S1211Q Introduction to Statistics
Lecture 12

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Expectation of Functions

- Recall how we compute E[*h*(X)]. A similar result also holds for a function *h*(X, Y) of two jointly distributed rv's.

- Let X and Y be jointly distributed rv's with pmf *p*(*x*, *y*), if they are discrete; or pdf *f* (*x*, *y*), if they are continuous. The expected value of a function *h*(X, Y), denoted by E[*h*(X, Y)] is given by

$$\mathrm{E}[h(\mathrm{X},\mathrm{Y})] = \begin{cases} \sum_x \sum_y h(x,y)\cdot p(x,y) & \text{if X and Y are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\cdot f(x,y)\,dx\,dy & \text{if X and Y are continuous} \end{cases}$$

- This result can also be extended to multiple (>2) rv case.

Examples

Ex. (Important! Linearity of expectations) Show that for any two random variables X and Y, $E(X+Y) = E(X) + E(Y)$.

Example

Ex. If two random variables X and Y are independent, what is E(XY)? What about E(g(X)h(Y))?

Expectation of Linear Function of Multiple RV's

- Linearity is well preserved in expectation.

$$E(a \cdot X + b \cdot Y + c) = a \cdot E(X) + b \cdot E(Y) + c$$

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- Unlike the linear case, expectation of product in general doesn't equal to the product of expectations

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$$\begin{aligned} E(XY) &= \int \int xyf(x,y)dxdy = \int \int xyf_X(x)f_Y(y)dxdy \\ &= \int xf_X(x)dx \int yf_Y(y)dy = E(X)E(Y) \end{aligned}$$

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- And for independent RV's, in general

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Covariance

- When two random variables X and Y are not independent, it is often of interest to assess how strongly they are related to one another.

- A popular measurement to characterize the dependence of two rv's is called correlation. To calculate correlation of two rv's, we'll have calculate the covariance of the two rv's.

- The covariance between two rv's X and Y is

$$
\begin{aligned}
\mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= \mathrm{E}[(\mathrm{X}-\mu_X)(\mathrm{Y}-\mu_Y)] \\
&= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y)\cdot p(x,y) & \mathrm{X},\ \mathrm{Y}\ \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)\cdot f(x,y)\,dx\,dy & \mathrm{X},\ \mathrm{Y}\ \text{continuous} \end{cases}
\end{aligned}
$$

Short cut

- Proposition:

$$\boxed{\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)}$$

- What happens if we set Y=X?

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- This is a special case of

$$\begin{aligned} Cov(aX + bY, cZ + dW) =& ac \cdot Cov(X, Z) + ad \cdot Cov(X, W) \\ &+ bc \cdot Cov(Y, Z) + db \cdot Cov(Y, W) \end{aligned}$$

Example

Ex. Suppose the joint distribution of X and Y are

$$f(x,y) = \begin{cases} 24xy & 0 \le x \le 1, 0 \le y \le 1, x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the covariance of X and Y?

$$f_X(x) = \int_y f(x,y)dy = \int_0^{1-x} 24xydy = 12x(1-x)^2$$

$$f_Y(y) = 12y(1-y)^2$$

$$\mathrm{E(X)} = \int_0^1 x \cdot 12x(1-x)^2 dx = \frac{2}{5} = \mathrm{E(Y)}$$

$$\mathrm{E(XY)} = \int\int_{x,y} xyf(x,y)dxdy = \int_0^1 \int_0^{1-y} 24x^2y^2 dxdy = \frac{2}{15}$$

$$\mathrm{Cov(X,Y)} = \mathrm{E(XY)} - \mathrm{E(X)E(Y)} = \frac{2}{15} - \left(\frac{2}{5}\right)^2 = -\frac{2}{75}$$

Correlation

- The correlation coefficient of X and Y, denoted by Corr(X, Y) or $\rho_{X,Y}$ is defined by

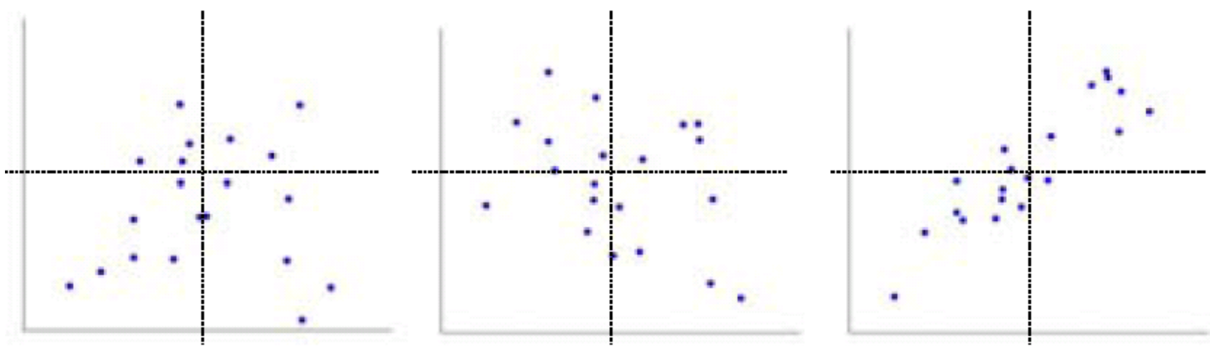
$$\rho_{X,Y} = \frac{\mathrm{Cov}(\mathrm{X}, \mathrm{Y})}{\sigma_X \cdot \sigma_Y}$$

