10

Discrete-Time Fourier Series

In this and the next lecture we parallel for discrete time the discussion of the last three lectures for continuous time. Specifically, we consider the representation of discrete-time signals through a decomposition as a linear combination of complex exponentials. For periodic signals this representation becomes the discrete-time Fourier series, and for aperiodic signals it becomes the discrete-time Fourier transform.

The motivation for representing discrete-time signals as a linear combination of complex exponentials is identical in both continuous time and discrete time. Complex exponentials are eigenfunctions of linear, time-invariant systems, and consequently the effect of an LTI system on each of these basic signals is simply a (complex) amplitude change. Thus with signals decomposed in this way, an LTI system is completely characterized by a spectrum of scale factors which it applies at each frequency.

In representing discrete-time periodic signals through the Fourier series, we again use harmonically related complex exponentials with fundamental frequencies that are integer multiples of the fundamental frequency of the periodic sequence to be represented. However, as we discussed in Lecture 2, an important distinction between continuous-time and discrete-time complex exponentials is that in the discrete-time case, they are unique only as the frequency variable spans a range of 2π. Beyond that, we simply see the same complex exponentials repeated over and over. Consequently, when we consider representing a periodic sequence with period N as a linear combination of complex exponentials of the form $e^{jk\Omega_0 n}$ with $\Omega_0 = 2\pi/N$, there are only N distinct complex exponentials of this type available to use, i.e., $e^{jk\Omega_0 n}$ is periodic in k with period N. (Of course, it is also periodic in n with period N.) In many ways, this simplifies the analysis since for discrete time the representation involves only N Fourier series coefficients, and thus determining the coefficients from the sequence corresponds to solving N equations in N unknowns. The resulting analysis equation is a summation very similar in form to the synthesis equation and suggests a strong duality between the analysis and synthesis equations for the discrete-time Fourier transform. Because the basic complex exponentials repeat periodically in frequency, two alternative interpretations arise for the behavior of the Fourier series coefficients. One interpretation is that there are only N coefficients. The second is that the sequence representing the Fourier series coefficients can run on indefinitely but repeats periodically. Both interpretations, of course, are equivalent because in either case there are only N unique Fourier series coefficients. Partly to retain a duality between a periodic sequence and the sequence representing its Fourier series coefficients, it is typically preferable to think of the Fourier series coefficients as a periodic sequence with period N, that is, the same period as the time sequence $x(n)$. This periodicity is illustrated in this lecture through several examples.

Partly in anticipation of the fact that we will want to follow an approach similar to that used in the continuous-time case for a Fourier decomposition of aperiodic signals, it is useful to represent the Fourier series coefficients as samples of an envelope. This envelope is determined by the behavior of the sequence over one period but is not dependent on the specific value of the period. As the period of the sequence increases, with the nonzero content in the period remaining the same, the Fourier series coefficients are samples of the same envelope function with increasingly finer spacing along the frequency axis (specifically, a spacing of $2\pi/N$ where N is the period). Consequently, as the period approaches infinity, this envelope function corresponds to a Fourier representation of the aperiodic signal corresponding to one period. This is, then, the Fourier transform of the aperiodic signal.

The discrete-time Fourier transform developed as we have just described corresponds to a decomposition of an aperiodic signal as a linear combination of a continuum of complex exponentials. The synthesis equation is then the limiting form of the Fourier series sum, specifically an integral. The analysis equation is the same one we used previously in obtaining the envelope of the Fourier series coefficients. Here we see that while there was a duality in the expressions between the discrete-time Fourier series analysis and synthesis equations, the duality is lost in the discrete-time Fourier transform since the synthesis equation is now an integral and the analysis equation a summation. This represents one difference between the discrete-time Fourier transform and the continuous-time Fourier transform. Another important difference is that the discrete-time Fourier transform is always a periodic function of frequency. Consequently, it is completely defined by its behavior over a frequency range of 2π in contrast to the continuous-time Fourier transform, which extends over an infinite frequency range.

Suggested Reading

Section 5.0, Introduction, pages 291–293

Section 5.1, The Response of Discrete-Time LTI Systems to Complex Exponentials, pages 293–294

Section 5.2, Representation of Periodic Signals: The Discrete-Time Fourier Series, pages 294–306

Section 5.3, Representation of Aperiodic Signals: The Discrete-Time Fourier Transform, pages 306–314

Section 5.4, Periodic Signals and the Discrete-Time Fourier Transform, pages 314–321

MARKERBOARD 10.1

$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t)$

$x[n] \qquad\qquad y[n]$

Decompose Input as

$$x = a_1\phi_1 + a_2\phi_2 + \cdots$$

Then $\downarrow \quad \downarrow$

$$y = a_1\psi_1 + a_2\psi_2 + \cdots$$

Choose $\phi_k(t)$ or $\phi_k[n]$ so that:

- Broad Class of Signals can be constructed
- Response to ϕ_k's easy to compute

CONTINUOUS-Time

$$\phi_k(t) = e^{j\omega_k t}$$

唯一变化的是复振幅

$$e^{j\omega_k t} \longrightarrow e^{j\omega_k t}\underbrace{\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega_k \tau}\,d\tau}$$

