2

Signals and Systems: Part I

In this lecture, we consider a number of basic signals that will be important building blocks later in the course. Specifically, we discuss both continuous-time and discrete-time sinusoidal signals as well as real and complex exponentials.

Sinusoidal signals for both continuous time and discrete time will become important building blocks for more general signals, and the representation using sinusoidal signals will lead to a very powerful set of ideas for representing signals and for analyzing an important class of systems. We consider a number of distinctions between continuous-time and discrete-time sinusoidal signals. For example, continuous-time sinusoids are always periodic. Furthermore, a time shift corresponds to a phase change and vice versa. Finally, if we consider the family of continuous-time sinusoids of the form $A \cos \omega_0 t$ for different values of ω_0, the corresponding signals are distinct. The situation is considerably different for discrete-time sinusoids. Not all discrete-time sinusoids are periodic. Furthermore, while a time shift can be related to a change in phase, changing the phase cannot necessarily be associated with a simple time shift for discrete-time sinusoids. Finally, as the parameter Ω_0 is varied in the discrete-time sinusoidal sequence $A\cos(\Omega_0 n + \phi)$, two sequences for which the frequency Ω_0 differs by an integer multiple of 2π are in fact indistinguishable.

Another important class of signals is exponential signals. In continuous time, real exponentials are typically expressed in the form ce^{at}, whereas in discrete time they are typically expressed in the form $c\alpha^n$.

A third important class of signals discussed in this lecture is continuous-time and discrete-time complex exponentials. In both cases the complex exponential can be expressed through Euler's relation in the form of a real and an imaginary part, both of which are sinusoidal with a phase difference of $\pi/2$ and with an envelope that is a real exponential. When the magnitude of the complex exponential is a constant, then the real and imaginary parts neither grow nor decay with time; in other words, they are purely sinusoidal. In this case for continuous time, the complex exponential is periodic. For discrete time the complex exponential may or may not be periodic depending on whether the sinusoidal real and imaginary components are periodic.

In addition to the basic signals discussed in this lecture, a number of additional signals play an important role as building blocks. These are introduced in Lecture 3.

Suggested Reading

Section 2.2, Transformations of the Independent Variable, pages 12–16

Section 2.3.1, Continuous-Time Complex Exponential and Sinusoidal Signals, pages 17–22

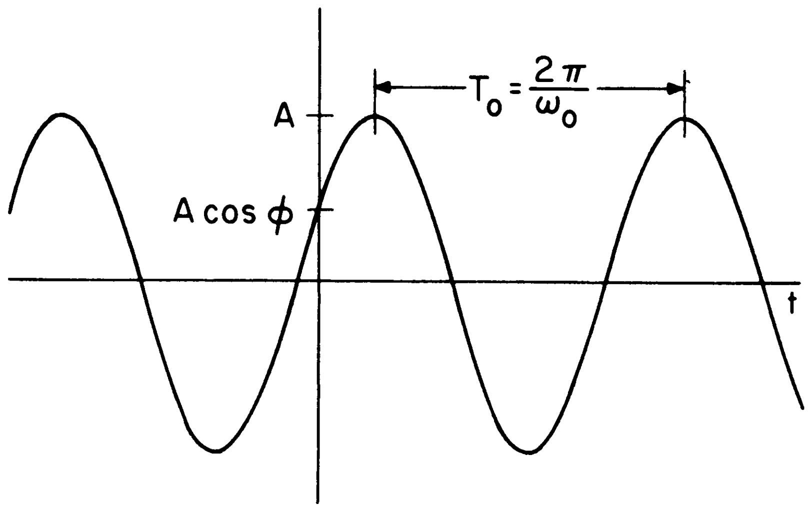
Section 2.4.2, Discrete-Time Complex Exponential and Sinusoidal Signals, pages 27–31

Section 2.4.3, Periodicity Properties of Discrete-Time Complex Exponentials, pages 31–35

CONTINUOUS-TIME SINUSOIDAL SIGNAL

$$x(t) = A\cos(\omega_o t + \phi)$$

amp freq phase



TRANSPARENCY 2.1
Continuous-time sinusoidal signal indicating the definition of amplitude, frequency, and phase.

