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Cluster Consensus for Nonlinear Multi-Agent Systems

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Abstract
A cluster consensus algorithm for nonlinear multi-agent systems under directed graph topology is proposed in this paper. Cluster consensus is the convergence of states/outputs of agents in the same cluster to consistent values which are different from those of other clusters. Cluster consensus has been obtained based on Lyapunov stability and matrix theory in terms of some sufficient conditions. A feedback control law is provided using Linear Matrix Inequality (LMI) to achieve cluster consensus for multi-agent systems. Moreover, cluster consensus for nonlinear multi-agent systems in the presence of time delay has been studied in this paper. Finally, simulation results are presented for different number of clusters to validate theoretical analysis. Examples are provided for first-order and second-order and also general high-order systems. Furthermore, first-order system with time delay is simulated for a single-link flexible joint manipulator.

Keywords Multi-agent systems · Cluster consensus · Nonlinear systems · General high-order systems

1 Introduction

Recently, multi-agent systems have attracted so much attention because of their applications such as consensus [1], formation [2] and flocking [3]. In the other words, consensus is one of the attractive problems in multi-agent systems which means all the agents can converge to certain quantities [4]. A kind of consensus which all the followers can track one or a group of leaders is called Leader-follower consensus [5].

In most researches, complete consensus is presented which means all agents achieve a common state [6–10]. But in the real world, there are different tasks that agents in the network may be divided into several groups for doing these. This is the concept of cluster consensus which agents agree on different values in different clusters because of the changes in the environment, situation, cooperative tasks or even time. The applications of cluster consensus are obstacle avoidance for a team of predators or animal herds, etc.

Recently, several researchers studied cluster consensus [11–16]. Cluster consensus for multi-agent systems was discussed in [12, 17, 18] and with time varying topology was proposed in [12]. Cluster consensus for linear multi-agent systems was studied in [15] where the topology is switching. Multi-agent system with fixed topology and communication delays was discussed in [19]. In [20], cluster consensus was proposed for multi-agent systems which are discrete-time.

The multi-consensus for multi-agent systems which have first-order and second-order systems under fixed topology with two balanced subnetworks was studied in [21, 22]. In [23–25], to reach consensus, some conditions for multi-agent systems have been derived via pinning control approach where the topology doesn't have the in-degree balance condition. In [26], cluster consensus for multi-agent systems which are discrete-time has been studied for both fixed and stochastic switching topologies. The multi-consensus control with pinning control techniques for multi-agent systems which are generic linear was discussed for both fixed and switching directed graph based on matrix analysis and Lyapunov stability theory in [27]. In [28], multi-consensus problem for nonlinear multi-agent systems was studied for directed graph which is weighted, and nonlinearity exists in control algorithm and systems are linear. A multi-consensus problem for the heterogeneous agents that are governed by the Euler–Lagrange system with parametric uncertainty and the double-integrator system was investigated in [29].

Most of existing works have studied cluster consensus for linear multi-agent systems. However, in this paper, the multi-agent system is nonlinear. Cluster consensus is achieved when the topologies satisfy conditions such as each cluster has a

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directed spanning tree and each cluster is balanced, which will be defined, bellow.

The main contributions of this paper are divided into three parts. First, each agent with multi-agent system model is described with general nonlinear and any order of dynamics is a special case of this general form. Second, cluster consensus is proved for general form nonlinear multi-agent systems which communication topology has a directed spanning tree and the information exchange between two clusters is balanced. Finally, time delays are considered for the cluster consensus control of nonlinear multi-agent system.

Many mechanical systems are modeled as nonlinear systems. For example, robotic systems are practical applications general nonlinear systems which are investigated in this paper.

This paper is organized as follows. In section 2, the background of graph theory has been presented. Lyapunov theory is proposed in section 3. Problem formulation and sufficient condition for cluster consensus have been studied in section 4. Simulation examples are shown in section 5. Finally, conclusion is presented in Section 6.

2 Background of Graph Theory

In this section, a brief description of graph theory is proposed. The weighted directed graph is defined with $G = (\nu, \varepsilon)$ in which $\nu = \{\nu_1, \ldots, \nu_n\}$ is the node set and $\varepsilon \subseteq \nu \times \nu$ is the edge set. $\varepsilon_{ij} = (\nu_j, \nu_i)$ is used to show the information exchange where agent i receive information from agent j. $\mathcal{A} = (a_{ij})$ is the adjacency matrix where $a_{ii} = 0$ for all i and $a_{ij} \neq 0$ if $\varepsilon_{ij} \in \varepsilon$. $\mathcal{N}_i = \{\nu_j | \varepsilon_{ij} \in \varepsilon\}$ represents the set of neighbors of agent i.

One of the important concepts in graph theory is directed spanning tree. It defines a directed graph that every node or agent has exactly one parent or in the other words, it is a directed tree that there exists a directed path from one node to other nodes.

Another important concept is the balanced directed graph (digraph) which is defined as $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ for all $i \in n$.

3 Lyapunov Candidate Function for Using Lyapunov Stability

In this section, the stability of nonlinear systems is investigated using the Lyapunov candidate function $V(t, x(t)) = \frac{1}{2} x^T(t) M\, x(t)$, where M is a symmetric positive definite matrix that will be defined. Later, we start with the Lyapunov candidate function lemma.

Lemma 1. $x = 0$ is a stable equilibrium point of $\dot{x} = f(x, t)$, $(f(0, t) = 0)$ if a differentiable function $V(t, x(t))$ exists with the following conditions which is continuous [30]:

$$\alpha_1 \|x\|^2 \leq V(t, x(t)) \leq \alpha_2 \|x\|^2, \; \dot{V}(t, x(t)) \leq -\alpha_3 \|x\|^2.$$

where $V(t, x(t)) : [0, +\infty) \times D \to R$ is the locally Lipschitz function on x, $D \in R^n$, which contains the origin; $t \geq 0$, α_1, α_2 and α_3 are any positive constants.

4 Problem Formulation

4.1 First-Order System

In this section, first-order nonlinear systems describe agents. There are n agents which the model of agent i, $i = 1, \ldots, n$, can be given as:

$$\dot{x}_i = f(x_i, t) + u_i \tag{1}$$

where x_i is the position and u_i is the control input of agent i, and $f(x_i, t)$ describes the nonlinear dynamics of agent i. A cluster consensus algorithm for system (1) is given as follows:

$$u_i(t) = k \sum_{j \in \mathcal{N}_i} a_{ij} \left[x_j(t) - x_i(t) \right] \tag{2}$$

where a_{ij}, $i, j = 1, \ldots, n$, is the (i,j) element of the weighted adjacency matrix, a_{ij} is permitted to be negative between clusters, and $k > 0$ is the feedback gain.

4.1.1 Two Clusters

In this part, cluster consensus problem is explained, where the graph of agents is divided into two parts and each part reaches a different consistent value. Consider a graph with n agents divided like $l_1 = \{1, 2, \ldots, N_1\}$, $l_2 = \{N_1 + 1, \ldots, N_1 + N_2\}$ where $n = N_1 + N_2$.

The control algorithm (2) solves the cluster consensus problem for system (1) asymptotically if

$$\lim_{t \to \infty} (x_i - x_j) = 0, \qquad \forall i, j \in l_1$$

$$\lim_{t \to \infty} (x_i - x_j) = 0, \qquad \forall i, j \in l_2$$

Some assumptions for analyzing the cluster consensus problem are expressed in this section.

Assumption 1. The Lipschitz condition for function $f(x, t)$ and the Lipschitz constant l is assumed:

$$|f(x_2, t) - f(x_1, t)| \leq l |x_2 - x_1|, \qquad \forall x_1, x_2 \in R, \forall t \geq 0$$

Assumption 2.

(a) Each subnetwork has a directed spanning tree.

(b) Two clusters graph, satisfy the following conditions: $\sum_{j=N_1+1}^{N_1+N_2} a_{ij} = 0$, for all $i \in l_1$, $\sum_{j=1}^{N_1} a_{ij} = 0$, for all $i \in l_2$.

Assumption 2-b states that the information exchange between two subnetworks should be balanced.

Given these assumptions, we start to analyze consensus problem. By substituting control action (2), system (1) is rewritten as follows:

$$\dot{x_i} = f(x_i, t) + k\sum_{j\in\mathcal{N}_i} a_{ij}\left[x_j(t) - x_i(t)\right], \qquad i = 1, \ldots, n \quad (3)$$

Let X be defined as $X = [x_1, x_2, \ldots, x_{N_1}, x_{N_1+1}, \ldots, x_{N_1+N_2}]^T$ and $F(X, t) = [f(x_1, t), \ldots, f(x_{N_1}, t), f(x_{N_1+1}, t), \ldots, f(x_{N_1+N_2}, t)]^T$, now system (3) can be written as:

$$\dot{X} = F(X, t) - kHX \quad (4)$$

where H is the Laplacian matrix of G and described for two clusters as follows:

$$H = \begin{bmatrix} L_1 & \Omega_{12} \\ \Omega_{21} & L_2 \end{bmatrix}$$

where L_i, $i = 1, 2$ are the Laplacian matrix of the subnetworks and Ω_{ij}, $i, j = 1, 2$ are the information exchange between the two subnetworks as:

$$\Omega_{12} = -\begin{bmatrix} a_{1(N_1+1)} & \cdots & a_{1(N_1+N_2)} \\ \vdots & \ddots & \vdots \\ a_{N_1(N_1+1)} & \cdots & a_{N_1(N_1+N_2)} \end{bmatrix}$$

$$\Omega_{21} = -\begin{bmatrix} a_{(N_1+1)1} & \cdots & a_{(N_1+1)N_1} \\ \vdots & \ddots & \vdots \\ a_{(N_1+N_2)1} & \cdots & a_{(N_1+N_2)N_1} \end{bmatrix}$$

To investigate system (4), we introduce error matrices as follows:

$$e_i = x_i - x_{N_1}, \quad i = 1, \ldots, N_1 - 1$$

$$e_j = x_j - x_{N_1+N_2}, \quad j = N_1 + 1, \ldots, N_1 + N_2 - 1$$

$$E = \left[e_1, \ldots, e_{N_1-1}, x_{N_1}, e_{N_1+1}, \ldots, e_{N_1+N_2-1}, x_{N_1+N_2}\right]^T$$

$$\begin{aligned} e &= \left[e_1, \ldots, e_{N_1-1}, e_{N_1+1}, \ldots, e_{N_1+N_2-1}\right]^T \\ &= \left[x_1 - x_{N_1}, \ldots, x_{N_1-1} - x_{N_1}, x_{N_1+1} - x_{N_1+N_2}, \ldots, x_{N_1+N_2-1} - x_{N_1+N_2}\right]^T \end{aligned}$$

