Potential function, constructions

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Hori-Vafa mirror for toric Fano manifold

$X = \mathbb{C}P^2$ (A-model). Mirror (B-model) Landau-Ginzburg superpontential $(\mathbb{C}^*)^2 \to \mathbb{C}$:

$$W = z_1 + z_2 + \frac{1}{z_1 z_2}.$$

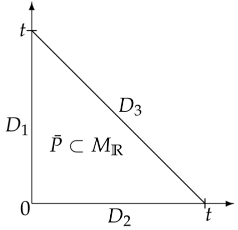
And this one corresponds to the Clifford torus in $P^2 = P^2(1,1,1)$.

The idea of Hori-Vafa mirror:

1. The potential is defined by Maslov index 2 holomorphic disks bounded by the toric fiber,
2. $QH^*(X) = Jac(W) := \mathbb{C}[z_1 \cdots z_n]/ < \partial z_1, \cdots, \partial z_n >$. (In particular, # of critical point of W has to equal dim of $QH^*(X)$.)

For toric Fano case this is verified by [Cho-Oh], toric case by [FOOO].
Note: $\mathbb{C}P^2$ can be viewed as a toric Fano manifold, Flag variety, or Grassmannian. We'll first focus on $\mathbb{C}P^2$ and see generalizations (failure) on other examples later.

Base diagram for almost toric fibration



$W = z_1 + z_2 + \frac{q}{z_1 z_2}$ where q is a Novikov variable.

- $QH^*(\mathbb{C}P^2) = \mathbb{C}[D_1, D_2, D_3]/ < D_1 - D_3, D_2 - D_3, D_1 * D_2 * D_3 = q > \simeq \mathbb{C}[H]/ < H^3 = q >$
- Rewriting $Z_1 = z_1, Z_2 = z_2, Z_3 = 1/z_1 z_2$, $Jac(W) = \mathbb{C}[Z_1, Z_2, Z_3]/ < Z_1 - Z_3, Z_2 - Z_3, Z_1 \times Z_2 \times Z_3 = q > \simeq \mathbb{C}[Z]/ < Z^3 = q >$,
- There are 3 critical points of W, and dim $H^*(\mathbb{C}P^2) = 3$.

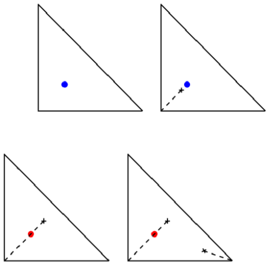
Base diagram for almost toric fibration, mutation on A-side

Eigen-direction given by the monodromy of the singular fiber.

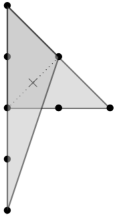
Nodal trade: trade a corner fixed point with an interior focus-focus singularity.

Nodal slide: Move the focus-focus singularity in the direction of the eigenvector for its affine monodromy.

Representing the toric degeneration to $P^2(1,1,4)$



From the Clifford torus on the top left base diagram, to the Chekanov torus



Mutate the lower left edges, obtaining $P^2(1,1,4)$

More toric degenrations

More generally, the projective plane admits degenerations to weighted projective spaces $\mathbb{CP}(a^2, b^2, c^2)$, where (a, b, c) is a Markov triple, ie it satisfies the Markov equation

$$a^2 + b^2 + c^2 = 3abc.$$

All Markov triples are obtained from (1,1,1) by a sequence of 'mutations' of the form

$$(a, b, c) \to (a, b, c' = 3ab - c).$$

Open question: are these all monotone Lagrangian tori up to Hamiltonian isotopy?

Mutation on the potential (B-) side.

[Galkin-Usnich]
Define a mutation in the direction $(m,n)$where m,n are coprime integers given by the formula:

$$\mu_{(m,n)} : x^a y^b \to x^a y^b (1 + x^n y^{-m})^{an-bm}.$$

An example (1st step of the Markov tree):

$$W = x + y + 2x^{-2} + x^{-4} y^{-1}.$$

This is obtained by mutation along (1,2).

Note: mutation can be viewed as a birational automorphism of $\mathbb{C}P^2$.

