Dehn twist exact sequences through Lagrangian Cobordisms

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Overview:

(1) Review of Lag. Cobordisms. Surgeries via flow handles.

(2) Examples & Proofs of Seidel's exact seq's. including the fixed point and family Dehn twist version.

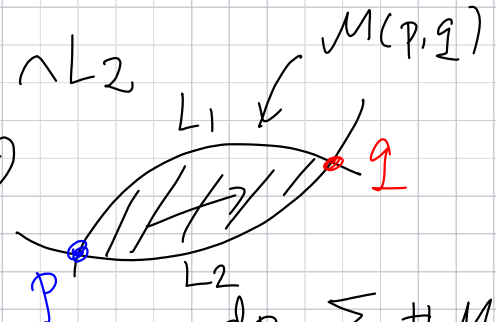
(3) Further Results, $\mathbb{P}^n$-objects & projective twists.

(4) Immersed Floer theory & Proof of Proj. twist exact seq. when $L \pitchfork \mathbb{C}P^n = \{pt\}$.

Lagrangian Floer cohomology:

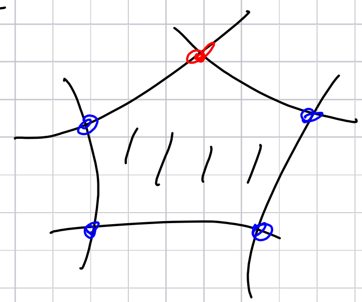
Def: $CF(L_1, L_2) = L_1 \cap L_2$

$d : CF(L_1, L_2) \circlearrowleft$



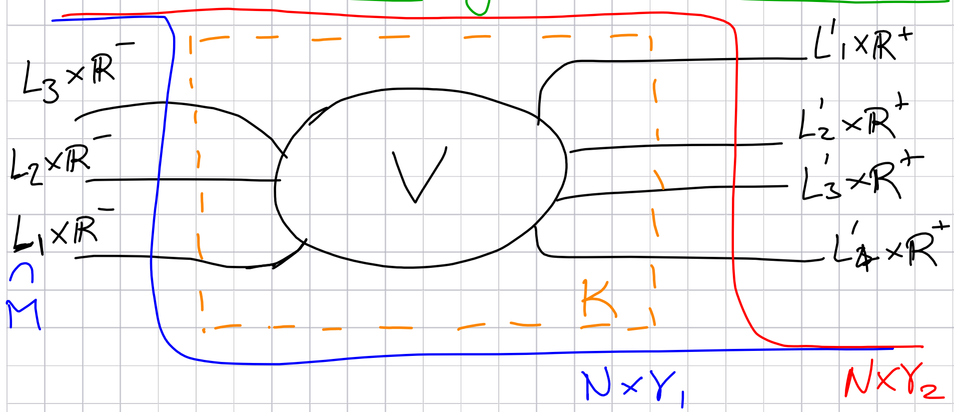
$$dp = \sum \#\mathcal{M}(p,q)\cdot q$$

A_∞-structures:



$\Rightarrow \mu^4(-,-,-,-) =$

Biran - Cornea's Lag. Cobordism Formalism



$\subset M \times \mathbb{C}$

Def: An (exact) Lag. Cobordism $V \subset M \times \mathbb{C}$ is an exact Lag. s.t.

(1) $\pi(V) \subset \mathbb{C}$ Consist of disjoint half lines outside cpt set $K \subset \mathbb{C}$.

(2) Over the half lines, $V = L \times \mathbb{R}^{\pm}$, $L \subset M$ exact Lag.

(3) V satisfies usual constraints @ infinity bdries (cylindrical etc.)

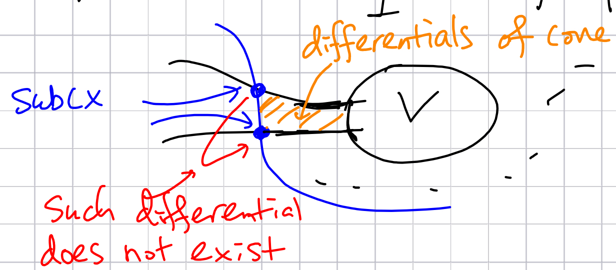
<u>Key</u> <u>Point:</u> Take $N \times \gamma_i$, $i=1,2$, compactly supp. isotopic, then

$CF(N \times \gamma_i, V)$ are chain homotopic.