Rewriting (4) in terms of e, we have

$$\dot{e} = f_e(E, t) - kH_G e \quad (5)$$

where

$$H_G = \begin{bmatrix} \widetilde{L}_1 & \widetilde{\Omega}_{12} \\ \widetilde{\Omega}_{21} & \widetilde{L}_2 \end{bmatrix}$$

$$\widetilde{L}_1 = \begin{bmatrix} l_{11} - l_{N_1 1} & \cdots & l_{1(N_1-1)} - l_{N_1(N_1-1)} \\ \vdots & \ddots & \vdots \\ l_{(N_1-1)1} - l_{N_1 1} & \cdots & l_{(N_1-1)(N_1-1)} - l_{N_1(N_1-1)} \end{bmatrix}$$

$$\widetilde{L}_2 = \begin{bmatrix} l_{(N_1+1)1} - l_{(N_1+N_2)(N_1+1)} & \cdots & l_{(N_1+1)(N_1+N_2-1)} - l_{(N_1+N_2)(N_1+N_2-1)} \\ \vdots & \ddots & \vdots \\ l_{(N_1+N_2-1)1} - l_{(N_1+N_2)(N_1+1)} & \cdots & l_{(N_1+N_2-1)(N_1+N_2-1)} - l_{(N_1+N_2)(N_1+N_2-1)} \end{bmatrix}$$

$$\widetilde{\Omega}_{12} = \begin{bmatrix} a_{N_1(N_1+1)} - a_{1(N_1+1)} & \cdots & a_{N_1(N_1+N_2-1)} - a_{1(N_1+N_2-1)} \\ \vdots & \ddots & \vdots \\ a_{N_1(N_1+1)} - a_{(N_1-1)(N_1+1)} & \cdots & a_{N_1(N_1+N_2-1)} - a_{(N_1-1)(N_1+N_2-1)} \end{bmatrix}$$

and

$$\widetilde{\Omega}_{21} = \begin{bmatrix} a_{(N_1+N_2)1} - a_{(N_1+1)1} & \cdots & a_{(N_1+N_2)(N_1-1)} - a_{(N_1+N_2)(N_1-1)} \\ \vdots & \ddots & \vdots \\ a_{(N_1+N_2)1} - a_{(N_1+N_2-1)1} & \cdots & a_{(N_1+N_2)(N_1-1)} - a_{(N_1+N_2-1)(N_1-1)} \end{bmatrix}$$

Now, define:

$$\begin{aligned} f_e(E, t) &= \left[f(x_1, t) - f(x_{N_1}, t), \ldots, f(x_{N_1-1}, t) - f(x_{N_1}, t), f(x_{N_1+1}, t) - f(x_{N_1+N_2}, t), \ldots, f(x_{N_1+N_2-1}, t) - f(x_{N_1+N_2}, t)\right]^T \\ &= \left[f(x_{N_1} + e_1, t) - f(x_{N_1}, t), \ldots, f(x_{N_1} + e_{N_1-1}, t) - f(x_{N_1}, t), \ldots, f(x_{N_1+N_2} + e_{N_1+N_2-1}, t) - f(x_{N_1+N_2}, t)\right]^T \end{aligned}$$

Note that $H1_{n*1} = 0$ because of the properties of Laplacian matrix and Assumption 2-b. The cluster consensus is achieved for system (4) if and only if system given by (5) is asymptotically stable. Cluster consensus is achieved with the conditions in following theorem for systems with two clusters.

Theorem 1. Assume that Assumption1 and Assumption2 are satisfied. The control algorithm (2) solves the cluster consensus problem for system (1) if the following condition by selecting a proper value for k is satisfied:

$$-k\left[H_G^T P + PH_G\right] + l^2\lambda_{max}(P)I_{n-2} + P < 0 \quad (6)$$

where P is a positive definite matrix and satisfies the following condition:

$$H_G^T P + PH_G = I_{n-2}.$$

Proof.

First, select a Lyapunov function as follows to prove the cluster consensus and show that system (5) is asymptotically stable:

$$V(t) = e^T(t)Pe(t) \quad (7)$$

Now, differentiating $V(t)$ yields in:

$$\dot{V}(t) = -e^T k\left[H_G^T P + PH_G\right]e + 2e^T(t)Pf_e(E,t) \tag{8}$$

Consider Lemma 2 to continue the proof of Lyapunov stability.

Lemma 2. For any two real vectors $a,b \in R^n$, we have:

$$a^T b \le a^T \Phi a + b^T \Phi^{-1} b \tag{22}$$

where Φ is a positive definite matrix with an appropriate dimension [31].

In (8) $a^T = e^T$, $b = Pf_e(E,t)$, $\Phi = P$. Now, using Lemma 2 and Assumption 1, (9) is given as follows:

$$\begin{aligned}\dot{V}(t) &\le -e^T k\left[H_G^T P + PH_G\right]e + {f_e}^T(E,t)Pf_e(E,t)\\ &\quad + e^T Pe \le -e^T k\left[H_G^T P + PH_G\right]e + l^2\lambda_{max}(P)e^T e\\ &\quad + e^T Pe\end{aligned} \tag{9}$$

Now, using (6), $\dot{V}(t) \le 0$ we can conclude that asymptotically stability results in (5).

4.1.2 N Clusters Case

Consider a graph with N clusters and n agents divided as $l_1 = \{1, 2, \ldots, N_1\}$, $l_2 = \{N_1+1, \ldots, N_1+N_2\}$, …, $l_N = \{N_1 + N_2 + \ldots + N_{N-1} + 1, \ldots, N_1 + N_2 + \ldots + N_N\}$, where $n = N_1 + N_2 + \ldots + N_N$.

Now, using (3) and defining X as $X = [x_1, x_2, \ldots, x_{N_1}, x_{N_1+1}, \ldots, x_{N_1+N_2}, \ldots, x_{N_1+\ldots+N_{N-1}+1}, \ldots, x_n]^T$ and $F(X,t) = [f(x_1,t), \ldots, f(x_{N_1},t), f(x_{N_{1+}1},t), \ldots, f(x_{N_{1+}N_2},t), \ldots, f(x_{N_1+\ldots+N_{N-1}+1},t), \ldots, f(x_n,t)]^T$. System (3) can be written as:

$$\dot{X} = F(X,t) - kHX \tag{10}$$

where H is the Laplacian matrix of G described for N clusters as follows:

$$H = \begin{bmatrix} L_1 & \Omega_{12} & \ldots & \Omega_{1N}\\ \Omega_{21} & L_2 & \ldots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \Omega_{N1} & \ldots & \ldots & L_N \end{bmatrix}$$

where L_i, $i = 1, \ldots, N$ are the Laplacian matrix of subnetworks and Ω_{ij}, $i, j = 1, \ldots, N$ are the information exchange between n subnetworks as:

$$\Omega_{ij} = \begin{bmatrix} a_{(N_{i-1}+1)(N_1+\ldots+N_{j-1}+1)} & \cdots & a_{(N_{i-1}+1)(N_1+\ldots+N_j)}\\ \vdots & \ddots & \vdots\\ a_{(N_1+\ldots+N_i)(N_1+\ldots+N_{j-1}+1)} & \cdots & a_{(N_1+\ldots+N_i)(N_1+\ldots+N_j)} \end{bmatrix}$$

The error matrix is defined as:

$$e_{ij} = x_j - x_{\sum_{k=1}^{i} N_k}$$

$$E = \left[e_{ij}, x_{N_1}, x_{N_1+N_2}, \ldots, x_{\underbrace{N_1+N_2+\ldots+N_N}_{n}}\right]$$

$$\begin{aligned} e = \big[&e_{11}, \ldots, e_{1(N_1-1)}, e_{2(N_1+1)}, \ldots, e_{2(N_1+N_2-1)}, \ldots,\\ &e_{N(N_1+N_2+\ldots+N_{N-1}+1)}, \ldots, e_{N(N_1+N_2+\ldots+N_N-1)}\big]^T \end{aligned}$$

where $i = 1, \ldots, N$ and j^{th} agent is in the i^{th} cluster.

Rewriting (5) in terms of e, we have

$$\dot{e} = f_e(E,t) - kH_G \tag{11}$$

where

$$f_e(E,t) = \begin{bmatrix} f(x_1,t) - f(x_{N_1},t), f(x_2,t) - f(x_{N_1},t), \ldots, f(x_{N_1-1},t) - f(x_{N_1},t),\\ f(x_{N_1+1},t) - f(x_{N_1+N_2},t), \ldots,\\ f(x_{N_1+N_2+\ldots+N_{N-1}+1},t) - f(x_{N_1+N_2+\ldots+N_N},t) \end{bmatrix}^T$$

$$H_G = \begin{bmatrix} \widetilde{L}_1 & \widetilde{\Omega}_{12} & \ldots & \widetilde{\Omega}_{1N}\\ \widetilde{\Omega}_{21} & \widetilde{L}_2 & \ldots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \widetilde{\Omega}_{N1} & \ldots & \ldots & \widetilde{L}_N \end{bmatrix}$$

where $\widetilde{L}_1$ and $\widetilde{\Omega}_{ij}$ are defined in (12).

$$\begin{aligned} \widetilde{L}_i &= \begin{bmatrix} l_{(N_1+N_2+\ldots+N_{i-1}+1)1} - l_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_{i-1}+1)} & \cdots & l_{(N_1+N_2+\ldots+N_{i-1}+1)(N_1+N_2+\ldots+N_i-1)} - l_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_i-1)}\\ \vdots & \ddots & \vdots\\ l_{(N_1+N_2+\ldots+N_i-1)1} - l_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_{i-1}+1)} & \cdots & l_{(N_1+N_2+\ldots+N_i-1)(N_1+N_2+\ldots+N_i-1)} - l_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_i-1)} \end{bmatrix}\\ \widetilde{\Omega}_{ij} &= \begin{bmatrix} a_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_{j-1}+1)} - a_{(N_1+N_2+\ldots+N_{i-1}+1)(N_1+N_2+\ldots+N_{j-1}+1)} & \cdots & a_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_j-1)} - a_{(N_1+N_2+\ldots+N_{i-1}+1)(N_1+N_2+\ldots+N_j-1)}\\ \vdots & \ddots & \vdots\\ a_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_{j-1}+1)} - a_{(N_1+N_2+\ldots+N_i-1)(N_1+N_2+\ldots+N_{j-1}+1)} & \cdots & a_{(N_1+N_2+\ldots+N_i)(N_1+N_2+\ldots+N_j-1)} - a_{(N_1+N_2+\ldots+N_i-1)((N_1+N_2+\ldots+N_j-1)} \end{bmatrix} \end{aligned} \tag{12}$$

The cluster consensus is achieved for system (10) if and only if the system given by (11) is asymptotically stable that can be a method similar to two clusters.

