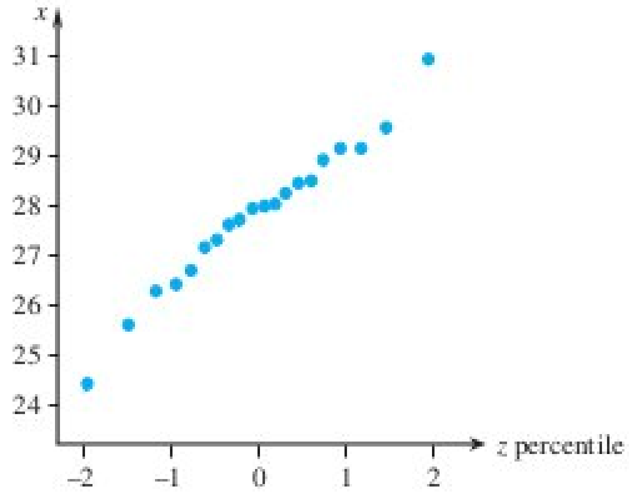


- Two Non-normal Samples

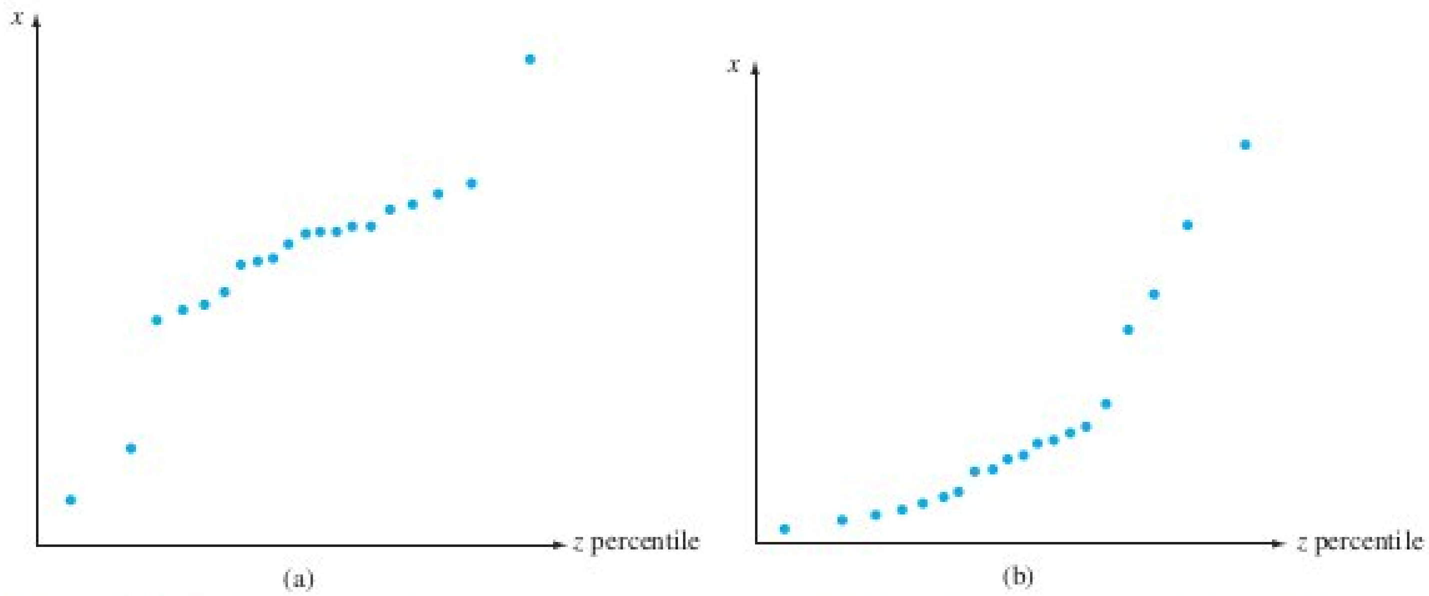


Figure 4.37 Probability plots that suggest a nonnormal distribution: (a) a plot consistent with a heavy-tailed distribution; (b) a plot consistent with a positively skewed distribution

Joint Distribution

- How can we model two rv's using probability models? For example, if we are interested in both weight and height.

