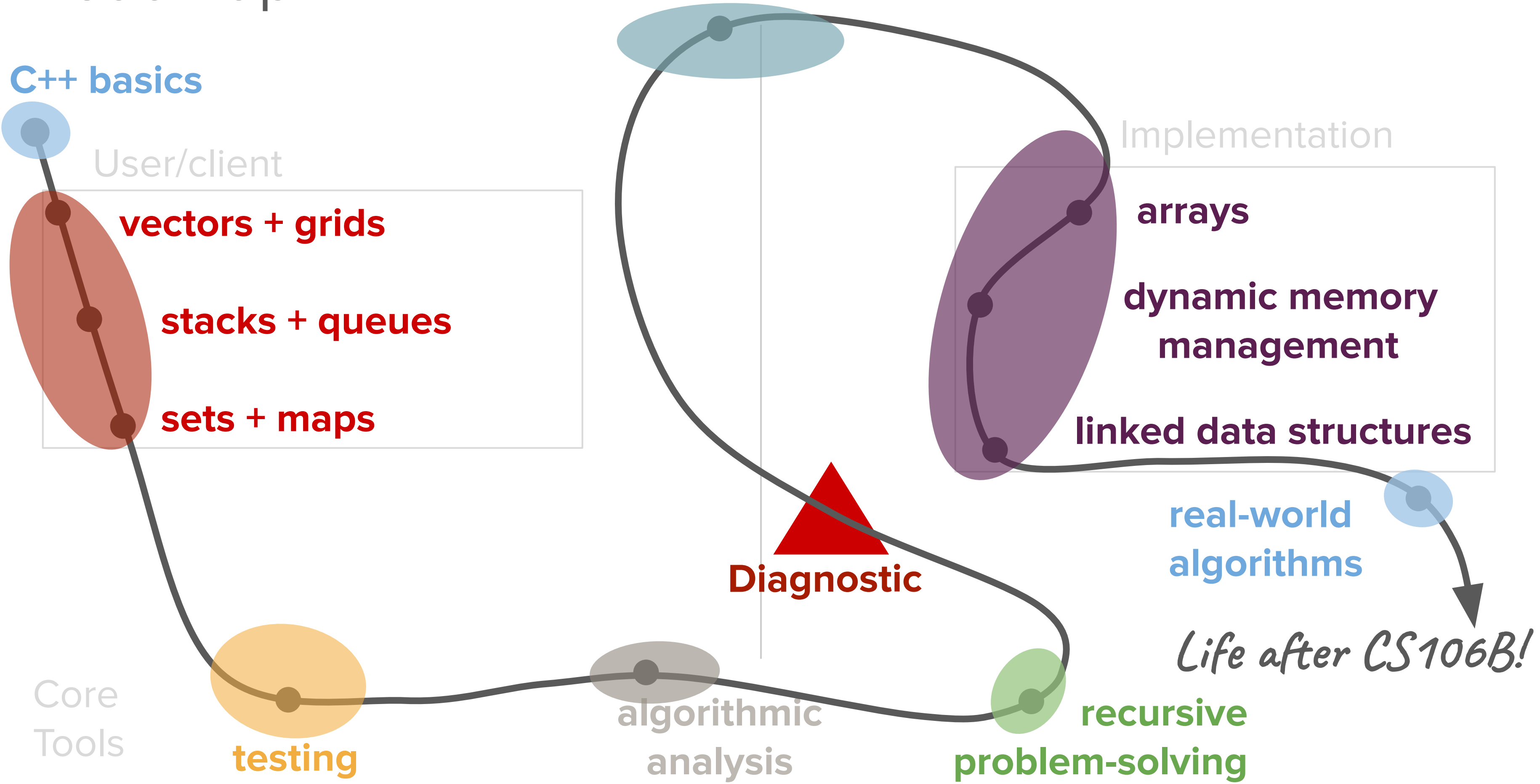
Big-O Notation and Algorithmic Analysis

What do you think makes some algorithms "faster" or "better" than others?

(put your answers the chat)

Roadmap



C++ basics

User/client

vectors + grids

stacks + queues

sets + maps

Core Tools

testing

algorithmic analysis

recursive problem-solving

Diagnostic

Object-Oriented Programming

Implementation

arrays

dynamic memory management

linked data structures

real-world algorithms

Life after CS106B!

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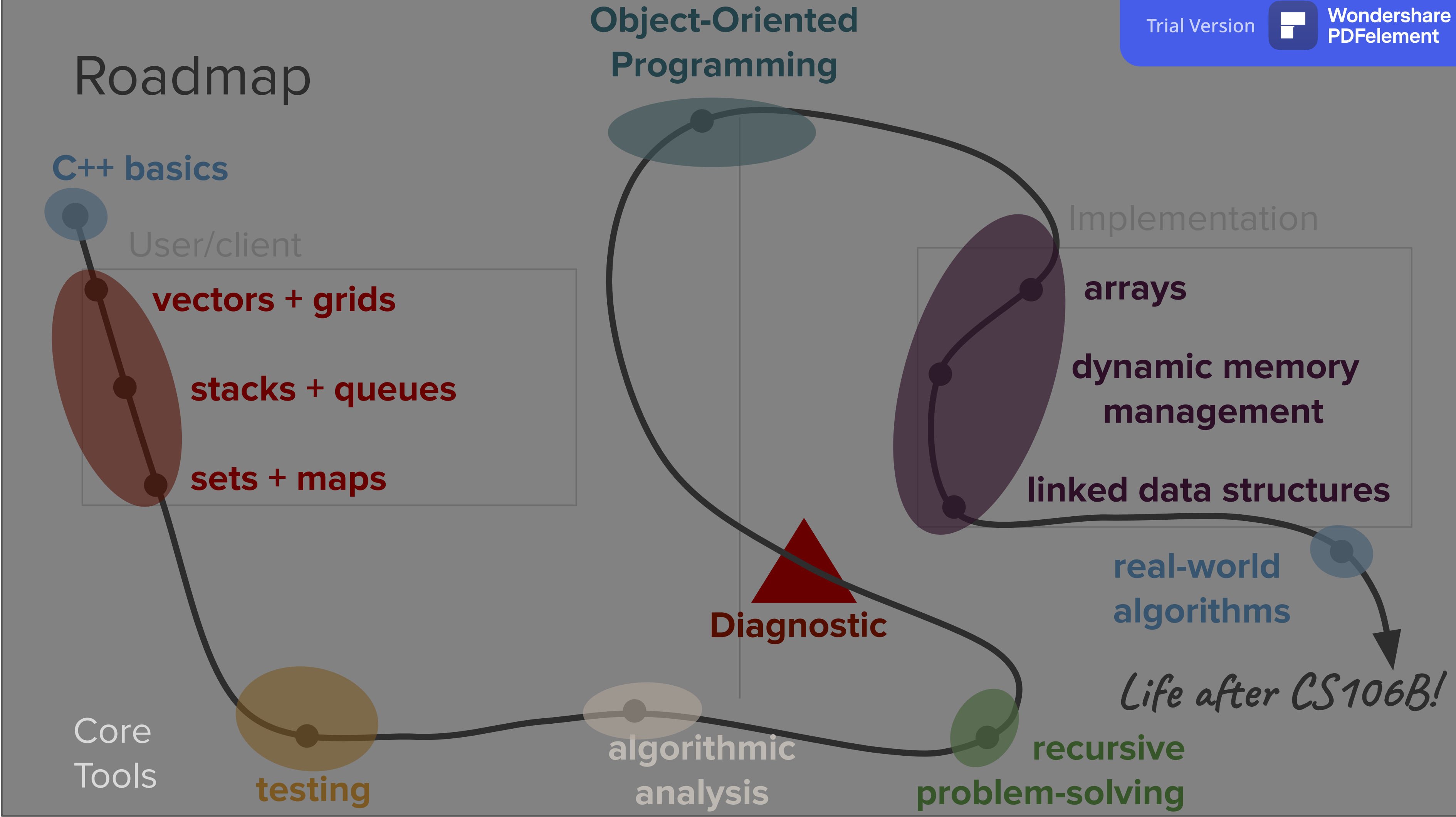

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Life after CS106B!

