S1211Q Introduction to Statistics
Lecture 19

Wei Wang

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Notes on Normal Probability Plot

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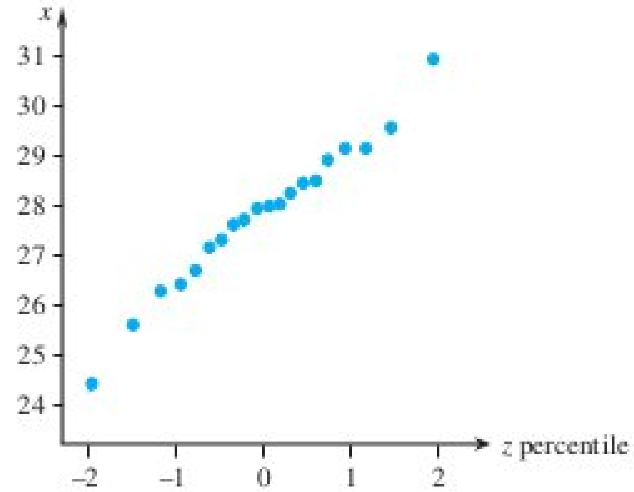
- Because of the important role that Normal Distribution plays in statistical inference, we often want to assess whether a given sample is roughly normal distributed. Normal Probability Plot is used for this purpose.
- The basic strategy is to compare sample features with population features. In probability plot, we are using sample percentile(quantile) and population percentile(quantile), so it is also known as Q-Q plot.
- The definition of a normal probability plot

> A plot of the n pairs
>
> $$([100(i - .5)/n]\text{th } z \text{ percentile, } i\text{th smallest observation})$$
>
> on a two-dimensional coordinate system is called a **normal probability plot.** If the sample observations are in fact drawn from a normal distribution with mean value μ and standard deviation σ, the points should fall close to a straight line with slope σ and intercept μ. Thus a plot for which the points fall close to some straight line suggests that the assumption of a normal population distribution is plausible.

Examples of Normal Probability Plot

- A Normal Sample



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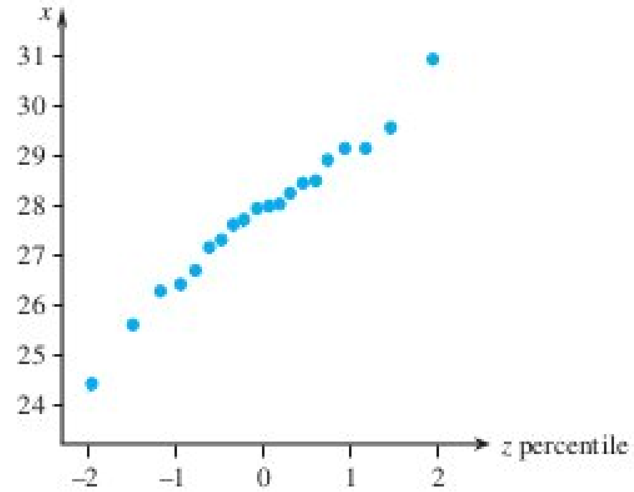


