[Talk] Examples of SYZ and superpotential

2020年7月7日 14:01

SYZ: (X,ω,J) = kahler mfd, D = anti-canonical divisor.

$$\begin{array}{ccccc} L \to X\setminus D & \longleftrightarrow & M \leftarrow L^\vee = (L,\nabla) = \mathrm{Hom}(H_1(L),S^1) \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

, $W: M \to \mathbb{C}$ holo function, Super potential.

M $\quad$ $\underbrace{W = \sum_{\mu(\beta)=2} z_\beta \cdot n_\beta(L), \quad z_\beta = \exp(-\int_\beta \omega)\,\mathrm{hol}_\nabla(\partial\beta)}$

Nice case: (1) all Maslov 2 discs are regular.

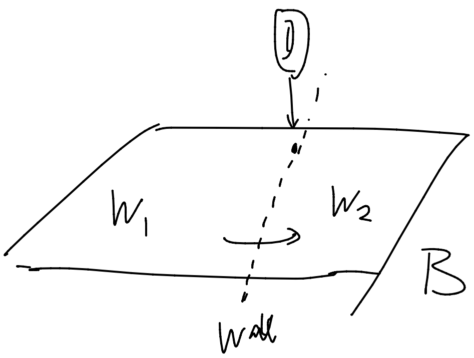
Eg. Fano toric (2) No nonconstant Maslov 0 disc exists. (3) No holo spheres w/ $c_1\cdot[S^2]\le 0$.

$M\subset(\mathbb{C}^*)^n \Rightarrow n_\beta(L)$ is well-defined and locally constant.

then $(\tilde{M}, W)$ is LG mirror.

completion.

Less nice case: L bound Maslov index 0 disc $\sim$ codim 1



change of coordinate.

$$W = \sum_{\mu(\beta)=2} n_\beta z_\beta$$

$(X^\vee, W)$

$\parallel$

$M = M_1 \cup M_2$

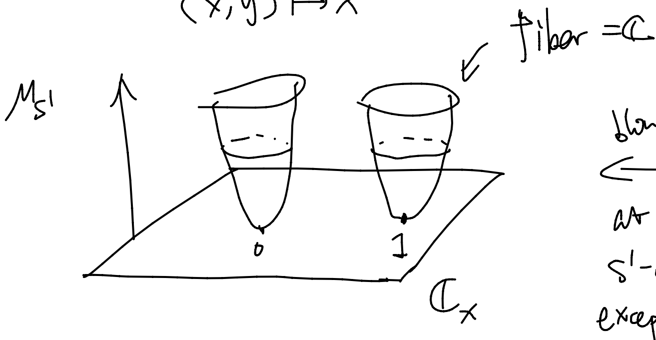
holo $\downarrow$ $\quad\downarrow$

$X = \mathbb{C}^2 \# \overline{\mathbb{CP}^2}$

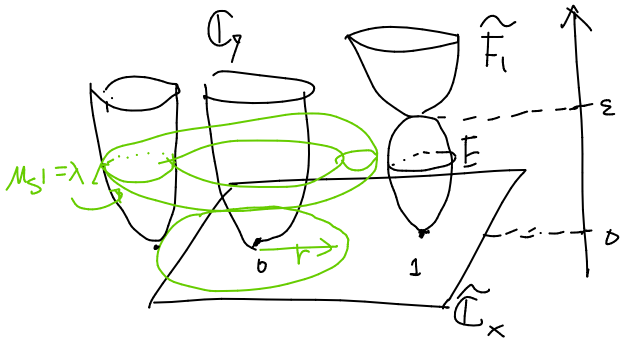
$\mathbb{C}^2_{(x,y)}$, $D = \mathbb{C}_x \cup \mathbb{C}_y$, $\Omega = d\log x \wedge d\log y$ on $\mathbb{C}^2\setminus D$