Potential function for a generic fiber of a toric moment polytope

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \geq 0,\ \lim_{i\to\infty} \lambda_i = \infty \right\}$$

is the Novikov ring and its valuation

$$\begin{array}{cccc} \mathfrak{v}: & \Lambda_0 & \to & \mathbb{R} \\ & \Cup & & \Cup \\ & \sum_{i=1}^{\infty} a_i T^{\lambda_i} & \mapsto & \min_i \{\lambda_i\}_{i=1}^{\infty} \end{array}$$

The maximal ideal of Λ_0 is denoted by Λ_+ and Λ_0 is the local ring.

Potential function for a generic fiber of a toric moment polytope

$P \subset M_{\mathbb{R}} = \mathbb{R}^n$ is a convex polytope defined by the affine functions $l_i(u) =< u, v_i > -\lambda_i \geq 0$ for $i = 1, \cdots, n$.
Then the following hold from construction: 1)$l_i = 0$ on $\partial_i P$;
2)$P = \{u \in M_{\mathbb{R}} | l_i(u) \geq 0, i = 1, \cdots, n\}$.

Theorem

Once a choice of the family of bases $\{e_i\}$ on $H^1(L(u); \mathbb{Z})$ for $u \in Int(P)$, then we can regard this function as a function of $(x_1, \cdots, x_n) \in (\Lambda_+)^n$ and $(u_1, \cdots, u_n) \in P$ and the potential function $\mathfrak{PO}^u$:$H^1(L(u); \Lambda_+) \to \Lambda_+$ is given by

$$\mathfrak{PO}^u(x) = \sum_{i=1}^{n} e^{\langle v_i, x\rangle} T^{\ell_i(u)}. \tag{1}$$

As a Laurent polynomial

$$\mathfrak{PO}^u \in [Q_1^{\pm 1}, \ldots, Q_{r+1}^{\pm 1}][y_1^{\pm 1}, \ldots, y_N^{\pm 1}]$$

where

$$y_k = e^{x_k}, \qquad k = 1, \ldots, N$$

are combinations of the variable $x \in H^1(L(u); \Lambda_+)$ by $Q_j = T^{\lambda_{n_j}}, \qquad j = 1, \ldots, r+1.$

A simple example, F_1

one point blow up X of P^2. We choose its Kähler form so that the moment polytope is

$$P = \{(u_1, u_2) \mid 0 \le u_1, u_2,\ u_1 + u_2 \le 1,\ u_2 \le 1 - \alpha\},$$

$0 < \alpha < 1$. The potential function is

$$\mathfrak{PO} = y_1 T^{u_1} + y_2 T^{u_2} + (y_1 y_2)^{-1} T^{1-u_1-u_2} + y_2^{-1} T^{1-\alpha-u_2}.$$

It has 4 critical points, dim H^*=4.

Hirzebruch surface F_n, $n \ge 3$. We take its Kähler form so that the moment polytope is

$$P = \{(u_1, u_2) \mid 0 \le u_1, u_2,\ u_1 + nu_2 \le n,\ u_2 \le 1 - \alpha\},$$

$0 < \alpha < 1$. The leading order potential function is

$$\mathfrak{PO}_0 = y_1 T^{u_1} + y_2 T^{u_2} + y_1^{-1} y_2^{-n} T^{n-u_1-nu_2} + y_2^{-1} T^{1-\alpha-u_2}.$$

A quick note on the non-Fano case ($F_n, n \geq 2$)

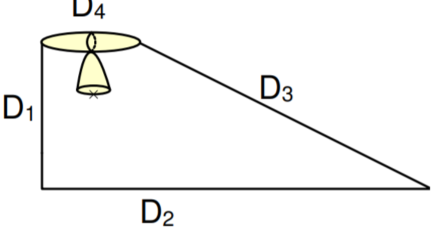


Figure: Holomorphic disk may have a sphere bubble with negative index

For the potential function of $F_n, n \geq 3$, there are $m+2$ critical points, but dim H^* is always 4.
There has to be some (infinitely many) correction terms.

A digression into integrable systems

An integrable system is a symplectic manifold M^{2n} equipped with n linearly independent Poisson commuting functions $\{f_1, \cdots, f_n\}$ (i.e. the corresponding Hamiltonian vector fields are linearly independent almost everywhere).