- Two Non-normal Samples

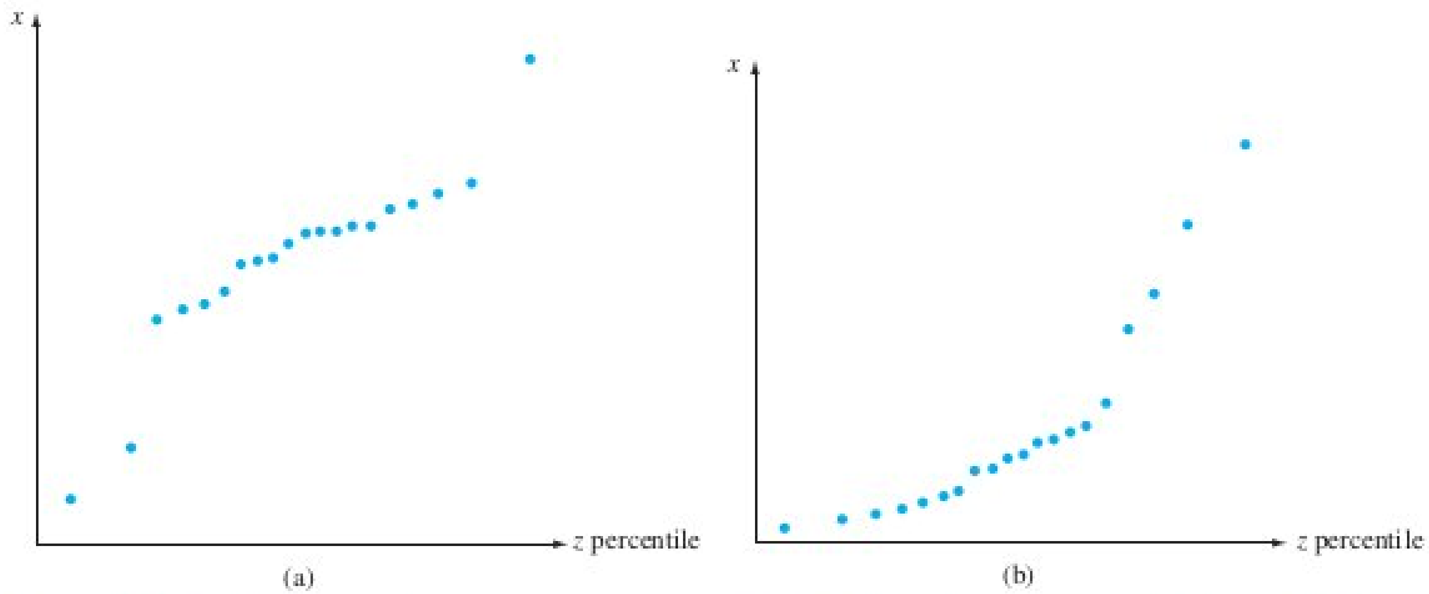


Figure 4.37 Probability plots that suggest a nonnormal distribution: (a) a plot consistent with a heavy-tailed distribution; (b) a plot consistent with a positively skewed distribution

Hypothesis Testing for a Population Proportion

- Let p denote the proportion of individuals or objects in a population who possess a specified property (probability of success). In order to make inference about p, naturally we would look at the sample proportion, which is X/n. X is the number of Successes in the sample. In practice, X should follow a binomial distribution, and when n is large, it can further be approximated by a normal distribution.
- We first consider large sample tests.

Large-sample tests

- Thanks to the Central Limit Theorem, we have

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} \sim N(0, 1)$$

  under the null hypothesis.

- Thus the rejection region is determined by

1. H_a: $p > p_0$: $Z > z_\alpha$
2. H_a: $p < p_0$: $Z < -z_\alpha$
3. H_a: $p \neq p_0$: $Z > z_{\alpha/2}$ or $Z_0 < -z_{\alpha/2}$

- The test procedures are valid provided that $np_0 \geq 10$ and $n(1-p_0) \geq 10$.

Example

Ex. (Defective rate cont.) A factory claims that less than 10% of the components they produce are defective. A consumer group is skeptical of the claim and checks a random sample of 300 components and finds that 39 are defective. Is there evidence that 10% of all components made at the factory are defective?

$H_0 : p = 0.10$ $\quad$ $H_a: p > 0.10$

$$\hat{p} = \frac{39}{300} = 0.13 \quad Z = \frac{0.13 - 0.1}{\sqrt{0.1(1-0.1)/300}} = 1.72$$

$z_{0.05} = 1.645$. $Z > z_{0.05}$, thus we would reject H_0 at level $\alpha=0.05$.