• **Periodic:**

$$x(t) = x(t + T_o) \qquad \text{period} \stackrel{\triangle}{=} \text{smallest } T_o$$

$$A\cos[\omega_o t + \phi] = A\cos[\omega_o t + \underbrace{\omega_o T_o}_{2\pi m} + \phi]$$

$$T_o = \frac{2\pi m}{\omega_o} \Rightarrow \text{period} = \frac{2\pi}{\omega_o}$$ m=1也就是可能的最小值，即为周期

时移产生了相变，相变也导致了时移

• **Time Shift <=> Phase Change**

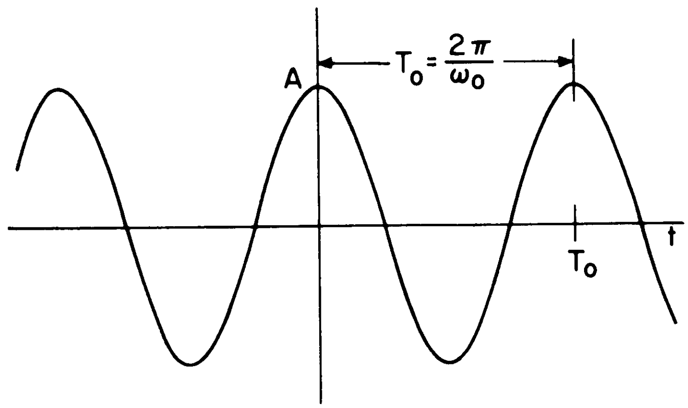
$$A\cos[\omega_o(t + t_o)] = A\cos[\omega_o t + \underbrace{\omega_o t_o}_{\Delta\phi}]$$

$$A\cos[\omega_o(t + t_o) + \phi] = A\cos[\omega_o t + \omega_o t_o + \phi]$$

TRANSPARENCY 2.2
Relationship between a time shift and a change in phase for a continuous-time sinusoidal signal.

TRANSPARENCY 2.3
Illustration of the signal $A \cos \omega_0 t$ as an even signal.

$\phi = 0 \qquad x(t) = A \cos \omega_0 t$

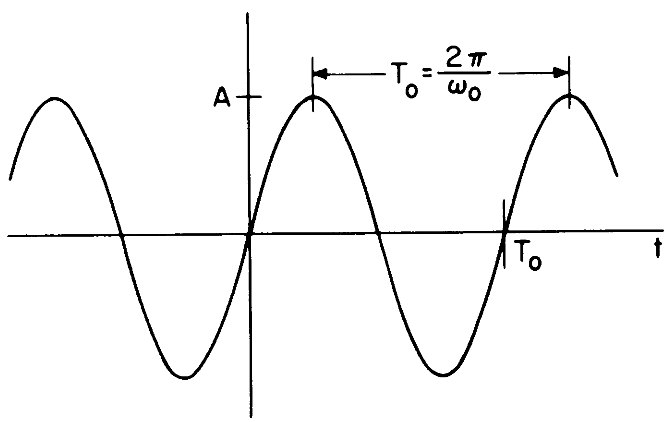


Periodic: $x(t) = x(t + T_o)$

Even: $x(t) = x(-t)$

TRANSPARENCY 2.4
Illustration of the signal $A \sin \omega_0 t$ as an odd signal.

$$\phi = -\frac{\pi}{2} \qquad x(t) = \begin{cases} A \cos (\omega_o t - \frac{\pi}{2}) \\ A \sin \omega_o t \\ A \cos [\omega_o (t - \frac{T_o}{4})] \end{cases}$$