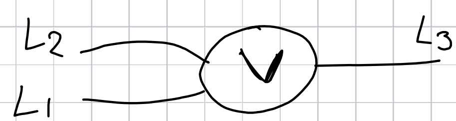
$$\begin{gathered} \| \\ \mathrm{Cone}\big(CF(N, L_1) \to CF(N, L_2) \to CF(N, L_3) \big) \end{gathered}$$

$$\mathrm{Hom}^0 \ni \left(\begin{gathered} \mathrm{Cone}(A \xrightarrow{c} B) \\ = A[1] \oplus B \end{gathered} \right) \qquad \begin{pmatrix} d_A & 0 \\ c & d_B \end{pmatrix}$$

This is a consequence of open mapping theorem:



Hence, when V looks like

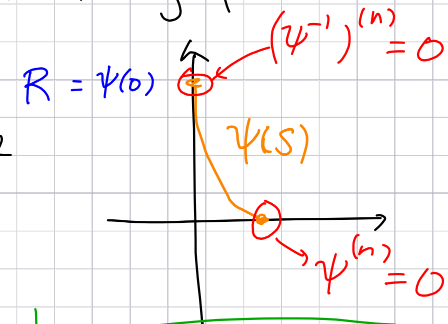


$$\Rightarrow L_3 = \mathrm{Cone}(L_1 \to L_2) \Rightarrow HF(N, L_1) \to HF(N, L_2) \to HF(N, L_3)$$

LES. $[1]$

Surgeries through geodesic flow. (generalizing Lalonde-Polterovich)

Def: (1) Let $\psi(s)$ be a function so that its graph "looks like" admissible curve.



(2) $D \subset L_1 \overset{\text{Lag}}{\subset} M$, N^*_D conormal bdle

(3) ϕ_t is the Ham flow gen by $\|p\|$

Then $\quad H_\psi := \phi_{\psi(\|p\|)}(N^*_D \setminus D)$ $\quad \boxed{\text{flow handle}}$

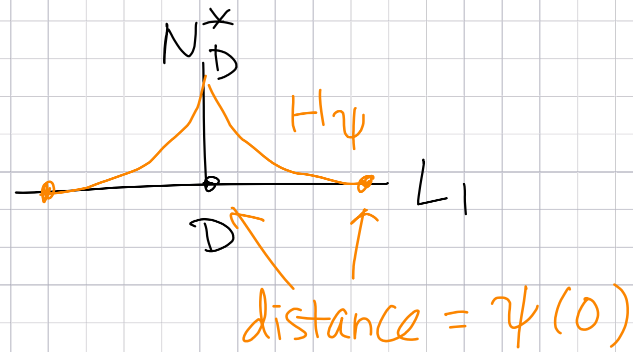
Geometric Description of ϕ_t:

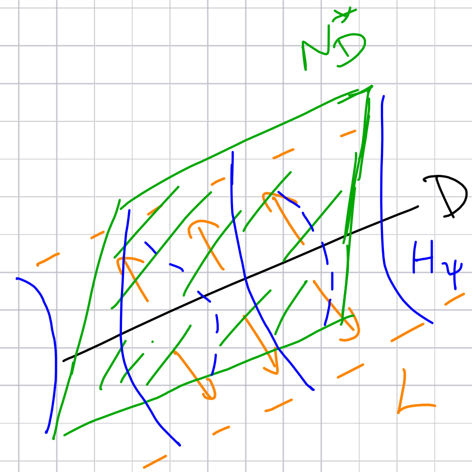
1) point (x, p) travels along direction pointed by p

2) $\phi_t(p)$ stays (dual of) tangent vector of geod.

3) $\|\phi_t(p)\| = \|p\|$

Picture of H_ψ:





Lemma: If $\text{graph}(\psi) = (b(t), a(t))$

$$\gamma(t) = (a(t), b(t)),$$

then $H_\gamma = H_\psi$

Flow handles may go beyond inj. radius!

Motivation for flow handles:

1. Easy to define on clean intersection
2. Easy to construct Lag. Cobordism
3. Easy to compare with Dehn twists

Def: (Model Dehn Twist on $S \cong S^n$)

In T^*S, consider admissible $\psi(0) = 2\pi$ ($= \pi$ when $S = S^n$) ← length of simple geod.

$\phi_{\psi(\|p\|)}$ smoothly extend to zero section

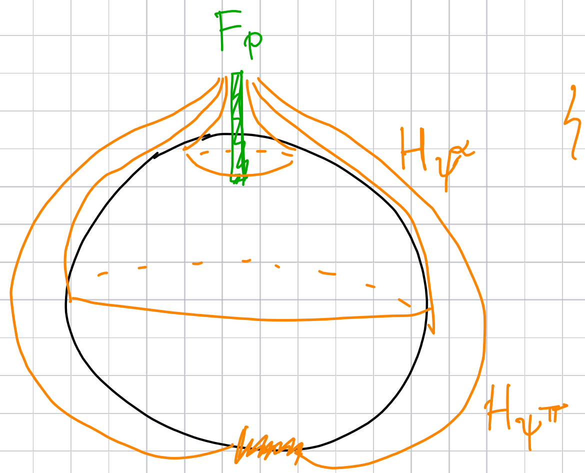
then τ_S is the extension.

ex. next page

$\Rightarrow$ flow handle is exactly the image of same flow

Example: $L = S \subset T^*S$, $S = S^n, \mathbb{RP}^n, \mathbb{CP}^n$

exotic sphere ...



$\{\psi^\alpha\}_{\alpha \in [0,\pi]}$,

$\psi^\alpha(0) = \alpha$

When $S = S^n$, $H_{\psi^\pi} = \tau_S(F_p)$

$\Rightarrow$ Dehn. twist is isotopic to a surgery.