Type II Error

- We can calculate Type II Error based on the large sample normal approximation

$$
\begin{aligned}
\beta(p') &= \mathrm{P}(H_0 \text{ is not rejected when } p = p') \\
&= \mathrm{P}\left(\frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \le z_\alpha | p = p'\right) \\
&= \mathrm{P}\left(\frac{\hat{p} - p'}{\sqrt{p_0(1-p_0)/n}} \le z_\alpha + \frac{p_0 - p'}{\sqrt{p_0(1-p_0)/n}} | p = p'\right) \\
&= \mathrm{P}\left(\frac{\hat{p} - p'}{\sqrt{p'(1-p')/n}} \le \frac{z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}} + \frac{(p_0 - p')}{\sqrt{p'(1-p')/n}} | p = p'\right) \\
&= \Phi\left(\frac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)
\end{aligned}
$$

Determining sample size

- If we specify a particular alternative *p*’ and specify a *β* value that can be tolerated (e.g. 0.1). Then from

$$\beta = \Phi\left(\frac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right) \Longrightarrow -z_\beta = \frac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}$$

- Therefore, in order to achieve the specified type I and type II error, one has to have a sample size of at least

$$n = \left(\frac{z_\alpha\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right)^2$$

- For two sided test, we have to change z_α to $z_{\alpha/2}$ in the above formula.

- Difference between the sample size calculation formula in chapter 7 and the one above.

Type II Error and Sample Size calculation

- In general Type II Error and Sample Size formulas are give below

Alternative Hypothesis	$\beta(p')$
H_a: $p > p_0$	$\Phi\left[\dfrac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$
H_a: $p < p_0$	$1 - \Phi\left[\dfrac{p_0 - p' - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$
H_a: $p \neq p_0$	$\Phi\left[\dfrac{p_0 - p' + z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$ $-\Phi\left[\dfrac{p_0 - p' - z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$

The sample size n for which the level α test also satisfies $\beta(p') = \beta$ is

$$n = \begin{cases} \left[\dfrac{z_\alpha\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right]^2 & \text{one-tailed test} \\ \left[\dfrac{z_{\alpha/2}\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right]^2 & \text{two-tailed test (an approximate solution)} \end{cases}$$

Example

Ex. A package-delivery service advertises that at least 90% of all packages brought to its office by 9 a.m. for delivery in the same city are delivered by noon that day. Let p denote the true proportion of such packages that are delivered as advertised and consider the hypothesis $H_0: p = 0.9$ versus $H_a: p < 0.9$. If only 80% of the packages are delivered, how likely is it that a level .01 test based on n=225 packages will detect such departure from H_0? What should the sample size be to ensure that $\beta(0.8) = 0.01$? With $\alpha = .01$, $p_0 = .9$, $p' = .8$, and $n = 225$.

Type II error:
$$\begin{aligned}\beta(p') &= 1 - \Phi\left(\frac{p_0 - p' - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right) \\ &= 1 - \Phi\left(\frac{.9 - .8 - 2.33\sqrt{(.9)(.1)/225}}{\sqrt{(.8)(.2)/225}}\right) \\ &= 1 - \Phi(2.00) = .0228\end{aligned}$$

Example cont.

- Using $z_{.01}$=2.33, the sample size can then be calculated from

$$n = \left(\frac{z_\alpha\sqrt{p_0(1-p_0)/n} + z_\beta\sqrt{p'(1-p')/n}}{p'-p_0}\right)^2$$
$$= \left(\frac{2.33\sqrt{(.9)(.1)} + 2.33\sqrt{(.8)(.2)}}{.8-.9}\right)^2 \approx 266$$

- 1-β is often referred to as the power of a test. It is the probability that the test can actually detect the alternative given the alternative is true! For α-level tests, the bigger the power the better!

Small sample tests

- For testing population proportions, when the sample size is small, the normal approximation is no longer appropriate. Thus a more accurate test should be used.

- As mentioned before, the sample proportion is X/n. X is the number of *S*’s in the sample and can be treated as a binomial random variable. Thus a rejection region can be constructed using binomial cdf/pmf.

- Can we get an exact α-level test using binomial?