- Is it enough if we just use a normal model for weight and another normal model for height?

- We need to introduce joint probability distribution in order to model multiple rv's.

Joint PMF

- Let X and Y be two discrete rv's defined on the sample space. The joint probability mass function $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = P(X=x, Y=y).$$

- As in the single rv case, we must have $p(x, y) \geq 0$ and $\sum_x \sum_y p(x, y) = 1$.

Example

Ex. We randomly put two different balls into 3 numbered (numbered as {1,2,3}) boxes. Let X be the number of empty boxes left; let Y be the minimum of the box number that has balls in it. What is the joint distribution of (X, Y)?

X can take values from {1, 2};

Y can take values from {1, 2, 3};

It's not hard to see we have the following (why?):

$p(2, j) = P(X=2, Y=j) = 1/9$, for $j = 1, 2, 3$.

$p(1, 3) = P(X=1, Y=3) = 0$.

$p(1, 1) = P(X=1, Y=1) = 4/9$.

$p(1, 2) = P(X=1, Y=2) = 2/9$.

p_{ij}	1	2	3
1	4/9	2/9	0
2	1/9	1/9	1/9

Marginal PMF

- The marginal probability mass functions of X and Y, denoted by $p_X(x)$ and $p_Y(y)$, respectively, are given by

$$p_X(x) = \sum_y p(x,y) \qquad p_Y(y) = \sum_x p(x,y)$$

Ex.

p_{ij}	1	2	3	$p_X(x)$
1	$4/9$	$2/9$	0	2/3
2	$1/9$	$1/9$	$1/9$	1/3
$p_Y(y)$	5/9	1/3	1/9	

- Notice that the marginal probability mass functions are automatically proper pmf's. (why?)

Two continuous rv's

- We would like to extend the same ideas to the continuous case. Let X and Y be continuous rv's. A joint probability density function $f(x, y)$ for these two variables is a function satisfying $f(x, y) \geq 0$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

- The marginal probability density function of X and Y, denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_{\mathrm{X}}(x) = \int_{-\infty}^{\infty} f(x,y)dy \quad \text{for} \quad -\infty < x < \infty$$

$$f_{\mathrm{Y}}(y) = \int_{-\infty}^{\infty} f(x,y)dx \quad \text{for} \quad -\infty < y < \infty$$

Remarks

- In the continuous case, roughly speaking, $f(x, y)\mathrm{d}x\mathrm{d}y$ can be treated as $\mathrm{P}(\mathrm{X}=x,\mathrm{Y}=y)$.
- $\mathrm{P}(a < \mathrm{X} < b, c < \mathrm{Y} < d) = \int_a^b \int_c^d f(x, y)dxdy$
- As in the discrete case, $f_X(x)$ and $f_Y(y)$ calculated from the joint distribution are automatically proper pdf's.
- Marginal distributions are, in fact, the distributions of the marginal random variables when they are treated as univariate random variables.

Example

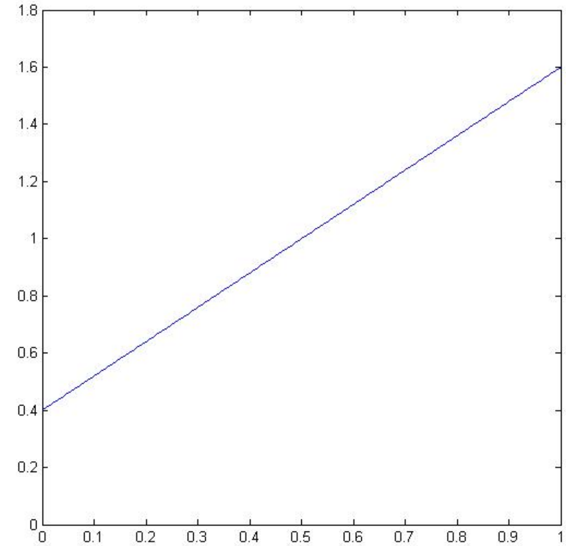
Ex. Suppose the joint pdf of the pair (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that this is a proper joint pdf.
2. What is $P(0 \leq X \leq 1/4, 0 \leq Y \leq 1/4)$?
3. What is $P(0 \leq Y \leq 1/4)$