Further Extensions to Lagrangian Surgeries

$D^m \subset L^n \overset{\text{Lag}}{\subset} M^{2n}$, $T^*L = E_1^{n-m} \oplus E_2^m$ (example: L = product mfd)

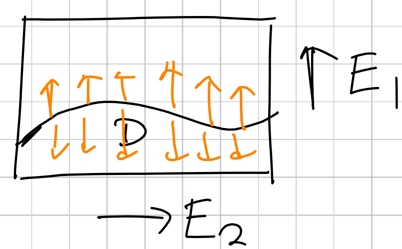
(implicitly identified with TL)

Assume $E_1|_D$ transverse to D.

$\Rightarrow$ flow handle along E_1.

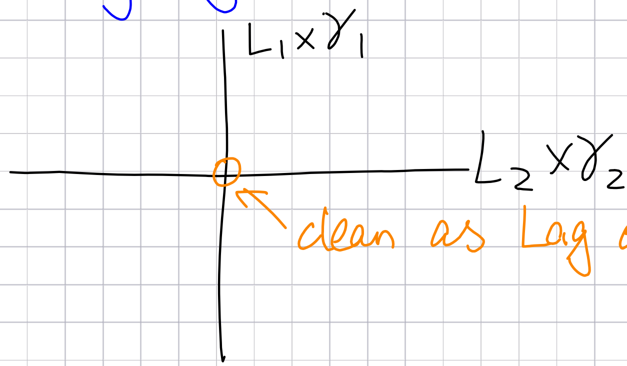
Concretely, assume $E_1 \perp E_2$, take Ham function

$$\tilde{\Psi}(\underset{\underset{L^n}{\in}}{x}, \underset{\underset{T_x^*L}{\in}}{p}) = \Psi(\|\pi_{E_1}(p)\|)$$



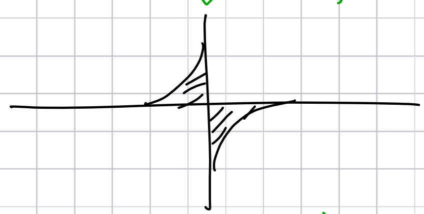
Constructing Lag. Cobordism through flow handles.

$L_1 \pitchfork L_2 = D$



clean as Lag on $M \times \mathbb{C}$

$+$ flow handle, $\mathbb{R}$-direction use product metric.

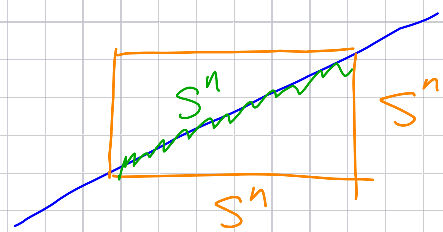


take upper half, straighten up

Biran-Cornea's trick, very adaptable.

First Main Example + Punchline

Consider $M \times \underbrace{M^-}_{(M,-\omega)}$, $L_1 = \Delta$, $L_2 = S^n \times (S^n)^-$



$T^*S^n \ni$

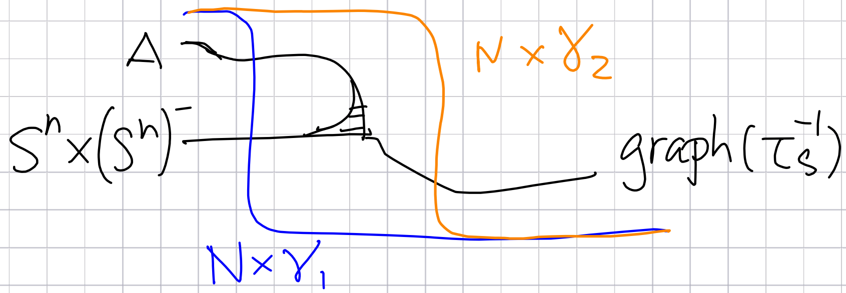
point $(\underline{(x,p)}, (x,p)) \in \Delta$ flows according to

$\Psi(\|p\|) = \Psi(\|\pi_1(p,p)\|)$ along E_1 direction.

$\Rightarrow$ image on flow handle is $((x,p), \tau_{S^n}^{-1}(x,p))$

$\Rightarrow \Delta \# (S^n \times (S^n)^-) = \operatorname{graph}(\tau_{S^n}^{-1})$

