Target Capturability Using Agents in Cyclic Pursuit

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In the literature, several variants of the conventional cyclic pursuit law have been discussed. In this paper, one such variant, a modified heterogeneous cyclic pursuit scheme, has been proposed to capture a moving target. As a special case, when the target is stationary, the problem of capturing the target becomes the same as the rendezvous problem. The control laws proposed here ensure that such fixed targets (points) can always be collectively captured (reached) and a maneuvering target can be captured, provided that the agents can rendezvous at its initial position. Agents with double-integrator dynamics have also been considered, and a suitable cyclic pursuit law has been proposed to ensure global reachability and target capturability for bounded target maneuver. The theoretical findings are backed by simulation results.

I. Introduction

THE classical n-bugs problem has attracted considerable interest from researchers [1–3]. There have been several interesting results that have emerged from analyses of this pursuit problem. Briefly, the cyclic pursuit problem may be described as a problem where n bugs close in on each other, with each bug chasing its leader, that is, bug i chases bug $i+1$ (modulo n). Inspired by the basic concepts of cyclic pursuit, there have been several studies related to stable vehicular formations when a group of vehicles (also called agents) undergo cyclic pursuit [4,5]. The use of multi-agent systems to perform tasks such as surveillance, formation maneuvers, perimeter tracking, and simultaneous rendezvous to a point is well known [6]. In all these problems, the agents must arrive at an agreement or consensus (in position, orientation, or velocity, etc.). At the same time, this task must be achieved in a decentralized fashion. It is also well known that if the communication graph (directed or undirected) containing the agents at the vertices and their connections as the edges is a connected graph, this consensus is achievable [6]. This connectedness implies that there is a path that connects any agent i to any other agent j. The cyclic pursuit scheme is the simplest directed graph that remains connected at all times, while each agent shares information with exactly one other agent in the network. Hence, it is of interest to investigate tasks that can be achieved by using this cyclic pursuit paradigm and some of its variants.

In general, the velocity of any agent i, in cyclic pursuit, is proportional to the distance separating the agent from its leader (agent $i+1$, modulo n) and is along the vector directed from agent i to agent $i+1$. This is pictorially depicted in Fig. 1. The constant of proportionality (also called the gain) is usually chosen to be the same for all the agents (homogeneous). For positional consensus, the agents must converge to a point, at steady state, and remain there. When the gains are homogeneous, the point of convergence (rendezvous point) is the centroid of the initial positions of the agents. In [7], it was shown that by choosing heterogeneous gains for the agents, the point of convergence may be varied. It was also shown that the system will retain stability even when, at most, one of the gains is negative, subject to a lower bound. By choosing this negative gain, the set of reachable points could be significantly extended. However, some parts of the two-dimensional space are still not reachable for certain initial configurations of the agents, even with this negative gain. In [8], it was shown that by choosing suitable angles of deviation for each agent, the reachability set could be expanded further to include points that were hitherto unreachable using heterogeneous gains. However, global reachability was not achievable even with these deviations. It may be noted that, in practice, the velocity of every agent is bounded above. However, this involves saturation functions, and it has been shown in [9] that under saturation limits, if all the gains are positive, stability and consequent rendezvous of agents is guaranteed. But incorporation of such saturation terms implies that the solution of the cyclic pursuit system cannot be expressed in the closed form, and neither can stability be guaranteed when one gain is negative. The stability and the rendezvous point will then be functions of the maximum speed limit, the initial positions, and the gains. In this paper, such constraints have not been imposed.

A considerable amount of work has been done on formation control of agents in cyclic pursuit [10,11]. The focus of these works is on performing formation maneuvers about certain designated goal points in space. Several variants and modified versions of the cyclic pursuit algorithm have been used to meet this objective. If this goal point is extended to a target trajectory, then target capture is possible using cyclic pursuit. The inherent advantages of cyclic pursuit, such as minimal communication requirements to retain connectivity, can then be gainfully used to track a moving target and possibly neutralize the threat. Some researchers have combined online path-generator design methodologies with cyclic pursuit to enclose a target via geometric formations around the target [12]. Yet others have used vision-based cyclic pursuit strategies to capture a moving target in the same sense [13,14]. Some others have used a combination of unmanned air vehicles (UAVs) and unmanned ground vehicles to detect targets [15]. This certainly calls for heterogeneity among agents, as indicated in [16]. In [17], a target capturing strategy, in three-dimensional space, is outlined based on local information about the target and one leader agent. In [18], a cyclic-pursuit-based strategy is proposed to capture a target in minimum time using multiple micro air vehicles (MAVs). However, all these works consider a formation maneuver around the target as a successful capture or monitoring of the target and do not consider a salvo attack scenario to neutralize the threat.

In this paper, two problems, namely one of capturing a moving target cooperatively and the other of ensuring global reachability, are addressed. The same control law, in two different forms, is shown to successfully tackle both the problems. Instead of requiring both target velocity and position measurements at all time instants, as in [13], it is shown that velocity measurements alone suffice to capture a moving target asymptotically. Besides, unlike in [13], this paper does not use any angle of deviation. Thus, the agents follow their leader along the line of sight. This implies that the motion along the x and y directions is decoupled. The concept of capturability is defined in a manner similar to the interceptor–target problem in this paper. This implies that instead of moving in a formation around the target, as in most of the existing literature, the agents are required to ensure zero miss

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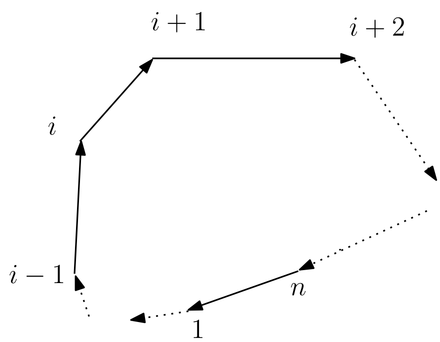


Fig. 1 Cyclic pursuit scheme.

distance. Of course, for asymptotic convergence, this implies an infinite time engagement. However, for all practical purposes, negligible distances between target and interceptors may be considered as successful capture. The salvo attack scenario is particularly useful when one interceptor alone is incapable of capturing a target and may require multiple allies to neutralize the threat. All of them must capture the target under such circumstances. This means that they must come within a certain proximity of the target at some time to capture it. It has been shown that if the initial position of the target can be reached, then the target (performing bounded maneuver) can be captured. Thus, reachability leads to capturability because the initial position of the target can always be reached in this case.

The second problem addressed in this paper is one of global reachability. This implies that the agents must be able to rendezvous at any point in the two-dimensional space. The rendezvous problem has been addressed in some recent work, such as [19]. To address this problem, a fictitious target velocity may be incorporated in the control law to steer the agents to rendezvous at any desired point. Thus, although there is no real target, the agents perceive a fictitious target having a known velocity (which converges to zero) and initial position. These parameters (velocity and initial position), along with the gains of each of the agents, are chosen a priori. Based on these choices, the agents may reach any specified point in the two-dimensional space at steady state.

For agents with double-integrator dynamics, a control strategy has been proposed that ensures capturability of any target performing bounded maneuvers. Similarly, a modified cyclic pursuit algorithm, proposed in this paper, ensures global reachability for such agents. This part, dealing with global reachability of double-integrator agents, was initially proposed in [20].

The organization of the paper is as follows. Section II reviews some known results in cyclic pursuit that are used in this paper. Section III states and proves the main results and includes an analysis of the proposed control law for agents with single-integrator dynamics in cyclic pursuit. It also provides an analysis of the control law proposed for agents with double-integrator dynamics in cyclic pursuit and proves that any target can be captured using this law. In Sec. IV, simulations are provided to support the theoretical results in the previous section. Finally, Sec. V concludes the paper.

II. Background and Motivation

In this section, the mathematical frameworks for conventional homogeneous and heterogeneous cyclic pursuit are provided. Then, the definition for capturability, as used in this paper, is introduced along with the motivation behind the same.

A. Mathematical Preliminaries on Cyclic Pursuit

In conventional heterogeneous linear cyclic pursuit, where the position of agent i is given by (x_i, y_i), the kinematics are given by

$$\dot{x}_i = k_i(x_{i+1} - x_i) \tag{1}$$

$$\dot{y}_i = k_i(y_{i+1} - y_i) \tag{2}$$

Because the kinematics are identical and decoupled along the two directions, it is sufficient to consider motion along one direction only. Therefore, throughout this paper, only the kinematics along the x direction has been considered. The same results hold along the y direction. When $k_i = k > 0, \forall i$, it is the case of homogeneous cyclic pursuit. For this case, the kinematics may then be written as

$$\dot{x} = Ax \tag{3}$$

$$A = \mathrm{circ}(-k, k, 0, \ldots, 0); \qquad A \in R^{n\times n}, \quad x \in R^n \tag{4}$$

where circ(., . . . , .) denotes a circulant matrix, whose elements can be specified by those of the first row only. In other words, if the first row of a circulant matrix of order $n \times n$ is given by $(a_1 a_2 \ldots a_n)$, the second row would be a right-shifted and folded back version of the first, such as $(a_n a_1 \ldots a_{n-1})$, and so on with each subsequent row. A typical circulant matrix is shown next:

$$C = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ a_n & a_1 & \ldots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \ldots & a_1 \end{pmatrix} \tag{5}$$

The characteristic equation of this system is given by

$$(s + k)^n - k^n = 0 \tag{6}$$

The roots of this equation are

$$p_r = -k + ke^{j(2\pi/n)r}; \qquad r = 0, 1, 2, ..., n-1 \tag{7}$$

Thus, for positive values of k, the system has only one root at the origin and the others in the left half of the complex plane (LHP). It has been proved in [4] that the root at the origin ensures positional consensus of the agents and can be disregarded for stability analysis.

