Random variables

- A random variable is a variable whose value is a numerical outcome of a random phenomenon.

- For a given sample space *S* of some experiment, a random variable (rv) is any rule that associates a number with each outcome in *S*.

- To put it more mathematically, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

- Remark: a random variable is NOT a sample space.

Discrete vs. Continuous

- X is a discrete random variable if its possible values either constitute a finite set or else can be listed in an infinite sequence in which there is a first element, a second element, and so on ("countably" infinite).

- X is a continuous random variable if it takes all possible values in an interval of numbers or all numbers in a disjoint union of such intervals. No possible value of the variable has positive probability, that is, $P(X=c) = 0$ for any possible value c.

- X can also be a random variable with a mixture distribution of both discrete and continuous components.

PMF

- The probability model for a discrete random variable X, lists its possible values and their probabilities.

Value of X	x_1	x_2	……..	x_k
Probability	p_1	p_2	……..	p_k

- Every probability, p_i, is a number between 0 and 1.
- $p_1+p_2+\ldots+p_k=1$

- The probability distribution or probability mass function (pmf) of a discrete rv is defined for every number x by $p(x) = P(X=x) = P(\text{all } s \in S\text{: } X(s)=x)$.

- How to check if some function $p(x)$ is a proper PMF?

Bernoulli RV

- The arguably simplest probability model is Bernoulli. Any random variable whose possible values are only 0 and 1 is called a Bernoulli random variable.

Ex. Flip a coin. S={H, T}. X is a Bernoulli random variable. X(H)=1, X(T)=0.

$$P(X=1) = 0.5,\ P(X=0) = 0.5.$$

Ex. Roll a die. S={1, 2, 3, 4, 5, 6}. X is a Bernoulli random variable. X(1)=1, X(2)=1, X(3)=0, X(4)=0, X(5)=0, X(6)=0.

$$P(X=1) = 1/3,\ P(X=0) = 2/3.$$

Example

<u>Ex.</u> Flip three fair coins. (*Binomial*)

S = {HHH, HHT, HTH, HTT, THT, THH, TTH, TTT}. Let's define random variable X to be the number of heads in the experiment, i.e., X(HHH)=3, X(THT)=1, etc.

X	
0	TTT
1	TTH THT HTT
2	THH HTH HHT
3	HHH

Value of X	0	1	2	3
Probability	0.125	0.375	0.375	0.125

One can calculate the probability of an event by adding the probabilities p_i of the particular values of x_i that make up the event. For example, if we want to know the probability of getting less than 2 heads, we can use

$$P(X<2) = P(X=0) + P(X=1) = 0.125 + 0.375 = 0.5$$

Note: $P(X\leq 2) = P(X=0) + P(X=1) + P(X=2) = 0.875$

CDF

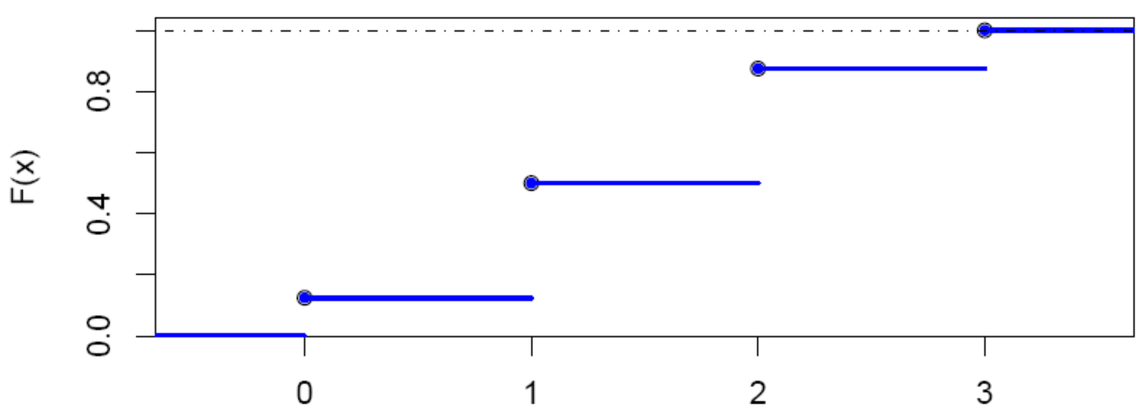
- The cumulative distribution function (cdf) F(*x*) of a discrete rv variable X with pmf *p*(*x*) is defined for every number *x* by

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \leq x) = \sum_{y:y \leq x} p(y).$$

For any number *x*, F(*x*) is the probability that the observed value of X will be at most *x*.