Example cont.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{6}{5}(x+y^2)dy = \frac{6}{5}x + \frac{2}{5}$$



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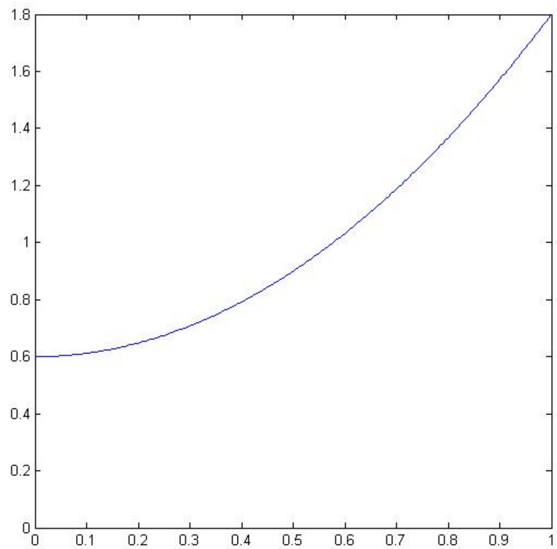
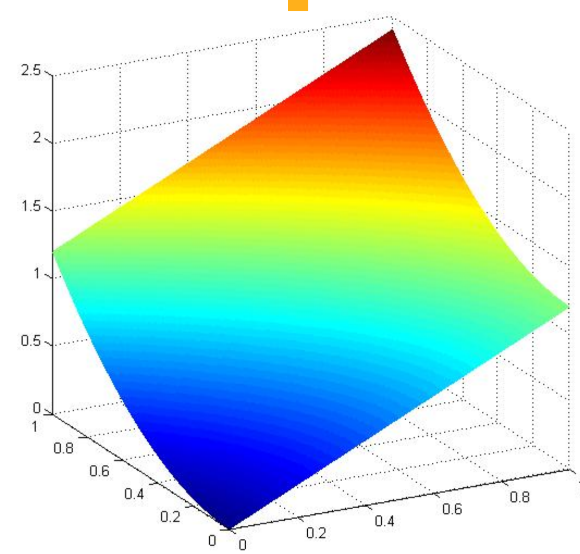
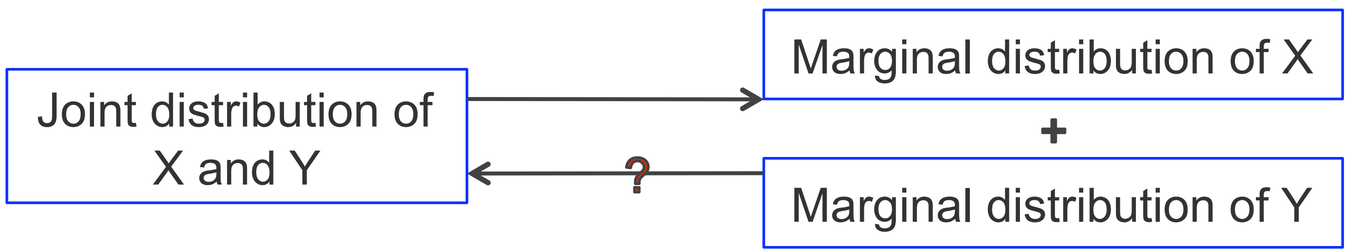




Joint and Marginal

- Now we have



- In general, we CANNOT go the other way around. Further information about the dependence structure of X and Y is needed to determine the joint distribution.

