More on Steenrod squares via Morse theory

Monday, June 1, 2020 11:14 AM

<u>Recall</u>: Definition of Steenrod square:

$$C^* \longrightarrow \left(C^* \otimes C^*[[\hbar]],\ d_{eq} = d_{C\otimes C} + \hbar(id + \iota)\right)$$

ι: flipping factors

$$x \longrightarrow x \otimes x \qquad x \otimes y$$

Not a chain map, but induces maps on $H^*(C^*) \to H^*_{\mathbb{Z}/2}(C^* \otimes C^*)$

Def. X = smooth mfd. closed.

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\otimes 2} & H^*_{\mathbb{Z}/2}(X \times X) \\ \downarrow St & \circlearrowleft & \downarrow \text{restrict} \\ H^*(X)[[\hbar]] & \xleftarrow{\sim} & H^*_{\mathbb{Z}/2}(\Delta_X) \end{array}$$

$\iota : X \to X$ $\mathbb{Z}/2$-action

$X \times X \to X \times X$

$(p, q) \longmapsto (\iota(q), \iota(p))$

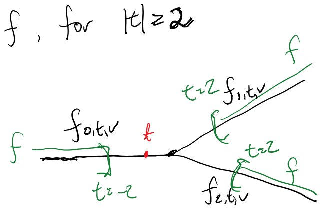
$x \circ$

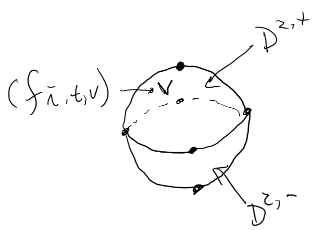
$y \circ$

<u>Morse definition of Steenrod squares</u>:

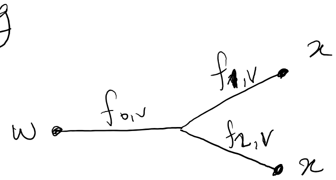
Take a generic S^∞-family of auxiliary data $(f_{0,t,v}, f_{1,t,v}, f_{2,t,v})$ $t \in S^\infty$, and $(f_{i,t,v})$ is a triple of Morse functions depending on t, s.t.

$$\begin{cases} f_{1,t,v} = f_{2,t,-v}, \quad f_{0,t,v} = f_{0,t,-v} \\ f_{i,t,v} = f, \text{ for } |t| \geq 2 \end{cases}$$





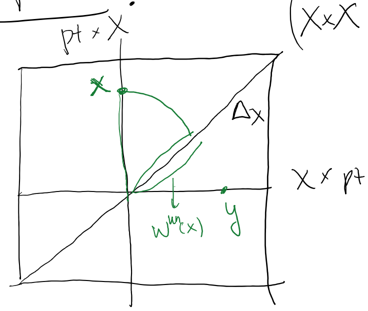
Consider $Sq(x) = \sum Sq_{|x|-i}(x)\hbar^i$, where $Sq_{|x|-i}(x)$ is obtained by counting

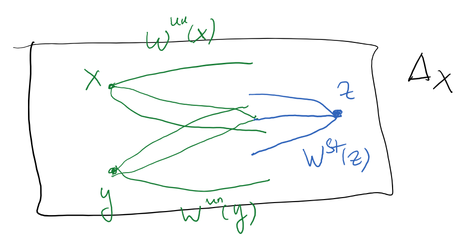


where $\{f_{i,v}\}_{v \in S^i}$, the family on the i^{th}-cells.

<u>How do the two definitions coincide?</u>

<u>Usual Cup product</u>: $(X \times X, f_1 + f_2)$





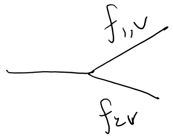
$\Longleftrightarrow \quad x \otimes y \in C^*(f_1) \otimes C^*(f_2)$ restricted to Δ_X

Equivariant cup product:

Do the above fiberwisely $\left(H^*(X) \to H^*_{eq}(X \times X) \xrightarrow{\text{rest}} H^*_{eq}(\Delta_X) \right)$

$H^*_{eq}(X \times X) = H^*((X \times X) \times (B\mathbb{Z}/2))$, $\quad H^*_{eq}(\Delta_X) = H^*(\Delta_X \times B\mathbb{Z}/2)$

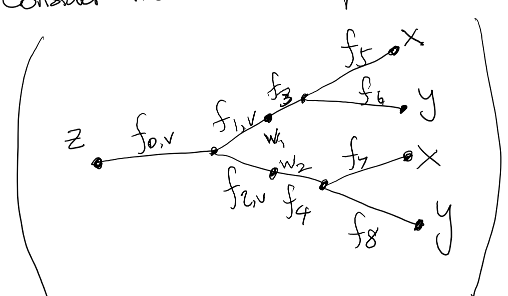
$\Rightarrow$ Over each fiber, choose a set of auxiliary data,



$f_{1,v} = f_{2,-v}$ comes from the equivariance flipping the two $X \times pt$ & $pt \times X$