S^1-action on $\mathbb{C}^2$ by $e^{i\theta}(x,y) = (x, e^{i\theta}y)$

$f: \mathbb{C}^2 \to \mathbb{C}$
$(x,y) \mapsto x$



blow up
at $(1,0)$
S^1-equivariant
exception sphere has area ε



$X = \mathbb{C}^2 \# \overline{\mathbb{CP}}^2$

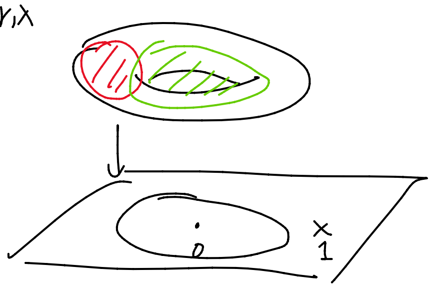
$D = \mathbb{C}_y \cup \tilde{\mathbb{C}}_x$, $\quad X \setminus D$ special Lag fibration.

$(r, \lambda) \in \mathbb{R}^2$. $\quad L_{r,\lambda} = \{ |x| = r, \ \mu_{S^1} = \lambda \}$

w/ singularity $(1, \varepsilon)$ $\qquad$ $L_{1,\lambda}$

Lemma: There is nontrivial Maslov 0 disc $\iff$ $r = 1$ ← wall

$r < 1$ $\quad L_{r,\lambda}$



bound two Maslov 2 discs.

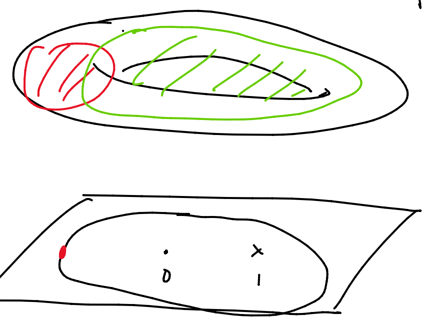
$\delta = \{ \mu_{S^1} \le \lambda \} \subset F_x = \mathbb{C}$

$\beta = \{ y = \text{const}, \ |x| \le r \}$

$$W = \sum_{\mu(\beta)=2} n_\beta(L) \underbrace{\exp\left(-\int_\beta \omega\right) \mathrm{hol}_\nabla(\partial\beta)}_{z_\beta}$$

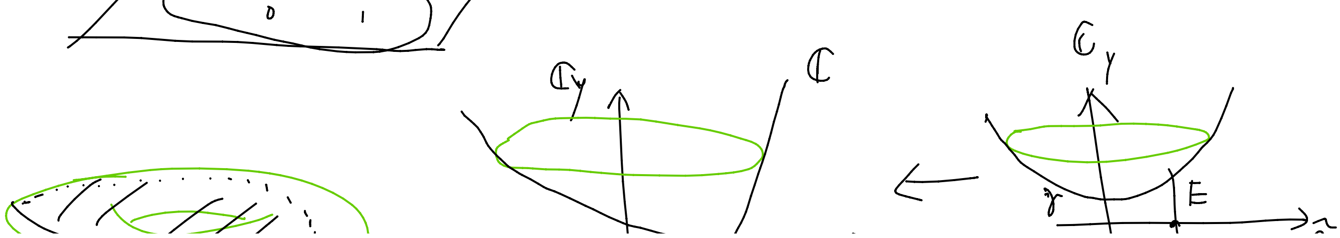
$$= z_\beta + z_\delta = u + z \qquad \text{holo function on } M = \{(L,\nabla)\}_{r<1}$$

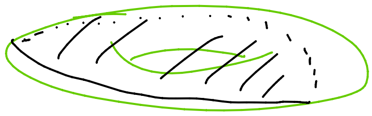
$r > 1$:



3 Maslov 2 discs.

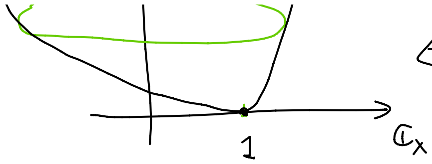
$[\gamma] = [\beta + \delta - E]$

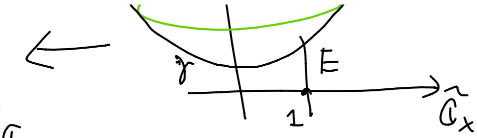




$\partial(\text{disc}) = \partial\delta + \partial\beta$

Maslov 4





$\mathbb{C}^2$

$$W = z_\beta + z_\delta + z_\gamma$$
$$= u + z + e^\varepsilon u z$$

Recall: Special Lagrangian fibration on $\mathbb{C}^2 \# \overline{\mathbb{C}P}^2 \setminus (\tilde{\mathbb{C}}_x \cup \mathbb{C}_y)$