- For X a discrete rv, the graph of F(*x*) will have a jump at every possible value of X and will be flat between possible values. Such a graph is called a step function.

The three coin flips example



Parameter and Family

- Suppose $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a parameter of the distribution. The collection of all probability distributions for different values of the parameter is called a family of probability distributions.

Ex. For Bernoulli rv's, the parameter is the probability of being 1 (or 0), that is,

$$p = P(X=1)$$

Expectation and Variance

- Random variables have distributions, so they have centers and spreads.

- The expected value (mean value or expectation) of a random variable describes its theoretical long-run average value.

- We typically use μ or $E(X)$ to denote the mean, $Var(X)$ to denote the variance and σ or $SD(X)$ to denote the standard deviation of a rv X.

Motivating examples

Ex. How many heads would you expect if you flipped a fair coin twice?

S = {HH, HT, TH, TT}.

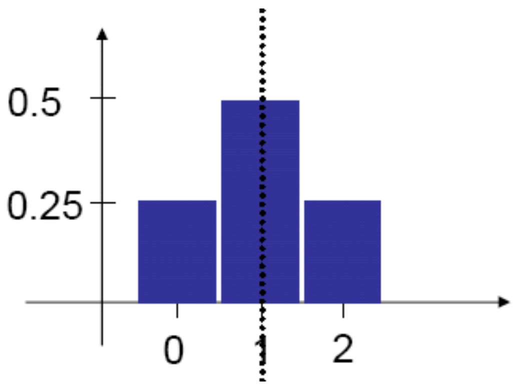
X = number of heads.

0 TT

1 HT TH

2 HH

$p(X=0) = 0.25$; $p(X=1) = 0.5$; $p(X=2) = 0.25$.



Each outcome is weighted by its probability.

$\mu = 0 \times 0.25 + 1 \times 0.5 + 2 \times 0.25 = 1$

Example

<u>Ex.</u> How many heads would you expect if you flipped a coin three times?

$$\mu = 0 \times 0.125 + 1 \times 0.375 + 2 \times 0.375 + 3 \times 0.125 = 1.5$$

This can never occur in a single trial of 3 flips. However, on average we would expect to get 1.5 heads if we repeated the experiment many times.

Definition

- Suppose X is a discrete random variable whose probability model is given by

Value of X	x_1	x_2	……..	x_k
Probability	p_1	p_2	……..	p_k

The expected value of X is given by

$$\mathrm{E(X)} = \mu_X = \sum_{x \in D} x \cdot p(x) = x_1 p_1 + x_2 p_2 + \cdots x_k p_k$$

Example

<u>Ex.</u> Expectation of a Bernoulli rv.

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & x \neq 0,1 \end{cases}$$

$\mu = 0 \times (1-p) + 1 \times p = p$.

Example

Ex. The general form for the pmf of X = number of children born up to and including the first boy is,

$$p(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

1. Verify that this is a proper pmf.
2. Calculate the expected value of X.

The Expected Value of a Function

- A bookstore purchases ten copies of a books at \$60 each to sell at \$120, and any unsold copies after three months can be redeemed for \$20. If the number of copies sold is X, what is the profit of the bookstore?

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- The profit is $h(X) = 100X - 400$. Is $h(X)$ a Random Variable? Then what is the expectation of $h(X)$?
- If Random Variable X has range D and pmf $p(x)$, then the expected value of function $h(X)$ is given by

$$E(h(X)) = \sum_{x \in D} \left(h(x) \cdot p(x) \right)$$

The Linear Function Case

- In the case of linear function, we have a much more convenient formula
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- To prove,

$$E(aX+b) = \sum_{x \in D} (ax+b) \cdot p(x) = a \sum_{x \in D} x \cdot p(x) + b \sum_{x \in D} p(x) = aE(X) + b$$

What does $P(X = x)$ mean?

- We only defined probability on events, so what does we mean when we say the probability that $X = x$, or $P(X = x)$?
- In fact, $X = x$ is a shorthand for the event $\{s \in S : X(s) = x\}$, which is consisted of outcomes s in sample space S that are mapped to value x under random variable X.