Periodic: $x(t) = x(t + T_o)$

Odd: $x(t) = -x(-t)$

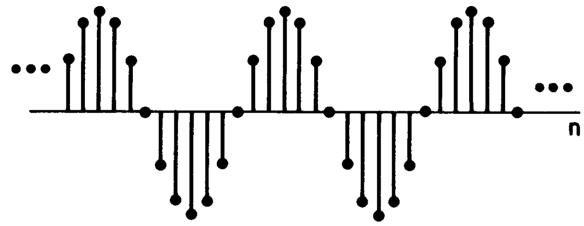
DISCRETE-TIME SINUSOIDAL SIGNAL

amp freq phase

$$x[n] = A\cos(\Omega_o n + \phi)$$

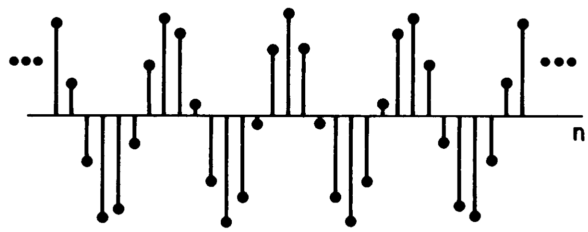
$\Omega_o = \frac{2\pi}{12}$

$\phi = 0$



$\Omega_o = \frac{8\pi}{31}$

$\phi = 0$



TRANSPARENCY 2.5
Illustration of discrete-time sinusoidal signals.