- Because of Cauchy-Schwarz inequality, we have

$$\mathrm{Cov}^2(\mathrm{X}, \mathrm{Y}) \leq \mathrm{Var}(\mathrm{X})\mathrm{Var}(\mathrm{Y}) \Longrightarrow |\rho_{X,Y}| \leq 1$$

- The correlation coefficient $\rho_{X,Y}$ is NOT a completely general measure of the strength of a relationship. $\rho_{X,Y}$ is actually a measure of the degree of *linear* relationship between X and Y.

Remarks

- If X and Y are independent, then $\rho_{X,Y} = 0$ (why?). But $\rho_{X,Y} = 0$ does NOT imply independence.
- $\rho_{X,Y} = 1$ or -1 iff $Y = aX+b$ for some numbers a and b with $a \neq 0$.
- 

$r \approx 0$ $\quad$ $r \approx -0.3$ $\quad$ $r \approx 0.9$

Relationship Between Correlation and Independence

- Independence leads to uncorrelatedness.

$$Cov(X, Y) = E(XY) - E(X)E(Y) = E(X)E(Y) - E(X)E(Y) = 0$$

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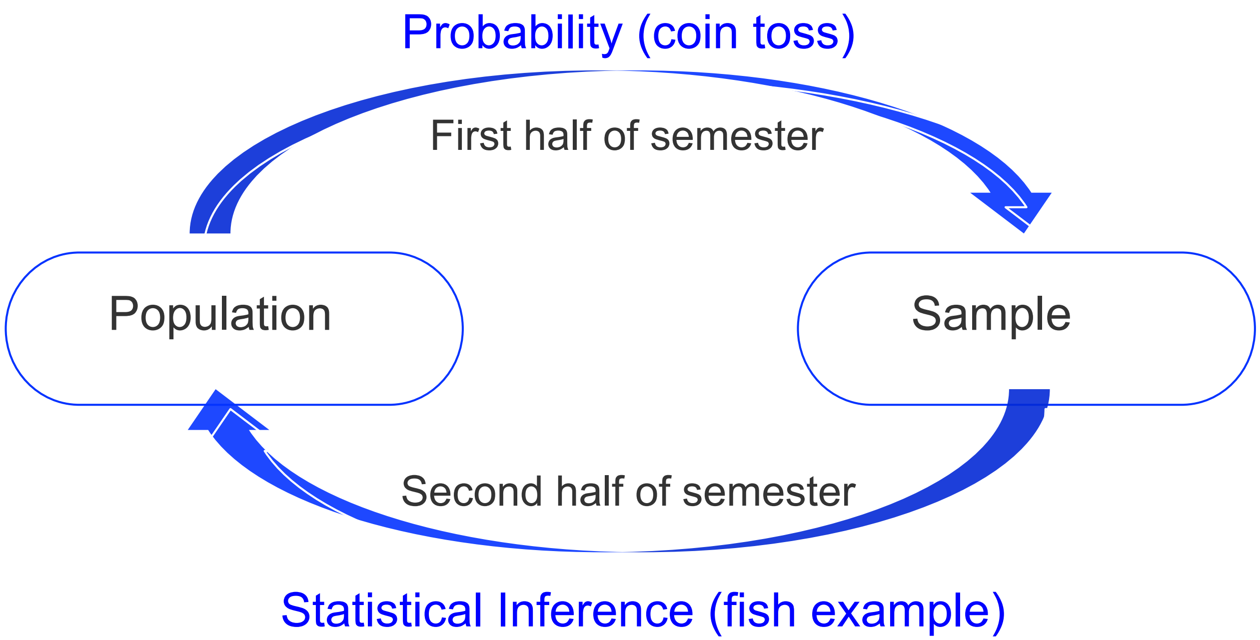
- But not vice versa!
- We will talk about this more in regression.

Population and Sample

- We will start changing our discussion from probability to statistics, which means we need to think about samples and how they relate to the underlying population.
- Recall the relationship between population and sample (probability and inference) that we visualized in the first lecture.

Probability and Inference



RV or a Particular Number

- In the first chapter, we use lowercase letters to represent the sample, $x_1, x_2, x_3, \ldots$. That means we have already observed the data and each of the letters can be replaced by a particular number.
- Before the data becoming available, there is uncertainty as to what value we will observe, so we view each observation as a RV, thus denoted by uppercase letter $X_1, X_2, X_3, \ldots$

Sample and Statistics

- A statistic is any quantity whose value can be calculated from sample data, such as Sample Mean and Sample Variance.
- Before obtaining data, a statistic is also a RV. The bulk of statistical inference is to find the distribution of the statistics, or the so-called *Sampling Distributions*.
- To make things easier, we often need to assume the observed data are *Simple Random Samples*, which means they are IID (Independently Identically Distributed).

