Quantum steenrod squares

Let X be a closed monotone symp mfld

$N :=$ min chern number

$J :=$ w compatible a.c.s

Want to define a degree doubling operation

$$QS : QH^*(M) \longrightarrow QH^*(M)[h]$$

f'

Moduli space

$$M^{def}_{i,j}(x,z) := \left\{ (u,v) \;\middle|\; \begin{array}{l} u : S^2 \to M \quad u(\infty) \xrightarrow{\partial} z \\ c_1(u) = 2Nj \ , \ u(1) \xrightarrow{f^2_{-v}} X \\ v \in S^i \quad u(0) \xrightarrow{f^2_{v}} X \end{array} ,\ v \in S^i \right\} \Big/ (u,v) \sim (u \circ R, -v)$$

$= \Big\{ z \longrightarrow$  $, \pm v \in S^i \Big\} \Big/ (u,v) \sim (u \circ R, -v)$

Definition. the i,j-th component, $QS_{i,j}(x)$ at z is:

$$\langle QS_{i,j}(x), z \rangle = \begin{cases} \# M^{def}_{i,j}(x,z) & \text{if } \boxed{|z| - 2|x| + i + 2jN = 0} \\ 0 & \text{if otherwise.} \end{cases}$$

$\langle QS(x), z\rangle = \sum_{i,j\ge 0} \langle QS_{i,j}(x,z)\rangle h^i T^j$

where $h \in H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ is the generator

Remark: $QS_{i,0}(x) = Sq^{|x|-i}(x)$

$\sum_{j\ge 0} QS_{0,j}(x) T^j = x * x$

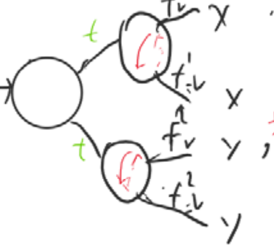
$Sq^i(x) = x \cup x$ if $|x| = i$

Quantum Cartan relation

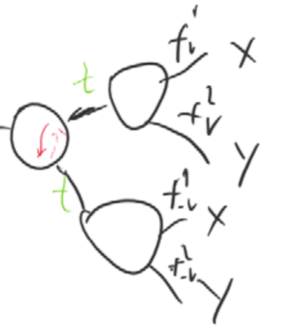
$Sq(x) \cup Sq(y) = Sq(x \cup y)$

$QS(x) * QS(y) \neq QS(x * y)$

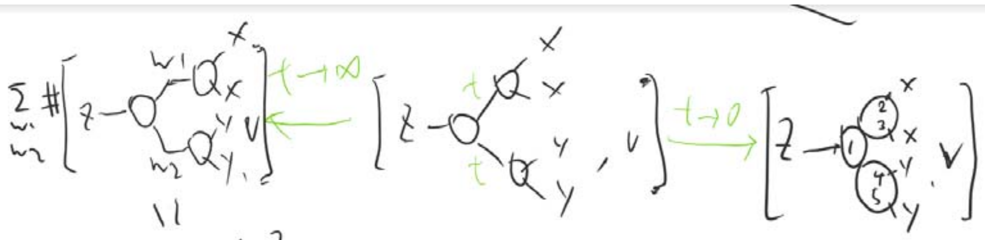
To apply cobordism argument, define:

$M^t_{i,j}(z, x, x, y, y) = \{ (t, u, v) \mid z \to$ 

$, \pm v \in S^i \}$

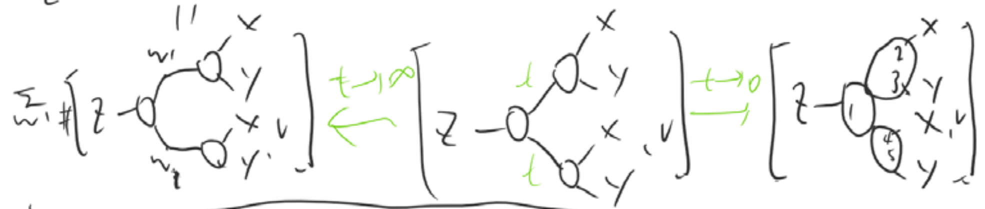
$M^t_{i,j}(z, x, y, x, y) = \{ (t, u, v) \mid z -$ 

$, \pm v \in S^i \}$



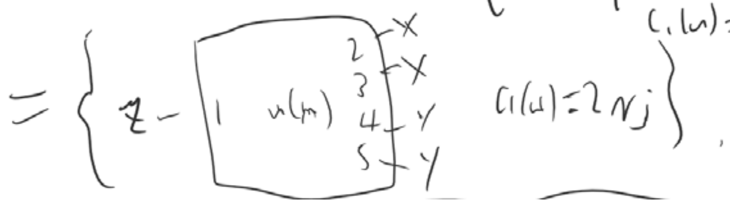
$[QS(x) * QS(y)]_{i,j}$

$[QS(x * y)]_{i,j}$



To describe the differences on the RHS, Define:

$$\widetilde{M}_j^{stab}(x,y,z) := \left\{ (u,m) \;\middle|\; \begin{array}{l} m \in \overline{M}_{0,5} \\ u: m \to M \\ c_1(u) = 2N_j \end{array},\ \begin{array}{l} z \to z_1 \\ z_2, z_3 \to x \\ z_4, z_5 \to y \end{array} \right\}$$