In [7], it has been shown that even if the gains corresponding to each agent were chosen to be different as in Eqs. (1) and (2), but positive, the system will still retain stability. The proof is based on Gershgorin's theorem [21], which states that the eigenvalues of a matrix must lie within the union of the Gershgorin discs. Each Gershgorin disc corresponds to a row (or column) of a matrix, with its center at the diagonal and radius equal to the sum of the absolute values of the non-diagonal elements of the corresponding row (or column). The typical Gershgorin discs for an nth-order cyclic pursuit system, with all positive (but heterogeneous) gains, as in Eqs. (1) and (2), is shown in Fig. 2. In [7], it has also been shown that even if at most one gain (say k_p) were negative and bounded below by the relation

$$k_p > -\frac{\prod_{i\neq p} k_i}{\sum_i \prod_{l\neq i,p} k_l}$$

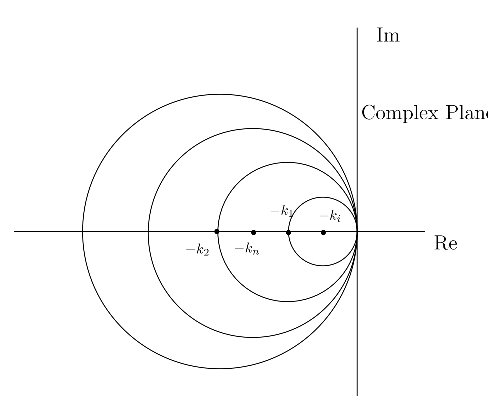


Fig. 2 Gershgorin discs corresponding to the rows of an *n*th-order system.

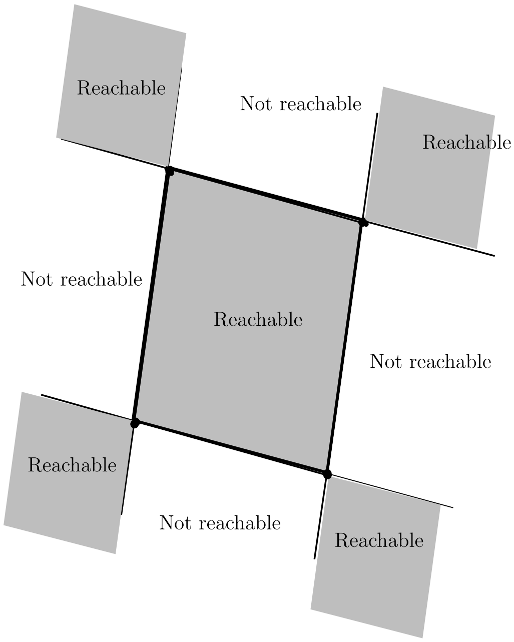


Fig. 3 Reachable regions in heterogeneous cyclic pursuit.

the system remains stable. This negative gain was used to rendezvous at points outside the convex hull of the initial coordinates of the agents. In general, it was shown that the point of convergence, (X_f, Y_f) is given by

$$X_f = \frac{\sum_{i=1}^{n} x_{i0}/k_i}{\sum_{i=1}^{n} 1/k_i}, \qquad Y_f = \frac{\sum_{i=1}^{n} y_{i0}/k_i}{\sum_{i=1}^{n} 1/k_i} \tag{8}$$

where k_i is the gain corresponding to agent i, and (x_{i0}, y_{i0}) is the initial position of the same agent. It had been remarked in [7] and explicitly proved with an example in [8] that certain regions in the two-dimensional space cannot be reached even with this negative gain. As an example of such a region, the unshaded portions in Fig. 3 may be observed. The agents are assumed to be initially located on the vertices of a convex polygon, as shown by the dots in Fig. 3. Any point within the unshaded portion of Fig. 3 cannot be a point of convergence for the agents [7] and is hence unreachable.

B. Capturability

For the tracking of moving targets using cyclic pursuit, formation control is generally used to enclose a target. It is assumed that the position and velocity of the target are available to, or can be measured by, each agent. However, the agents cannot proceed directly toward the target based on this information because they would then reach the target at widely varying instants of time and cannot move around the target in formations. Thus, there would be a lack of coordination among the agents. Some approaches based on coordination variable architecture are available in the literature to tackle this type of formation control problem [22] and simultaneous arrival or loose and tight sequencing among agents [23]. A time-critical cooperative control strategy for path following by a group of UAVs is presented in [24]. However, no target is considered in [24]. In [17], some momentary occlusion of target for some agents is considered. However, position and angle measurements of the agents are required [17]. In some cases, the velocity measurement is also required [13]. Much of the work related to target tracking has focused on monitoring the trajectory of the target based on local measurement where the agents are mostly required to travel in uniformly spaced circular or polygonal formations around the target.

In this paper, the agents converge on the target in such a way that the target is either always within the convex hull of the agents' positions (if the target starts from inside the convex hull of the agents' initial positions) or the distance between the aforementioned convex hull and the target shrinks, just as the convex hull itself shrinks with time (if the target starts from outside the convex hull of the agents' initial positions). The information about the target's maneuver is available to all the agents, as in [25], and this avoids the dependence on any implicit leader. Here, capture, implying annihilation of a hostile target, is the main objective. All of the agents may be equipped with certain resources to achieve this objective, but they might have to unite to effectively neutralize the threat posed by the target because each agent may not have sufficient resources to eliminate the threat alone [26]. It may be remarked that, though the target may be captured by each agent individually without any coordination, provided they all know the target's velocity and initial position, the advantage of using cyclic pursuit is that they will converge on the target in a coordinated fashion, with the convex hull of the agents' positions converging on the target trajectory. Moreover, the use of cyclic pursuit ensures that the time-to-go computations are not required for coordinated convergence on the target. The following definition of capture may be introduced in view of the preceding discussion.

Definition 1: Let the position of agent i and the target be given by (x_i, y_i) and (x_T, y_T) respectively, $\forall i$. The target is said to be captured by the agents 1 through n starting from time $t = t_0$; if $\forall \epsilon > 0$, there exists $t_f > t_0$, such that $|x_i(t_c) - x_T(t_c)| < \epsilon$, and $\forall t_c > t_f$.

III. Global Reachability and Target Capturability

Although the main focus of this paper is on capturability of moving targets, it turns out that the concept of reachability to a point is crucial in this context. To determine if a target is capturable, it is important to determine if the target's initial position lies within the reachable set (as illustrated in Fig. 3). This section also proposes a control law, for agents in cyclic pursuit, to ensure global reachability. Subsequently, this reachability is used to ensure the capturability of a moving target, using a suitable control law. Both single and double-integrator dynamics are considered.

A. Agents with Single-Integrator Dynamics

In the heterogeneous cyclic pursuit, the control law for agent i, with single-integrator dynamics, may be written as Eqs. (1) and (2), that is,

$$\dot{x}_i = u_i \tag{9}$$

$$u_i = k_i(x_{i+1} - x_i) \tag{10}$$

which, in compact matrix notation, can be described by

$$\dot{x} = Ax; \qquad x \in \mathbb{R}^n \tag{11}$$

$$A = \begin{pmatrix} -k_1 & k_1 & \dots & 0 \\ 0 & -k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_n & 0 & \dots & -k_n \end{pmatrix} \tag{12}$$

It has been established that if $k_i > 0$, $\forall i$, then all eigenvalues of A lie in the LHP, except for one that lies at the origin. Even with one negative gain (bounded below), this condition can be ensured. This, in turn, leads to stable dynamics and positional consensus, with the point of convergence being given by Eq. (8).

In this paper, the control law u_i, as given by Eq. (10), is augmented by an additional term that represents the velocity of the moving target to be captured. The control law is thus given by

$$u_i = k_i(x_{i+1} - x_i) + \dot{x}_T \tag{13}$$

where $\dot{x}_T$ denotes the velocity of the target. Throughout this paper, capture is defined in accordance with definition 1. It may be remarked that the control law in Eq. (13) requires only relative velocity (with respect to the target) information of the agents, and this can be obtained using a Doppler radar. If the agents were required to achieve certain formations around the target, instead of capturing the same as defined in definition 1, then Eq. (13) could be modified to $u_i = k_i(x_{i+1} - x_i + d_{x_i}) + \dot{x}_T$, with $[k_1 d_{x_1} k_2 d_{x_2} \ldots k_n d_{x_n}]^T$ belonging to the column space of the system matrix A of Eq. (12). A similar reasoning holds along the y direction. The d_{x_i} and d_{y_i} serve as relative spacing terms between the agents. An optimal choice of these terms and the gains that ensures minimal distance of each agent from the target may be an interesting future direction of research. However, in this paper, such spacing is not considered.

1. Tracking a Moving Target

Consider a choice of states given by $e_i = x_i - x_T$, $\forall i$. This is the distance between the agent i and the target. It must be emphasized that this choice is made only for the purpose of analysis and that no information about the target's current position is required in the control law given by Eq. (13). It will be shown that only the initial position of the target needs to be known for successful capture. The subsequent system, upon combining Eq. (13) with Eq. (9), may then be written as

$$\dot{e}_i = k_i(e_{i+1} - e_i) \Rightarrow \dot{e} = Ae, \qquad e \in \mathbb{R}^n \tag{14}$$

with A given by Eq. (12). With the choice of states proposed previously, it may be seen that the system given by Eqs. (11), (12), and (14) is effectively the same. Hence, the results of [7] may be carried over and applied to the new states e_i. The following theorem may now be stated.

Theorem 1: Consider a system of agents with single-integrator dynamics given by Eq. (9). If the initial position of the target is reachable using the control law given by Eq. (10), the target can be captured using the modified cyclic pursuit law, with heterogeneous gains, given by Eq. (13).