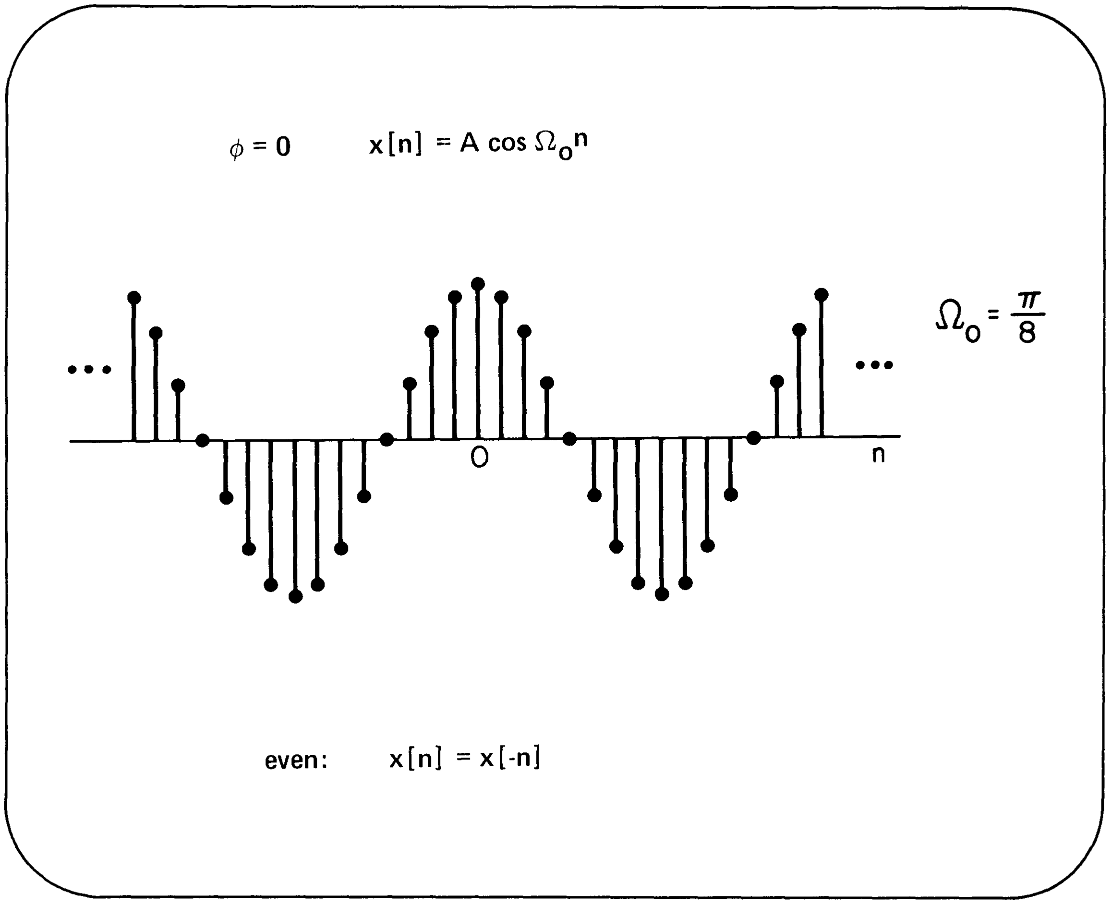
Time Shift => Phase Change

$\angle\phi$

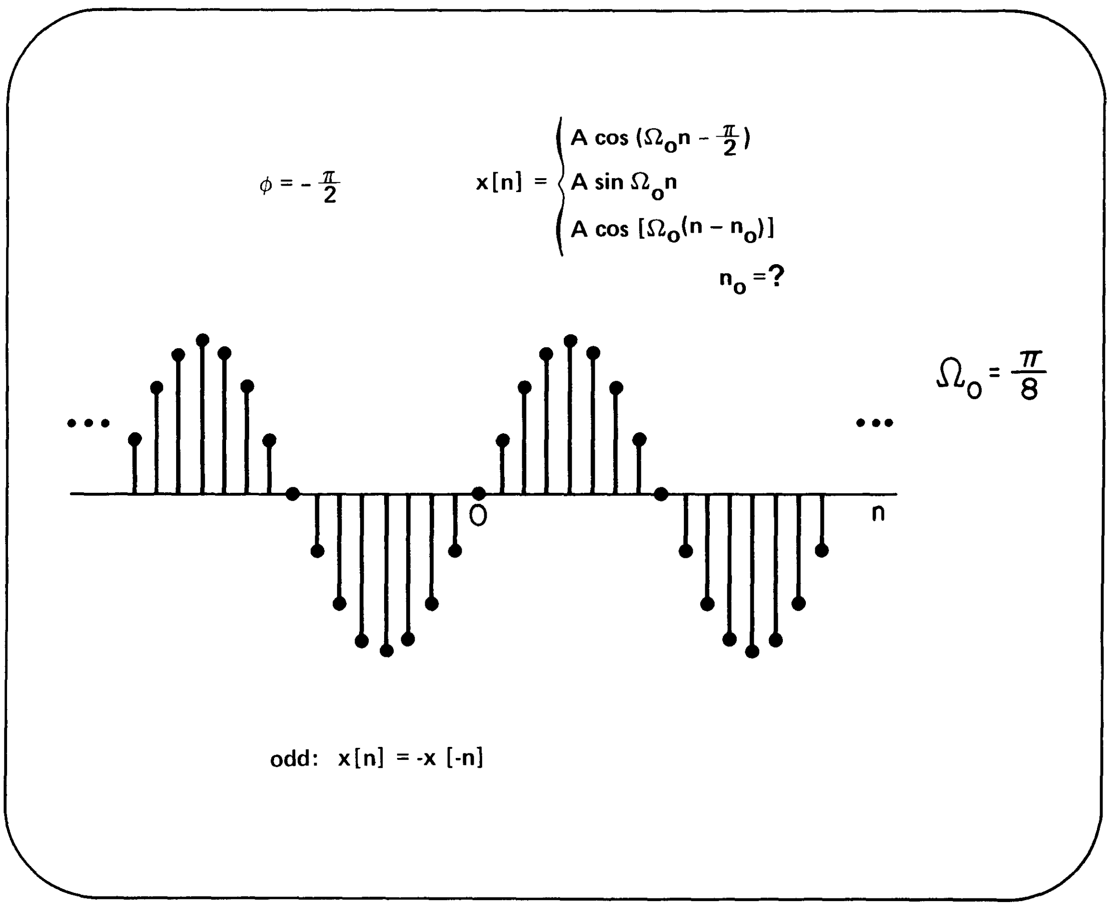
$$A\cos[\Omega_o(n+n_o)] = A\cos[\Omega_o n + \Omega_o n_o]$$

TRANSPARENCY 2.6
Relationship between a time shift and a phase change for discrete-time sinusoidal signals. In discrete time, a time shift always implies a phase change.

TRANSPARENCY 2.7
The sequence $A \cos \Omega_0 n$ illustrating the symmetry of an even sequence.