Theorem 2. The control algorithm (2) solves the cluster consensus problem with n-cluster for system (1) if the following condition by selecting a proper value for k is satisfied:

$$-k\left[H_G^T P + PH_G\right] + l^2\lambda_{max}(P)I_{n-N} + P < 0 \tag{13}$$

where P is a positive definite matrix and satisfies the following condition:

$$H_G^T P + PH_G = I_{n-N}.$$

4.2 Second-Order System

In this subsection, the cluster consensus problem for agents with second-order dynamics for two clusters is discussed. Agent i, $i = 1, \ldots, n$, is modeled by a second-order system as follows:

$$\dot{x}_i = v_i \qquad \dot{v}_i = f(x_i, v_i, t) + u_i \tag{14}$$

For system (14), the cluster consensus algorithm is considered as follows:

$$u_i = k\sum_{j\in\mathcal{N}_i} a_{ij}\left[\left(x_j - x_i\right) + \left(v_j - v_i\right)\right] \tag{15}$$

We assume that the function $f(x, v, t)$ in (14) satisfies the Lipschitz condition in x_i and v_i as follows:

$$|f(x_2, v_2, t) - f(x_1, v_1, t)| \le l\sqrt{(x_2 - x_1)^2 + (v_2 - v_1)^2} \tag{16}$$

Now, system (14) can be written as:

$$\dot{\varepsilon} = \begin{bmatrix} 0 & I_n \\ -k\,H & -k\,H \end{bmatrix}\varepsilon + \begin{bmatrix} 0 \\ \overline{f}(\varepsilon, t) \end{bmatrix} \tag{17}$$

where $\overline{f}(\varepsilon, t) = [f(x_1, v_1, t), \ldots, f(x_n, v_n, t)]$ and $\varepsilon = \left[X^T, \widehat{v}^T\right]^T$ with $X = [x_1, \ldots, x_n]^T$, $\widehat{v} = [v_1, \ldots, v_n]^T$.

The error matrices are defined as:

$$\eta = \left[e^T, \overline{v}^T\right]^T$$

$$e_{ij} = x_j - x_{\sum_{k=1}^{i} N_k},\ i = 1, 2$$

$$e = \left[e_{11}, \ldots, e_{1(N_1-1)}, e_{2(N_1+1)}, \ldots, e_{2(N_1+N_2-1)}\right]^T$$

$$\overline{v}_{ij} = v_j - v_{\sum_{k=1}^{i} N_k},\ i = 1, 2$$

$$\overline{v} = \left[\overline{v}_{11}, \ldots, \overline{v}_{1(N_1-1)}, \overline{v}_{2(N_1+1)}, \ldots, \overline{v}_{2(N_1+N_2-1)}\right]^T$$

Rewriting (17) in terms of η, we have

$$\dot{\eta} = \begin{bmatrix} 0 & I_{n-2} \\ -kH_G & -kH_G \end{bmatrix}\eta + \begin{bmatrix} 0 \\ f_\eta(\eta, t) \end{bmatrix} \tag{18}$$

where

$$f_\eta(\eta, t) = \begin{bmatrix} f(x_1, v_1, t) - f(x_{N_1}, v_{N_1}, t), \ldots, \\ f(x_{N_1-1}, v_{N_1-1}, t) - f(x_{N_1}, v_{N_1}, t), \\ f(x_{N_1+1}, v_{N_1+1}, t) - f(x_{N_1+N_2}, v_{N_1+N_2}, t), \ldots, \\ f(x_{N_2-1}, v_{N_2-1}, t) - f(x_{N_1+N_2}, v_{N_1+N_2}, t) \end{bmatrix}^T$$

Theorem 3. The cluster consensus problem for system (14) is solved by the algorithm (15) if the following condition is satisfied by selecting a proper k:

$$\begin{bmatrix} -kI_{n-2} + P + 2\,l^2\,\lambda_{max}(P)I_{n-2} & \mu P - kI_{n-2} \\ \mu P - kI_{n-2} & 3P - kI_{n-2} + 2\,l^2\,\lambda_{max}(P)I_{n-2} \end{bmatrix} < 0 \tag{19}$$

where P is a positive definite matrix and satisfies the following condition:

$$H_G^T P + PH_G = I_{n-2}.$$

Proof.

Now, choose a Lyapunov function as follows to prove the cluster consensus and show that system (18) is asymptotically stable:

$$V = \eta^T\overline{P}\,\eta, \overline{P} = \begin{bmatrix} \mu P & \upsilon P \\ \upsilon P & \gamma P \end{bmatrix} \tag{20}$$

where υ, γ and μ are positive scalars and $\mu\gamma > \upsilon^2$ is satisfied. Now, differentiating $V(t)$ yields in:

$$\begin{aligned} \dot{V}(t) &= \eta^T\left[\begin{bmatrix} 0 & I_{n-2} \\ -kH_G & -kH_G \end{bmatrix}^T\overline{P} + \overline{P}\begin{bmatrix} 0 & I_{n-2} \\ -kH_G & -kH_G \end{bmatrix}\right]\eta + 2\eta^T(t)\overline{P}\begin{bmatrix} 0 \\ f_\eta(\eta, t) \end{bmatrix} \\ &= \eta^T\left[\begin{bmatrix} 0 & I_{n-2} \\ -kH_G & -kH_G \end{bmatrix}^T\overline{P} + \overline{P}\begin{bmatrix} 0 & I_{n-2} \\ -kH_G & -kH_G \end{bmatrix}\right]\eta + 2\eta^T(t)\ \mathrm{diag}(\upsilon P, \gamma P)\begin{bmatrix} f_\eta(\eta, t) \\ f_\eta(\eta, t) \end{bmatrix} \end{aligned} \tag{21}$$

For simplicity, we choose $\upsilon = \gamma = 1$ and $\mu = \beta\upsilon$ with the positive constant β.

$$\dot{V}(t) = \eta^T \begin{bmatrix} -k\left(H_G^T P + PH_G\right) & \mu P - k\left(H_G^T P + PH_G\right) \\ \mu P - k\left(H_G^T P + PH_G\right) & 2P - k\left(H_G^T P + PH_G\right) \end{bmatrix} \eta$$
$$+ 2\eta^T(t)\ \mathrm{diag}(P,P) \begin{bmatrix} f_\eta(\eta,t) \\ f_\eta(\eta,t) \end{bmatrix} \le \eta^T \begin{bmatrix} -k\left(H_G^T P + PH_G\right) & \mu P - k\left(H_G^T P + PH_G\right) \\ \mu P - k\left(H_G^T P + PH_G\right) & 2P - k\left(H_G^T P + PH_G\right) \end{bmatrix} \eta$$
$$+ \left[f_\eta{}^T(\eta,t)\ \ f_\eta{}^T(\eta,t) \right] \mathrm{diag}(P,P) \begin{bmatrix} f_\eta(\eta,t) \\ f_\eta(\eta,t) \end{bmatrix} + \eta^T \mathrm{diag}(P,\mathrm{P})\eta \tag{22}$$

where P is chosen such that $H_G^T P + PH_G = I_{n-2}$, hence (22) can be rewritten as follows:

$$\dot{V}(t) \le \eta^T \begin{bmatrix} -kI_{n-2} & \mu P - kI_{n-2} \\ \mu P - kI_{n-2} & 2P - kI_{n-2} \end{bmatrix} \eta + \eta^T \mathrm{diag}(P,P)\eta +$$
$$2L\ \lambda_{max}(P)\ \eta^T\eta = \eta^T \begin{bmatrix} -kI_{n-2} + P + 2\ l^2\ \lambda_{max}(P) I_{n-2} & \mu P - kI_{n-2} \\ \mu P - kI_{n-2} & 3P - kI_{n-2} + 2\ l^2\ \lambda_{max}(P) I_{n-2} \end{bmatrix} \eta \tag{23}$$

Now, using (19), $\dot{V}(t) \le 0$ we can conclude that asymptotically stability results in (18).

4.3 General High-Order System

In this section, a group of n agents is considered with general high-order nonlinear dynamics. The i^{th} agent is described by following dynamics [32]:

$$\dot{x}_i = A\, x_i + C\, f(x_i,t) + Bu_i \tag{24}$$

where $x_i \in R^p$ are the states of i^{th} agent for $i = 1, 2, \ldots, n$, $f(x_i, t) = (f_1(x_i, t), f_2(x_i, t), \ldots, f_p(x_i, t))^T$ is a nonlinear vector-valued function which satisfied Lipschitz condition, u_i is a control input and A, B, C are constant matrices.

A cluster consensus algorithm for system (24) is given as follows:

$$u_i(t) = \alpha F \sum_{j \in \mathcal{N}_i} a_{ij}\left[x_j(t) - x_i(t)\right], \qquad i = 1, \ldots, n \tag{25}$$

where $\alpha > 0$, $F \in R^{1 \times p}$ is the feedback gain matrix.

By substituting control action (25), system (24) is rewritten as follows:

$$\dot{x}_i = A\, x_i + C\, f(x_i,t) + \alpha BF \sum_{j \in \mathcal{N}_i} a_{ij}\left[x_j(t) - x_i(t)\right], \quad i = 1, \ldots, n \tag{26}$$

Like section 4.1.1, error matrix for two clusters is defined as follows:

$$\dot{e} = (I_{n-2} \otimes A)\ e + (I_{n-2} \otimes C)\ f_e(E,t) - \alpha(H_G \otimes BF)e \tag{27}$$

where H_G, $f_e(E, t)$ and e are defined like (5). $\otimes$ is Kronecker product.

Theorem 4. The cluster consensus problem for system (24) is achieved by the algorithm (25) if P is a positive definite matrix and satisfies the following condition:

$$\begin{bmatrix} AP + PA^T + l^2 CC^T - \alpha\lambda_{min}(H_G)BB^T + \beta P & P \\ P & -I_p \end{bmatrix} < 0 \tag{28}$$

where $\beta > 0$ and take $F = \frac{1}{2} B^T P^{-1}$.

Proof.

First, select a Lyapunov function as follows to prove the cluster consensus and show that system (27) is asymptotically stable:

$$V(t) = e^T(t)\left(I_{n-2} \otimes P^{-1}\right)e(t) \tag{29}$$

Now, differentiating $V(t)$ yields in:

$$\dot{V}(t) = e^T\left[I_{n-2} \otimes \left(P^{-1}A + A^T P^{-1}\right)\right]e$$
$$+ 2e^T\left(I_{n-2} \otimes P^{-1}C\right)f_e(E,t) - 2\alpha e^T\left(H_G \otimes P^{-1}BF\right)e \tag{30}$$

Consider Lemma 3 to continue the proof of Lyapunov stability.