Proof: At steady state, when $\dot{e}_i = 0$, $\forall i$, $e_i = e_j$, $\forall i$, $j \in 1, 2, 3, \ldots, n$. This is because the vector $[1 \quad 1 \quad 1 \quad \ldots \quad 1]^T$ belongs to the null space of A, thereby implying an equilibrium set for the given system. This may be verified from the theory of differential equations [27]. Let $e_i = e_f$, $\forall i$ at steady state, that is $\lim_{t_f \to \infty} e_i(t_f) = e_f$, $\forall i$. Now, summing over $\dot{e}_i/k_i$ (with index determined as modulo n) and integrating from $t = 0$ to $t = t_1 > 0$ yields the following expression:

$$\sum_{i=1}^{n} \frac{e_i(t_1)}{k_i} = \sum_{i=1}^{n} \frac{e_i(0)}{k_i} \tag{15}$$

Because of the stability of the matrix A in Eq. (14), it may be concluded that, for a $\delta > 0$ and $t_l = 0 + l\delta$, $|e_i(t_l) - e_f| \to 0$ as $l \to \infty$, $\forall i$. Now, if the initial position of the target is reachable, this implies that there exists a set of gains k_i such that

$$\frac{\sum_{i=1}^{n} e_i(0)/k_i}{\sum_{i=1}^{n} 1/k_i} - e_T(0) = 0 \tag{16}$$

where $e_T(0) = x_T(0) - x_T(0) = 0$, by definition. On substitution in Eq. (15), it leads to the following condition:

$$\sum_i \frac{e_i(t_1)}{k_i} = 0, \qquad \forall t_1 > 0 \tag{17}$$

Using the facts that $|e_i(t_l) - e_f| \to 0$ as $l \to \infty$, $\forall i$, and $\sum_i 1/k_i \neq 0$ in Eq. (17), it is apparent that

$$e_f = 0 \tag{18}$$

Clearly, this is ensured by the reachability condition in Eq. (16). Now, given any $\epsilon > 0$, there exists N_i, for any agent i, such that $|e_i(t_l) - e_f| < \epsilon$, $\forall l > N_i$. Let $N = \max_i N_i$ and $t_f = 0 + N\delta$. Hence, $\forall t_c > t_f$, $|x_i(t_c) - x_T(t_c)| < \epsilon$, $\forall i$, which is the definition of successful capture. Similar arguments hold along the y direction, hence the proof. □

To meet the condition $X_f = x_T(0)$, with X_f as defined in Eq. (8), one requires information about the target's initial position so as to choose the gains k_i for the agents. This initial position is not used otherwise as part of the control law explicitly. Hence, the control law is dependent on the target's actual velocity measurements $\dot{x}_T(t)$ only.

The analysis carried out here is similar to the one in [13]. However, the system matrix is different here because it admits heterogeneous gains. Moreover, the control law in [13] required a damping term for stable formation, thereby necessitating the measurement of target position.

2. Global Reachability

Now, a strategy for global reachability is proposed for agents with single-integrator dynamics. To this end, a phantom target is used. The same control law, as suggested in Eq. (13), may be used to ensure global reachability. Although (X_f, Y_f), as defined in Eq. (8), may not include the entire two-dimensional space, for all possible permissible values (from the point of view of stability) of the gains k_i, $i = 1, 2, \ldots, n$, and initial positions of the agents, a suitable choice of $\lim_{t_f \to \infty}(x_P(t_f) - x_P(0))$, when added to X_f, can cover the entire two-dimensional space. This depends on a choice of gains k_i and a continuous function $\dot{x}_P(t)$ so that $\lim_{t_f \to \infty} x_P(t_f)$ exists and is bounded. Note that $\dot{x}_P(t)$ in this case is an assumed function of time with certain properties and has nothing to do with a real target. One class of functions that can be chosen is the exponential functions with negative exponents. This term, $\dot{x}_P(t)$, acts like the velocity of a phantom target and results in the agents pursuing this phantom target. This, in turn, ensures that the agents converge on the desired point of rendezvous while approaching each other as well. This may be interpreted in the following way. Just as the agents capture a real target, here the agents pursue a phantom target and the information about the phantom's velocity results in the agents converging to any desired point, depending on the choice of this velocity. In view of this discussion, the following result may now be formalized as a theorem.

Theorem 2: Consider the multi-agent system of agents with single-integrator dynamics given by Eqs. (9) and (13). With a suitably chosen set of gains k_i and a continuous function $\dot{x}_p(t)$, the agents can rendezvous at any arbitrary prespecified point.

Proof: Consider the required point of rendezvous to be $(X, Y))$. The proof will be illustrated only for the x coordinate because a similar reasoning holds in the y direction as well. There always exists a choice of gains (which could be either heterogeneous or homogeneous) such that the agents rendezvous at a point (X_f, Y_f) inside the convex hull of the initial positions of the agents, as per Eq. (8). Let $\dot{x}_P = ae^{bt}$, $a \neq 0$, $b < 0$ be the velocity of the phantom target. Thus, $\lim_{t_f \to \infty}(x_P(t_f) - x_P(0)) = -a/b$. If a and b are so chosen that $X - X_f = -a/b$, then from Eq. (18) it follows that $X = x_f$. This implies rendezvous at the desired point. Thus, global reachability is ensured. □

Thus, the agents with single-integrator dynamics can be made to converge to a desired point. The velocity function could be chosen in many different ways, only one of which is used in the preceding proof. The choice of a and b are not unique. As long as $-a/b$ equals the difference between desired rendezvous point (X) and the rendezvous point obtained without the use of the phantom velocity (the point X_f) according to Eq. (8), the agents will converge to X, the desired point. Naturally, the choices of a and b also depend on the choice of gains because the gains in turn determine X_f.

An interesting observation follows. Even if the initial coordinates of the target $(x_T(0), y_T(0))$ are not reachable using heterogeneous cyclic pursuit in Eq. (10), the target can still be captured by adding a phantom velocity to the actual target velocity. Thus, $\dot{x}_T$, the actual target velocity, can be added to $\dot{x}_P$, which is the phantom velocity. The role of this phantom is to ensure that $x_T(0)$ is reachable. In other words, $x_T(0) - X_f = -a/b$.

B. Agents with Double-Integrator Dynamics

The control laws presented thus far in this paper have dealt only with agents having single-integrator dynamics. In these agents, the commanded input is the velocity. However, one may also consider agents with double-integrator dynamics, where the commanded input is the acceleration.

A control law for agents with double-integrator dynamics where the motion of agent i (modulo n) is given by

$$\dot{z}_i = v_i \tag{19}$$

$$\dot{v}_i = u_i \tag{20}$$

$$u_i = k_i(z_{i+1} - z_i) - 2q_i v_i; \qquad k_i > 0 \quad \forall\, i \tag{21}$$

where $z_i \in \mathbb{C}$ denotes the position of agent i, and $v_i \in \mathbb{C}$ is its velocity. The damping term q_i ensures that the agents come to rest. Without the damping term, the agents reach a consensus in position, but they do not attain zero velocity. This implies that the agents converge to a point but do not come to rest there, but keep moving. However, for rendezvous, the agents are required to come to rest. This is achieved by the damping term. As with conventional cyclic pursuit, where each agent has information about only one of its neighbors (depicted in Fig. 1), here also the same scheme of information exchange holds. Because the motion along the x and y directions is decoupled and similar, it suffices to replace z_i by $x_i \in \mathbb{R}$ and v_i by $v_{i_x} \in \mathbb{R}$ in Eqs. (19) and (21), for the purpose of investigation. The same governing equation is valid along the y direction. The system equation can thus be written in compact form as

$$\dot{w} = Aw, \qquad w \in \mathbb{R}^{2n}, \qquad A \in \mathbb{R}^{2n\times 2n} \tag{22}$$

$$w = [x_1\ x_2\ \dots\ x_n\ \ v_{1_x}\ v_{2_x}\ \dots\ v_{n_x}]^T \tag{23}$$

$$A = \begin{pmatrix} 0 & I_n \\ P & Q \end{pmatrix} \tag{24}$$

$$P = \begin{pmatrix} -k_1 & k_1 & \dots & 0 \\ 0 & -k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_n & 0 & \dots & -k_n \end{pmatrix} \tag{25}$$

$$Q = \mathrm{diag}(-2q_1 - 2q_2 \dots - 2q_n) \tag{26}$$

where $\mathrm{diag}(.,\ \dots,.)$ represents the elements along the diagonals of a diagonal matrix. The choice of $q_i = \sqrt{k_i}$ as a damping term ensures that a system of order $2n$ can be characterized by n parameters only, and this makes the analysis tractable. Furthermore, these n parameters (that is, $k_i, \forall i$) are sufficient to ensure rendezvous at any desired point in the two-dimensional space, as will be illustrated later. To investigate the stability of the control law proposed previously, the characteristic polynomial corresponding to the system given by Eqs. (22–26) needs to be studied. Let $r(s)$ be the characteristic polynomial given by

$$r(s) = \det(sI_{2n} - A) = \det\begin{pmatrix} sI_n & -I_n \\ -P & sI_n - Q \end{pmatrix} \tag{27}$$

Because the blocks in the first row block of $(sI_{2n} - A)$ commute, from [28], the expression for $r(s)$ can be written as

$$r(s) = \det[(sI_n - Q)sI_n - PI_n] = \prod_i (s + \sqrt{k_i})^2 - \prod_i k_i \tag{28}$$

From Eq. (28), it is obvious that the characteristic equation $r(s) = 0$ has a root at the origin. If the nullity of A is unity, it implies that the nontrivial null space of A is spanned by the vector $[\underbrace{11\dots1}_{n}\ \ \underbrace{00\dots0}_{n}]^T$ because A has only one eigenvalue at the origin. Furthermore, if all the other eigenvalues of A are in the open left half-plane, then, by the same arguments as presented in [4,7], the system is stable, and positional consensus will be reached asymptotically. From the null vector, it can also be deduced that the velocities of the agents will be zero at steady state. To investigate the stability of A, another matrix having the same characteristic equation, $r(s)$ as in Eq. (28), given by $H \in \mathbb{R}^{2n\times 2n}$ is considered:

$$H = \begin{pmatrix} -\sqrt{k_1} & \sqrt{k_1} & 0 & \dots & 0 \\ 0 & -\sqrt{k_1} & \sqrt{k_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\sqrt{k_n} & \sqrt{k_n} \\ \sqrt{k_n} & 0 & \dots & \dots & -\sqrt{k_n} \end{pmatrix} \tag{29}$$

The fact that H has the same characteristic equation, $r(s)$, can be verified by the principle of mathematical induction by checking for $n = 1, 2$ and subsequently assuming the pattern to hold for $n = m$; it is verified that it holds for $n = m + 1$. Upon a direct application of Gershgorin's theorem (stated in [21]), it is apparent that, except for one eigenvalue at the origin, all the other eigenvalues of H must lie within the open left half-plane. Because H and A have the same characteristic equations, their eigenvalues must be the same. Thus, the matrix A must also have exactly one eigenvalue at the origin and all other eigenvalues in the open left half-plane. Therefore, stability of the system is guaranteed with the control law proposed in Eq. (21). This ensures asymptotic rendezvous. In view of the preceding discussion, the following theorem may now be stated.