Example

Ex. Consider the following two joint distributions of X and Y.

p_{ij}	0	1
0	$3/10$	$3/10$
1	$3/10$	$1/10$

p_{ij}	0	1
0	$9/25$	$6/25$
1	$6/25$	$4/25$

It is easy to see that the marginal distributions of X and Y are the same in both cases. P(X=0) = P(Y=0) = 3/5; P(X=1) = P(Y=1) = 2/5.

This is the example that *different* joint distributions may have the *same* marginal distributions.

Independent rv's

- Recall the definition of independence of two random events A and B.

$$P(A \cap B) = P(A)\, P(B)$$

- We say two random variables X and Y are independent if and only if

$$P(X=x, Y=y) = P(X=x)\, P(Y=y), \text{ for any } x \text{ and } y.$$

- More specifically, two random variables X and Y are said to be independent if for every pair x and y values,

$$p(x, y) = p_X(x)\, p_Y(y), \text{ when X and Y are discrete;}$$

  or

$$f(x, y) = f_X(x)\, f_Y(y), \text{ when X and Y are continuous.}$$

Ex. The second case of the previous example.

Multiple Random Variables

- If $X_1, X_2, \ldots, X_n$ are all discrete random variables, the joint pmf of the variables is the function

$$p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

  If the variables are continuous, the joint pdf of $X_1, X_2, \ldots, X_n$ is the function $f(x_1, x_2, \ldots, x_n)$ such that for any n intervals $[a_1, b_1], \ldots, [a_n, b_n]$,

$$P(a_1 \leq X_1 \leq b_1, \ldots, a_n \leq X_n \leq b_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

- What should be the regularity conditions for $p(x_1, x_2, \ldots, x_n)$ and $f(x_1, x_2, \ldots, x_n)$?

- How do get the marginal distributions of $X_1, X_2, \ldots$ by using $p(x_1, x_2, \ldots, x_n)$ and $f(x_1, x_2, \ldots, x_n)$?

Independence

- Proposition:

The random variables $X_1, X_2, \ldots, X_n$, are said to be independent if for every subset $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$, of the variables (each pair, each triple, and so on), the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

- $p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$

- $f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$

Example

Ex. Two people each arrive independently at the station at some random time between 5:00 am and 6:00 am (arrival time for either person is uniformly distributed). They stay exactly five minutes and then leave. What is the probability they will meet on a given day.

Conditional dist.

- Using the marginal distributions, one can calculate the conditional distribution of one rv given the other.

- Let X and Y be two conditional rv's with joint pdf $f(x, y)$ and marginal X pdf $f_X(x)$. Then for any X value x for which $f_X(x)>0$, the conditional probability density function of Y given that X=x is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad -\infty < y < \infty.$$

- If X and Y are discrete, replace pdf's by pmf's in this definition gives the conditional probability mass function of Y when X=x.

Expectation of Functions

- Recall how we compute E[*h*(X)]. A similar result also holds for a function *h*(X, Y) of two jointly distributed rv's.

- Let X and Y be jointly distributed rv's with pmf *p*(*x*, *y*), if they are discrete; or pdf *f* (*x*, *y*), if they are continuous. The expected value of a function *h*(X, Y), denoted by E[*h*(X, Y)] is given by

$$\mathrm{E}[h(\mathrm{X}, \mathrm{Y})] = \begin{cases} \sum_x \sum_y h(x,y) \cdot p(x,y) & \text{if X and Y are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y) dx dy & \text{if X and Y are continuous} \end{cases}$$

- This result can also be extended to multiple (>2) rv case.

Examples

Ex. (Important! Linearity of expectations) Show that for any two random variables X and Y, E(X+Y) = E(X) + E(Y).

Example

<u>Ex.</u> If two random variables X and Y are independent, what is E(XY)? What about E(g(X)h(Y))?