Introduction to IID

- A sequence of random variables, X_1, X_2, …, X_n, is independent and identically distributed (i.i.d.) if each random variable has the same probability distribution as the others and all are mutually independent.

- In statistical analysis, we often assume the sampled data X_1, X_2, …, X_n, are i.i.d. from a common distribution *f*(x). And usually, we end up analyzing a linear combination of the X_i's, that is

$$\mathrm{Y} = a_1 \mathrm{X}_1 + \cdots + a_n \mathrm{X}_n = \sum_{i=1}^{n} a_i \mathrm{X}_i$$

Sample Mean***

- Let X_1, X_2, …, X_n, be an i.i.d. sequence of rv's from a distribution with mean value μ and standard deviation σ.

- Notice that the sample mean or the sample total ($T = X_1+X_2+\ldots+X_n$) can also be viewed as a special case of linear combination of X_1, X_2, …, X_n. In the i.i.d. case,

$$E(T) = E(X_1)+E(X_2)+\ldots+E(X_n) = n\mu$$

$$Var(T) = Var(X_1)+Var(X_2)+\ldots+Var(X_n) = n\sigma^2$$

- It is also easy to verify that for sample mean,

$$\mathrm{E}(\bar{\mathrm{X}}) = \mu_{\bar{X}} = \mu$$

$$\mathrm{Var}(\bar{\mathrm{X}}) = \sigma^2_{\bar{X}} = \sigma^2/n \Longrightarrow \sigma_{\bar{X}} = \sigma/\sqrt{n}$$

Invariance of Normal RV under Linear Transformation

- When $X_1, X_2, X_3, X_4, \ldots$ are normal random variables, then the linear combination of them

$$a_1 X_1 + a_2 X_2 + \ldots + a_n X_n = \sum_{i=1}^{n} a_i X_i$$

  is still a normal random variable.
- In particular, sample mean $\bar{X}$ is still a random variables.
- Remark: 1. No IID assumption is necessary; 2. This property is for Normal only.

Sample Mean of IID Normal

- If $X_1, X_2, \ldots, X_n$ IID $\sim N(\mu, \sigma^2)$, then what is the distribution of $\bar{X}$?

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Sample Mean of IID Normal

- If $X_1, X_2, \ldots, X_n$ IID $\sim N(\mu, \sigma^2)$, then what is the distribution of $\bar{X}$?
- $$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$
- But how can we derive the distribution of Sample Mean when the sample are not normal? We need to use Central Limit Theorem.

CLT

- Theorem:

The Central Limit Theorem (CLT)

Let X_1, X_2, …, X_n, be an i.i.d. sequence from a distribution with mean μ and variance σ^2. Then if n is sufficiently large, the sample mean $\bar{X}$ has approximately a normal distribution with $\mu_{\bar{X}} = \mu$ and $\sigma^2_{\bar{X}} = \sigma^2/n$;
And the sample total has approximately a normal distribution with $\mu_T = n\mu$, $\sigma_T^2 = n\sigma^2$. The larger the value of n, the better the approximation.

- Rule of Thumb: if n>30, the CLT can be used.

Distribution of a Linear Combination

- Sample mean is a particular case of linear combinations.
- The expectation and variance of a general linear combination

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

  is given by the following result.

A key result ***

Let $X_1, X_2, \ldots, X_n$, have mean values $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively.

- Whether or not the Xi's are independent,

$$\begin{aligned}\mathrm{E}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) &= a_1\mathrm{E}(\mathrm{X}_1) + a_2\mathrm{E}(\mathrm{X}_2) + \cdots + a_n\mathrm{E}(\mathrm{X}_n) \\ &= a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n\end{aligned}$$

- For any $X_1, X_2, \ldots, X_n$,

$$\mathrm{Var}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}(\mathrm{X}_i, \mathrm{X}_j)$$

If they are independent, then

$$\begin{aligned}&\mathrm{Var}(a_1\mathrm{X}_1 + a_2\mathrm{X}_2 + \cdots + a_n\mathrm{X}_n) \\ =\ & a_1^2\mathrm{Var}(\mathrm{X}_1) + a_2^2\mathrm{Var}(\mathrm{X}_2) + \cdots + a_n^2\mathrm{Var}(\mathrm{X}_n) \\ =\ & a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2\end{aligned}$$

Special Cases

- $E(X+Y) = E(X) + E(Y)$;
- $E(X-Y) = E(X) - E(Y)$;
- $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- $Var(X-Y) = Var(X) + Var(Y) -2Cov(X, Y)$
- If X and Y are independent, then $Cov(X, Y) = 0$, and
  $Var(X+Y) = Var(X) + Var(Y)$
  $Var(X - Y) = Var(X) + Var(Y)$

Example

Ex. Show that if X ~ Bin(n, p), then E(X) = np, and Var(X) = $np(1 – p)$.

Ex. Show that if X is a negative binomial rv with pmf $nb(x; r, p)$, then E(X) = $r(1–p)/p$, Var(X) = $r(1 – p)/p^2$.