TRANSPARENCY 2.8
The sequence $A \sin \Omega_0 n$ illustrating the antisymmetric property of an odd sequence.



Time Shift => Phase Change

$$A \cos[\Omega_o(n+n_o)] = A\cos[\Omega_o n + \Omega_o n_o]$$

Time Shift $\overset{?}{\Leftarrow}$ Phase Change

$$A \cos[\Omega_o(n+n_o)] \overset{?}{=} A\cos[\Omega_o n + \phi]$$

TRANSPARENCY 2.9
For a discrete-time sinusoidal sequence a time shift always implies a change in phase, but a change in phase might not imply a time shift.

$$x[n] = A\cos(\Omega_o n + \phi)$$

Periodic?

$x[n] = x[n+N]$ smallest integer $N \triangleq$ period

$$A\cos[\Omega_o(n+N)+\phi] = A\cos[\Omega_o n + \underbrace{\Omega_o N}_{\text{integer multiple of } 2\pi\ ?} + \phi]$$

Periodic $\Rightarrow \Omega_o N = 2\pi m$

$$N = \frac{2\pi m}{\Omega_o}$$

N,m must be integers

smallest N (if any) = period

TRANSPARENCY 2.10
The requirement on Ω_0 for a discrete-time sinusoidal signal to be periodic.

TRANSPARENCY 2.11
Several sinusoidal sequences illustrating the issue of periodicity.

$\frac{2\pi}{\Omega_0} = 12$ $\Omega_0 = \frac{2\pi}{12}$

连续/离散 $\phi = 0$

时间序列均为周期函数

$\frac{2\pi}{\Omega_0} = \frac{31}{4}$ $\Omega_0 = \frac{8\pi}{31}$

↓

$T = 31$ $\phi = 0$

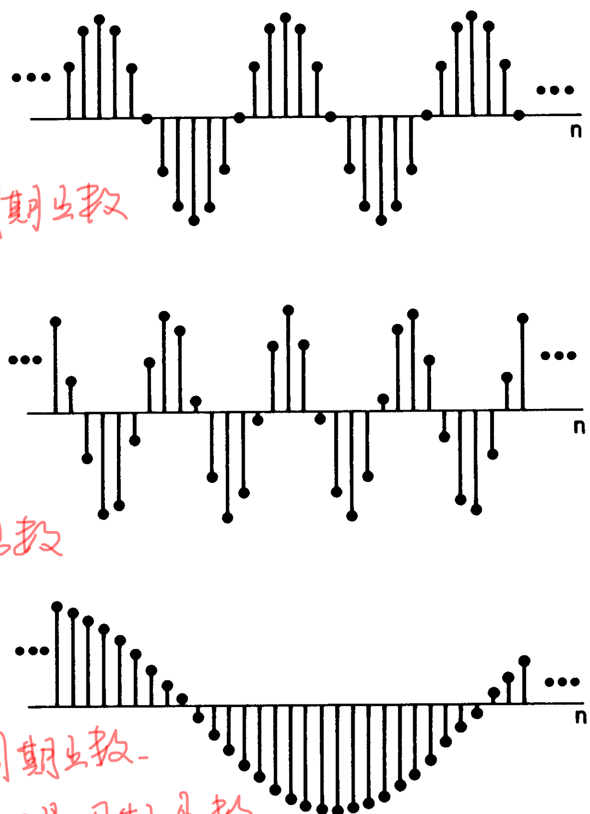
离散 × 4 => 周期函数

$\frac{2\pi}{\Omega_0} = 12\pi$ $\Omega_0 = \frac{1}{6}$

$\phi = 0$

连续时间序列为周期函数。

但离散时间序列不是周期函数。



TRANSPARENCY 2.12
Some important distinctions between continuous-time and discrete-time sinusoidal signals.