Cartan Relation: $\quad Sq(x \cup y) = Sq(x) \cup Sq(y)$

Consider the moduli space

$\mathcal{M}$ 

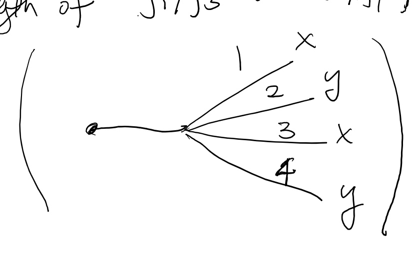
$f_{i,v} = f_{j,-v}$, $f_{0,v} = f_{0,-v}$

for $(i,j) = (1,2), (3,4), (5,7)$
$(6,8)$

This counts $\langle Sq_{|x|+|y|-i}(x \cup y), z \rangle$, or equivalently,

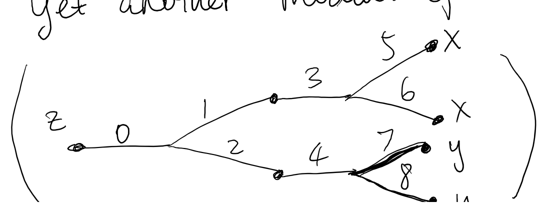
the $z \cdot h^i$ - component of $Sq(x \cup y)$, on the top cell of $\mathbb{RP}^{|x|+|y|-i}$

A gluing argument shows (and shinking the length of f_1/f_3 and f_2/f_4)

from ∞ to zero, the above moduli $= \#$ 

where $f_{1,v} = f_{3,-v}$, $f_{2,v} = f_{4,-v}$.

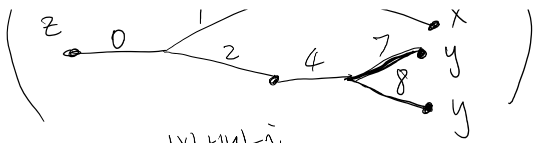
Consider yet another moduli space



$f_{1,2,3,4}$: independent of v

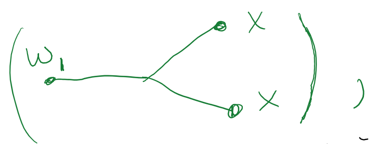
$f_{5,v} = f_{6,-v}$

$f_{7,v} = f_{8,-v}$



$f_{7,v} = f_{8,-v}$

Note that $v \in S^{|x|+|y|-i}$ is in a cell with bigger dimension than the definition of Steenrod square, both in x-subtree and y-subtree.

Focus on 

$|w_1| = |x| + j$, $j + k = i$;

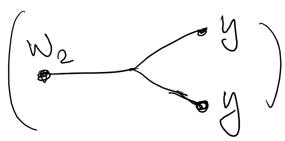
When v varies in $\mathbb{R}P^{|x|+|y|-i}$, this moduli space has virtual dimension $= |y| - k$. Its projection through forgetful map

$$\text{forget}: (u: T \to M, (f_{i,v})) \mapsto v \in \mathbb{R}P^{|x|+|y|-i}$$

gives a $(|y|-k)$-dimensional cycle $[C_{x,w_1}] \in H^{|y|-k}(\mathbb{R}P^{|x|+|y|-i})$

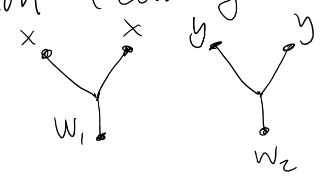
Intuition Remark: By definition, $\langle [C_{x,w_1}], [\mathbb{R}P^{|x|-j}] \rangle = \langle Sq_{|x|-j}(x), w_1 \rangle$.

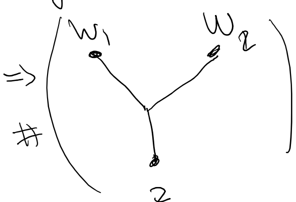
In other words, $\langle Sq_{|x|-j}(x), w_1 \rangle$ is the multiplicity of $[C_{x,w_1}]$

The same procedure applies to 

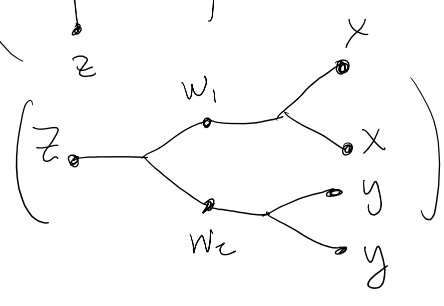
$\Rightarrow [C_{y,w_2}] = \langle Sq_{|y|-k}(y), w_2 \rangle \cdot [\mathbb{R}P^{|x|-j}]$

Therefore, at each intersection (counting multiplicity) of

$C_{x,w_1} \cap C_{y,w_2} \rightsquigarrow$ 

exists for the same data $\Rightarrow$ 

$= \langle w_1 \smile w_2, z \rangle$ over the same data set.

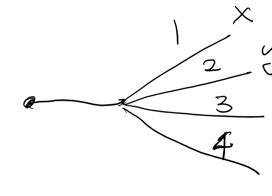
This implies $\#$ 

$$\# \left(\cdots \right)_{v \in \mathbb{R}P^{|x|+|y|-i}} = \langle Sq_{|y|-k}(y), w_2 \rangle \cdot \langle Sq_{|x|-j}(x), w_1 \rangle .$$

$\langle w_1 \cap w_2 , z \rangle$

Summing over all w_1, w_2 with appropriate $\hbar$-powers

$$\Rightarrow \left\langle \sum_{j+k=i} Sq_{\hbar}^{|x|-j}(x) \cdot Sq_{\hbar}^{|y|-k}(y) \, , \, z \right\rangle$$

Again by shrinking the middle edges length $\Rightarrow$ $\#\Big($  $\Big)$

$$\left\langle \sum_{j+k=i} Sq_{\hbar}^{|x|-j}(x) \cdot Sq_{\hbar}^{|y|-k}(y) \, , z \right\rangle = \left\langle Sq_{\hbar}^{|x|+|y|-i}(x \cup y), z \right\rangle,$$ as desired.