Let $\mathbb{Z}/2$ act on $\tilde{M}^{stab}_{ij}(x,y,z)$ by $(23)(45)$ on marked pts.

$$\boxed{\tilde{M}^{stab}_{ij} := \tilde{M}^{stab} \times_{\mathbb{Z}_2} S^i}$$

$$\begin{array}{ccc} \tilde{M}^{stab}_{j}(x,y,z) \times_{\mathbb{Z}/2} S^i & & [u, m, v] \\ \downarrow \pi_{x,y,z} & & \downarrow \\ \bar{M}_{0,5} \times_{\mathbb{Z}/2} S^i & & [stab(m), v] \end{array}$$

Forgetful map

Now, $\forall$ submfd $W \subseteq \bar{M}_{0,5} \times_{\mathbb{Z}/2} S^i$ we can define homomorphisms

$$q_{i,j}(W) : QH^l \times QH^k \longrightarrow QH^{l+k-\dim W}$$

$$\int_{\pi(u,m)} ev^*_a \cup \boxed{ev_c^* \beta}$$

$$q_{ij}(W)(x,y) = \sum_z \# | \pi^{-1}_{x,y,z}(W) | \, z T^j$$

This map only depends on homology class of W

Also $q_{ij}(W) + q_{ij}(W') = q_{ij}(W + W')$

Since it is intersection of pseudocycles $\pi_{x,y,z}$ and W

Let $m_1 =$  $\in \overline{M}_{0,5}$

$m_2 =$  $\in \overline{M}_{0,5}$

m_1 and m_2 are fixed by the action $(23)(45)$ on $\overline{M}_{0,5}$

$[QS(x)*QS(y)]_{i,j} \overset{\text{Cob}}{=} \#[z$  $, v] = q_{i,j}(\{m_1\}\times D^{i,+})(x,y)$

$[QS(x*y)]_{i,j} \overset{\text{Cob}}{=} \#[z$  $, v] = q_{i,j}(\{m_2\}\times D^{i,+})(x,y)$

In fact

$$[QS(x)*QS(y)]_{i,j} - [QS(x*y)]_{i,j}$$

$$= q_{i,j}(\{m_1\}\times D^{i,+} + \{m_2\}\times D^{i,+})(x,y)$$

$$\overset{\star}{=} q_{i,j}(W_0\times D^{i-2,+})(x,y).$$

where $W_0 \subseteq \overline{M}_{0,5} \cong Bl_{(0,0),(1,1),(\infty,\infty)}\, \mathbb{CP}^1\times\mathbb{CP}^1$

is the exceptional divisor over $(0,0)$

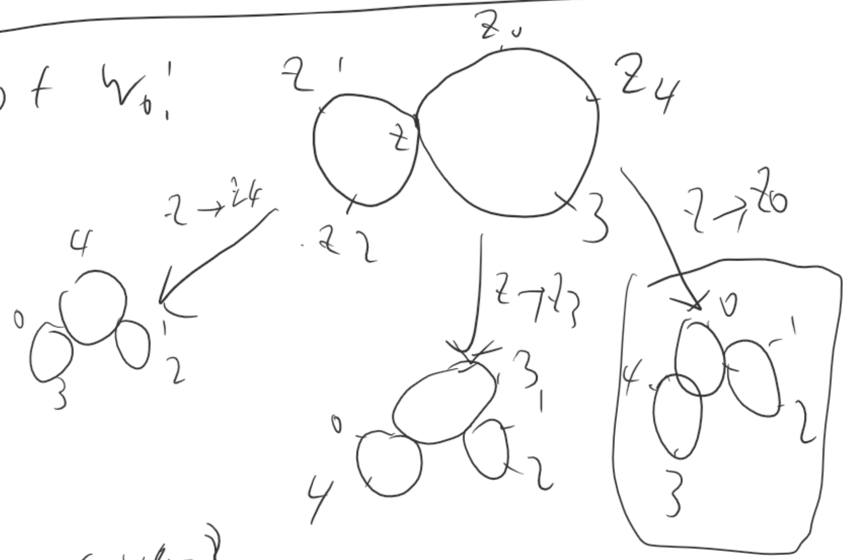
Lemma: let M be smooth cntd mfld with $\mathbb{Z}/2$ action $i: M \to M$,

and $W^n, L^{n-1} \subset M$ are submflds, invariant

and $W = L \cup V \cup iV$ for some open subm fld V, s.t. $\partial \bar{V} = L$

Then $[W \times D^{1+k,+}] = [L \times D^{i,+}]$

elements of W_0:



Identify $W_0 \cong \mathbb{C} \cup \{\infty\}$.

Identify $h_0 \cong \mathbb{C} \cup \{\infty\}$.

Then the $\mathbb{Z}/2$ action is $z \mapsto \frac{z}{z-1}$

$L = \mathbb{R} \subseteq \mathbb{C} \cup \{\infty\}$

$$\boxed{[h_0 \times D^{i-2,t}] = [L \times D^{i-1,t}]}$$

L contain 2 fixed pts, $\{m_1, m_3\}$

corresponding to $\{0, 2\} \subseteq \mathbb{R}$

Then $\boxed{[L \times D^{i-1,t}] = [\{m_1, m_3\} \times D^{i,t}]}$

But $[\{m_3\} \times D^{i,t}] = [\{m_2\} \times D^{i,t}]$

since m_2 and m_3 lie in same fixed pts set.