系统的频率响应 $= H(\omega_k) =$ 系统脉冲响应的傅里叶变换、连续时间

Discrete-Time

$$\phi_k[n] = e^{j\Omega_k n}$$

$$e^{j\Omega_k n} \longrightarrow e^{j\Omega_k n}\underbrace{\sum_{r=-\infty}^{+\infty} h[r]e^{-j\Omega_k r}}$$

频率响应 $= H(\Omega_k) =$ 系统脉冲响应的离散时间傅里叶变换

Discrete-Time Fourier Series

$x[n]$ periodic

period N

fundamental freq. $\Omega_0 = \frac{2\pi}{N}$

$$e^{jk\Omega_0 n} = e^{j(k+N)\Omega_0 n}$$

$$x[n] = \sum_k a_k e^{jk\Omega_0 n}$$

$k = 0, 1, 2, \ldots N-1$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$$

Note that dt should be added at the end of the last equation in column 1.

MARKERBOARD 10.2

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$$

N equations in N unknowns

$$a_k = \frac{1}{N}\sum_{\langle N\rangle} x[n]\,e^{-jk\Omega_0 n}$$

Continuous-Time:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$$

Synthesis

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$$

Analysis

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_0 n}$$

C-T

$x[n]$ periodic in n

$e^{jk\Omega_0 n}$ periodic in n

$e^{jk\Omega_0 n}$ periodic in k

a_k periodic in k

~~C-T~~

Convergence of Fourier Series

Continuous-Time:

$x(t)$ square integrable

or

Dirichlet conditions

Discrete-Time

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$$

$$\hat{x}[n] = \sum_{p\ \text{terms}} a_k e^{jk\Omega_0 n}$$

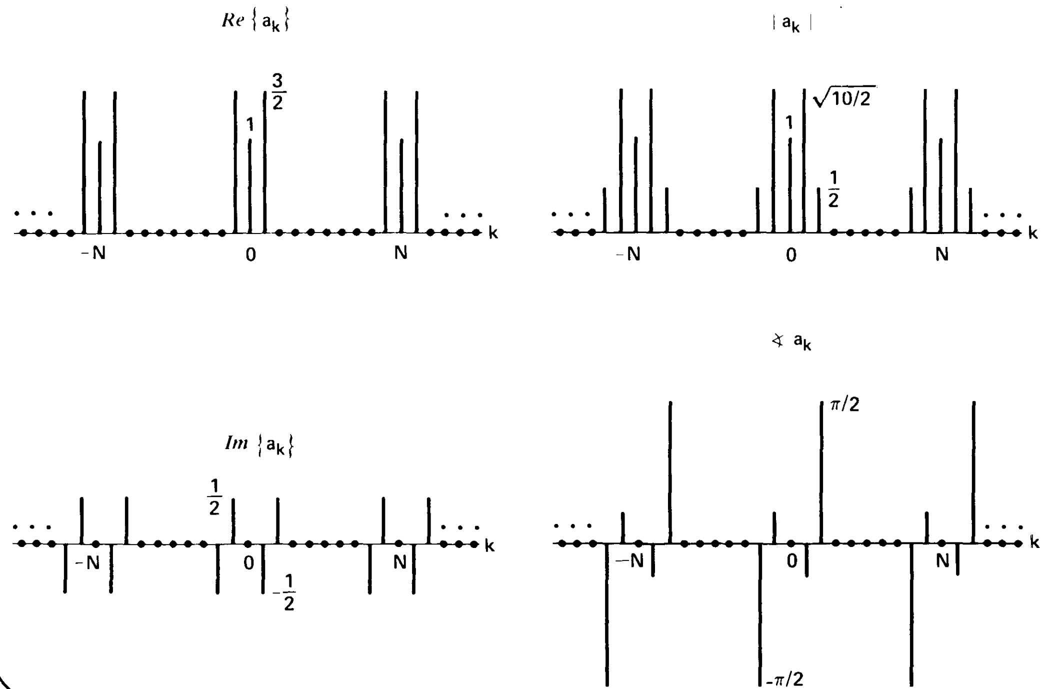
$p = N$

$$\hat{x}[n] \equiv x[n]$$

TRANSPARENCY 10.1
Example of the Fourier series coefficients for a discrete-time periodic signal.

Example 5.2:

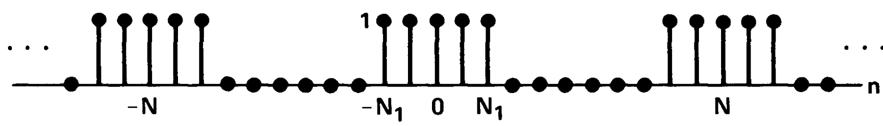
$$x[n] = 1 + \sin \Omega_o n + 3 \cos \Omega_o n + \cos\left(2\Omega_o n + \frac{\pi}{2}\right)$$



TRANSPARENCY 10.2
Comparison of the Fourier series coefficients for a discrete-time periodic square wave and a continuous-time periodic square wave.