Theorem 3: Consider the system of double-integrator agents given by Eqs. (22–26) with the control law as in Eq. (21). The system has exactly one eigenvalue at the origin and all other eigenvalues in the open left half-plane, thereby ensuring stability and positional consensus of the multi-agent system.

Proof: Consider the matrix H in Eq. (29). Corresponding to each row of H, a Gershgorin disc may be constructed with its center at $(-\sqrt{k_i}, 0)$ and radius equal to $\sqrt{k_i}$ as shown in Fig. 4. Because each gain k_i appears in two rows, there are effectively n distinct discs at most (assuming all the gains are different). Now, by Gershgorin's theorem [21], all the eigenvalues of the matrix H must lie within the union of these discs. Because all these discs lie within the left half-plane (except for the origin), the eigenvalues of H must be in the left half-plane, too, and thus the characteristic equation of H must have all its roots in the same region. This same characteristic equation $r(s)$ is shared by the matrix A, and so the same conclusions can be drawn about the eigenvalues of A. An inspection of $r'(s)$, the derivative of $r(s)$ with respect to s, reveals that it does not have a root at zero. Thus, $r(s)$ has only one root at the origin and no repeated roots there. According to [27], each coordinate of the solution to the initial value problem [Eq. (22)] is a linear combination of functions of the form $t^k e^{a_i t} \cos b_i t$ or $t^k e^{a_i t} \sin b_i t$, where $\lambda_i = a_i + jb_i$ is an eigenvalue

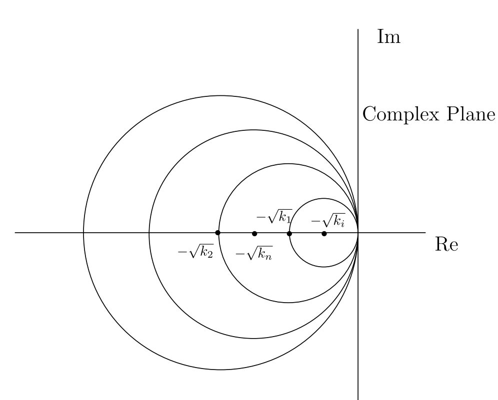


Fig. 4 Gershgorin discs corresponding to each row of H.

of the matrix A, and k ($0 \le k \le n-1$) is the algebraic multiplicity of the eigenvalue λ_i. Thus, all these components will decay at steady state because $a_i < 0$, $\forall i$, except for the component corresponding to the zero eigenvalue. Hence, the null vector (eigenvector corresponding to the zero eigenvalue) determines the steady-state behavior. From the null vector $[\underbrace{11\ldots1}_{n}\ \underbrace{00\ldots0}_{n}]^T$, it is clear that, at steady state, positional consensus is achieved with zero velocity. This is also in accordance with the results in [4,7]. □

From Eqs. (19–21), the following may be written after a simple algebraic manipulation (replacing $z_i \in \mathbb{C}$ by $x_i \in \mathbb{R}$):

$$\frac{\ddot{x}_i + 2\sqrt{k_i}\dot{x}_i}{k_i} = x_{i+1} - x_i \tag{30}$$

Summing both sides, the following expression is obtained:

$$\sum_{i=1}^{n} \frac{\ddot{x}_i + 2\sqrt{k_i}\dot{x}_i}{k_i} = 0 \tag{31}$$

Integrating Eq. (31) over the interval t_0 (initial time) to t_f (final time), it is apparent that

$$\sum_{i=1}^{n} \frac{\dot{x}_i(t_f) + 2\sqrt{k_i}x_i(t_f)}{k_i} = \sum_{i=1}^{n} \frac{\dot{x}_i(t_0) + 2\sqrt{k_i}x_i(t_0)}{k_i} \tag{32}$$

Now, at $t = t_f$, $\dot{x}_i(t_f) = 0$, $\forall i$, and $x_i(t_f) = X_f$, $\forall i$, as are evident from the null vector obtained earlier. Thus, the point of convergence is given by

$$X_f = \frac{\sum_{i=1}^{n} \dot{x}_i(t_0) + 2\sqrt{k_i}x_i(t_0)/k_i}{\sum_{i=1}^{n} 2/\sqrt{k_i}} \tag{33}$$

It may be noted that this point of convergence is a sum of two terms, one of which is a point inside the convex hull of the initial positions of the agents

$$\sum_{i=1}^{n} \frac{x_i(t_0)}{\sqrt{k_i}} \Big/ \sum_{i=1}^{n} \frac{1}{\sqrt{k_i}}$$

and the other term

$$\sum_{i=1}^{n} \frac{\dot{x}_i(t_0)}{2k_i} \Big/ \sum_{i=1}^{n} \frac{1}{\sqrt{k_i}}$$

is a function of the initial velocities of the agents and the gains chosen. If the initial velocity of at least one agent may be chosen at will, it is clear that X_f may assume any value, even outside the convex hull. Hence, it is this second term that enables global reachability. The following theorem may now be stated.

Theorem 4: Consider the system of double-integrator dynamics given by Eqs. (22–26), with the control law as in Eq. (21). The system of agents can be made to converge to any point in the two-dimensional space using a suitable choice of gains k_i and initial velocities $\dot{x}_i(t_0)$ and $\dot{y}_i(t_0)$. The point of convergence (X_f, Y_f) is given by

$$X_f = \frac{\sum_{i=1}^{n} x_i(t_0)/\sqrt{k_i}}{\sum_{i=1}^{n} 1/\sqrt{k_i}} + \frac{\sum_{i=1}^{n} \dot{x}_i(t_0)/2k_i}{\sum_{i=1}^{n} 1/\sqrt{k_i}} \tag{34}$$

$$Y_f = \frac{\sum_{i=1}^{n} y_i(t_0)/\sqrt{k_i}}{\sum_{i=1}^{n} 1/\sqrt{k_i}} + \frac{\sum_{i=1}^{n} \dot{y}_i(t_0)/2k_i}{\sum_{i=1}^{n} 1/\sqrt{k_i}} \tag{35}$$

Proof: From Theorem 3, rendezvous of the agents is guaranteed. Thus, the point of convergence derived previously is the point where the agents rendezvous. It only remains to be shown that the entire two-dimensional space can be reached. Consider a desired point $(X_d,\ Y_d)$ where the agents must rendezvous. Let the point of convergence with zero initial velocities of all agents be given by (X_1, Y_1) where

$$X_1 = \sum_{i=1}^{n} \frac{x_i(t_0)}{\sqrt{k_i}} \Big/ \sum_{i=1}^{n} \frac{1}{\sqrt{k_i}}$$

and

$$Y_1 = \sum_{i=1}^{n} \frac{y_i(t_0)}{\sqrt{k_i}} \Big/ \sum_{i=1}^{n} \frac{1}{\sqrt{k_i}}$$

Considering motion only along the x direction (the same reasoning can be extended along the y direction), let the initial velocities of all but one agent be zero. The agent l with nonzero initial velocity may have an initial velocity given by $(X_d - X_1)\sum_{i=1}^{n}(1/\sqrt{k_i})2k_l$. □

Plugging this expression for $\dot{x}_l(t_0)$ in Eq. (34), it is easy to see that $X_f = X_d$. This completes the proof. □

The velocities $\dot{x}_i(t_0)$ and $\dot{y}_i(t_0)$ are, in general, zero if the agents start from rest. However, if the motion of the agents is divided into two phases and the agents are already in motion during the first phase, when they switch to the cyclic pursuit, the initial velocities for the latter phase are not necessarily zero. To ensure global reachability (implying that any point in the two-dimensional space can be a point of convergence of the agents), this nonzero initial velocity is important. This key idea is exploited in the following subsection.

1. Global Reachability

Here, an algorithm for ensuring global reachability of agents with double-integrator dynamics, in cyclic pursuit, is outlined, based on the system described in Eqs. (19–21). It was shown that any point on the two-dimensional space may be reached, provided at least one of the agents has a suitably chosen nonzero initial velocity. However, in practice, the agents generally start from rest, and if they have a nonzero velocity, then their position will vary in accordance with the governing equations. Thus, the initial position may not be as desired. One way of circumventing this problem is to give one of the agents a dithering motion before the start of the cyclic pursuit phase. It is proposed to split the reachability problem into two phases. In the first phase, one of the agents executes small oscillations of suitably chosen frequency about its mean position. In the next phase, the cyclic pursuit law is executed so as to rendezvous at a desired location. The algorithmic steps are outlined next.