Lemma 3. For the vectors x, y and matrices P, D and S, the following inequality is established:

$$2x^T DSy \le x^T DPD^T x + y^T S^T P^{-1} Sy \tag{31}$$

Using Lemma 3 and Assumption 2, (32) is given as:

$$\dot{V}(t) \le\ e^T\left[I_{n-2} \otimes \left(P^{-1}A + A^T P^{-1}\right)\right]e$$
$$+ e^T\left[I_{n-2} \otimes \left(l^2 P^{-1} CC^T P^{-1} + I_p\right)\right]e - 2\alpha e^T\left(H_G \otimes P^{-1}BF\right)e \tag{32}$$

By substituting $F = \frac{1}{2} B^T P^{-1}$, (32) is rewritten as follows:

$$\dot{V}(t) \le\ e^T\left[I_{n-2} \otimes \left(P^{-1}A + A^T P^{-1}\right)\right]e \tag{33}$$
$$+ e^T\left[I_{n-2} \otimes \left(l^2 P^{-1} CC^T P^{-1} + I_p\right)\right]e - \alpha e^T\left(H_G \otimes P^{-1}BB^T P^{-1}\right)e$$

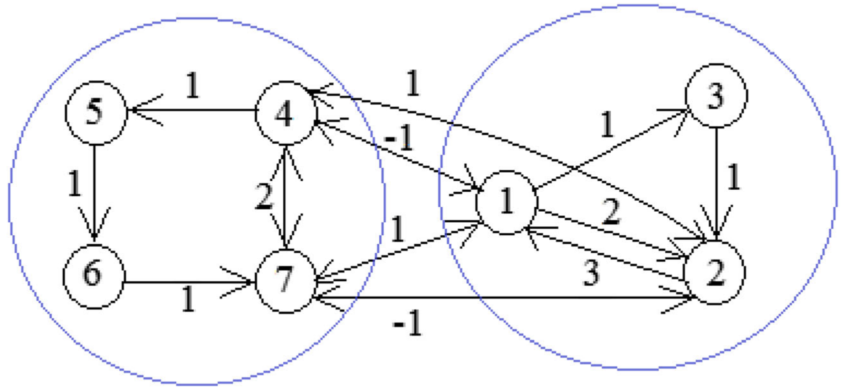

Fig. 1 The topology interaction of the agents

Now, (33) can be written as:

$$\dot{V}(t) \leq \varepsilon^T \left[I_{n-2} \otimes (AP + PA^T + l^2 CC^T + P^T P)\right] \varepsilon - \alpha \varepsilon^T (H_G \otimes BB^T) \varepsilon \quad (34)$$

where $\varepsilon = (I_{n-2} \otimes P^{-1})e$.

$$\dot{V}(t) \leq \varepsilon^T \left[I_{n-2} \otimes \left(AP + PA^T + l^2 CC^T + P^T P\right)\right]$$
$$\varepsilon - \alpha \lambda_{min}(H_G) \varepsilon^T \left(I_{n-2} \otimes BB^T\right) \varepsilon \leq -\beta \varepsilon^T (I_{n-2} \otimes P) \varepsilon$$
$$= -\beta e^T \left(I_{n-2} \otimes P^{-1}\right) e = -\beta\, V(t) \quad (35)$$

Using (28) and Schur complement lemma [33], (35) is rewritten as follows:

$$\dot{V}(t) \leq -\beta \varepsilon^T (I_{n-2} \otimes P) \varepsilon = -\beta e^T \left(I_{n-2} \otimes P^{-1}\right) e$$
$$= -\beta\, V(t) \quad (36)$$

So, with $\dot{V}(t) \leq 0$ we can conclude that asymptotically stability results in (27).

4.4 Cluster Consensus of Nonlinear Multi-Agent Systems with Time-Delays Via Impulsive Control

In this subsection, the consensus tracking problem for the followers that are nonlinear is investigated. The nonlinear dynamics for the followers are given as follows:

$$\dot{x}_i(t) = f_{\hat{i}}(x_i(t), x_i(t-\tau_1), t) + u_i(t), \quad i = 1, .., n \quad (37)$$

where $f_{\hat{i}}(x_i(t), t)$ is a nonlinear function for each cluster, which satisfies the Lipschitz condition like Assumption 1. $\hat{i}$ =1 for $i \in l_1$ and $\hat{i}$ =2 for $i \in l_2$. τ_1 is the time delay which occurs inside the agent and $u_i(t)$ is a controller.

The dynamics of the $\hat{i}^{\text{th}}$ leader of the multi-agent systems is given by:

$$\dot{x}_{r\hat{i}}(t) = f_{\hat{i}}\left(x_{r\hat{i}}(t), x_{r\hat{i}}(t-\tau_1), t\right), \hat{i} = 1, 2 \quad (38)$$

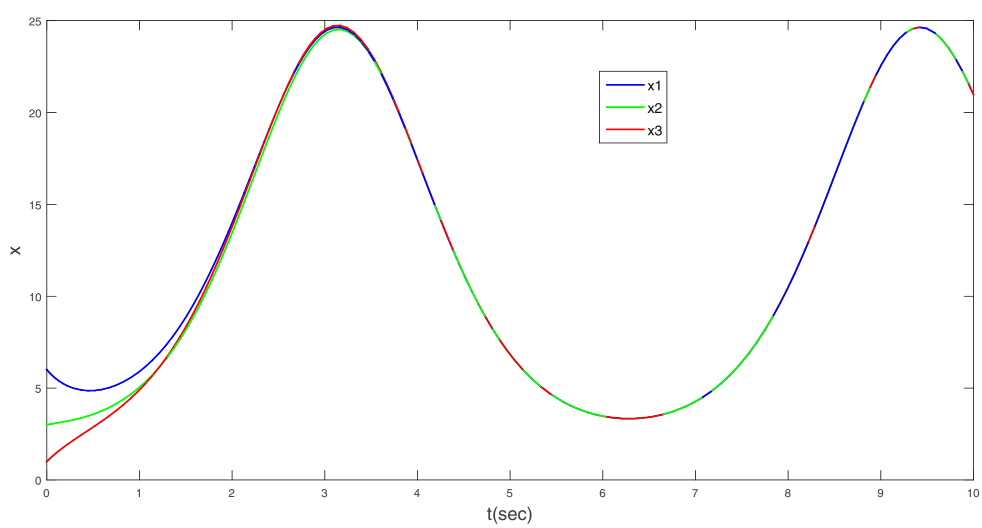

Fig. 2 Cluster consensus for cluster1

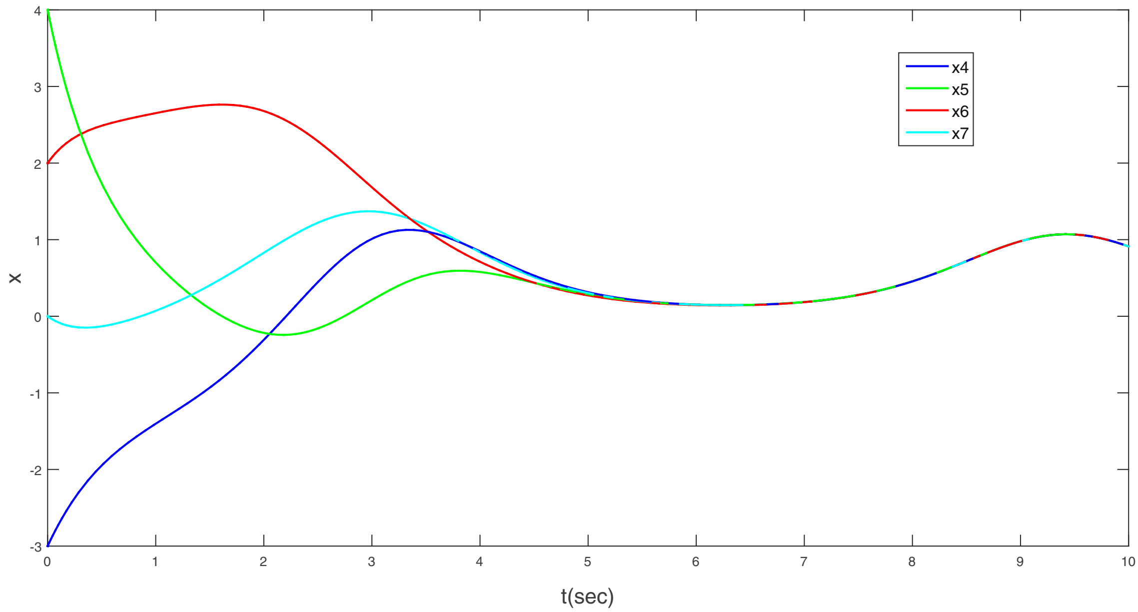

Fig. 3 Cluster consensus for cluster2

When the following condition is satisfied for any initial condition, the cluster consensus is achieved.

$$\lim_{t\to+\infty}\left\|x_i - x_{\hat{ri}}\right\| = 0, \qquad \forall i \in l_1$$
$$\lim_{t\to+\infty}\left\|x_i - x_{\hat{ri}}\right\| = 0, \qquad \forall i \in l_2 \tag{39}$$

A cluster consensus algorithm for system (37) is given as follows:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\left[x_j(t-\tau_2) - x_i(t-\tau_2)\right] + d_i\left[x_i(t-\tau_3) - x_{ri}(t-\tau_3)\right] \tag{40}$$

where $D = diag(d_1, d_2, \ldots, d_n)$ is the leader adjacency matrix. d_i is used for describing the leader information exchange to agent i. $d_i = 1$, if information is transmitted from leader to agent i and otherwise $d_i = 0$. For each cluster, $d_i = 1$ for only one follower. τ_2 is the time delay when a signal is sent from the j^{th} agent to the i^{th} agent and τ_3 is the time delay when the signal is sent from the leader to the i^{th} agent.