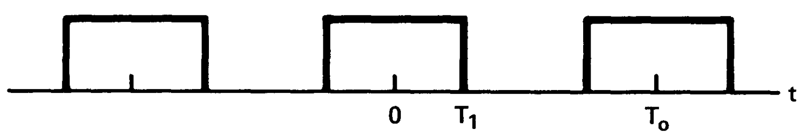
$\frac{\sin NX}{\sin X}$离散时间周期方波的傅立叶级数的包络线

Example 5.3:

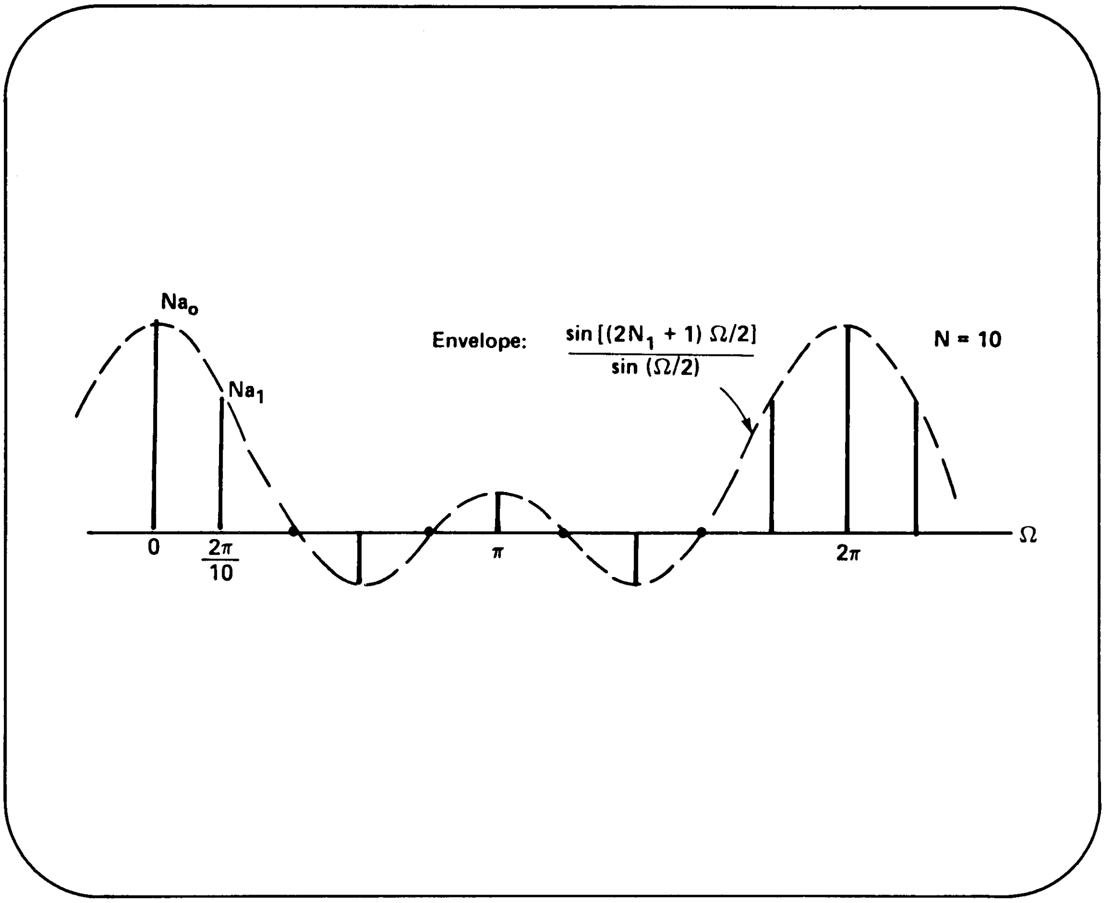


$$N a_k = \frac{\sin[(2N_1+1)\,\Omega/2]}{\sin(\Omega/2)} \quad \Bigg| \quad \Omega = \frac{2\pi k}{N}$$