Conclusion: $\exists$ Cobordism



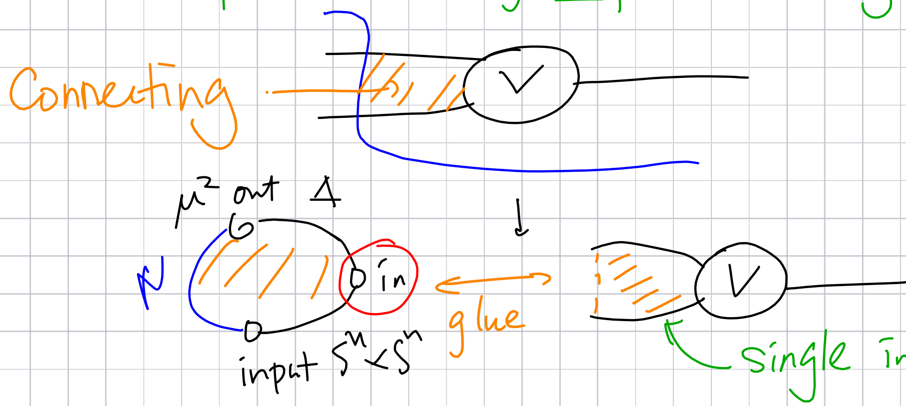
(1) $N = L_1 \times L_2^-$, $\Rightarrow$ Seidel's exact sequence.

(2) $N = \text{graph}(\phi^{-1})$: $HF(\Delta, \text{graph}(\phi)) = HF(\phi)$

$$\Rightarrow HF(S^n, \phi(S^n)) \to HF(\phi) \to HF(\phi \circ \tau_S)$$

(Appeared in early works by Seidel)

Can compute connecting map! (usually hard for cobordisms)



single int. computed by F000 Chapt 10.

☆ **Fact:** $\mathrm{Cone}(A \xrightarrow{[C]} B) \underset{q\text{-iso}}{\simeq} \mathrm{Cone}(A \xrightarrow{[t.C]} B)$

when t is invertible.

$\mathrm{rk}(HF^0(\Delta, S^n \times (S^n)^-)) = 1$! $\Rightarrow$ only need to verify connecting map $\neq 0$.

A few remarks:

1) New info: In monotone cases, Seidel's exact seq. holds for $\mathbb{Z}_2$ ($\mathbb{Z}$)-coefficient.
Before only holds for Novikov coeff.

2) Rank-1 trick recovers FOOO's Surgery exact seq:

Thm: (FOOO) $L_1 \pitchfork L_2 = \{pt\}$, then $\exists$ cone in monotone exact cases

$$L_1 \xrightarrow{[pt]} L_2 \longrightarrow L_1 \# L_2 \xrightarrow{[1]} L_1$$

3) Seidel's exact sequence for general symp. mfds:

Ongoing → i) Lag. Cobordism for general symp. mfds

FOOO ⇒ ii) Isom. $HF(\Delta, L_1 \times L_2) = HF(L_1, L_2)$

4) Also for V = spherical fibered coisotropic:

$$\begin{array}{ccccc} S^d & \longrightarrow & V^{2m+d} & \overset{i}{\hookrightarrow} & M^{2m+2d} \\ & & {\scriptstyle\pi}\downarrow & & \\ & & B^{2m} & & \end{array}$$

$\pi^*\omega_B = i^*\omega_M$

(Think of $S^n \times B \subset \overbrace{N \times B}^{M}$, $S^n \subset N$)

$\Rightarrow$ Cobordism with neg ends $-$ $\hat{V} = \{(x,y) \in V \times V \subset M \times M$
$\pi(x) = \pi(y)\}$

$- \ \Delta$

positive ends $-$ graph(τ_V^{-1})

$\Rightarrow$ Wehrheim–Woodward's family version.

Again, with better transversality freedom of WW's theory

+ issues $\Rightarrow$ family Dehn twist seq for general Symp. mfds.

5) (General remarks regarding surgeries)

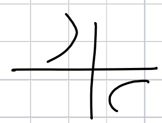
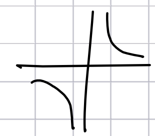
Algebraic "surgery" $\approx$ Cone of chain cx / objects:

$$\mathrm{Cone}(A \xrightarrow{c} B) = A[1] \oplus B$$

deg = 0 !

c

i) This means the only categorically meaningful Lag surgeries are @ deg = 0 intersections ($CF^0(L_1, L_2)$)

ii) Two Lag. surgeries  & 

correspond to $CF(L_1, L_2)$ and $CF(L_2, L_1)$, resp.

$\Rightarrow$ to get meaningful objects needs grading shifts

$\Rightarrow$ resolving two intersections of $\neq$ deg creates problems (obstructions on $L_1 \# L_2$)

Combining with MWW-functor (A_∞-level)

$$\Phi : Fuk(M \times M^-) \xrightarrow{\sim} Funct(Fuk(M), Fuk(M))$$

$\Rightarrow$ functor-level cone $\Rightarrow$ object cone $\left(\begin{array}{c} S^n \times S^n \to Id \to \tau_S \\ \wr \text{ Feed } L \\ CF(S^n, L) \otimes S^n \to L \to \tau_S L \end{array} \right)$

Corollary : $Symp_c(A_n)_* \subset Aut(Fuk(A_n))$

is split generated by Dehn twists along

standard spheres.

$\divideontimes$ In dim=4, $\pi_0(Symp_c(A_n))$ is generated (as group)

by Dehn twists.

slightly diff. sense.