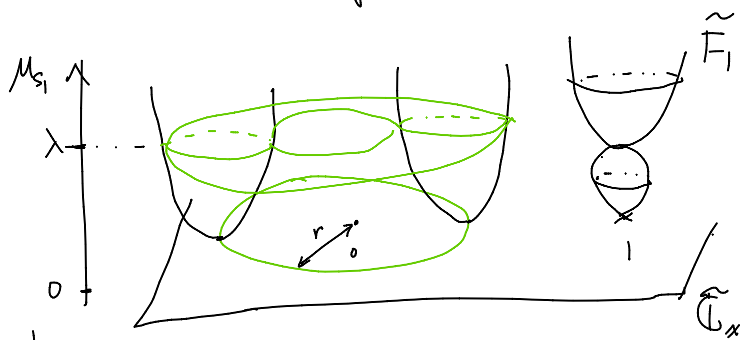
$L_{r,\lambda} = \{ |x| = r,\ \mu_{S^1} = \lambda \}$

$M = \{ (L_{r,\lambda}, \nabla) \}$

$W = \sum_{\mu(\beta)=2} n_\beta z_\beta$

$z_\beta = \exp(-\int_\omega \beta)\, \mathrm{hol}_\nabla(\partial\beta)$ holomorphic.

$n_\beta = \deg(ev : \mathcal{M}_1(\beta) \to L)$



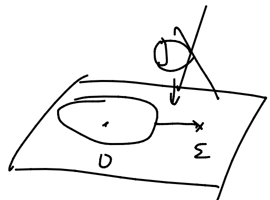
$\mathbb{C}^2 - \{xy = \varepsilon\} \supset D$

$\downarrow xy-\varepsilon$

$\mathbb{C}$

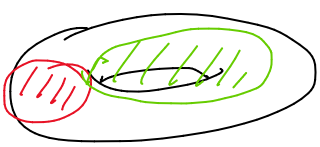
Fact:

- Maslov index $\mu(\beta) = 2(\beta \cdot (\tilde{\mathbb{C}}_x + \mathbb{C}_y))$
- $L_{r,\lambda}$ bounds Maslov index 0 disc $\Leftrightarrow$ $r = 1$ $\Leftarrow$ wall



$r < 1$

$L_{r,\lambda}$ bounds 2 family of maslov 2 disc.



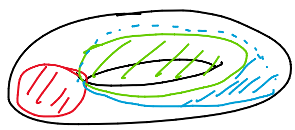
$\delta \Rightarrow z$

$\beta \Rightarrow u$

$W = z + u$

$r > 1$

$L_{r,\lambda}$ bounds 3 family of Maslov 2 disc



$\delta' \Rightarrow z'$

$\beta' \Rightarrow u'$

$\gamma' \Rightarrow e^\varepsilon u' z'$

$[\gamma'] = [\beta' + \delta' - E]$

$W = z' + u' + e^\varepsilon u' z'$

$$\begin{cases} z = z' \\ u = u' + e^\varepsilon u' z' \end{cases} \Rightarrow W = z + u$$

Maslov 0 disc $\Leftrightarrow$ $r = 1$ Wall

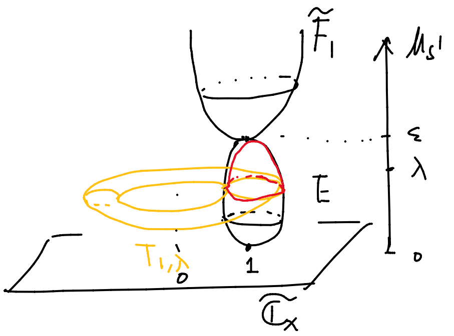
$\underline{\lambda < \varepsilon}$:

$\alpha = E - \delta$

$\mu(\alpha) = 0$

$\{\lambda \le \mu_{S^1} \le \varepsilon\}$

$x = 1$



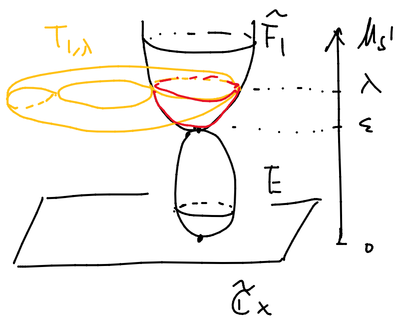
$\underline{\lambda > \varepsilon}$:

$-\alpha = \delta - E$

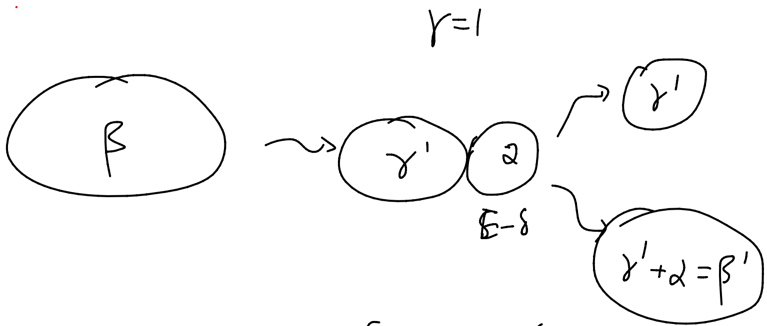
$\mu(-\alpha) = 0$

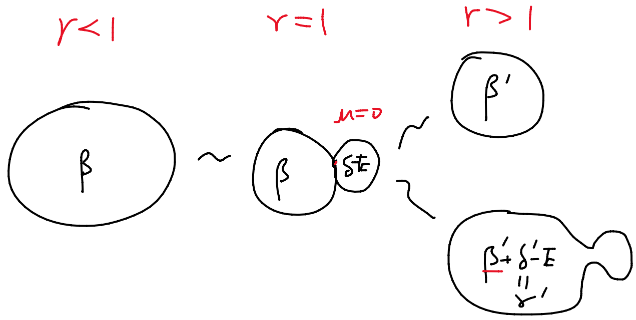
$\{\varepsilon \le \mu_{S^1} \le \lambda\}$

$x = 1$



$$Z_\alpha = \exp\left(-\int_\omega \alpha\right) \mathrm{hol}_\nabla(\partial\alpha)$$





$$u \longmapsto e^{\varepsilon} u' z' + e^{\varepsilon} u' z' \cdot \underbrace{(e^{-\varepsilon}(z')^{-1})}$$
$$= e^{\varepsilon} u' z' + u'$$

$$u \longmapsto u' + e^{\varepsilon} u' z'$$

$$X^\vee = \{(L_{r,\lambda}, \nabla) \mid r < 1\} \cup \{(L_{r,\lambda}, \nabla) \mid r > 1\}$$

(first set $\approx (\mathbb{C}^*)^2$ with coordinates (u, z); second set $\approx (\mathbb{C}^*)^2$ with coordinates (u', z'))

glue by $\begin{cases} z = z' \\ u = u' + e^{\varepsilon} u' z' \end{cases}$

Compare w/ Hori–Vafa Mirror, $\xrightarrow{\text{completion}}$ $X^\vee$ = affine variety.

$$X^\vee = \operatorname{Spec} \mathbb{C}[u^{\pm}, u'^{\pm}, z^{\pm}, z'^{\pm}] / (z = z', u = u' + e^{\varepsilon} u' z')$$
$$= \{(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}^* \mid uv = 1 + e^{\varepsilon} z\}$$

Fano Toric Variety

X = Fano toric variety, $\pi : X \to \mathbb{R}^n$ moment map w/ polytope Δ.