P-Value

- To report the result of a hypothesis-testing analysis is to simply say whether the null hypothesis was rejected at a specified level of significance. This type of statement is somewhat inadequate because it says nothing about whether the conclusion was a very close call or quite clear cut.

- P-value is a quantity that conveys much information about the strength of evidence against H_0 and allows an individual decision maker to draw a conclusion at any specified level α.

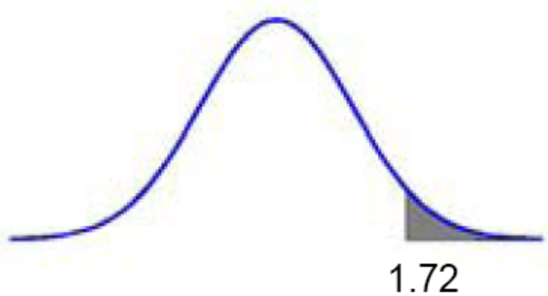
- The P-value (*observed significance level*) is the probability, under the null hypothesis, that the test statistic is more ***extreme*** than the observed statistic.

Example cont.

Ex. (Defective rate cont.) A factory claims that less than 10% of the components they produce are defective. A consumer group is skeptical of the claim and checks a random sample of 300 components and finds that 39 are defective. Is there evidence that 10% of all components made at the factory are defective?

If H_0 is true, $\mathrm{Z} = \dfrac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \sim \mathrm{N}(0,1)$

$$\hat{p} = \frac{39}{300} = 0.13 \quad \mathrm{Z} = \frac{0.13 - 0.1}{\sqrt{0.1(1-0.1)/300}} = 1.72$$



P(Z>1.72) = $\boxed{0.0416}$ ← *P-value*

Remarks

- P-value is corresponding to the smallest level of significance at which H_0 would be rejected when a specified test procedure is used on a given data set. The smaller the P-value, the more contradictory is the data to H_0.

- Once the P-value has been determined, the conclusion at any particular level α results from comparing the P-value to α:
  1. P-value $\leq \alpha \rightarrow$ reject H_0 at level α.
  2. P-value $> \alpha \rightarrow$ do not reject H_0 at level α.

- To calculate P-value:
  1. Calculate the test statistic as before.
  2. Compute probability that we will reject the null if the threshold is the test statistic obtained from 1.

- Question: what is the relationship of P-value of the one-sided test and the P-value of the two-sided test?

Two sample tests

- A new drug is claimed to significantly reduce the blood pressure for high blood pressure patients. What kind of tests can we use to verify the claim?

- A new drug is claimed to perform much better in terms of reducing blood pressure than an old drug. What kind of tests can we use to verify the claim?

Things to cover

- As in the one sample testing problem, we will cover the following cases:
  1. Two normal populations with known variance.
  2. Two populations with unknown distribution and large sample size.
  3. Two normal populations with unknown variance.
  4. Two population proportions with large sample size.
  5. Tests about variances. (NOT required.)

- Basic assumptions for comparing population means:
  1. $X_1, X_2, \ldots, X_m$ is a random sample (i.i.d.) from a population with mean μ_1 and variance σ_1^2.
  2. $Y_1, Y_2, \ldots, Y_n$ is a random sample (i.i.d.) from a population with mean μ_2 and variance σ_2^2.
  3. The X and Y samples are independent of one another.

Test statistics

- Since we are comparing the population means, a natural test statistic to use would be the difference of two sample means. Because of independence we have,

$$
\begin{aligned}
E(\bar{X} - \bar{Y}) &= \mu_1 - \mu_2 \\
Var(\bar{X} - \bar{Y}) &= \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}
\end{aligned}
$$

Case I: normal, known variance

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

Test statistic: $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{m} + \dfrac{\sigma_2^2}{n}}} \sim \mathrm{N}(0,1)$

vs Alternative Hypothesis:

$H_a : \mu_1 - \mu_2 > \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{m} + \dfrac{\sigma_2^2}{n}}} > Z_\alpha$