Short Remark:

Previous example shows an instance when getting a Wehrheim-Woodward functor on A_∞-level is useful.

More generally, an easy corollary of WW's A_∞-functor

$$H^*_{Funct}(Id_{Fuk}, Id_{Fuk}) \simeq HH^*(Fuk(X)) \quad \text{(definition)}$$

$$\simeq HF^*(\Delta, \Delta) \quad \text{(WW's formalism)}$$

$$\simeq HF^*(M) \simeq QH^*(M)$$

Potentially, this is actually $Fuk^{\#}$, which apriori contains more objects.

More general Dehn twists, $\mathbb{P}^n$-objects.

Note: τ_S is def via geodesic flows. This works when:

(*) All geodesics of S are closed and have the same length.

focus.

Natural extension: $S = \mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ etc.

Question: What is the effect on $\mathcal{F}(M)$ for general τ_S?

Huybrechts-Thomas: ($\mathbb{P}^n$-objects) $\mathcal{E} \in D^b(X)$, $\mathrm{Ext}(\mathcal{E},\mathcal{E}) \simeq H^*(\mathbb{P}^n)$, $\mathcal{E} \otimes \omega = \mathcal{E}$

Expectation: A $\mathbb{P}^n$-object is the mirror of Lag. $\mathbb{C}P^n$.

Where to find $\mathbb{P}^n$-objects: HyperKähler mfds

ex 1: $\mathbb{P}^n \subset X^{2n}$, $\Rightarrow$ $\mathcal{O}_{\mathbb{P}^n}$ is $\mathbb{P}^n$-object.

ex 2: $\pi : X \to \mathbb{P}^n$ Lag. fib. of irreducible hol. symp.
$+ H^*(X, \mathcal{O}_X) \cong H^*(\mathbb{P}^n, \mathbb{C})$.

$\mathcal{E}$ $\quad \bigoplus \mathrm{Ext}^p(\mathcal{E}, \mathcal{E} \otimes \Omega^1) = \bigoplus H^p(\mathbb{P}^n, \Omega^1)$

Then $\pi^*\mathcal{E}$ is $\mathbb{P}^n$-object.

Upshot: Interesting Lag. $\mathbb{C}P^n$'s should be found in mirrors of hyperkählers. (SYZ is usually nicely behaved) (hyperkähler rotation?)

$\mathbb{P}^n$-twists: (Huybrechts–Thomas)

$$\mathcal{F} \mapsto \mathrm{Cone}\big(\, \mathrm{Cone}(\mathrm{Ext}^{*-2}(\mathcal{E},\mathcal{F}) \otimes \mathcal{E} \to \mathrm{Ext}^{*}(\mathcal{E},\mathcal{F}) \otimes \mathcal{E}) \to \mathcal{F} \big)$$

upgraded to functor cone:

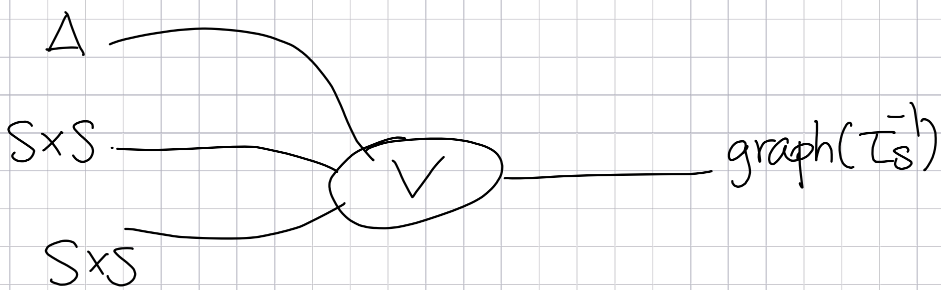
$$P_{\mathcal{E}} = \mathrm{Cone}\big(\, \mathrm{Cone}((\mathcal{E}^\vee \boxtimes \mathcal{E})[-2] \to \mathcal{E}^\vee \boxtimes \mathcal{E}) \to \mathrm{Id} \big)$$

This defines a new auto-equivalence in $D^b(X)$.

Question: Mirror to Lag $\mathbb{CP}^n$-twist ?

Theorem: (Mak–W.)

Given $S = \mathbb{C}P^n$, $\exists$ Lag. Cobordism for $M \times M^- \times \mathbb{C}$



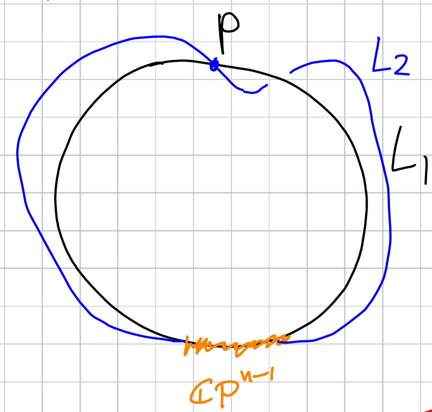
$$\Rightarrow \mathrm{Cone}\,(\begin{array}{c} S \times S[-2] \\ \updownarrow \\ \mathcal{E}^\vee \boxtimes \mathcal{E}[-2] \end{array} \to \begin{array}{c} S \times S \\ \updownarrow \\ \mathcal{E}^\vee \boxtimes \mathcal{E} \end{array} \to \begin{array}{c} \Delta \\ \updownarrow \\ \mathrm{Id} \end{array}) = \begin{array}{c} \mathrm{graph}(\tau_S^{-1}) \\ \updownarrow \\ P_{\mathcal{E}} \end{array}$$