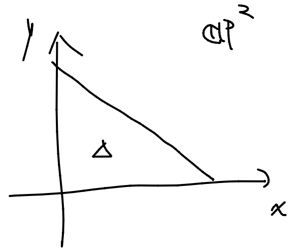
D = toric divisor, $X \setminus D \cong (\mathbb{C}^*)^n \longleftarrow$ coordinates $(x_1, \cdots x_n)$

holomorphic volume form on $X \setminus D$ $\quad \Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$

$\pi : X \setminus D \to \mathbb{R}^n$ is special Lag fibration holomorphic volume form on $X \setminus D$ $\quad \Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$

$\pi : X \setminus D \to \mathbb{R}^n$ is special Lag fibration

$L = S^1(r_1) \times \cdots \times S^1(r_n)$, $\quad S^1(r_i) = \{ |x_i| = r_i \} \subset \mathbb{C}_{x_i}$



$H_1(L) = \mathbb{Z}\{ \gamma_i = [S^1(r_i)] \}$

$\nabla \leftrightsquigarrow \{ hol_\nabla(\gamma_j) = e^{i\theta_j} \} \quad \{\theta_1, \cdots \theta_n\} \cong T^n$

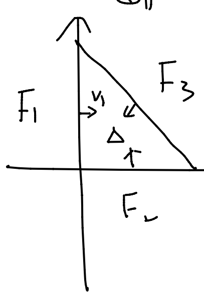
$\mu(\beta) = 2\beta \cdot D \quad \Rightarrow \quad \beta$ Maslov 2 disc $\Leftrightarrow \beta \cdot D_F = 1 \qquad D_F = \pi^{-1}(F)$

<u>Polytope</u>: $\quad \Delta = \bigcap_F \{ \langle \phi, \nu(F) \rangle \geq -\alpha(F) \}, \quad -\alpha(F) \in \mathbb{R}$

$\pi(L) = (\phi_1, \cdots \phi_n)$

$\nu(F)$ = primitive inward pointing normal of F

$\mathbb{CP}^2$



$\nu_1 = (1,0)$, $\quad \alpha(F_1) = 0$

$\nu_2 = (0,1) \quad \alpha(F_2) = 0$

$\nu_3 = (-1,-1) \quad \alpha(F_3) = \lambda$

<u>Prop</u> [Cho-Oh] $\quad W = \sum_F e^{-2\pi\alpha(F)} z^{\nu(F)}$, $\quad z_j = \exp(-2\pi\phi_j(L)) \, hol_\nabla(\gamma_j)$

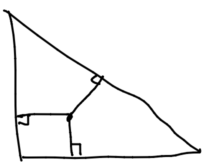
$z^{\nu(F)} = z_1^{\nu_1(F)} \cdots z_n^{\nu_n(F)}$

① No Maslov 0 disc.

② $\forall$ codim 1 face F, $\Rightarrow \exists!$ Maslov 2 disc.

$u : \quad w \mapsto (w^{\nu_1(F)} x_1, \cdots, w^{\nu_n(F)} x_n)$

$w \in D^2 \setminus 0 \qquad (x_1, \cdots x_n)$ coordinate for $(\mathbb{C}^*)^n$



$u : \quad 0 \mapsto D_F \qquad \partial u = \sum \nu_i(F) \gamma_i$

③ Area: [Guillemin] $\quad \int u^* \omega = 2\pi \langle \nu(F), \phi(L) \rangle + 2\pi\alpha(F)$

$W = \sum_F \exp(-2\pi \langle \nu(F), \phi(L) \rangle) \exp(-2\pi\alpha(F)) \, hol_\nabla(\partial[u])$

$= \sum_F \exp(-2\pi\alpha(F)) \cdot z^{\nu(F)}$

$$= \sum_F \exp(-2\pi\lambda(F)) \cdot z^{v(F)}$$

$\mathbb{CP}^2$:

$$v_1 = (1,0) \qquad \lambda(F_1) = 0$$
$$v_2 = (0,1) \qquad \lambda(F_2) = 0$$
$$v_3 = (-1,-1) \qquad \lambda(F_3) = \lambda$$

$$\Longrightarrow W = z_1 + z_2 + \frac{e^{-2\pi\lambda}}{z_1 z_2}.$$