A toric manifold is an integrable system for which the functions $\{f_1, \cdots, f_n\}$ may be chosen in such a way that the Hamiltonian flows of the Poisson commuting functions are periodic with period 1 almost everywhere.

(partial) Flag variety

Fix a sequence $0 = n_0 < n_1 < \cdots < n_r < n_{r+1} = n$ of integers, and set $k_i = n_i - n_{i-1}$ for $i = 1, \ldots, r+1$. The partial flag manifold $F = F(n_1, \ldots, n_r, n)$ is a complex manifold parameterizing subspaces

$$0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n, \quad \dim V_i = n_i.$$

Examples: $CP^2 = Gr(1,3)$ partial flag
full flag manifold $F(1, 2, \ldots, n)$.
$Gr(2,4)$
Guillemin-Sternberg constructed an integrable system, with the Gelfand-Celtin polytope.

Gelfand-Celtin pattern and polytope

$$\begin{array}{ccccccccc}
a_{1,1} & & a_{1,2} & & a_{1,3} & & \cdots & & a_{1,n} \\
& a_{2,2} & & a_{2,3} & & \cdots & & a_{2,n} & \\
& & a_{3,3} & & \cdots & & a_{3,n} & & \\
& & & \ddots & & & & & \\
& & & & a_{n,n} & & & &
\end{array}$$

such that $a_{i,j} \geq a_{i+1,j+1} \geq a_{i,j+1}$.

The first line is a set of real number $\lambda_1, \cdots, \lambda_n$ given by the flag pattern.

The 3-d (full) flag variety $F(1,2,3)$ is the embedded of $P^1 \times P^2$ into $P^2 \times P^2$ as a quadratic hypersurface

$$\begin{array}{ccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 \\
 & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & \\
 & & u_1 & & & & u_2 & & \\
 & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
 & & & & u_3 & & & &
\end{array}$$

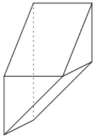
and the Gelfand-Cetlin polytope Δ_λ is defined by six inequalities

$$\Delta_\lambda = \{u = (u_1, u_2, u_3) \in \mathbb{R}^3 \mid \ell_i(u) = \langle v_i, u\rangle - \tau_i \geq 0, \quad i = 1, \ldots, 6\}$$

.

Potential function of F(1,2,3)

$$\ell_1(u) = \langle(-1,0,0), u\rangle + \lambda_1,$$
$$\ell_2(u) = \langle(1,0,0), u\rangle - \lambda_2,$$
$$\ell_3(u) = \langle(0,-1,0), u\rangle + \lambda_2,$$
$$\ell_4(u) = \langle(0,1,0), u\rangle - \lambda_3,$$
$$\ell_5(u) = \langle(1,0,-1), u\rangle,$$
$$\ell_6(u) = \langle(0,-1,1), u\rangle.$$



Only one non-Delzant point

Each equation comes from an inequality of the G-C pattern.

The only non-toric fiber is an embedded S^3.

Theorem

Nishinou-Nohara-Ueda08: For a Gelfand-Celtin toric fiber, one has the same potential function as the toric case.

Potential function of F(1,2,3)

The potential function of $F(1,2,3)$:

$$\begin{aligned}\mathfrak{PO} &= e^{-x_1}T^{-u_1+\lambda_1} + e^{x_1}T^{u_1-\lambda_2} + e^{-x_2}T^{-u_2+\lambda_2} \\ &\quad + e^{x_2}T^{u_2-\lambda_3} + e^{x_1-x_3}T^{u_1-u_3} + e^{-x_2+x_3}T^{-u_2+u_3} \\ &= \frac{Q_1}{y_1} + \frac{y_1}{Q_2} + \frac{Q_2}{y_2} + \frac{y_2}{Q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2}.\end{aligned}$$