1) Select an agent whose initial position Z_{i0}, given by (X_{i0}, Y_{i0}) is closest to the desired point Z_f, given by (X_f, Y_f) of convergence, that is, i corresponding to which $\|Z_{i0} - Z_f\|_2$ is minimum. Suppose, this corresponds to agent l, without any loss of generality.

2) Assuming the initial velocities to be zero, the set of gains may be chosen to rendezvous at a point (X_1, Y_1) arbitrarily close to the initial coordinates of agent l. Clearly, this choice is nonunique, and so the gains may be suitably scaled by the designer. As a good rule of thumb, $\sum_i (1/\sqrt{k_i}) = 1$ may be an imposed constraint. Of course, any other value of the sum would have served the purpose, too.

3) Next, the initial velocity of agent l needs to be chosen. Suppose the velocity of agent l along the x direction, before the initiation of the cyclic pursuit scheme, is given by

$$v_{lx} = A_{lx}\,\sin(\omega_{lx}t + \phi_{lx}) \tag{36}$$

This is chosen to be a sinusoid to ensure that the agent l does not drift away from its initial position. Rather, it executes a simple harmonic motion about its mean position. A similar choice is made for velocity along the y direction.

4) Depending on the permissible level of oscillations in position, the choice of the ratio A_{lx}/ω_{lx} is made because this ratio corresponds

to the maximum shift in position for the agent l about its mean position.

5) The parameters A_{lx}, ω_{lx}, and ϕ_{lx} (correspondingly A_{ly}, ω_{ly}, and ϕ_{ly}) need to be chosen. At $t = t_0$, the velocity must be given by

$$A_{lx} \sin(\omega_{lx} t_0 + \phi_{lx}) = (X_f - X_1)\left(\sum_{i=1}^{n} \frac{1}{\sqrt{k_i}}\right) 2k_l \qquad (37)$$

Thus, for a given t_0 (starting instant), A_{lx}, ω_{lx}, and ϕ_{lx} may be chosen to satisfy Eq. (37) and ensure small oscillations about the mean position of agent l.

It should be noted that this algorithm works with $n + 6$ decision variables (n gains and three terms each corresponding to velocities along x and y directions). But the system order is $2n$ due positions and velocities corresponding to the n agents. The terms ϕ_{lx} and ϕ_{ly} ensure that even if $\sin(\omega_{lx} t_0)$ or $\sin(\omega_{ly} t_0)$ are zeros, the initial velocity of agent l does not become zero. This could also be ensured by suitably modifying ω_{lx} or ω_{ly} as per requirement, but the additional terms ϕ_{lx} and ϕ_{ly} enable an independent choice of the ratio A_{lx}/ω_{lx} to ensure small oscillations. It may be remarked that other forms of velocity (besides dithering motion) will also have the same effect on the initial velocity and thus contribute to the final point of convergence. But in such cases, the initial position will also be far from the actual initial position, if the agents were at rest.

2. Tracking a Moving Target

A strategy is now proposed for agents with double-integrator dynamics, in cyclic pursuit, to ensure global capturability. The nonzero initial velocity of agents is gainfully employed therein to rendezvous at any point within the two-dimensional space. In this work, the control law in Eq. (21) is modified as follows, to include the target velocity and acceleration:

$$u_i = k_i(z_{i+1} - z_i) - 2q_i(v_i - v_T) + \dot{v}_T; \qquad k_i > 0, \quad \forall i \quad (38)$$

where v_T and $\dot{v}_T$ are the target's velocity and acceleration, respectively, and the damping term $q_i = \sqrt{k_i}$. The following theorem may now be stated.

Theorem 5: Consider the system given by Eqs. (19), (20), and (38). A target starting from any initial position can be captured using the cyclic pursuit law proposed in Eq. (38).

Proof: Let the choice of states be given by

$$e_i = z_i - z_T \qquad (39)$$

where z_i and z_T are the positions of agent i and the target, respectively, denoted by complex numbers in compact form. The system [Eqs. (19), (20), and (38)] may be rewritten as

$$\ddot{e}_i = k_i(e_{i+1} - e_i) - 2\sqrt{k_i}\dot{e}_i \qquad (40)$$

This is similar to the system in Eqs. (19–21), and hence the stability results of the system, described by Eqs. (19–21), are valid. It follows that

$$\frac{\ddot{e}_i + 2\sqrt{k_i}\dot{e}_i}{k_i} = e_{i+1} - e_i \qquad (41)$$

Summing over i and integrating over time $t = 0$ to t_f, the following expression results:

$$\sum_{i=1}^{n} \frac{\dot{e}_i(t_f) + 2\sqrt{k_i} e_i(t_f)}{k_i} = \sum_{i=1}^{n} \frac{\dot{e}_i(t_0) + 2\sqrt{k_i} e_i(t_0)}{k_i} \qquad (42)$$

Because of the stability of the system, at steady state, $\dot{e}_i(t_f) = 0$, $\forall i$as $t_f \to \infty$. Because the system results in consensus, $e_i(t_f) = e_f$, $\forall i$ as $t_f \to \infty$. This implies that $z_i(t_f) = z_T(t_f) + e_f$ at steady state or in other words, $z_i(t_f) = z_f$, $\forall i$. Upon some algebraic manipulations, and using the fact that $\dot{z}_T(0) = 0$, the following expression is obtained:

$$z_f - z_T(t_f) = \frac{\sum_i \dot{z}_i(0)/k_i}{\sum_i 2/\sqrt{k_i}} + \frac{\sum_i z_i(0)/\sqrt{k_i}}{\sum_i 1/\sqrt{k_i}} - z_T(0) - \dot{z}_T(0)\frac{\sum_i 1/k_i}{\sum_i 2/\sqrt{k_i}} \qquad (43)$$

It has been proved [20] that, using the algorithm suggested in Sec. III.B.1, a suitable choice of initial velocities of the agents makes the sum of the first three terms on the right-hand side of Eq. (43) zero, that is, the sum of the first two terms equals the target's initial position. For this, a suitable choice of any one agent's velocity suffices. Using the same reasoning, a suitable choice of initial velocity for another agent would nullify $\dot{z}_T(0)(\sum_i 1/k_i)(\sum_i 2/\sqrt{k_i})$.

This implies that, for a given $\epsilon > 0$, there exists some t_f such that $\forall t_c > t_f$, $|z_i(t_c) - z_T(t_c)| < \epsilon$, $\forall i$, by using the same line of arguments as in the proof of theorem 1. This last condition is the definition of a successful capture, hence the proof. □

If the target starts from within the convex hull of the agents' initial positions, then the nonzero velocity is not required, and merely using the heterogeneity in the gains of cyclic pursuit would suffice to capture the target for agents with double-integrator dynamics. It should be remarked that, in both theorems 1 and 5, the target velocity and acceleration should be bounded from practical considerations, that is, they should not attain infinite values within a finite time. This is because the target's velocity and acceleration appear in the control laws for the agents. Therefore, an unbounded target maneuver will imply an unbounded control signal for the agent. However, for any realistic target, these quantities are always bounded. Hence, this condition is not a limitation as such.

C. Implementability in Complex Dynamic Models

The main focus of this paper is on proposing and analyzing the reachability of the newly presented strategies. Although the implementation of the same, with complex dynamic models, is beyond the scope of this paper, it is important to discuss the feasibility of implementing these strategies in realistic systems. It is also an interesting direction for future work. It may be pointed out that, although the basic cyclic pursuit law is simple in nature, the present work considers both the single-integrator and double-integrator dynamics. Further, it has been mentioned in [29], and has also been shown in other works, that most of the major conclusions drawn about the linearized kinematic model carry forward to the nonlinear unicycle model. In a realistic setup, instead of coming to rest, the unicycles will move in a formation around the target in a circle. Note that the possibility of modifying the proposed control law in the present paper to capture the target by moving in a formation around it has been discussed in Sec. III.A. In [25], hardware-in-the loop simulation results have been presented, with realistic MAV dynamics, based on target-centric cyclic pursuit, to demonstrate the capture of fixed and slowly moving targets. In [30,31], cyclic pursuit strategies have been used for wheeled mobile robots with significant success. In particular, Marshall et al. [31] state that these pursuit-based coordination algorithms are robust in the presence of unmodeled dynamics and delays due to sensing and information processing. In [13], Ma and Hovakimyan have used the nonholonomic dynamics of wheeled robots for vision-based guidance using cyclic pursuit laws. The results presented in these and other similar works evoke confidence about the implementability of the cyclic-pursuit-based strategies presented in this paper.

IV. Simulation Results

In this section, several simulation examples are provided, which illustrate the results on global reachability and capturability of a moving target that have been presented in the preceding section. The examples contain both agents with single-integrator dynamics and double-integrator dynamics, in cyclic pursuit. It may be remarked that the choice of gains in the following simulations is not unique. Because the paper does not address saturation of the agent velocity, the choice of gains has been made arbitrarily to achieve successful rendezvous or target capture, based on the solution of an underdetermined system of linear equations obtained based on the desired rendezvous point or the target's initial position. Because the problem of determining the set of gains that ensures stability (when one gain is negative), subject to an upper limit on the agent velocity, is an open problem, it has not been addressed in these simulations. Moreover, another guideline for choosing gains could be dictated by the need to maintain some spacing between the agents. This has been briefly discussed earlier. However, such considerations have not been taken into account in this paper. Hence, the gains in most cases have been chosen arbitrarily to satisfy the underdetermined system of linear equations, based on Eq. (16) or similar equations related to the rendezvous point.

Example 1: Consider the case of four agents with single-integrator dynamics starting from the vertices of a square at (0,0), (3,0), (3,3), and (0,3). Suppose the target initially starts off from the point (1.5,1.5). This then helps us determine the gains of the agents. Clearly, the choice $k_i = 4, \forall i$, in both directions, according to Eq. (8), will cancel out the effect of the target's initial position. This also ensures that $\sum_i 1/k_i = 1$.