$\Rightarrow$ matching except for connecting maps

Similar situation for $S = \mathbb{R}P^n, \mathbb{H}P^n, \mathbb{O}P^n, \ldots$

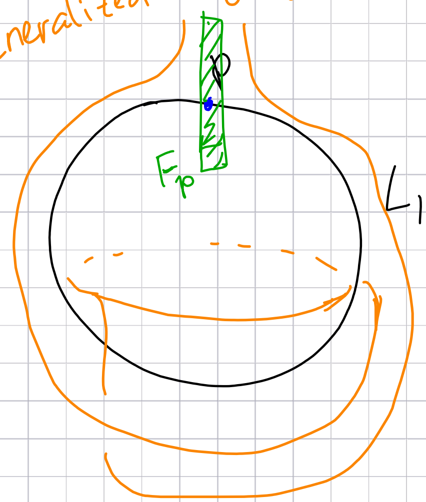
Example: $L_1 = \mathbb{CP}^n \ni \{p\}$, $L_2 = \mathrm{graph}(d(p,-))$ ← local pert. near p.

$M = T^*\mathbb{CP}^n$.

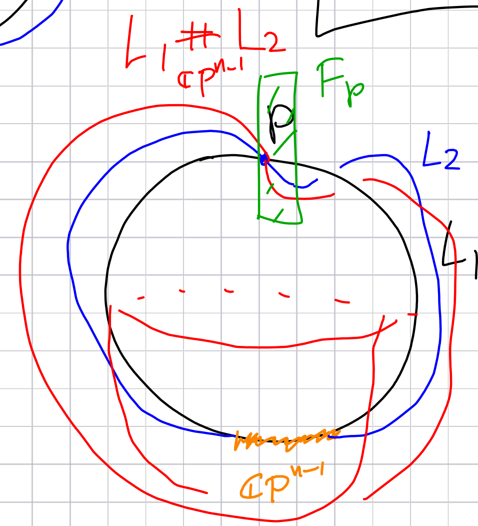


Pozniak:
clean-int.
nbh $\simeq N_D^*$

$\tau_S(F_p)$
$\|$
"generalized surgery"



handle beyond inj rad.



Claim: $F_p \#_p (\underbrace{L_1 \#_{\mathbb{CP}^{n-1}} L_2}_{S_{\mathcal{L}\to}}) = \underbrace{\text{generalized surgery of } F_p}_{= \tau_{\mathbb{CP}^n} F_p}$

Theorem: (Mak-W.) when $L \pitchfork S = \{pt\}$

$$\begin{tikzcd} S_{\mathcal{L}\to} \arrow[r] & L \arrow[d] \\ & \tau_S L \arrow[ul, "{[1]}"] \end{tikzcd}$$

* Works for all $S = \mathbb{RP}^n, \mathbb{CP}^n \ldots$

This matches Huybrechts-Thomas also for connecting maps.

When $S=\mathbb{RP}^n$, LES takes a "non-immersed" form

<u>Theorem</u>: (Mak–W.) $L \pitchfork \mathbb{RP}^n = \{pt\}$

loc. system

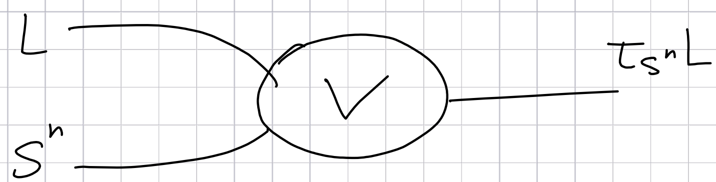
$$
\begin{tikzcd}
(\mathbb{RP}^n, \mathbb{Z}_2) \arrow[r] & L \arrow[d] \\
 & \tau_S L \arrow[ul, "{[1]}"]
\end{tikzcd}
$$

* Interestingly, all involved components can be $\mathbb{Z}$-graded

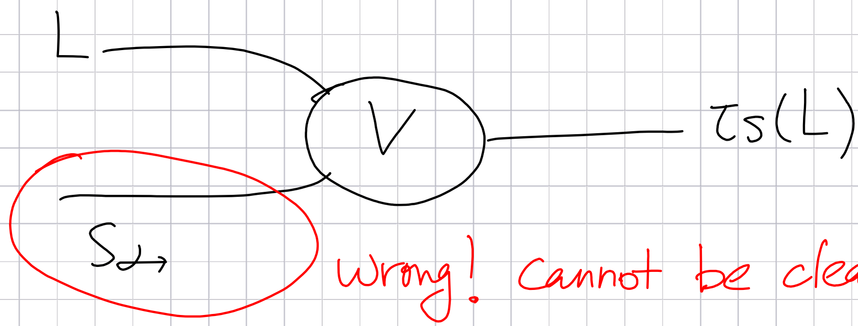
EVEN WHEN $\mathbb{RP}^n$ IS NOT!