By equating the partial derivatives $\frac{\partial\mathfrak{PO}}{\partial y_1} = -\frac{Q_1}{y_1^2} + \frac{1}{Q_2} + \frac{1}{y_3} = \frac{\partial\mathfrak{PO}}{\partial y_2} = -\frac{Q_2}{y_2^2} + \frac{1}{Q_3} - \frac{y_3}{y_2^2} = \frac{\partial\mathfrak{PO}}{\partial y_3} = -\frac{y_1}{y_3^2} + \frac{1}{y_2} = 0$ one obtains

$$\begin{aligned} y_1 &= \frac{y_3^2}{y_2}, \\ y_2 &= \pm\sqrt{Q_3(y_3+Q_2)}, \\ y_3 &= \sqrt[3]{Q_1Q_2Q_3},\ \omega\sqrt[3]{Q_1Q_2Q_3},\ \omega^2\sqrt[3]{Q_1Q_2Q_3}, \end{aligned}$$

where $\omega = \exp(2\pi\sqrt{-1}/3)$ is a primitive cubic root of unity. Since $\dim H^*(F(1,2,3),\Lambda)$ is six, one has as many critical point as $\dim H^*(F(1,2,3))$ in this case.

$Gr(2,4)$ is a partial flag variety, and via the Plücker embedding into P^5 as a quadratic hypersurface.
Choose $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$:

$$\begin{array}{ccccccc}
\lambda_1 & & \lambda_1 & & \lambda_3 & & \lambda_3 \\
& \rotatebox{-45}{$=$}\ \rotatebox{45}{$=$} & & \rotatebox{-45}{$\geq$}\ \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$=$}\ \rotatebox{45}{$=$} & \\
& \lambda_1 & & u_1 & & \lambda_3 & \\
& & \rotatebox{-45}{$\geq$}\ \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$}\ \rotatebox{45}{$\leq$} & & \\
& & u_2 & & u_3 & & \\
& & & \rotatebox{-45}{$\geq$}\ \rotatebox{45}{$\leq$} & & & \\
& & & u_4 & & &
\end{array}$$

so that the Gelfand-Cetlin polytope Δ_λ is defined by six inequalities

$$\Delta_\lambda = \{u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \mid \ell_i(u) = \langle v_i, u\rangle - \tau_i \geq 0,$$

Only 6 effective inequality of the G-C pattern.

$$\begin{aligned}
\ell_1(u) &= \langle (0,-1,0,0), u\rangle + \lambda_1, \\
\ell_2(u) &= \langle (-1,1,0,0), u\rangle, \\
\ell_3(u) &= \langle (1,0,-1,0), u\rangle, \\
\ell_4(u) &= \langle (0,0,1,0), u\rangle - \lambda_3, \\
\ell_5(u) &= \langle (0,1,0,-1), u\rangle, \\
\ell_6(u) &= \langle (0,0,-1,1), u\rangle,
\end{aligned}$$

More complicated non-toric fibers, culd be $S^1 \times S^3$ or immerced spheres.

Potential function of $Gr(2,4)$

$$\begin{aligned}\mathfrak{PO} &= e^{-x_2}T^{-u_2+\lambda_1} + e^{-x_1+x_2}T^{-u_1+u_2} + e^{x_1-x_3}T^{u_1-u_3} \\ &\quad + e^{x_3}T^{u_3-\lambda_3} + e^{x_2-x_4}T^{u_2-u_4} + e^{-x_3+x_4}T^{-u_3+u_4} \\ &= \frac{Q_1}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_3}{Q_3} + \frac{y_2}{y_4} + \frac{y_4}{y_3}.\end{aligned}$$

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We have only 4 critical points

$$\begin{aligned} y_1 &= \pm\sqrt{Q_1 Q_3}, \\ y_2 &= Q_1 Q_3 / y_3, \\ y_3 &= \pm\sqrt{2 Q_3 y_1}, \\ y_4 &= y_1. \end{aligned}$$

while $\dim H^*(Gr(2,4), \Lambda) = 6$.

A disscussion on the "failure" of matching QH with Jac

Fact [NNU08]: Full flag manifold, $QH(X) = Jac(W)$.

Partial flag manifolds of higher dimension, critical points of the disk potential of a regular fiber is in general smaller than the dimension of the quantum cohomology ring.

2.There are not enough regular fibers to generate the Fukaya category and more critical points are contributed by the disk potential of the non-toric fibers, e.g. product of (immersed) sphere.

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What's next?

How to correct the Mirror?

Ristch-Williams for Grassmannian. Use cluster charts $Gr(k, n) \setminus D$.

Are there Lag tori that's not Hamiltonian isotopic to the Gelfand-Celtin fiber?

Yes, recently discovered by the mutation on the B-side.

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