Today's question

How can we formalize the notion of efficiency for algorithms?

Today's topics

1. Review(-ish)
2. Big-O Notation
3. Algorithmic Analysis

Review(-ish)

(using nested data structures)

Note on Breadth-First Search (and learning overall)

- We covered the intuition, data structure choices, and pseudocode of breadth-first search yesterday ***and*** coded it up to make a working program
    - This is a lot to take in!
- As Kylie mentioned at the beginning of the quarter, we want to normalize struggle in this class.
    - We cover content very quickly in this class!
    - If you leave lecture feeling you don't understand the algorithm/concept covered that day, don't worry.
    - Lecture is always your first exposure to content – very few people can build deep understanding upon the first exposure
    - The assignments (and section and office hours and LaIR) are your chance to revisit lecture, practice, and really nail down the concepts!
    - Struggling along the way means that you are really ***learning***.

Nested Data Structures

Nested Data Structures

- We've already seen one example of nested data structures when we used the **`Queue<Stack<string>>`** to keep track of our search for word ladders.

- Nesting data structures (using one ADTs as the data type inside of another ADT) is a great way of organizing data with complex structure.

- You will thoroughly explore nested data structures (specifically nested Sets and Maps) in Assignment 2!

Nested Data Structures Example

- Imagine we are designing a system to keep track of feeding times for the different animals at a zoo
- Requirements: We need to be able to quickly look up the feeding times associated with an animal if we know it's name. We need to be able to store multiple feeding times for each animal. The feeding times should be stored in the order in which the feedings should happen.
- Data Structure Declaration
    - `Map<string, Vector<string>>`

Quick lookup by animal name

Nested Data Structures Example

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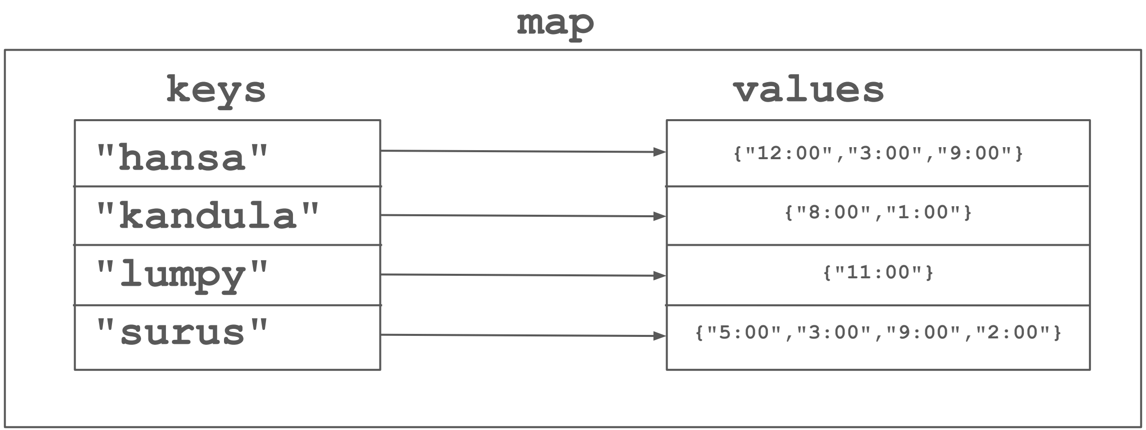
- Requirements: We need to be able to quickly look up the feeding times associated with an animal if we know it's name. We need to be able to store multiple feeding times for each animal. The feeding times should be stored in the order in which the feedings should happen.
- Data Structure Declaration
  - `Map<string, Vector<string>>`

Store multiple, ordered feeding times per animal

Nested Data Structures Example

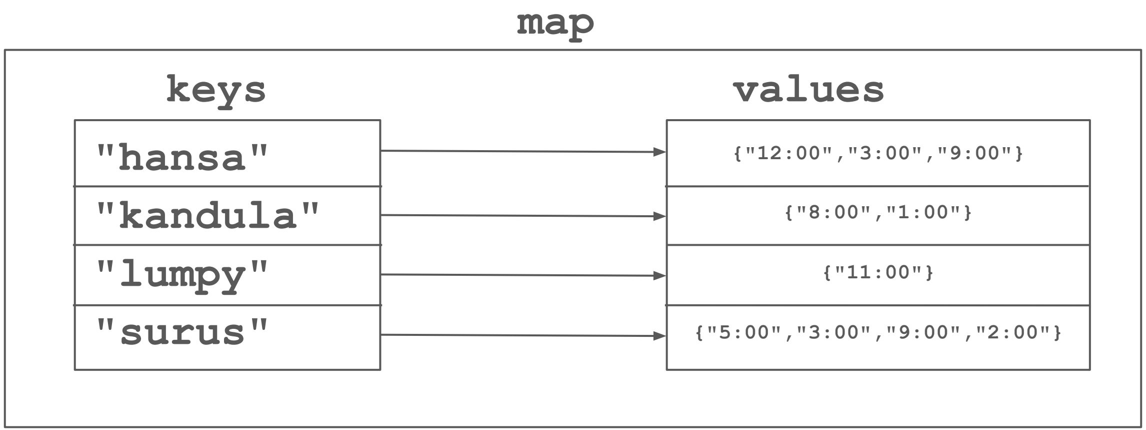


map

keys	values
"hansa"	{"12:00","3:00","9:00"}
"kandula"	{"8:00","1:00"}
"lumpy"	{"11:00"}
"surus"	{"5:00","3:00","9:00","2:00"}

Wonderful diagram and animal naming borrowed from Sonja Johnson-Yu

Nested Data Structures Example



map

keys	values
"hansa"	{"12:00","3:00","9:00"}
"kandula"	{"8:00","1:00"}
"lumpy"	{"11:00"}
"surus"	{"5:00","3:00","9:00","2:00"}

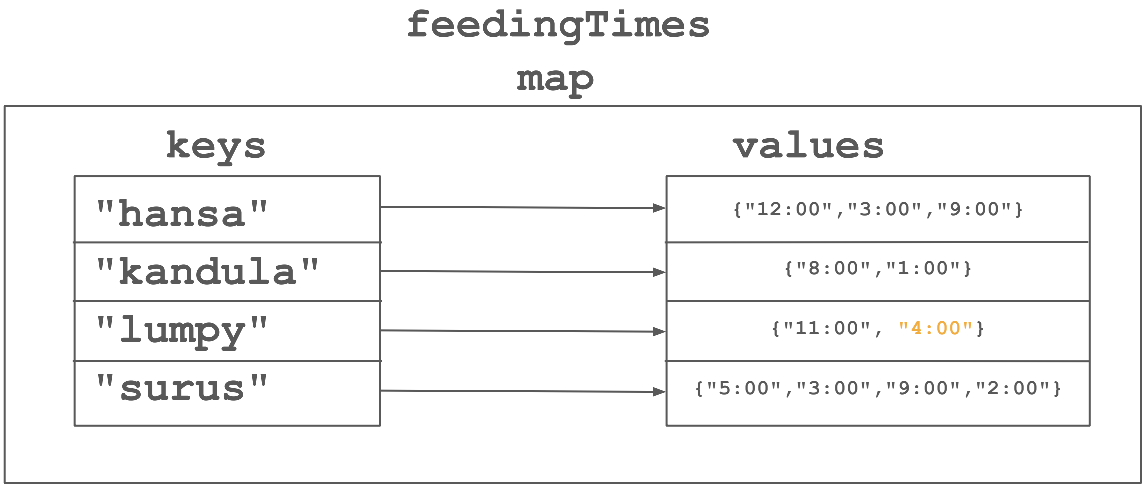
How do we use modify the internal values of this map?