The agents are expected to converge on the target's trajectory asymptotically, according to theorem 1, following Eq. (13). The target is assumed to move on an expanding spiral with the components of its velocity along the x and y directions being given by $5e^t\ \cos(5t)$ and $5e^t\ \sin(5t)$, respectively. It may be seen from Fig. 5a that the agents finally capture the target. The square shows the convex hull of the initial coordinates of the agents. It should be noted that, because the target starts from within this convex hull, positive gains suffice to ensure zero miss distance. Thus, successful capture is ensured. The positional consensus of the agents in both directions is illustrated in Fig. 5b. Even if the target velocity measurement, available to one of the agents (agent 1 in this case), is noisy, successful interception occurs. This can be seen in Figs. 6a and 6b, where the level of uniform random noise is about 10% of the target velocity.

Example 2: In this example, the target's velocity is given by t^2 and $2t$ along the x and y directions, respectively, whereas the agents start from the same position as in example 1. However, the target starts from the point $(-8, -12)$, which, though outside the convex hull of the initial positions of the agents, is nevertheless reachable by the agents, using a negative gain. The agents have single-integrator dynamics and follow the control law in Eq. (13). The choice of the gains is guided by stability considerations, and they are given by $[-3 \quad 12 \quad 20 \quad 6.67]^T$. It may be seen from Figs. 7a and 7b that the agents succeed in ensuring capture. This is in accordance with the

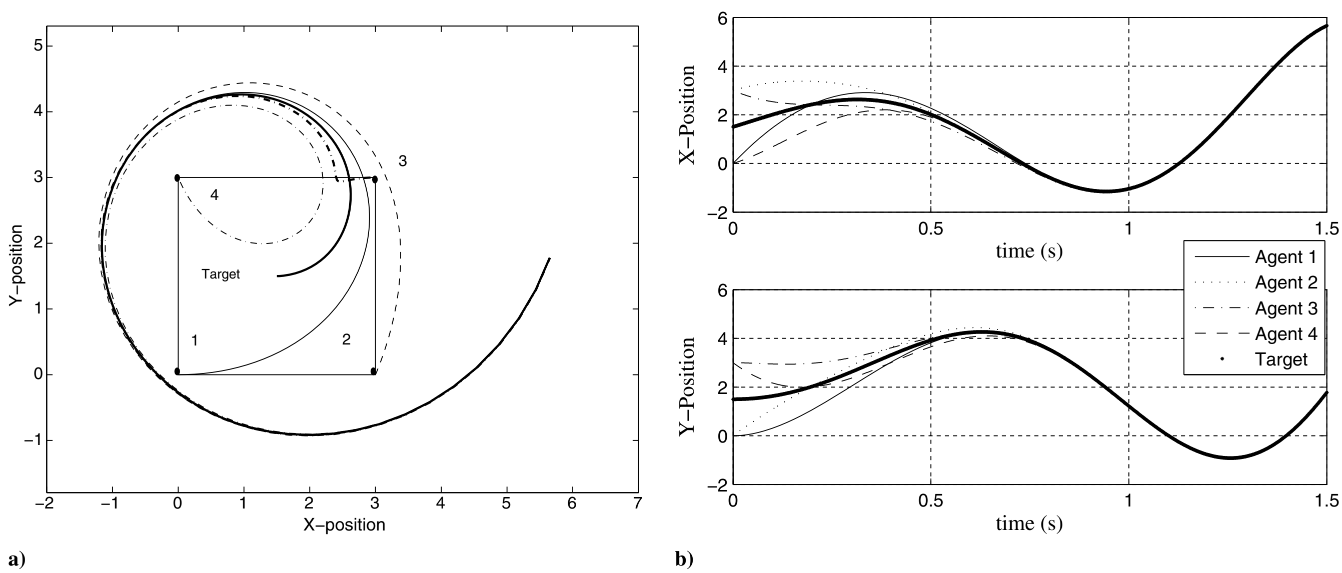


Fig. 5 Representations of a) four agents capturing a moving target in example 1, and b) positional consensus while capturing moving target (single integrator).

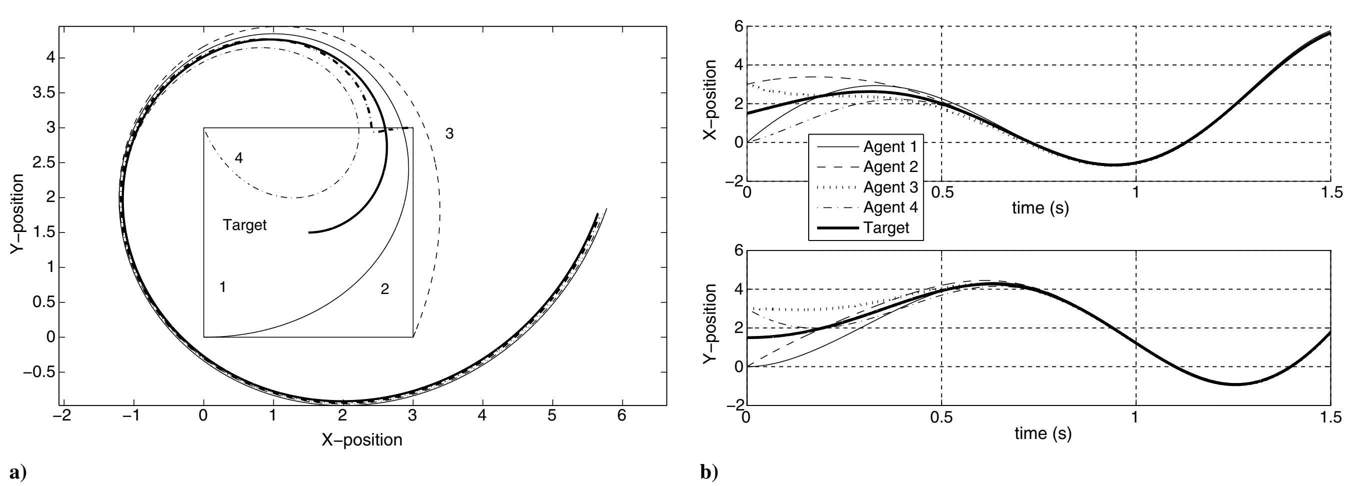
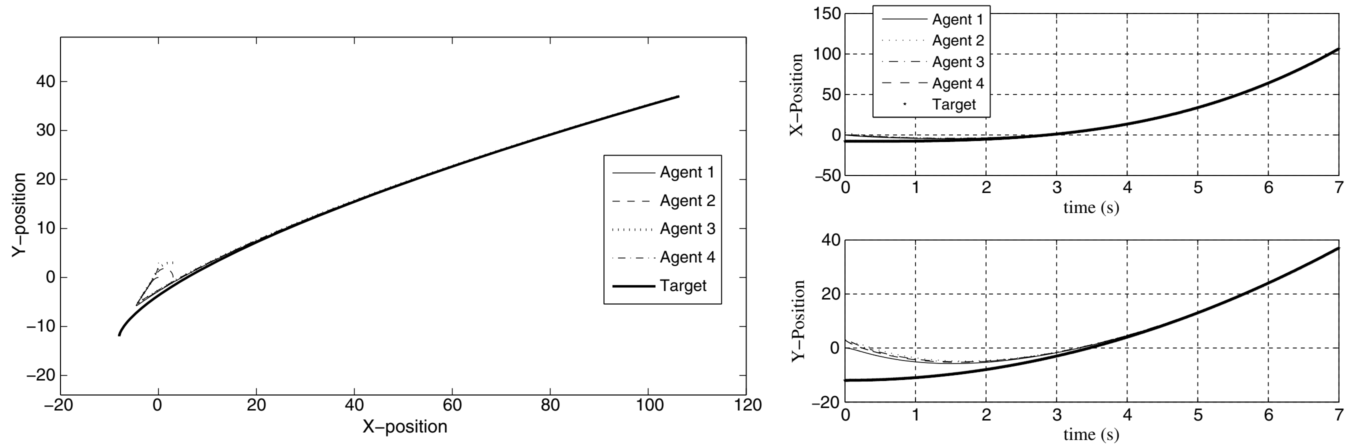


Fig. 6 a) Four agents capturing a moving target in example 1 (target velocity information for agent 1 is noisy), and b) positional consensus (single integrator).



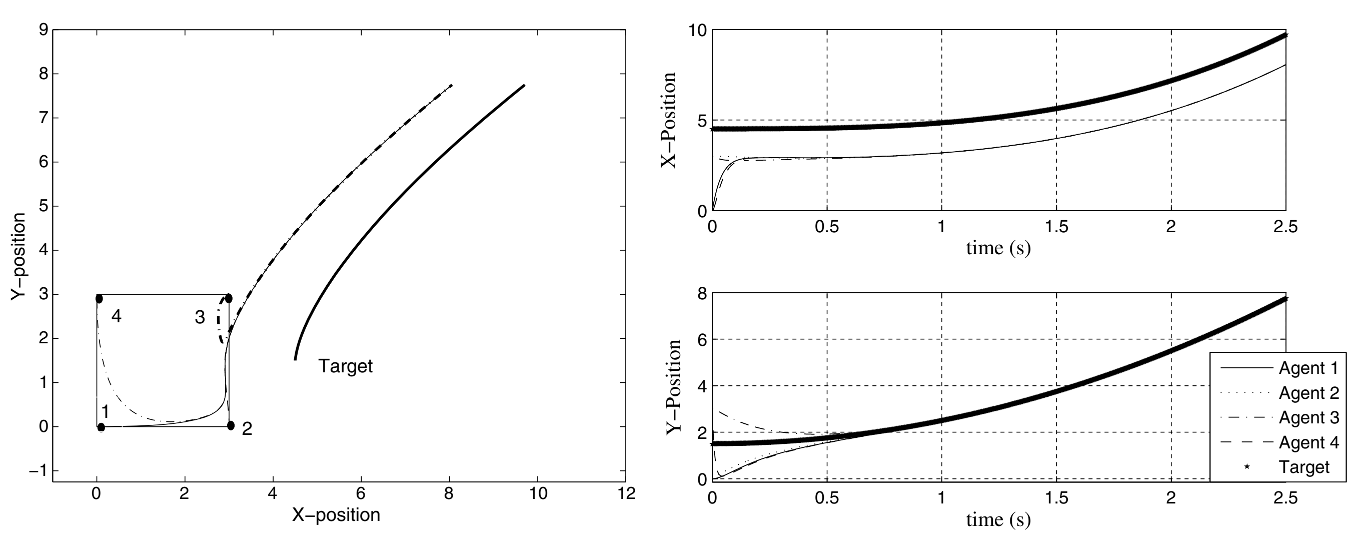
a) b)

Fig. 7 Representations of a) target starts from outside convex hull (reachable region) in example 2, and b) positional consensus in case of successful interception (single integrator).

result that if the initial position of the target is reachable, the target can be captured. The apparent sharp turn in the trajectories of the agents is due to the fact that once the agents come close to each other, their trajectories are mainly controlled by the target's velocity, rather than the purely cyclic pursuit part of the control signal.