$A\cos(\omega_0 t + \phi)$	$A\cos(\Omega_0 n + \phi)$
Distinct signals for distinct values of ω_0	Identical signals for values of Ω_0 separated by 2π
Periodic for any choice of ω_0	Periodic only if $\Omega_0 = \frac{2\pi m}{N}$ for some integers $N > 0$ and m

SINUSOIDAL SIGNALS AT DISTINCT FREQUENCIES:

Continuous time:

$x_1(t) = A\cos(\omega_1 t + \phi)$ If $\omega_2 \neq \omega_1$

$x_2(t) = A\cos(\omega_2 t + \phi)$ Then $x_2(t) \neq x_1(t)$

Discrete time:

$x_1[n] = A\cos[\Omega_1 n + \phi]$ If $\Omega_2 = \Omega_1 + 2\pi m$

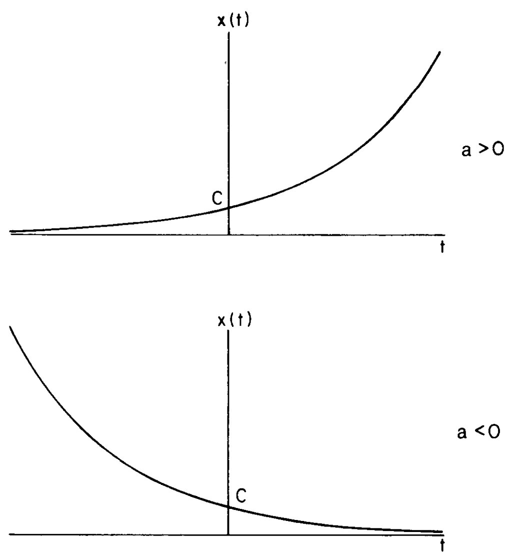
$x_2[n] = A\cos[\Omega_2 n + \phi]$ Then $x_2[n] = x_1[n]$

TRANSPARENCY 2.13
Continuous-time sinusoidal signals are distinct at distinct frequencies. Discrete-time sinusoidal signals are distinct only over a frequency range of 2π.

REAL EXPONENTIAL: CONTINUOUS-TIME

$x(t) = Ce^{at}$

C and a are real numbers



Time Shift <=> Scale Change

$Ce^{a(t+t_0)} = Ce^{at_0}\,e^{at}$

TRANSPARENCY 2.14
Illustration of continuous-time real exponential signals.

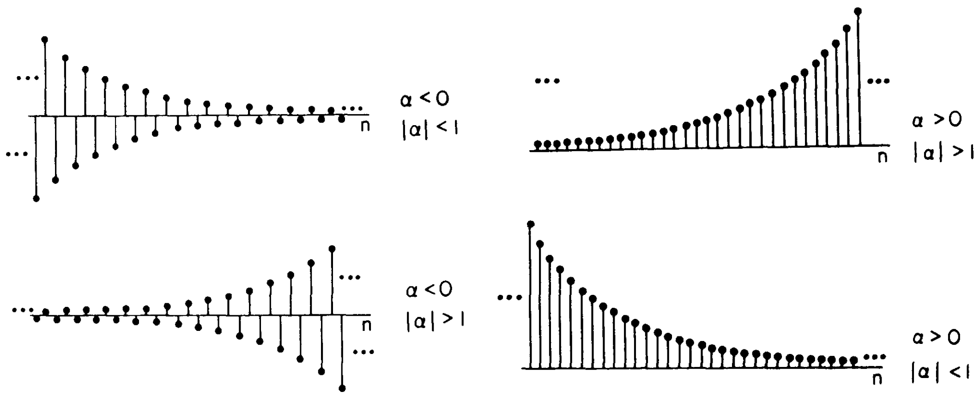
TRANSPARENCY 2.15
Illustration of discrete-time real exponential sequences.

REAL EXPONENTIAL: DISCRETE-TIME

α为正数

$$x[n] = Ce^{\boxed{\beta} n} = C\boxed{\alpha}^n \qquad \beta = \ln\alpha$$

C, α are real numbers



TRANSPARENCY 2.16
Continuous-time complex exponential signals and their relationship to sinusoidal signals.