Nested Data Structures Example

Goal: We want to add a second feeding time of 4:00 for "lumpy".

Which of the following three snippets of code will correctly update the state of the map?

```
1.   feedingTimes["lumpy"].add
     ("4:00");
2.   Vector<string> times =
     feedingTimes["lumpy"];
     times.add("4:00");
3.   Vector<string> times =
     feedingTimes["lumpy"];
     times.add("4:00");
     feedingTimes["lumpy"] =
     times;
```



feedingTimes
map

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"hansa"	{"12:00","3:00","9:00"}
"kandula"	{"8:00","1:00"}
"lumpy"	{"11:00", "4:00"}
"surus"	{"5:00","3:00","9:00","2:00"}

[ ] Operator and = Operator Nuances

- When you use the [] operator to access an element from a map, you get a reference to the map, which means that any changes you make to the reference will be persistent in the map.

  -
    ```
    feedingTimes["lumpy"].add("4:00");
    ```
- However, when you use the = operator to assign the result of the [] operator to a variable, you get a copy of the internal data structure.

  -
    ```
    Vector<string> times = feedingTimes["lumpy"]; // this makes a copy
    times.add("4:00"); // modifies the copy, not the actual map value!!!
    ```
- If you choose to store the internal data structure in a variable, you must do an explicit reassignment to get your changes to persist

  -
    ```
    Vector<string> times = feedingTimes["lumpy"]; // this makes a copy
    times.add("4:00"); // modifies the copy
    feedingTimes["lumpy"] = times; // stores the modified copy in the map
    ```

Nested ADTs Summary

- Powerful
  - Can express highly structured and complex data
  - Used in many real-world systems
- Tricky
  - With increased complexity comes increased cognitive load in differentiating between the levels of information stored at each level of the nesting
  - Specifically in C++, working with nested data structures can be tricky due to the fact that references and copies show up at different points in time. Follow the correct paradigms presented earlier to stay on track!

One Final Note… Const Reference

- Passing a large object (e.g. a million-element Vector) by value makes a copy, which can take a lot of time.
- Taking parameters by reference avoids making a copy, but risks that the object gets tampered with in the process.

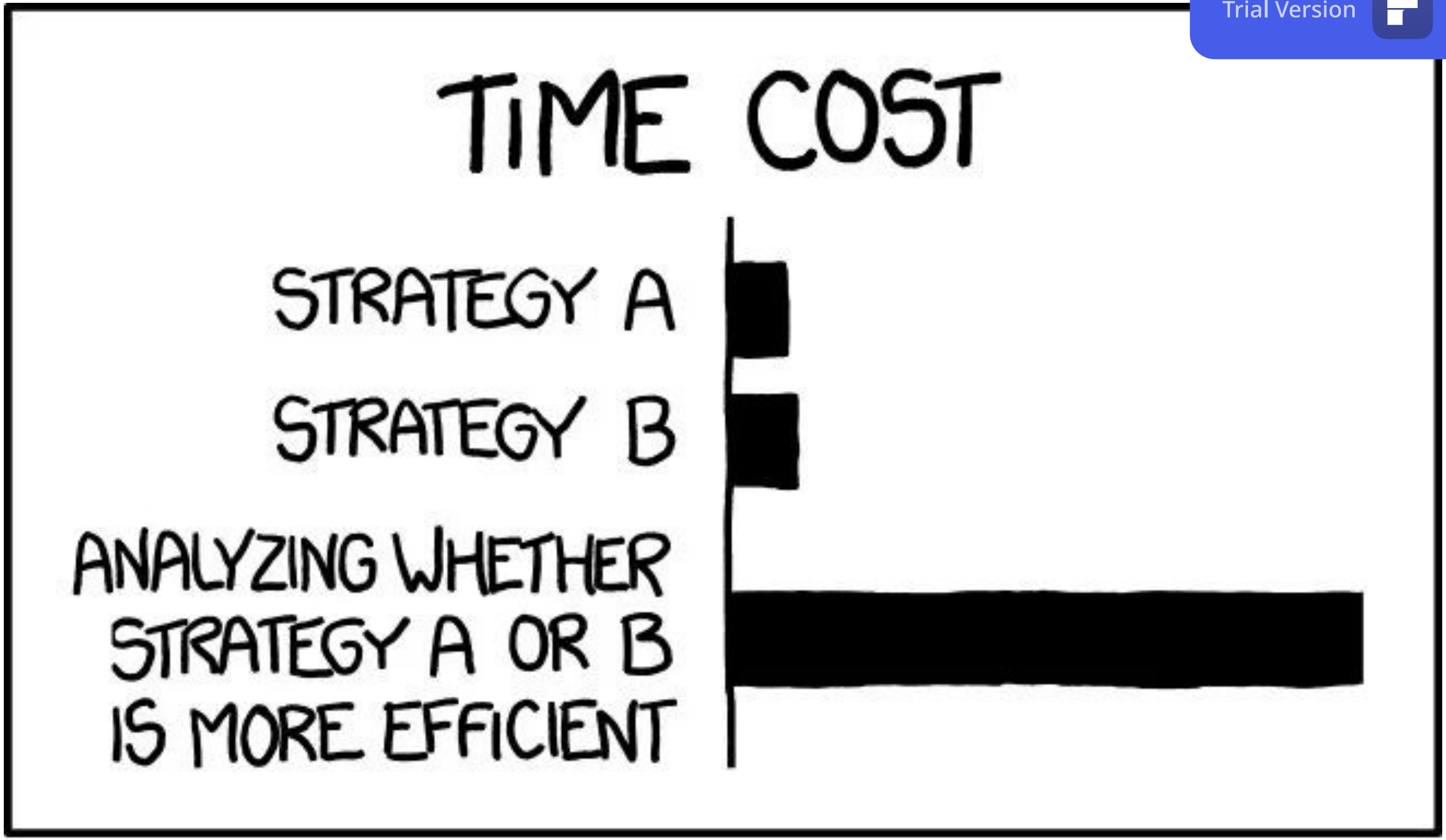
篡改

- As a result, it’s common to have functions that take objects as parameters take their argument by const reference:
  - The “by reference” part avoids a copy.
  - The “const” (constant) part means that the function can’t change that argument.
- For example:

```
void proofreadLongEssay(const string& essay) {
    /* can read, but not change, the essay. */
}
```

Example from slides made by Keith Schwarz

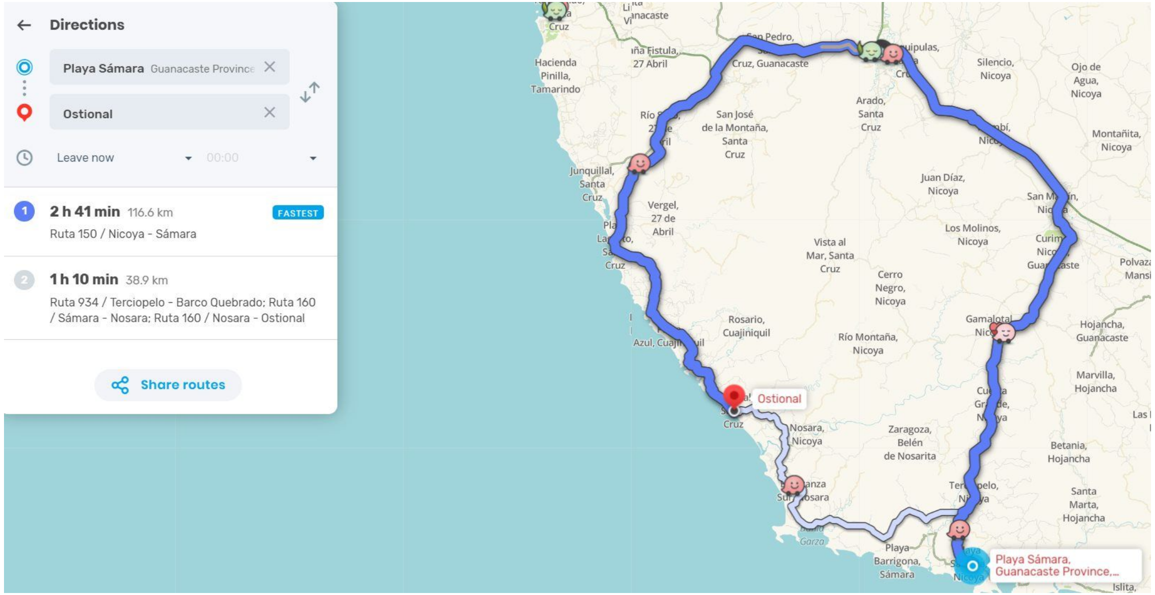
How can we formalize the notion of efficiency for algorithms?



THE REASON I AM SO INEFFICIENT

Why do we care about efficiency?

- Implementing inefficient algorithms may make solving certain tasks impossible, even with unlimited resources



Directions

Playa Sámara Guanacaste Province

Ostional

Leave now 00:00

1 2 h 41 min 116.6 km FASTEST
Ruta 150 / Nicoya - Sámara

2 1 h 10 min 38.9 km
Ruta 934 / Terciopelo - Barco Quebrado; Ruta 160 / Sámara - Nosara; Ruta 160 / Nosara - Ostional

Share routes

Why do we care about efficiency?

- Implementing inefficient algorithms may make solving certain tasks impossible, even with unlimited resources

- Implementing efficient algorithms allows us to solve important problems, often with limited resources available

- If we can quantify the efficiency of an algorithm, we can understand and predict its behavior when we apply it to unseen problems

Assignment 1 Redux

- In Assignment 1, you implemented three different algorithms for finding perfect numbers
  - Exhaustive Search
    - Runtime predictions to find 5th perfect number: Anywhere from 25-100+ days
  - Smarter Search
    - Runtime predictions to find 5th perfect number: Anywhere from a couple minutes to 1 hour
  - Euclid's Algorithm
    - Actual runtime to predict 5th perfect number: Less than a second!
- Core idea: Although each individual experienced dramatically different real runtimes for these three algorithms, there is a clear distinction here between "fast"/"efficient" and "slow"/"inefficient" algorithms

Estimating Quantities

[breakout rooms]

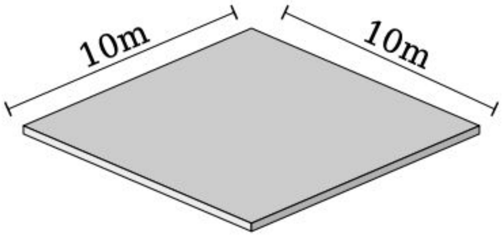
Leveraging Intuition

- Today's activity is going to look a little bit different than usual. There's no code, no pseudocode, and nothing that resembles C++.

- Instead, you're going to be presented with a set of 5 scenarios, where you have two similar items of different magnitudes, one small and one larger. You know the exact magnitude of the smaller item – can you predict what the magnitude of the larger item will be based on the intuitive visual relationship?

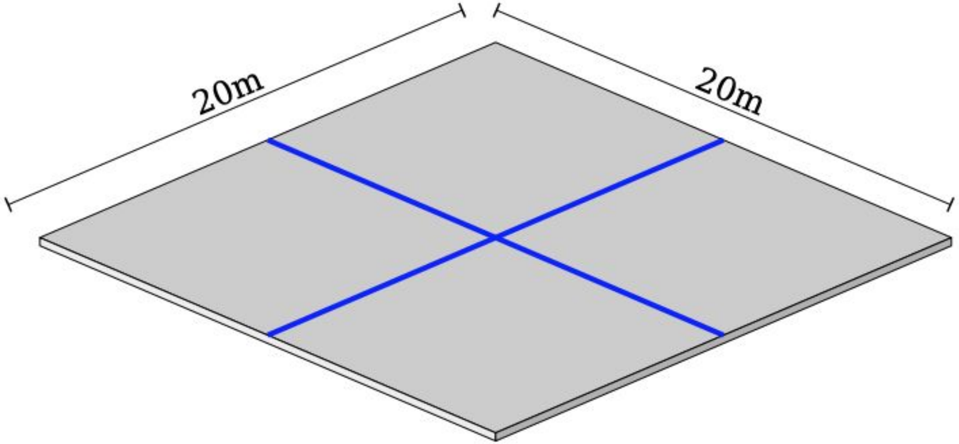
- Answer the questions in the next 5 slides by discussing with your groupmates. Note down your answers (and your reasoning) in the `instructions.txt` file in Ed.

Example 1



Mass: 100kg

Mass is about 400kg (4 smaller squares make up the larger square)

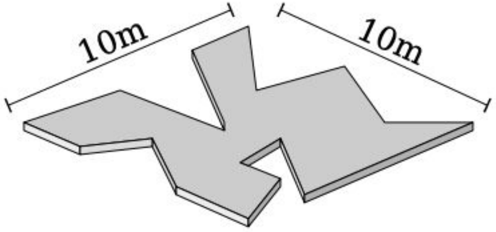


These two square plates are made of the same material.

They have the same thickness.

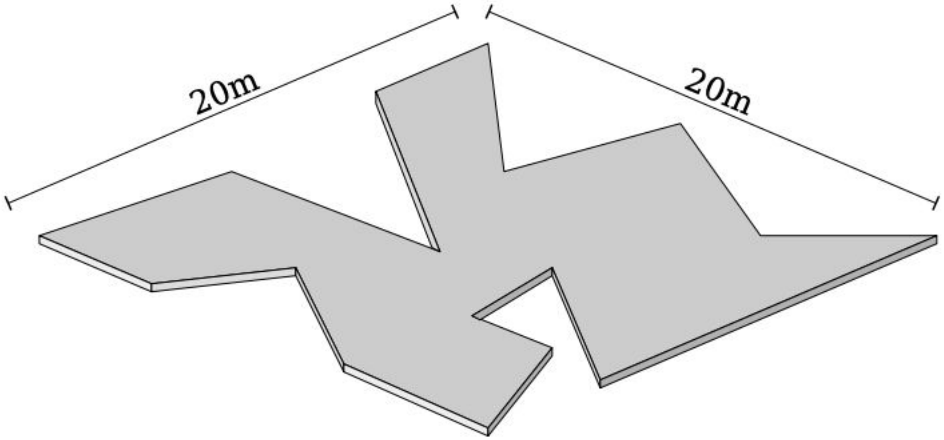
What’s your best guess for the mass of the second square?

Example 2



Mass: 60kg

Mass is about 240kg (side length is doubled, overall are increases by factor of 4)



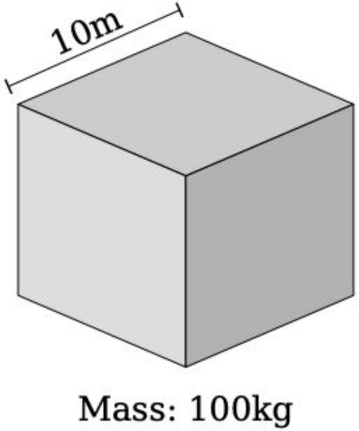
These two square plates are made of the same material.

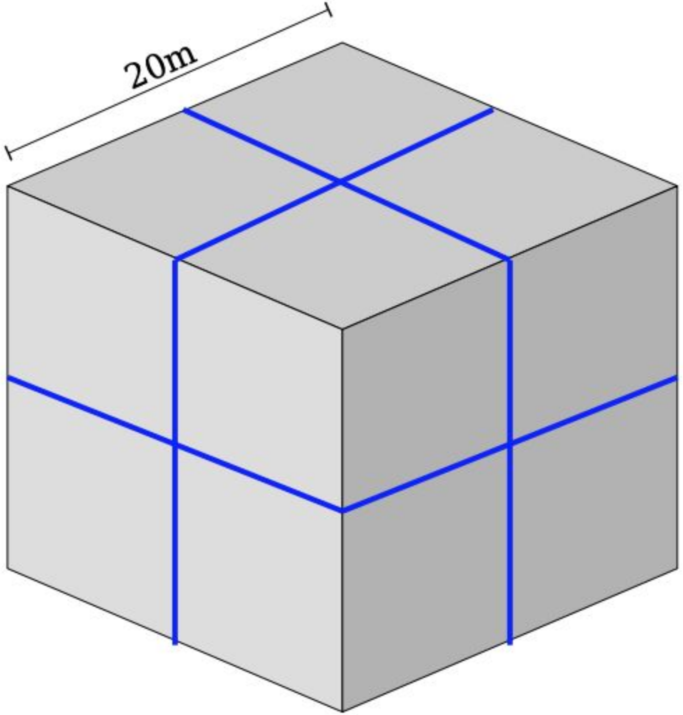
They have the same thickness.

What’s your best guess for the mass of the second square?

Example 3

Mass is about 800kg (8 smaller cubes make up the larger cube)



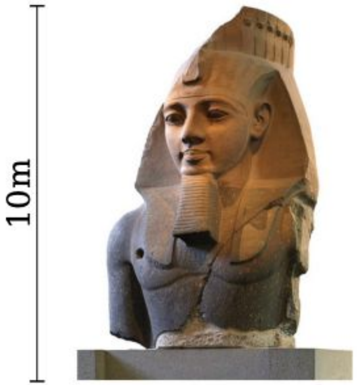


These two cubes are made of the same material.

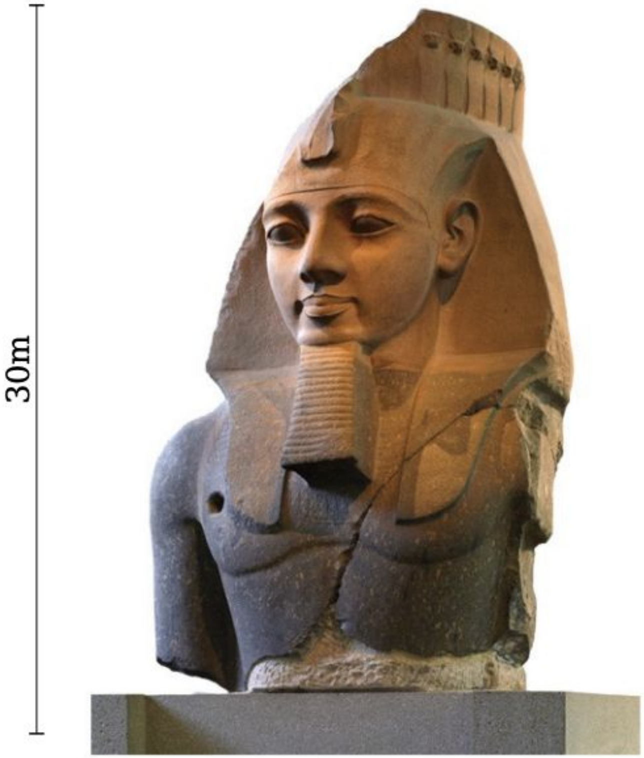
What’s your best guess for the mass of the second cube?

Example 4

Mass is about 27000kg (statue dimensions increased by factor of 3, and volume increases by factor of 27)



Mass: 1,000kg



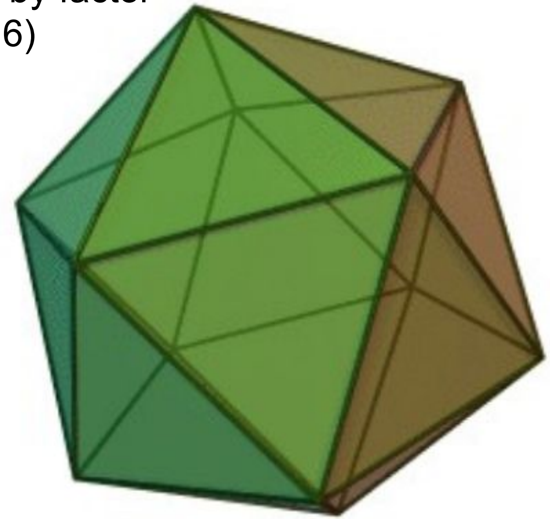
These two statues are made of the same material.

What’s your best guess for the mass of the second statue?

Example 5

All sides of each triangle are $10m$ long.

Paint required:
90L



All sides of each triangle are $40m$ long.

How much paint is needed to paint the surface of the larger icosahedron?

Paint Required is about 1440L (side length grows by factor of 4, area increases by factor of 4^2 = 16)

Key Takeaway