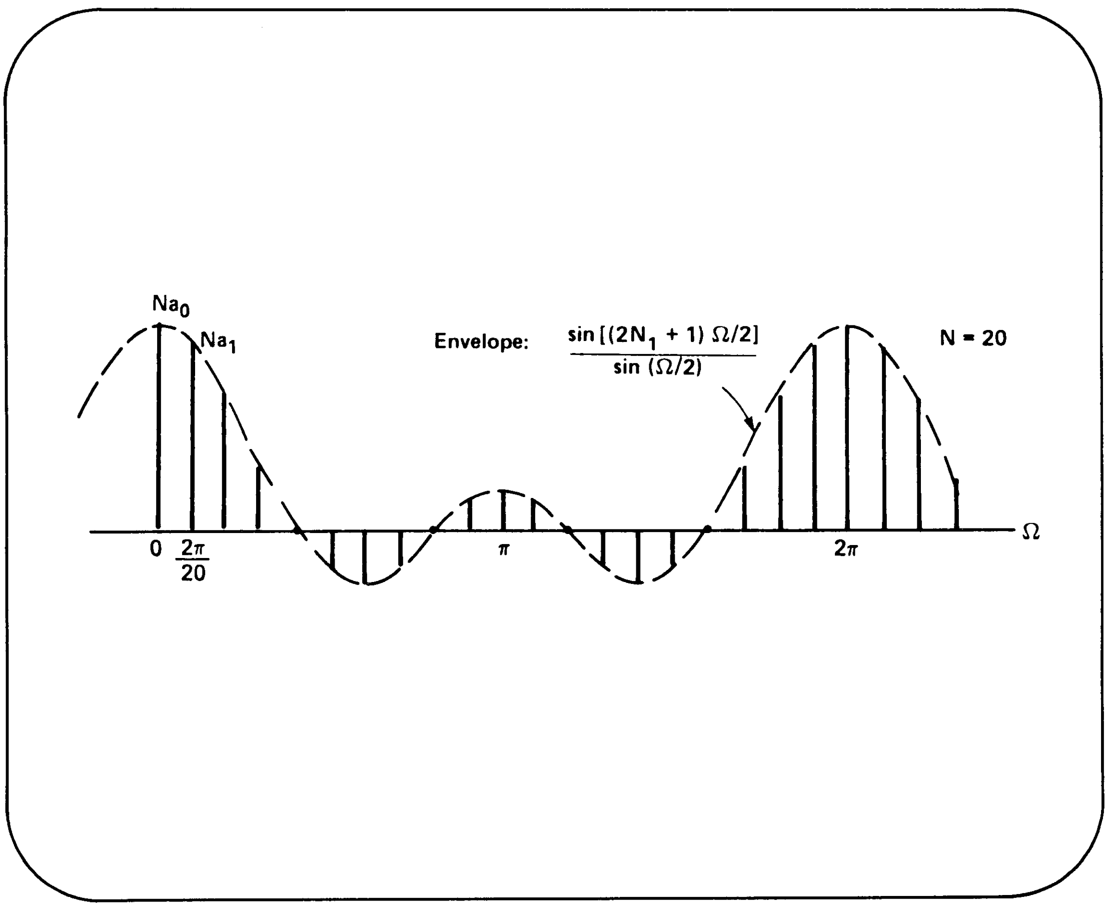
Example 4.5 and 4.12:



$$T_o a_k = \frac{2 \sin \omega T_1}{\omega} \quad \Bigg| \quad \omega = \frac{2\pi k}{T_o}$$

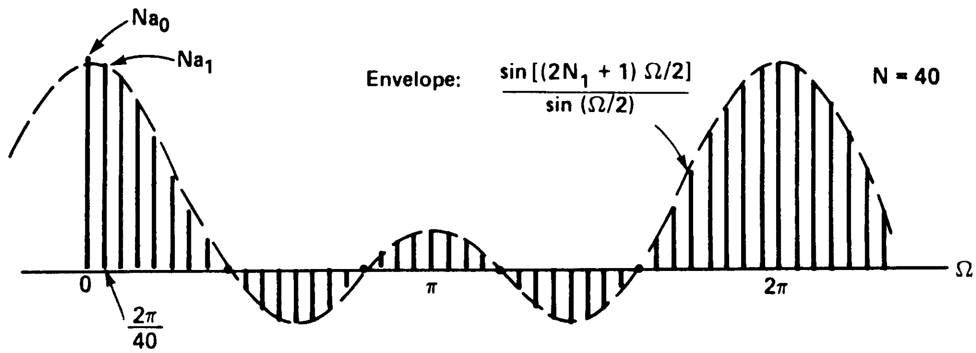


TRANSPARENCY 10.3
Illustration of the discrete-time Fourier series coefficients as samples of an envelope. Transparencies 10.3–10.5 demonstrate that as the period increases, the envelope remains the same and the samples representing the Fourier series coefficients become more closely spaced. Here, $N = 10$.



TRANSPARENCY 10.4
$N = 20$.

TRANSPARENCY
10.5
$N = 40.$



TRANSPARENCY
10.6
A review of the approach to developing a Fourier representation for aperiodic signals.

1. x(t) APERIODIC

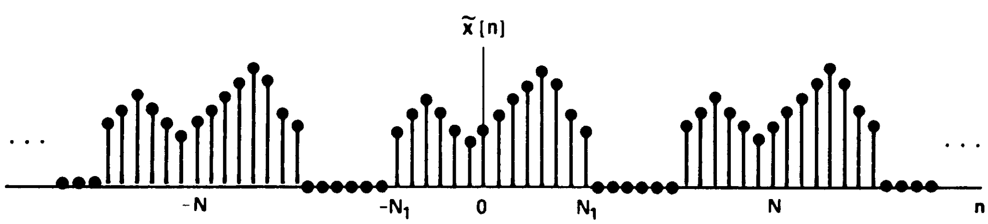
- construct periodic signal $\tilde{x}(t)$ for which one period is x(t)
- $\tilde{x}(t)$ has a Fourier series
- as period of $\tilde{x}(t)$ increases, $\tilde{x}(t) \rightarrow x(t)$ and Fourier series of $\tilde{x}(t) \rightarrow$ Fourier Transform of x(t)

1. x[n] APERIODIC

- construct periodic signal $\tilde{x}[n]$ for which one period is $x[n]$
- $\tilde{x}[n]$ has a Fourier series
- as period of $\tilde{x}[n]$ increases, $\tilde{x}[n] \rightarrow x[n]$ and Fourier series of $\tilde{x}[n] \rightarrow$ Fourier Transform of $x[n]$

TRANSPARENCY 10.7
A summary of the approach to be used to obtain a Fourier representation of discrete-time aperiodic signals.

