OLG Model in Asset Pricing

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There are multiple versions of overlapping generation model (hereafter "OLG"). Asset pricing literature uses Blanchard-Yaari perpetual youth model the most. A typical feature is that the probability of death conditional on the agent's age is a constant, which is not realistic. But it fits the general population demographic features well.

If there is an instantaneous probability v of dying, the probability (as viewed by a person alive in period s) of still being alive (not dead) in period t is $exp(-v(t-s))$. Each instant there is a mass of v entrants coming into the economy.

Proposition 1: *The mass of investors is a constant 1.*

Proof. Suppose the mass of investors at time t is m. At time $t+dt$, $m(1-\exp(-vdt)$ mass of existing investors exit the market and v mass of investors enter the market. When the economy is stationary, $m(1-exp(-vdt)) \approx m * vdt = vdt$. Then $m = 1$.

Another way of proof is like this: at time t, the mass of investors equals to those who can live until time t: $\int_{-\infty}^{t} v \exp(-v(t-s))ds = v$ □

For an investor born at time s receives endowment $y_{s,t}$ at time t. Define the aggregate endowment: $Y_t = \int_{-\infty}^{t} v \exp(-v(t-s))y_{s,t}ds$. For simplicity, I assume that $y_{s,t}$ is not random and equal to Y_t. Then the aggregate endowment is Y_t. Later on, I will introduce uncertainty.

A typical question for the investor born at time s is to maximize lifetime utility:

$$\max_{C_{s,t}} E_s[\int_s^{\tilde{\tau}} e^{-\rho(t-s)} log(c_{s,t})]dt = E_s[\int_s^{+\infty} e^{-(\rho+v)(t-s)} log(c_{s,t})]dt \tag{1}$$

subject to the constraint $E_s\left[\int_s^{+\infty} \exp(-v(t-s))\frac{M_t}{M_s}c_{s,t}dt\right] = E_s\left[\int_s^{+\infty} \exp(-v(t-s))\frac{M_t}{M_s}y_{s,t}dt\right] = E_s\left[\int_s^{+\infty} \exp(-v(t-s))\frac{M_t}{M_s}Y_t dt\right]$ The FOC is $c_{s,t} = \exp(-\rho(t-s))\frac{1}{\kappa_s}\frac{M_s}{M_t}$. It is straightforward that $c_{s,s} = \frac{1}{\kappa_s}$. Then we have $c_{s,t} = \exp(-\rho(t-s))c_{s,s}\frac{M_s}{M_t}$. Substitute this equation back to the budget constraint:

$$\int_s^{+\infty} \exp(-v(t-s))\frac{M_t}{M_s}\exp(-\rho(t-s))\frac{1}{\kappa_s}\frac{M_s}{M_t}dt = c_{s,s}\frac{1}{v+\rho} = W_s \tag{2}$$

*Wei Wang collected the notes.

where $W_{s,s} = E_s \left[\int_s^{+\infty} \exp(-v(t-s)) \frac{M_t}{M_s} Y_t dt \right]$. Then we have $c_{s,s} = (\rho + v) W_{s,s}$. Then $c_{s,t} = \exp(-\rho(t-s)) c_{s,s} \frac{M_s}{M_t} = \exp(-(\rho+v)(t-s)) \int_s^{+\infty} \exp(-v(k-t)) \frac{M_k}{M_t} y_k dk = \exp(-(\rho+v)(t-s)) W_{s,t}$.

The ratio of the endowment to wealth is constant in this scenario:

$$Y_t = \int_{-\infty}^{t} v \exp(-v(t-s)) c_{s,t} ds \tag{3}$$

$$= \int_{-\infty}^{t} v \exp(-v(t-s))(\rho + v) W_{s,t} ds \tag{4}$$

$$= (\rho + v) \int_{-\infty}^{t} v \exp(-v(t-s)) W_{s,t} ds = (\rho + v) W_t \tag{5}$$

To fully solve the model, we need to get the expression of M_t. Notice that

$$Y_t = \int_{-\infty}^{t} v \exp(-v(t-s)) c_{s,t} ds \tag{6}$$

$$= \int_{-\infty}^{t} v \exp(-(v+\rho)(t-s)) c_{s,s} \frac{M_s}{M_t} ds \tag{7}$$

We can rearrange it to get the expression of M_t:

$$M_t = \int_{-\infty}^{t} v \exp(-(v+\rho)(t-s)) c_{s,s} M_s ds \frac{1}{Y_t} \tag{8}$$

If we do the differentiation of M_t:

$$\frac{dM_t}{M_t} = \frac{dF_t}{F_t} - \frac{dY_t}{Y_t} \tag{9}$$

$$= (\frac{vC_{t,t}}{Y_t} - (\rho + v) - \mu_Y) dt \tag{10}$$

Define $\beta_t = \frac{C_{t,t}}{Y_t}$ as the consumption share of new born agents. Then the interest rate is $r_t = (\rho + v) + \mu_Y - v\beta_t = \rho + \mu_Y + v(1 - \beta_t)$

Proposition 2: *The interest rate is* $r_t = (\rho + v) + \mu_Y - v\beta_t = \rho + \mu_Y + v(1 - \beta_t)$

The interpretation of $\mu_Y + v(1 - \beta_t)$ is that if the new born agents consume more than the endowment, then effective growth rate is $\mu_Y + v(1 - \beta_t)$. But in this case, $\frac{c_{t,t}}{Y_t} = \frac{(\rho+v)W_{t,t}}{(\rho+v)W_t} = \frac{W_{t,t}}{W_t}$.