Knowing the rate at which some quantity scales allows you to predict its value in the future, even if you don't have an exact formula.

Announcements

Announcements

- Assignment 2 was released last night. It will be due on **Wednesday, July 8**.
    - After feedback from several students, we've decided to change the assignment deadlines to **11:59pm PDT** instead of your local timezone.
    - YEAH hours: **Hosted by Trip today, 7/2 at 7pm PDT**. The Zoom info is posted on the Zoom details page of the course website
    - Make sure to follow the instructions to customize the Qt debugger to work nicely with the Stanford C++ collections ***before*** starting on the assignment.
    - This assignment is a step-up in complexity compared to A1 – get started early!
- If you haven't already, come visit Trip, Kylie, and me at our office hours!

Big-O Notation

Big-O Notation

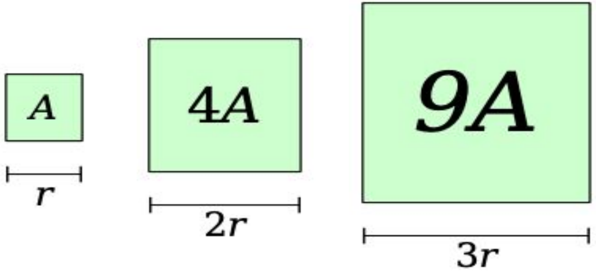
- **Big-O notation** is a way of quantifying the rate at which some quantity grows.
- Example:
  - A square of side length `r` has area `O(r`^{2}`)`.

The "O" stands for "on the order of", which is a growth prediction, not an exact formula

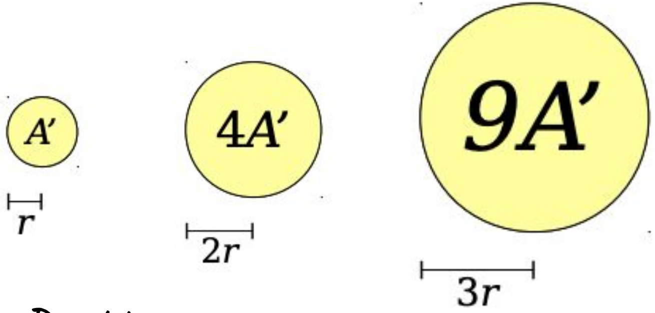
Big-O Notation

- **Big-O notation** is a way of quantifying the rate at which some quantity grows.
- Example:
  - A square of side length **r** has area $O(r^2)$.
  - A circle of radius **r** has area $O(r^2)$.

This just says that these quantities grow at the same relative rates. It does not say that they're equal!



Doubling r increases area 4x
Tripling r increases area 9x



Doubling r increases area 4x
Tripling r increases area 9x

Big-O in the Real World

Big-O Example: Network Value

- Metcalfe's Law
  - The value of a communications network with **n** users is $O(n^2)$.
- Imagine a social network has 10,000,000 users and is worth $10,000,000. Estimate how many users it needs to have to be worth $1,000,000,000.

1. 10,000,000
2. 50,000,000
3. [✓ 100,000,000]
4. 1,000,000,000

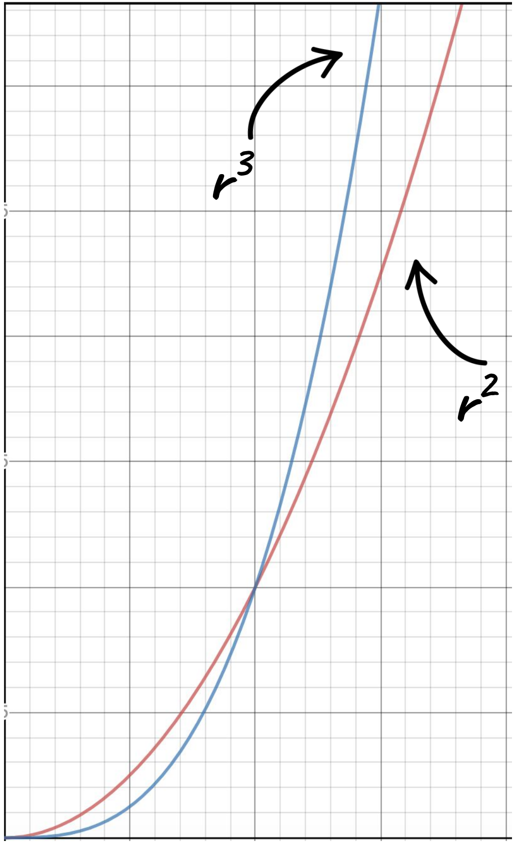
Big-O Example: Network Value

- Metcalfe's Law
  - The value of a communications network with n users is $O(n^2)$.
- Imagine a social network has 10,000,000 users and is worth $10,000,000. Estimate how many users it needs to have to be worth $1,000,000,000.
- **Reasonable guess:** The network needs to grow its value 100×. Since value grows quadratically with size, it needs to grow its user base 10×, requiring 100,000,000 users.

平方

Big-O Example: Cell Size

- Question: Why are cells tiny?
- Assumption: Cells are spheres
- A cell absorbs nutrients from its environment through its surface area.
  - Surface area of the cell: `O(r`^{2}`)`
- A cell needs to provide nutrients all throughout its volume
  - Volume of the cell: `O(r`^{3}`)`
- As a cell gets bigger, its resource *intake* grows slower than its resource *consumption*, so each part of the cell gets less energy.



Big-O Example: Manufacturing

- You’re working at a company producing cat toys. It costs you some amount of money to produce a cat toy, and there was some one-time cost to set up the factory.
- What data would you need to gather to estimate the cost of producing ten million cat toys?

This term grows as a function of n

This term does not grow

$$\begin{aligned}\mathtt{Cost(n)} &= \mathtt{n} \times \mathtt{costPerToy} + \mathtt{startupCost}\\ &= \mathtt{O(n)}\end{aligned}$$

Nuances of Big-O

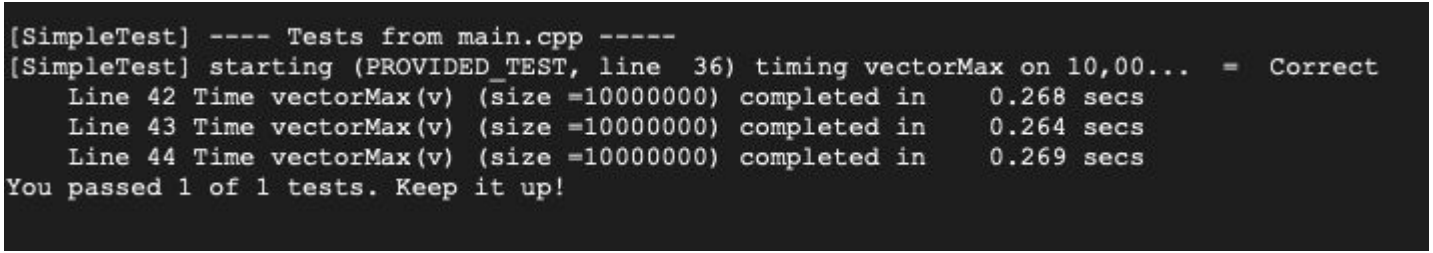
- Big-O notation is designed to capture **the rate at which a quantity grows.** It does not capture information about
    - leading coefficients: the area of a square and a circle are both O(r^2).
    - lower-order terms: there may be other factors contributing to growth that get glossed over.
- However, it's still a **very powerful tool for predicting behavior.**

Analyzing Code

How can we apply Big-O to computer science?

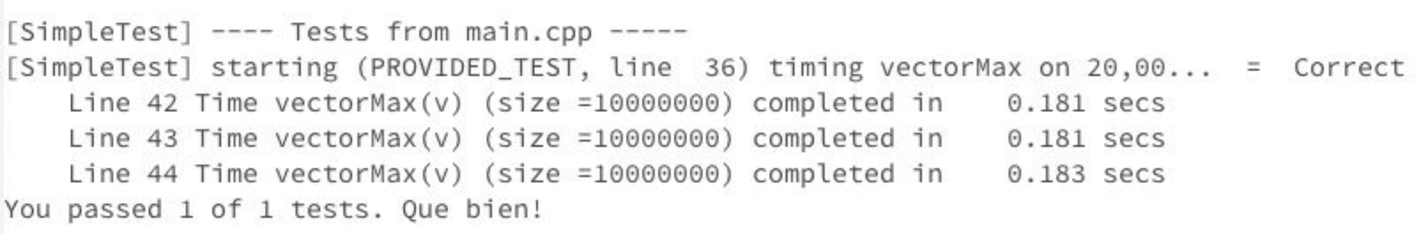
Why runtime isn't enough

- What is runtime?
    - Runtime is simply the amount of real time it takes for a program to run