FOURIER REPRESENTATION OF APERIODIC SIGNALS



$$\tilde{x}[n] = x[n] \qquad |n| < \frac{N}{2}$$

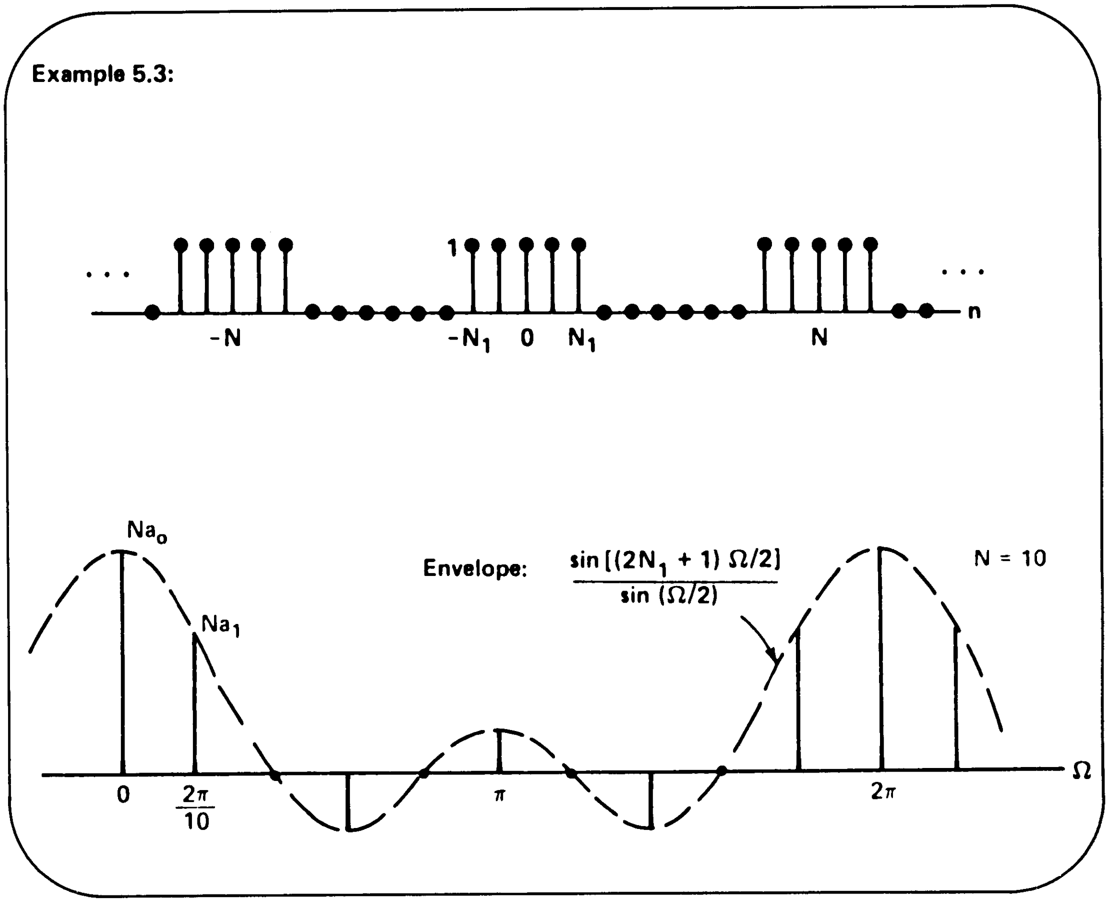
As $N \rightarrow \infty \qquad \tilde{x}[n] \rightarrow x[n]$

– let $N \rightarrow \infty$ to represent $x[n]$

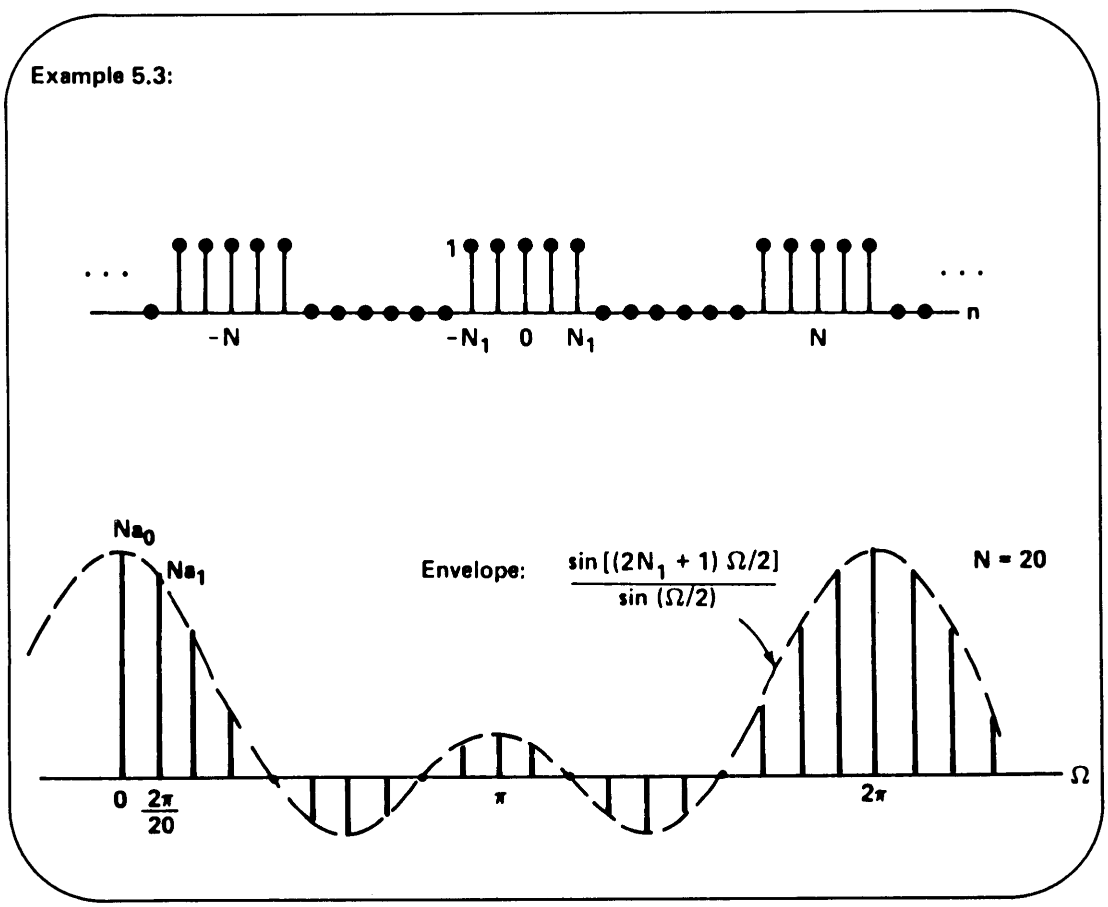
– use Fourier series to represent $\tilde{x}[n]$