COMPLEX EXPONENTIAL: CONTINUOUS-TIME

$$x(t) = Ce^{at}$$

C and a are complex numbers

$$C = |C|\, e^{j\theta}$$

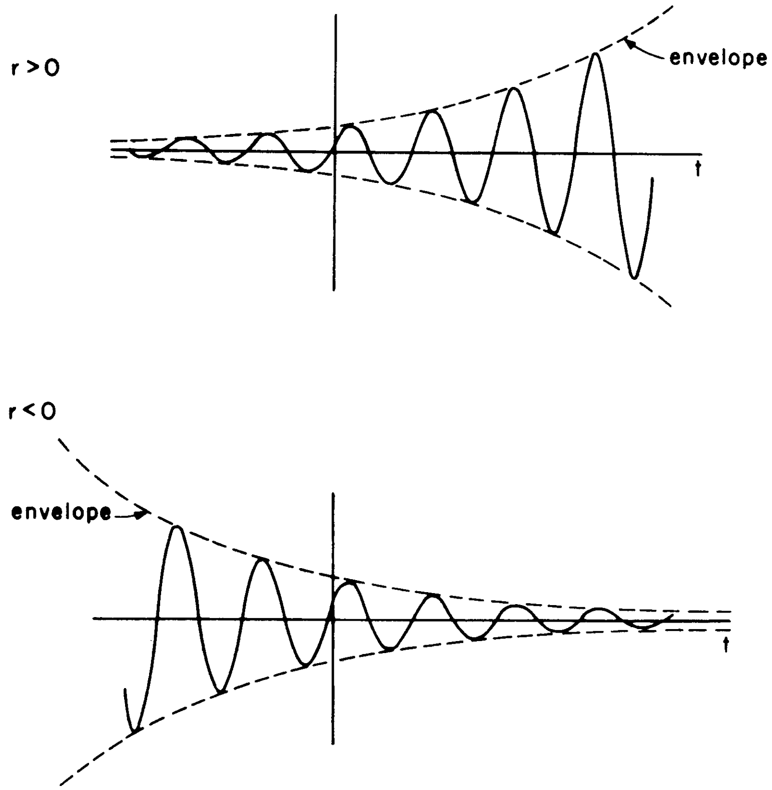
$$a = r + j\omega_o$$

$$x(t) = |C|\, e^{j\theta}\, e^{(r + j\omega_o)t} = |c|\, e^{j\theta + rt + jt\omega_0}$$

$$= |C|\, e^{rt}\, \underbrace{e^{j(\omega_o t + \theta)}}$$

$\boxed{\text{Euler's Relation:}}$ $\cos(\omega_o t + \theta) + j\sin(\omega_o t + \theta) = e^{j(\omega_o t + \theta)}$

$$x(t) = |C|\, e^{rt}\cos(\omega_o t + \theta) + j\,|C|\, e^{rt}\sin(\omega_o t + \theta)$$



TRANSPARENCY 2.17
Sinusoidal signals with exponentially growing and exponentially decaying envelopes.

COMPLEX EXPONENTIAL: DISCRETE-TIME

$$x[n] = C\alpha^n$$

C and α are complex numbers

$$C = |C|\, e^{j\theta}$$

$$\alpha = |\alpha|\, e^{j\Omega_o}$$

$$x[n] = |C|\, e^{j\theta} \left(|\alpha|\, e^{j\Omega_o}\right)^n$$

$$= |C|\, |\alpha|^n \underbrace{e^{j(\Omega_o n + \theta)}}$$

Euler's Relation: $\cos(\Omega_o n + \theta) + j \sin(\Omega_o n + \theta)$

$$x[n] = |C|\, |\alpha|^n \cos(\Omega_o n + \theta) + j\, |C|\, |\alpha|^n \sin(\Omega_o n + \theta)$$

$|\alpha| = 1$ => sinusoidal real and imaginary parts

$Ce^{j\Omega_o n}$ periodic ? 取决于 Ω_0

TRANSPARENCY 2.18
Discrete-time complex exponential signals and their relationship to sinusoidal signals.

TRANSPARENCY 2.19
Sinusoidal sequences with geometrically growing and geometrically decaying envelopes.