Quantum Steenrod squares

Let X be a closed monotone symp mfd
N = min Chern number
J = w com patible a.c.s
Want to define a degree doubling operation
$QS : QH^*(M) \to QH^*(M)[h]$

Moduli space

$$\overline{\mathcal{M}}^{\deg}_{i,j}(x,z) := \left\{ (u,v) \;\middle|\; \begin{array}{l} u : S^1 \to M \;\; u(p_0) = z \\ (i,0) \to (u,v) \;\; , \; u(p_1) = x \\ v \in S^1 \end{array} \right\} / (u,v) \sim (u \circ \tau, v \circ \tau)$$

$$= \left\{ z \to \; \text{ } \right\} / (u,v) \sim (u \circ \tau, v \circ \tau)$$

(Defining the ith component, $QS_{ij}(x)$ at z is:

$$\langle QS_{ij}(x), z \rangle = \begin{cases} \# \overline{\mathcal{M}}^{\deg}_{i,j}(x,z) & \text{if } |x| - 2|z| + i + j = 2 \\ 0 & \text{if otherwise.} \end{cases}$$

$$\langle QS(x), z \rangle = \sum_{i \geq 0} \langle QS_{i,i}(x), z \rangle h^i T^i$$

where $h \in H^*(\mathbb{P}^1_{\mathbb{R}}, \mathbb{Z}_2)$ is the generator

Remark $QS^0_{i,0}(x) = \{ q^{|x|-1}(x) \}$ $\quad Sq^i(x) = x \cup x$ if $|x| = i$

$\sum_{i \geq 0} QS_{i,j}(x) T^i = x \star x$

Quantum Cartan relation

$Sq_i(x) \cup Sq_i(y) = Sq_i(x \cup y)$

$QS_i(x) \star QS_i(y) \neq QS_i(x \star y)$

To apply cobordism argument: define

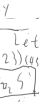
$$\overline{\mathcal{M}}^{\deg}_{i,j}(z, x, y, x \cdot y) = \left\{ (t, u, v) \;\middle|\; z \to \; \text{ } \right\}$$

$$\overline{\mathcal{M}}^{\deg}_{i,j}(z, x, y, x \times y) = \left\{ (t, u, v) \;\middle|\; z \to \; \text{ } \right\}$$

$$\sum \# \left\{ z \to \; \text{ } \right\} \xleftarrow{t \to \infty} \left\{ z \to \; \text{ } \right\} \xrightarrow{t \to 0} \left\{ z \to \; \text{ } \right\}$$

$[QS(x) \star QS(y)]_{ij}$

$[QS(x \cdot y)]_{ij}$

$$\sum \# \left\{ z \to \; \text{ } \right\} \xleftarrow{t \to \infty} \left\{ z \to \; \text{ } \right\} \xrightarrow{t \to 0} \left\{ z \to \; \text{  } \right\}$$

To describe the differences on the RHS, Define:

$$\widetilde{\mathcal{M}}^{stab}_j(x, y, z) := \left\{ (u, m) \;\middle|\; \begin{array}{l} u : \overline{\mathcal{M}}_{0,3} \\ u(m) = \mathcal{M} \\ (i,0) \to u_j \end{array} \;\; \begin{array}{l} z \to z_1 \\ z_2, z_3 \to x \\ z_4, z_5 \to y \end{array} \right\}$$

$$= \left\{ z \to \; \text{ } \right\}$$

Let $\mathbb{Z}_2$ act on $\widetilde{\mathcal{M}}^{stab}(x,y,z)$ by $(23)(01)$ on marked pts

$$\widetilde{\mathcal{M}}^{stab}_j / \mathbb{Z}_2 = \widetilde{\mathcal{M}}^{stab} \times_{\mathbb{Z}_2} S^1$$

$$\widetilde{\mathcal{M}}^{stab}_j(x,y,z) \times_{\mathbb{Z}_2} S^1 \qquad [(stab(m), v]$$

$$\downarrow \pi_{1,2,3}$$

$$\overline{\mathcal{M}}_{0,3} \times_{\mathbb{Z}_2} S^1$$

Forgetful map

Now, $\forall$ submanifold $W \subseteq \overline{\mathcal{M}}_{0,3} \times_{\mathbb{Z}_2} S^1$ we can define homomorphism (abstract) $\boxed{\left(\begin{array}{c} \text{genus} = 0 \\ \text{Mixing} \end{array} \right) \neq 0}$

$$q_{ij}(W) : QH^*(X \times H^k \to QH^*(X)$$

$$q_{ij}(W)(x \cdot y) = \sum_z \# \overline{\mathcal{M}}^{\deg}_{i,j}(W) \, z T^j$$

This map only depends on homology class of W

Also $q_{ij}(W) + q_{ij}(W') = q_{ij}(W + W')$

Since it is intersection of pseudo cycles $\pi_{1,2,3}$ and W

Let $m_1 = \; \text{ } \in \overline{\mathcal{M}}_{0,5}$

m_1 and m_2 are fixed by the action $(23)(14)$ on $\overline{\mathcal{M}}_{0,5}$

$m_2 = \; \text{ } \in \overline{\mathcal{M}}_{0,5}$

$$[QS(x \star y) QS(y)]_{ij} \overset{\text{def}}{=} \# \left[z \to \; \text{ } , v \right] = q_{ij}((m_1 \times D^1)(x,y)$$

$$[QS(x \cdot y)]_{ij} \overset{\text{def}}{=} \# \left[z \to \; \text{ } , v \right] = q_{ij}((m_2 \times D^1)(x,y)$$

In fact

$$[QS(x) \star QS(y)]_{ij} - [QS(x \cdot y)]_{ij}$$

$$= q_{ij}((m_1 \times D^1 + (m_2 \times D^1)(x,y)$$

$$\overset{\text{?}}{=} q_{ij}(W_0 \times D^{i,+})(x,y).$$

where $W_0 \subseteq \overline{\mathcal{M}}_{0,5} \cong Bl_{\{(x,1),(x,\infty)\}} \mathbb{P}^1 \times \mathbb{P}^1$

is the exceptional divisor over $(x,0)$

Lemma Let M be smooth $\mathbb{Z}_2$ manifold with $\mathbb{Z}_2$ action $i : M \to M$.

and $W^+, L^+ \subset M$ are submanifolds, invariant

and $W = L \cup V \cup iV$ for some open sub mfd V, s.t. $\partial \overline{V} = L$

Then $[W \times D^{i,+}] = [L \times D^{i,+}]$

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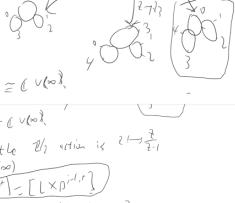
elements of W_0:



Identify $W_0 \cong \mathbb{C} \cup \{\infty\}$

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Then the $\mathbb{Z}_2$ action is $z \mapsto \frac{1}{\bar{z}}$

$L = \mathbb{R} \subset \mathbb{C} \cup \{\infty\}$

$$\boxed{[W_0 \times D^{i,+}] = [L \times D^{i,+}]}$$

L contains 2 fixed pts $\{\infty, m_1, m_2\}$ corresponding to $\{0, 1\} \subset \mathbb{R}$

Then $[L \times D^{i,+}] = [\{m_1\} \times D^{i,+}]$

But $[\{m_2\} \times D^{i,+}] = [\{m_1\} \times D^{i,+}]$

since m_1 and m_2 lie in same fixed pt set.