In this section, the impulsive control for cluster consensus is used. In the impulsive control method, the control signals are received by systems only in discrete times. The impulsive control protocol is described as follows [34]:

$$x_i\left(t_k^+\right) - x_i\left(t_k^-\right) = \mu_{ik}\left(x_i\left(t_k^-\right) - x_{ri}\left(t_k^-\right)\right) \tag{41}$$

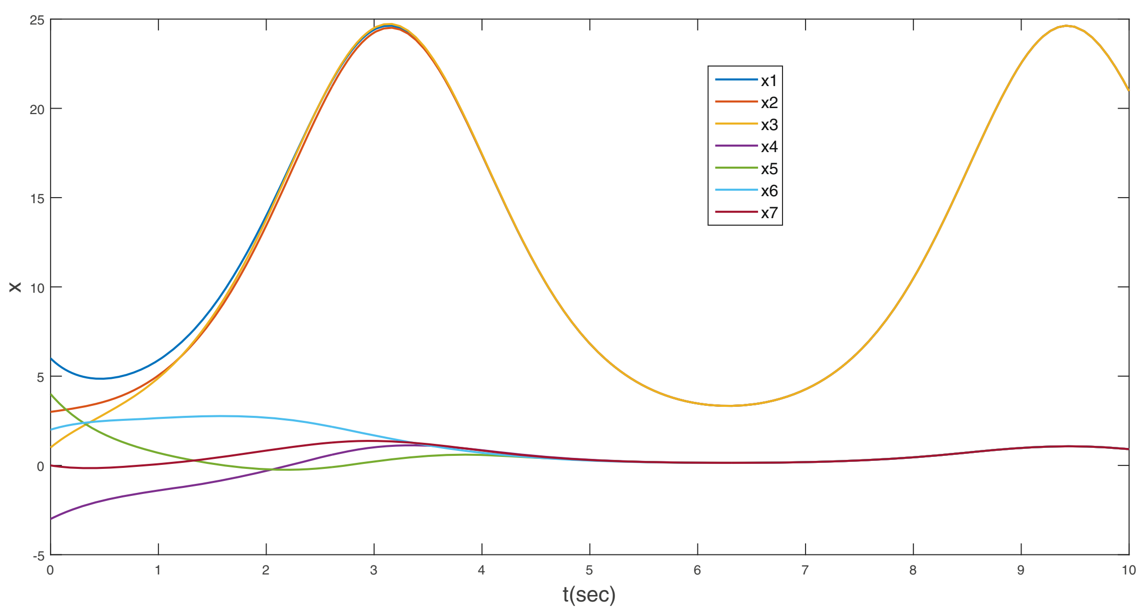

Fig. 4 Cluster consensus for the graph in Fig.1

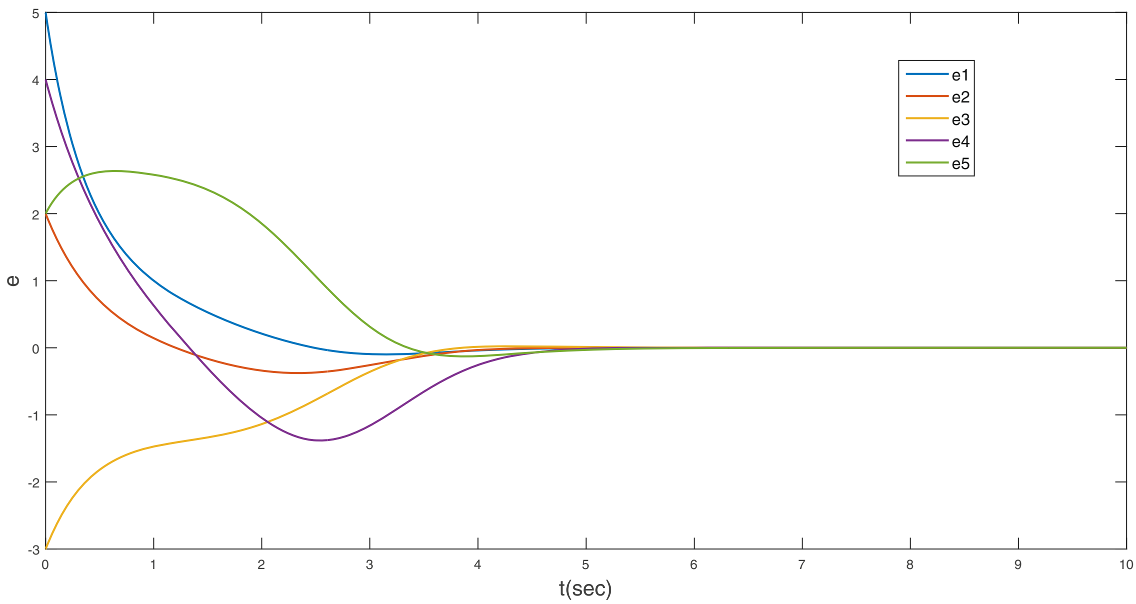

Fig. 5 Error of system

where μ_{ik} are impulsive gains and $\{t_k\}$ is the discrete set which $t_0 < t_1 < \ldots < t_k$ and $t_k - t_{k-1} < T$, $k = 1, 2, 3\ldots$ $x_i\left(t_k^+\right) = \lim_{h\to 0^+} x_i(t_k + h)$ and $x_i\left(t_k^-\right) = \lim_{h\to 0^+} x_i(t_k - h)$.

Assumption 3. The Lipschitz condition is defined as follow for function $f(x, x(t-\tau), t)$.

$$(x_2 - x_1)^T (f(x_2, x_2(t-\tau_1), t) - f(x_1, x_1(t-\tau_1), t)) \le \beta_1 \|x_2 - x_1\|^2 + \beta_2 \|x_2(t-\tau_1) - x_1(t-\tau_1)\|^2, \qquad \forall x_1, x_2 \in R, \forall t \ge 0$$

where β_1 and β_2 are the Lipschitz constants.

By substituting control action (40), system (37) is rewritten as follows:

$$\dot{x}_i(t) = f_{\hat{i}}(x_i(t), x_i(t-\tau_1), t) + \sum_{j\in\mathcal{N}_i} a_{ij}\left[x_j(t-\tau_2) - x_i(t-\tau_2)\right] + d_i[x_i(t-\tau_3) - x_{ri}(t-\tau_3)] \tag{42}$$

where $X = [x_1, x_2, \ldots, x_{N_1}, x_{N_1+1}, \ldots, x_{N_1+N_2}]^T$ and $F(X, t) = \left[f_{\hat{i}}(x_1, t), \ldots, f_{\hat{i}}(x_{N_1}, t), f_{\hat{i}}(x_{N_1+1}, t), \ldots, f_{\hat{i}}(x_{N_1+N_2}, t)\right]^T$ now system (42) can be written as:

$$\dot{X} = F(X, X(t-\tau_1), t) - HX(t-\tau_2) - DX(t-\tau_3) \tag{43}$$

where H is described like (4).

Let $\widetilde{x}_i = x_i - x_{ri}$, so we can write as follows:

$$\widetilde{X}(t) = F(X, X(t-\tau_1), t) - F(X_r, X_r(t-\tau_1), t) - H\widetilde{X}(t-\tau_2) - D\widetilde{X}(t-\tau_3) \tag{44}$$

where, $\widetilde{X} = [\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_{N_1}, \widetilde{x}_{N_1+1}, \ldots, \widetilde{x}_n]^T$.

Theorem 5. The cluster consensus problem for system (37) is achieved by the algorithm (40) and impulsive control protocol (41), if there exists positive constants ε_1, ε_2 and the following conditions are established:

$$\sigma = \frac{\ln\frac{1}{\rho}}{T} - (p+q) > 0$$

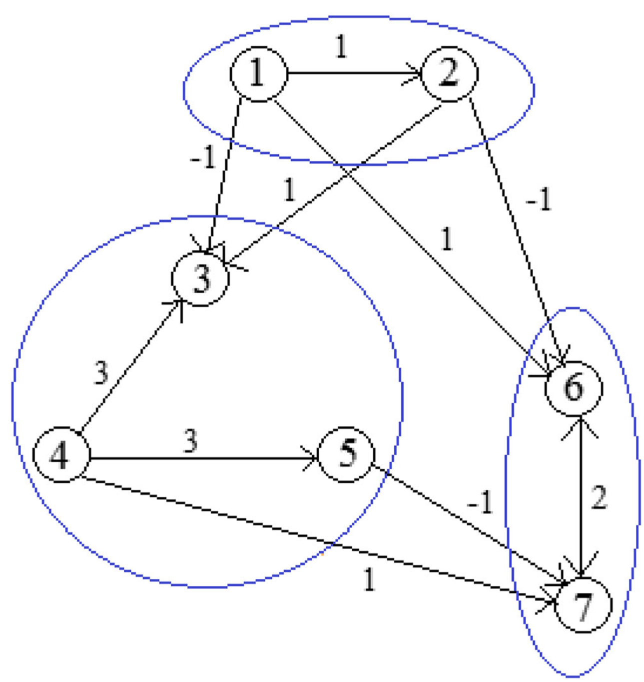

Fig. 6 The topology interaction of the agents

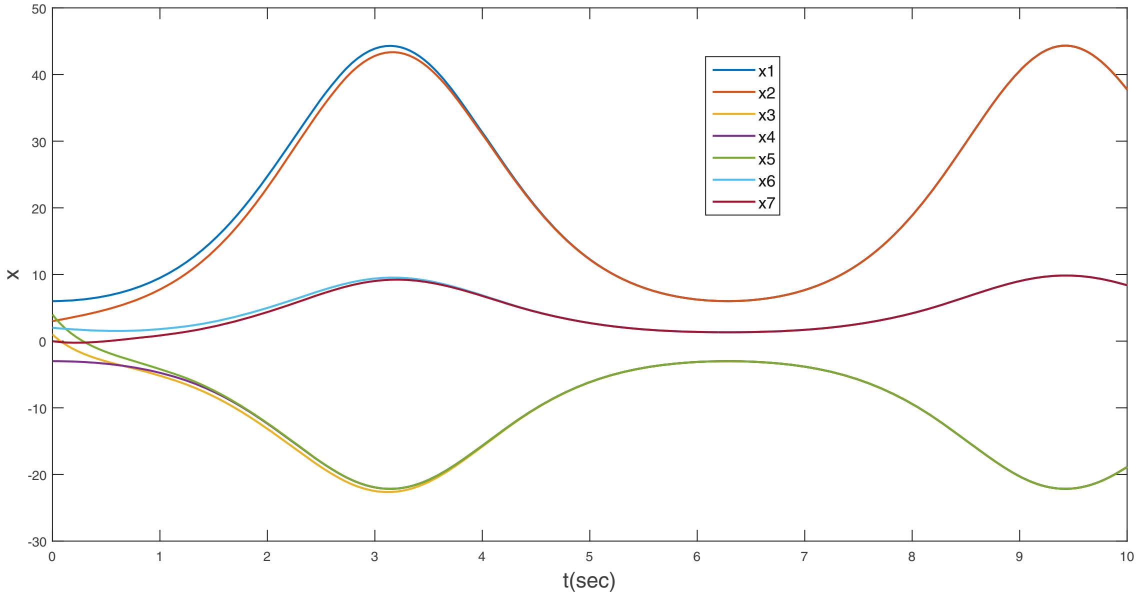

Fig. 7 Cluster consensus for the graph in Fig.6

where $\rho = \lambda_{max}(\Phi^T\Phi)e^{(p+q)\ T}$, $\Phi = diag\ \{1+\mu_{1k}, 1+\mu_{2k}, \ldots, 1+\mu_{nk}\}$, $p = \beta_1 + \frac{\varepsilon_1}{2}\lambda_{max}\left(HH^T\right) + \frac{\varepsilon_2}{2}\lambda_{max}\left(DD^T\right) > 0$ and $q = q_1 + q_2 + q_3 > 0$, $q_1 = \beta_2$, $q_2 = \frac{1}{2\varepsilon_1}$, $q_3 = \frac{1}{2\varepsilon_2}$.