$H_a : \mu_1 - \mu_2 < \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{m} + \dfrac{\sigma_2^2}{n}}} < -Z_\alpha$

$H_a : \mu_1 - \mu_2 \neq \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{m} + \dfrac{\sigma_2^2}{n}}} < -Z_{\alpha/2}$ or $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{m} + \dfrac{\sigma_2^2}{n}}} > Z_{\alpha/2}$

Questions

- How to compute P-value for case I?
- How to compute type II errors for case I?
- In a balanced design, derive the sample size calculation formula (for alternative “>”):

$$m = n = \frac{(\sigma_1^2 + \sigma_2^2)(Z_\alpha + Z_\beta)^2}{(\Delta' - \Delta_0)^2}$$

Case II: large sample

$H_0 : \mu_1 - \mu_2 = \Delta_0$

Test statistic: $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} \sim A\mathrm{N}(0,1)$

vs Alternative Hypothesis:

$H_a : \mu_1 - \mu_2 > \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} > Z_\alpha$

$H_a : \mu_1 - \mu_2 < \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} < -Z_\alpha$

$H_a : \mu_1 - \mu_2 \neq \Delta_0$, reject if $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} < -Z_{\alpha/2}$ or $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} > Z_{\alpha/2}$

Questions

- How to construct confidence interval for $\mu_1 - \mu_2$ in case II?

Case III: normal, unknown variance

$H_0 : \mu_1 - \mu_2 = \Delta_0$

Test statistic: $\dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} \sim t_\nu$, ν is the df of the t-distribution and it's approximately estimated

by the sampled data: $\nu = \dfrac{\left(\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}\right)^2}{\dfrac{(s_1^2/m)^2}{m-1} + \dfrac{(s_2^2/n)^2}{n-1}}$, and round ν town to the nearest integer.

Case III cont.

vs Alternative Hypothesis:

$$H_a : \mu_1 - \mu_2 > \Delta_0 \text{, reject if } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} > t_{\alpha,\nu}$$

$$H_a : \mu_1 - \mu_2 < \Delta_0 \text{, reject if } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} < -t_{\alpha,\nu}$$

$$H_a : \mu_1 - \mu_2 \neq \Delta_0 \text{, reject if } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} < -t_{\alpha/2,\nu} \text{ or } \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} > t_{\alpha/2,\nu}$$

Questions

- How to compute P-values of the test?
- How to construct confidence interval for $\mu_1 - \mu_2$ in case III?
- What if we know that $\sigma_1^2 = \sigma_2^2$?

  The *pooled estimator* of $\sigma^2 = \sigma_1^2 = \sigma_2^2$ is given by

$$S_p^2 = \frac{m-1}{m+n-2} \cdot S_1^2 + \frac{n-1}{m+n-2} \cdot S_2^2$$

Case IV

$H_0 : p_1 - p_2 = 0$

Test statistic: $\dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{m}+\dfrac{1}{n}\right)}}$, $\hat{p} = \dfrac{m}{m+n}\hat{p}_1 + \dfrac{n}{m+n}\hat{p}_2$ (the *weighted* average of $\hat{p}_1$

and $\hat{p}_2$)

Case IV cont.

vs Alternative Hypothesis:

$$H_a : p_1 - p_2 > 0 \text{, reject if } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} > Z_\alpha$$

$$H_a : p_1 - p_2 < 0 \text{, reject if } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} < -Z_\alpha$$

$$H_a : p_1 - p_2 \neq 0 \text{, reject if } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} > Z_{\alpha/2} \text{ or } \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m}+\frac{1}{n}\right)}} < -Z_{\alpha/2}$$

Paired t-test

- As in the previous example, the data is paired, the two scores (before and after) recorded for each individual are dependent, but the between individuals the pairs are independent.

- Thus in order to test H_0: $\mu_1 - \mu_2 = 0$, one has to look at the difference of each pair. The problem eventually becomes a one sample t-test problem.