Similar point of view appeared in Damian, Sheridan, Alston-Bao etc.

$S = S^n$, construction is straightforward:



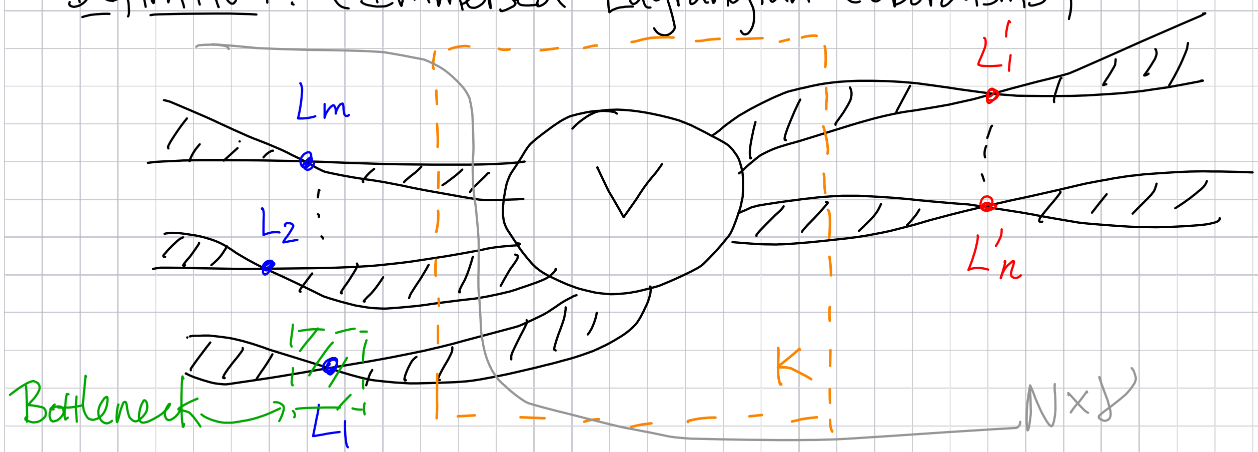
$S \neq S^n$, construction is harder:



wrong! cannot be clean self-int. !

exact!

Definition: (Immersed Lagrangian Cobordisms)



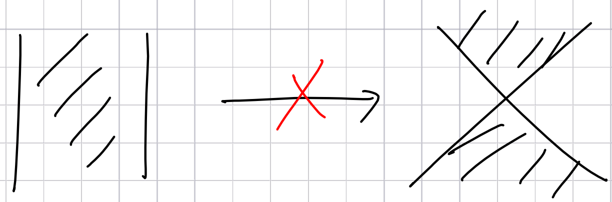
☆ Outside compact set K, each end of V should look like a bottleneck.

↳ An extension of Biran–Cornea's trick in embedded case

Lemma: $HF(N \times Y, V)$ is well-defined and invariant under choices if

1) isotopy lies inside bottleneck.
2) near bottleneck, take product cx structures.

Key point:



$\Rightarrow$ compactness

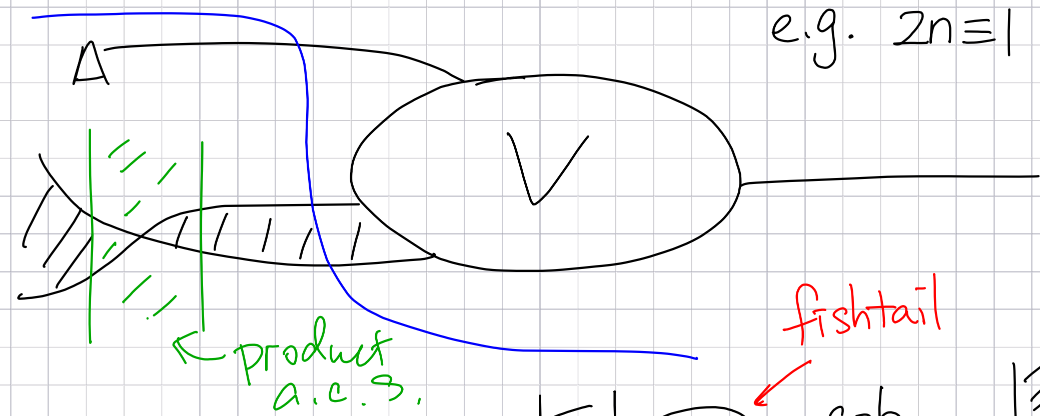
exclude bubbles

Theorem: (Mak-W.) when L_i satisfy certain deg. rest.