```
[SimpleTest] ---- Tests from main.cpp -----
[SimpleTest] starting (PROVIDED_TEST, line  36) timing vectorMax on 10,00...  =  Correct
    Line 42 Time vectorMax(v) (size =10000000) completed in    0.268 secs
    Line 43 Time vectorMax(v) (size =10000000) completed in    0.264 secs
    Line 44 Time vectorMax(v) (size =10000000) completed in    0.269 secs
You passed 1 of 1 tests. Keep it up!
```

Nick's 2012 MacBook



```
[SimpleTest] ---- Tests from main.cpp -----
[SimpleTest] starting (PROVIDED_TEST, line  36) timing vectorMax on 20,00...  =  Correct
    Line 42 Time vectorMax(v) (size =10000000) completed in    0.181 secs
    Line 43 Time vectorMax(v) (size =10000000) completed in    0.181 secs
    Line 44 Time vectorMax(v) (size =10000000) completed in    0.183 secs
You passed 1 of 1 tests. Que bien!
```

Ed's powerful computers

Why runtime isn’t enough

- Measuring wall-clock runtime is less than ideal, since
  - It depends on what computer you’re using,
  - What else is running on that computer,
  - Whether that computer is conserving power,
  - Etc.

- Worse, **individual runtimes can’t predict future runtimes**.

- Let's develop a computer-independent efficiency metric using big-O!

Analyzing Code:
`vectorMax()`

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
      if (currentMax < v[i]) {
          currentMax = v[i];
      }
  }
  return currentMax;
 }
```

Assume any individual statement takes one unit of time to execute.