Proof:

The following Lyapunov function is considered:

$$V = \frac{1}{2}\widetilde{X}^T\widetilde{X} \tag{45}$$

The derivation of V is given as follows by using Assumption 3:

$$\begin{aligned}\dot{V} &= \widetilde{X}^T\dot{\widetilde{X}} \\ &= \widetilde{X}^T\left[F(X,X(t-\tau_1),t)-F(X_r,X_r(t-\tau_1),t)-H\widetilde{X}(t-\tau_2)-D\widetilde{X}(t-\tau_3)\right] \\ &\le \beta_1\widetilde{X}^T\widetilde{X} + \beta_2\widetilde{X}^T(t-\tau_1)\widetilde{X}(t-\tau_1)-\widetilde{X}^T H\widetilde{X}(t-\tau_2)-\widetilde{X}^T D\widetilde{X}(t-\tau_3)\end{aligned} \tag{46}$$

Lemma 4. The following inequality with vectors x, y and constant ε is established:

$$x^Ty = y^Tx = \frac{1}{2}\left(x^Ty + y^Tx\right) \le \frac{\varepsilon}{2}\ x^Tx + \frac{1}{2\varepsilon}y^Ty \tag{47}$$

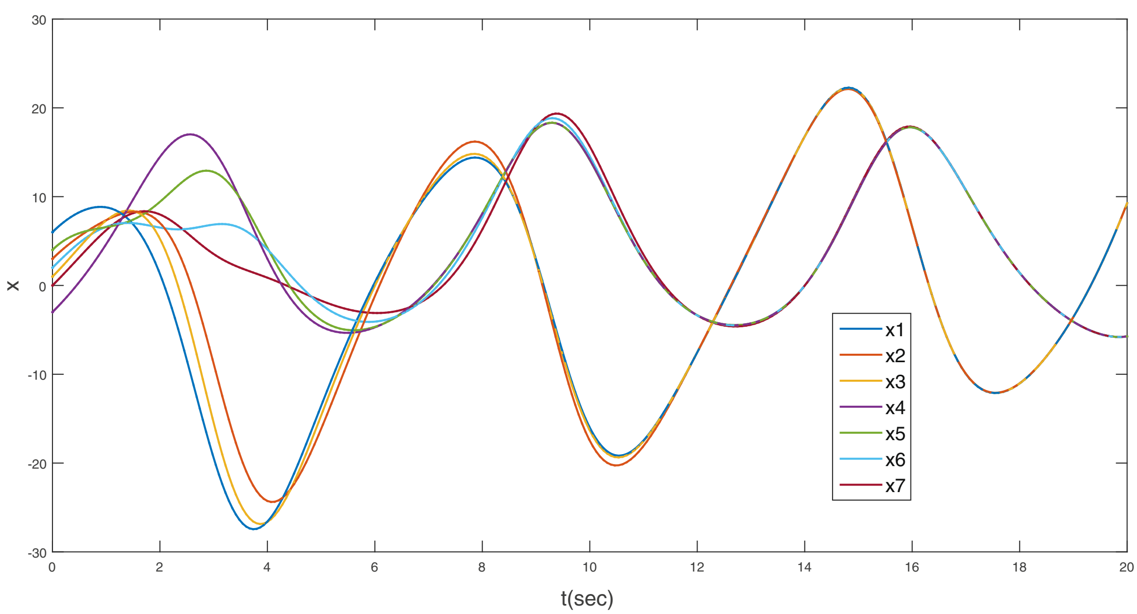

Fig. 8 Cluster consensus for X with the graph in Fig.1

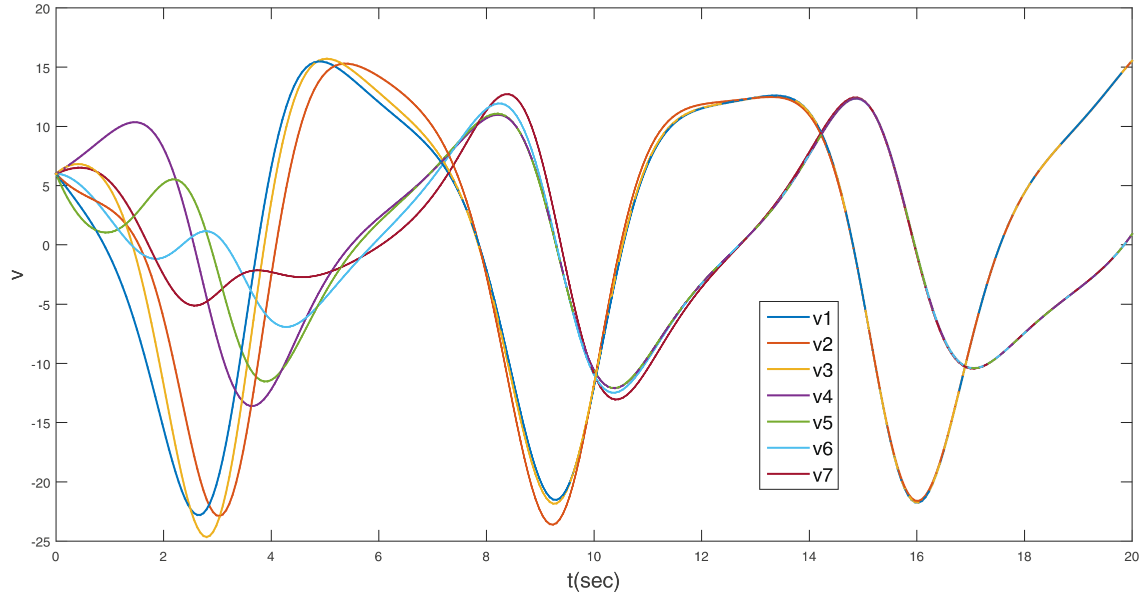

Fig. 9 Cluster consensus for V with the graph in Fig.1

Using Lemma 4 and Assumption 2, (48) is given as:

$$\begin{aligned}\dot{V} \le\ & \beta_1 \widetilde{X}^T \widetilde{X} + \beta_2 \widetilde{X}^T (t-\tau_1) \widetilde{X}(t-\tau_1) + \frac{\varepsilon_1}{2} \widetilde{X}^T HH^T \widetilde{X} + \frac{1}{2\varepsilon_1} \widetilde{X}^T (t-\tau_2) \\ & \widetilde{X}(t-\tau_2) + \frac{\varepsilon_2}{2} \widetilde{X}^T DD^T \widetilde{X} + \frac{1}{2\varepsilon_2} \widetilde{X}^T (t-\tau_3) \widetilde{X}(t-\tau_3) \\ \le\ & \left(\beta_1 + \frac{\varepsilon_1}{2} \lambda_{max}(HH^T) + \frac{\varepsilon_2}{2} \lambda_{max}(DD^T) \right) \widetilde{X}^T \widetilde{X} + \beta_2 \widetilde{X}^T (t-\tau_1) \\ & \widetilde{X}(t-\tau_1) + \frac{1}{2\varepsilon_1} \widetilde{X}^T (t-\tau_2) \widetilde{X}(t-\tau_2) + \frac{1}{2\varepsilon_2} \widetilde{X}^T (t-\tau_3) \widetilde{X}(t-\tau_3)\end{aligned} \tag{48}$$

Let $p = \beta_1 + \frac{\varepsilon_1}{2} \lambda_{max}(HH^T) + \frac{\varepsilon_2}{2} \lambda_{max}(DD^T)$, $q_1 = \beta_2$, $q_2 = \frac{1}{2\varepsilon_1}$, $q_3 = \frac{1}{2\varepsilon_2}$, so (49) is as follows:

$$\begin{aligned}\dot{V} \le\ & pV + q_1 V(t-\tau_1) + q_2 V(t-\tau_2) \\ & + q_3 V(t-\tau_3) \le (p+q) \overline{V}\end{aligned} \tag{49}$$

where $q = q_1 + q_2 + q_3$, $\overline{V} = \max\limits_{t-\tau \le s \le t} V(s)$, $\tau = \max\{\tau_1, \tau_2, \tau_3\}$, so (50) is as follows:

$$V(t) \le e^{(p+q)\ (t-t_{k-1})}\ V\left(t_{k-1}^+\right), \qquad t \in (t_{k-1}, t_k] \tag{50}$$

Using (41), The impulsive control law can be rewritten as follows [34]:

$$\widetilde{X}(t^+) = \Phi \widetilde{X}(t^-) \tag{51}$$

where $\Phi = diag\{1+\mu_{1k}, 1+\mu_{2k}, \ldots, 1+\mu_{nk}\}$. Therefore,

$$\begin{aligned}V\left(t_k^+\right) &= \frac{1}{2} \widetilde{X}^T \left(t_k^+\right) \widetilde{X}\left(t_k^+\right) \\ &= \frac{1}{2} \widetilde{X}^T \left(t_k^-\right) \Phi^T \Phi \widetilde{X}\left(t_k^-\right) \le \frac{1}{2}\ \lambda_{max}\left(\Phi^T \Phi\right)\ \widetilde{X}^T \left(t_k^-\right) \widetilde{X}\left(t_k^-\right) \\ &= \lambda_{max}\left(\Phi^T \Phi\right)\ V\left(t_k^-\right)\end{aligned} \tag{52}$$

All agents are left continuous, so $x_i\left(t_k^-\right) = x_i(t_k)$. Then from (52) and (50) we have:

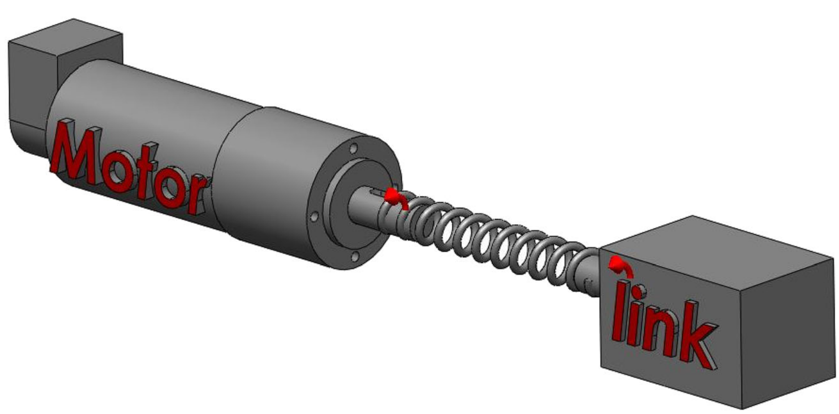

Fig. 10 single-link manipulator with a flexible joint

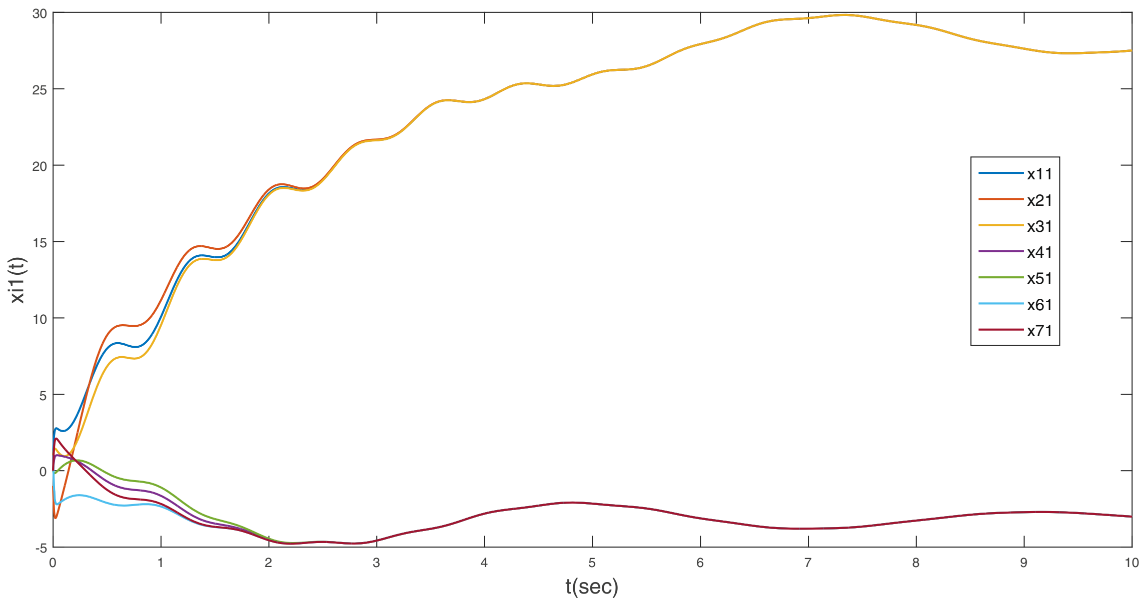

Fig. 11 Cluster consensus for $\mathbf{x_{i1}(t), i = 1, \ldots, 7}$

$$
\begin{aligned}
V(t) &\le e^{(p+q)\ (t-t_1)}\ V\left(t_1^+\right) \\
&\le \lambda_{max}\left(\Phi^T\Phi\right)e^{(p+q)\ (t-t_1)}\ \ V\left(t_1^-\right) \\
&= \lambda_{max}\left(\Phi^T\Phi\right)e^{(p+q)\ (t-t_1)}\ \ V(t_1) \\
&\le \lambda_{max}\left(\Phi^T\Phi\right)e^{(p+q)\ (t-t_1)}e^{(p+q)\ (t-t_0)}\ V\left(t_0^+\right), \qquad t\in(t_1, t_2]
\end{aligned} \tag{53}
$$

Thus

$$
V(t) \le \rho^m\ e^{(p+q)\ (t-t_0)} V\left(t_0^+\right), \qquad t > t_0 \tag{54}
$$

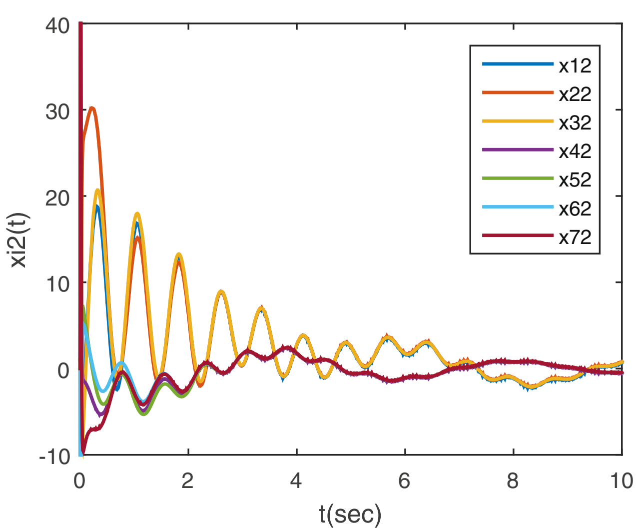

Fig. 12 Cluster consensus for $\mathbf{x_{i2}(t), i = 1, \ldots, 7}$

where $\frac{t-t_0}{T} < m$ and $\rho = \lambda_{max}(\Phi^T\Phi)e^{(p+q)\ T}$.

(54) is rewritten using Lemma 6 and exponential properties as follows:

$$
\begin{aligned}
V(t) &\le e^{\frac{\ln\rho}{T}\ (t-t_0)}\ e^{(p+q)\ (t-t_0)} V(t_0) \\
&= e^{-\left(\frac{\ln\frac{1}{\rho}}{T}-(p+q)\right)\ (t-t_0)} V(t_0) = e^{-\sigma(t-t_0)} V(t_0)
\end{aligned} \tag{55}
$$

where $\sigma = \frac{\ln\frac{1}{\rho}}{T} - (p+q)$, therefore:

$$
\left\|\widetilde{X}\right\| \le e^{-\frac{\sigma(t-t_0)}{2}} \left\|\widetilde{X}(t_0)\right\|, \qquad t \ge t_0 \tag{56}
$$

If Theorem 5. is satisfied, the system (44) is exponentially stable, which means $\lim\limits_{t\to+\infty} \|x_i(t) - x_{ri}(t)\| = 0, \quad i = 1, \ldots, n$ and the cluster consensus for the system (37) is achieved using the controller (40).

5 Simulation

In this section, numerical examples for showing the efficiency of the theoretical results are given.

Example 1. The nonlinear dynamics is given by $f(x, t) = x \sin(t)$. A topology with seven agents is shown in Fig. 1. The graph consists of two clusters and each cluster has spanning tree.

The cluster consensus problem is solved by the control algorithm (2) for (1) under the fixed topology. Initial condition is considered as $X(0) = [6\ \ 3\ \ 1\ \ -3\ \ 4\ \ 2\ \ 0]^T$. The matrix H is given as

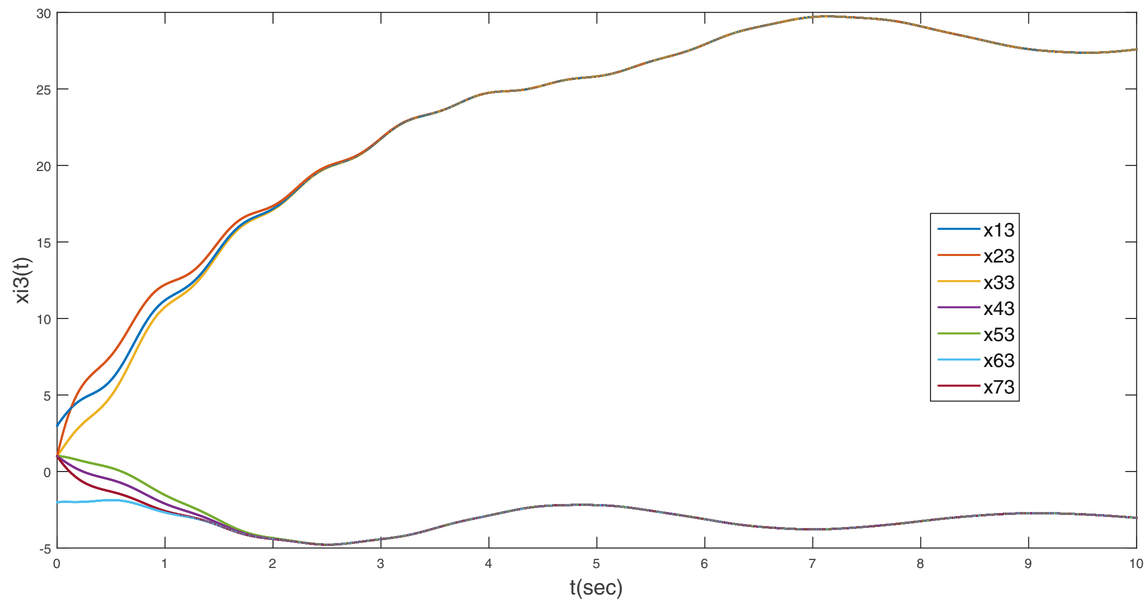

Fig. 13 Cluster consensus for $\mathbf{x_{i3}(t), i = 1, \ldots, 7}$

$$H = \begin{bmatrix} 3 & -3 & 0 & 1 & 0 & 0 & -1 \\ -2 & 3 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & -2 & 0 & -1 & 3 \end{bmatrix}$$

The cluster consensus for both clusters with $k = 1.1$ are shown in Figs.2 and 3. We can see two clusters in same figure in Fig.4 and the error of system is shown in Fig.5. The position trajectories of agents are shown in Fig.4 and x_1 to x_3 converge to one consensus value which is time varying and x_4 to x_7 converge to another consensus value. So cluster consensus is achieved for two clusters.