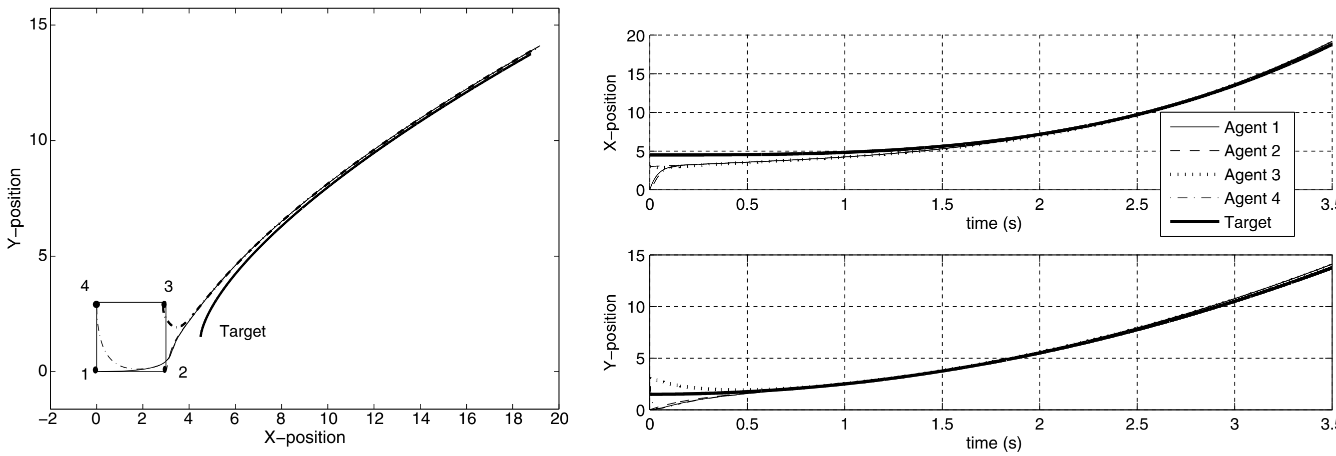
Example 3: In this example, the agents start off from the same positions as in examples 1and 2. But the target starts off from the point (4.5,1.5), and its velocities along the x and y directions are given by t^2 and $2t$, respectively. Clearly, the initial position of the target is not reachable using heterogeneous gains, even with a single negative gain, because it corresponds to the unshaded region of Fig. 3. The closest one can get to the initial position of the target is (3,1.5), unless heterogeneous deviations, as in [8], are used. For a practical choice, the gains are given by $[25 \quad 2.1739 \quad 2.0408 \quad 100]^T$, which would normally lead to a rendezvous at (2.85,1.50). The results in Figs. 8a and 8b illustrate that the miss distance in this case is close to 1.65, which is approximately equal to the minimum distance of the target's initial position from the convex hull, indicated by a square in Fig. 8a. The thick line indicates the target's trajectory. This is to be expected because the miss distance is the same as the distance between the target's initial position and the rendezvous point for the cyclic pursuit law in Eq. (8). However, by adding a suitable phantom target velocity to the actual measured target velocity, the target can be captured. This is because the initial position (4.5,1.5) can be reached by using a phantom velocity function $1.65e^{-t}$ along with the set of gains chosen earlier. Unless this phantom velocity is used, the target's initial position is unreachable, and hence in Fig. 8a, the target cannot be captured. Addition of the phantom velocity described previously implies that $-a/b = 1.65$. Now, $2.85 + 1.65 = 4.5$, which is the target's initial x coordinate. Hence, this phantom velocity $\dot{x}_P(t)$ when added to the actual target velocity $\dot{x}_T(t)$, in Eq. (13), leads to successful capture. From Figs. 9a and 9b, it may be observed that even when two agents receive noisy measurements about the target's velocity (agents 1 and 2), the target is successfully captured. The level of noise is at most 10% of the signal level and is a uniform random noise.

Example 4: In this example, the agents with single-integrator dynamics are required to rendezvous at the point (6.5,1.5) starting from the vertices of a square given by (0,0), (3,0), (3,3), and (0,3). Clearly, it may be seen from Fig. 3 and the example in [8] that the rendezvous at this point is not possible using any known cyclic pursuit strategy such as Eq. (1). In this case, the gains of all the agents are chosen to be 25. The choice could be heterogeneous, in which case the velocity function of the phantom target must be suitably chosen. Here, the velocities of the phantom target in the x and y directions are given by $5e^{-t}$ and 0, respectively. This phantom velocity $\dot{x}_P(t)$ replaces $\dot{x}_T(t)$ in Eq. (13). This choice of phantom velocity, along with the choice of gains, ensures that the agents rendezvous at the desired point because $\lim_{t_f\to\infty}\int_0^{t_f} 5e^{-t}\,\mathrm{d}t = 5$ and $\lim_{t_f\to\infty}\int_0^{t_f} 0\,\mathrm{d}t = 0$, whereas $X_f = Y_f = 1.5$ as per Eq. (8). The successful rendezvous of the agents is demonstrated in Figs. 10a and 10b.



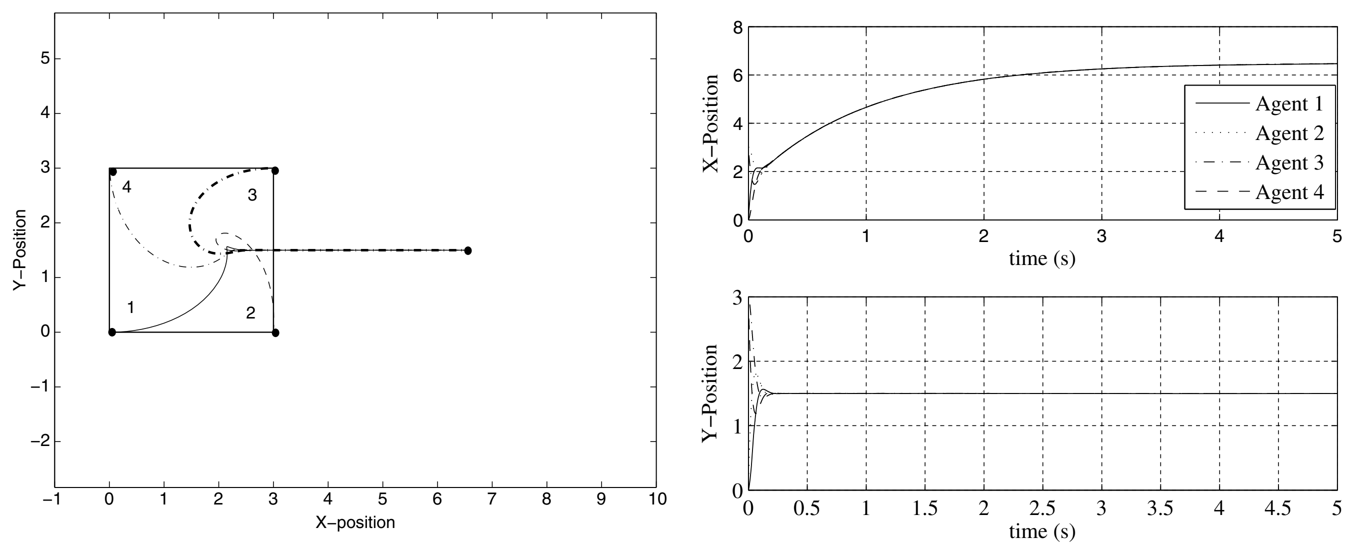
a) b)

Fig. 8 Representations of a) target starts from unreachable region in example 3, resulting in Fig. 8b, failure to capture moving target (single integrator) in the absence of phantom velocity.



a) b)

Fig. 9 a) Target captured due to global reachability (target velocity information is corrupted with noise for two agents), and b) positional consensus achieved in example 3 (using phantom velocity).



a) b)

Fig. 10 Representations of a) rendezvous at a point outside the convex hull for four agents in example 4, and b) positional consensus for four agents (single integrator).