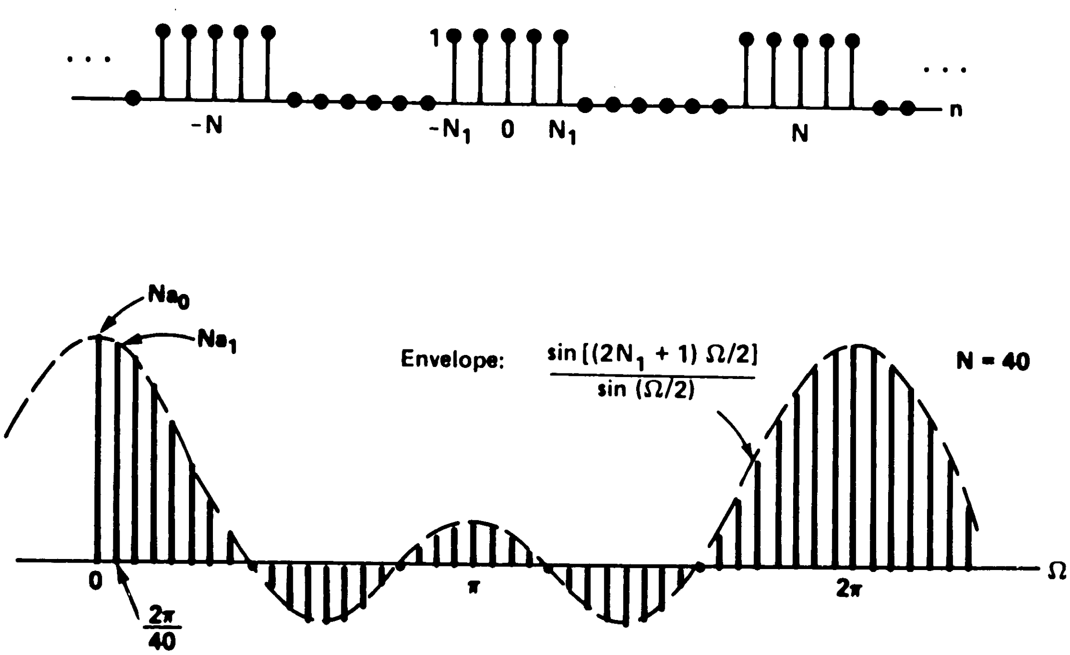
TRANSPARENCY 10.8
Representation of an aperiodic signal as the limiting form of a periodic signal with the period increasing.

TRANSPARENCY 10.9
Transparencies 10.9–10.11 illustrate how the Fourier series coefficients for a periodic signal approach the continuous envelope function as the period increases. Here, $N = 10$. [Example 5.3 from the text.]



TRANSPARENCY 10.10
$N = 20$. [Example 5.3 from the text.]





TRANSPARENCY 10.11
$N = 40$. [Example 5.3 from the text.]

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{2\pi} X(k\Omega_o)\, e^{jk\Omega_o n}\, \Omega_o$$

$$X(k\Omega_o) = N\, a_k = \sum_{n=-N/2}^{n=N/2} \tilde{x}[n]\, e^{-jk\Omega_o n}$$

As $N \longrightarrow \infty$

Fourier Transform:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n}$$

TRANSPARENCY 10.12
Limiting form of the Fourier series as the period approaches infinity. [The upper limit in the summation in the second equation should be $n = (N/2) - 1$.]

TRANSPARENCY 10.13
The analysis and synthesis equations for the discrete-time Fourier transform. [As corrected here, $x[n]$, not $x(t)$, has Fourier transform $X(\Omega)$.]

