

OLG Model in Asset Pricing

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There are multiple versions of overlapping generation model (hereafter "OLG"). Asset pricing literature uses Blanchard-Yaari perpetual youth model the most. A typical feature is that the probability of death conditional on the agent's age is a constant, which is not realistic. But it fits the general population demographic features well.

If there is an instantaneous probability v of dying, the probability (as viewed by a person alive in period s) of still being alive (not dead) in period t is $\exp(-v(t-s))$. Each instant there is a mass of v entrants coming into the economy.

Proposition 1: *The mass of investors is a constant 1.*

Proof. Suppose the mass of investors at time t is m . At time $t + dt$, $m(1 - \exp(-vdt))$ mass of existing investors exit the market and v mass of investors enter the market. When the economy is stationary, $m(1 - \exp(-vdt)) \approx m * vdt = vdt$. Then $m = 1$.

Another way of proof is like this: at time t , the mass of investors equals to those who can live until time t : $\int_{-\infty}^t v \exp(-v(t-s))ds = v$ □

For an investor born at time s receives endowment $y_{s,t}$ at time t . Define the aggregate endowment: $Y_t = \int_{-\infty}^t v \exp(-v(t-s))y_{s,t}ds$. For simplicity, I assume that $y_{s,t}$ is not random and equal to Y_t . Then the aggregate endowment is Y_t . Later on, I will introduce uncertainty.

A typical question for the investor born at time s is to maximize lifetime utility:

$$\max_{C_{s,t}} E_s \left[\int_s^{\tilde{\tau}} e^{-\rho(t-s)} \log(c_{s,t}) dt \right] = E_s \left[\int_s^{+\infty} e^{-(\rho+v)(t-s)} \log(c_{s,t}) dt \right] \quad (1)$$

subject to the constraint $E_s \left[\int_s^{+\infty} \exp(-v(t-s)) \frac{M_t}{M_s} c_{s,t} dt \right] = E_s \left[\int_s^{+\infty} \exp(-v(t-s)) \frac{M_t}{M_s} y_{s,t} dt \right] = E_s \left[\int_s^{+\infty} \exp(-v(t-s)) \frac{M_t}{M_s} Y_t dt \right]$ The FOC is $c_{s,t} = \exp(-\rho(t-s)) \frac{1}{\kappa_s} \frac{M_s}{M_t}$. It is straightforward that $c_{s,s} = \frac{1}{\kappa_s}$. Then we have $c_{s,t} = \exp(-\rho(t-s)) c_{s,s} \frac{M_s}{M_t}$. Substitute this equation back to the budget constraint:

$$\int_s^{+\infty} \exp(-v(t-s)) \frac{M_t}{M_s} \exp(-\rho(t-s)) \frac{1}{\kappa_s} \frac{M_s}{M_t} dt = c_{s,s} \frac{1}{v + \rho} = W_s \quad (2)$$

*I collect the notes from my advisor Christian's board writing. All the errors are my own.

where $W_{s,s} = E_s \left[\int_s^{+\infty} \exp(-v(t-s)) \frac{M_t}{M_s} Y_t dt \right]$. Then we have $c_{s,s} = (\rho + v)W_{s,s}$. Then $c_{s,t} = \exp(-\rho(t-s))c_{s,s} \frac{M_s}{M_t} = \exp(-(\rho+v)(t-s)) \int_s^{+\infty} \exp(-v(k-t)) \frac{M_k}{M_t} y_k dk = \exp(-(\rho+v)(t-s))W_{s,t}$.

The ratio of the endowment to wealth is constant in this scenario:

$$Y_t = \int_{-\infty}^t v \exp(-v(t-s)) c_{s,t} ds \quad (3)$$

$$= \int_{-\infty}^t v \exp(-v(t-s)) (\rho + v) W_{s,t} ds \quad (4)$$

$$= (\rho + v) \int_{-\infty}^t v \exp(-v(t-s)) W_{s,t} ds = (\rho + v) W_t \quad (5)$$

To fully solve the model, we need to get the expression of M_t . Notice that

$$Y_t = \int_{-\infty}^t v \exp(-v(t-s)) c_{s,t} ds \quad (6)$$

$$= \int_{-\infty}^t v \exp(-(v+\rho)(t-s)) c_{s,s} \frac{M_s}{M_t} ds \quad (7)$$

We can rearrange it to get the expression of M_t :

$$M_t = \int_{-\infty}^t v \exp(-(v+\rho)(t-s)) c_{s,s} M_s ds \frac{1}{Y_t} \quad (8)$$

If we do the differentiation of M_t :

$$\frac{dM_t}{M_t} = \frac{dF_t}{F_t} - \frac{dY_t}{Y_t} \quad (9)$$

$$= \left(\frac{vC_{t,t}}{Y_t} - (\rho + v) - \mu_Y \right) dt \quad (10)$$

Define $\beta_t = \frac{C_{t,t}}{Y_t}$ as the consumption share of new born agents. Then the interest rate is $r_t = (\rho + v) + \mu_Y - v\beta_t = \rho + \mu_Y + v(1 - \beta_t)$

Proposition 2: *The interest rate is $r_t = (\rho + v) + \mu_Y - v\beta_t = \rho + \mu_Y + v(1 - \beta_t)$*

The interpretation of $\mu_Y + v(1 - \beta_t)$ is that if the new born agents consume more than the endowment, then effective growth rate is $\mu_Y + v(1 - \beta_t)$. But in this case, $\frac{c_{t,t}}{Y_t} = \frac{(\rho+v)W_{t,t}}{(\rho+v)W_t} = \frac{W_{t,t}}{W_t}$.