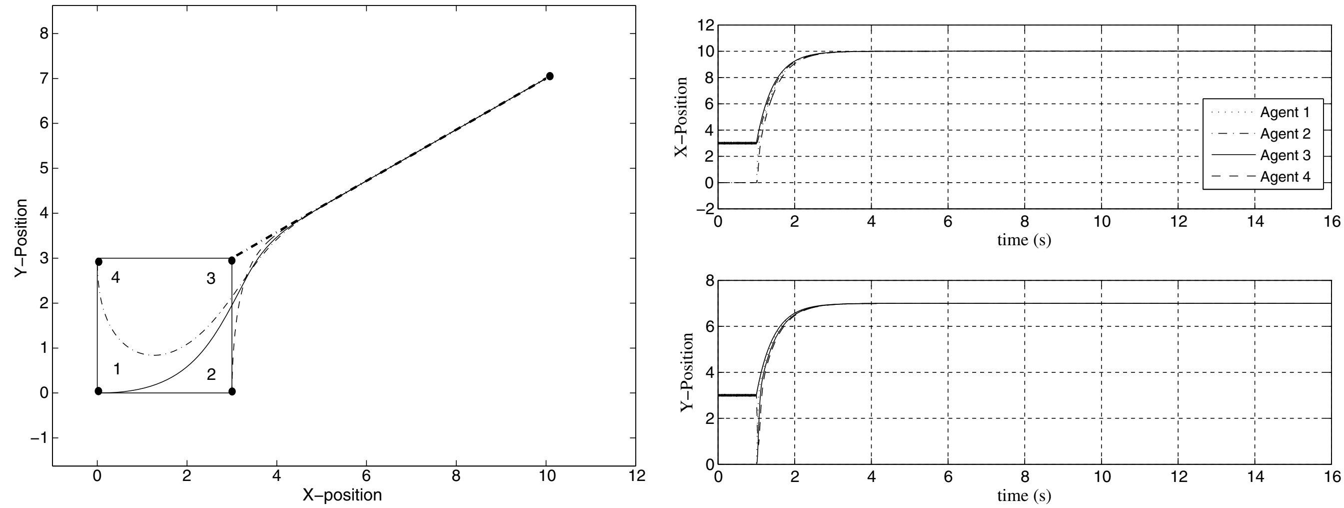
Example 5: Consider a system of four agents with double-integrator dynamics starting from the vertices of a square (0,0), (3,0), (3,3), (0,3). It is desired that the agents should rendezvous at the point (10, 7), using the control law [Eq. (21)] and the algorithm outlined in Sec. III.B.1. According to the first step of the algorithm for global reachability, agent 3 is closest to the desired point of convergence. Now, the gains are chosen assuming zero initial velocities. Thus, the point (X_1, Y_1) of step 2 may be chosen as (2.9,2.9). One choice of gains is $[3600 \quad 3600 \quad 1.108 \quad 3600]^T$, which results in $\sum_i 1/\sqrt{k_i} = 1$.

As pointed out in the previous section, the choice of the gains is not unique, and the same result may be obtained using a different choice in this example. Now, agent 3 must execute a sinusoidal motion about its initial position (3,3) before the initiation of the cyclic pursuit phase while the other agents are stationary. If the amplitude of oscillation is restricted to 0.1 for agent 3, the ratio of amplitude to frequency (A_{3x}/ω_{3x} and A_{3y}/ω_{3y}) must be less than 0.1. If $t_0 = 1$ s, it is clear that $A_{3x} = 15.734$, $A_{3y} = 9.086$, $\omega_{3x} = \omega_{3y} = 200$, and $\phi_{3x} = \phi_{3y} = \pi/2–200$ satisfy Eq. (37). This particular choice also ensures that, at $t = t_0$, the position of agent 3 is (3,3). The additional terms ϕ_{3x} and ϕ_{3y} have been used to this end. Figure 11a shows the trajectories of the four agents, which converge at the desired point. The square shown in bold lines is the convex hull of the initial positions of the agents. Figure 11b shows that, after 1 s, cyclic pursuit is initiated, and before that, only agent 3 executes small oscillations about its mean position. It may be noted that even with positive gains, it is possible to converge outside the convex hull of the initial coordinates of the agents.

Example 6: In this example (for agents with double-integrator dynamics), all conditions and design parameters are kept the same as in example 5except the value of A_{3y}, which is changed to -2.437. This results in a rendezvous at the point (10,1.8), as shown in Fig. 12. It may be noted that this point will not be reachable using conventional cyclic pursuit with single-integrator dynamics, even if a negative gain is used. This is because this point does not belong to the union of the convex hull of initial coordinates and the cones formed by them as defined in [7]. However, the present scheme enables rendezvous at any point. Here, too, the control law used is Eq. (21).

Example 7: In this example, the target's acceleration is given by $10e^{-2t}\cos(t)$ and $10e^{-2t}\sin(t)$ along the x and y directions, respectively. The target starts with zero velocity from the point (10,7), which can be reached following the control law in Eq. (38), for agents with double-integrator dynamics. The agents start from the vertices of the square, given by (0,0), (3,0), (3,3), and (0,3) respectively. The gains required for this are $[3600 \quad 3600 \quad 1.108 \quad 3600]^T$. This particular choice leads to $\sum_i 1/\sqrt{k_i} = 1$.

However, other choices are also admissible. All the agents, other than agent 3, have zero initial velocity, and the initial velocities of agent 3 along the x and y directions are 15.734 and 9.086, respectively. It may



a) **b)**

Fig. 11 Representations of a) agents converging to a desired point in example 5, and b) consensus in X and Y coordinates (double integrator).

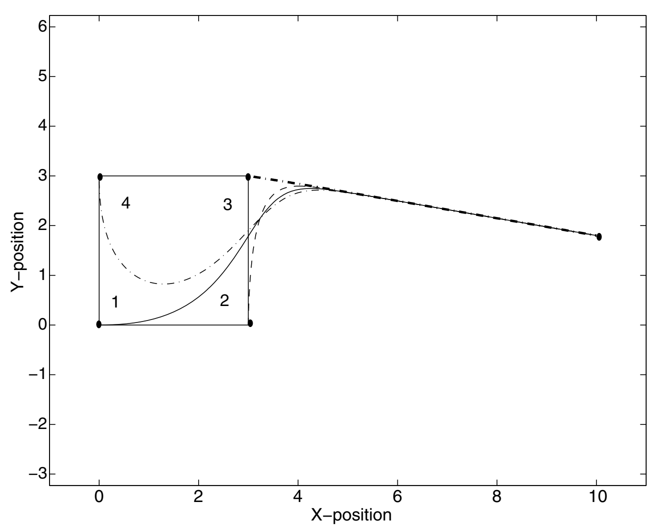


Fig. 12 Agents converging to a point outside convex hull or cones as defined in [7] in example 6 (double integrator).

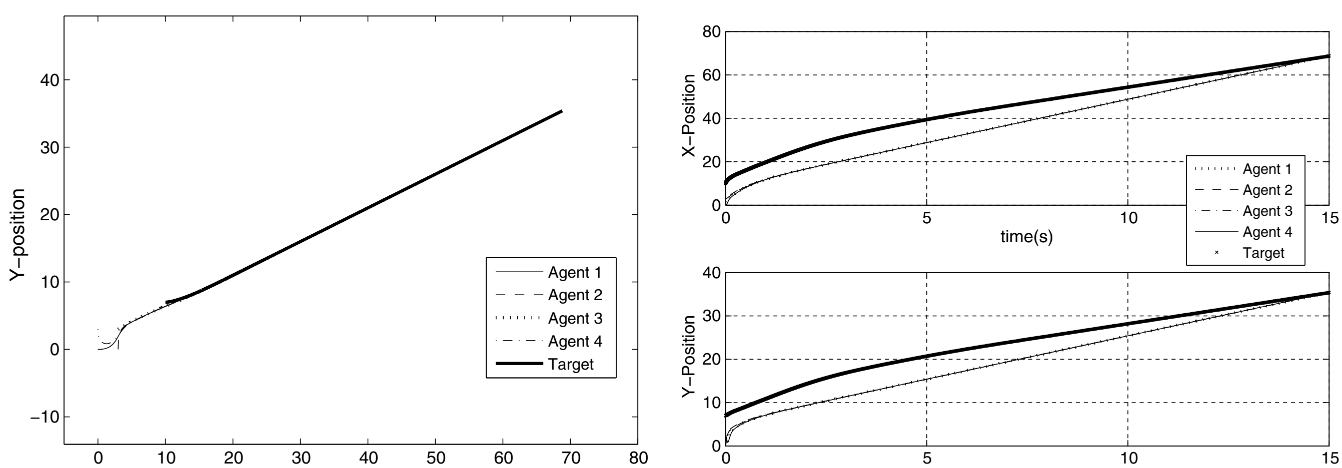
be seen from Figs. 13a and 13b that these parameters ensure successful capture of the target. This is only to be expected because, according to theorem 5, the control law in Eq. (38) ensures that the target will be captured no matter where it starts from.

V. Conclusions

In this paper, a modified cyclic pursuit scheme has been proposed and analyzed. A new notion of capturability of the target has been introduced, which does not require formation maneuvers around target points.

Collision avoidance among the agents has not been explicitly addressed here, although collisions can be avoided by choosing different elevations for the agents as the consensus is achieved only in the x and y directions. The control law has been shown to be stable in terms of achieving positional consensus. There are primarily two applications of the proposed scheme. First, the control law has been used to capture and neutralize a target provided that its initial position is reachable. Furthermore, this law has been suitably tailored to achieve global reachability of the agents, that is, the agents may rendezvous at any point in the two-dimensional space, by assuming a suitable fictitious target velocity. This global reachability is then used to ensure that a target starting from an unreachable region can also be captured. The strategy performs satisfactorily in the presence of noisy target measurements, as shown through simulations.

A control law has also been proposed for agents with double-integrator dynamics in cyclic pursuit, and this has been shown to be capable of capturing any target performing a bounded maneuver. Besides, an algorithm to ensure global reachability of agents with double-integrator dynamics has been proposed, which makes use of nonzero initial velocities of the agents.



a) **b)**

Fig. 13 Representations of a) target capture for agents with double-integrator dynamics in example 7, and b) positional consensus on target trajectory.

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