Poisson Distribution

- Poisson Distribution is for describing outcomes that come in the form of count data, e.g., visits to a particular website during a time interval
- But unlike Binomial or Hypergeometric Distribution, there is no simple experiment that Poisson Distribution is based on.
- A random variable X is said to have Poisson Distribution with parameter $\mu(> 0)$ if the pmf of X is

$$p(x; \mu) = e^{-\mu}\frac{\mu^x}{x!}, x = 0, 1, 2, \ldots$$

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- So

$$p(0;\mu) + p(1;\mu) + p(2;\mu) + \cdots = e^{\mu} \times e^{-\mu} = 1$$

Example

- Let X denote the number of creatures of a particular type captured in a trap during a given time period. Suppose that X has a Poisson distribution with =4.5, so on average traps will contain 4.5 creatures. Then the probability that a trap contains exactly five creatures is

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- The probability that the a trap has at most five creatures is

$$P(X \leq 5) = \sum_{x=0}^{5} \frac{e^{-4.5}(4.5)^x}{x!} = .7029$$

Poisson Distribution as a Limit

- Suppose that in the binomial pmf $b(x; n; p)$, we let $n \to \infty$ and $p \to 0$ in such a way that np approaches a value $\mu > 0$. Then $b(x; n; p) \to p(x; \mu)$.
- So in any binomial experiment in which n is large and p is small, , then Binomial can be approximated by Poisson Distribution with parameter $\mu = np$.

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- With Poisson Approximation $\mu = np = 3$

$$P(X \leq 2) \approx e^{-3} + 3e^{-3} + \frac{3^2 e^{-3}}{2} = .4232$$

Mean and Variance of Poisson Distribution

- If X has a Poisson Distribution with parameter μ, then $E(X) = Var(X) = \mu$.
- It can be derived directly from the pmf, or through the Binomial limit argument.
- If X is $b(x; n; p)$, then

$$E(X) = np \to \mu, \; Var(X) = np(1-p) \to \mu$$