If the input **`Vector`** *has* **`n`** *elements, how many time units will this code take to run?*

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
      if (currentMax < v[i]) {
          currentMax = v[i];
      }
  }
  return currentMax;
 }
```

Total time based on # of repetitions

1 time unit

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
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  }
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 }
```

Total time based on # of repetitions

1 time unit

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      }
  }
  return currentMax;
}
```

Total time based on # of repetitions

1 time unit

1 time unit

1 time unit

vectorMax()

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          currentMax = v[i];
      }
  }
  return currentMax;
 }
```

Total time based on # of repetitions

1 time unit

1 time unit

1 time unit

N time units

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
      if (currentMax < v[i]) {
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      }
  }
  return currentMax;
 }
```

Total time based on # of repetitions

1 time unit

1 time unit

1 time unit

N time units

N-1 time units

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
      if (currentMax < v[i]) {
          currentMax = v[i];
      }
  }
  return currentMax;
}
```

Total time based on # of repetitions

1 time unit

1 time unit

1 time unit

N time units

N-1 time units

N-1 time units

vectorMax()

```
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  int currentMax = v[0];
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      }
  }
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 }
```

Total time based on # of repetitions

1 time unit

1 time unit

1 time unit

N time units

N-1 time units

N-1 time units

(up to) N-1 time units

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
      if (currentMax < v[i]) {
          currentMax = v[i];
      }
  }
  return currentMax;
}
```

Total time based on # of repetitions

1 time unit

1 time unit

1 time unit

N time units

N-1 time units

N-1 time units

(up to) N-1 time units

1 time unit

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
      if (currentMax < v[i]) {
          currentMax = v[i];
      }
  }
  return currentMax;
}
```

Total amount of time

$$4N + 1$$

Is this useful?

What does this tell us?

vectorMax()

```
int vectorMax(Vector<int> &v) {
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++) {
      if (currentMax < v[i]) {
          currentMax = v[i];
      }
  }
  return currentMax;
 }
```

Total amount of time

O(n)

More practical: Doubling the size of the input roughly doubles the runtime. Therefore, the input and runtime have a linear (O(n)) relationship.

Analyzing Code: `printStars()`

printStars()

```
void printStars(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            cout << '*' << endl;
        }
    }
}
```

How much time will it take for this code to run, as a function of n?

Answer using big-O notation.

printStars()

```
void printStars(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            // do a fixed amount of work
        }
    }
}
```

How much time will it take for this code to run, as a function of n?

Answer using big-O notation.

printStars()

```
void printStars(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            // do a fixed amount of work
        }
    }
}
```

How much time will it take for this code to run, as a function of n?

Answer using big-O notation.

printStars()

```
void printStars(int n) {
    for (int i = 0; i < n; i++) {

        // do O(n) time units of work

    }
}
```

How much time will it take for this code to run, as a function of n?

Answer using big-O notation.

printStars()

```
void printStars(int n) {
    for (int i = 0; i < n; i++) {

        // do O(n) time units of work

    }
}
```

How much time will it take for this code to run, as a function of n?

Answer using big-O notation.

printStars()

```
void printStars(int n) {

    // do O(n^2) time units of work

}
```

How much time will it take for this code to run, as a function of n?
Answer using big-O notation.

printStars()

```
void printStars(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            cout << '*' << endl;
        }
    }
}
```

$O(n^2)$

A final analyzing code example

hmmThatsStrange()

```
void hmmThatsStrange(int n) {
    cout << "Mirth and Whimsy" << n << endl;
}
```

The runtime is *completely independent* of the value **n**.

hmmThatsStrange()

```
void hmmThatsStrange(int n) {
    cout << "Mirth and Whimsy" << n << endl;
}
```

How much time will it take for this code to run, as a function of n?
Answer using big-O notation.

hmmThatsStrange()

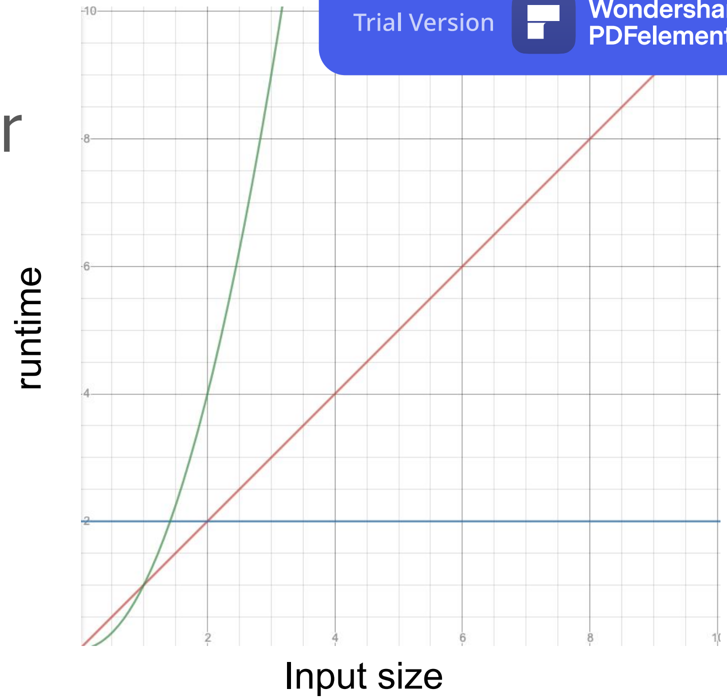
```
void hmmThatsStrange(int n) {
    cout << "Mirth and Whimsy" << n << endl;
}
```

$O(1)$

Applying Big-O to ADTs

Efficiency Categorizations So Far

- Constant Time – O(1)
  - Super fast, this is the best we can hope for!
  - Euclid's Algorithm for Perfect Numbers
- Linear Time – O(n)
  - This is okay, we can live with this
- Quadratic Time – O(n^2)
  - This can start to slow down really quickly
  - Exhaustive Search for Perfect Numbers
- How do all the ADT operations we've seen so far fall into these categories?



ADT Big-O Matrix

- Vectors
  - .size() – O(1)
  - .add() – O(1)
  - v[i] – O(1)
  - .insert() – O(n)
  - .remove() – O(n)
  - .clear() – O(n)
  - traversal – O(n)
- Grids
  - .numRows()/.numCols() – O(1)
  - g[i][j] – O(1)
  - .inBounds() – O(1)
  - traversal – O(n^2)
- Queues
  - .size() – O(1)
  - .peek() – O(1)
  - .enqueue() – O(1)
  - .dequeue() – O(1)
  - .isEmpty() – O(1)
  - traversal – O(n)
- Stacks
  - .size() – O(1)
  - .peek() – O(1)
  - .push() – O(1)
  - .pop() – O(1)
  - .isEmpty() – O(1)
  - traversal – O(n)
- Sets
  - .size() – O(1)
  - .isEmpty() – O(1)
  - .add() – ???
  - .remove() – ???
  - .contains() – ???
  - traversal – O(n)
- Maps
  - .size() – O(1)
  - .isEmpty() – O(1)
  - m[key] – ???
  - .contains() – ???
  - traversal – O(n)