DISCRETE-TIME FOURIER TRANSFORM

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \qquad \text{synthesis}$$

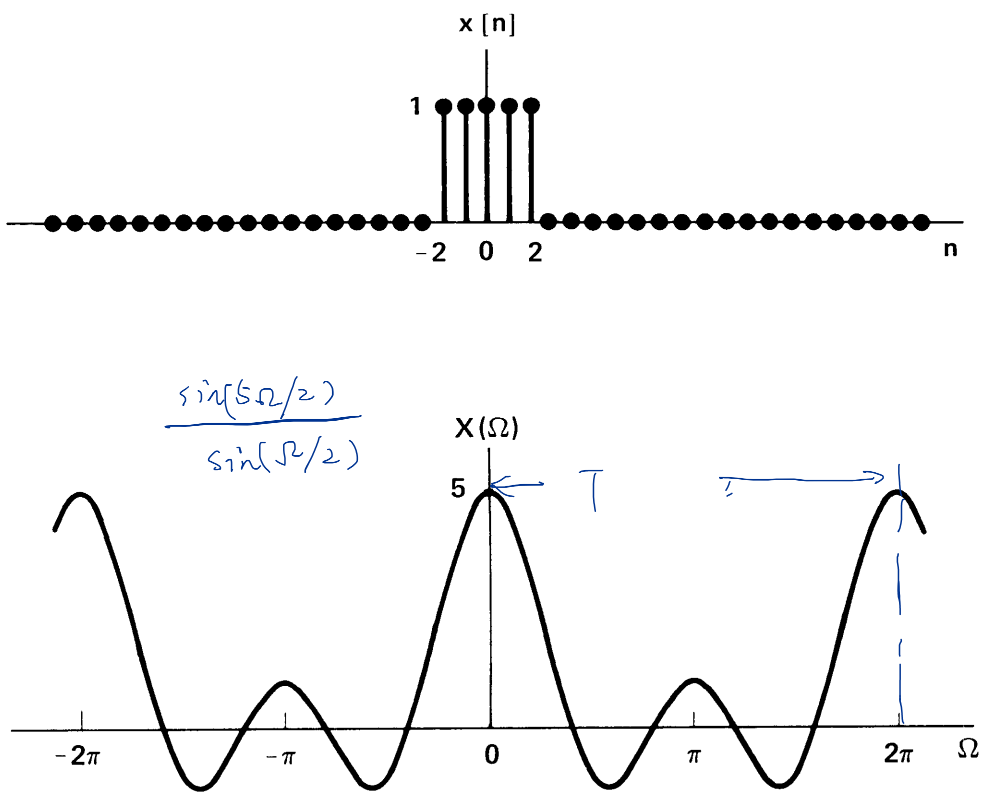
$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n} \qquad \text{analysis}$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)$$

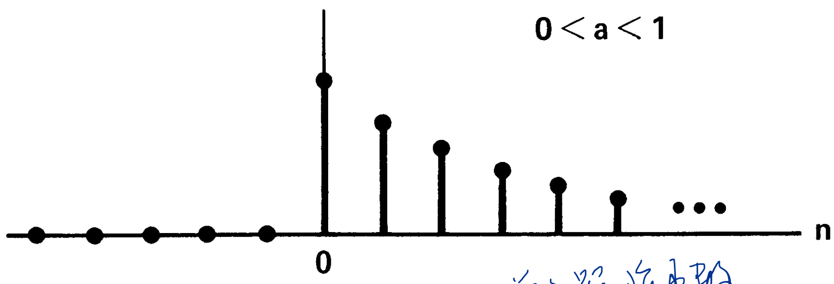
$$X(\Omega) = \mathit{Re}\{X(\Omega)\} + j\,\mathit{Im}\{X(\Omega)\}$$

$$= |X(\Omega)|\, e^{j\sphericalangle X(\Omega)}$$

TRANSPARENCY 10.14
The discrete-time Fourier transform for a rectangular pulse.



$$x[n] = a^n u[n]$$



单位跃阶函数

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\Omega n}$$

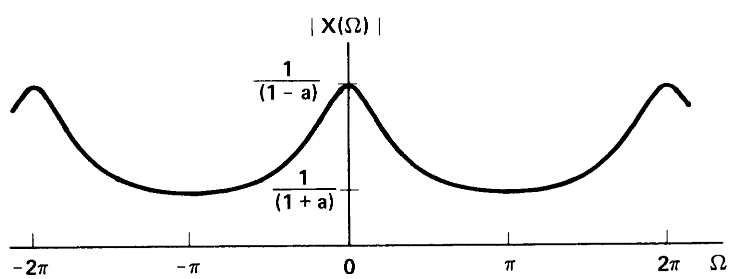
$$= \sum_{n=0}^{\infty} a^n e^{-j\Omega n}$$

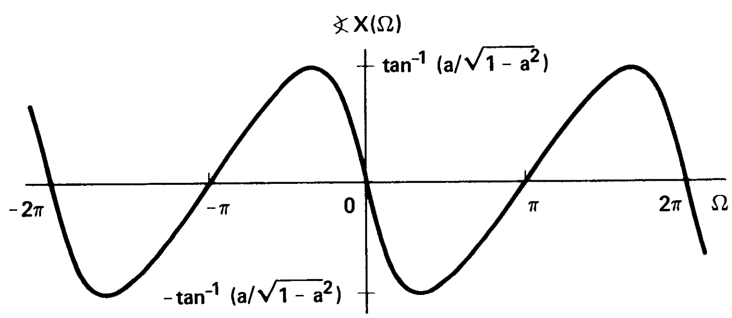
$$= \frac{1}{1 - ae^{-j\Omega}}$$

TRANSPARENCY 10.15
The discrete-time Fourier transform for an exponential sequence.

$$x[n] = a^n u[n] \quad 0 < a < 1$$

$$X(\Omega) = \frac{1}{1 - a e^{-j\Omega}}$$





TRANSPARENCY 10.16
Illustration of the magnitude and phase of the discrete-time Fourier transform for an exponential sequence. [Note that a is real.]