Example 2. Now, A topology with seven agents is shown in Fig. 6. The graph consists of three clusters and each cluster has spanning tree. The dynamics and initial condition are the

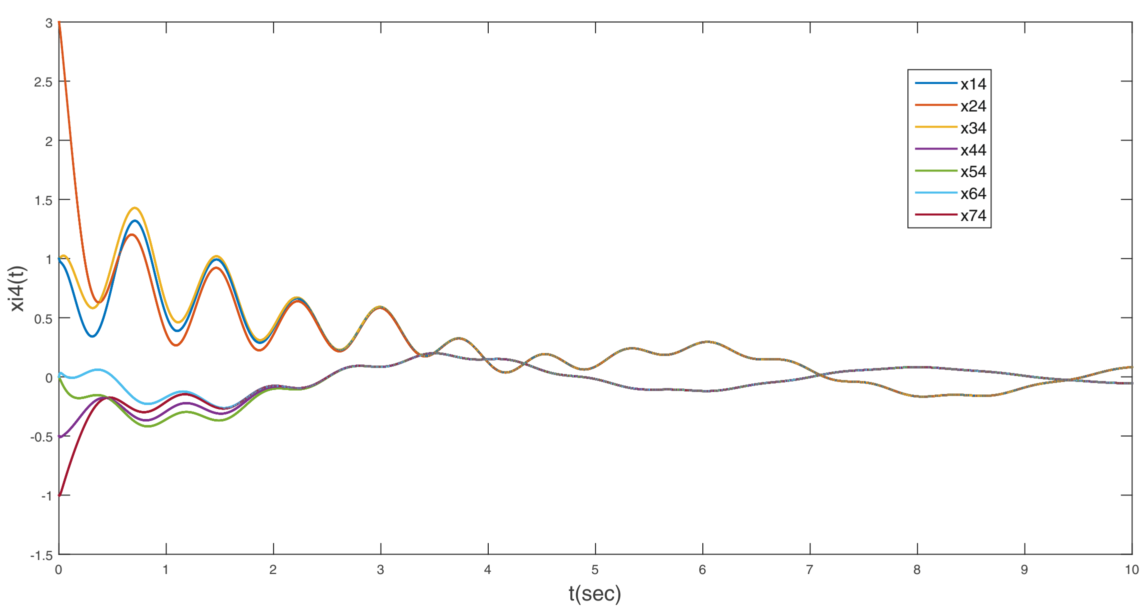

Fig. 14 Cluster consensus for $\mathbf{x_{i4}(t), i = 1, \ldots, 7}$

Table. 1 Comparison of examples

Order of system	Graph	Number of clusters	Consensus time
first	7 agents	2	2 s
first	7 agents	3	4 s
second	7 agents	2	11 s
general	7 agents	2	2 s

same as those given in previous simulation with control algorithm (2). Simulation results for cluster consensus with these conditions are shown in Fig.7.

The matrix H for Fig.6 is given as

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 3 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -1 & 1 & -2 & 2 \end{bmatrix}$$

Simulation results for cluster consensus with $k = 1.2$ are shown in Fig.7. The position trajectories of agents are shown in Fig.7 and x_1 to x_2 converge to one consensus value which is time varying, x_3 to x_5 converge to other consensus value and x_6 to x_7 converge to other consensus value. So cluster consensus is achieved for three clusters.

Example 3. The nonlinear dynamics is given by $f(x, v, t) = -x + v \sin(t)$. A topology with seven agents is shown in Fig. 1 and system (14) with control algorithm (15). Initial conditions

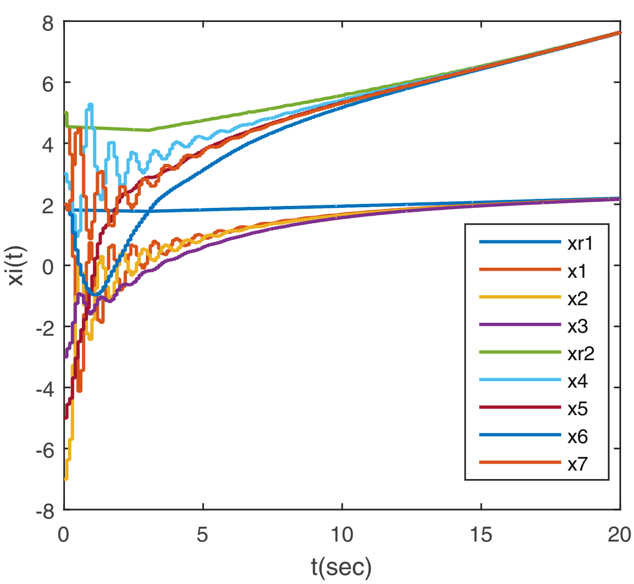

Fig. 15 Cluster consensus for the graph in Fig.1 with time delay under impulsive controller with $\boldsymbol{T}$ = **0.1**

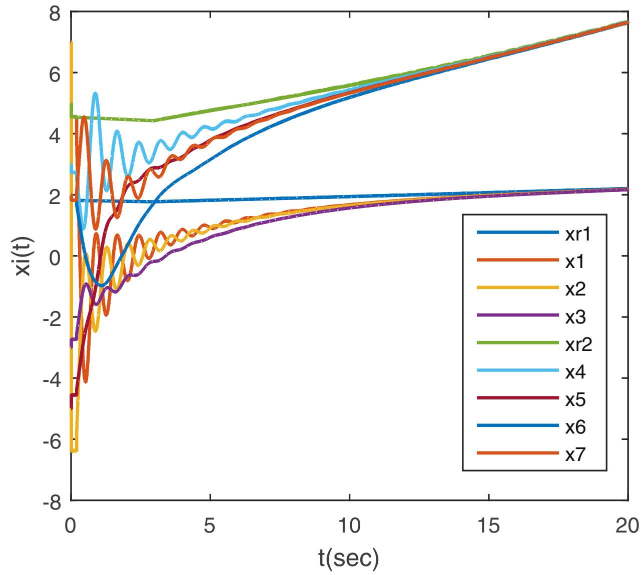

Fig. 16 Cluster consensus for the graph in Fig.1 with time delay under impulsive controller with $\boldsymbol{T}$ = **0.01**

are considered as $X(0) = [6 \quad 3 \quad 1 \quad -3 \quad 4 \quad 2 \quad 0]^T$ and $V(0) = [6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6]^T$. The cluster consensus for both clusters and both states (x, v) with $k = 2.1$ are shown in Figs.8 and 9

The position trajectories of agents are shown in Fig.8 and the velocity trajectories of agents are shown in Fig.9. In Fig.8, x_1 to x_3 converge to one consensus value which is time varying and x_4 to x_7 converge to another consensus value. In Fig.9, v_1 to v_3 converge to one consensus value which is time varying and v_4 to v_7 converge to another consensus value which means cluster consensus is achieved.

Example 4. In this example, each agent is a single-link manipulator which is shown in Fig.10. The revolving joints are actuated by a DC motor and a linear spring is used to model the elasticity of the joint. The state- space model in which the states of the system are position and velocity of motor and link is as follows

$$\begin{aligned} \dot{\theta}_m &= \omega_m \\ \dot{\omega}_m &= \frac{k}{J_m}(\theta_l - \theta_m) - \frac{k}{J_m}\omega_m + \frac{K_T}{J_m}u \\ \dot{\theta}_l &= \omega_l \\ \dot{\omega}_l &= -\frac{k}{J_l}(\theta_l - \theta_m) - \frac{m\,g\,h}{J_l}\sin(\theta_l) \end{aligned}$$

where J_m and J_l are the inertia of the motor and link, θ_m and θ_l are the angular rotation of the motor and the angular position of the link, respectively; and ω_m and ω_l are the angular velocity of the motor and link.

The dynamics of manipulator is in the form of (24) and parameters are defined as follows [35]:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.26 & 48.6 & 0 \\ 0 & 0 & 0 & 10 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, C = I_4$$

$x_i(t) = [\theta_{mi}, \omega_{mi}, \theta_{li}, \omega_{li}]^T = [x_{i1}(t), x_{i2}(t), x_{i3}(t), x_{i4}(t)]^T$,

$f(x_i, t) = [0, 0, 0, 0.333\sin(x_{i3}(t))]^T$.

Initial conditions are considered as $x_1(0) = [1 \quad 0 \quad 3 \quad 1]^T$, $x_2(0) = [-1 \quad 1 \quad 1 \quad 3]^T$, $x_3(0) = [1 \quad 0 \quad 1 \quad 1]^T$, $x_4(0) = [0 \quad 1 \quad 1 \quad -0.5]^T$, $x_5(0) = [0 \quad 1 \quad 1 \quad 0]^T$, $x_6(0) = [1 \quad 0 \quad -2 \quad 0]^T$, $x_1(0) = [0 \quad 0 \quad 1 \quad -1]^T$.

Cluster consensus for the manipulator is achieved for the graph in Fig.1 with $l = 0.333$, $\beta = 0.7$, $\alpha = 28.75$ and P is achieved by LMI (28) as follows:

$$P = \begin{bmatrix} 0.5283 & -0.9689 & 0.4636 & -0.0826 \\ -0.9689 & 122.4712 & -0.0074 & 0.0009 \\ 0.4636 & -0.0074 & 0.5134 & -0.0482 \\ -0.0826 & 0.0009 & -0.0482 & 0.0195 \end{bmatrix}$$

Simulation results are shown in Figs.11, 12, 13 and 14.

The state trajectories of agents are shown in Figs.11, 12, 13 and 14. In Fig.11, first state of agents x_{11} to x_{31} converge to one consensus value which is time varying and x_{41} to x_{71} converge to another consensus value. In Fig.12, second state of agents x_{12} to x_{32} converge to one consensus value which is time varying and x_{42} to x_{72} converge to another consensus value. In Fig.13, first state of agents x_{13} to x_{33} converge to one consensus value which is time varying and x_{43} to x_{73} converge to another consensus value. In Fig.14, first state of agents x_{14} to x_{34} converge to one consensus value which is time varying and x_{44} to x_{74} converge to another consensus value. So cluster consensus is achieved for all states of agents. A comparison of these examples is shown in Table.1.

Example5. A directed graph with seven followers and two leaders which communicated with first and fourth agents, is shown in Fig.1. We choose $f_1(x_i(t)) = \frac{1}{4}sin\left(\frac{x_i(t)}{10}\right)$, $f_2(x_i(t)) = sin\left(\frac{x_i(t)}{20}\right)$, $\tau_1 = 0.8$, $\tau_2 = 0.19$, $\tau_3 = 0.19$, $i = 1, .., n$. Initial condition is considered as $X(0) = [5 \quad -7 \quad -3 \quad 3 \quad -5 \quad 2 \quad 2]^T$ and $x_{r1}(0) = 2$, $x_{r2}(0) = 5$. The condition of Theorem 5. with $\mu_{ik} = -0.78$ and $\rho = 0.2$ is satisfied and simulation result with $t_k - t_{k-1} = 0.1$ is shown in Fig.15. The condition of Theorem 5. with $\mu_{ik} = -0.137$ and $\rho = 0.86$ is satisfied and simulation result with $t_k - t_{k-1} = 0.01$ is shown in Fig.16

Figs.15 and 16 show the position trajectories of the agents with time delays for $T = 0.1$ and $T = 0.01$ and the state of agents x_1 to x_3 follow the first leader and x_4 to x_7 follow the second leader after 15 s.

6 Conclusion

This paper studied the cluster consensus for multi-agent systems which are governed by first-order, second-order, general high-order nonlinear systems and multi-agent systems in the presence of time delay under directed topologies. The Lyapunov stability and matrix theories are employed to achieve cluster consensus. A feedback control low has been designed to reach cluster consensus which means convergence of states/outputs of all agents in same cluster to certain quantities which are different from those of other clusters. Simulation examples illustrated the efficiency of theoretical analysis. A single link flexible-joint manipulator has been used to show the validation of cluster consensus problem for general high-order multi-agent systems.

Future work will focus on investigation of cluster consensus problem for high-order nonlinear multi-agent systems in the presence of communication delay and switching topologies.

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