ADT Big-O Matrix

- Vectors
  - .size() – O(1)
  - .add() – O(1)
  - v[i] – O(1)
  - .insert() – O(n)
  - .remove() – O(n)
  - .clear() – O(n)
  - traversal – O(n)
- Grids
  - .numRows()/.numC
    – O(1)
  - g[i][j] – O(1)
  - .inBounds() – O(1)
  - traversal – O(n^2)

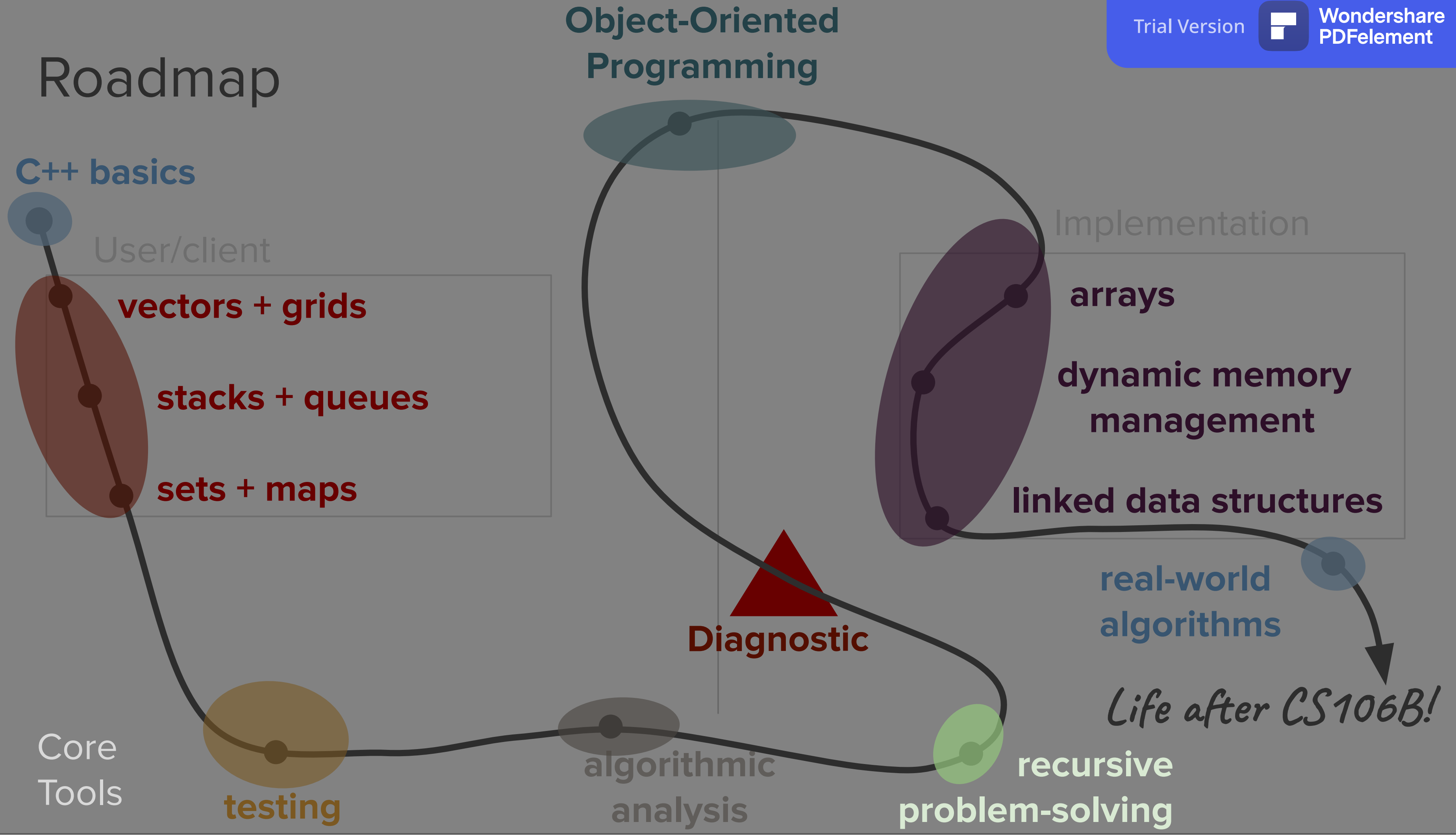
- Queues
  - .size() – O(1)
  - .pop() – O(1)
  - .isEmpty() – O(1)
  - traversal – O(n)

- Sets
  - size() – O(1)
  - Empty() – O(1)
  - d() – ???
  - move() – ???
  - ntains() – ???
  - versal – O(n)
- aps
  - ze() – O(1)
  - Empty() – O(1)
  - key] – ???
  - .contains() – ???
  - traversal – O(n)

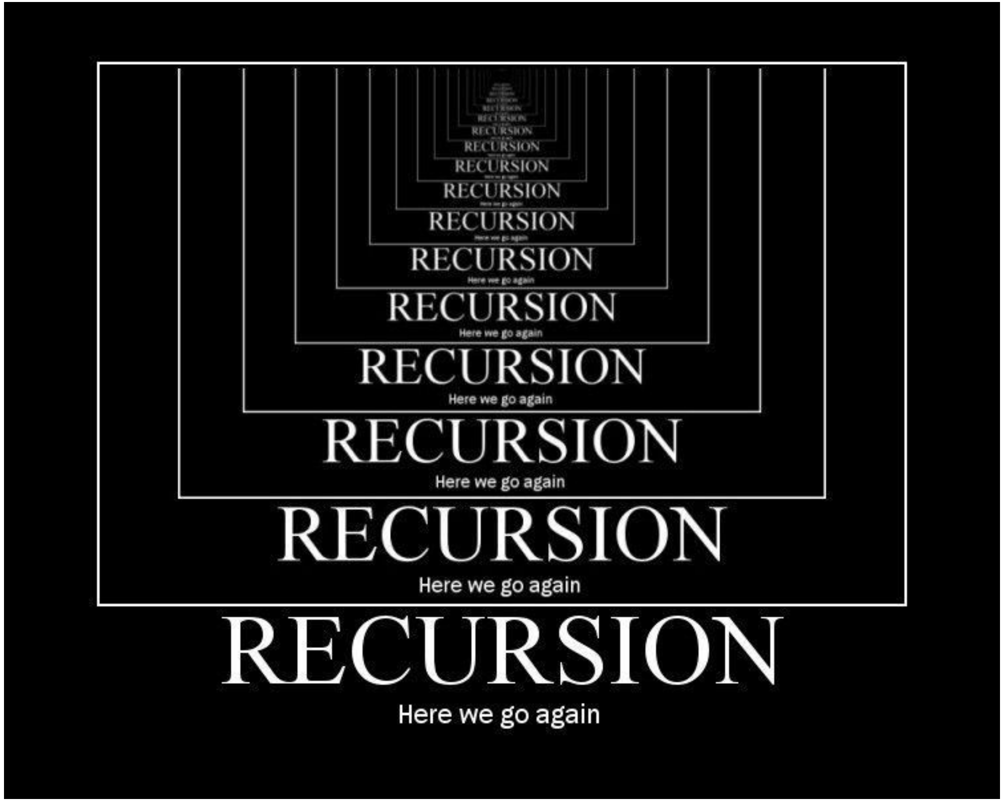
How can we achieve faster than O(n) runtime when searching/storing n elements?

What’s next?

Roadmap


Object-Oriented Programming
C++ basics
User/client
vectors + grids
stacks + queues
sets + maps
Implementation
arrays
dynamic memory management
linked data structures
real-world algorithms
Diagnostic
Life after CS106B!
Core Tools
testing
algorithmic analysis
recursive problem-solving

Recursion


RECURSION
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RECURSION
Here we go again
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Here we go again
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Here we go again