TRANSPARENCY 10.17
A review of some relationships for the Fourier transform associated with periodic signals.

2. $\tilde{x}(t)$ PERIODIC, $x(t)$ REPRESENTS ONE PERIOD

- Fourier series coefficients of $\tilde{x}(t)$

  = $(1/T_o)$ times samples of Fourier transform of $x(t)$

3. $\tilde{x}(t)$ PERIODIC

- Fourier transform of $\tilde{x}(t)$ defined as impulse train:

$$\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_o)$$

TRANSPARENCY 10.18
A summary of some relationships for the Fourier transform associated with periodic sequences.

2. $\tilde{x}[n]$ PERIODIC, $x[n]$ REPRESENTS ONE PERIOD

- Fourier series coefficients of $\tilde{x}[n]$

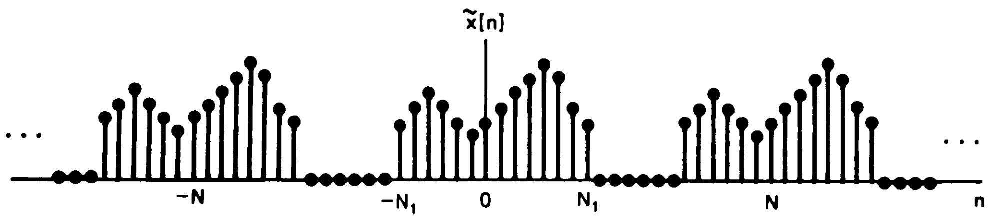
  = $(1/N)$ times samples of Fourier transform of $x[n]$

3. $\tilde{x}[n]$ PERIODIC

- Fourier transform of $\tilde{x}[n]$ defined as impulse train:

$$\tilde{X}(\Omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\Omega - k\Omega_o)$$

Fourier series coefficients equal $\frac{1}{N}$ times samples of Fourier transform of one period



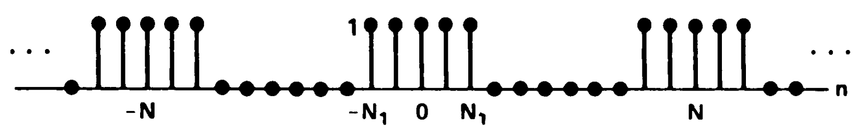
$x[n] \triangleq$ one period of $\tilde{x}[n]$

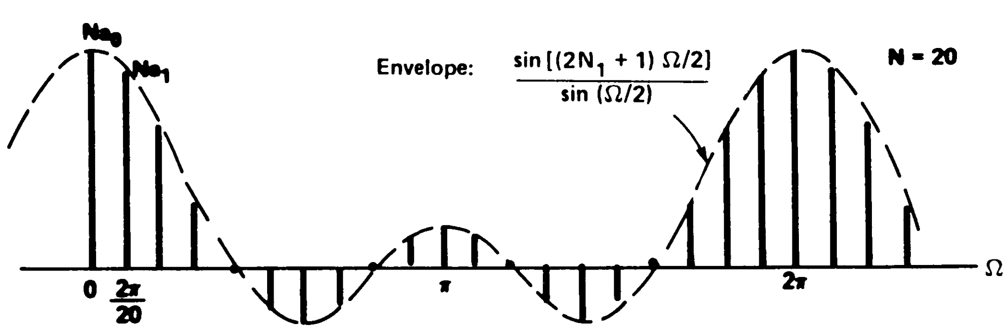
$\tilde{x}[n] \longleftrightarrow a_k$

$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)$

$$\boxed{a_k = \frac{1}{N} X(\Omega)\Big|_{\Omega = \frac{2\pi k}{N}}}$$

TRANSPARENCY 10.19
The relationship between the Fourier series coefficients of a periodic signal and the Fourier transform of one period.





TRANSPARENCY 10.20
Illustration of the relationship in Transparency 10.19.

TRANSPARENCY 10.21
A summary of some relationships for the Fourier transform associated with periodic sequences. [Transparency 10.18 repeated]

2. $\tilde{x}[n]$ PERIODIC, $x[n]$ REPRESENTS ONE PERIOD

- Fourier series coefficients of $\tilde{x}[n]$

  = (1/N) times samples of Fourier

  transform of $x[n]$

3. $\tilde{x}[n]$ PERIODIC

- Fourier transform of $\tilde{x}[n]$ defined as

  impulse train:

$$\tilde{X}(\Omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \, \delta(\Omega - k\Omega_0)$$

TRANSPARENCY 10.22
Illustration of the Fourier series coefficients and the Fourier transform for a periodic square wave.