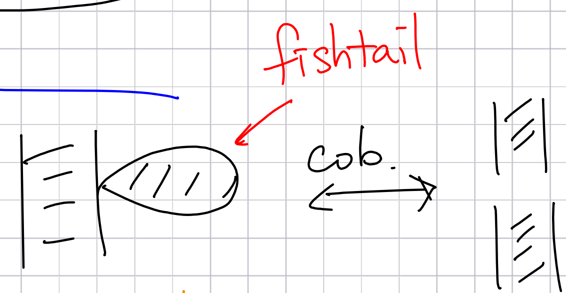
$$\mathrm{Cone}(L_1 \to L_2 \cdots \to L_m) \simeq \mathrm{Cone}(L_1' \to \cdots \to L_n')$$

Monotone Case: (no grading obstructions to fishtails, e.g. $2n \equiv 1 \bmod \underline{N}$)

min. maslov



Potential Problem:



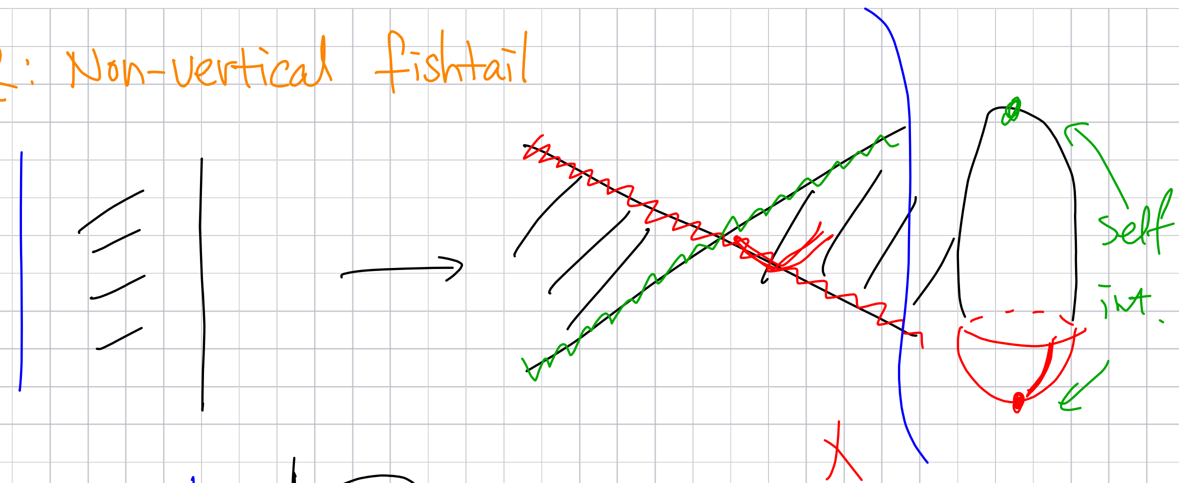
Case 1: Vertical fishtails

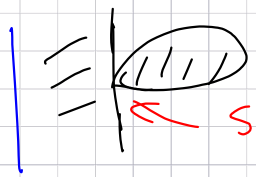
(1) exits $T^*\mathbb{C}P^n$ $\Longrightarrow$ Neck-Stretching

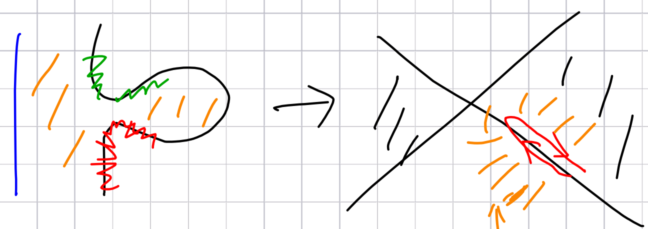
$\Longrightarrow$ high virtual dim

(2) Inside $T^*\mathbb{C}P^n$ $\Longrightarrow$ absolute index

Case 2: Non-vertical fishtail



Assume  self-int ⇒ switch branch.

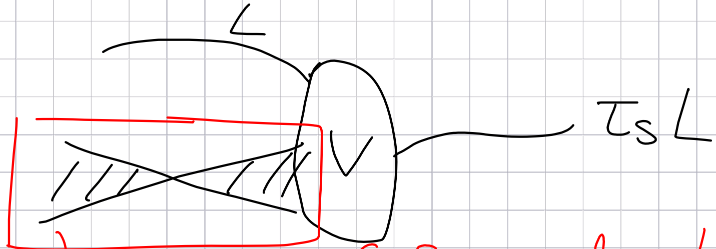
Before bubbling 

Contradiction!

⇒ No such fishtails!

More difficulties when continuation maps & chain homotopies are involved. $\not\Rightarrow$ OK for 1 intersection.

In general,



$rk(CF(L,S))$ – copies of immersed spheres

Extremely difficult Problems:

gap #1 Well-def HF (no fishtail) but NOT

Well-def continuation map (—) but NOT

gap #2 Well-def chain homotopy (—)

Without appropriate grading gaps ⇒ needs explicit counts

Thank You !!